Darboux-Halphen-Ramanujan Vector Field on a Moduli of Calabi-Yau Manifolds

Younes Nikdelan
Instituto de Matemática Pura e Aplicada (IMPA),
Estrada Dona Castorina, 110,
22460-320, Rio de Janeiro, RJ, Brazil,
nikdelan@impa.br

Abstract

In this paper we obtain an ordinary differential equation H from a Picard-Fuchs equation associated with a nowhere vanishing holomorphic n-form. We work on a moduli space T constructed from a Calabi-Yau manifold W together with a basis of the middle complex de Rham cohomology of W. We verify the existence of a unique vector field H on T such that its composition with the Gauss-Manin connection satisfies certain properties. The ordinary differential equation given by H is a generalization of differential equations introduced by Darboux, Halphen and Ramanujan.

Kywords Darboux-Halphen-Ramanujan vector field. Hodge structure. Picard-Fuchs equation. Gauss-Manin connection.

Mathematics Subject Classification (2010) 14H10. 34M45. 37F75.

1 Introduction

The system of differential equations

$$\begin{cases} \frac{dt_1}{dz} + \frac{dt_2}{dz} = t_1t_2 \\ \frac{dt_2}{dz} + \frac{dt_3}{dz} = t_2t_3 \\ \frac{dt_1}{dz} + \frac{dt_3}{dz} = t_1t_3 \end{cases}, \tag{1.1}$$

appeared in 1878 in the work of Gaston Darboux [9], where he was treating on the curvilinear coordinates and orthogonal systems. The problem that he was working on it is as follow: *Let A and B be two fixed surfaces in 3-dimensional Euclidean space $\mathbb{R}^3$. Suppose that Σ is a family of surfaces parallel to A, and Σ' is another family of surfaces parallel to B. Is there a third family of surfaces parameterized by u such that intersects Σ and Σ' orthogonally?* The more interesting case of this problem is when the family (u) is of the second degree and Darboux proved that in this case this family is given by

$$\frac{x_1^2}{t_1(u)} + \frac{x_2^2}{t_2(u)} + \frac{x_3^2}{t_3(u)} = 1,$$

in which x_1, x_2, x_3 are coordinates of $\mathbb{R}^3$, and t_1, t_2, t_3 are functions of u given by the following equation

$$t_3(\frac{dt_1}{du} + \frac{dt_2}{du}) = t_2(\frac{dt_1}{du} + \frac{dt_3}{du}) = t_1(\frac{dt_2}{du} + \frac{dt_3}{du}). \tag{1.2}$$

Therefore, the system of equations (1.1) is a particular case of the equation (1.2).

In 1881, G. Halphen [17] studied the system of differential equations (1.1) in $\mathbb{C}^3$. He proved that this system satisfies an important invariant property. To express this invariant property, for the constants a, b, a', b', let

$$w = \frac{az+b}{a'z+b'} \qquad \& \qquad t_i = -\frac{2a'}{a'z+b'} + \frac{ab'-ba'}{(a'z+b')^2} s_i, \;\; i = 1, 2, 3. \tag{1.3}$$

By substituting (1.3) in the system (1.1), we have

$$\begin{cases} \frac{ds_1}{dw} + \frac{ds_2}{dw} = s_1 s_2 \\ \frac{ds_2}{dw} + \frac{ds_3}{dw} = s_2 s_3 \\ \frac{ds_1}{dw} + \frac{ds_3}{dw} = s_1 s_3 \end{cases}, \tag{1.4}$$

from which it follows that the system (1.1) is invariant under the change of variables (1.3). Therefore, to find a general solution of (1.1), it is enough to apply (1.3) to a particular solution of (1.4). Halphen gave a solution of the system (1.1) in terms of the logarithmic derivatives of the null theta functions; namely

$$\begin{aligned} t_1 &= 2(\ln \theta_4(0|z))', \\ t_2 &= 2(\ln \theta_2(0|z))', \qquad ' = \frac{\partial}{\partial z}, \\ t_3 &= 2(\ln \theta_3(0|z))'. \end{aligned}$$

where

$$\begin{cases} \theta_2(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(0|z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \; q = e^{2\pi i z}, \; \mathrm{Im}(z) > 0.$$

F. Brioschi [3] in 1881 studied the following extension of the system (1.1)

$$\begin{cases} \frac{dt_1}{dz} + \frac{dt_2}{dz} = t_1 t_2 + \varphi(z) \\ \frac{dt_2}{dz} + \frac{dt_3}{dz} = t_2 t_3 + \varphi(z) \\ \frac{dt_1}{dz} + \frac{dt_3}{dz} = t_1 t_3 + \varphi(z) \end{cases}, \tag{1.5}$$

in which $\varphi(z)$ is a function of z. Again in 1881, Halphen in [18] introduced and investigated a class of differential equations with (1.5) as a particular case. In the case of three variables, he showed that this class is given by

$$\begin{cases} \frac{dt_1}{dz} = a_1 t_1^2 + (\lambda - a_1)(t_1 t_2 + t_1 t_3 - t_2 t_3) \\ \frac{dt_2}{dz} = a_2 t_2^2 + (\lambda - a_2)(t_2 t_3 + t_2 t_1 - t_3 t_1) \\ \frac{dt_3}{dz} = a_3 t_3^2 + (\lambda - a_3)(t_3 t_1 + t_3 t_2 - t_1 t_2) \end{cases}, \tag{1.6}$$

where, a_1, a_2, a_3, λ are constants. He proved that the system (1.6) also satisfies the invariant property and it is in a direct relationship with the Gauss hypergeometric equation (see [16]). One can see that the system (1.6) is equivalent to the system (1.1), when $a_1 = a_2 = a_3 = 0$ and $\lambda = 1$. If we look at the system (1.6) as a vector field in $\mathbb{C}^3$, then it is a semi-complete vector field. In this context, an extension of Halphen vector field, namely Halphen type vector field, was introduced by Adolfo Guillot in [14, 15].

In 1916 Ramanujan [24] introduced another system of differential equations as follow

$$\mathrm{R}: \begin{cases} \frac{dr_1}{d\tau} = r_1^2 - \frac{1}{12} r_2 \\ \frac{dr_2}{d\tau} = 4 r_1 r_2 - 6 r_3 \\ \frac{dr_3}{d\tau} = 6 r_1 r_3 - \frac{1}{3} r_2^2 \end{cases}, \tag{1.7}$$

that is in a close relationship with Darboux-Halphen differential equation (1.1). He verified that the Eisenstein series $\frac{2\pi i}{12}E_2(\tau)$, $12(\frac{2\pi i}{12})^2E_4(\tau), 8(\frac{2\pi i}{12})^3E_6(\tau)$ satisfy (1.7), where

$$E_{2j}(\tau) := 1 - \frac{4j}{B_{2j}}\sum_{r=1}^{\infty}\sigma_{2j-1}(r)q^r, \quad q = e^{2\pi i\tau},$$

$$\sigma_i(n) := \sum_{d|n} d^i,$$

and B_k's are Bernoulli's numbers. In (1.9) we will see a relationship between the systems of equations (1.1) and (1.7).

Calabi-Yau manifolds are defined as compact connected Kähler manifolds whose canonical bundle is trivial, though many other equivalent definitions are sometimes used. They were named "Calabi-Yau manifold" by Candelas et al. (1985) [7] after E. Calabi (1954) [4, 5], who first studied them, and S. T. Yau (1976) [27], who proved the Calabi conjecture that says Calabi-Yau manifolds accept Ricci flat metrics. In this text, we suppose that for an n-dimensional Calabi-Yau manifold $\mathsf{h}^{p,0} = 0$, $0 < p < n$, where $\mathsf{h}^{p,q}$ refers to (p,q)-th Hodge number of Calabi-Yau manifold (see §3). It is clear that the connectedness of a Calabi-Yau manifold and the triviality of its canonical bundle imply that $\mathsf{h}^{0,0} = \mathsf{h}^{n,0} = 1$. In order to explain the generalization of Darboux-Halphen-Ramanujan vector fields, DHR for short, we consider the family of 1-dimensional Calabi-Yau manifolds, which are elliptic curves, and for more details the reader refers to [23].

Let E be an elliptic curve over $\mathbb{C}$. Then the Hodge filtration $F^\bullet H^1$ of the first de Rham cohomology group $H^1_{\rm dR}(E)$ is given as follow,

$$\{0\} = F^2 \subset F^1 \subset F^0 = H^1_{\rm dR}(E), \quad \dim F^i = 2 - i,$$

where $F^1 \subset H^1_{\rm dR}(E)$ includes classes of holomorphic closed 1-forms on E. Let T be the moduli of the pair $(E, [\alpha_1, \alpha_2])$, in which $\alpha_1 \in F^1$, $\alpha_2 \in F^0 \setminus F^1$, and the intersection form matrix in α_i's is as follow

$$(\langle\alpha_i, \alpha_j\rangle)_{1\le i,j\le 2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To be more precise, let $E_0 := E \setminus \{\infty\}$ be the affine curve that its Weierstrass presentation is given as follow

$$E_0 = \{(x,y) \in \mathbb{C}^2 |\, f(x,y) := y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3 = 0\}.$$

Then α_1 and α_2, resp., are induced by $[\frac{dx}{y}]$ and $[\frac{xdx}{y}]$, resp., where $[\frac{dx}{y}]$ and $[\frac{xdx}{y}]$ are generators of the first de Rham cohomology $H^1_{\rm dR}(E_0)$ of the affine curve E_0. Since $H^1_{\rm dR}(E) \cong H^1_{\rm dR}(E_0)$, it follows that α_1 and α_2 are generators of $H^1_{\rm dR}(E)$; and hence $\alpha_1 \wedge \alpha_2 \neq 0$. It is seen that T is a 3-dimensional space, and there exist a unique vector field H on T such that the composition of Gauss-Manin connection (see §3.1)

$$\nabla : H^1_{\rm dR}(E/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H^1_{\rm dR}(E/\mathsf{T}),$$

with H satisfies the following:

$$\nabla_{\mathsf{H}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Roughly speaking, for $y^2 = 4(x-t_1)(x-t_2)(x-t_3)$ we have $\mathsf{T} = \mathsf{T}_{\mathrm{DH}} := \{(t_1,t_2,t_3) \in \mathbb{C}^3 | t_1 \neq t_2 \neq t_3\}$ and $\mathsf{H} = \mathrm{DH}$ is given by the following system

$$\mathrm{DH}: \begin{cases} \frac{dt_1}{dz} = t_1(t_2+t_3) - t_2t_3 \\ \frac{dt_2}{dz} = t_2(t_1+t_3) - t_1t_3 \\ \frac{dt_3}{dz} = t_3(t_1+t_2) - t_1t_2 \end{cases}, \tag{1.8}$$

which is an special case of the system (1.6) introduced by Darboux-Halphen with $a_1 = a_2 = a_3 = 0$ and $\lambda = 1$. Or equivalently for $y^2 = 4(x-r_1)^3 + t_2(x-r_1) + r_3$ we obtain $\mathsf{T} = \mathsf{T}_{\mathrm{R}} := \{(r_1,r_2,r_3) \in \mathbb{C}^3 | 27r_3^2 - r_2^3 = 0\}$, and $\mathsf{H} = \mathrm{R}$ is presented by system (1.7) introduced by Ramanujan (for details see [23, Proposition 3.8]). The algebraic morphism $\phi : \mathsf{T}_{\mathrm{DH}} \to \mathsf{T}_{\mathrm{R}}$ defined by

$$\phi : (t_1,t_2,t_3) \mapsto (T, 4\sum_{1\leq i<j\leq 3} (T-t_i)(T-t_j), 4(T-t_1)(T-t_2)(T-t_3)), \tag{1.9}$$

where $T := \frac{1}{3}(t_1+t_2+t_3)$, connects two systems (1.7) and (1.8), i.e., $\phi_*\mathrm{DH} = \mathrm{R}$.

After these works, H. Movasati [22] considered a one parameter family of Calabi-Yau 3-folds, which is known as the family of mirror quintic 3-folds, and studied on it. If W is a mirror quintic 3-fold, then the Hodge filtration of $H^3_{\mathrm{dR}}(W)$ is given as follow

$$\{0\} = F^4 \subset F^3 \subset \ldots \subset F^0 = H^3_{\mathrm{dR}}(W), \quad \dim F^i = 4-i.$$

The complex moduli of W is one dimensional that we parameterize it by z. There is a nowhere vanishing holomorphic 3-form $\omega \in F^3$ such that the Picard-Fuchs equation associated with it is given by

$$\mathsf{L} = \vartheta^5 - 5^5 z(\vartheta + \frac{1}{5})(\vartheta + \frac{2}{5})(\vartheta + \frac{3}{5})(\vartheta + \frac{4}{5}),$$

in which $\vartheta := \nabla_{z\frac{\partial}{\partial z}}$ is the composition of Gauss-Manin connection ∇ with the vector field $z\frac{\partial}{\partial z}$. Movasati treated on the moduli space T of the pair $(W, [\alpha_1,\alpha_2,\alpha_3,\alpha_4])$, where $\alpha_i \in F^{4-i} \setminus F^{5-i}$, and the intersection form matrix in α_i's is given as follow

$$(\langle \alpha_i, \alpha_j \rangle)_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

He proved that T is a 7-dimensional space and there is a unique vector field H and a unique meromorphic function y on T such that,

$$\nabla_{\mathsf{H}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

Indeed he expressed H and y explicitly, and he showed that y is related with the normalized Yukawa coupling, whose was introduced by Candelas et al. in [6]. They computed the coefficients of the q-expansion of the normalized Yukawa coupling for quintic 3-folds in $\mathbb{P}^4$, that are conjectured to be the Gromov-Witten invariants of rational curves on a quintic 3-fold in $\mathbb{P}^4$.

