Generalizations of the Black-Litterman Approach to Skew Normal Markets

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Abstract

Abstract. In this technical report, we extend the Black-Litterman model for the skew normal market by applying conditional value-at-risk as an alternative risk measure to obtain the optimal portfolio. Furthermore, we modify the model of the location parameter L by using the covariance matrix of the market and $L \sim N(\mu_0, \tau Cov(X))$. In this case, we introduce a non-orthogonal formulation to the skew normal case, which correlates the prior model and the views.

Illustrative examples of the approach are developed for Brazilian stock market portfolios using publicly available data of some of the major traded assets leading to a robust analysis of some the main risk indicators such as Value at Risk and the Conditional Value at Risk.

Keywords. Black-Litterman model, skew normal distribution, conditional value-at-risk, non-orthogonal formulation.

Contents

1 Introduction

In 1992, Black and Litterman [BL91] introduced a model to combine the market equilibrium with the views of the investor. It is called the Black-Litterman model (hereafter, BL model). Specifically, they used the CAPM equilibrium market portfolio as a starting point and 'reverse optimization' to generate a stable distribution of returns. Then they gave a way to specify investors' views and used Bayes' formula to blend them with the prior model to obtain a posterior distribution of the portfolio. Finally, a new optimal portfolio is obtained by using mean-variance approach. For more details on the BL model, see the survey written by Walters [Wal11].

However, the Black-Litterman model is based on many assumptions, such as the assumptions of normality to value risk, etc, and hence, questioned by many practitioners. To quantify the views involves a lot of uncertainties and errors. The parameters in this model, such as τ and Ω are still under discussion. Critics also doubt that this model depends too much on the input data and may result in a useless output. Therefore, many researchers extended the model to be more general and suitable for more types of portfolios.

Firstly, one extension is to deal with non-normally distributed markets in reality. Giacometti, Bertocchi, Rachev and Fabozzi [GBRF07] used the t-distribution or the stable distribution to model the market. Xiao and Valdez [XV13] considered the case when returns in the market fall within the class of the elliptical distribution. Meucci [Meu06] used copula-opinion pooling (COP) approach to extend the BL methodology to non-normally distributed markets and views. Blasi [Bla09] gave an example of a very simple volatility trading strategy producing skew normal returns and provided the optimization problem to embed the BL model in the skew normal market case.

Another extension is to use newly defined risk measures, such as value-at-risk (VaR) and conditional value-at-risk (CVaR). Accordingly, in the work of Giacometti, Bertocchi, Rachev and Fabozzi [GBRF07], they established a frame work by applying the t and the stable distribution for asset returns. For each case, they gave the formulas of using value-at-risk and conditional value-at-risk as risk measures. Correspondingly, Meucci [Meu06] minimized the CVaR subject to the constraint of a minimum target expected return.

Finally, many researchers have investigated to embed other models in the BL model. Beach and Orlov [BO07] used GARCH-derived views as an input into the Black-Litterman model. Fabozzi, Focardi and Kolm [FFK06] used cross-sectional momentum strategy as an input view and combine the strategy with the Black-Litterman model. Ogliaro, Rice, Becker and de Carvalho [ORBdC12] introduced a non-orthogonal approach into the BL model. By using a matrix Γ, they characterized the relation between the prior model and the views.

Inspired by these works, in this report, we will explore further along the three directions. Specifically, in Section 2, we will consider the BL model in skew-normal market. We firstly follow the same idea in [Bla08] to model the market, however, we provide another different way to model the location parameter L. Instead of using the scale matrix, we use the covariance matrix of the market and $L \sim N(\mu_0, \tau Cov(X))$. In Subsection 2.2, we convert the views on the expected returns to the location parameters. Correspondingly, we obtain two posterior models based on how we model the location parameter L. Moreover, for the second model, we apply a matrix Γ as in [ORBdC12] to correlate the prior model and the view. In this way, we introduce the non-orthogonal approach into the skew-normal market. We call the two models the extended Black-Litterman model 1 and 2 , in brief, EBL1 and EBL2. Section 3 serves two purposes, we will use the data of 8 stocks in the Brazilian stock market, BM&F Bovespa, to illustrate how our model works and confirm our insight. Finally, conclusions and more future work will be discussed.

2 The Black-Litterman Model in Skew Normal Markets

2.1 The Market

Consider a portfolio of N securities or asset classes, whose returns have a multivariate skew normal distribution:

$$X|_{L=\mu} \sim SN(\mu, \Sigma, \alpha), \tag{1}$$

where μ is the location parameter, and is considered to be random and normally distributed with $\mu \sim N(\mu_0, \Sigma_0)$. Note that Σ is a positive definite matrix and α is the shape parameter. For more properties of the skew normal family, see [Azz05], [AC99] and [ADV96].

From the work of Blasi [Bla08] and [Bla09], we have the following lemma:

Lemma 2.1 *(Blasi [Bla08] and [Bla09]) If the returns of assets follow a multivariate skew normal distribution as (1), and the location parameter has a normal distribution, which is,*

$$L \sim N(\mu_0, \Sigma_0). \tag{2}$$

Then the marginal density function of X is:

$$f_X(x) = 2\varphi(x; \mu_0, \Sigma + \Sigma_0)\Phi\big(\alpha'\sigma_1^{-1}(x - \mu_0)\big), \tag{3}$$

where

$$\begin{aligned}
\alpha &= \alpha_L'\sigma^{-1}\Sigma(\Sigma + \Sigma_0)^{-1}(1 + \alpha_1'\overline{\Delta}\alpha_1)^{-1/2}\sigma_1 \\
\Delta &= (\Sigma^{-1} + \Sigma_0^{-1})^{-1} \\
\overline{\Delta} &= d^{-1}\Delta d^{-1} \\
\alpha_1 &= -\alpha_L\sigma^{-1}d,
\end{aligned}$$

and d is the diagonal matrix of standard deviations of Δ, and σ_1 is the diagonal matrix of standard deviations of $\Sigma + \Sigma_0$.

In particular, if $\Sigma_0 = \tau\Sigma$*, where* $\tau \in (0,1]$*, we have*

$$L \sim N(\mu_0, \tau\Sigma). \tag{4}$$

We can simplify (3) and yield:

$$X \quad \sim \quad SN\big(\mu_0, (1+\tau)\Sigma, \alpha\big), \tag{5}$$

where $\alpha = \frac{\frac{\alpha_L}{\sqrt{1+\tau}}}{\sqrt{1+\frac{\tau}{1+\tau}\alpha_L'\overline{\Sigma}\alpha_L}}$.

