DARBOUX-HALPHEN-RAMANUJAN VECTOR FIELD ON A MODULI OF CALABI-YAU MANIFOLDS

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF MATHEMATICS
AT
INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA (IMPA)
RIO DE JANEIRO, BRAZIL
SEPTEMBER 2014

INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA
(IMPA)

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Dated: September 2014

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Date: **September 2014**

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Title: **Darboux-Halphen-Ramanujan Vector Field on a Moduli of Calabi-Yau Manifolds**

Degree: **Ph.D.** Convocation: **September** Year: **2014**

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To the Memory of My Beloved Brother "SADEGH"...

Table of Contents

Abstract

This research was intended as an attempt to obtain an ordinary differential equation H from a linear differential equation L. We work on a Calabi-Yau n-fold W whose complex deformation is one dimensional and middle complex de Rham cohomology $H^n_{\mathrm{dR}}(W;\mathbb{C})$ is $(n+1)$-dimensional. Moreover, we suppose that the Picard-Fuchs equation L associated with the unique nowhere vanishing holomorphic n-form ω on W is of order $n+1$. As a first result, we prove that L is self-dual.

Next, we define T to be the moduli of W together with a basis of $H^n_{\mathrm{dR}}(W;\mathbb{C})$, where the basis is required to be compatible with the Hodge filtration of $H^n_{\mathrm{dR}}(W;\mathbb{C})$; furthermore, the intersection form matrix in this basis is constant. Then in our second result, we verify the existence of a unique vector field H on T that satisfies certain properties. Indeed, the ordinary differential equation given by H is a generalization of differential equations introduced by Darboux, Halphen and Ramanujan.

Keywords: Darboux-Halphen-Ramanujan vector field, Hodge structure, Picard-Fuchs equation, Gauss-Manin connection.

Acknowledgements

First I would like to express my very great appreciation to Prof. Hossein Movasati, my supervisor, who introduced to me this area and always was available and I used his valuable and constructive suggestions and helps during the planning and development of this research.

I wish to thank Khosro M. Shokri, for constructive conversations that we had during my research. I also was benefited by the supports of Professors Alcides Lins Neto and Marcos Dajczer that I express my gratitude to them.

I had the pleasure to do some courses with Professors Alcides Lins Neto, Marcelo Viana, Marcos Dajczer, Jorge Vitrio Pereira, Henrique Bursztyn and Luis Adrian Florit, and I thank them for teaching and motivating me more.

The author is greatly indebted to his master supervisor Prof. S. M. B. Kashani, for his continuous supports.

I would like to thank IMPA and its staffs for preparing such an excellent academic environment, and also I am grateful to have economic supports of "CNPq-TWAS Fellowships Programme".

A special thanks to my family. My sincerest thanks goes to my mum and my dad for their patience, love and unconditional supports and also to my brothers Akbar, Asghar, Ahmad, Amin and to my sisters Ashraf, Fatemeh, Tayebeh. Words can not express how grateful I am to my beloved wife Jhoana, who was always there cheering me up and stood by me through the happiness and sadness.

Finally, I wish to thank all my friends who I spent gratifying and delightful time with them. In particular: Ali Golmakani, Kyara, Felipe Macias, Kari Marin, Sajad, Vahideh, Hamed, Ana, Vivi, Marco, Francisco, Vandi, Ruben, Leandro, Anita, Alex, Roberto, Mehdi, Mahdi, Sina and Ali.

Rio de Janeiro, Brasil Younes Nikdelan
July 27, 2014

Introduction

Calabi-Yau manifolds are defined as compact connected Kähler manifolds whose canonical bundle is trivial, though many other equivalent definitions are sometimes used (see Theorem 4.3). They were named "Calabi-Yau manifold" by Candelas et al. (1985) [9] after E. Calabi (1954) [6, 7], who first studied them, and S. T. Yau (1976) [42], who proved the Calabi conjecture that says Calabi-Yau manifolds accept Ricci flat metrics. In this text, we suppose that for an n-dimensional Calabi-Yau manifold $h^{p,0} = 0$, $0 < p < n$, where $h^{p,q}$ refers to (p, q)-th Hodge number of Calabi-Yau manifold. It is clear that the connectedness of a Calabi-Yau manifold and the triviality of its canonical bundle imply that $h^{0,0} = h^{n,0} = 1$.