After what we saw about the family of Calabi-Yau 1-folds and the family of mirror quintic 3-folds, it is natural to ask whether there exist such a moduli space T and such a vector field H in higher dimensions. In the present paper we give a positive answer to this question. To do this, we fix an n-dimensional Calabi-Yau manifold W. We suppose that the complex deformation of W is given by a one parameter family $\pi : \mathcal{W} \to P$ of n-dimensional Calabi-Yau manifolds, where P is a 1-dimensional quasi-projective variety parameterized by z. Moreover, we assume that the n-th relative de Rham cohomology group $H^n_{\rm dR}(\mathcal{W}/P)$ is $n+1$-dimensional and the Picard-Fuchs equation L associated with the unique nowhere vanishing holomorphic n-form $\omega \in \mathcal{F}^n$ is given by

$$\mathsf{L} = \vartheta^{n+1} - a_n(z)\vartheta^n - \ldots - a_1(z)\vartheta - a_0(z), \tag{1.10}$$

where $\mathcal{F}^\bullet H^n$ is the Hodge filtration of $H^n_{\rm dR}(\mathcal{W}/P)$, $\vartheta := \nabla_{z\frac{\partial}{\partial z}}$ and $a_i(z) \in \mathbb{Q}(z)$, $i = 0, 1, \ldots, n$. We provide the first result in the following proposition.

Proposition 1.1. *The Picard-Fuchs equation* L *is self-dual.*

Before stating the main theorem of this paper, we fix the $(n+1)\times(n+1)$ matrix Φ as follow. If n is an *odd* integer, then set

$$\Phi := \begin{pmatrix} 0_{\frac{n+1}{2}} & J_{\frac{n+1}{2}} \\ -J_{\frac{n+1}{2}} & 0_{\frac{n+1}{2}} \end{pmatrix}, \tag{1.11}$$

where $0_k, k \in \mathbb{N}$, denotes a $k\times k$ block of zeros, and J_k is the following $k \times k$ block

$$J_k := \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{1.12}$$

If n is an *even* integer, then $\Phi := J_{n+1}$. Also we suppose that $F^\bullet H^n$ denotes the Hodge filtration of $H^n_{\rm dR}(W;\mathbb{C})$ given as follow:

$$F^\bullet H^n : \ \{0\} = F^{n+1} \subset F^n \subset \ldots \subset F^1 \subset F^0 = H^n_{\rm dR}(W;\mathbb{C}).$$

Theorem 1.1. *Let W be the Calabi-Yau n-fold given above and T be the moduli of $(W, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, where $\{\alpha_i\}_{i=1}^{n+1}$ is a basis of $H^n_{\rm dR}(W;\mathbb{C})$ satisfying*

$$\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}, \ i = 1, 2, \ldots, n+1, \tag{1.13}$$

and the intersection form matrix in α_i's is subject to the condition:

$$(\langle \alpha_i, \alpha_j \rangle)_{1\le i,j\le n+1} = \Phi. \tag{1.14}$$

Then there exist a unique vector field H and unique meromorphic functions y_i, $i = 1, 2, \ldots, n-2$, on T such that the composition of Gauss-Manin connection ∇ with the vector field H satisfies:

$$\nabla_{\mathsf{H}}\alpha = Y\alpha, \tag{1.15}$$

in which

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \end{pmatrix}^{\mathsf{t}},$$

and

$$Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & y_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & y_{n-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{1.16}$$

Moreover we have,

$$\dim \mathsf{T} = \begin{cases} \frac{(n+1)(n+3)}{4} + 1; & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{4} + 1; & \text{if } n \text{ is even} \end{cases}.$$

As we saw above, the system of ordinary differential equations given by H is an extension of systems of differential equations introduced by Darboux, Halphen and Ramanujan.

Definition 1.1. The vector field H introduced in Theorem 1.1, is called *Darboux-Halphen-Ramanujan*, DHR for brevity, *vector field.*

The structure of this article is prepared as follow. First, in Section 2 we give an algorithm to find the existence relationships among coefficients of a self-dual linear differential equation of an arbitrary degree. In particular we provide these relationships in degrees three and five. Section 3 contains a brief summary of some basic facts. After fixing some notations and assumptions, the proof of Proposition 1.1 is given in §3.4. Finally, Section 4 is devoted to the proof of Theorem 1.1. In this section the proof is divided to the even case and odd case depending to the dimension of Calabi-Yau manifold. And also we present DHR vector field explicitly in dimensions three and five.

Remark 1.1. *As we will see in §4, to prove Theorem 1.1 we introduce several matrices and matrix equations. Recently we discovered that they are in a close relationship with Birkhoff factorization given in quantum cohomology (see* [13]*). In fact, if we talk in physicists language, our work is in B-model and the Birkhoff factorization is discussed in A-model and mirror symmetry gives the existence relationships between them.*

Acknowledgment. Here I would like to express my very great appreciation to Hossein Movasati, my Ph.D. supervisor, who always was available and I used his valuable and constructive suggestions and helps during the planning and development of this work. I wish to thank IMPA for preparing such an excellent academic environment. This work has been done during my Ph.D. and I am grateful to have economic supports of "CNPq-TWAS Fellowships Programme" during this period.

2 Self-Dual Linear Differential Equation

In this section by R we mean the simple commutative differential ring $\mathbb{C}[z]$, with quotient field $k := \mathbb{C}(z)$ and derivative $(.)'$; and $R[\partial]$ is the ring of differential operators where ∂ is the usual derivation $\frac{\partial}{\partial z}$ or logarithmic derivation $z\frac{\partial}{\partial z}$. It is not difficult to check that $k[\frac{\partial}{\partial z}]$ and $k[z\frac{\partial}{\partial z}]$ are isomorphic, hence we can freely switch between these two differential rings. The pair (M, ∂) refers to a differential R-module, i.e., M is a finitely generated R-module and $\partial : M \to M$ is a map satisfying $\partial(m+n) = \partial(m) + \partial(n)$ for every $m, n \in M$; and $\partial(fm) = f'm + f\partial(m)$ for every $f \in R$ and every $m \in M$. For more details the reader can see [25].

Definition 2.1. Let (M, ∂) be a differential k-module. Then for each $m \in M$ we define the *evaluation map* $ev_m : k[\partial] \to M$ by $\sum_{i=0}^{n} a_i \partial^i \mapsto \sum_{i=0}^{n} a_i \partial^i m$. The monic generator of the kernel of ev_m as a left ideal is called the *minimal operator* of m over $k[\partial]$. Furthermore, we call m a *cyclic vector* of M if the degree of its minimal operator equals the k-dimension of M, i.e. the set $\{m, \partial m, ..., \partial^{\dim_k(M)-1} m\}$ is a k-basis of M. We call a pair (M, e) consisting of a differential module M and a cyclic vector $e \in M$ a *marked differential module*.

By a result due to N. Katz (see [25, § 2.1]), there is a one to one correspondence between monic differential equations $L \in k[\partial]$ and marked differential modules (M, e). More precisely, each differential k-module M has a cyclic vector, and in particular there is a differential equation $L \in k[\partial]$ such that M is isomorphic to $k[\partial]/k[\partial]L$. Thus we can assume $L = \partial^{n+1} + \sum_{i=0}^{n} a_i \partial^i \in \mathbb{Q}(z)[\partial]$, is an irreducible monic differential equation and

$$(M_L, e) \cong (\mathbb{C}(z)[\partial]/\mathbb{C}(z)[\partial]L, [1]), \tag{2.1}$$

is its corresponding marked differential $\mathbb{C}(z)$-module. The *dual equation* $\check{L}$ of L is defined as follow

$$\check{L} = \sum_{i=0}^{n+1} (-1)^{n-i} \partial^i a_i, \quad a_{n+1} = 1. \tag{2.2}$$

Definition 2.2. It is said that L satisfies *property* (P), if there is a non-degenerate form $\langle .,. \rangle : M_L \times M_L \to \mathbb{C}(z)$ such that

(i) $\langle .,. \rangle$ is a $(-1)^n$-symmetric form, i.e. $\langle .,. \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]}(\mathrm{Sym}^2 M_L, \mathbb{C}(z))$ if n is even, and $\langle .,. \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]}(\bigwedge^2 M_L, \mathbb{C}(z))$ if n is odd.

(ii) $\langle e, \partial^i e \rangle = 0$ for $i = 0, 1, \ldots, n-1$.

We state a proposition that gives an equivalence condition for property (P) and for a proof see [2]. Note that for $\psi \in \mathbb{C}(z)$, the operator $\partial\psi$ is given as $\partial\psi = \partial(\psi) + \psi\partial$, and for convenient we denote by $\psi' = \partial(\psi)$, so $\psi^{(i)} = \underbrace{\partial(\partial(\ldots(\partial}_{i-\text{times}}(\psi))\ldots))$.

Proposition 2.1. *The equation L satisfies the property* (P) *if and only if L is self-dual, i.e., there is an $0 \neq \psi \in \mathbb{C}(z)$, such that*

$$L\psi = \psi\check{L}. \tag{2.3}$$

Using Proposition 2.1, we give an algorithm to find the relationships that exist among coefficients a_i's. Let $L = \sum_{i=0}^{n+1} a_i \partial^i$, with $a_{n+1} = 1$, be a linear differential equation satisfying property (P). Suppose that $n = 2m$ or $2m+1$, for a positive integer m. Then coefficients $a_{n-2}, a_{n-4}, \ldots, a_{n-2m}$ depend to the rest of the coefficients and their derivations. First using the induction, one can easily verify that for $\psi \in \mathbb{C}(z)$

$$\partial^j \psi = \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} \psi^{(j-i)} \partial^i.$$

Therefore, it follows that

$$\check{L} = \sum_{i=0}^{n+1} \left(\sum_{j=i}^{n+1} (-1)^{n+1-j} \left(\begin{array}{c} j \\ i \end{array} \right) a_j^{(j-i)} \right) \partial^i, \tag{2.4}$$

and

$$L\psi = \sum_{i=0}^{n+1} \left(\sum_{j=i}^{n+1} \left(\begin{array}{c} j \\ i \end{array} \right) a_j \psi^{(j-i)} \right) \partial^i. \tag{2.5}$$

If we substitute (2.4) and (2.5) in $L\psi = \psi\check{L}$, then we have

$$\sum_{i=0}^{n+1} \left(\sum_{j=i}^{n+1} \left(\begin{array}{c} j \\ i \end{array} \right) a_j \psi^{(j-i)} \right) \partial^i = \sum_{i=0}^{n+1} \psi \left(\sum_{j=i}^{n+1} (-1)^{n+1-j} \left(\begin{array}{c} j \\ i \end{array} \right) a_j^{(j-i)} \right) \partial^i. \tag{2.6}$$

Now by comparing the coefficient of ∂^n in (2.6), we express ψ' and $\psi^{(i)}$'s in terms of ψ, a_n and derivations of a_n as follows

$$\begin{aligned} \psi' &= -\frac{2}{n+1} a_n \psi, \\ \psi'' &= \left((-\frac{2}{n+1})^2 a_n^2 - \frac{2}{n+1} a_n' \right) \psi, \\ \psi''' &= \left((-\frac{2}{n+1})^3 a_n^3 + 3(-\frac{2}{n+1})^2 a_n a_n' - \frac{2}{n+1} a_n'' \right) \psi, \\ &\vdots \end{aligned} \tag{2.7}$$

and we substitute $\psi^{(i)}$' in the left side of (2.6). In order to state a_{n-2k}, $k = 1, 2, \ldots, m$, as an equation of $a_n, a_{n-1}, a_{n-3}, \ldots, a_{n-(2k-1)}$ and their derivations it is enough to compare the coefficient of ∂^{n-2k} of both sides of (2.6), which yields

$$\sum_{j=n-2k}^{n+1} \left(\begin{array}{c} j \\ n-2k \end{array} \right) a_j \psi^{(j-(n-2k))} = \left(\sum_{j=n-2k}^{n+1} (-1)^{n+1-j} \left(\begin{array}{c} j \\ n-2k \end{array} \right) a_j^{(j-(n-2k))} \right) \psi.$$

Therefore,

$$\begin{aligned} 2a_{n-2k} = & \sum_{j=n-2k+1}^{n} (-1)^{n+1-j} \left(\begin{array}{c} j \\ n-2k \end{array} \right) a_j^{(j-(n-2k))} \\ & - \sum_{j=n-2k+1}^{n+1} \left(\begin{array}{c} j \\ n-2k \end{array} \right) a_j (\psi^{(j-(n-2k))}/\psi). \end{aligned}$$

For example if $k = 1$, then a_{n-2} is given as follow

$$a_{n-2} = \frac{n-1}{n+1} a_{n-1} a_n - \frac{n(n-1)}{2(n+1)} a_n a_n' - \frac{n(n-1)}{3(n+1)^2} a_n^3 + \frac{(n-1)}{2} a_{n-1}' - \frac{1}{12} n(n-1) a_n''.$$

As a result of this algorithm we provide the following lemma.