Note that the positive definite matrix Σ is not the covariance matrix of X, which is equal to

$$Cov(X) = \Sigma - \frac{2}{\pi}(\sigma\tilde{\alpha})(\sigma\tilde{\alpha})', \tag{6}$$

where σ is the diagonal matrix of standard deviations of Σ and

$$\tilde{\alpha} = \frac{\overline{\Sigma}\alpha}{\sqrt{1+\alpha'\overline{\Sigma}\alpha}}, \ \overline{\Sigma} = \sigma^{-1}\Sigma\sigma^{-1}.$$

Naturally, we can alternatively model the covariance matrix of location as $\tau \cdot Cov(X)$, such that

$$L \sim N\big(\mu_0, \tau Cov(X)\big). \tag{7}$$

Substituting the assumption (7) to Lemma 2.1, the marginal density function of X is

$$f_X(x) = 2\varphi\big(x; \mu_0, \Sigma + \tau Cov(X)\big)\Phi\big(\alpha'\sigma_1^{-1}(x-\mu_0)\big), \tag{8}$$

where

$$\begin{aligned}
\alpha &= \alpha_L'\sigma^{-1}\Sigma\big[\Sigma + \tau Cov(X)\big]^{-1}(1+\alpha_1'\overline{\Delta}\alpha_1)^{-1/2}\sigma_1 \\
\Delta &= \Big[\Sigma^{-1} + \big[\tau Cov(X)\big]^{-1}\Big]^{-1} \\
\overline{\Delta} &= d^{-1}\Delta d^{-1} \\
\alpha_1 &= -\alpha_L\sigma^{-1}d,
\end{aligned}$$

and d is the diagonal matrix of standard deviations of Δ, and σ_1 is the diagonal matrix of standard deviations of $\Sigma + \tau Cov(X)$.

2.2 The Views

In order to put the views in the prior distribution of asset returns, we use the same technique with the standard BL model. That is, given the expected returns of the assets, the views expressed on the expected returns are normally distributed:

$$V|_{E(X)} \sim N(v, \Omega). \tag{9}$$

However in this model, the distribution of X is conditioned on the location parameter L described in Section 2.1. We, therefore, try to 'move' our views from the expectations of X to L. This is an alternative way to connect the market and the views. Note we have

$$E(X) = L + \sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}.$$

Suppose P is the $K \times N$ pick matrix: the k-th row of the pick matrix determines the weights of the k-th view. That is, $P = (P_1, P_2, \ldots, P_K)'$, where P_i is a $1 \times N$ matrix. So (9) is equivalent with

$$V|_{L=\mu} \sim N(P\Big(\mu + \sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}\Big), \Omega). \tag{10}$$

2.3 The Posterior

2.3.1 Model One of $L|V$

Let f_L be the density functions of L satisfied (4) and $f_{V|_{L=\mu}}$ is the density function of $V|_{L=\mu}$ satisfied (10). Based on Bayes' rule and and obtain

$$\begin{aligned} f_{L|V}(\mu|v) &\propto f_{V|\mu}(v|\mu)f_L(\mu) \\ &\propto |\tau\Sigma|^{\frac{1}{2}}|\Omega|^{\frac{1}{2}} \cdot \\ &\quad e^{-\frac{1}{2}\left[(\mu-\mu_0)'(\tau\Sigma)^{-1}(\mu-\mu_0)+\left(v-P\left(\mu+\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}\right)\right)'\Omega^{-1}\left(v-P\left(\mu+\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}\right)\right)\right]} \\ &\propto |(\tau\Sigma)^{-1} + P'\Omega^{-1}P|^{\frac{1}{2}} \cdot \\ &\quad e^{-\frac{1}{2}(\mu-\mu^L_{BL})'((\tau\Sigma)^{-1}+P'\Omega^{-1}P)(\mu-\mu^L_{BL})}, \end{aligned} \tag{11}$$

where

$$
\begin{aligned}
\mu_{BL}^L &= \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\mu_0 + P'\Omega^{-1}\left(v - P\left(\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}\right)\right)\right] \\
&= \mu_0 + (\tau\Sigma)P'\left[P(\tau\Sigma)P' + \Omega\right]^{-1}\left[v - P\left(\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}\right) - P\mu_0\right], \qquad (12)
\end{aligned}
$$

with the covariance matrix

$$
\begin{aligned}
\Sigma_{BL}^L &= \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1} \qquad (13) \\
&= (\tau\Sigma) - (\tau\Sigma)P'\big(P(\tau\Sigma)P' + \Omega\big)^{-1}P(\tau\Sigma). \qquad (14)
\end{aligned}
$$

Eventually, the posterior distribution of locations given the views is a normal distribution similar with the standard BL model:

$$
L|V \sim N(\mu_{BL}^L, \Sigma_{BL}^L) \qquad (15)
$$

2.3.2 Model Two of $L|V$

As is discussed in Section 2.1, an alternative way to model the location parameter L is to use the covariance matrix $Cov(X)$ of X as in (7). The views also follow the distribution as in (10). Moreover, as in the working paper of Ogliaro, Rice, Becker and de Carvalho [ORBdC12], we follow the same idea of the theorem in [ORBdC12] and suppose the prior model and the views are correlated by a matrix Γ. Then the posterior density of $L|V$ is a normal distribution $N(\mu_{BL}^L, \Sigma_{BL}^L)$, where

$$
\begin{aligned}
\Sigma_{BL}^L &= \left[\Sigma_0^{-1} + \big(P - \Gamma\Sigma_0^{-1}\big)'\big(\Omega - \Gamma\Sigma_0^{-1}\Gamma'\big)^{-1} \cdot \big(P - \Gamma\Sigma_0^{-1}\big)\right]^{-1} \qquad (16) \\
\mu_{BL}^L &= \Sigma_{BL}^L \cdot \left[\Sigma_0^{-1}\mu_0 + \big(P - \Gamma\Sigma_0^{-1}\big)'\big(\Omega - \Gamma\Sigma_0^{-1}\Gamma'\big)^{-1} \cdot \right. \\
&\qquad \left. \left(v - P\left(\sigma\tilde{\alpha}\sqrt{\frac{2}{\pi}}\right) - \Gamma\Sigma_0^{-1}\mu_0\right)\right] \qquad (17)
\end{aligned}
$$

In [ORBdC12], the authors define $\Gamma = \gamma\hat{\Gamma}$, where $\hat{\Gamma}$ is calculated through the data following a certain rule. They introduced an index forecast error to calculate γ through an optimization model. However, the model cannot guarantee that the optimal γ is always valid. By modifying the model, we provide two methods rather than in paper [ORBdC12].

