impa **Instituto Nacional de Matemática Pura e Aplicada**

Doctoral Thesis

LOW DIMENSIONAL POLAR ACTIONS

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Rio de Janeiro
2014

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Thesis presented to the Post-graduate Program in Mathematics
at Instituto Nacional de Matemática Pura e Aplicada as par-
tial fulfillment of the requirements for the degree of Doctor in
Philosophy in Mathematics.

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Rio de Janeiro
1st December 2014

A mi familia,
por compartir la distancia
y apoyarme siempre.

Agradecimentos

Quero expressar minha gratidão aos meus orientadores Luis A. Florit e Wolfgang Ziller por compartilhar comigo seu tempo, seu conhecimento matemático e sua visão da profissão. Me considero afortunado por contar com duas pessoas íntegras, exímios matemáticos, bem dispostos à me ouvir, explicar e corrigir inúmeras vezes.

Agradeço aos professores Detang Zhou, Marcos M. Alexandrino Da Silva, Reimundo Heluani, Michael Belolipetsky e Miguel Domínguez Vázquez por participar da banca examinadora e colaborar com seus comentários.

Aos demais professores do IMPA e convidados com quem aprendi e tive a oportunidade de falar de matemática, particularmente a Marcos Dajczer, Henrique Burztyn e Bernhard Hanke. Gostaria também de agradecer a Universidade de Buenos Aires e seus professores, por me dar a base para continuar com meus estudos de doutorado, lembrando especialmente de Jimmy Petean, Fernando Cukiermann, Pablo Solernó, Alicia Dickenstein e Guillermo Cortinhas.

Ao IMPA por me acolher ao longo deste tempo e providenciar o apoio do CNPq. Destaco em especial a labor diária e dedicada dos seus funcionários, dando ouvidos aos pedidos tanto pessoais como coletivos, visando sempre melhorar as condições de trabalho na medida dos recursos disponíveis. Agradeço também a UPenn por um semestre muito produtivo, e as outras instituições que tive a oportunidade de visitar.

Aos amigos e colegas geômetras, destacando a Vanderson Lima, Marco A. Méndez Guaraco, Lucas Ambrozio, Rafael Montezuma, Lázaro Rodriguez, Guilherme Machado Freitas, Fabio Simas, Dima Yeroshkin, Marco Radeschi, Fernando Galaz García e Renato Bettiol.

A todos os alunos do IMPA, aos colegas da APG, companheiros de moradia e amigos do Rio, nomeando a Ricardo Paleari, Leandro Cruz, Lucas Backes, Luiz Freire, Karina Marin, Felipe Macias, Younes Nikdelan, Martina Negrin, Lenka Ptackova e Luciano Araujo Pereira.

A Clara, protagonista essencial, em cuja companhia escrevo estas linhas e os meus dias.

Finalmente quero agradecer à minha família por seu incentivo e apoio, e dedicar este trabalho ao meu irmão Nicolás, à minha tia Alejandra, à María Luján e Rina, aos meus pais Mariana e Fernando, à Abu e à memoria de Fanny, Martha, Tatalo e Focho.

"He who desires but acts not, breeds pestilence."
William Blake, "The Proverbs of Hell".

Resumo

Uma ação polar é um tipo de ação isométrica própria definida pelo requerimento da existência de uma subvariedade imersa completa intersecando todas as órbitas ortogonalmente. Esta classe de ações aparece naturalmente em questões geométricas e generaliza as ações de cohomogeneidade um, assim como a ação por conjugação de um grupo de Lie compacto em si mesmo.

Este trabalho estuda sistematicamente as possíveis ações polares em variedades compactas e simplesmente conexas de dimensão cinco ou menos. Como resultado se classificam os tipos equivariantes de ações assim como as variedade subjacentes a menos de um difeomorfismo. Os exemplos são construídos a partir de ações lineares e outras ações naturais em bi-quocientes, seguidas de operações de cirurgia equivariante. Como aplicação se determinam aquelas ações que admitem uma métrica invariante de curvatura seccional não negativa.

Palavras-chave: ações isométricas, polar, curvatura não-negativa.

Abstract

Polar actions are a special kind of proper isometric actions defined by the existence of an immersed complete submanifold intersecting every orbit orthogonally. This class of actions appears naturally in geometric questions and generalizes actions of cohomogeneity one, as well as the adjoint action of a compact Lie group on itself.

The aim of this work is to study polar actions on simply connected compact manifolds of dimension at most five. As a result we obtain an equivariant classification as well as a classification of the underlying manifolds up to diffeomorphism. There is a great variety of examples, most of them obtained from linear actions and natural actions on biquotients followed by equivariant surgery operations. Finally, as an application, it is shown which actions admit an invariant metric of non-negative curvature.

Key words: isometric actions, polar, non-negative curvature.

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1 Introduction

The concept of symmetry has played an important role in geometry since its inception. Within the field of Riemannian geometry we find the general theory of proper isometric actions by Lie groups, and as a special case of interest that of polar actions.

A complete Riemannian manifold M together with a proper isometric action of a Lie group G is said to be *polar* if it admits a *section*, i.e., an immersed complete submanifold Σ of M intersecting every G-orbit and doing so orthogonally. The existence of Σ is equivalent to the integrability of the horizontal distribution along the regular part; see [Bou95]. The section is preserved by the so called polar group Π which is a discrete group acting properly and discontinuously on Σ with quotient Σ/Π isometric to M/G. Hence, the orbit space is a good orbifold. The existence of a section enables the reduction of a geometric problem on M invariant under G to a possibly simpler one on Σ invariant under Π, as e.g. illustrated by the generalization of the Chevalley Restriction Theorem in [PT87] which states that smooth G-invariant functions on M are in one to one correspondence with Π-invariant functions on Σ, $C^\infty(M)^G \cong C^\infty(\Sigma)^\Pi$.

The knowledge of the orbit space together with certain isotropy data is enough to reconstruct the equivariant diffeomorphism type of the action, as shown in [GZ12]. This feature provides flexibility in the construction of new examples in terms of abstract polar data and facilitates their equivariant classification. It also proves useful in adapting general tools, as e.g. equivariant surgeries, to the polar context.

Among classical examples are the adjoint action of a compact Lie group on itself, equipped with a bi-invariant metric, or more generally the left action of K on G/K, associated to a symmetric pair (G, K). The isotropy representations of K on $T_{[eK]}G/K$, so called *s-representations*, are of particular importance since they agree with the class of linear polar actions up to orbit equivalence; see [Dad85]. In fact, such *s*-representations completely describe the local structure of a general polar action.

Trivial examples are given by the product of a homogeneous space with a manifold. Another simple yet rich source of examples is that of cohomogeneity one actions, where a section is given by a geodesic orthogonal to a regular orbit. In this case, polar data is given by a group diagram $H \subseteq K_-, K_+ \subseteq G$, and the reconstruction of the manifold is achieved as a gluing of two disc bundles $G \times_{K_\pm} \mathbb{D}^{l_\pm}$ along their common boundary G/H. In this framework, compact simply-connected cohomogeneity one manifolds were classified up to dimension 7 in [Hoe10]. In particular, in dimension 5 the author shows that M is diffeomorphic to either $\mathbb{S}^5$, $\mathrm{SU}(3)/\mathrm{SO}(3)$, $\mathbb{S}^3 \times \mathbb{S}^2$ or the non-trivial $\mathbb{S}^3$-bundle over $\mathbb{S}^2$, denoted by $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$. In addition, he classifies the actions up to equivariant diffeomorphism.

In this paper we classify all closed simply-connected polar manifolds of dimension

5 or less by addressing the case of cohomogeneity at least two. We assume that the group acting is connected, since a polar action by a Lie group G naturally restricts to its identity component G_0 in a polar fashion and with the same cohomogeneity. In order to facilitate the description of the actions we shall assume they are almost-effective, i.e., at most a finite subgroup acts trivially. Polar manifolds of dimension 3 or 4 are not hard to classify (see Theorems 4.5 and 4.6) where in dimension 4 the main case of T^2 actions follows from [OR74, GZ12]. Our main contribution is thus the classification of polar 5-manifolds. Among these, the effective polar T^2-actions are the most important and difficult case. We describe both the equivariant type of the actions as well as the diffeomorphism type of the underlying manifolds.

Many of the actions discussed later on can be described as follows.

Main Example. Consider $\mathbb{S}^3 \times \mathbb{S}^3 \subset \mathbb{C}^4$ where the circle acts linearly by multiplication as $\theta \star (z_1, z_2, z_3, z_4) = (z_1\theta, z_2\theta^{-1}, z_3\theta^k, z_4)$, $\theta \in \mathbb{C}, |\theta| = 1, k \in \mathbb{Z}$. We denote this circle by $\mathrm{S}^1_{(1,-1,k,0)}$ and observe that it acts freely on $\mathbb{S}^3 \times \mathbb{S}^3$. Passing to the quotient, we retrieve one of the two $\mathbb{S}^3$ bundles over $\mathbb{S}^2$ with map to the base induced from the projection onto the first factor of $\mathbb{S}^3 \times \mathbb{S}^3$. In fact, the outcome manifold depends on the parity of k, as we may trivialize such bundle over two hemispheres $\mathbb{D}^2_\pm \subset \mathbb{S}^2$ and observe that the gluing along their common boundary $\mathbb{S}^1$ is determined by $k \in \pi_1(\mathrm{SO}(4)) \cong \mathbb{Z}_2$. We have:

$$\mathbb{S}^3 \times_k \mathbb{S}^2 := (\mathbb{S}^3 \times \mathbb{S}^3)/\mathrm{S}^1_{(1,-1,k,0)} \cong \begin{cases} \mathbb{S}^3 \times \mathbb{S}^2 & k \text{ even} \\ \mathbb{S}^3 \, \tilde{\times} \, \mathbb{S}^2 & k \text{ odd.} \end{cases}$$

We may construct natural actions on the quotients $\mathbb{S}^3 \times_k \mathbb{S}^2$, induced from actions on $\mathbb{S}^3 \times \mathbb{S}^3$ that either extend or commute with the previous right action by $\mathrm{S}^1_{(1,-1,k,0)}$:

1. A circle action by $\mathrm{S}^1_{(0,0,0,1)}$ acting as $\theta \star (z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, z_4\theta)$;

2. The standard action of $\mathrm{SU}(2)$ on the first factor of $\mathbb{S}^3 \times \mathbb{S}^3$;

3. An $\mathrm{SU}(2) \times \mathrm{S}^1$ action by $(A, \theta) \star (z_1, z_2, z_3, z_4) = (A \cdot (z_1, z_2), z_3\theta, z_4)$;

4. A linear T^3 action by $(\theta_1, \theta_2, \theta_3, 1) \star (z_1, z_2, z_3, z_4) = (z_1\theta_1, z_2\theta_2, z_3\theta_3, z_4)$, which induces an effective action by $T^2 \cong T^3 / \mathrm{S}^1_{(1,-1,k,0)}$ on $\mathbb{S}^3 \times_k \mathbb{S}^2$;

5. The maximal torus $T^4 \subset \mathrm{SO}(4) \times \mathrm{SO}(4)$ which induces an effective T^3 action on $\mathbb{S}^3 \times_k \mathbb{S}^2$.

As we will see, the induced actions are all polar with respect to the submersion metric on $\mathbb{S}^3 \times_k \mathbb{S}^2$, inherited from the standard metric on $\mathbb{S}^3(1) \times \mathbb{S}^3(1)$. In particular, the metrics have non-negative curvature.

We now proceed with the description of all polar T^2 actions on 5-manifolds, where the above T^2 actions on $\mathbb{S}^3 \times_k \mathbb{S}^2$ play a key role as basic building blocks of our construction.

Theorem A. *A closed simply-connected 5-manifold equipped with an effective polar action by* T^2 *is obtained, by equivariant surgery operations, from linear polar actions on spheres and products of spheres, or one of the above* T^2-*actions on* $\mathbb{S}^3 \times_k \mathbb{S}^3$. *Moreover, the equivariant surgeries correspond either to connected sums at fixed points, or surgeries along regular orbits.*

The question as to which surgeries are allowed is more subtle, see Figure 8 for some examples and Section 5 for a detailed discussion.

For the diffeomorphism type of the manifolds themselves, we use the Barden-Smale classification of 5-manifolds. It turns out that for polar S^1, T^2, and T^3 actions the possible diffeomorphism types agree.

Theorem B. *A closed simply-connected 5-manifold that admits a polar action by an abelian Lie group is diffeomorphic to either* $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$, *or their connected sums.*

The case of T^3 actions in dimension 5 is treated in [Oh83]. More generally, all cohomogeneity two tori actions are automatically polar, as observed in [GZ12]. The case of polar S^1 actions is fairly simple, and can also be treated in general for any dimension; see Proposition 4.4.

For the classification of the non-abelian case we have:

Theorem C. *Let* G *be a compact connected non-abelian Lie group, with a non-trivial and almost-effective polar action on a compact simply-connected 5-manifold* M *of cohomogeneity at least two. Then* M *is equivariantly diffeomorphic to one of the following:*

a) *The linear polar actions on* $\mathbb{S}^5$;

b) *The* $\mathrm{SU}(2)$ *and* $\mathrm{SU}(2) \times \mathrm{S}^1$ *actions on* $\mathbb{S}^3 \times_k \mathbb{S}^2$ *as described in the Main Example;*

c) *The factor-wise linear actions by* $\mathrm{SO}(3) \times \mathrm{S}^1$ *and* $\mathrm{SO}(3) \times \{1\}$ *on* $\mathbb{S}^3 \times \mathbb{S}^2$, *and their connected fixed-point sums* $\#_n \mathbb{S}^3 \times \mathbb{S}^2$;

d) *The unique* $\mathrm{SO}(3)$-*actions on the Wu manifold* $\mathcal{W} = \mathrm{SU}(3)/\mathrm{SO}(3)$ *and on the Brieskorn variety* $\mathcal{B}$ *of type* $(2, 3, 3, 3)$, *or their respective fixed-point connected sums,* $\#_k \mathcal{B}$ *and* $\#_l \mathcal{W}$.

In [Sim] general actions by $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ were classified. It follows from our work that all but 2 of the actions admitting singular orbits are in fact polar.

We will see that further fixed point connected sums $\#_k \mathcal{B} \#_l \mathcal{W}$ are equivariantly diffeomorphic to $\#_k \mathcal{B}$ or $\#_l \mathcal{W}$ themselves, and that $\#_k \mathcal{B}$ has a unique polar action by $\mathrm{SO}(3)$ whereas $\#_l \mathcal{W}$ has finitely many.

As an immediate consequence of the previous theorems we have the following,

Corollary. *A closed simply-connected polar 5-manifold is diffeomorphic to either* $\mathbb{S}^5$, $\#_\alpha(\mathbb{S}^3 \widetilde{\times} \mathbb{S}^2) \#_{n_0}(\mathbb{S}^3 \times \mathbb{S}^2)$, $\#_{n_1}\mathcal{B}$ *or* $\#_{n_2}\mathcal{W}$, *for* $n_i \in \mathbb{N}_0$ *and* $\alpha = 0, 1$.