Lemma 2.1. *Let $L = \sum_{i=0}^{n+1} a_i \partial^i$, with $a_{n+1} = 1$, be a linear differential equation satisfying property* (P). *Then followings hold:*

(i) *If $n=3$, then*

$$a_1 = \frac{1}{2}a_2a_3 - \frac{3}{4}a_3a_3' - \frac{1}{8}a_3^3 + a_2' - \frac{1}{2}a_3''.$$

(ii) *If $n=5$, then*

$$\begin{aligned}
a_3 &= \frac{2}{3}a_4a_5 - \frac{5}{3}a_5a_5' - \frac{5}{27}a_5^3 + 2a_4' - \frac{5}{3}a_5'', \\
a_1 &= a_2' - a_4''' + a_5^{(4)} - a_4^{(2)}a_5 - a_4'a_5' + \frac{5}{3}a_5(a_5')^2 + \frac{1}{3}a_2a_5 \\
&\quad - \frac{1}{27}a_4a_5^3 + \frac{10}{27}a_5^3a_5' + \frac{1}{81}a_5^5 - \frac{1}{3}a_4'a_5^2 - \frac{1}{3}a_4a_5a_5' + \frac{10}{9}a_5^2a_5'' \\
&\quad + \frac{10}{3}a_5'a_5'' - \frac{1}{3}a_4a_5'' + \frac{5}{3}a_5a_5'''.
\end{aligned}$$

3 Picard-Fuchs Equation as a Self-Dual Linear Differential Equation

In this section $\pi : \mathcal{W} \to P$ refers to a family of n-dimensional compact Kähler manifolds, i.e., π is a holomorphic proper submersion of complex manifolds $\mathcal{W}$ and P such that for any $z \in P$, $W_z := \pi^{-1}(z)$ is an n-dimensional compact Kähler manifold. If we denote the k-th de Rham cohomology group of W_z by $H^k_{\mathrm{dR}}(W_z)$, then *de Rham Lemma* gives the isomorphism $H^k_{dR}(W_z) \cong H^k(W_z, \mathbb{R})$, or equivalently $H^k_{dR}(W_z;\mathbb{C}) \cong H^k(W_z,\mathbb{C})$ where $H^k_{dR}(W_z;\mathbb{C})$ denotes the complexified de Rham cohomology group. Here $\mathsf{b}_k(W_z) := \dim H^k_{dR}(W_z;\mathbb{C})$ stands for the k-th betti number of W_z. Also by *Hodge decomposition theorem* we have

$$H^k_{dR}(W_z;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(W_z), \tag{3.1}$$

in which $H^{p,q}(W_z)$ is (p,q)-th Dolbeault cohomology and by *Dolbeault's theorem* we have the isomorphism $H^{p,q}(W_z) \cong H^q(W_z, \Omega^p_{W_z})$. We denote by $\mathsf{h}^{p,q}(W_z) := \dim H^{p,q}(W_z)$, that is called *$(p,q)$-th Hodge number* of W_z. By defining

$$F^p(W_z) := \bigoplus_{p\le r\le n} H^{r,n-r}(W_z),\ 0 \le p \le n,$$

we yield the following decreasing filtration which is known as the *Hodge filtration* of $H^k_{\mathrm{dR}}(W_z)$,

$$F^\bullet H^k(W_z):\ \{0\} = F^{k+1}(W_z) \subset F^k(W_z) \subset \ldots \subset F^0(W_z) = H^k_{\mathrm{dR}}(W_z;\mathbb{C}). \tag{3.2}$$

We can consider the family $\mathcal{W}$ as a complex deformation of $W := W_0$, $0 \in P$. As one can find in standard texts of complex geometry, e.g. [26], up to replacing P by a neighborhood of the base point 0, $\mathsf{b}_k(W_z) = \mathsf{b}_k(W)$ and $\mathsf{h}^{p,q}(W_z) = \mathsf{h}^{p,q}(W)$ for any $z \in P$. Hence simply we can write b_k and $\mathsf{h}^{p,q}$ instead of $\mathsf{b}_k(W_z)$ and $\mathsf{h}^{p,q}(W_z)$. Also one can see that $\mathsf{b}_k = \sum_{p+q=k} \mathsf{h}^{p,q}$, $\mathsf{h}^{p,q} = \mathsf{h}^{q,p}$ and $\mathsf{h}^{p,q} = \mathsf{h}^{n-q,n-p}$.

3.1 Gauss-Manin Connection and Griffiths Transversality

Consider the sheaf $R^k\pi_*\underline{\mathbb{C}_{\mathcal{W}}}$ on P, where $\underline{\mathbb{C}_{\mathcal{W}}}$ is the constant sheaf on $\mathcal{W}$ with fibers $\mathbb{C}$ and $R^k\pi_*$ refers to k-th derived functor of the pushforward. For any $z \in P$, we have the following presentation of the stalks of $R^k\pi_*\underline{\mathbb{C}_{\mathcal{W}}}$

$$(R^k\pi_*\underline{\mathbb{C}_{\mathcal{W}}})_z \simeq H^k(W_z,\mathbb{C}) \simeq H^k_{\mathrm{dR}}(W_z;\mathbb{C}).$$

Hence $R^k\pi_*\underline{\mathbb{C}_{\mathcal{W}}}$ is a locally constant sheaf on P. Formally speaking, $R^k\pi_*\underline{\mathbb{C}_{\mathcal{W}}}$ is the sheaf associated to the presheaf $U \mapsto H^k(\pi^{-1}(U),\mathbb{C})$ (see [12]). In fact, for a contractible open subset $U \subset P$, by Ehresmann Lemma $\pi^{-1}(U) \cong U \times W_z$ for some $z \in P$, so $H^k(\pi^{-1}(U),\mathbb{C}) \simeq H^k(W_z,\mathbb{C})$. By defining

$$H^k_{\mathrm{dR}}(\mathcal{W}/P) := R^k\pi_*\underline{\mathbb{C}_{\mathcal{W}}} \otimes_{\mathbb{C}} \mathcal{O}_P, \tag{3.3}$$

which is a holomorphic vector bundle on P, then for any $z \in P$, $H^k_{\mathrm{dR}}(\mathcal{W}/P)_z \cong H^k_{\mathrm{dR}}(W_z;\mathbb{C})$.

Definition 3.1. The holomorphic vector bundle $H^k_{\mathrm{dR}}(\mathcal{W}/P)$ defined in (3.3) is called *k-th relative de Rham cohomology group*. The unique integrable connection

$$\nabla^{\mathsf{GM}} : H^k_{\mathrm{dR}}(\mathcal{W}/P) \to \Omega^1_P \otimes_{\mathcal{O}_P} H^k_{\mathrm{dR}}(\mathcal{W}/P),$$

whose flat sections coincides with $R^k\pi_*\underline{\mathbb{C}_{\mathcal{W}}}$ is known as *Gauss-Manin connection*.

For a vector field v on P, consider the map $v \otimes \mathrm{Id} : \Omega^1_P \otimes_{\mathcal{O}_P} H^k_{\mathrm{dR}}(\mathcal{W}/P) \to H^k_{\mathrm{dR}}(\mathcal{W}/P)$. Then by composing the Gauss-Manin connection ∇^{GM} with $v \otimes \mathrm{Id}$ we define

$$\begin{aligned} \nabla^{\mathsf{GM}}_v :& H^k_{\mathrm{dR}}(\mathcal{W}/P) \to H^k_{\mathrm{dR}}(\mathcal{W}/P) \\ \nabla^{\mathsf{GM}}_v :=& (v \otimes \mathrm{Id}) \circ \nabla^{\mathsf{GM}}. \end{aligned} \tag{3.4}$$

From now on, if no confusion arises, we denote the Gauss-Manin connection by ∇ instead of ∇^{GM}.

Remark 3.1. The k-th relative de Rham cohomology group $H^k_{\mathrm{dR}}(\mathcal{W}/P)$ is locally free of finite rank, say m. Let $\{\omega_j\}_{j=1}^m$ be a local frame of $H^k_{\mathrm{dR}}(\mathcal{W}/P)$ and $\varpi := (\omega_1 \quad \omega_2 \quad \ldots \quad \omega_m)^{\mathsf{t}}$ be the matrix presentation of this frame, where t refers to the matrix transpose. Then we define the *matrix of Gauss-Manin connection*, which is denoted by GM_ϖ, as follow

$$\nabla\varpi := (\nabla\omega_1 \quad \nabla\omega_2 \quad \ldots \quad \nabla\omega_m)^{\mathsf{t}} = \mathsf{GM}_\varpi \otimes \varpi.$$

We are noting that for any $z \in P$ and any $j \in \{1,2,\ldots,m\}$, $\omega_j(z) \in H^k_{\mathrm{dR}}(W_z;\mathbb{C})$ and we can present it by a k-form on W_z that we denote it also by $\omega_j(z)$.

Each fiber $H^k_{\mathrm{dR}}(W_z;\mathbb{C})$ of $H^k_{\mathrm{dR}}(\mathcal{W}/P)$ has a Hodge filtration, and this yields a decreasing filtration of $H^k_{\mathrm{dR}}(\mathcal{W}/P)$ by holomorphic subbundles

$$\mathcal{F}^\bullet H^k : \ \{0\} = \mathcal{F}^{k+1} \subset \mathcal{F}^k \subset \ldots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = H^k_{\mathrm{dR}}(\mathcal{W}/P), \tag{3.5}$$

such that for any $z \in P$ and any $p \in \{0,1,2,\ldots,k\}$

$$\mathcal{F}^p_z \cong F^p(W_z) = \bigoplus_{p \le r \le k} H^{r,k-r}(W_z).$$

The filtration $\mathcal{F}^\bullet H^k$ given in (3.5), is also called *Hodge filtration of* $H^k_{\mathrm{dR}}(\mathcal{W}/P)$.

Theorem 3.1. (Griffiths transversality) *Under above terminologies, following holds:*

$$\nabla\mathcal{F}^p \subset \Omega^1_P \otimes \mathcal{F}^{p-1}, \ \ p = 1,2,\ldots k.$$

3.2 Picard-Fuchs Equation

Here we consider the Hodge filtration $\mathcal{F}^\bullet H^n$ of $H^n(\mathcal{W}/P)$ and fix the local section $\omega \in \mathcal{F}^n$; indeed for any $z \in P$, $\omega(z) \in H^{n,0}(W_z)$ is a holomorphic n-form. Let $\mathcal{D}$ be the ring of linear differential operators on P. If $\dim P = r$ and $z_1, z_2, \ldots, z_r$ is a local coordinate of $(P, 0)$, then we have $\mathcal{D} = \mathbb{C}(z_1, z_2, \ldots, z_r)[\partial_1, \partial_2, \ldots, \partial_r]$, where $\mathbb{C}(z_1, z_2, \ldots, z_r)$ is the ring of convergent power series of $z_1, z_2, \ldots, z_r$ and $\partial_i = \frac{\partial}{\partial z_i}$. We define the $\mathcal{O}_P$-homomorphism $\Psi : \mathcal{D} \to H^n_{\mathrm{dR}}(\mathcal{W}/P)$, which for vector fields $v_1, v_2, \ldots, v_k$ on P is determined by

$$\Psi(v_1 v_2 \ldots v_k) = \nabla_{v_1} \nabla_{v_2} \ldots \nabla_{v_k} \omega.$$

By this definition, Ψ gives the structure of a $\mathcal{D}$-module to $H^n_{\mathrm{dR}}(\mathcal{W}/P)$.

Definition 3.2. The ideal $\mathcal{I} = \ker \Psi$, consist of differential operators that annihilate ω, by definition is called *Picard-Fuchs ideal* and any $L \in \mathcal{I}$ is called a *Picard-Fuchs equation.*

Assumption 3.1. In what follows in this section, we suppose that $\mathcal{W}$ is a one parameter family of n-dimensional compact Kähler manifolds, i.e., $\dim P = 1$.

Let z be a coordinate of $(P, 0)$ and define the differential operator $\vartheta := \nabla_{z\frac{\partial}{\partial z}}$. Then $(H^n_{\mathrm{dR}}(\mathcal{W}/P), \vartheta)$ is a differential $\mathbb{C}(z)$-module. Considering the terminologies introduced in §2, we present the following definition of Picard-Fuchs equation.

Definition 3.3. Let $\mathcal{W}$ be a one parameter family of n-dimensional compact Kähler manifolds and $\omega \in H^n_{\mathrm{dR}}(\mathcal{W}/P)$ be a fixed non-zero element. Then the minimal operator of ω is called the *Picard-Fuchs equation associated with ω.*

Assumption 3.2. From now on, we suppose that there exists a nowhere vanishing holomorphic n-form $\omega \in \mathcal{F}^n$ such that the Picard-Fuchs equation L associated with it is of order $n+1$ given as follow

$$\mathsf{L} = \vartheta^{n+1} - a_n(z)\vartheta^n - \ldots - a_1(z)\vartheta - a_0(z), \tag{3.6}$$

where $a_i(z) \in \mathbb{Q}(z)$, $i = 0, 1, \ldots, n$. Therefore, by definition $\mathsf{L}\omega = 0$.