For the first one, denote Υ as the correlation matrix of each asset with the portfolio formed by each view. Namely,

$$\Upsilon_{ij} = corr\big(X_j, P_i(X_1, X_2, \ldots, X_N)'\big).$$

By multiplying the matrices of standard deviations, we have

$$\Gamma = \gamma \times \Big(diag(\Omega)\Big)^{\frac{1}{2}} \Upsilon \Big(diag(Cov(X))\Big)^{\frac{1}{2}},$$

where $\gamma \geq 0$ is a parameter specified by users to make the matrix

$$\begin{pmatrix} \tau Cov(X) & \Gamma' \\ \Gamma & \Omega \end{pmatrix}$$

positive definite and can be viewed as a covariance matrix. One obvious drawback is the correlation we calculate is Pearson's correlation, this cannot characterize the correlation precisely when the data is skewed. We can calculate the Spearman's or Kendall's covariance instead. But in any way, the magnitude of every row of Γ is small. When γ gets larger, the views and the prior model are more correlated.

The alternative method is simple. We set $\Gamma = \gamma P \Sigma_0$, where $\gamma \in [0, 1]$.

For this model, if we suppose $\Gamma = \gamma P \Sigma_0$, we have

$$\begin{aligned}
\Sigma_{BL}^{L} &= \Big[\Sigma_0^{-1} + \big(P - \Gamma\Sigma_0^{-1}\big)'\big(\Omega - \Gamma\Sigma_0^{-1}\Gamma'\big)^{-1}\big(P - \Gamma\Sigma_0^{-1}\big)\Big]^{-1} \\
&= \Sigma_0 - (P\Sigma_0 - \Gamma)'\big[(P\Sigma_0 - \Gamma)(P - \Gamma\Sigma_0^{-1})' + \Omega - \Gamma\Sigma_0^{-1}\Gamma'\big]^{-1}(P\Sigma_0 - \Gamma) \\
&= \Sigma_0 - (P\Sigma_0 - \Gamma)'(P\Sigma_0 P' - P\Gamma' - \Gamma P' + \Omega)^{-1}(P\Sigma_0 - \Gamma) \\
&= \Sigma_0 - (1-\gamma)^2 \Sigma_0 P'\big[(1 - 2\gamma)P\Sigma_0 P' + \Omega\big]^{-1} P\Sigma_0 \qquad (18)
\end{aligned}$$

Equation (18) is because we substitute $\Gamma = \gamma P \Sigma_0$. When we choose γ such that $(1 - 2\gamma)P\Sigma P' + \Omega$ is close to a zero matrix, the inverse of it would get very large, and hence, $\det(\Sigma_{BL}^{L})$ goes to infinity. If we assume $\gamma \neq 1$ and $\Omega = P\Sigma_0 P'$, for this special case we have

$$\begin{aligned}
\Sigma_{BL}^{L} &= \Sigma_0 - \Sigma_0 P'\Big[\frac{1-2\gamma}{(1-\gamma)^2} P\Sigma_0 P' + \frac{1}{(1-\gamma)^2}\Omega\Big]^{-1} P\Sigma_0 \\
&= \Sigma_0 - \Sigma_0 P'\Big[P\Sigma_0 P' + \frac{1+\gamma}{1-\gamma} P\Sigma_0 P'\Big]^{-1} P\Sigma_0 \qquad (19)
\end{aligned}$$

We notice that the function

$$f(\gamma) = \frac{1+\gamma}{1-\gamma}$$

is an increasing function and $f(0) = 1$. Let

$$\Omega_1 = \frac{1+\gamma}{1-\gamma} P \Sigma_0 P'.$$

From Equation (19) and comparing with Equation (14), we will analyse in three cases about the effects caused by changing γ.

- when $0 < \gamma < 1$, we have $1 < \frac{1+\gamma}{1-\gamma}$. So when γ tends to 1, it has the same effect with changing Ω to diminish the confidence.
- when $-1 < \gamma < 0$, we have $0 < \frac{1+\gamma}{1-\gamma} < 1$, and therefore, this has the same effect with enhancing the confidence of the views.
- Clearly, when γ tends to zero, (16) tends to (13) and (17) tends to (12) with $\tau\Sigma$ replaced by Σ_0.

2.3.3 The Posterior Model for X

By substituting Equation (12) and (13), or (16) and (17) to Lemma 2.1 we get:

$$f_{X|V}(x|v) = 2\phi(x; \mu^L_{BL}, \Sigma + \Sigma^L_{BL}) \Phi\big(\alpha'_{BL} \sigma^{-1}_{BL} (x - \mu^L_{BL})\big) \tag{20}$$

where σ_{BL} is the diagonal matrix of the standard deviations of $\Sigma + \Sigma^L_{BL}$. To make it satisfy the form of a multivariate skew normal distribution, the parameter α_{BL} is given by:

$$\begin{aligned}
\alpha'_{BL}\sigma^{-1}_{BL} &= \alpha' \sigma^{-1} \Sigma \big(\Sigma + \Sigma^L_{BL}\big)^{-1} \big(1 + \alpha'_\Delta \overline{\Delta}_{BL} \alpha_\Delta\big)^{-1/2} \\
\alpha'_\Delta &= -\alpha' \sigma^{-1} d_{BL} \\
\Delta_{BL} &= \Big[\Sigma^{-1} + \big(\Sigma^L_{BL}\big)^{-1}\Big]^{-1},
\end{aligned}$$

where $\overline{\Delta}_{BL}$ is the correlation matrix of Δ_{BL} and d_{BL} is the diagonal matrix of standard deviations of Δ_{BL}. Obviously, we have $d_{BL}\overline{\Delta}_{BL} d_{BL} = \Delta_{BL}$.