As an application, we obtain an equivariant classification of all polar actions on 5-manifolds which admit invariant metrics with non-negative curvature.

Theorem D. *Let M be a closed simply-connected 5-manifold with a polar action of cohomogeneity at least two whose sectional curvature is non-negative. Then M^5 is equivariantly diffeomorphic to a product of linear actions on spheres, the left action of* $\mathrm{SO}(3)$ *on* $\mathrm{SU}(3)/\mathrm{SO}(3)$, *or one of the actions on* $\mathbb{S}^3 \times_k \mathbb{S}^2$ *constructed in the Main Example above.*

In contrast, cohomogeneity one actions on 5-manifolds all admit non-negative curvature, see [Hoe10]. The added rigidity when the cohomogeneity is ≥ 2 is analogous to the case of positive curvature, where it was shown that a polar action is equivariantly diffeomorphic to a linear action on a compact rank one symmetric space; see [FGT].

This work is organized as follows. In Section 2 we summarize the basic definitions and results on polar actions and describe the equivariant surgeries on such actions. The third section is devoted to the construction of various examples that will appear throughout the paper. In Section 4, we discuss all actions other than the ones by T^2 in dimension 5, which are treated in Section 5. The proof of Theorem B is the content of Section 6, while the last section is reserved for applications in non-negative curvature.

2 Preliminaries on Polar Manifolds

We begin by recalling general facts about isometric group actions (see e.g. [Bre72]) and fixing some notations. Next, we discuss polar manifolds as described in [GZ12], together with the equivariant "cut and paste" operations needed for the construction of large families of relevant examples.

Let G denote a compact Lie group acting by isometries on a Riemannian manifold M. The orbit through a point $p \in M$ is an embedded homogeneous submanifold $G \cdot p \cong G/G_p$, where $G_p = \{g \in G \mid g.p = p\}$ is the *isotropy subgroup* at p. This group acts on the normal sphere to the orbit in $T_p(G \cdot p)^\perp$, and by the Slice Theorem we have:

$$\mathcal{T}_\epsilon(G \cdot p) \cong (G \times S_p)/G_p =: G \times_{G_p} S_p, \tag{2.1}$$

where S_p is a *slice* at p. The neighborhood of a point in the orbit space, $[p] \in M/G$, is homeomorphic to the quotient S_p/G_p. The conjugacy class (G_p) of an isotropy subgroup is called an *isotropy type*. Orbits corresponding to a minimal isotropy type are called *regular*, and we reserve H to denote a choice of one of this regular or *principal* isotropy subgroups. Orbits with bigger isotropy type are called *singular*, unless they have the same dimension as the principal ones in which case they are called *exceptional*.

The orbit space is not a manifold in general but a disjoint collection of strata, which are smooth manifolds corresponding to the projection of each $M_{(K)} := \{x \in M \mid (G_x) = (K)\}$. The partial order between isotropy types induces an inverse relation between the strata, where a smaller isotropy type $(K_1) \leq (K_2)$ corresponds to a bigger stratum containing the smaller one in its closure, $M_{(K_2)} \subset \overline{M_{(K_1)}}$.

In the case of polar actions we obtain an optimal reduction in dimension for the understanding of the orbit space M/G and its stratification by isotropy types. In fact, the action of G on M induces an action of a discrete group $\Pi = N(\Sigma)/Z(\Sigma)$, called the *polar group*, where $N(\Sigma) = \{g \in G | g \cdot \Sigma = \Sigma\}$ and the subgroup $Z(\Sigma)$ is the point-wise stabilizer of the section. The latter agrees with the principal isotropy group of a regular point in the section, i.e., $Z(\Sigma) = H$, while Π acts properly and discontinuously on Σ with isometric quotient $\Sigma/\Pi \cong M/G$. In this way, the section provides an orbifold cover of the orbit space.

Inside Π we identify *reflections*, i.e., elements which have a hypersurface or *mirror* $\mathcal{M}^{k-1} \subset \Sigma^k$ inside their fixed-point set. We consider the complement in Σ of all possible mirrors $\{\mathcal{M}_i\}$ and define a *chamber* C to be the closure of a connected component. By construction, the boundary of $C \subseteq \Sigma$ has a stratification by totally geodesic faces, given by the mirrors and their intersections. More precisely, we define *faces* F_i of C to be the closure of each connected component of the open intersections with a mirror,

$$F_i = \overline{int(\mathcal{M}_i \cap C)} \subset \mathcal{M}_i.$$

Deeper faces $F_{i_1 \cdots i_k}$ are defined as the connected intersections of the previous ones, $F_{i_1 \cdots i_k} = F_{i_1} \cap \cdots \cap F_{i_k}$ The length of the sub-index is referred to as the *depth* at which the face lies. To each face we can assign a unique generic isotropy subgroup.

Recall that by definition every orbit intersects the section, and thus a given chamber as well. Moreover, an orbit may possibly intersect C more than once, implying the existence of a non trivial stabilizer subgroup preserving the chamber, $\Pi_C := \{ g \in \Pi \, | \, g \cdot C = C \} \subset \Pi$.

From now on, we shall restrict ourselves to the following class of actions, which is a natural setting in our context since a polar action by a connected group on a simply-connected manifold is always of this type (see Remark 2.10).

Definition 2.2. A polar action without exceptional orbits and with trivial stabilizer group Π_C is called *Coxeter polar*.

One of the main properties about Coxeter polar actions is that the inclusion $C \to \Sigma \xrightarrow{\sigma} M$ induces an isometry between the orbit spaces and the chamber,

$$C \cong \Sigma / \Pi \cong M/G. \tag{2.3}$$

In the case of a general isometric group action we may not have a cross section to the projection map $M \to M/G$, while for a Coxeter polar action this is given by the chamber C. Furthermore, the faces have constant isotropy group along the interior, and (2.3) identifies the open strata in C with the Π-isotropy type strata in Σ / Π, and the corresponding G-isotropy type strata in M/G. We may encode this information in a marking of the open strata F_I of C forming a partially ordered graph of G-isotropy subgroups K_I associated to each one. The subgraph below a given vertex K_I is called the *history* of K_I and corresponds to the labeling of strata at a point $p \in F_I$. We say that the marking of C is *compatible* if the history at a point $p \in F_I$ comes from a Coxeter polar representation, and the orbifold group at $p \in C$ coincides with the polar group of such representation.

The isotropy subgroups of Π are Coxeter groups, in fact given $p \in F_{i_1 \cdots i_k}$, we have that Π_p is generated by the reflections r_i defined by the corresponding mirrors through p, and satisfying relations $(r_i . r_j)^{m_{ij}} = 1$ depending on the angle between them. The whole polar group Π is a quotient of a Coxeter group, as it is generated by the reflections along mirrors in Σ, but there may be more relations than the local ones just described. We shall regard the orbit space as a *Riemannian Coxeter orbifold*, i.e., a Riemannian orbifold modeled on finite reflection groups. It is a good orbifold, with only manifold points along the interior.

Definition 2.4. A connected Riemannian Coxeter orbifold C together with a compatible group graph marking the strata defines *Coxeter polar data*, denoted by $(C, G(C))$.

We point out that the slice representations do not need to be specified as part of the Coxeter polar data. Indeed, they are determined by the group marking due to the following lemma that will be proved in general in a forthcoming paper. Observe that this is a weaker statement than that of item (b) of Lemma 2.4 in [GZ12], whose proof seems to have a gap for reducible representations. An ad-hoc argument for representations of dimension five or less follows easily from the work of [Ber01, EH99] and would also be enough for the scope of this paper.

Lemma 2.5. *A Coxeter polar representation by a Lie group L is determined up to linear equivalence by its history and dimension. Furthermore, L is generated by the face isotropy groups, as well as by L_0 and H.*

In particular, the group graph of a Coxeter polar action must satisfy the following algebraic conditions:

$$K_{i_0\cdots i_l} = \mathrm{gen}_{0\leq k\leq l}\{K_{i_k}\}$$

$$H = K_i \cap K_j, \ \forall i \neq j \text{ with } F_{ij} \neq \emptyset. \tag{2.6}$$

The first equation rephrases part of the previous Lemma. The second one is valid for adjacent faces, i.e., those with non-empty intersection. In fact, by definition $K_i \supset H$, and $K_i \cap K_j \subset H$ since it acts on the slice at $p \in F_{ij}$ with fixed point set including the tangent space to each face, and therefore spanning all of $T_p\Sigma$.

The face isotropy group K_i acts transitively on the sphere orthogonal to its strata,

$$K_i/H \cong \mathbb{S}^{l_i}, \ l_i > 0. \tag{2.7}$$

Because of (2.6) the group graph is completely determined by the isotropy groups K_i marking the maximal faces. Furthermore, the reflections $r_i \in \Pi$ along these maximal faces generate the polar group. These elements can also be thought of as the unique involutions in the normalizer $N^K(H)/H \subset N(H)/H$.

As an application of the fact that Π acts simply-transitively on the set of chambers, thus *tessellating* the section, we can try to orient Σ by prescribing an orientation on C and declaring each reflection along a maximal face to be orientation reversing. The reader may see 3.2 for an illustrative example having non-orientable section. This leads to the following criterion for orientability.

Lemma 2.8. *The section of a Coxeter polar manifold is orientable if and only if the chamber is, and if there are no relations of odd order among the reflections corresponding to maximal faces.*

Coxeter polar data is an invariant of the action which is enough to construct and identify Coxeter polar manifolds:

Theorem 2.9 ([GZ12]). *Coxeter polar data $(C, G(C))$ determines a Coxeter polar manifold $M(C, G(C))$ up to equivariant diffeomorphism.*

We point out though that 2 different Coxeter polar data $(C, G(C))$ and $(C, G'(C))$ can be equivariantly diffeomorphic since the diffeomorphism may not respect the metric. This can also change the polar group and section, in fact, a closed section can become non compact. It is an interesting question whether via such a change of metric the section for any Coxeter polar action can be made compact, as is the case for cohomogeneity one manifolds. See [GWZ08] for a discussion in the case of cohomogeneity one actions.

As an immediate application of Theorem 2.9 observe that each strata in M gets uniquely reconstructed as a trivial bundle $M_{(K)} = M_{(K)}^* \times G/K$, since it corresponds to data given by the manifold $M_{(K)}^*$ constantly marked by K.

Remark 2.10. A general polar manifold can be described as the quotient of a Coxeter polar manifold by a group acting freely and discontinuously which commutes with the action and preserves orbit types; see [GZ12]. In particular, it follows that a simply-connected manifold with a polar action by a connected group is Coxeter polar.

The following lemmas show how to construct new polar actions as quotients of given ones.

Lemma 2.11. *Let M be a polar G-manifold, and L be a Lie group acting freely on M, such that the two actions commute. Then the induced G action on M/L is polar if G and L have orthogonal orbits, i.e., $T_p(G \cdot p) \perp T_p(L \cdot p)$.*

Proof. The assumption implies that $T_p(L \cdot p) \subset T_p\Sigma$ and hence Σ is L-invariant. Furthermore, it projects to an immersed complete submanifold in M/L. The polarity of the action is clear once we observe that Σ/L has the right dimension and integrates the horizontal distribution along the regular part. $\square$

Lemma 2.12. *Let M be a (Coxeter) polar G-manifold with section Σ, and $L \lhd G$ be a normal subgroup which acts freely on M. Then the induced G/L action on the quotient M/L is (Coxeter) polar with the submersion metric on M/L. A section is given as a proper and discontinuous quotient of Σ by the action of $\Pi \cap L$. Moreover, in the Coxeter polar case, chambers are isometrically identified and the marking by G/L-isotropy groups is obtained as the projection along $G \to G/L$ of the previous marking of G-isotropy groups. This gives an isomorphism of isotropy groups. In addition, the new polar group is obtained as the quotient $\Pi / \Pi \cap L$.*

Proof. The section $\sigma : \Sigma \to M$ is horizontal with respect to the projection map $\mathrm{p} : M \to M/L$, therefore we have by composition a section in the quotient given by $\mathrm{p} \circ \sigma : \Sigma \to M/L$. The orbit spaces are isometric to each other, $M/G = (M/L)/(G/L)$. It then follows, that in the Coxeter polar case p identifies chambers isometrically. The isotropy group at a point $\mathrm{p}(x) \in M/L$ corresponds to the projection of G_x under the map $\Phi : G \to G/L$. In fact, the projection Φ induces

an isomorphism between isotropy groups since $L \cap G_x = \{e\}$. This shows that we have retrieved a Coxeter polar action associated to data $(C, G/L(C))$.

Notice in general that we have a unique minimal section up to covers, which admits possible self-intersections but does not have any self-tangencies since it is totally geodesic. In our case, Σ covers the new section in M/L up to the discrete action of $N(\Sigma) \cap L$. As before, we may identify $N(\Sigma) \cap L =: \Pi \cap L$ since $L \cap H = \{e\}$. Finally, in the Coxeter polar case, the polar group is generated by the same reflections (as elements in G) thanks to the coincidence of the chambers, but is possibly smaller and equals $\Pi / \Pi \cap L$. $\qquad\square$

Remark 2.13. Given a (Coxeter) polar L-action on M', and $L \subset G$, we can construct a (Coxeter) polar G-manifold M as the quotient of $M' \times G$ by the diagonal action of L,

$$M \cong M' \times_L G, \qquad (2.14)$$

where the action of G is given by right multiplication on the second factor. Furthermore, the inclusion of a section Σ' of M' as $\Sigma \times \{e\} \to M' \times_L G$ gives a section for the G-action on M. In particular, the action on M is *imprimitive*, i.e., it admits an equivariant map $M \to G/L$ for $L \subsetneq G$.

Conversely, consider a Coxeter polar G-manifold M such that all the isotropy subgroups in the data $G(C)$ are contained in a closed subgroup $L \subseteq G$. Then the same Coxeter polar data yields, by restriction, compatible data $(C, L(C))$ for an L-manifold M'. By uniqueness of the reconstruction process we have that M agrees with $M' \times_L G$ as above, up to an equivariant diffeomorphism.

Finally, there are equivariant "*cut and paste*" operations on G-manifolds that can be adapted to work on polar manifolds. Given two G-manifolds M_1, M_2, we may glue them along orbits of the same *slice type*, i.e., same isotropy type and isomorphic slice representations. In this case, orbits will not only be diffeomorphic, but will also have equivariantly isomorphic tubular neighborhoods $\mathcal{T}_1 := \mathcal{T}(G \cdot p_1) \cong \mathcal{T}(G \cdot p_2) =: \mathcal{T}_2$. We can then remove these tubular neighborhoods and glue the remaining pieces along the equivariantly identified boundary. We thus obtain a new G-manifold, denoted by

$$M_1 \#_{p_1 \sim p_2} M_2 := M_1 \backslash \mathcal{T}_1 \cup_{\partial \mathcal{T}} M_2 \backslash \mathcal{T}_2. \qquad (2.15)$$

Notice that the orbit space of $M_1 \#_{p_1 \sim p_2} M_2$ corresponds to a connected sum of M_1/G and M_2/G along $[p_1] \sim [p_2]$. This motivates the following.