3.3 Intersection Form

For any $\alpha, \xi \in H^n_{\mathrm{dR}}(\mathcal{W}/P)$, the *intersection form* of α and ξ by definition is

$$\langle \alpha, \xi \rangle(z) := \mathrm{Tr}(\alpha(z) \smile \xi(z)), \ \forall z \in P,$$

in which "$\smile$" refers to the cup product. In de Rham cohomology, the cup product of differential forms is induced by the wedge product, hence in the family $\mathcal{W}$ the intersection form is defined as follow

$$\langle \alpha, \xi \rangle(z) = \int_{W_z} \alpha(z) \wedge \xi(z). \tag{3.7}$$

We state below a lemma that follows easily from properties of wedge product.

Lemma 3.1. *Followings hold:*

(i) $\langle \alpha, \xi \rangle = (-1)^n \langle \xi, \alpha \rangle$*, for any* $\alpha, \xi \in H^n_{\mathrm{dR}}(\mathcal{W}/P)$.

(ii) *If* $\mathcal{F}^\bullet H^n$ *is the Hodge filtration of* $H^n_{\mathrm{dR}}(\mathcal{W}/P)$*, then*

$$\langle \mathcal{F}^i, \mathcal{F}^j \rangle = 0, \ \text{for} \ \ i + j \geq n+1. \tag{3.8}$$

3.4 Self-Duality

Here we give the proof of Proposition 1.1. First we fix following notation.

Notation 3.1. By notation, for $i = 1, \ldots, n+1$, we define $\omega_i := \vartheta^{i-1}\omega$.

We know that $\omega_1 = \omega \in \mathcal{F}^n$, hence by Griffiths transversality $\omega_i \in \mathcal{F}^{(n+1)-i}$. Therefore, Lemma 3.1**(ii)** implies that

$$\langle \omega_1, \omega_i \rangle = 0, \ i = 1, 2, \ldots, n.$$

One can find in [1, § 4.5] that

$$\langle \omega_1, \omega_{n+1} \rangle(z) = c_0 \exp\left(-\frac{2}{n+1} \int_0^z a_n(v) \frac{dv}{v} \right), \tag{3.9}$$

for some nonzero constant c_0. If we denote by $\tilde{a}(z) := c_0 \exp\left(-\frac{2}{n+1} \int_0^z a_n(v) \frac{dv}{v} \right)$, then for any $i \in \{1, \ldots, n\}$

$$\langle \omega_i, \omega_{n+2-i} \rangle = (-1)^{i-1} \tilde{a}. \tag{3.10}$$

To see this, first note that by Lemma 3.1**(ii)** we have $\langle \omega_{j+1}, \omega_{n-j} \rangle = 0, \ j = 0, 1, \ldots, n-1$. On the other hand we know that

$$\vartheta \langle \omega_{j+1}, \omega_{n-j} \rangle = \langle \vartheta\omega_{j+1}, \omega_{n-j} \rangle + \langle \omega_{j+1}, \vartheta\omega_{n-j} \rangle = \langle \omega_{j+2}, \omega_{n-j} \rangle + \langle \omega_{j+1}, \omega_{n-j+1} \rangle = 0,$$

where in the first side of above equation by ϑ we mean the usual derivation operator $z\frac{\partial}{\partial z}$. Thus we obtain $\langle \omega_{j+2}, \omega_{n-j} \rangle = -\langle \omega_{j+1}, \omega_{n-j+1} \rangle$, from which follows (3.10).

Proposition 3.1. *Let $\mathcal{F}^\bullet H^n$ be the Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$. Then $\dim H^n_{\mathrm{dR}}(\mathcal{W}/P) = n+1$ if and only if $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1$ for any $i \in \{0, 1, \ldots, n\}$.*

Proof. If $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1$, then it is evident that $\dim H^n_{\mathrm{dR}}(\mathcal{W}/P) = n+1$. Conversely suppose that $\dim H^n_{\mathrm{dR}}(\mathcal{W}/P) = n+1$. Then it is enough to prove that $\dim \mathcal{F}^i/\mathcal{F}^{i+1} \neq 0$. By (3.9) we know that $\langle \omega_1, \omega_{n+1} \rangle \neq 0$, hence Lemma 3.1**(ii)** implies that $\omega_{n+1} \in \mathcal{F}^0 \setminus \mathcal{F}^1$. Now to prove $\dim \mathcal{F}^i/\mathcal{F}^{i+1} \neq 0$, by contradiction suppose that there is a $j \in \{1, 2, 3, \ldots, n-1\}$ such that $\dim \mathcal{F}^j/\mathcal{F}^{j+1} = 0$, and hence $\mathcal{F}^{j+1} = \mathcal{F}^j$. We know that $\omega_{(n+1)-j} \in \mathcal{F}^j$, thus by Griffiths transversality $\omega_{(n+1)-j+1} = \vartheta\omega_{(n+1)-j} \in \mathcal{F}^{j+1} = \mathcal{F}^j$. Again by using of Griffiths transversality we obtain that $\omega_{(n+1)-j+2} \in \mathcal{F}^j$. By continuing this process it follows that $\omega_{n+1} \in \mathcal{F}^j$, which contradicts $\omega_{n+1} \in \mathcal{F}^0 \setminus \mathcal{F}^1$. ■

Assumption 3.3. In the rest of this section we assume that for any $i \in \{0, 1, \ldots, n\}$, $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1$, or equivalently $\dim H^n_{\mathrm{dR}}(\mathcal{W}/P) = n+1$.

Remark 3.2. Assumption 3.3 yields that $\dim \mathcal{F}^i = (n+1) - i, \ \ i = 0, 1, \ldots, n+1$. It is equivalent to say $\dim \mathsf{h}^{i,j}(W_z) = 1$ for any $z \in P$ and any non-negative integers i, j with $i + j = n$.

Proposition 3.2. *The set $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ construct a frame for $H^n_{\mathrm{dR}}(\mathcal{W}/P)$ such that for any $i \in \{1, 2, \ldots, n+1\}$,*

$$\omega_i \in \mathcal{F}^{(n+1)-i} \setminus \mathcal{F}^{(n+2)-i}. \tag{3.11}$$

Proof. We know that $\dim H^n_{\mathrm{dR}}(\mathcal{W}/P) = n+1$, hence it is enough to show that for any z, the set $\{\omega_1(z), \omega_2(z), \ldots, \omega_{n+1}(z)\}$ is linearly independent. To this end, suppose that there are constants $b_1, b_2, \ldots, b_{n+1}$ such that $b_1\omega_1(z) + b_2\omega_2(z) + \ldots + b_{n+1}\omega_{n+1}(z) = 0$. If we set $k := \max\{i \,|\, b_i \neq 0,\ i = 1, 2, \ldots, n+1\}$, then we can write

$$\omega_k(z) = c_1\omega_1(z) + c_2\omega_2(z) + \ldots + c_{k-1}\omega_{k-1}(z),$$

in which $c_i = \frac{b_i}{b_k}$. By intersecting ω_k with ω_{n+2-k}, and using Lemma 3.1**(ii)** we have

$$\langle \omega_k, \omega_{n+2-k}\rangle(z) = c_1(z)\langle \omega_1, \omega_{n+2-k}\rangle(z) + \ldots + c_{k-1}(z)\langle \omega_{k-1}, \omega_{n+2-k}\rangle(z) = 0.$$

On account of (3.10) we get $\langle \omega_k, \omega_{n+2-k}\rangle(z) \neq 0$, which is an contradiction. Thus for any $i \in \{1, 2, \ldots, n+1\}$, $b_i = 0$.
To prove (3.11), first note that Griffiths transversality implies that $\omega_i \in \mathcal{F}^{(n+1)-i}$, $i = 1, 2, \ldots, n+1$. On the other hand, since $\dim \mathcal{F}^{(n+2)-i} = i - 1$ and $\{\omega_1, \omega_2, \ldots, \omega_i\}$ is an independent subset of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$, it follows that $\omega_i \notin \mathcal{F}^{(n+2)-i}$. ■

Finally in the following proposition we give the proof of Proposition 1.1.

Proposition 3.3. *The picard-Fuchs equation* L *satisfies the property* (P)*, or equivalently* L *is self-dual.*

Proof. Consider the intersection form defined as follow

$$\langle ., . \rangle : H^n_{\mathrm{dR}}(\mathcal{W}/P) \times H^n_{\mathrm{dR}}(\mathcal{W}/P) \to \mathbb{C}(z).$$

Equation (3.10) implies that $\langle ., . \rangle$ is non-degenerate, and the Lemma 3.1**(i)** verifies that $\langle ., . \rangle$ is a $(-1)^n$-symmetric form. Lemma 3.1**(ii)** guaranties that in frame $\{\omega, \vartheta\omega, \ldots, \vartheta^n\omega\}$ we have,

$$\langle \omega, \vartheta^i\omega\rangle = 0, \text{ for } i = 0, 1, \ldots, n-1.$$

Hence by Definition 2.2, L satisfies the property (P). This is equivalent with self-duality of L by Proposition 2.1. ■

4 Darboux-Halphen-Ramanujan Vector Field

In this section $\pi : \mathcal{W} \to P$ refers to a one parameter family of n-dimensional Calabi-Yau manifolds, or equivalently it is a complex deformation of an n-dimensional Calabi-Yau manifold $W := W_0$. Let z be a local coordinate of $(P, 0)$. Then for any $z \in P$, W_z is a Calabi-Yau n-fold. We know that Calabi-Yau manifold W is a compact Kähler manifold whose, up to multiplication by a constant, has a unique nowhere vanishing holomorphic n-form $\omega \in H^{n,0}(W)$. Thus, there is a holomorphic section of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$ that at 0 coincides with ω and we denote it also by ω. Hence $\omega \in \mathcal{F}^n$, where $\mathcal{F}^\bullet H^n$ is the Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$, and $\omega(z) \in H^{n,0}(W_z)$ is a nowhere vanishing holomorphic n-form of W_z for any z. In this section we fix $\omega \in \mathcal{F}^n$ and suppose that ω satisfies Assumption 3.2 and $\mathcal{F}^\bullet H^n$ satisfies Assumption 3.3. Throughout this section, we employ the notations of pervious sections.

Example 4.1. *By now, as I know, there are* 14 *examples of one parameter families of Calabi-Yau 3-folds satisfying the hypothesis of* $\mathcal{W}$ *given above. Any of these* 14 *families is mirror symmetry of* 14 *structures given in* Table 1. *In this table* $X(d_1, d_2, \ldots, d_r) \subset \mathbb{P}^s(l_0, l_1, \ldots, l_s)$ *refers to the complete intersection of* r *hypersurfaces of degree* $d_1, d_2, \ldots, d_r$ *in weighted projective space* $\mathbb{P}^s(l_0, l_1, \ldots, l_s)$ *with* $r \leq s$*, such that* $\sum_{i=1}^r d_i = \sum_{j=0}^s l_j$*. The Picard-Fuchs equation* L *associated with the nowhere vanishing holomorphic* 3*-form of any of these families is a hypergeometric equation given as follow:*

$$\mathsf{L} = \vartheta^4 - cz(\vartheta + r_1)(\vartheta + r_2)(\vartheta + 1 - r_2)(\vartheta + 1 - r_1), \tag{4.1}$$

where r_1, r_2, c *are given in* Table 1. *Note that* $\sharp 1$ *is the family of quintic 3-folds that we pointed it out in §1. For more details one can see the references given in* Table 1.