Therefore, Equation (20) is a multivariate skew normal distribution density function. We have

$$X|V \sim SN(\mu^L_{BL}, \Sigma + \Sigma^L_{BL}, \alpha_{BL}). \tag{21}$$

2.4 The Allocation

Since the posterior distribution is also a multivariate skew normal distribution, as is shown in (21), we cannot use the mean-variance optimization or CAPM freely. However, what we concern is the riskiness, especially the loss in some extreme scenarios, such as economic recession or financial crisis; therefore we choose to use CVaR to measure the risk.

In the paper by Rockafellar and Uryasev [RU00], they considered minimizing CVaR, with a given expected return. The model can also be found in Pflug [Pfl00]. Suppose we have a portfolio with N assets and $X = (X_1, \ldots, X_N)'$ is the return vector of X. Let $w = (w_1, w_2, \ldots, w_N)'$ be the weight vector. Then we have the return $R(w, X)$ of the portfolio as a function of w and X. Let $\mathrm{VaR}_\beta(w, X)$ and $\mathrm{CVaR}_\beta(w, X)$ be the value-at-risk and conditional value-at-risk of the portfolio. Let $R_{S\times N} = (R_1, R_2, \ldots, R_S)$ be the panel data with S simulated joint scenarios of returns and R_i, $i = 1, 2, \ldots, S$ be vectors of N-dimension. $\hat{R}_{1\times N} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_N)'$ is a vector of mean values of all the assets, or more generally, the expected returns of the assets. Denote $d_i = \max\{-w'R_i - \mathrm{VaR}_\beta(w, X), 0\}$, $i = 1, 2, \ldots, S$ and $R_{\min}$ is the target return. Given a target expected return, we minimize the CVaR to obtain the optimal portfolio.

The model can be formalized in the following way:

$$\begin{aligned}
\min_w \; & CVaR_\beta(w, X) = \mathrm{VaR}_\beta(w, X) + \frac{1}{S(1-\beta)} \sum_{n=1}^{S} \max\{-w'R_n - \mathrm{VaR}_\beta(w, X), 0\} \\
s.t. \; & d_i \geq -w'R_i - \mathrm{VaR}_\beta(w, X), i = 1, 2, \ldots, S \\
& w'\hat{R}_{1\times N} \geq R_{\min} \\
& \sum_{j}^{N} w_j = 1 \\
& w_j \geq 0, j = 1, 2, \ldots, N. \\
& d_i \geq 0, i = 1, 2, \ldots, S
\end{aligned} \tag{22}$$

3 Implementation

Based on the analysis, we call the Black-Litterman model using the skew normal distribution to fit the data and CVaR portfolio optimization to choose the optimal portfolio as an *extended Black-Litterman model* (EBL model). Specifically, using model 1 and 2 for the location parameter is called EBL1 and EBL2, respectively. The data is comprised of eight stocks in BM&F Bovespa. These stocks are contained in the Ibovespa index and are mid-large cap. We will perform an initial check for evidence of the skewness in the data. Then, we express the same views on returns and proceed with the study of the two models.

3.1 The Data

To implement our model, we select eight stocks in BM&F Bovespa, ITUB4, PETR4, VALE5, BRFS3, ITSA4, BBAS3, GGBR4, EMBR3. These stocks are contained in the Ibovespa index and all together make up a great percentage. We summarize some information of the stocks in Table 1.

Table 1: The information of the stocks

Code	Sector	Part. %
ITUB4	Financial / Financial Intermediaries	7.036%
PETR4	Oil, Gas and Biofuels	7.820%
VALE5	Basic Materials Mining	8.278%
BRFS3	Consumer Non Cyclical / Food Processors	2.292%
ITSA4	Financial / Financial Intermediaries	2.869%
BBAS3	Financial / Financial Intermediaries	2.599%
GGBR4	Basic Materials / Steel and Metalurgy	1.994%
EMBR3	Capital Goods and Services / Transportation Equipment and Co	1.355%

The data series starts from 01/01/2004 to 01/01/2014, with weekly observations. This time period includes the 2008 finance crisis, the European debt crisis and also other events that may affect the Brazilian market. Short-term trading is always conducted in stock market or futures market. We calculate the compound 5-day returns of every stock. The statistic characteristics of the data are shown in Table 2, where we can see that BRFS3 has the biggest mean value and positive skewness. This means BRFS3 has more extreme values to the right. On the contrary, GGBR4 has the biggest negative skewness and the largest kurtosis and GGBR4 has more values on the left tail, which can be proved in the following plots.

Table 2: The statistic characteristics of the data

Variable	Mean$\times 10^{-3}$	Std Dev	Skewness	Kurtosis	Min & Max
ITUB4	2.2	0.052	0.07	4.65	-0.27 & 0.29
PETR4	2.6	0.040	-0.14	1.46	-0.17 & 0.21
VALE5	2.2	0.047	-0.11	1.17	-0.17 & 0.16
BRFS3	4.8	0.097	0.43	8.14	-0.41 & 0.57
ITSA4	2.0	0.052	0.04	7.01	-0.29 & 0.34
BBAS3	1.9	0.054	-0.24	2.85	-0.31 & 0.23
GGBR4	0.5	0.065	-1.42	10.61	-0.55 & 0.26
EMBR3	0.2	0.050	-0.34	3.46	-0.29 & 0.17

From the statistics of the returns, we make an insight that the distribution of the data is skewed and has a fat tail. Take GGBR4 as an example, the histogram of the returns of GGBR4, Figure 1, has a left fat tail. We fit the normal and the skew normal distributions to the data, respectively. Note that in Figure 1 the solid line represents the skew normal density function, whereas the dotted line is fitted by a normal density. Figure 2 is the QQ-plot. Almost all the points in the plot for the skew normal distribution are on a straight line.

Finally, based on the results of MLE in Table 3, we can conclude that GGBR4 is skew normally distributed.

Table 3: The test of normality

	likelihood ratio test (test.normality)
LRT	33.48
p-value	0

For all the 8 stocks, we plot the Gaussian CVaR and modified CVaR, and the latter is calculated using higher moments. In Figure 3, it shows that for each stock, the Gaussian CVaR and modified CVaR are quite different especially when the confidence-level gets higher. This means that the higher moments cannot be ignored. We, therefore, fit the data to a skew normal distribution. The MLE estimations for 8 stocks are listed in Table 4.