Definition 2.16. Given points $p_i \in C_i \subset M_i$, $i = 1, 2$, lying on respective chambers of Coxeter polar G-manifolds, having the same isotropy subgroups and isomorphic slice representations, we define their *orbit sum* as a (marking preserving) connected sum of the chambers $C := C_1 \#_{p_1 \sim p_2} C_2$. For this, we cut out the points p_i together with their neighborhoods in C_i, and then glue along the boundary via the identification induced from an isomorphism between the slice representations.

As usual, the choice of compatible orientations on the chambers determines the gluing. Moreover, we can make the identification isometric if we first modify the metrics on each chamber, so that it is a product near the boundary of each ball $B_\epsilon(p_i) \subset C_i$.

By Theorem 2.9 we have a unique Coxeter polar manifold associated to this new compatible data $(C, G(C))$. It will be referred to as the *orbit sum* of M_1 and M_2 along the prescribed orbits, and denoted as before by $M_1 \#_{p_1 \sim p_2} M_2$.

From now on we focus on the case where the cohomogeneity is at least two. Taking advantage of the fact that the reconstructed manifold is not only polar but Coxeter polar, we can determine the polar group out of the polar data $(C, G(C))$ thanks to the fact that this group is generated by the reflections along the maximal (depth one) faces. Indeed, as only 0-dimensional strata can be deleted by this operation, the assumption on the cohomogeneity implies that the gluing will not erase any maximal face, thus preserving the corresponding reflections. As a consequence we have that the new polar group Π will be the subgroup generated by the respective polar groups Π_i of the original manifolds M_i, when seen as subgroups of the effective normalizer to the common principal isotropy subgroup H,

$$\Pi = gen\{\Pi_1, \Pi_2\} \subset N(H)/H. \tag{2.17}$$

Next, we discuss the two extreme cases of orbit sum, that is, along either regular orbits or fixed points. These will be of interest in our description of T^2 polar actions on 5-dimensional manifolds.

Proposition 2.18. *The manifold, chamber and section coming from a fixed point sum are the connected sum of the corresponding pieces, and the new polar group coincides with the original:*

$$M_1 \#_{fix} M_2 \cong M_1 \# M_2$$

$$\Sigma \cong \Sigma_1 \# \Sigma_2$$

$$\Pi = \Pi_1 = \Pi_2 \,.$$

Proof. Recalling the identity (2.15) above, together with the fact that the tubular neighborhoods to the orbits are just balls around fixed points, it is clear that the resulting manifold will be the connected sum of the original ones.

Notice that at corresponding fixed points p_i we have $\Pi_i = (\Pi_i)_{p_i}$, and in turn $(\Pi_1)_{p_1} = (\Pi_2)_{p_2}$ as they have isomorphic slice representations. From (2.17) it follows that the polar group remains the same, $\Pi_1 = \Pi_2 = \Pi$.

As the polar group acts freely and transitively on the set of chambers, we can reflect our given chamber $C = C_1 \# C_2$ by the new polar group and retrieve the section, i.e.,

$$\Pi \cdot C = \Sigma.$$

In particular, when reflecting the chamber minus a small neighborhood of a fixed point, we retrieve the section minus a disc $\mathbb{D}^k$. For the section of the orbit sum, we have

$$
\begin{aligned}
\Sigma \;=\; \Pi \cdot (C_1 \# C_2) \;&=\; \Pi \cdot [(C_1 - B_\epsilon(p_1)) \cup_{(\partial \mathbb{D}^k / \Pi)} (C_2 - B_\epsilon(p_2))] \\[2mm]
&=\; [\Pi_1 \cdot (C_1 - B_\epsilon(p_1))] \cup_{\partial \mathbb{D}^k} [\Pi_2 \cdot (C_2 - B_\epsilon(p_2))] \\[2mm]
&=\; (\Sigma_1 - \mathbb{D}^k) \cup_{\mathbb{S}^{k-1}} (\Sigma_2 - \mathbb{D}^k) \;=\; \Sigma_1 \# \Sigma_2.
\end{aligned}
$$

$\square$

Proposition 2.19. *The section associated to the polar action given by the regular orbit sum $M_1 \#_{p_1 \sim p_2} M_2$ is a connected manifold obtained by connected sums of copies of the corresponding sections Σ_i of M_i, $i = 1, 2$, together with additional 0-surgeries.*

Proof. Analogously to the proof of the previous proposition, reflecting the chamber $C_1 \# C_2$ retrieves the new section.

Recall that Π is generated by the previous polar groups Π_1 and Π_2. An element $[g] \in \Pi - \Pi_1$ does not preserve the section Σ_1, and in fact it gives a traslated copy of it. Notice that the points p_i, along with their neighborhoods where the gluing takes place, are completely embedded in the regular part.

The spheres that bound the disks which were removed in the respective chambers C_1 and C_2, get reflected along with the new chamber as many as $|\Pi|$ times. As they lie over the regular part they do not interfere with possible self-intersections. Therefore, reflecting $C_1 - \epsilon.\mathbb{D}^k$ by the polar group Π, will give $|\Pi / \Pi_1|$ disjoint copies of the original section Σ_1, with each copy having as many as $|\Pi_1|$ discs removed. We proceed to compute the new section:

$$
\begin{aligned}
\Sigma \;=\; \Pi \cdot (C_1 \# C_2) \;&=\; \Pi \cdot [(C_1 - \mathbb{D}^k) \cup_{\mathbb{S}^{k-1}} (C_2 - \mathbb{D}^k)] \\[2mm]
&=\; |\,\Pi / \Pi_1\,| \cdot [\Pi_1 \cdot (C_1 - \mathbb{D}^k)] \cup_{\Pi \cdot \mathbb{S}^{k-1}} |\,\Pi / \Pi_2\,| \cdot [\Pi_2 \cdot (C_2 - \mathbb{D}^k)] \\[2mm]
&=\; |\,\Pi / \Pi_1\,| \cdot [\Sigma_1 - \Pi_1 \cdot \mathbb{D}^k] \cup_{\Pi \cdot \mathbb{S}^{k-1}} |\,\Pi / \Pi_2\,| \cdot [\Sigma_2 - \Pi_2 \cdot \mathbb{D}^k] \\[2mm]
&=\; |\,\Pi / \Pi_1\,| \cdot \Sigma_1 \;\sqcup\; |\,\Pi / \Pi_2\,| \cdot \Sigma_2 \;\;/\;|\Pi| \text{ identifications.}
\end{aligned}
$$

The removal of open discs by pairs for later gluing along the bounding spheres, corresponds to the definition of a 0-surgery, which is the usual connected sum when the pieces are disjoint. Keeping track of the gluings we see that the section we construct is connected, as each copy of Σ_1 is connected to any of Σ_2, and reciprocally. In particular, only $|\Pi / \Pi_1| + |\Pi / \Pi_2| - 1$ such surgeries are enough to

connect the different copies. This gives a possibly non-trivial remnant of 0-surgeries to be performed on the connected sum of all the copies. We have:

$$\Sigma \;\; = \;\; \left(\#_{n_1} \Sigma_1 \#_{n_2} \Sigma_2 \right) / m \;\; \text{0-surgeries.}$$

where $n_i = |\Pi \,/\, \Pi_i|$ and $m = |\Pi| - |\Pi \,/\, \Pi_1| - |\Pi \,/\, \Pi_2| + 1$. $\qquad\qquad$ $\square$

3 Examples of polar actions

We now construct several examples that will appear in our classification.

Example 3.1. A linear polar representation gives rise to a polar action on the sphere. Because of Lemma 2.5 these actions are Coxeter polar whenever the group acting is connected. We have the following examples.

1. The action of a torus $T^n \subset SO(2n) \subset SO(2n + N)$ on $\mathbb{S}^{2n+N-1} \subset \mathbb{C}^n \oplus \mathbb{R}^N$ is polar, since it comes from a trivially extended product of cohomogeneity one circle actions of S_i^1 on $\mathbb{C}_i$, for $i = 1, \ldots, n$. The section is $\{(\oplus_{i=1}^n \mathbb{R}_i) \oplus \mathbb{R}^N\} \cap \mathbb{S}^{2n+N-1}$, with polar group $\mathbb{Z}_2^n$ generated by reflections through the origin in each $\mathbb{R}_i$ factor, and corresponding chamber given by $\{(\prod_{i=1}^n \mathbb{R}_{\geq 0}) \times \mathbb{R}^N\} \cap \mathbb{S}^{2n+N-1}$. Isotropy subgroups are coordinate sub-tori $T_{i_1,\ldots,i_l}^l = \Pi_{j \in \{i_1,\ldots,i_l\}} S_j^1$.

2. The irreducible 5-dimensional SO(3) representation gives a linear action of cohomogeneity one on the sphere $\mathbb{S}^4$ with discrete principal isotropy group $H = S(O(1)^3) \subset SO(3)$ and singular isotropy groups $K_\pm$ given by two different block embeddings of $O(2)$ in $SO(3)$ in the ij coordinates. The polar group is D_3. Other examples are the transitive linear actions of SO(3) on $\mathbb{S}^2$ and SU(2) on $\mathbb{S}^3$, which are trivially polar.

3. Other linear polar actions on $\mathbb{S}^5$ come from the $SO(3) \times S^1$ product action on $\mathbb{R}^3 \oplus \mathbb{R}^2$, the standard 4-dimensional representation of SO(4), or its subgroup U(2), when trivially extended to $\mathbb{R}^6$. The latter two are double suspensions of transitive actions on $\mathbb{S}^3$, and therefore have Coxeter polar data given by a 2-disk with fixed points of the action along the boundary and interior marked by corresponding principal isotropy groups $SO(3) \subset SO(4)$ and $S^1 \subset U(2)$, respectively. The first case has a bi-angle (2-gon) for its quotient, marked with fixed points at the vertices, sides labeled by $K = SO(3)$ and $SO(2) \times S^1$, and interior with principal isotropy group $H = SO(2) \subset SO(3)$.

Before we give more detail about the main examples presented in the introduction, let us set the following convention.

Notation. A circle subgroup of a given Lie group G lying inside a fixed maximal torus identified as $T^k = S^1 \times \cdots \times S^1$, will be specified by its (Lie algebra) *slope* and denoted $S_v^1 \subset T^k \subset G$ where $Lie(S_v^1) = <v> \subset \mathbb{R}^k \cong Lie(T^k)$, e.g. $S_{(v_1,v_2)}^1 \subset S^1 \times S^1 \subset G$. Notice that v can be chosen to have coprime integer entries.

Example 3.2. The linear $G = T^3$ action on $\mathbb{S}^3 \times \mathbb{S}^3 \subset \mathbb{C}^2 \oplus \mathbb{C}^2$ given by $(\theta_1, \theta_2, \theta_3) \star (z_1, z_2, z_3, z_4) = (z_1\theta_1, z_2\theta_2, z_3\theta_3, z_4)$ is clearly Coxeter polar since it is a product of polar actions. It admits $\mathbb{S}^1 \times \mathbb{S}^2$ as section, with polar group $\Pi \cong (\mathbb{Z}_2)^3$ given by all the involutions in T^3 and chamber equal to $[-1,1] \times \mathbb{D}^2$, i.e., a solid "can". The boundary of C is composed of three faces with S^1 isotropy and two closed

circles with isotropy group isomorphic to T^2. Its interior is marked by the trivial group. Secondly, if we consider the quotient of $\mathbb{S}^3 \times \mathbb{S}^3$ by the action of the circle $L = \mathrm{S}^1_{(1,-1,k)}$ we obtain a Coxeter polar action on $\mathbb{S}^3 \times_k \mathbb{S}^2$, as described in Lemma 2.12. In particular the associated Coxeter polar data for both actions have the same (isometric) chamber. Furthermore, the marking of the induced action can be obtained as a relabeling of the isotropy group marking given by the projection $G \to G/L$. In order to facilitate its description we choose the subgroup $T^2 \cong \mathrm{S}^1_{(0,1,0)} \times \mathrm{S}^1_{(0,0,-1)} \subset T^3$, which is a complement to $\mathrm{S}^1_{(1,-1,k)} \subset T^3$ and therefore projects isomorphically onto its image under the map $\Phi : T^3 \to T^2 \cong T^3 / \mathrm{S}^1_{(1,-1,k)}$. We illustrate this in the following diagram:
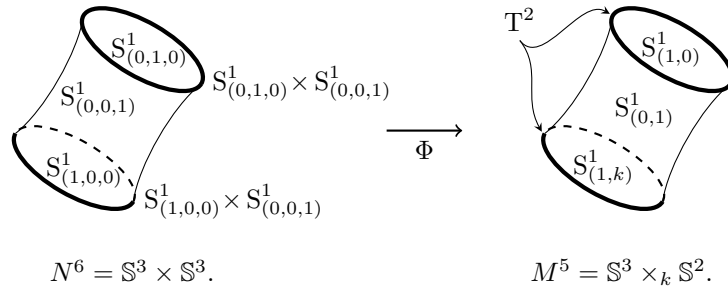

$$N^6 = \mathbb{S}^3 \times \mathbb{S}^3. \qquad\qquad M^5 = \mathbb{S}^3 \times_k \mathbb{S}^2.$$

Figure 1: A T^3 action on $\mathbb{S}^3 \times \mathbb{S}^3$ and the induced T^2 action on $\mathbb{S}^3 \times_k \mathbb{S}^2$ obtained from the quotient by a circle subgroup $\mathrm{S}^1_{(1,-1,k)}$.

The reflections along the maximal faces for the T^3 action are $(1,1,\pm 1)$, $(1,\pm 1,1)$ and $(\pm 1,1,1) \in T^3$ which, by Lemma 2.12, project to reflections for the induced T^2 on $\mathbb{S}^3 \times_k \mathbb{S}^2$ given by $r_1 = (1,\pm 1)$, $r_2 = (\pm 1,1)$ and $r_3 = (-1,(-1)^k)$, respectively. In addition, the new section corresponds to $\mathbb{S}^1 \times \mathbb{S}^2/\tau_k$ where $\tau_k = (-1,-1,(-1)^k)$ is the unique involution in L which belongs to the polar group of the T^3 action on $\mathbb{S}^3 \times \mathbb{S}^3$. This element reflects the $\mathbb{S}^2$ factor in Σ along a 2-plane when k is odd and acts trivially when k is even, while it acts as the antipode along the $\mathbb{S}^1$ factor. It is thus orientation preserving in the even case only. The resulting section depends on the parity of k:

$$\mathbb{S}^1 \times \mathbb{S}^2/\tau_k \cong \begin{cases} \mathbb{S}^1 \times \mathbb{S}^2 & k \text{ even} \\ (\mathbb{S}^1 \times \mathbb{S}^2)/\tau_k & k \text{ odd.} \end{cases}$$

Notice that the group Π is generated by the reflections r_1, r_2 with only even relations among them, but that when k is odd $r_3 = r_1 \cdot r_2$ gives an odd relation. This shows that in Lemma 2.8 one needs to include the reflections for all faces.