$\sharp$	r_1	r_2	c	Structure	References
1	1/5	2/5	5^5	$X(5) \subset \mathbb{P}^4$	[6, 11]
2	1/6	2/6	$2^5 3^6$	$X(6) \subset \mathbb{P}^4(2,1,1,1,1)$	[21]
3	1/8	3/8	2^{18}	$X(8) \subset \mathbb{P}^4(4,1,1,1,1)$	[21]
4	1/10	3/10	$2^9 5^6$	$X(10) \subset \mathbb{P}^4(5,2,1,1,1)$	[21]
5	1/3	1/3	3^6	$X(3,3) \subset \mathbb{P}^5$	[20]
6	1/4	2/4	2^{10}	$X(2,4) \subset \mathbb{P}^5$	[20]
7	1/3	1/2	$2^4 3^3$	$X(2,2,3) \subset \mathbb{P}^6$	[20]
8	1/2	1/2	2^8	$X(2,2,2,2) \subset \mathbb{P}^7$	[20]
9	1/4	1/4	2^{12}	$X(4,4) \subset \mathbb{P}^5(2,2,1,1,1,1)$	[19]
10	1/6	1/6	$2^8 3^6$	$X(6,6) \subset \mathbb{P}^5(3,3,2,2,1,1)$	[19]
11	1/4	1/3	$2^6 3^3$	$X(3,4) \subset \mathbb{P}^5(2,1,1,1,1,1)$	[19]
12	1/6	3/6	$2^8 3^3$	$X(2,6) \subset \mathbb{P}^5(3,1,1,1,1,1)$	[19]
13	1/6	1/4	$2^{10} 3^3$	$X(4,6) \subset \mathbb{P}^5(3,2,2,1,1,1)$	[19]
14	1/12	5/12	12^6	$X(2,12) \subset \mathbb{P}^5(6,4,1,1,1,1)$	[10]

Table 1: Calabi-Yau 3-folds [8]

Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$ is as follow

$$\mathcal{F}^\bullet H^n: \ \{0\} = \mathcal{F}^{n+1} \subset \mathcal{F}^n \subset \ldots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = H^n_{\mathrm{dR}}(\mathcal{W}/P), \ \ \dim \mathcal{F}^i = (n+1) - i, \tag{4.2}$$

and as we saw in Proposition 3.2, $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ construct a frame of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$ such that

$$\omega_i \in \mathcal{F}^{(n+1)-i} \setminus \mathcal{F}^{(n+2)-i}. \tag{4.3}$$

By using of Picard-Fuchs equation (3.6) we have

$$\vartheta^{n+1}\omega = \vartheta\omega_{n+1} = a_0\omega_1 + a_1\omega_2 + \ldots + a_n\omega_{n+1}. \tag{4.4}$$

Hence, considering Remark 3.1, if we apply the Gauss-Manin connection to the column of n-forms $\varpi = \begin{pmatrix} \omega_1 & \omega_2 & \ldots & \omega_{n+1} \end{pmatrix}^{\mathsf{t}}$, then

$$\nabla\varpi = \mathsf{GM}_\varpi \otimes \varpi, \tag{4.5}$$

where

$$\mathsf{GM}_\varpi = \frac{1}{z}\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} & a_n \end{pmatrix} dz. \tag{4.6}$$

To see this, for $j = 1, 2, \ldots, n$, we have

$$z\nabla_{\frac{\partial}{\partial z}}\omega_j = \nabla_{z\frac{\partial}{\partial z}}\omega_j = \vartheta\omega_j = \omega_{j+1} \Longrightarrow \nabla\omega_j = \frac{1}{z}dz \otimes \omega_{j+1}.$$

Analogously, on account of (4.4), for ω_{n+1} we obtain $\nabla\omega_{n+1} = \frac{1}{z}\sum_{i=0}^{n} a_i dz \otimes \omega_{i+1}$.

In this section we are going to prove Theorem 1.1. In order to do this, we will treat with intersection form, but because of different behaviors of intersection form for odd or even integer n, see Lemma 3.1**(i)**, we separate the cases for odd and even integers. First, we state the results in the odd case in §4.1. In particular for $n = 3, 5$ we give an explicit computation of results in §4.4 and §4.3.

4.1 Odd Case

In the whole of this subsection n is considered to be an odd positive integer. If we define the intersection form matrix as follow

$$\Omega = (\Omega_{ij})_{1\leq i,j\leq n+1} := (\langle\omega_i, \omega_j\rangle)_{1\leq i,j\leq n+1}, \tag{4.7}$$

then Lemma 3.1**(i)** implies that $\Omega^{\mathsf{t}} = -\Omega$, and hence $\Omega_{ii} = 0$, $i = 1, 2, \ldots, n+1$. Lemma 3.1**(ii)** yields $\Omega_{ij} = \langle\omega_i, \omega_j\rangle = 0$ for $i+j \leq n+1$, and by (3.10) we find $\Omega_{i(n+2-i)} = (-1)^{i-1}\tilde{a}$ for any $i = 1, 2, \ldots, n+1$. Therefore, we can state the matrix Ω as follow

$$\Omega = \begin{pmatrix} 0 & 0 & \ldots & 0 & \tilde{a} \\ 0 & 0 & \ldots & -\tilde{a} & \Omega_{2(n+1)} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \tilde{a} & \ldots & 0 & \Omega_{n(n+1)} \\ -\tilde{a} & -\Omega_{2(n+1)} & \ldots & -\Omega_{n(n+1)} & 0 \end{pmatrix}. \tag{4.8}$$

Definition 4.1. We say that a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ of $H^n_{\mathrm{dR}}(W;\mathbb{C})$ is *compatible with its Hodge filtration*, if for any $i \in \{1, 2, \ldots, n+1\}$

$$\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}. \tag{4.9}$$

Next we introduce a special moduli space of Calabi-Yau manifold W that in the rest of this text will be in interest. To do this, we first provide an equivalence relation.

Definition 4.2. Let W_1, W_2 be two Calabi-Yau n-folds and $\{\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_{n+1}\}$ be a basis of $H^n_{\mathrm{dR}}(W_i;\mathbb{C})$, $i = 1, 2$, compatible with its Hodge filtration. Then we write

$$(W_1, [\alpha^1_1, \alpha^1_2, \ldots, \alpha^1_{n+1}]) \sim (W_2, [\alpha^2_1, \alpha^2_2, \ldots, \alpha^2_{n+1}]) \tag{4.10}$$

if and only if there exist a biholomorphism $\varphi : W_1 \to W_2$ such that $\varphi^*(\alpha^2_j) = \alpha^1_j$, $j = 1, 2, \ldots, n+1$. It is obvious that "$\sim$" is an equivalence relation. For the Calabi-Yau n-fold W, and a basis $\{\alpha_i\}_{i=1}^{n+1}$ of $H^n_{\mathrm{dR}}(W;\mathbb{C})$ compatible with its Hodge filtration, the *moduli space* $\tilde{\mathsf{T}}$ of pair $(W, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$ is defined under above equivalence relation (4.10).

Remark 4.1. We know that the family $\pi : \mathcal{W} \to P$ is the complex deformation of W. Hence for any different $z_1, z_2 \in P$, W_{z_1} and W_{z_2} are not biholomorph. We thus have two different members $(W_{z_1}, [\omega_1(z_1), \omega_2(z_1), \ldots \omega_{n+1}(z_1)])$ and $(W_{z_2}, [\omega_1(z_2), \omega_2(z_2), \ldots \omega_{n+1}(z_2)])$

of moduli space $\tilde{\mathsf{T}}$. Also suppose that $\{\mu_i\}_{i=1}^{n+1}$ and $\{\nu_i\}_{i=1}^{n+1}$ are two bases of $H^n_{\rm dR}(W;\mathbb{C})$ compatible with its Hodge filtration. If for any

$$\varphi \in \mathrm{Aut}(W) := \{f : W \to W |\, f \text{ is a biholomorphism}\},$$

it does not preserve the bases, i.e., there exist a $j \in \{1, 2, \dots, n+1\}$ such that $\varphi^*\nu_j \neq \mu_j$, then $(W, [\mu_1, \mu_2, \dots, \mu_{n+1}])$ and $(W, [\nu_1, \nu_2, \dots, \nu_{n+1}])$ yield two different elements of moduli space $\tilde{\mathsf{T}}$.

As we fixed in the beginning of this section, for any $z \in P$, $\{\omega_1(z), \omega_2(z), \dots, \omega_{n+1}(z)\}$ construct a basis for $H^n_{\rm dR}(W_z, \mathbb{C})$ that is compatible with its Hodge filtration. By abuse of notation, we remove the letter z from this basis and denote it by $\{\omega_1, \omega_2, \dots, \omega_{n+1}\}$, and hence $(W_z, [\omega_1, \omega_2, \dots \omega_{n+1}]) \in \tilde{\mathsf{T}}$. Let S be the change of basis matrix $\alpha = S\varpi$, where $\{\alpha_i\}_{i=1}^{n+1}$ is a basis of $H^n_{\rm dR}(W_z;\mathbb{C})$ compatible with its Hodge filtration, and $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \end{pmatrix}^{\mathsf{t}}$. Then (4.3) and (4.9) imply that S is a lower triangular matrix which we consider it as follow

$$S = \begin{pmatrix} s_{11} & 0 & 0 & \dots & 0 \\ s_{21} & s_{22} & 0 & \dots & 0 \\ s_{31} & s_{32} & s_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{(n+1)1} & s_{(n+1)2} & s_{(n+1)3} & \dots & s_{(n+1)(n+1)} \end{pmatrix}. \tag{4.11}$$

Hence the entries of S present coordinates of a chart of $\tilde{\mathsf{T}}$ that we will employ it soon.

Lemma 4.1. *Let $\{\alpha_i\}_{i=1}^{n+1}$ be a frame of $H^n_{\rm dR}(\mathcal{W}/P)$ compatible with its Hodge filtration.*

(i) *If we define $\Psi := (\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le n+1}$, then $\Psi = S\Omega S^{\mathsf{t}}$.*

(ii) *If we set $\nabla\alpha = \mathsf{GM}_\alpha \otimes \alpha$, then*

$$\mathsf{GM}_\alpha = (dS + S.\mathsf{GM}_\varpi)S^{-1}, \tag{4.12}$$

where

$$dS = \begin{pmatrix} ds_{11} & 0 & 0 & \dots & 0 \\ ds_{21} & ds_{22} & 0 & \dots & 0 \\ ds_{31} & ds_{32} & ds_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ds_{(n+1)1} & ds_{(n+1)2} & ds_{(n+1)3} & \dots & ds_{(n+1)(n+1)} \end{pmatrix}. \tag{4.13}$$

Proof.

(i) By using of $\alpha = S\varpi$, verifying $\Psi = S\Omega S^{\mathsf{t}}$ is an easy exercise of linear algebra.

(ii) If we apply the Gauss-Manin connection to the equation $\alpha = S\varpi$, and considering $\nabla\varpi = \mathsf{GM}_\varpi \otimes \varpi$, then we have

$$\begin{aligned}\nabla\alpha &= dS \otimes \varpi + S\nabla\varpi = (dS + S.\mathsf{GM}_\varpi) \otimes \varpi \\ &= (dS + S.\mathsf{GM}_\varpi)S^{-1} \otimes \alpha,\end{aligned}$$

which completes the proof. ■

Following proposition give a more important step of the proof of Theorem 1.1.

Proposition 4.1. *Let $\tilde{\mathsf{T}}$ be the moduli of $(W,[\alpha_1,\alpha_2,\ldots,\alpha_{n+1}])$, where $\{\alpha_i\}_{i=1}^{n+1}$ is a basis of $H^n_{\rm dR}(W;\mathbb{C})$ compatible with its Hodge filtration. Then there exist a unique vector field $\tilde{\mathsf{H}}$ and unique meromorphic functions y_i, $i=1,2,\ldots,n-2$, on $\tilde{\mathsf{T}}$ such that*

$$\nabla_{\tilde{\mathsf{H}}}\alpha = Y\alpha, \tag{4.14}$$

in which $\alpha = \begin{pmatrix}\alpha_1 & \alpha_2 & \ldots & \alpha_{n+1}\end{pmatrix}^{\mathsf{t}}$, and Y is given by (1.16).

Proof. The idea of the proof is to present the vector field $\tilde{\mathsf{H}}$ explicitly in a chart of $\tilde{\mathsf{T}}$. It is easily seen that the dimension of $\tilde{\mathsf{T}}$ is $k+1$, where $k=\frac{(n+1)(n+2)}{2}$. For any $(W_z,[\alpha_1,\alpha_2,\ldots,\alpha_{n+1}])\in\tilde{\mathsf{T}}$, let S be the change of basis matrix $\alpha=S\varpi$ given in (4.11). We consider the chart $t=(t_0,t_1,\ldots,t_k)$ of $\tilde{\mathsf{T}}$, for which the coordinates are defined as $t_0=z, t_1=s_{11}, t_2=s_{12},\ldots,t_k=s_{(n+1)(n+1)}$. We suppose that the vector field $\tilde{\mathsf{H}}$ is given as follow

$$\tilde{\mathsf{H}}=\sum_{i=0}^{k}\tilde{\mathsf{H}}_i(t)\frac{\partial}{\partial t_i},$$

where $\tilde{\mathsf{H}}_i$'s, $i=0,1,\ldots,k$, are meromorphic functions on $\tilde{\mathsf{T}}$. Since $\tilde{\mathsf{H}}$ satisfies $\nabla_{\tilde{\mathsf{H}}}\alpha=Y\alpha$, Lemma 4.1**(ii)** implies that

$$(dS+S.\mathsf{GM}_\varpi)S^{-1}(\tilde{\mathsf{H}})=Y. \tag{4.15}$$

We have $S.\mathsf{GM}_\varpi(\tilde{\mathsf{H}})=\dot{z}S.\widehat{\mathsf{GM}}_\varpi$, where $\dot{z}(t):=\tilde{\mathsf{H}}_0(t)$ and $\widehat{\mathsf{GM}}_\varpi$ is defined by $\mathsf{GM}_\varpi=\widehat{\mathsf{GM}}_\varpi dz$. Also if we define $\dot{s}_{11}(t):=\tilde{\mathsf{H}}_1(t)$, $\dot{s}_{21}(t):=\tilde{\mathsf{H}}_2(t)$, $\ldots$, $\dot{s}_{(n+1)(n+1)}(t):=\tilde{\mathsf{H}}_k(t)$, then we have $dS(\tilde{\mathsf{H}})=\dot{S}$, where

$$\dot{S}=\begin{pmatrix}\dot{s}_{11} & 0 & 0 & \ldots & 0\\ \dot{s}_{21} & \dot{s}_{22} & 0 & \ldots & 0\\ \dot{s}_{31} & \dot{s}_{32} & \dot{s}_{33} & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \dot{s}_{(n+1)1} & \dot{s}_{(n+1)2} & \dot{s}_{(n+1)3} & \ldots & \dot{s}_{(n+1)(n+1)}\end{pmatrix}. \tag{4.16}$$

Therefore (4.15) gives $(\dot{S}+\dot{z}S.\widehat{\mathsf{GM}}_\varpi)S^{-1}=Y$, which yields

$$\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_\varpi. \tag{4.17}$$

Consequently we can find $\dot{z}$ (or $\tilde{\mathsf{H}}_0$) and y_i's (that we state them in Lemma 4.2 below). Hence all the terms of the right hand side of (4.17) are determined, from which we can find $\tilde{\mathsf{H}}_i$'s, $i=1,2,\ldots,k$. Thus, the existence of vector field $\tilde{\mathsf{H}}$ that satisfies (4.14) is verified. The uniqueness of $\tilde{\mathsf{H}}$ and y_i's follow from Lemma 4.2**(i)**,**(ii)**. ■

The proof of Proposition 4.1, implies more results about entries of Y that we express them in a lemma below. Before that, we provide the following fact as a remark.