Table 4: The MLE estimations for 8 stocks

	ITUB4	PETR4	VALE5	BRFS3	ITSA4	BBAS3	GGBR4	EMBR3
$\mu_0 \times 10^{-3}$	16.8	7.7	22.2	-19.3	14.7	15.6	54.1	19.3
α	-0.16	0.58	-0.12	0.58	0.47	-0.03	-2.10	-0.30

The matrix of Σ is

$$1000 \times \begin{pmatrix} 2.8 & 1.2 & 1.4 & 0.3 & 2.5 & 2.0 & 2.5 & 1.0 \\ & 2.3 & 1.3 & 0.2 & 1.3 & 1.3 & 1.8 & 0.6 \\ & & 2.5 & 0.2 & 1.3 & 1.3 & 3.1 & 1.1 \\ & & & 9.9 & 0.4 & 0.6 & -0.3 & 0.0 \\ & & & & 2.7 & 1.9 & 2.4 & 1.0 \\ & & & & & 3.2 & 2.4 & 1.0 \\ & & & & & & 7.1 & 2.1 \\ & & & & & & & 2.7 \end{pmatrix}$$

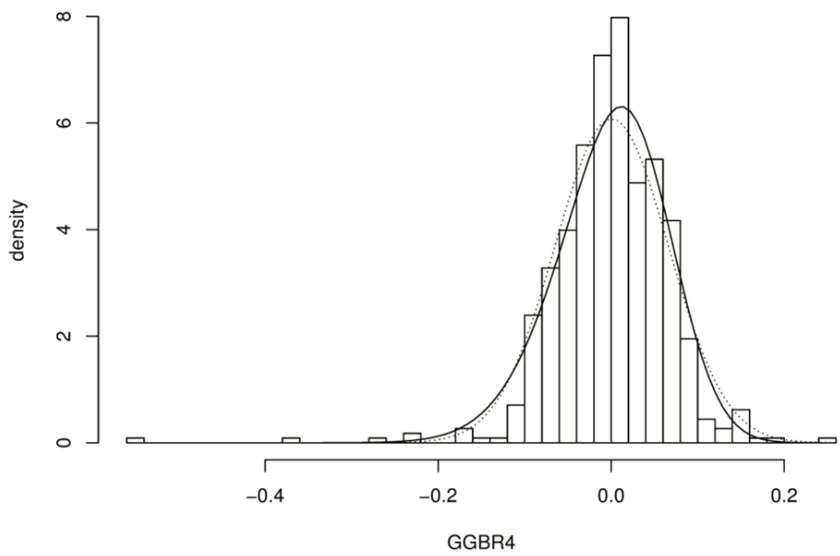

Figure 1: Fit the normal and the skew normal distribution to the compound returns of GGBR4.

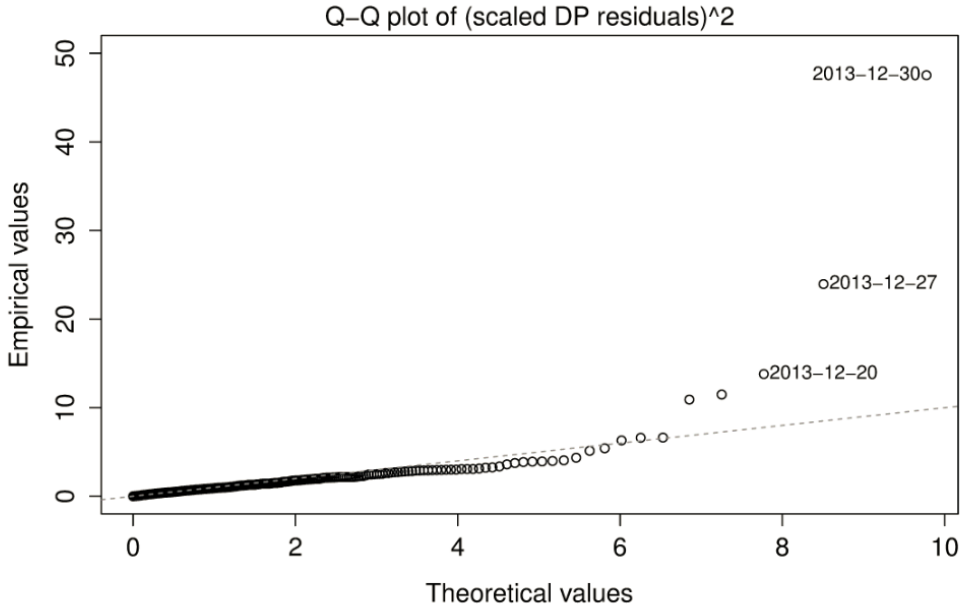

Figure 2: QQ-plot for GGBR4

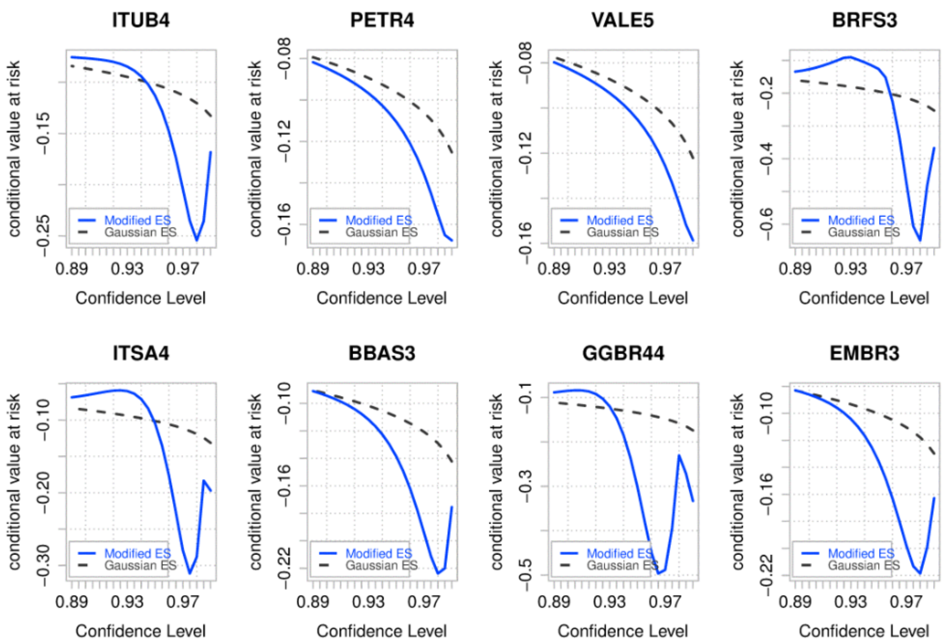

Figure 3: Conditional Value-at-Risk with $\beta \in [0.9, 0.99]$

3.2 Methodology

For this particular problem, we will process the EBL1 and EBL2 model, respectively. We set the risk confidence $\beta = 0.95$. To avoid the allocation focuses on some certain assets, we constrain $w_i \in [5\%, 30\%]$, $i = 1, 2, \ldots, 8$. Next, we blend our views with the two models respectively. Assume that our views are:

> 'GGBR4 will have a weekly return of 1%'

and

> 'BBAS3 will outperform ITUB4 by 2%'.