More complicated T^2 actions can be constructed combining the previous ones by regular orbit sums or fixed point sums; see Figure 8. These will be discussed in more detail in Section 5.

Example 3.3. General effective cohomogeneity two torus actions were previously studied in [OR74] and [Oh83, Oh82]. From their work it follows that a T^n action on a compact simply-connected manifold M^{n+2} which admits singular orbits has a topological $\mathbb{D}^2$ as quotient, with boundary stratified by segments of isotropy $K_i \cong S^1$, and vertices of isotropy $K_{i,i+1} \cong T^2$. Furthermore, they show that the marking determines the equivariant diffeomorphism type of the action. In their terminology this marking has to be *legally weighted*, in the sense that consecutive circle isotropy subgroups give independent generators of the common vertex group T^2, which is equivalent to (2.6). Hence, such a marking also gives compatible Coxeter polar data and hence another Coxeter polar 5-manifold. From [OR74] it follows that these actions are equivariantly diffeomorphic. Thus all these manifolds turn out to be polar, as pointed out in [GZ12].

In the case of T^2 actions in dimension 4, a constructive description and classification is achieved as connected fixed point sums of basic actions on $\mathbb{S}^4, \pm\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ (see [OR74]). Here, the action on $\mathbb{S}^4$ is linear, while the one on $\mathbb{CP}^2$ is obtained from the linear action of $G = T^3 \subset SO(6)$ on $\mathbb{S}^5$ under the quotient by the subgroup $L = S^1_{(1,1,1)}$; see Lemma 2.12. There is also an infinite family of inequivalent actions on $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# - \mathbb{CP}^2$. Coxeter polar data for these actions are illustrated in Figure 2 below.
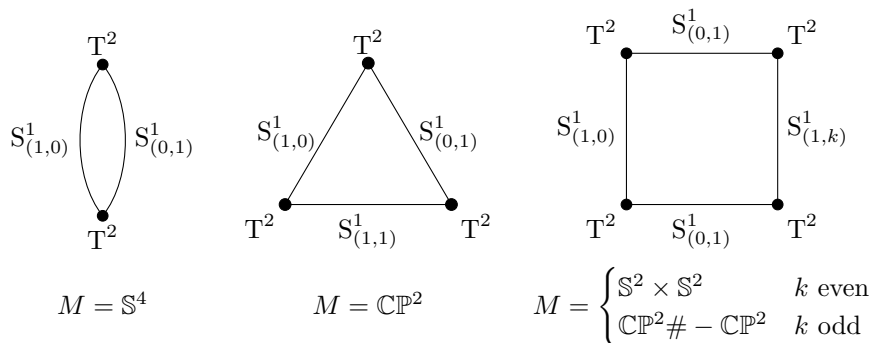

Figure 2:   Coxeter polar data for the basic T^2 actions on M^4.

In dimension 5, there is no finite set of actions such that all T^3 actions are obtained as equivariant surgeries along orbits of S^1 or T^2 isotropy ([Oh83]). Nevertheless, in addition to the equivariant description in terms of data, Oh shows that the underlying manifolds are diffeomorphic to $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$ or connected sums of these.

Example 3.4. There is a polar $SU(2)$ action on the biquotient $\mathbb{S}^3 \times_k \mathbb{S}^2$. Consider quaternion notation, $SU(2) \cong Sp(1)$, and let this group act on itself by multiplication from the left, identified as the first sphere factor of $M = \mathbb{S}^3 \times \mathbb{S}^3$. The latter action commutes with that of $S^1_{(1,-1,k,0)}$, and thus allows us to consider the product

action of $\mathrm{Sp}(1) \times \mathrm{S}^1_{(1,-1,k,0)} \ni (u,\theta)$ on $M = \mathrm{Sp}(1) \times \mathbb{S}^3$ by $(u,\theta) \star (z_1 + z_2 j, z_3, z_4) = (u \cdot (z_1 + z_2 j) \cdot \theta, z_3 . \theta^k, z_4)$. This is easily seen to be polar with $\Sigma = \{p\} \times \mathbb{S}^2 \subset \mathbb{S}^3 \times \mathbb{S}^3$, since it is orbit equivalent to the product of the standard action of $\mathrm{SU}(2)$ on the first factor with a polar circle action on the second. It is hence Coxeter polar by Remark 2.10.

We may consider the quotient under the normal subgroup $L = \mathrm{S}^1_{(1,-1,k,0)}$ to obtain an $\mathrm{SU}(2)$ action on $M/L = \mathbb{S}^3 \times_k \mathbb{S}^2$, as in Lemma 2.12. Coxeter polar data for the induced action is given in Figure 3 below. It is an imprimitive $\mathrm{SU}(2)$ polar action on $\mathbb{S}^3 \times_k \mathbb{S}^2$.

Example 3.5. Analogous to Example 3.2 is the case of the linear Coxeter polar $\mathrm{SU}(2) \times \mathrm{S}^1$ action on $\mathbb{S}^3 \times \mathbb{S}^3$ given by $(A,\theta) \star (z_1, z_2, z_3, z_4) = (A \cdot (z_1, z_2), z_3\theta, z_4)$ and the corresponding induced action on the quotient by $\mathrm{S}^1_{(1,-1,k,0)}$. In particular, we retrieve an imprimitive Coxeter polar $\mathrm{SU}(2) \times \mathrm{S}^1$ action on the manifold $M = \mathbb{S}^3 \times_k \mathbb{S}^2$. Choosing a maximal torus of G to be given by $\mathrm{U}(1) \times \mathrm{S}^1 \subset \mathrm{SU}(2) \times \mathrm{S}^1$, we have Coxeter polar data as shown in Figure 3.

$$H = \mathbb{Z}_k \qquad K = \mathrm{S}^1 \qquad\qquad H = \mathrm{S}^1_{(1,k)} \qquad K = \mathrm{T}^2$$

$$G = \mathrm{SU}(2) \supset K \supset H \qquad\qquad G = \mathrm{SU}(2) \times \mathrm{S}^1 \supset K \supset H$$
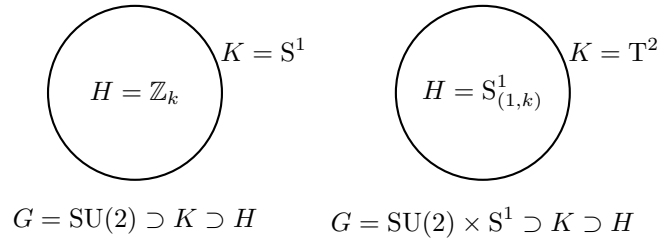
Figure 3:   Coxeter polar data for actions on $M^5 = \mathbb{S}^3 \times_k \mathbb{S}^2$.

Notice that the diagrams in Figure 3, when considered as Coxeter polar data for K actions and later made effective, agree with the standard polar circle action on $\mathbb{S}^3$.

Example 3.6. Consider the polar action of the circle $\mathrm{S}^1_{(0,0,0,1)}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ which, as before, commutes with that of the circle $\mathrm{S}^1_{(1,-1,k,0)}$. The polarity of the induced S^1 action on $\mathbb{S}^3 \times_k \mathbb{S}^2$ follows from Lemma 2.11. Notice that the original circle action has section $\Sigma = \mathbb{S}^3 \times \mathbb{S}^2 \subset \mathbb{S}^3 \times \mathbb{S}^3$ with chamber $\mathbb{S}^3 \times \mathbb{D}^2$, and that in this case the action of $\mathrm{S}^1_{(1,-1,k,0)}$ restricts to Σ. In fact, it restricts to the chamber $\mathbb{S}^3 \times \mathbb{D}^2$ preserving boundary and interior. In particular, the 4-dimensional chamber $C^4 = (\mathbb{S}^3 \times \mathbb{D}^2)/\mathrm{S}^1_{(1,-1,k,0)}$ associated to the action on $\mathbb{S}^3 \times_k \mathbb{S}^2$ is a smooth manifold with unstratified boundary consisting of fixed points of the action, and interior with trivial principal isotropy group. Notice that C^4 has the structure of a 2-disc bundle over $\mathbb{S}^2$ with projection onto the base $\mathbb{S}^2$ induced from that of $\mathbb{S}^3 \times \mathbb{D}^2$ onto its first factor. Such bundles are classified (up to bundle isomorphism) by $\pi_1(\mathrm{S}^1) \cong \mathbb{Z}$, and we may further restrict to non-negative integers by allowing an orientation

reversing diffeomorphism of the base. Finally, they are all topologically distinct and easily identifiable by the fundamental group of their boundary:

$$\partial C = (\mathbb{S}^3 \times \mathbb{S}^1)/\mathrm{S}^1_{(1,-1,k)} \cong \begin{cases} \mathbb{S}^2 \times \mathbb{S}^1 & k = 0 \\ L_k^3 & k \neq 0 \end{cases}$$

where $L_k^3 \cong \mathbb{S}^3/\mathbb{Z}_k$ is a Lens space. The polar group is $\mathbb{Z}_2$ given by a unique reflection, which reconstructs Σ as the double of C.

Example 3.7. The $\mathrm{SO}(3)$ action from the left on $\mathrm{SU}(3)/\mathrm{SO}(3) =: \mathcal{W}^5$ associated to the symmetric pair $(\mathrm{SU}(3), \mathrm{SO}(3))$ is naturally polar with section T^2 and hence Coxeter polar due to Remark 2.10.

Notice that the base point $[e \cdot \mathrm{SO}(3)] \in \mathrm{SU}(3)/\mathrm{SO}(3)$ is a fixed point of the action, where the slice representation is the unique irreducible 5-dimensional $\mathrm{SO}(3)$ representation of Example 3.1. Hence, the principal isotropy group is $H = \mathrm{S}(O(1)^3)$ and any face isotropy group has to be $O(2)$ since K/H is a sphere. Furthermore the embedding of $O(2)$ in $\mathrm{SO}(3)$ is determined by an axis, when thinking of $\mathrm{SO}(3)$ as acting on $\mathbb{R}^3$, which is also fixed by the principal isotropy $H = \mathrm{S}(O(1)^3)$. It must hence coincide with one of the coordinate axis, and $O(2)$ be given by one of the three coordinate block embeddings into $\mathrm{SO}(3)$. Moreover, two consecutive sides of the chamber correspond to different embeddings of $O(2)$ since they need to generate the isotropy group at the vertex of intersection. Vertices are thus fixed points of the actions. Finally, there are only three vertices, since fixed points $[g \cdot \mathrm{SO}(3)] \in \mathrm{SU}(3)/\mathrm{SO}(3)$ correspond to $[g \cdot \mathrm{SO}(3)] \in N^{\mathrm{SU}(3)}(\mathrm{SO}(3))/\mathrm{SO}(3) \cong \mathbb{Z}_3$. We conclude that the chamber is a triangle marked by each of the three block embeddings of $O(2)$ along the sides, as shown in Figure 4.
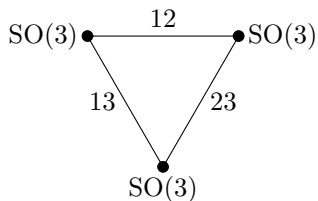

Figure 4:   Coxeter polar data for the $\mathrm{SO}(3)$ action on the Wu manifold. The labeling of the sides corresponds to the block embeddings of $O(2) \to \mathrm{SO}(3)$ in the ij coordinates.

Example 3.8. There is one case among the manifolds listed in Theorem C whose polarity is only established in an indirect way. This is the case of the $\mathrm{SO}(3)$ action on the Brieskorn variety of type $(2, 3, 3, 3)$, which is the zero locus of the polynomial $z_0^2 + z_1^3 + z_2^3 + z_3^3$ intersected with the corresponding sphere $\mathbb{S}^7(1) \subset \mathbb{C}^4$. This action was constructed in [Hud79] in the classification of $\mathrm{SO}(3)$ actions on

5-manifolds, with prescribed orbit space and isotropy data as in Figure 5. Using the Barden-Smale classification of 5-manifolds the author showed that the manifold is diffeomorphic to the Brieskorn variety $\mathcal{B}$. Furthermore, it was shown that there is a unique SO(3) action with such data up to equivariant diffeomorphism. In turn, we identify this as compatible Coxeter polar data to which there corresponds a unique Coxeter polar SO(3)-manifold, thus establishing the polarity of his construction. Notice that the polar group agrees with polar group at a fixed point and is hence D_3. Σ is a compact surface determined by its Euler characteristic and orientability. In fact, it follows from Lemma 2.8 that Σ is orientable since any presentation of D_3 with generators given by involutions has even relations only. Furthermore, we can regard the quotient map $\Sigma \to C$ under the action of Π as an orbifold cover (see [Dav11]), and compute the Euler characteristic of the section from the *orbifold Euler characteristic* of C by the formula:

$$\chi(\Sigma)/|\Pi| = \chi_{orb}(C).$$

In this case C has four points with orbifold group D_3, four sides fixed by an involution and one regular face, thus giving $\chi_{orb}(C) = 4 \cdot 1/6 - 4 \cdot 1/2 + 1 = -1/3$. Hence, the section is a bi-torus, $\Sigma = T^2 \# T^2$.

It is interesting to compare the data for this action with the connected fixed point sum of two copies of the Wu manifold $\mathcal{W} \# \mathcal{W}$; see Figure 5. For both $\mathcal{B}$ and $\mathcal{W} \# \mathcal{W}$ the associated section, polar group and chamber coincide, but the action of Π on Σ is different.

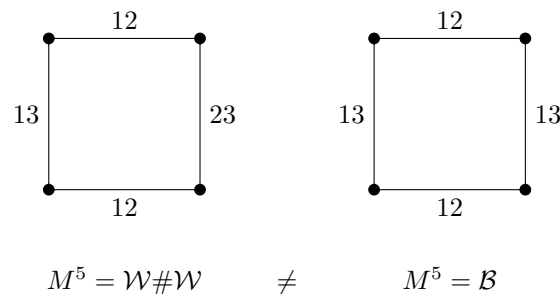$$M^5 = \mathcal{W} \# \mathcal{W} \qquad \neq \qquad M^5 = \mathcal{B}$$

Figure 5: Coxeter polar data for SO(3) actions on M^5. Vertices are fixed points of the action and the labeling of the sides corresponds to the ij coordinate embedding of $O(2)$ in SO(3).