Remark 4.2. Since the matrix S is the change of basis matrix, it is invertible; thus for any $1\leq i\leq n+1$, $s_{ii}\neq 0$.

Lemma 4.2. *The equation*

$$\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_\varpi, \tag{4.18}$$

implies that:

(i) $\dot{z} = \frac{zs_{22}}{s_{11}} = \frac{zs_{(n+1)(n+1)}}{s_{nn}}$.

(ii) $y_{i-1} = \frac{s_{22}s_{ii}}{s_{11}s_{(i+1)(i+1)}}$, *for all* $i = 2, 3, \dots, n-1$.

(iii) *Moreover, if* $S\Omega S^{\mathsf{t}} = \Phi$, *then* $y_{i-1} = -y_{n-i}$, *for* $i \neq \frac{n+1}{2}$; *and*

$$y_{\frac{n-1}{2}} = (-1)^{\frac{n+3}{2}} \frac{\tilde{a}s_{22}s^2_{\frac{n+1}{2}\frac{n+1}{2}}}{s_{11}}.$$

In the other word

$$Y\Phi = -\Phi Y^{\mathsf{t}}. \tag{4.19}$$

Proof. Let's define $B = (b_{ij})_{1\leq i,j\leq n+1} := YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$.

(i) The equation (4.18) implies that $b_{12} = s_{22} - \frac{\dot{z}}{z}s_{11} = 0$ and $b_{n(n+1)} = s_{(n+1)(n+1)} - \frac{\dot{z}}{z}s_{nn} = 0$, which prove **(i)**.

(ii) The proof of **(ii)** follows from **(i)** and $b_{i(i+1)} = y_{i-1}s_{(i+1)(i+1)} - \frac{\dot{z}}{z}s_{ii} = 0$, $i = 2, 3, \dots, n-1$.

(iii) Let's define $C = (c_{ij})_{1\leq i,j\leq n+1} := S\Omega S^{\mathsf{t}}$. Then equation $C = \Phi$ yields $c_{i(n+2-i)} = (-1)^{i+1}\tilde{a}s_{ii}s_{(n+2-i)(n+2-i)} = 1$, $i = 1, 2, \dots, \frac{n+1}{2}$, from which we obtain

$$s_{(n+2-i)(n+2-i)} = (-1)^{i+1}\frac{1}{\tilde{a}s_{ii}},\ i = 1, 2, \dots, \frac{n+1}{2}. \tag{4.20}$$

Thus,

$$\frac{s_{ii}}{s_{(i+1)(i+1)}} = -\frac{s_{(n+1-i)(n+1-i)}}{s_{(n+2-i)(n+2-i)}},\ i = 1, 2, \dots, \frac{n-1}{2}.$$

Therefore, on account of **(ii)** the proof of **(iii)** is complete.

■

Lemma 4.3. *Let* $A := z\widehat{\mathsf{GM}}_{\varpi}$. *Then following equation holds:*

$$\vartheta\Omega = A\Omega + \Omega A^{\mathsf{t}}. \tag{4.21}$$

Proof. By using of the fact $\vartheta\langle\omega_i, \omega_j\rangle = \langle\vartheta\omega_i, \omega_j\rangle + \langle\omega_i, \vartheta\omega_j\rangle$ and Picard-Fuchs equation (4.4), the proof is an easy exercise of linear algebra. ■

Finally, we are now in a position that can prove Theorem 1.1.

Proof of Theorem 1.1. Let $\tilde{\mathsf{T}}$ be the moduli space introduced in Proposition 4.1, and suppose that $(W_z, [\alpha_1, \alpha_2, \dots, \alpha_{n+1}]) \in \tilde{\mathsf{T}}$ is an arbitrary element. As we saw in the proof of Proposition 4.1, there exist the matrix S such that

$$(\langle\alpha_i, \alpha_j\rangle)_{1\leq i,j\leq n+1} = S\Omega S^{\mathsf{t}}.$$

Define the vector subspace $M \subset \mathrm{Mat}_{n+1}(\mathbb{C})$ to be

$$M := \{B = (b_{ij})_{1\leq i,j\leq n+1} \in \mathrm{Mat}_{n+1}(\mathbb{C})|\ b_{ij} = 0,\ if\ i \leq n+1-j\}.$$

If we define the map f as follow

$$\begin{aligned}&f:\tilde{\mathsf{T}}\to M\\&f(\ W_z,[\alpha_1,\alpha_2,\ldots,\alpha_{n+1}])=S\Omega S^{\mathsf{t}},\end{aligned}$$

then $\mathsf{T}=f^{-1}(\Phi)$. Hence T is a subspace of $\tilde{\mathsf{T}}$ and to prove the existence of vector field H on T, it is enough to show that the vector field $\tilde{\mathsf{H}}$, which was introduced in Proposition 4.1, is tangent to T and define $\mathsf{H}:=\tilde{\mathsf{H}}|_{\mathsf{T}}$. To demonstrate the tangency of $\tilde{\mathsf{H}}$ to T , it suffices to prove that $df|_{\mathsf{T}}(\tilde{\mathsf{H}})=0$, or equivalently verify that

$$(\dot{S}\Omega S^{\mathsf{t}}+S\dot{\Omega}S^{\mathsf{t}}+S\Omega\dot{S}^{\mathsf{t}})|_{\mathsf{T}}=0, \tag{4.22}$$

in which $\dot{\Omega}=d\Omega(\tilde{\mathsf{H}})$. Since Ω just depends to z, it follows that $\dot{\Omega}=\dot{z}\frac{\partial}{\partial z}\Omega$. By using of Lemma 4.3 it is deduced that

$$\dot{\Omega}=\frac{\dot{z}}{z}(A\Omega+\Omega A^{\mathsf{t}})=\dot{z}(\widehat{\mathsf{GM}}_{\varpi}\Omega+\Omega\widehat{\mathsf{GM}}_{\varpi}^{\mathsf{t}}).$$

On the other hand as we saw in (4.17), $\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$, hence

$$\dot{S}\Omega S^{\mathsf{t}}+S\dot{\Omega}S^{\mathsf{t}}+S\Omega\dot{S}^{\mathsf{t}}=YS\Omega S^{\mathsf{t}}+S\Omega S^{\mathsf{t}}Y^{\mathsf{t}}.$$

Since $S\Omega S^{\mathsf{t}}|_{\mathsf{T}}=\Phi$, by using of Lemma 4.2**(iii)** we get

$$(\dot{S}\Omega S^{\mathsf{t}}+S\dot{\Omega}S^{\mathsf{t}}+S\Omega\dot{S}^{\mathsf{t}})|_{\mathsf{T}}=(YS\Omega S^{\mathsf{t}}+S\Omega S^{\mathsf{t}}Y^{\mathsf{t}})|_{\mathsf{T}}=(Y\Phi+\Phi Y^{\mathsf{t}})|_{\mathsf{T}}=0,$$

and the proof of existence of H is complete.

To prove the uniqueness, first notice that Lemma 4.2(ii) guaranties the uniqueness of y_i's. Hence we just need to prove that the vector field H is unique. Suppose that there are two vector fields H_1 and H_2 such that $\nabla_{\mathsf{H}_i}\alpha=Y\alpha$, $i=1,2$. If we set $\mathrm{R}:=\mathsf{H}_1-\mathsf{H}_2$, then

$$\nabla_{\mathrm{R}}\alpha=0. \tag{4.23}$$

We need to prove that $\mathrm{R}=0$, and to do this it is enough to verify that any integral curve of R is a constant point. Assume that γ is an integral curve of R given as follow

$$\gamma:(\mathbb{C},0)\to\mathsf{T};\qquad x\mapsto\gamma(x).$$

Let's denote $\mathcal{C}:=\gamma(\mathbb{C},0)\subset\mathsf{T}$. We know that the members of T are in the form of the pairs $(\widehat{W},[\alpha_1,\alpha_2,\ldots,\alpha_{n+1}])$, where $\widehat{W}$ is a Calabi-Yau manifold of the family $\mathcal{W}$, and $\{\alpha_i\}_{i=1}^{n+1}$ form a basis of $H^n_{\mathrm{dR}}(\widehat{W};\mathbb{C})$ that is compatible with its Hodge filtration and has constant intersection form matrix Φ. Thus, for any $x\in(\mathbb{C},0)$, we have $\gamma(x)=(\widehat{W}(x),[\alpha_1(x),\alpha_2(x),\ldots,\alpha_{n+1}(x)])$, and the vector field R on $\mathcal{C}$ is reduced to $\frac{\partial}{\partial x}$ as well. We know that $\widehat{W}(x)$ depends only on the parameter z, and hence x holomorphically depends to z. From this we obtain a holomorphic function f such that $x=f(z)$. We now proceed to prove that f is constant. Otherwise, by contradiction suppose that $f'\neq 0$. Then we get

$$\nabla_{\frac{\partial}{\partial x}}\alpha_1=\frac{\partial z}{\partial x}\nabla_{\frac{\partial}{\partial z}}\alpha_1. \tag{4.24}$$

Equation (4.23) gives that $\nabla_{\frac{\partial}{\partial x}}\alpha_1=0$, but since $\alpha_1=s_{11}\omega_1$, it follows that the right hand side of (4.24) is not zero, which is a contradiction. Thus f is constant and $\widehat{W}(x)$

does not depend on the parameter x. Since $\widehat{W}(x)=\widehat{W}$ does not depend on x, we can write the Taylor series of $\alpha_i(x)$, $i=1,2,3,\ldots,n+1$, in x at some point x_0 as $\alpha_i(x)=\sum_j(x-x_0)^j\alpha_{i,j}$, where $\alpha_{i,j}$'s are elements in $H^n_{\rm dR}(\widehat{W};\mathbb{C})$ independent of x. In this way the action of $\nabla_{\frac{\partial}{\partial x}}$ on α_i is just the usual derivation $\frac{\partial}{\partial x}$. Again according to (4.23) we yield $\nabla_{\frac{\partial}{\partial x}}\alpha_i=0$, and we conclude that α_i's also do not depend on x. Therefore, the image of γ is a point.

To prove that $\dim \mathsf{T}=\frac{(n+1)(n+3)}{4}+1$, it is enough to observe that $S\Omega S^{\mathsf{t}}=\Phi$ gives $\frac{(n+1)(n+3)}{4}$ independent equations and that $\mathcal{W}$ is a one parameter family. ■

Remark 4.3. Let $m:=\frac{(n+1)(n+3)}{4}$ and fix m entries of S. By notation we denote them by $t_1,t_2,\ldots,t_m$ and call them *independent entries* of S. The matrix equation $S\Omega S^{\mathsf{t}}=\Phi$ yields $\frac{(n+1)^2}{4}$ independent equations that express the rest of entries of S, what we shall call *dependent entries*, in terms of t_i's. For instance, suppose that s_{ab} is a dependent entry of S and $S\Omega S^{\mathsf{t}}=\Phi$ gives the expression $s_{ab}=\varphi(t_1,t_2,\ldots,t_m)$. We can obtain $\dot{s}_{ab}$ in the following two ways:

(i) On account of $s_{ab}=\varphi(t_1,t_2,\ldots,t_m)$, we first get

$$\dot{s}_{ab}=\sum_{i=1}^{m}\dot{t}_i\frac{\partial\varphi}{\partial t_i}, \tag{4.25}$$

and then substitute $\dot{t}_i$'s from $\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ in (4.25).

(ii) We first find $\dot{s}_{ab}$ directly from $\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$, and then using $S\Omega S^{\mathsf{t}}=\Phi$ to express $\dot{s}_{ab}$ just in terms of t_i's.

We say that *the equations $S\Omega S^{\mathsf{t}}=\Phi$ and $\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ are compatible* if **(i)** and **(ii)** give the same result for $\dot{s}_{ab}$. We are now in a position to introduce a chart of T, where t_i's are its coordinates. In order to this, let $t_0:=z$. Then $t=(t_0,t_1,\ldots,t_m)$ gives a chart for T that we will work explicitly with it in §4.3 and §4.4.