Therefore the pick matrix reads

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Accordingly, the views vector becomes $V = (1\%, 2\%)'$, and the confidence matrix of the views is

$$\Omega = \begin{pmatrix} 0.03^2 & 0 \\ 0 & 0.05^2 \end{pmatrix}.$$

Hereafter, we denote the results of the two models by the following symbols:

- μ^1_{EBL}, α^1_{EBL} and Σ^1_{EBL}: parameters of the posterior distribution of the EBL1 model;
- μ^2_{EBL}, α^2_{EBL} and Σ^2_{EBL}: parameters of the posterior distribution of the EBL2 model;

3.3 The Results

3.3.1 EBL1

The prior model is obtained directly in Figure 4. We blend our views and calculate the posterior distribution and the optimal portfolio. The parameters of the posterior distribution are shown in Table 5.

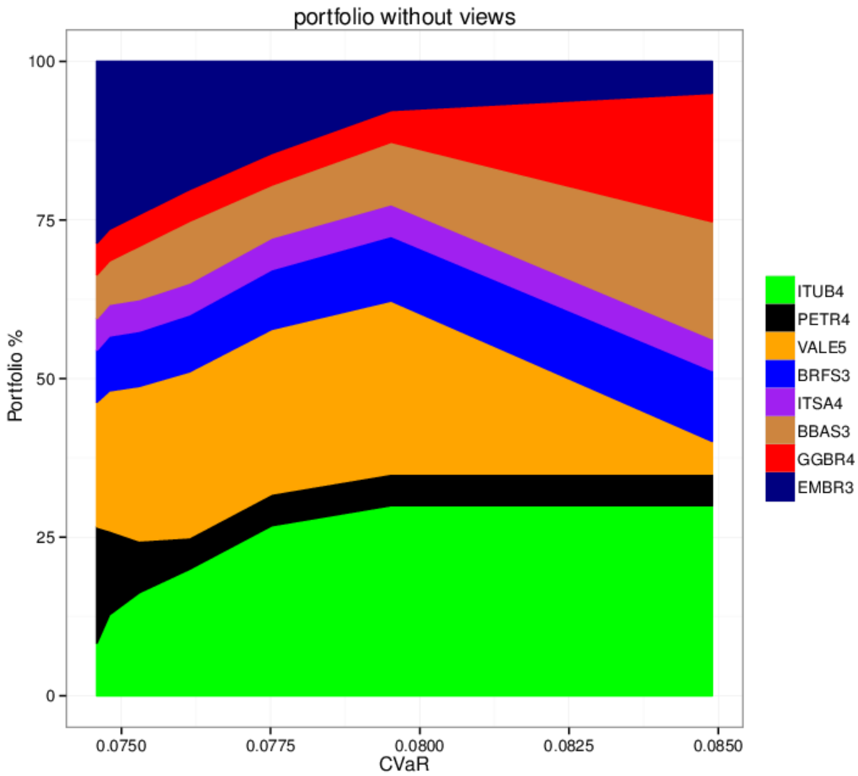

Figure 4: The transition map of the prior model

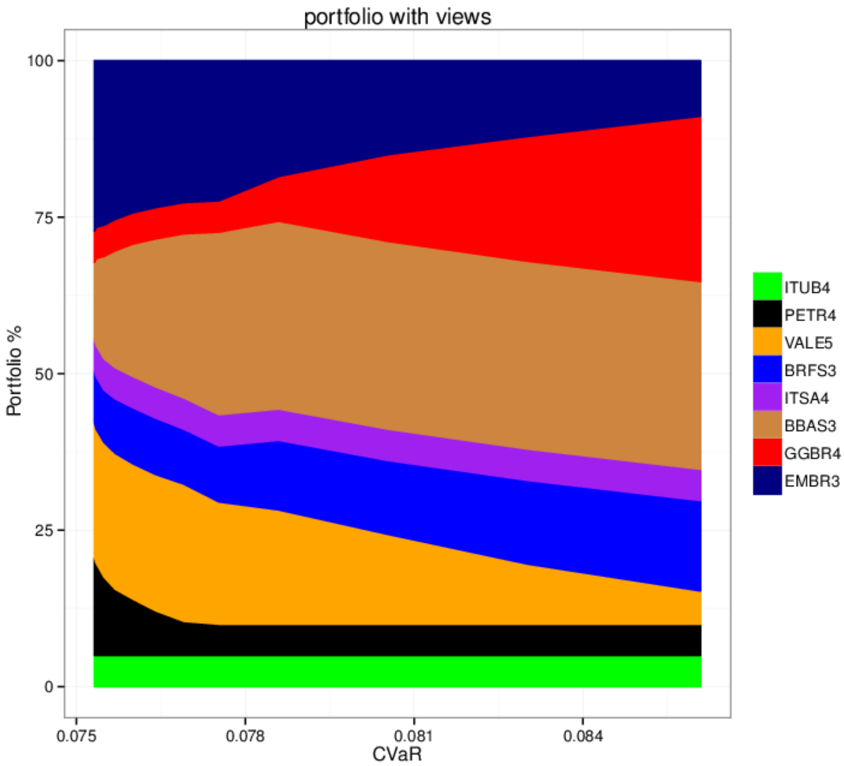

Figure 5: The transition map for EBL model

3.3.2 EBL2

The prior model of EBL2 is the same with EBL1. Likewise, the posterior model has the parameters in Table 5.