Further fixed point connected sums give rise to polar actions on $\#_k \mathcal{B} \#_l \mathcal{W}$. The polar group is always the local polar group at a fixed point, $\Pi = D_3$, hence the section is orientable and corresponds to a connected sum of tori $\#_n T^2$. Coxeter polar data can be computed by Lemma 2.18, thus obtaining a polygon marked with principal isotropy group $H = S(O(1)^3)$ along the interior, vertices given by fixed points of the action, and sides labeled by either of the three coordinate block

embeddings of $O(2)$ in SO(3) without consecutive repetitions. Conversely, any such labeling of a polygon gives compatible Coxeter polar data for an SO(3) action. With only two sides this is the linear action on $\mathbb{S}^5$ given by the trivial extension of the irreducible 5-dimensional SO(3) representation. For three sides, the compatibility conditions force all three block embeddings to appear once and this corresponds to the isotropy action of SO(3) on the Wu manifold. With 4 sides, we have two possibilities for the markings, up to conjugation by a single element in $N(H)$, and these correspond either to the connected fixed point sum $\mathcal{W}\#\mathcal{W}$ or to the Brieskorn variety $\mathcal{B}$, as in Figure 5. One may continue with this kind of reasoning and prove that there is a unique marking with five sides and four possibilities for the case of six sides. Finally, an inductive argument on the number of sides shows that the fixed point sums of $\mathcal{W}$ and $\mathcal{B}$ exhaust all possibilities. For this, notice first that if only two block embeddings of $O(2)$ are present in the data, then the number of sides must be even and it corresponds to a repeated fixed point connected sum of $\mathcal{B}$ with itself. On the other hand, if three different block embeddings appear in the data, we can find a consecutive sequence of three sides with distinct labels. Erasing the middle side of these three and merging two consecutive vertices into one, we again have compatible polar data to which we can apply the inductive hypothesis. Thus this manifold can be written as a fixed point sum $\#_k\mathcal{B}\#_l\mathcal{W}$. Finally we can perform an additional fixed point sum with the Wu manifold at the vertex where we erased a side, which gives back our original data and thus establishes the claim. Notice that when we have only one side which has a different block embedding than the rest, we can choose not to remove it but choose a neighboring side instead, thus keeping the three different block embeddings of $O(2)$ on the resulting data. In this way the induction continues and proves that the original manifold can be written as a repeated fixed point sum of Wu manifolds only, $\#_n\mathcal{W}$. We conclude that the SO(3) actions constructed out of such Coxeter polar data are connected fixed point sums $\#_n\mathcal{B}$ or $\#_n\mathcal{W}$. However, while there is a unique action on $\#_n\mathcal{B}$ up to conjugation, this is not the case for $\#_n\mathcal{W}$ since it depends on the possible markings, thus giving rise to inequivalent actions on the same manifold.

Example 3.9. The $\text{SO}(3) \times \text{S}^1$ factor-wise action on $\mathbb{S}^3 \times \mathbb{S}^2$ is a product of cohomogeneity one actions. Coxeter polar data is given by a rectangle with four fixed points, and sides labeled by singular isotropy groups $K_{2i+1} = \text{SO}(3)$ and $K_{2i} = \text{T}^2$. The principal isotropy group is $H = \text{S}^1_{1,0} =: \text{SO}(2)$.

We may perform fixed point sums of the latter action with itself, obtaining new polar manifolds. A generic picture for the Coxeter polar data is given in Figure 6. Notice that the polar group is always $(\mathbb{Z}_2)^2$ and the section $\text{T}^2 \# \cdots \# \text{T}^2$.

A diagram with only two vertices is also possible and corresponds to the suspension of the $\text{SO}(3) \times \text{S}^1$ action on $\mathbb{S}^4$, i.e., the action on $\mathbb{S}^5$ given by the trivial extension of the product $\text{SO}(3) \times \text{S}^1$ representation on $\mathbb{R}^3 \oplus \mathbb{R}^2$.
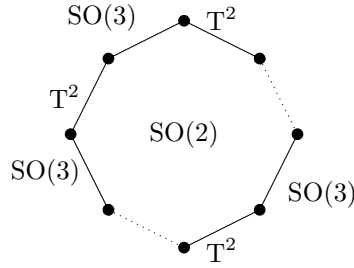
Figure 6: Polar data for the $SO(3) \times S^1$ action on $\#_n \mathbb{S}^3 \times \mathbb{S}^2$, where the vertices correspond to fixed points.

4 Classification in low dimensions

We shall consider compact simply-connected smooth manifolds M^n, $n \leq 5$, on which a compact connected Lie group G acts polarly, thus Coxeter polar, and classify them up to equivariant diffeomorphism. The actions are assumed to be almost effective, i.e., with a discrete ineffective kernel. As in the previous section, H will denote the principal isotropy group along a section $\sigma : \Sigma^k \to M^n$ with polar group Π and chamber C.

The strategy in order to achieve a classification is to solve the equivalent problem of determining all sets of inequivalent Coxeter polar data (see Theorem 2.9). To do so, we will consider the possible groups, chambers and compatible group graphs marking the stratifications. We will then identify these data with the actions in Section 3.

To begin, we notice that the bound on the dimension of the manifold restricts both the cohomogeneity k and the dimension of the group, as follows from the formulas,

$$n - k = dim(G) - dim(H)$$

$$\tag{4.1}$$

$$dim(H) \leq dim(O(n - k)).$$

The first is a simple dimension count on the decomposition $T_p M = T_p(G \cdot p) \oplus S_p$ at a regular point. The second comes from the fact that the representation of the principal isotropy group in the orbit direction corresponds to the restriction of the adjoint representation of G to H, and its kernel to the ineffective kernel of the G-action. Therefore, we have $H \subset O(n - k)$.

Moreover, any compact connected Lie group G can be taken, up to a finite cover, to be of the form

$$G = T^l \times G_1 \times \cdots \times G_k \tag{4.2}$$

for a torus T^l and simply-connected simple factors G_i. Together with the bound on the dimension, this yields finitely many possibilities for the group.

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Notice that if the action does not have singular orbits, the section will agree with the chamber, $C = \Sigma$, which is thus a smooth manifold without boundary constantly marked by a principal isotropy group H. This is an example of an imprimitive action for which the reconstructed manifold is a product of the section with a regular orbit, $M = G/H \times \Sigma$. These Coxeter polar actions will be called *trivial*.

Since M is simply-connected, the orbit space M/G is compact and simply-connected as well. Furthermore, it is isometric to a chamber $C^k \subset \Sigma$ which has the structure of a smooth Coxeter orbifold, i.e., one modeled on the quotient of euclidean k-space by finite Coxeter groups. In particular, there will be no singularities along the interior and we may "round the corners" along the boundary to obtain a homeomorphism of C with a topological manifold with boundary.

Proposition 4.3. *A compact simply-connected n-manifold with boundary is homeomorphic to $\mathbb{D}^2$ if $n=2$, or a sphere with finitely many discs removed $\mathbb{S}^3 \setminus \sqcup_{i=0}^r \mathbb{D}_i^3$, $r \in \mathbb{N}$, if $n=3$.*

The first case follows from the classification of closed surfaces, when capping the circles in the boundary with closed discs. The case $n = 3$ is a consequence of Poincaré duality, as shown in [Sim].

Once the orbit space is determined as a topological manifold with boundary, we must still address its stratification and what the markings of the strata by isotropy subgroups could be. This is particularly simple in the case of polar actions of maximal cohomogeneity, i.e., those with codimension one section.

Proposition 4.4. *A polar action on a closed n-manifold of cohomogeneity $n-1$ is a circle action with orbit space given by a smooth manifold with boundary marked as fixed points. Conversely, to each smooth manifold with boundary corresponds a unique polar circle action on a closed manifold.*

The associated manifold is simply-connected if and only if the orbit space is also simply-connected.

Proof. If the cohomogeneity of the action is $n-1$ we have by (4.1) that the principal isotropy has to be discrete, since $O(1)$ is, and the group G one-dimensional. When made effective this is an S^1-action with trivial principal isotropy group and $K = \mathrm{S}^1$ as the only other possible singular isotropy subgroup. This corresponds to fixed points on M, which constitute the boundary of the chamber.

As a converse, any (simply-connected) smooth manifold with boundary C^{n-1} gives rise to a unique Coxeter polar S^1-manifold $M^n = M(C, \mathrm{S}^1(C))$ by Theorem 2.9. In particular, if this boundary were empty we are in the trivial case $M = M' \times \mathrm{S}^1$, which is not simply connected. Furthermore, if the boundary is not trivial, Seifert-Van Kampen's theorem easily implies that the fundamental group of the reconstructed manifold is that of the chamber C. Indeed, notice that the interior of

C gets reconstructed as $\mathrm{int}(C) \times \mathrm{S}^1$, while an ϵ-collar neighborhood of a boundary component of C gives rise to $\partial_0 C \times \mathbb{D}^2 \cong D_\epsilon(\partial_0 C) \subset M$, from the uniqueness of the reconstruction process. Each such piece is to be glued with the interior along the common intersection $(0, \epsilon) \times \partial_0 C \times \mathrm{S}^1$, which cancels the fundamental group of the S^1 factor. We then continue by induction on the boundary components. $\square$

Remark. A polar circle action is *fixed point homogeneous*, i.e., the group S^1 acts transitively on the sphere orthogonal to the fixed point stratum.

The polar group is $\mathbb{Z}_2$, generated by the unique reflection in S^1 and the section is the double of the chamber, $\Sigma = \mathcal{D}(C)$.

Corollary 4.5. *There is only one effective compact simply-connected cohomogeneity two polar 3-manifold, given by a linear S^1-action on $\mathbb{S}^3$.*

We now describe the classification in dimension 4.

Theorem 4.6. *Let G be a compact connected Lie group, with a non-trivial and and almost-effective polar action on a compact simply-connected 4-manifold M of cohomogeneity $k \geq 2$. Then M is equivariantly diffeomorphic to one of the following:*

  a) T^2-actions as described in Example 3.3;

  b) The linear S^1 and $\mathrm{SO}(3)$ actions on $\mathbb{S}^4$ as in Example 3.1;

  c) The linear circle action on the first factor of $\mathbb{S}^2 \times \mathbb{S}^2$, or their connected fixed-point sums $\#_n \mathbb{S}^2 \times \mathbb{S}^2$.

Proof. The circle actions corresponds to the case $k = 3$ which was already considered in Proposition 4.4. Additionally we identify the reconstructed manifolds to be $\mathbb{S}^4$ when $C = \mathbb{D}^3$, $\mathbb{S}^2 \times \mathbb{S}^2$ for $C = \mathbb{D}^1 \times \mathbb{S}^2$, and fixed point sums of the latter in the remaining cases. In fact, each fixed point connected sum of the previous $\mathbb{S}^2 \times \mathbb{S}^2$ adds one sphere component to the boundary of the chamber (see Proposition 2.18), allowing to retrieve any possible chamber $C = \mathbb{S}^3 \setminus \sqcup_{i \leq N} (\mathbb{D}^3)_i$.

If the cohomogeneity is 2, either the principal isotropy group is discrete and hence $dim(G) = 2$, which implies $G = \mathrm{T}^2$, or we have a one dimensional principal isotropy group and $G = \mathrm{SU}(2)$. For this last case, since there are no subgroups of $\mathrm{SU}(2)$ of intermediate dimension, the chamber has to be a disc $\mathbb{D}^2$ with un-stratified boundary marked by $K = \mathrm{SU}(2)$. As one can check, this coincides with the polar data for the effective $\mathrm{SO}(3)$ action on $\mathbb{S}^4$ given by its irreducible 3-dimensional representation plus two fixed directions. $\square$

Corollary 4.7. *A closed simply-connected 4-manifold admitting a polar action is diffeomorphic to $\mathbb{S}^4$ or a connected sum of copies of $\pm\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$.*

The previous corollary combines our classification and the work on cohomogeneity one manifolds in [Hoe10]. It should be mentioned that this is true even for non-polar actions as observed in [GR].

We now address the classification of polar actions by non-abelian groups in dimension 5, starting with those by $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$.

Theorem 4.8. *A non-trivial polar* $\mathrm{SU}(2)$ *action on a closed simply-connected 5-manifold is equivariantly diffeomorphic to one of the following:*

1. *The linear polar actions on* $\mathbb{S}^5$ *listed in Example 3.1;*

2. *The actions on* $\mathbb{S}^3 \times_k \mathbb{S}^2$ *described in Example 3.4;*

3. *The* $\mathrm{SO}(3)$-*action on the Brieskorn variety* $\mathcal{B}$ *of type* $(2,3,3,3)$, *the left* $\mathrm{SO}(3)$-*action on the Wu manifold* $\mathcal{W} = \mathrm{SU}(3)/\mathrm{SO}(3)$, *or connected fixed point sums* $\#_n\mathcal{B}$ *and* $\#_m\mathcal{W}$;

4. *The linear* $\mathrm{SO}(3)$ *action on the first factor of* $\mathbb{S}^3 \times \mathbb{S}^2$ *and repeated connected fixed point sums,* $\#_n\mathbb{S}^3 \times \mathbb{S}^2$.

Proof. We distinguish between two main cases depending on whether the principal isotropy group is discrete or one dimensional. In the latter case, we have a cohomogeneity three action, with singular isotropy given by the whole group since there are no subgroups of $\mathrm{SU}(2)$ of intermediate dimension. It follows that $H = \mathrm{S}^1$, and that the chamber is a smooth manifold with unstratified boundary marked as fixed points. A basic example is the $\mathrm{SO}(3)$ linear action on $\mathbb{S}^5$ given by the trivial extension of its 3-dimensional representation, which corresponds to chamber $C = \mathbb{D}^3$. The remaining cases correspond to the action on the first factor of $\mathbb{S}^3 \times \mathbb{S}^2$ with chamber $\mathbb{D}^1 \times \mathbb{S}^2$, or their connected fixed point sums which have $C = \mathbb{S}^3 \backslash \bigsqcup_i \epsilon \mathbb{D}^3_i$.

We now assume that H is discrete. Hence the action has cohomogeneity two and $C = \mathbb{D}^2$. The possible singular isotropy groups are $K = \mathrm{S}^1$, $\mathrm{Pin}(2)$ or $\mathrm{SU}(2)$. If the boundary is unstratified and constantly marked by $K = \mathrm{S}^1$ or $\mathrm{Pin}(2)$, this gives imprimitive actions which reconstruct S^3 as K-manifolds, while as $\mathrm{SU}(2)$-manifolds they are equivariantly diffeomorphic to

$$M = \mathrm{SU}(2) \times_K \mathbb{S}^3.$$

Since K acts freely on the product, we conclude from the associated long homotopy sequence that, if K is not connected, then its quotient M is not simply-connected. If we assume $K = \mathrm{S}^1$ and $H = \mathbb{Z}_k$, this is uniquely identified with the examples treated in 3.4, which correspond to actions on the biquotient $\mathbb{S}^3 \times_k \mathbb{S}^2$. If on the other hand, the whole boundary corresponds to fixed points of the $\mathrm{SU}(2)$ action, the principal isotropy group is then trivial and we recognize this as the linear 4-dimensional $\mathrm{SU}(2)$-representation extended to act on $\mathbb{S}^5$.