The corollary stated below, is an immediate result of Theorem 1.1.

Corollary 4.1. *The equations $S\Omega S^{\mathsf{t}}=\Phi$ and $\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ are compatible on T.*

Conversely, one can find that the compatibility of equations $S\Omega S^{\mathsf{t}}=\Phi$ and $\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ implies the existence and uniqueness of DHR vector field. We see this clearly in §4.3 and §4.4, where we compute DHR vector field explicitly.

4.2 Even Case

During this subsection n refers to an even positive integer. As we mentioned before, the difference of even case with the odd case is just the symmetry of intersection form. Lemma 3.1 implies that in the odd case the intersection form matrix is anti-symmetric, but in the even case it is symmetric. Hence, in this section we follow all the notations and definitions of §4.1, except the concepts related with intersection form. In particular the matrix $\Omega = (\Omega_{ij})_{1\leq i,j\leq n+1} := (\langle\omega_i,\omega_j\rangle)_{1\leq i,j\leq n+1}$, is given as follow,

$$\Omega = \begin{pmatrix} 0 & 0 & & \cdots & 0 & a \\ 0 & 0 & & \cdots & -a & \Omega_{2(n+1)} \\ & & & & \iddots & \\ \vdots & \vdots & & \Omega_{ll} = (-1)^{\frac{n}{2}} a & \vdots & \vdots \\ & & \iddots & & & \\ 0 & -a & & \cdots & \Omega_{nn} & \Omega_{n(n+1)} \\ a & \Omega_{2(n+1)} & & \cdots & \Omega_{n(n+1)} & \Omega_{(n+1)(n+1)} \end{pmatrix}, \tag{4.26}$$

in which $l = \frac{n}{2}+1$. Almost all results of odd case are valid in even case. More precisely, we can repeat Lemma 4.1, Proposition 4.1 and Lemma 4.3 exactly the same. But for Lemma 4.2, **(i)** and **(ii)** are valid, and **(iii)** holds with some changes that we rewrite it as follow.

Lemma 4.4. *The equation $\dot{S} = YS - \dot{z}S.\mathsf{GM}_{\varpi}$, implies that,*

(iii) *Moreover, if $S\Omega S^{\mathsf{t}} = \Phi$, then $y_{i-1} = -y_{n-i}$, for $i = 2,3,\ldots,\frac{n}{2}$. In the other word $Y\Phi = -\Phi Y^{\mathsf{t}}$.*

Therefore to prove Theorem 1.1 in the even case, we can proceed analogously to the proof of the odd case.

Remark 4.4. If we are more exact on the dimension of moduli space T in the even case and odd case, then we find a nice relationship between them. Let T be the moduli space given in Theorem 1.1 associated with a Calabi-Yau n-fold, where n is even, and T' be the moduli space associated with a Calabi-Yau $(n-1)$-fold. Then we have

$$\dim \mathsf{T} = \frac{n(n+2)}{4} + 1 = \frac{((n-1)+1)((n-1)+3)}{4} + 1 = \dim \mathsf{T}'.$$

Thus, one of my interest for future works is to find more relationships between structures of T and T'.

4.3 Five-Dimensional Case

In this subsection we give an explicit presentation of DHR vector field H, and in particular we verify its uniqueness by using of self-duality of Picard-Fuchs equation. Here we are following the notations and terminologies of §4.1 for $n = 5$.

The Picard-Fuchs equation (3.6) associated with the fixed nowhere vanishing holomorphic 5-form $\omega \in \mathcal{F}^5$ reduces to

$$\mathsf{L} = \vartheta^6 - a_0(z) - a_1(z)\vartheta - a_2(z)\vartheta^2 - a_3(z)\vartheta^3 - a_4(z)\vartheta^4 - a_5(z)\vartheta^5. \tag{4.27}$$

Lemma 4.5. *The coefficients a_i's of L given in* (4.27) *satisfy following equations:*

$$\begin{aligned} a_3 &= -\tfrac{2}{3}a_4a_5 + \tfrac{5}{3}a_5\vartheta a_5 - \tfrac{5}{27}a_5^3 - \tfrac{5}{3}\vartheta^2 a_5 + 2\vartheta a_4, \\ a_1 &= \vartheta a_2 - \vartheta^3 a_4 + \vartheta^4 a_5 + \vartheta^2 a_4 a_5 + \vartheta a_4 \vartheta a_5 + \tfrac{5}{3}a_5(\vartheta a_5)^2 - \tfrac{1}{3}a_2a_5 \\ &\quad + \tfrac{1}{27}a_4a_5^3 - \tfrac{10}{27}a_5^3\vartheta a_5 + \tfrac{1}{81}a_5^5 - \tfrac{1}{3}\vartheta a_4 a_5^2 - \tfrac{1}{3}a_4a_5\vartheta a_5 + \tfrac{10}{9}a_5^2\vartheta^2 a_5 \\ &\quad - \tfrac{10}{3}\vartheta a_5 \vartheta^2 a_5 + \tfrac{1}{3}a_4\vartheta^2 a_5 - \tfrac{5}{3}a_5\vartheta^3 a_5. \end{aligned} \tag{4.28}$$

Proof. By Proposition 3.3 the Picard-Fuchs equation (4.27) is self-dual, and the proof follows from Lemma 2.1**(ii)** . ∎

In the following proposition we compute all entries of the intersection matrix in the case $n = 5$.

Proposition 4.2. *The intersection form matrix* $\Omega := (\langle \omega_i, \omega_j \rangle)_{1 \le i,j \le 6}$, *is given by*

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \tilde{a} \\ 0 & 0 & 0 & 0 & -\tilde{a} & \Omega_{26} \\ 0 & 0 & 0 & \tilde{a} & \Omega_{35} & \Omega_{36} \\ 0 & 0 & -\tilde{a} & 0 & \Omega_{45} & \Omega_{46} \\ 0 & \tilde{a} & -\Omega_{35} & -\Omega_{45} & 0 & \Omega_{56} \\ -\tilde{a} & -\Omega_{26} & -\Omega_{36} & -\Omega_{46} & -\Omega_{56} & 0 \end{pmatrix}, \tag{4.29}$$

where $\tilde{a} = c_0 \exp\left(\frac{1}{3}\int_0^z a_5(v)\frac{dv}{v}\right)$ *for some nonzero constant* c_0, *and*

$$\Omega_{26} = -\tfrac{2}{3}\tilde{a}a_5, \qquad \Omega_{35} = \tfrac{1}{3}\tilde{a}a_5,$$

$$\Omega_{36} = \tilde{a}a_4 + \tfrac{4}{9}\tilde{a}a_5^2 - \tfrac{2}{3}\tilde{a}\vartheta a_5, \qquad \Omega_{45} = -\tilde{a}a_4 - \tfrac{1}{3}\tilde{a}a_5^2 + \tilde{a}\vartheta a_5,$$

$$\Omega_{46} = -\tilde{a}a_3 - \tilde{a}a_4a_5 - \tfrac{8}{27}\tilde{a}a_5^3 + \tilde{a}\vartheta a_4 + \tfrac{4}{3}\tilde{a}a_5\vartheta a_5 - \tfrac{2}{3}\tilde{a}\vartheta^2 a_5,$$

$$\begin{aligned}\Omega_{56} = {} & \tilde{a}a_2 + \tfrac{2}{3}\tilde{a}a_3a_5 + \tilde{a}a_4^2 + \tilde{a}a_4a_5^2 + \tfrac{16}{81}\tilde{a}a_5^4 - \tilde{a}\vartheta a_3 - \tfrac{5}{3}\tilde{a}a_5\vartheta a_4 - \tfrac{16}{9}\tilde{a}a_5^2\vartheta a_5 \\ & -2\tilde{a}a_4\vartheta a_5 + \tfrac{4}{3}\tilde{a}(\vartheta a_5)^2 + \tilde{a}\vartheta^2 a_4 + \tfrac{16}{9}\tilde{a}a_5\vartheta^2 a_5 - \tfrac{2}{3}\tilde{a}\vartheta^3 a_5.\end{aligned}$$

Proof. On account of (4.8) we get that the matrix Ω is given by (4.29), and we just need to find the entries $\Omega_{26}, \Omega_{35}, \Omega_{36}, \Omega_{45}, \Omega_{46}, \Omega_{56}$. In order to do this, we first easily see that $\vartheta\tilde{a} = \frac{1}{3}\tilde{a}a_5$. By Picard-Fuchs equation (4.27) we obtain

$$\vartheta\omega_6 = \vartheta^6\omega = a_0\omega_1 + a_1\omega_2 + a_2\omega_3 + a_3\omega_4 + a_4\omega_5 + a_5\omega_6. \tag{4.30}$$

Since $\langle \omega_1, \omega_6 \rangle = \tilde{a}$, by considering (4.30) and the fact that $\langle \omega_1, \omega_i \rangle = 0$, for $i = 1, 2, \ldots, 5$, we find Ω_{26} as follow:

$$\begin{aligned}\vartheta\langle \omega_1, \omega_6 \rangle &= \langle \omega_2, \omega_6 \rangle + \langle \omega_1, \vartheta\omega_6 \rangle \\ &\Rightarrow \Omega_{26} = \vartheta\tilde{a} - \tilde{a}a_5 \Rightarrow \Omega_{26} = -\frac{2}{3}\tilde{a}a_5.\end{aligned}$$

We can find the rest of entries similarly and we just do that for Ω_{56},

$$\begin{aligned}\Omega_{56} &= \vartheta\Omega_{46} - \langle \omega_4, \vartheta\omega_6 \rangle = \vartheta\Omega_{46} - a_2\Omega_{43} - a_4\Omega_{45} - a_5\Omega_{46} \\ &= \tilde{a}a_2 + \frac{2}{3}\tilde{a}a_3a_5 + \tilde{a}a_4^2 + \tilde{a}a_4a_5^2 + \frac{16}{81}\tilde{a}a_5^4 - \tilde{a}\vartheta a_3 - \frac{5}{3}\tilde{a}a_5\vartheta a_4 - \frac{16}{9}\tilde{a}a_5^2\vartheta a_5 \\ &\quad - 2\tilde{a}a_4\vartheta a_5 + \frac{4}{3}\tilde{a}(\vartheta a_5)^2 + \tilde{a}\vartheta^2 a_4 + \frac{16}{9}\tilde{a}a_5\vartheta^2 a_5 - \frac{2}{3}\tilde{a}\vartheta^3 a_5.\end{aligned}$$

∎

The task is now to present DHR vector field H explicitly. In order to do this, we use the chart t that we pointed it out in Remark 4.3. In the theorem below, we verify that H

as an ordinary differential equation is given as follow:

$$
(4.31)\qquad \begin{cases}
\dot{t}_0 = \frac{t_0t_3}{t_1}\\
\dot{t}_1 = t_2\\
\dot{t}_2 = \frac{t_3^2t_4}{t_1t_6}\\
\dot{t}_3 = \frac{-t_2t_3t_6+t_3^2t_5}{t_1t_6}\\
\dot{t}_4 = \frac{\tilde{a}t_3t_6^2t_7}{t_1}\\
\dot{t}_5 = \frac{-t_3t_4+\tilde{a}t_3t_6^2t_8}{t_1}\\
\dot{t}_6 = \frac{-t_3t_5+\tilde{a}t_3t_6^2t_9}{t_1}\\
\dot{t}_7 = \frac{-t_3^2t_{10}}{t_1t_6}\\
\dot{t}_8 = \frac{-t_3^2t_{11}-t_3t_6t_7}{t_1t_6}\\
\dot{t}_9 = \frac{3\tilde{a}t_3t_5t_9-6\tilde{a}t_3t_6t_8-3t_3a_4-t_3a_5^2+3t_3\vartheta a_5}{3\tilde{a}t_1t_6}\\
\dot{t}_{10} = -t_{12}\\
\dot{t}_{11} = \frac{27\tilde{a}t_2t_{11}-54\tilde{a}t_3t_{10}+27\tilde{a}t_4t_8-27\tilde{a}t_5t_7+27a_2+27a_3a_5-27\vartheta a_3+15a_4a_5^2}{27\tilde{a}t_1}\\
\qquad +\frac{-9a_4\vartheta a_5-54a_5\vartheta a_4+27\vartheta^2a_4+4a_5^4-42a_5^2\vartheta a_5+54a_5\vartheta^2a_5+18(\vartheta a_5)^2-18\vartheta^3a_5}{27\tilde{a}t_1}\\
\dot{t}_{12} = \frac{-t_3a_0}{\tilde{a}t_1^2}
\end{cases}.
$$