$$L \sim N(\mu_0, \tau Cov(X)).$$

Table 5: Posterior model parameters

	$\mu^1_{EBL} \cdot 10^{-3}$	α^1_{EBL}	$diag(\Sigma^1_{EBL}) \cdot 10^{-3}$	$\mu^2_{EBL} \cdot 10^{-3}$	α^2_{EBL}	$diag(\Sigma^2_{EBL}) \cdot 10^{-3}$
ITUB4	18.1	-0.12	3.3	17.9	-0.13	3.3
PETR4	9.3	0.44	2.6	9.6	0.49	2.8
VALE5	25.0	-0.09	3.0	24.6	-0.10	3.0
BRFS3	-19.1	0.43	12.4	-17.4	0.48	12.2
ITSA4	16.4	0.34	3.2	16.4	0.38	3.2
BBAS3	20.3	-0.02	3.7	20.0	-0.02	3.7
GGBR4	61.2	-1.67	7.7	59.9	-1.77	7.6
EMBR3	21.6	-0.21	3.3	21.0	-0.24	3.3

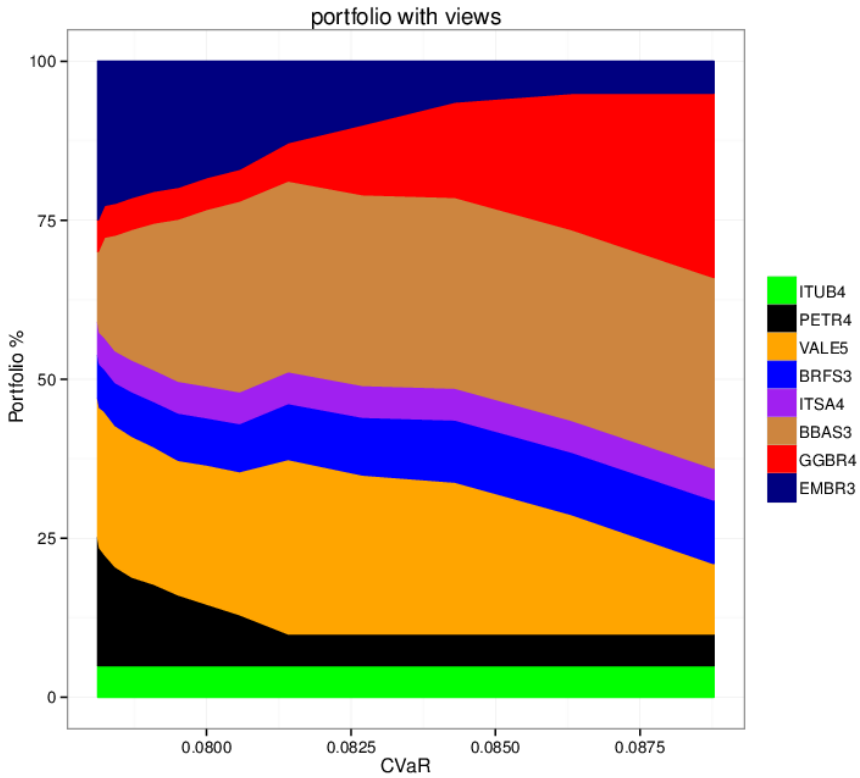

Figure 6: The transition map of the alternative approach

Now we will perturb the views a little in two ways. First, we want our views to have a little correlation with the prior model. We choose the first method to assign Γ. By setting

$\gamma = 0.2$, we have

$$\Gamma = 10^{-5} \times \begin{pmatrix} 16.67 & 14.38 & 17.51 & 8.56 & 16.86 & 15.01 & 39.36 & 9.99 \\ -18.64 & -3.78 & -2.94 & 6.47 & -12.67 & 26.32 & -4.03 & 1.37 \end{pmatrix}.$$

On the other hand, if we are less confident on our views, and we want to depend on the prior model mostly, we set γ close to 1. We use the second method with $\gamma = 0.95$, and we have

$$\Gamma = 10^{-5} \times \begin{pmatrix} 44.65 & 38.51 & 46.89 & 22.93 & 45.17 & 40.21 & 105.43 & 26.76 \\ -20.53 & -4.16 & -3.23 & 7.13 & -13.95 & 29.00 & -4.44 & 1.51 \end{pmatrix}$$

With the same method, we set $\gamma = -0.3$,

$$\Gamma = 10^{-5} \times \begin{pmatrix} -9.07 & -8.58 & -8.87 & -5.44 & -8.96 & -8.25 & -17.87 & -5.25 \\ 3.99 & 0.24 & 0.61 & -2.17 & 2.82 & -5.96 & 0.82 & -0.39 \end{pmatrix}$$

Correspondingly, the transition map is given in Figure 7 and 8, respectively.