For the remaining cases the chamber has stratified boundary, with isolated vertices corresponding to fixed points of the action, in order to satisfy (2.6). The slice representation at such points must be the irreducible 5-dimensional one of Example 3.1.2. These cases correspond to the action on $\mathbb{S}^5$ by the (trivial extension of the) SO(3) 5-dimensional representation, the isotropy action of SO(3) on the Wu manifold, the SO(3) action on the Brieskorn variety, and their connected fixed point sums. We refer to the discussion in Examples 3.7, 3.8 and to Figures 4 and 5. □

Remark 4.9. General SO(3) actions on 5-manifolds were previously studied in [Hud79] and more recently in [Sim] who corrected and extended the classification to that of (almost effective) SU(2) actions. An alternative proof of Theorem 4.8 follows from Theorem D in [Sim], after identifying the polar examples and discarding the non-polar ones. The latter correspond to the linear diagonal SO(3) action on $\mathbb{S}^5$ and to the left SU(2)-action on the Wu manifold. The SO(3) action on $\mathbb{S}^5 \subset \mathbb{R}^{3+3}$ has orbit space a disk with boundary marked by S^1 isotropy, and trivial principal isotropy along the interior. This gives compatible imprimitive Coxeter polar data for an SO(3) action, but corresponds to the (clearly polar) SO(3) action on $\mathbb{S}^3 \times_{k=2} \mathbb{S}^2 \cong \mathbb{S}^3 \times \mathbb{S}^2$, presented in Example 3.4 as an ineffective SU(2) action. In the same way, the left SU(2)-action on $\mathcal{W}$ has orbit space a disc with constant S^1 isotropy along the bounding circle and trivial isotropy in the interior, which gives rise to Coxeter polar data reconstructing $\mathbb{S}^3 \times \mathbb{S}^2$ instead of $\mathbb{S}^5$.

For general non-abelian group actions we have the following, which thus finishes the proof of Theorem C in the introduction.

Theorem 4.10. *Let G be a compact connected non-abelian Lie group, with a non-trivial and almost-effective polar action on a compact simply-connected 5-manifold M of cohomogeneity $k = 2, 3$. Then M is equivariantly diffeomorphic to one of the following:*

a) *The* SU(2) *polar actions listed in Theorem 4.8;*

b) *The family of* $\mathrm{SU}(2) \times S^1$*-actions on* $\mathbb{S}^3 \times_k \mathbb{S}^2$ *as in Example 3.5;*

c) *The linear factor-wise* $\mathrm{SO}(3) \times S^1$ *action on* $\mathbb{S}^3 \times \mathbb{S}^2$*, and their connected fixed-point sums;*

d) *Linear polar actions by* $\mathrm{U}(2)$ *and* $\mathrm{SO}(4)$ *on* $\mathbb{S}^5$*.*

Proof. From the same considerations as in the classification in dimension four, we have that a cohomogeneity $k = 3$ action corresponds to an SU(2) action with one dimensional principal isotropy group.

If $k = 2$ we have $C \cong \mathbb{D}^2$ and $H \subseteq O(3)$ up to a finite cover, leaving the following possibilities:

$$(G, H_0) = (\mathrm{SU}(2), e), (\mathrm{SU}(2) \times \mathrm{SU}(2), \Delta\mathrm{SU}(2)), (\mathrm{SU}(2) \times S^1, S^1). \tag{4.11}$$

Here, besides listing the possible groups by the dimension bound (4.1) for a given $H_0 \subset G$, we have discarded $G = \mathrm{SU}(2) \times \mathrm{T}^3$ with $H_0 = \mathrm{SU}(2)$, since it would not be almost-effective. The $\mathrm{SU}(2)$ actions of cohomogeneity 2 and 3 have been addressed in Theorem 4.8.

For $(G, H_0) = (\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$, notice that the effectiveness of the G-action implies that H_0 is embedded diagonally. Furthermore, there are no subgroups of intermediate dimension between G and H_0, implying that the boundary of the orbit space consists of fixed points by G. In particular, we have that H must be connected, since G corresponds to a face isotropy group and satisfies $G/H = \mathbb{S}^3$. Thus the polar data is a disc with interior regular isotropy $\Delta \mathrm{SU}(2)$ and boundary consisting of fixed points. If we pass to the quotient under the normal subgroup $\mathbb{Z}_2 \cong \{\pm 1\} \subset \Delta \mathrm{SU}(2)$, we have an effective $G = \mathrm{SO}(4)$ action with principal isotropy group $H = \mathrm{SO}(3)$. This equivalent effective data is recognized as the linear $\mathrm{SO}(4) \subset \mathrm{SO}(6)$ action on $\mathbb{S}^5$.

If $(G, H_0) = (\mathrm{SU}(2) \times \mathrm{S}^1, \mathrm{S}^1)$, we start by determining the ways in which H_0 may lie in G. This cannot be given by the inclusion of the second factor, as it would not be almost effective. It must hence be $H_0 = \mathrm{S}^1_{(1,k)} \subset \mathrm{SU}(2) \times \mathrm{S}^1$, for $k \in \mathbb{Z}$, since a general circle $\mathrm{S}^1_{(q,k)}$ would have normal ineffective kernel $\mathbb{Z}_q \cong < e^{i2\pi/q} > \subset \mathrm{S}^1$ which can be quotiented out and still retrieve an $\mathrm{SU}(2) \times \mathrm{S}^1$ action. We distinguish between the inclusion of the first factor only ($k = 0$) and the diagonal case ($k \neq 0$).

In the case of $H_0 = \mathrm{S}^1_{(1,k)}$, $k \neq 0$, the singular isotropy groups can be $K \cong \mathrm{T}^2$, $\mathrm{Pin}(2) \times \mathrm{S}^1$, or the whole group $\mathrm{SU}(2) \times \mathrm{S}^1$. We claim that, in any case, the boundary will have constant isotropy. Notice that there is only one choice for the maximal torus inside the different possible singular isotropy subgroups of a face, namely, the one given by the product of the circle subgroups obtained from factor-wise projection of $H_0 \cong \mathrm{S}^1 \subset \mathrm{SU}(2) \times \mathrm{S}^1$. This rules out the possibility of a further stratification of the boundary, since consecutive sides have to be labeled by different isotropy groups and generate the isotropy group of the vertex in common. In the case of unstratified boundary constantly marked by $K = \mathrm{T}^2$ or $\mathrm{Pin}(2) \times \mathrm{S}^1$ we have imprimitive actions which reconstruct $\mathbb{S}^3$ as a K-manifold, while as a $\mathrm{SU}(2) \times \mathrm{S}^1$-manifold they give

$$(\mathrm{SU}(2) \times \mathrm{S}^1) \times_K \mathbb{S}^3.$$

In particular, K has to be connected in order for M to be simply-connected. Thus we can assume $K = \mathrm{T}^2$ which is the case discussed in Example 3.5. The remaining case with unstratified boundary equal to the fixed point set can only happen if the principal isotropy group is $H = \mathrm{S}^1_{(1,1)}$, in order to satisfy $K/H \cong \mathbb{S}^3$. This is identified as the linear effective action on $\mathbb{S}^5$ of the group $\mathrm{U}(2) \cong \mathrm{SU}(2) \cdot \mathrm{S}^1$ acting as a subgroup of $\mathrm{SO}(4) \subset \mathrm{SO}(6)$.

Finally, we turn to the case where the principal isotropy is $H_0 = \mathrm{S}^1_{(1,0)}$, which corresponds to an effective $\mathrm{SO}(3) \times \mathrm{S}^1$ action (since we can quotient out the normal

subgroup $\{(\pm 1, 1)\} \subset H_0 \subset \mathrm{SU}(2) \times \mathrm{S}^1$). If the boundary is unstratified it is thus constantly marked by either $K = \mathrm{T}^2$, $\mathrm{SO}(3)$ or $\mathrm{SO}(3) \times \mathrm{S}^1$. The case with $K = \mathrm{SO}(3)$ gives an imprimitive action that reconstructs a non-simply connected manifold. The other two can be treated as before, and are easily identified as the action on $\mathbb{S}^3 \times_{k=0} \mathbb{S}^2 = \mathbb{S}^3 \times \mathbb{S}^2$ in Example 3.5, and the linear action on $\mathbb{S}^5$ given by the trivial extension of the $\mathrm{SO}(3) \times \mathrm{S}^1$ representation in $\mathbb{R}^3 \oplus \mathbb{R}^2$. Finally, if the boundary is stratified the marking of the sides must alternate between $K = \mathrm{T}^2$ and $\mathrm{SO}(3)$. These actions were the ones discussed in Example 3.9. $\qquad \square$

5 Torus actions

In this section we treat in detail the case of polar effective T^2-actions on compact simply-connected 5-manifolds. We will prove Theorem A by determining the admissible Coxeter polar data, addressing the possible isotropy groups, orbit spaces and marked stratifications. In a second step we identify the actions concretely, which requires the description of a few basic examples and the combination of them by means of equivariant surgeries.

First, observe that the possible isotropy types (G_x) are known and correspond to $\{e\} = H \subseteq \mathrm{S}^1 \cong K \subseteq \mathrm{T}^2$, where S^1 ranges over all the possible circle subgroups inside the torus, and we omit parenthesis as conjugation is trivial. In fact, the effectiveness of the action implies that the principal isotropy group is trivial since the group is abelian. A next to minimal isotropy subgroup has to be a circle, since K/H is a sphere. Moreover, since K_0 and H generate K, further isotropy groups will have to be connected leaving as only possibility $K = G = \mathrm{T}^2$, i.e., fixed points of the action.

Secondly, notice that at a fixed point the 5-dimensional slice representation coincides with the standard one by the maximal torus $\mathrm{T}^2 \subset \mathrm{SO}(4) \subset \mathrm{SO}(5)$. In particular, the existence of a fixed direction implies that fixed points are not isolated, but form closed geodesics. For a point with S^1 isotropy, the orbit is a circle and the slice must be 4-dimensional. The slice representation is the usual 2-dimensional one by S^1 extended trivially.

The possible orbit spaces are topological 3-manifolds with boundary, already characterized in Proposition 4.3 as the complement of a finite number of disjoint 3-discs in a 3-sphere,

$$C = M/G = \mathbb{S}^3 \backslash (\sqcup_N \mathbb{D}^3).$$

Each component of the boundary $\mathbb{S}^2 \subseteq \partial(M/G)$ will be the union of one or more faces with S^1-isotropy, separated by closed geodesics consisting of fixed points. We can encode the boundary stratification and marking information in an unoriented graph, defining vertices to be the 2-dimensional faces and edges to be the closed geodesics fixed by the action. It is well defined, since these fixed geodesics do not intersect each other, leaving only two adjacent faces corresponding to their endpoints. Also, as any simple closed curve separates the 2-sphere into two pieces, this graph is in fact a tree. The marking of the chamber labels each vertex with a pair of coprime integers $(m_i, n_i) \in \mathbb{Z}^2$, indicating the slope of the corresponding isotropy circle subgroup. The compatibility condition (2.6) corresponds to consecutive vertices having isotropy subgroups with trivial intersection, or equivalently that

$$\begin{vmatrix} m_i & n_i \\ m_{i+1} & n_{i+1} \end{vmatrix} = \pm 1.$$

We call this graph the *boundary tree*, which is a more convenient way of describing the Coxeter polar data, adapted to this case. In particular, any finite tree with

a compatible marking of slopes and possibly many connected components, will give rise to a unique connected Coxeter polar T^2-manifold of dimension five. This reconstructed manifold need not be simply-connected, a problem we shall address later.

We first focus on actions having only one boundary component for the orbit space, i.e., a connected boundary tree. The simplest case is the one without fixed points, and hence only one face marked by a circle isotropy subgroup K. It corresponds to a tree with only one vertex. This action is imprimitive and reconstructs $\mathbb{S}^4$ when restricted to $K = \mathrm{S}^1$ with the action by rotations in $\mathbb{R}^2$, trivially extended to $\mathbb{R}^5$. As a T^2 polar manifold it yields

$$\mathbb{S}^4 \times_K \mathrm{T}^2 = \mathbb{S}^4 \times_K (K \times \mathbb{S}^1) = \mathbb{S}^4 \times \mathbb{S}^1.$$

Although this manifold is not simply-connected it will be one of the building blocks in our construction.

A boundary tree with two vertices connected by a single edge corresponds to a pair of neighboring faces separated by one geodesic consisting of fixed points. We can assume that the circle isotropy subgroups marking the faces are the canonical ones of slopes $(1, 0)$ and $(0, 1)$, up to a T^2 automorphism. The manifold is recognized as the linear T^2 action on $\mathbb{S}^5$ given by the standard embeddings $\mathrm{T}^2 \subset \mathrm{SO}(4) \subset \mathrm{SO}(6)$.

The T^2 action on $\mathbb{S}^3 \times_k \mathbb{S}^2 = \mathbb{S}^3 \times \mathbb{S}^3 / \mathrm{S}^1_{(1,1,k,0)}$ as described in Example 3.2 has boundary tree given by a linear sequence of three vertices labeled as in Figure 7. Conversely, any T^2 Coxeter polar action with the same underlying boundary tree can be normalized to have the same marking by isotropy subgroups, as follows. First pick two consecutive vertices and assume that their isotropy groups are the circles of slopes $(1, 0)$ and $(0, 1)$, up to a T^2 automorphism. Furthermore, we can choose the middle vertex to be labeled $(0, 1)$, so that the remaining one will have isotropy subgroup of the form $(1, k)$, for some integer $k \in \mathbb{Z}$, since it is a singular isotropy subgroup with trivial intersection with the circle $(0, 1)$.

These manifolds can be combined by taking fixed point connected sums to obtain new T^2 polar manifolds. The resulting action may vary depending on the point where it is performed, as opposed to the usual topological connected sum. Once compatible points are specified the gluing is determined by the marking of isotropy groups, since orientations for the manifolds have to be chosen so that both have outward (or inward) pointing normal vector field along the maximal faces of the boundary. We illustrate these subtleties in Figure 8, where two almost identical pairs of manifolds are glued, and in fact only one summand differs from its analogous by a T^2 automorphism.

By means of fixed point connected sums of the basic actions in Figure 7 we obtain all Coxeter polar T^2 actions with orbit space $M^5 / \mathrm{T}^2 \cong \mathbb{D}^3$.

Proposition 5.1. *A Coxeter polar* T^2-*manifold of dimension* 5 *with orbit space homeomorphic to the* 3-*disc and* $N \geq 0$ *geodesics fixed by the action is either:*
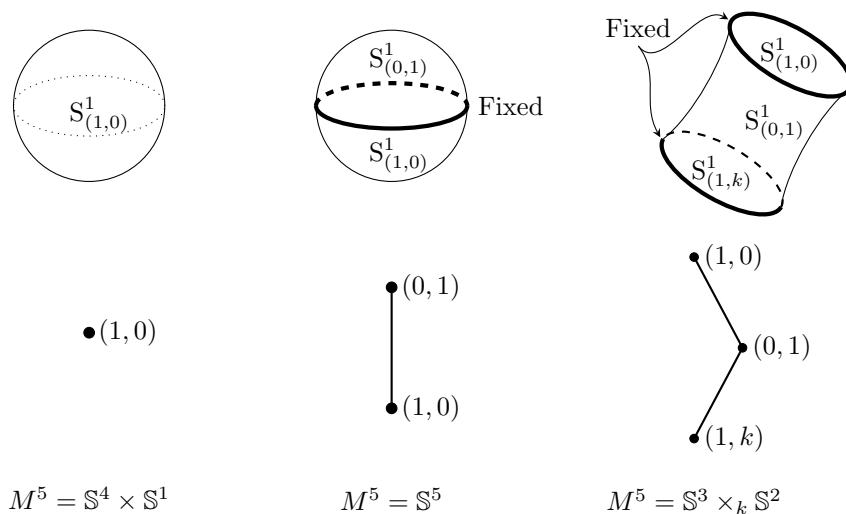
Figure 7: Coxeter polar data and their corresponding boundary trees for the basic T^2 actions on 5-manifolds with orbit space homeomorphic to the 3-disc.