Theorem 4.1. *Let T be the moduli space introduced in Theorem 1.1, for $n=5$. Then there is a chart $(t_0,t_1,\ldots,t_{12})$ for T such that in this chart we obtain*

$$
y_1 = \frac{t_3^2}{t_1t_6}, \qquad \& \qquad y_2 = \frac{\tilde{a}t_3t_6^2}{t_1},
$$

and DHR *vector field* H *is given by (4.31).*

Proof. By Theorem 1.1 we get that T is 13-dimensional. Using the equation $S\Omega S^{\mathsf{t}}=\Phi$, and considering $s_{11},s_{21},s_{22},s_{31},s_{32},s_{33},s_{41},s_{42},s_{43},s_{51},s_{52},s_{61}$ as independent entries of S, we thus can express dependent entries in terms of independent entries as follows:

$$
(4.32)\qquad
\begin{aligned}
&s_{44} = \frac{1}{\tilde{a}s_{33}}, \qquad s_{53} = \frac{-3\tilde{a}s_{32}s_{43}+3\tilde{a}s_{33}s_{42}+3a_4+a_5^2-3\vartheta a_5}{3\tilde{a}s_{22}},\\
&s_{54} = \frac{-3s_{32}+s_{33}a_5}{3\tilde{a}s_{22}s_{33}}, \qquad s_{55} = -\frac{1}{\tilde{a}s_{22}},\\
&s_{62} = \frac{-27\tilde{a}s_{21}s_{52}+27\tilde{a}s_{22}s_{51}-27\tilde{a}s_{31}s_{42}+27\tilde{a}s_{32}s_{41}-27a_2-27a_3a_5+27\vartheta a_3-15a_4a_5^2}{27\tilde{a}s_{11}}\\
&\qquad +\frac{9a_4\vartheta a_5+54a_5\vartheta a_4-27\vartheta^2a_4-4a_5^4+42a_5^2\vartheta a_5-54a_5\vartheta^2a_5-18(\vartheta a_5)^2+18\vartheta^3a_5}{27\tilde{a}s_{11}},\\
&s_{63} = \frac{27\tilde{a}s_{21}s_{32}s_{43}-27\tilde{a}s_{21}s_{33}s_{42}-27\tilde{a}s_{22}s_{31}s_{43}+27\tilde{a}s_{22}s_{33}s_{41}-27s_{21}a_4-9s_{21}a_5^2}{27\tilde{a}s_{11}s_{22}}\\
&\qquad +\frac{27s_{21}\vartheta a_5-27s_{22}a_3-9s_{22}a_4a_5+27s_{22}\vartheta a_4-2s_{22}a_5^3+18s_{22}a_5\vartheta a_5-18s_{22}\vartheta^2a_5}{27\tilde{a}s_{11}s_{22}},\\
&s_{64} = \frac{9s_{21}s_{32}-3s_{21}s_{33}a_5-9s_{22}s_{31}-9s_{22}s_{33}a_4-2s_{22}s_{33}a_5^2+6s_{22}s_{33}\vartheta a_5}{9\tilde{a}s_{11}s_{22}s_{33}},\\
&s_{65} = \frac{3s_{21}-2s_{22}a_5}{3\tilde{a}s_{11}s_{22}}, \qquad s_{66} = \frac{1}{\tilde{a}s_{11}}.
\end{aligned}
$$

We know that $\mathcal{W}$ is a family of one parameter 5-dimensional Calabi-Yau manifolds parameterized by z. Hence we present the chart $t=(t_0,t_1,\ldots,t_{12})$ for T, where $t_0=$

$z,\ t_1 = s_{11},\ t_2 = s_{21},\ t_3 = s_{22},\ t_4 = s_{31},\ t_5 = s_{32},\ t_6 = s_{33},\ t_7 = s_{41},\ t_8 = s_{42},\ t_9 = s_{43},\ t_{10} = s_{51},\ t_{11} = s_{52},\ t_{12} = s_{61}$. The same as the proof of Proposition 4.1, define $\mathsf{H} := \sum_{i=0}^{12} \mathsf{H}_i(t)\frac{\partial}{\partial t_i}$ and set $\dot{t}_i := \mathsf{H}_i(t)$. Then H_0, y_1 and y_2 follow from Lemma 4.2. Therefore, the right hand side of the equation

$$\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}, \tag{4.33}$$

is totally determined. We also substitute a_1, a_3 from Lemma 4.5, and dependent entries from (4.32) in the right hand side of (4.33). Consequently, the rest of H_i's follow directly from (4.33). Note that the compatibility of $S\Omega S^{\mathsf{t}} = \Phi$ and $\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ follow from substituting a_1 and a_3 from Lemma 4.5. ■

4.4 Three-Dimensional Case

Here we substitute the dimension $n = 5$ with $n = 3$, and will proceed analogously to §4.3.

The Picard-Fuchs equation L is given as

$$\mathsf{L} = \vartheta^4 - a_0(z) - a_1(z)\vartheta - a_2(z)\vartheta^2 - a_3(z)\vartheta^3, \tag{4.34}$$

where coefficients a_i's, by Lemma 2.1**(i)**, satisfy the following relationship:

$$a_1 = \frac{3}{4}a_3\vartheta a_3 + \vartheta a_2 - \frac{1}{2}\vartheta^2 a_3 - \frac{1}{8}a_3^3 - \frac{1}{2}a_2 a_3. \tag{4.35}$$

The intersection form matrix is as follow

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & \tilde{a} \\ 0 & 0 & -\tilde{a} & \Omega_{24} \\ 0 & \tilde{a} & 0 & \Omega_{34} \\ -\tilde{a} & -\Omega_{24} & -\Omega_{34} & 0 \end{pmatrix}, \tag{4.36}$$

in which

$$\tilde{a} = c_0 \exp\left(\frac{1}{2}\int_0^z a_3(v)\frac{dv}{v}\right), \tag{4.37}$$

and

$$\Omega_{24} = -\tfrac{1}{2}\tilde{a}a_3, \qquad \& \qquad \Omega_{34} = \tfrac{1}{4}\tilde{a}a_3^2 + \tilde{a}a_2 - \tfrac{1}{2}\tilde{a}\vartheta a_3.$$

The chart $t = (t_0, t_1, \ldots, t_6)$ of T is obtained by settin $t_0 = z,\ t_1 = s_{11},\ t_2 = s_{21},\ t_3 = s_{22},\ t_4 = s_{31},\ t_5 = s_{32},\ t_6 = s_{41}$. We compute below dependent entries:

$$\begin{aligned} s_{33} &= -\tfrac{1}{\tilde{a}s_{22}}, & s_{42} &= \tfrac{4\tilde{a}s_{22}s_{31} - 4\tilde{a}s_{21}s_{32} - a_3^2 - 4a_2 + 2\vartheta a_3}{4\tilde{a}s_{11}}, \\ s_{43} &= \tfrac{2s_{21} - s_{22}a_3}{2\tilde{a}s_{11}s_{22}}, & s_{44} &= \tfrac{1}{\tilde{a}s_{11}}. \end{aligned} \tag{4.38}$$

In the chart t, the meromorphic function y_1 is given by

$$y_1 = -\frac{\tilde{a}t_3^3}{t_1},$$

and DHR vector field H has following presentation:

$$
\begin{cases}
\dot{t}_0 = \frac{t_0 t_3}{t_1} \\
\dot{t}_1 = t_2 \\
\dot{t}_2 = -\frac{\tilde{a} t_3^3 t_4}{t_1} \\
\dot{t}_3 = -\frac{t_2 t_3 + \tilde{a} t_3^3 t_5}{t_1} \\
\dot{t}_4 = -t_6 \\
\dot{t}_5 = \frac{4\tilde{a} t_2 t_5 - 8\tilde{a} t_3 t_4 + a_3^2 + 4a_2 - 2\vartheta a_3}{4\tilde{a} t_1} \\
\dot{t}_6 = -\frac{t_3 a_0}{\tilde{a} t_1^2}
\end{cases} . \tag{4.39}
$$

Example 4.2. *In Example 4.1 we introduced 14 families of Calabi-Yau 3-folds for which Theorem 1.1 holds. We can rewrite the Picard-Fuchs equation (4.1) as follow*

$$\mathsf{L} = \vartheta^4 - a_0(z) - a_1(z)\vartheta - a_2(z)\vartheta^2 - a_3(z)\vartheta^3,$$

in which

$$a_0(z) = \frac{cz(r_1 r_2 - r_1^2 r_2 - r_1 r_2^2 + r_1^2 r_2^2)}{1 - cz}, \qquad a_1(z) = \frac{cz(r_1 + r_2 - r_1^2 - r_2^2)}{1 - cz},$$
$$a_2(z) = \frac{cz(1 + r_1 + r_2 - r_1^2 - r_2^2)}{1 - cz}, \qquad a_3(z) = \frac{2cz}{1 - cz}.$$

On account of (4.37), we obtain

$$\tilde{a} = \frac{c_0}{1 - cz}.$$

Now by replacing a_i's and $\tilde{a}$ in (4.39) we find DHR *vector field* H *on the moduli space associated with any of the families given in Table 1.*

References

[1] Batyrev, Victor V. and Duco van Starten: *Generalized Hypergeometric Functions and Rational Curves on Calabi-Yau Complete Intersections in Toric Variety*, Commun. Math. Phys. **168**, 169–178 (1995).

[2] Bogner, Michael: *On Differential Operators of Calabi-Yau Type*, ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–University of Minz, (2012).

[3] Brioschi, Fr.: *Sur un systéme d'équations differetielles*, Comptes Rendus des sánces de l'Académie des Sciences **92**, 1389–1393 (1881).

[4] Calabi, E.: *The space of Kähler metrics*, In Iroceeding of International Congrees of Mathematicians, Amesterdam, 1954 **2**, 206–207 (1956).

[5] Calabi, E.: *On Kähler manifolds with vanishing canonical class*, In algebraic geometry and topology, a simposium in honer of S. Lefschetz, Princeton University Press, 78–89 (1957).

[6] Candelas, P., de la Ossa, Xenia C., Green, Paul S. and Parkes, L.: *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359(1)**, 21–74 (1991).

[7] Candelas, P., Horowitz, G., Strominger, A. and Witten, E.: *Vacuum configurations for superstrings*, Nuclear Physics **258**, 47–74 (1985).

[8] Chen, Y., Yang, Y. and Yui, N: *Monodromy of PicardFuchs differential equations for CalabiYau threefolds*, J. Reine Angew. Math **258**, (2008).

[9] Darboux, G.: *Sur la théorie des coordonnées curvilignes et les systémes orthogonaux*, Ann Ecole Normale Supérieure **7**, 101–150 (1878).

[10] Doran, Charles F. and Morgan, John W.: *Mirror symmetry and integral variations of Hodge structure underlying one-parameter families of Calabi-Yau threefolds*, Mirror symmetry. V, AMS/IP Stud. Adv. Math. **38**, 517–537 (2006).

[11] Greene, B. R. and Plesser, M. R.: *Duality in Calabi-Yau moduli space*, Nuclear Physics **338(1)**, 15-37 (1990).

[12] Gross, Mark W., Huybrechts, D. and Joyce, Dominic D.: *Calabi-Yau manifolds and related geometries*, lectures at a summer school in Nordfjordeid, Norway, June 2001, Springer-Verlag, (2003).

[13] Guest, Martin A.: *From Quantum Cohomology to Integrable Systems*, Oxford University Press, (2008).

[14] Guillot, Adolfo: *Champs quadratiques uniformisables*, These de doctorat, Ecole Normale Suprieure de Lyon, (2001).

[15] Guillot, Adolfo: *Semicomplete meromorphic vector fields on complex surfaces*, J. reine angew. Math. **667**, 27–65 (2012).

[16] Halphen, G. H.: *Sur des fonctions qui proviennent de l'équation de Gauss*, C. R. Acad. Sci Paris **92**, 856–859 (1881).

[17] Halphen, G. H.: *Sur un systéme d'équations différetielles*, C. R. Acad. Sci Paris **92**, 1101–1103 (1881).

[18] Halphen, G. H.: *Sur certains systéme d'équations différetielles*, C. R. Acad. Sci Paris **92**, 1404–1407 (1881).

[19] Klemm, Albrecht and Theisen, Stefan: *Mirror maps and instanton sums for complete intersections in weighted projective space*, Modern Phys. Lett. **A9(20)**, 1807–1817 (1994).

[20] Libgober, A. and Teitelbaum, J.: *Lines on Calabi-Yau complete intersections, mirror symmetry, and Picard-Fuchs equations*, Internat. Math. Res. Notices **1**, 29–39 (1993).

[21] Morrison, David R.: *Picard-Fuchs equations and mirror maps for hypersurfaces*, Essays on mirror manifolds, Int. Press, Hong Kong, 241–264 (1992).

[22] Movasati, Hossein: *Eisenstein type series for Calabi-Yau varieties*, Nuclear Physics B **847**, 460–484 (2011).

[23] Movasati, Hossein: *Multiple Integrals and Modular Differential Equations*, 28th Brazilian Mathematics Colloquium, Instituto de Matemática Pura e Aplicada, IMPA, (2011).

[24] Ramanujan, S.: *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22**, 159–184 (1916).

[25] van der Put, Marius and Singer, Michael F.: *Galois theory of linear differential equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, (2003).

[26] Voisin, Claire: *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics Volume 76, Cambridge University Press, Cambridge, (2002) (Translated from the French original by Leila Schneps).

[27] Yau, S. T.: *On the Ricci curvature of a compct Kähler manifold and the complex Monge-Ampére equations I*, Communications on pure and applied mathematics **31**, 339–411 (1978).