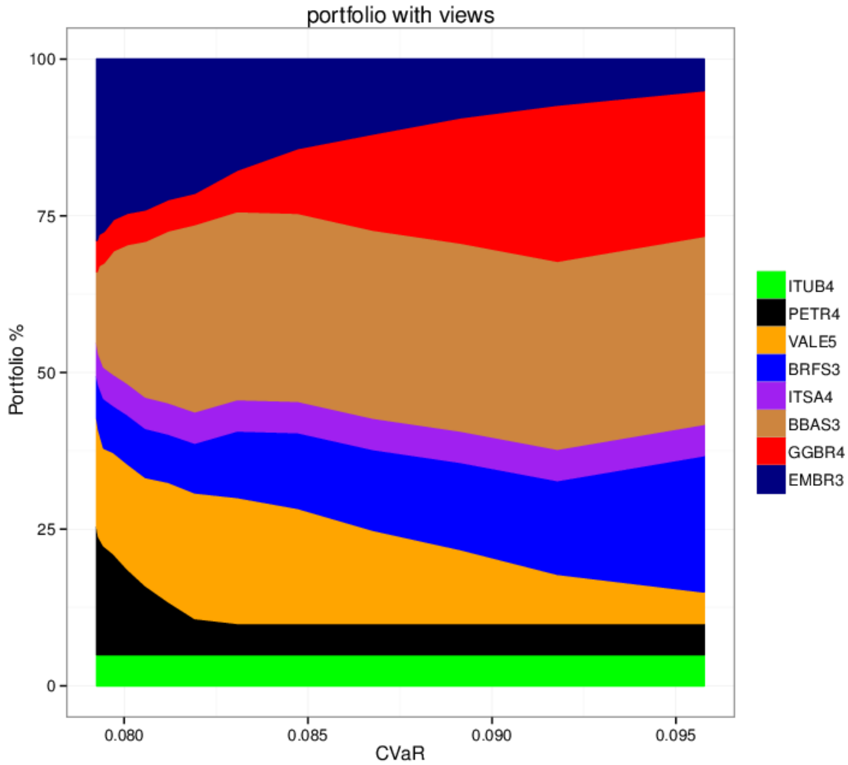

Figure 7: The transition map of the EBL2 with $\gamma = 0.2$

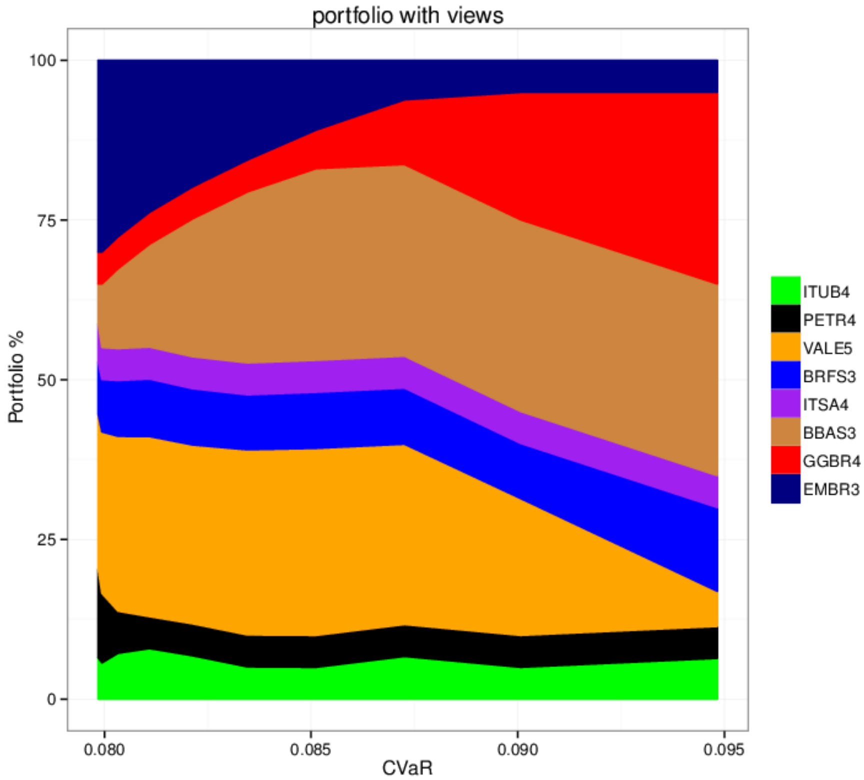

Figure 8: The transition map of EBL2 with $\gamma = 0.95$

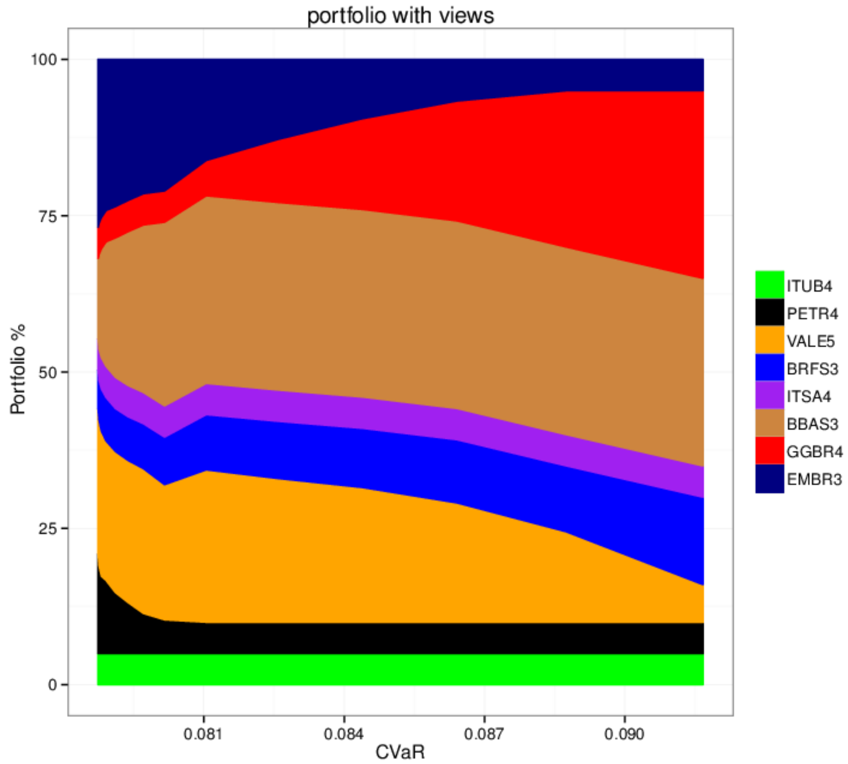

Figure 9: The transition map of EBL2 with $\gamma = -0.3$

3.4 Conclusions

In this report, we extend the Black-Litterman model based on the two models of the location parameter L, and we call the corresponding model EBL1 and EBL2. From the results of the implementation, the portfolios of the two models both change according to our views. That is, the weights of GGBR4 and BBAS4 increase, whereas ITUB4 decreases. However, it can be seen that GGBR4, PETR4 and BRFS3 change less in EBL2 than in EBL1. Moreover, for EBL2, we perturb the model a little by setting $\gamma = 0.2$, 0.95 and -0.3. In the case of $\gamma = 0.2$, the transition map in Figure 7 shows that GGBR4 increases more, whereas BBAS3 does not increase at all. This can be explained that we originally have more confidence on GGBR4 than BBAS3, nevertheless, ITUB4 decreases less. By setting γ as a small number close to zero, we want to reduce our confidence of our original views and make our views correlate with the prior model. On the other side, when γ is close to 1, our empirical study shows that the stable region is smaller than the one that γ is close to 0. Figure 8 shows that when γ is close to 1, the views and the prior model are almost perfect correlated and the resulting portfolios are almost the same. Therefore, for the Black-Litterman model in skew-normal market, we provide another way to reduce our confidence after we already fixed our views. For the last case, in Figure 9, we can see that both BBAS3 and GGBR4 increase more than any case we have discussed before.

CVaR optimization is a very powerful method in asset allocation. It does not depend on the distribution of the assets, and therefore can be used for any asset when it is not normally distributed. However, in Cont, Deguest and Scandolo [CDS10], they proved that CVaR is less robust than historical VaR in some circumstances. Therefore, considering other coherent risk measure to choose the portfolio would be interesting. Finally, some parts of the models can still be improved, such as assigning values for Γ, estimating covariance matrix for the data, and so forth.

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