1. The factor-wise $S^1 \times S^1$ polar action on $\mathbb{S}^4 \times \mathbb{S}^1$, for $N = 0$;

2. The linear T^2 action on $\mathbb{S}^5$ of Example 3.1, for $N = 1$;

3. The T^2-actions on $\mathbb{S}^3 \times_k \mathbb{S}^2$ described in Example 3.2 above, if $N = 2$;

4. Iterated $(N - 2)$ fixed-point connected sums of $\mathbb{S}^3 \times_k \mathbb{S}^2$, for different $k \in \mathbb{Z}$, in the case of $N \geq 3$.

Proof. Notice that the number of fixed point connected components is the number of edges in the boundary tree, and that we have already discussed the situation for $N \leq 2$. Observe as well that, if we consider two T^2-manifolds M_1 and M_2 as above that are associated to connected boundary trees $\mathcal{T}_i$, for $i = 1, 2$, and we perform an equivariant connected sum at respective fixed points, as in Proposition 2.18, we will retrieve a new manifold with connected boundary tree given by the gluing along corresponding edges (the ones associated to the fixed geodesics containing those fixed points):

$$\mathcal{T} = \mathcal{T}_1 \bigcup_{e_1 \sim e_2} \mathcal{T}_2.$$

In particular, when we glue a linear sequence of three vertices along an edge we are adding one more vertex to the original tree. We can do this in any way we like, thus being able to "grow" all trees by this repeated operation, proving part (4). □

Proof of Theorem A. If we are given two Coxeter polar T^2-manifolds of dimension five associated to polar data $(C_i, T^2(C_i))$, $i = 1, 2$, we may perform a surgery along

$S^1_{(q,1)}$

$S^1_{(1,0)}$

$S^1_{(0,1)}$ # $S^1_{(1,0)}$

$S^1_{(0,1)}$

$S^1_{(1,k)}$

$S^1_{(1,0)}$ # $S^1_{(1,0)}$

$S^1_{(0,1)}$

$S^1_{(1,k)}$ $S^1_{(0,1)}$

$S^1_{(1,q)}$

$\neq$

$S^1_{(q,1)}$

$S^1_{(1,0)}$

$S^1_{(0,1)}$

$S^1_{(1,k)}$

$S^1_{(1,0)}$

$S^1_{(0,1)}$

$S^1_{(1,k)}$ $S^1_{(1,q)}$

$(q,1)$

$(1,0)$
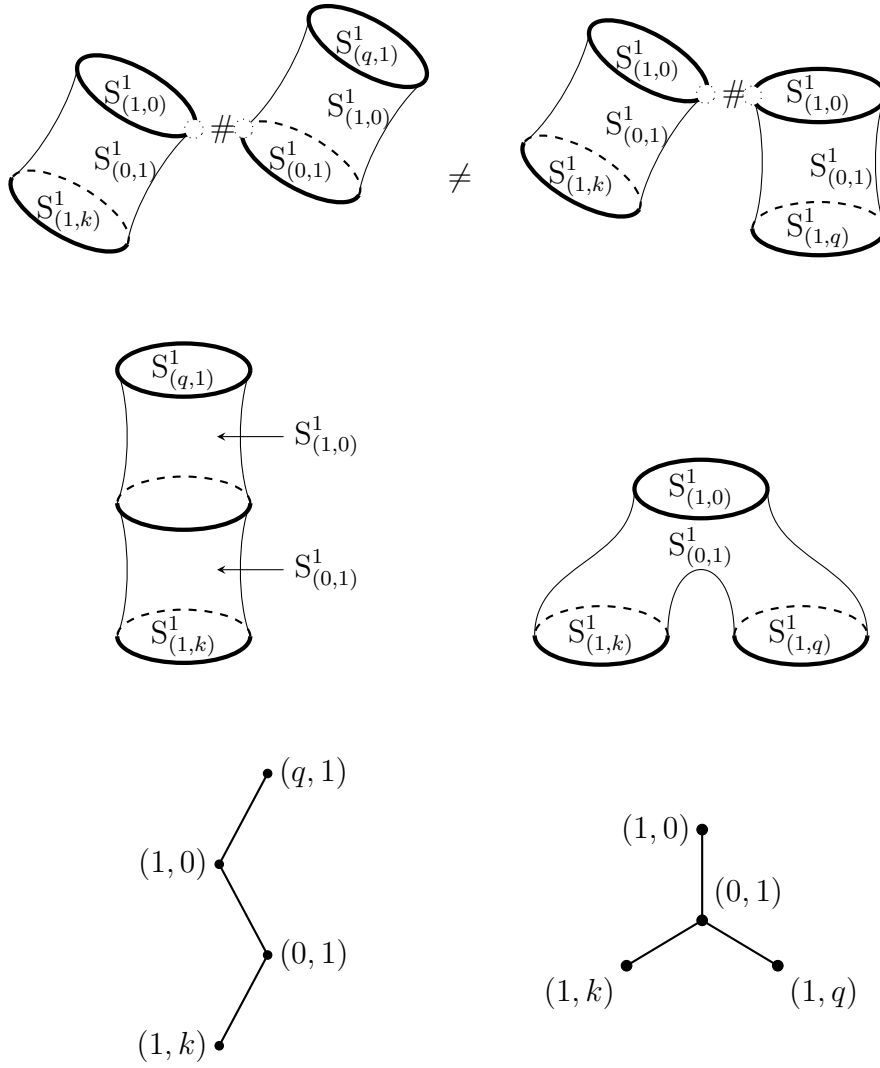
$(0,1)$

$(1,k)$

$(1,0)$

$(0,1)$

$(1,k)$ $(1,q)$

Figure 8: Each column shows a different fixed point connected sum on the level of Coxeter polar data, the resulting gluing and its corresponding boundary tree.

a regular orbit. By Definition 2.16, the new chamber C will be the connected sum through an interior point of the two given ones, $C = C_1 \# C_2$, and the new boundary will be the union of the preceding two, along with their unaffected marking. This can be translated as saying that the resulting boundary tree is the disjoint union of the two given ones.

In this way, we have shown how to retrieve all the admissible boundary trees or, equivalently, sets of Coxeter polar data, in terms of regular orbit sums of the actions listed in Proposition 5.1. □

However, we would like to say more, and establish a reciprocal statement.

Theorem 5.2. *A Coxeter polar* T^2*-manifold obtained from regular orbit sums of the actions in Proposition 5.1 is simply-connected unless each summand is* $\mathrm{S}^4 \times \mathrm{S}^1$. *In that case it will be simply-connected if and only if the circle isotropy groups that occur generate the homology lattice of* T^2.

Proof. Recall the decomposition of the manifold that defines a regular orbit sum

$$(M_1 \#_{reg} M_2) \cong (M_1 - (\mathrm{T}^2 \times \mathbb{D}^3)) \cup_{\mathrm{T}^2 \times \mathbb{S}^2} (M_2 - (\mathrm{T}^2 \times \mathbb{D}^3)) \qquad (5.3)$$

by using equation (2.15) for the case of a regular torus orbit. We can apply Seifert-Van Kampen's formula on the previous decomposition to compute the fundamental group. Notice first that removing a regular orbit does not affect the fundamental group of the manifold, because such orbits have codimension 3 in this case. We then have:

$$\pi_1(M_1 \#_{reg} M_2) \cong [\, \pi_1(M_1) \star \pi_1(M_2) \,] \, / \, i_* \pi_1(\mathbb{S}^2 \times \mathrm{T}^2). \qquad (5.4)$$

The manifolds we have to glue come from Proposition 5.1, and they are all simply-connected with the exception of $\mathbb{S}^4 \times \mathbb{S}^1$. For the latter, by the explicit description of the action we have that

$$(\mathbb{S}^4 \times \mathbb{S}^1) - (\mathrm{T}^2 \times \mathbb{D}^3) \cong (\mathbb{S}^4 - (\mathbb{S}^1 \times \mathbb{D}^3)) \times \mathbb{S}^1 \cong \mathbb{S}^2 \times \mathbb{D}^2 \times \mathbb{S}^1; \qquad (5.5)$$

and the induced map coming from the inclusion of its boundary $\mathbb{S}^2 \times \mathrm{T}^2$ is an epimorphism on the fundamental group level. Then, if we take regular orbit sums of $\mathbb{S}^4 \times \mathbb{S}^1$ with an already simply-connected manifold we will obtain a new simply-connected manifold. In particular, if the manifold admits fixed points, the boundary tree will have a connected component with at least one edge which corresponds to a simply-connected manifold we can begin with.

Otherwise, under the absence of fixed points, the boundary tree will have only isolated vertices marked by circle subgroups of slopes (m_i, n_i) corresponding to regular orbit sums of blocks $(\mathbb{S}^4 \times \mathbb{S}^1)_i$. For each of these, we observe that the

inclusion of a regular orbit induces an epimorphism in the fundamental groups, which leads to the following isomorphism

$$\pi_1(\ (\mathbb{S}^4 \times \mathbb{S}^1)_i) = \pi_1(\mathrm{T}^2/K_i) = \ \mathbb{Z}^2/<(m_i, n_i)>.$$

Fixing coordinates for T^2 will simplify the computation of the fundamental groups along the gluing. For instance, if M' is the regular orbit sum of two blocks $(\mathbb{S}^4 \times \mathbb{S}^1)_i$, $i = 1, 2$, then (5.4) implies that

$$\pi_1(M') \ \cong \ [\ \mathbb{Z}^2/<(m_1, n_1)>] \ \star \ [\mathbb{Z}^2/<(m_2, n_2)> \ ]/ \ \delta(\mathbb{Z}^2)$$

$$\cong \ \mathbb{Z}^2/<(m_1, n_1), (m_2, n_2)>,$$

where we have observed that the map corresponding to the inclusion of the boundary $i_*|_{\pi_1(\mathrm{T}^2)}$ is the diagonal map from $\mathbb{Z}^2$ onto each factor followed by the corresponding projections. Moreover, the inclusion of a regular orbit generates the fundamental group of this new manifold M'. In this way we may proceed inductively, where $\pi_1(M') \cong \ \mathbb{Z}^2/<(m_1, n_1), \cdots, (m_i, n_i)>$ is known, and we want to compute the fundamental group of $M = (\mathbb{S}^4 \times \mathbb{S}^1)_{i+1} \ \#_{reg} \ M'$. The inductive step is completely analogous, and so we have that the fundamental group of M will be $\mathbb{Z}^2/<(m_1, n_1), \cdots, (m_{i+1}, n_{i+1})>$.

In particular, the fundamental group will be trivial if and only if the isotropy subgroups generate the homology of T^2, or cyclic if they generate a sub-lattice. $\quad\square$

6 Diffeomorphism Classification

The goal of this section is to determine the diffeomorphism type of the underlying manifolds for the polar torus actions on simply-connected 5-manifolds. In the case of T^3 this was done in [Oh83] were it was proven that these are obtained as connected sums of $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$ and $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$. This will also be true for S^1 and T^2 actions, as stated in Theorem B. We separate the two cases, addressing the simpler one of S^1-actions first. In both cases the proof relies on our explicit description of the actions, together with the Barden-Smale classification of closed simply connected 5-manifolds.

Theorem 6.1. *A compact simply-connected 5-manifold admitting a polar circle action is diffeomorphic to a connected sum of $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, and $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$.*

Proof. Recall that these manifolds are reconstructed out of Coxeter polar data $(N, S^1(N))$ given by a smooth simply-connected 4-manifold N with non-trivial boundary, and a marking specifying the trivial subgroup along the interior and fixed points along the boundary.

The statement of the theorem follows from the classification of simply-connected closed 5-manifolds (see [Bar65]), together with the claim that our manifolds have free second homology group $H_2(M; \mathbb{Z})$. To compute these groups we will use the Mayer-Vietoris sequence for a decomposition of M given by

$$\partial N \times S^1 \subset (\partial N \times \epsilon \mathbb{D}^2) \cup (int(N) \times S^1) = M, \tag{6.2}$$

where $\partial N \times \mathbb{D}^2$ is a tubular neighborhood of ∂N inside M, and $int(N) \times S^1$ is the reconstructed manifold over the interior of N. Here, both the tubular neighborhood of ∂N and the reconstructed manifold over $int(N)$ are identified due to the uniqueness properties of the reconstructed polar actions with prescribed data. We then have the associated long exact sequence of homology with integer coefficients,

$$H_i(\partial N \times S^1) \xrightarrow{\alpha} H_i(\partial N) \oplus H_i(int(N) \times S^1) \xrightarrow{\beta} H_i(M) \xrightarrow{\delta} H_{i-1}(\partial N \times S^1). \tag{6.3}$$

We have by hypothesis that $\pi_1(N) = 0$, implying $H_1(N) = 0$. Also notice that the interior of N is a retract of it, and thus has the same homology.

By the Künneth formula, and thanks to the fact that S^1 has free homology, we have:

$$H_1(\partial N \times S^1) = H_1(\partial N) \oplus \mathbb{Z}^{\pi_0(\partial N)},$$

$$H_2(\partial N \times S^1) = H_2(\partial N) \oplus H_1(\partial N),$$

$$H_2(int(N) \times S^1) = H_2(N).$$

The map α in (6.3) gives isomorphisms between the corresponding terms in the i-th homology for $i = 1, 2$,

$$\alpha : H_i(\partial N) \subset H_i(\partial N \times \mathrm{S}^1) \xrightarrow{\sim} H_i(\partial N) \cong H_i(\partial N \times \mathbb{D}^2).$$

But there is another term coming from the common intersection in $H_2(\partial N \times \mathrm{S}^1)$, which is $H_1(\partial N) \otimes H_1(\mathrm{S}^1) \cong H_1(\partial N)$. In this case, its image via α will be trivial, as the generator of $H_1(\mathrm{S}^1)$ is zero in $H_1(\mathbb{D}^2)$, while the map from $H_1(\partial N)$ is trivial since $H_1(N) = 0$. Therefore, the kernel of δ in degree two is isomorphic to $H_2(int(N) \times \mathrm{S}^1) \cong H_2(N)$. It follows that $H_2(N)$ is free by the Poincaré duality for manifolds with boundary.

The image of δ is a submodule of $H_1(\partial N \times \mathrm{S}^1)$, and in particular of the summand $H_0(\partial N \times \mathrm{S}^1)$ which is free, as we have already seen how the term $H_1(\partial N)$ cancels out by applying itself isomorphically into its image via α. We conclude that $H_2(M)$ is a free $\mathbb{Z}$-module. $\qquad\square$

Remark 6.4. We may list the manifolds uniquely, with $\alpha = 0, 1$ and $n \in \mathbb{N}_0$, as

$$\#_\alpha \mathbb{S}^3 \tilde{\times} \mathbb{S}^2 \#_n \mathbb{S}^3 \times \mathbb{S}^2,$$

where the empty connected sum refers to the 5-sphere. These manifolds are uniquely identified by the rank of its second integer homology group, together with the property of being spin or not, which corresponds to $\alpha = 0, 1$ respectively. This is explicitly determined by the Coxeter polar data. In fact, we can compute the rank of the second homology $H_2(M)$ as

$$H_2(M) = H_2(N) \oplus \mathbb{Z}^{|\pi_0(\partial N)| - 1}.$$

Furthermore, M will be spin if and only if N is. First notice that, since N retracts onto its interior, one being spin implies so for the other. Then, observe that, if N is not spin, M will not be spin as well, for it contains the open subset $M_{reg} = int(N) \times \mathrm{S}^1$. On the other hand, if N is spin we fix a spin structure which induces one on its interior and boundary. We now can construct a spin structure on M by gluing along the decomposition (6.2). Each piece has a product spin structure by choosing for S^1 the unique spin structure that extends to $\mathbb{D}^2$. In this way, we ensure that the spin structures agree on the common open stripe where they are identified, allowing the gluing to be spin.

For polar T^2 actions, the identification of the underlying diffeomorphism types is analogous to the case of the circle actions, though significantly more involved. We rely on the general description of the actions achieved in Section 5.

Theorem 6.5. *A compact simply-connected 5-manifold admitting a polar effective* T^2*-action is diffeomorphic to a connected sum of* $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, *or* $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$.

Proof. Analogously to the proof of Theorem 6.1 we will show that the second integer homology groups $H_2(M;\mathbb{Z})$ are free.

Polar effective T^2-actions on compact simply-connected 5-manifolds were constructed by first forming connected sums of basic examples (this is the content of Proposition 5.1), and later performing surgeries along normally embedded tori. For the computations that follow, it is useful to revert the order in which these operations are done. We will begin by taking regular orbits sums of the actions on $\mathbb{S}^4 \times \mathbb{S}^1$ and $\mathbb{S}^5$ only, and later make fixed point sums with $\mathbb{S}^3 \times_k \mathbb{S}^2$. This last step will only add free summands to the second homology, so we only need to compute the second homology for the initial regular orbit sums. From the decomposition (5.3) we have the following Mayer-Vietoris long exact sequence for integer homology

$$\cdots \to H_2(\mathbb{S}^2 \times T^2) \xrightarrow{\alpha_2} H_2(M_1 - (T^2 \times \mathbb{D}^3)) \oplus H_2(M_2 - (T^2 \times \mathbb{D}^3)) \xrightarrow{\beta_2}$$

$$\xrightarrow{\beta_2} H_2(M_1 \#_{reg} M_2) \xrightarrow{\delta_2} H_1(\mathbb{S}^2 \times T^2) \xrightarrow{\alpha_1} \cdots \tag{6.6}$$

In order to compute the homology of a regular orbit sum $H_2(M_1\#_{reg}M_2)$, we first need to know what does each M_i look like when a tubular neighborhood around a regular orbit has been removed. For $M_i = \mathbb{S}^4 \times \mathbb{S}^1$ we have the equality (5.5), while for $\mathbb{S}^5$ we can make use of an auxiliary Mayer-Vietoris computation on the decomposition given by a tube $T^2 \times \mathbb{D}^3$ and its complement, and show that $H_2(\mathbb{S}^5 - (T^2 \times \mathbb{D}^3)) = \mathbb{Z}$. In both cases the second homology group is generated by the class of a 2-sphere bounding a slice $\mathbb{D}^3$ at a regular point. The group $H_2(M_1\#_{reg}M_2)$ will then be computed as the kernel to α_1 plus the image of β_2 from (6.6). This kernel is clearly free, as any subgroup of $\mathbb{Z}^2$, while the other summand requires that we control the map α_2 coming from the inclusion of the common intersection $\mathbb{S}^2 \times T^2$. Using the Künneth theorem for the product we have that $\mathbb{Z}^2 \cong H_2(\mathbb{S}^2 \times T^2) \cong \mathbb{Z}.[\mathbb{S}^2] \oplus \mathbb{Z}.[T^2]$. As mentioned above, the class of the sphere in the common intersection $[\mathbb{S}^2] \in H_2(T^2 \times \mathbb{S}^2)$ generates, via α_2, the second homology of the block that is being added, say M_2. Then, what is left for the image of β_2 is just the other term corresponding to M_1, $H_2(M_1 - (T^2 \times \mathbb{D}^3))$, when quotiented out by the image $\alpha_2(\mathbb{Z}.[T^2])$. This can be seen as the class of a regular orbit in the boundary of $(M_1 - (T^2 \times \mathbb{D}^3))$.

We claim that $\alpha_2([T^2]) \in H_2(M_1 - (T^2 \times \mathbb{D}^3))$ is trivial since we can construct an embedded 3-manifold with this torus for its boundary. For this, first consider the 3-dimensional chamber associated to the T^2-action in $M_1 - (T^2 \times \mathbb{D}^3)$, which is just the chamber that we already got for M_1 minus an interior disc. Then, take a segment inside this chamber joining the point that corresponds to this regular orbit in the boundary together with a point in a face (with circle isotropy subgroup), arriving orthogonally to this last stratum. The orbit of this segment will be smoothly embedded in M, and diffeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1$ with the required boundary, thus proving the claim.

In conclusion, if we can ensure that the piece $M_1 - (\mathrm{T}^2 \times \mathbb{D}^3)$ has free second homology, this will also be true for the sum of it with a basic block, $M_1 \#_{reg} M_2$. In particular, it is true for the sum of two basic blocks. For the induction to continue, we must see that the complement of a tube around a regular orbit in the resulting manifold has free second homology. A key observation in order to see this is that, instead of first gluing the pieces that define it and then removing a tube along a regular orbit, we may directly glue the two pieces that define it along $\mathrm{T}^2 \times \mathbb{D}^2$, obtaining

$$(M_1 \#_{reg} M_2) - (\mathrm{T}^2 \times \mathbb{D}^3) = (M_1 - (\mathrm{T}^2 \times \mathbb{D}^3)) \cup_{\mathrm{T}^2 \times \mathbb{D}^2} (M_2 - (\mathrm{T}^2 \times \mathbb{D}^3)).$$

Then, the homology can be computed via the corresponding Mayer-Vietoris sequence:

$$\cdots \to H_2(\mathrm{T}^2 \times \mathbb{D}^2) \xrightarrow{\alpha_2} H_2(M_1 - (\mathrm{T}^2 \times \mathbb{D}^3)) \oplus H_2(M_2 - (\mathrm{T}^2 \times \mathbb{D}^3)) \xrightarrow{\beta_2}$$

$$\xrightarrow{\beta_2} H_2((M_1 \#_{reg} M_2) - (\mathrm{T}^2 \times \mathbb{D}^3)) \xrightarrow{\delta_2} H_1(\mathbb{S}^2 \times \mathrm{T}^2) \xrightarrow{\alpha_1} \cdots .$$

As before, $H_2(M_1 \#_{reg} M_2)$ can be decomposed into the clearly free kernel of α_1 plus the image of β_2. The latter is isomorphic to the sum of $H_2(M_i - (\mathrm{T}^2 \times \mathbb{D}^3))$, for $i = 1, 2$, which are free by inductive hypothesis, once we observe that the map α_2 coming from the $[\mathrm{T}^2] \in H_2(\mathrm{T}^2 \times \mathbb{D}^2)$ is trivial by an analogous argument as in the previous claim. $\qquad\square$

Remark 6.7. The diffeomorphism type can be determined explicitly from given Coxeter polar data. Analogously to Remark 6.4, we may determine the rank of the second homology and characterize when it is spin. In fact, we have a spin structure on any regular orbit sum of copies of $\mathbb{S}^4 \times \mathbb{S}^1$ and $\mathbb{S}^5$. For this, it suffices to consider the given spin structure on $\mathbb{S}^5$, the unique spin structure on $\mathrm{T}^2 \times \mathbb{S}^2 \times (0, \epsilon) \subset \mathbb{S}^4 \times \mathbb{S}^1$ that extends to $\mathbb{S}^4 \times \mathbb{D}^2$ and finally observe that the gluing map agrees with the identity along the tubular neighborhood of the regular orbit, and is therefore spin. On the other hand, if we perform a fixed point connected with an action on $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$ the manifold will not be spin. Hence, the resulting manifold will be spin if and only if it is constructed taking fixed point connected sums with $\mathbb{S}^3 \times_k \mathbb{S}^2$ for even k only.

7 Applications to non-negative curvature

The interest in positively and non-negatively curved compact manifolds together with the abundance of symmetry for the relatively few known examples, has led to a program of studying manifolds with positive or non-negative curvature on which a Lie group acts by isometries (see for example the surveys [Gro02, Wil07, Zil07]).

Remarkable results have been accomplished in low dimensions, where topological classifications are obtained for prescribed curvature and symmetry conditions. For instance, compact simply-connected positively curved 4-manifolds with continuous symmetry, i.e., invariant under an isometric circle action, are equivariantly diffeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$ with a linear action ([HK89, Fin77, GW13]). If we consider non-negatively curved ones, we have in addition $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ ([Kle90, SY94]) and the circle action is induced by a biquotient action ([GGK14]).

In contrast, in dimension 5 the topological question is open if we only consider the action of a circle, unless this action is fixed point homogeneous (see [GGS12]). Otherwise, we must strengthen the symmetry hypothesis as in [Ron02], where it is proven that a positively curved (closed simply-connected) 5-manifold with an isometric T^2-action is diffeomorphic to the sphere $\mathbb{S}^5$. The list grows if we allow the metric to be non-negatively curved to include the Wu manifold $SU(3)/SO(3)$, $\mathbb{S}^3 \times \mathbb{S}^2$ and the non-trivial bundle $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$ (see [GGS]). However, equivariant classifications of non-negatively curved 5-manifolds are only obtained in the case of an $SU(2)$ action (see [Sim]) or of a T^3 action (see [Oh83, GGK14]).

Polar actions are expected to be rigid in this context. In fact, it was shown in [FGT] that a polar action with cohomogeneity at least two and positively curved polar metric is equivariantly diffeomorphic to a linear action on a CROSS (compact rank one symmetric space). However, the case of non-negative curvature is open in general. Here, as an application of our classification results together with a closer examination of what the sections are, we are able to classify polar actions with non-negative curvature in dimension 5.

Proof of Theorem D. First observe that the actions listed in the theorem were given with an invariant polar metric of non-negative curvature. We can then focus on discarding the other cases that appear in our classification, using the fact that the sections have to inherit non-negative curvature since they are totally geodesic submanifolds. In particular, by a classic result of J. Cheeger and D. Gromoll, their fundamental groups have to admit a lattice $\mathbb{Z}^r$ of finite index.

In general, the chamber C of a polar manifold is a convex submanifold with totally geodesic boundary. Moreover, if the curvature is non-negative, the proof of the Soul theorem [CG72] implies that C admits a soul S. Flowing along a gradient-like vector field adapted to the distance function to the soul, it follows that the chamber is homeomorphic to a disc bundle over S. The simply-connectedness of the manifold implies that of C and hence of S.

We separate the proof according to the Lie group being abelian or not. We begin with the S^1 and T^2 actions, and recall that the case of T^3 actions was treated in [Oh83, GGK14].

For a circle action, the soul can be a point, $\mathbb{S}^2$ or $\mathbb{S}^3$. In the first case we have $C = \mathbb{D}^4$ which is recognized as a linear action on $\mathbb{S}^5$. For the last case, the chamber will be $\mathbb{S}^3 \times I$, corresponding to the circle action on the second factor of $\mathbb{S}^3 \times \mathbb{S}^2$. Finally, when the soul is $\mathbb{S}^2$ we have infinitely many possible disk bundles which give rise to actions on $\mathbb{S}^3 \times_k \mathbb{S}^2$ as discussed in Example 3.6.

In the case of a T^2 action, the soul of the chamber must be a point or $\mathbb{S}^2$, giving rise to $C = \mathbb{D}^3$ or $\mathbb{S}^2 \times I$, respectively. We claim that only the basic actions on $\mathbb{S}^3 \times_k \mathbb{S}^2$, $\mathbb{S}^4 \times \mathbb{S}^1$ and $\mathbb{S}^5$ of Figure 7, and the linear T^2 action on the first factor of $\mathbb{S}^3 \times \mathbb{S}^2$ admit invariant non-negative curvature. When the orbit space is $\mathbb{D}^3$ the only other possible case is given by a repeated fixed point connected sum of $\mathbb{S}^3 \times_k \mathbb{S}^2$. Notice that the actions on $\mathbb{S}^3 \times_k \mathbb{S}^2$ have $\Sigma = \mathbb{S}^2 \times \mathrm{S}^1$ or a non-orientable one, $\mathbb{S}^2 \times \mathbb{S}^1/\tau_1$, as discussed in Example 3.2. Then, the fundamental group of the section associated to a fixed point connected sum of at least two copies of $\mathbb{S}^3 \times_k \mathbb{S}^2$ will have the free group on two generators as a subgroup and thus cannot admit non-negative curvature. When the chamber is $\mathbb{S}^2 \times I$ we have to perform one regular orbit sum between two of the manifolds listed in Proposition 5.1. As follows from Proposition 2.19, the fundamental group of such a gluing will be a free product of copies of the fundamental group of the summands plus one additional generator for each 0-surgery. A case by case analysis shows that the only admissible fundamental group corresponds to both summands being $\mathbb{S}^4 \times \mathbb{S}^1$. Furthermore, by Theorem 5.2 the gluing of two copies of $\mathbb{S}^4 \times \mathbb{S}^1$ will add to a simply-connected manifold if and only if they are marked by complementary circle isotropy groups, i.e., circles generating T^2 with trivial intersection. This is identified as the T^2 action on $\mathbb{S}^3 \times \mathbb{S}^2$, coming from the action on the first factor only.

Among the polar actions by non-abelian Lie groups there are two families of actions listed in Theorem C which are constructed as fixed point sums of linear polar actions on $\mathbb{S}^3 \times \mathbb{S}^2$ by the groups $\mathrm{SO}(3)$ and $\mathrm{SO}(3) \times \mathrm{S}^1$. These basic actions admit $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathrm{S}^1 \times \mathrm{S}^1$ for their respective sections. In either case, a fixed point sum of two or more copies has a section which can not be non-negatively curved.

Finally, among the $\mathrm{SO}(3)$ polar manifolds given by fixed point sums of the actions on the Brieskorn or Wu manifold $\#_k \mathcal{B}$ and $\#_l \mathcal{W}$, only the action on the Wu manifold alone supports non-negative curvature, because otherwise we have that $\Sigma = \#_n \mathrm{T}^2$, $n \geq 2$. $\qquad\square$

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