Since introducing Calabi-Yau manifolds, a lot of work has been done on these manifolds by mathematicians an physicists. The importance of Calabi-Yau manifolds were found more, after discovering the concept of *mirror symmetry* by physicists. Mirror symmetry is a conjecture that says there exist mirror pairs of Calabi-Yau manifolds. Quite roughly, we should think of W and $\tilde{W}$ as being a mirror pair if there is an isomorphism between $\mathcal{M}_{Kah}(W)$ and $\mathcal{M}_{cmplx}(\tilde{W})$, where $\mathcal{M}_{Kah}(X)$ and $\mathcal{M}_{cmplx}(X)$, resp., refer to Kähler and complex moduli, resp., of a Kähler manifold X. One of important and primary of these works in 1991 was given by Candelas et al. in [8], where they used the mirror symmetry to predict the number of rational curves on quintic 3-folds. In some other works, such as [1, 2, 11, 15, 27, 28, 29], authors construct new Calabi-Yau manifolds and their mirrors, and then they investigate other properties of these constructed manifolds.

The main work of this thesis, which is motivated by linear differential equations introduced by Darboux, Halphen and Ramanujan, is to present a vector field H with some special properties on a moduli space of Calabi-Yau manifolds and we call it DHR vector field. To know more about DHR vector field we start with 1-dimensional Calabi-Yau manifolds, which are elliptic curves. Let E be an elliptic curve over $\mathbb{C}$. Then the Hodge filtration (for definition see §2.3) $F^\bullet$ of the first de Rham cohomology group (for definition see §2.5) $H^1_{\rm dR}(E)$ is given as follow,

$$\{0\} = F^2 \subset F^1 \subset F^0 = H^1_{\rm dR}(E), \quad \dim_{\mathbb{C}}(F^i) = 2 - i.$$

Let T be the moduli of the pair $(E, [\alpha_1, \alpha_2])$, in which $\alpha_1 \in F^1$, $\alpha_2 \in F^0 \setminus F^1$, and the matrix of their intersection forms (for definition see §2.6) is as follow,

$$(\langle \alpha_i, \alpha_j \rangle)_{1\leq i,j\leq 2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To be more precise, if we consider the Weierstrass presentation of E in $\mathbb{P}^2$, then α_1 and α_2, resp., are induced by $[\frac{dx}{y}]$ and $[\frac{xdx}{y}]$, resp., where $[\frac{dx}{y}]$ and $[\frac{xdx}{y}]$ are generators of the first de Rham cohomology $H^1_{\mathrm{dR}}(E_0)$ of affine curve $E_0 := E \setminus \{\infty\}$. Since $H^1_{\mathrm{dR}}(E) \cong H^1_{\mathrm{dR}}(E_0)$, α_1 and α_2 are generators of $H^1_{\mathrm{dR}}(E)$ and hence $\alpha_1 \wedge \alpha_2 \neq 0$ (for more details see §1.1). Then T is a 3-dimensional space, and there exist a unique vector field H on T such that the composition of Gauss-Manin connection (for definition see §2.5) with H satisfies the following

$$\nabla_{\mathrm{H}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

As we will see in §1.1, by neglecting some details, if $t = (t_1, t_2, t_3)$ is a chart of T , then H is given by the following system

$$\begin{cases} \dot{t}_1 = t_1(t_2 + t_3) - t_2t_3 \\ \dot{t}_2 = t_2(t_1 + t_3) - t_1t_3 \\ \dot{t}_3 = t_3(t_1 + t_2) - t_1t_2 \end{cases}, \tag{0.1}$$

which for the first time appeared in the works of Darboux [12] and then Halphen [23] worked on its solutions and expressed a solution of this system in terms of the logarithmic derivatives of the null theta functions. Or equivalently, H can be presented by the following system of linear differential equations

$$\begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12}t_2 \\ \dot{t}_2 = 4t_1t_2 - 6t_3 \\ \dot{t}_3 = 6t_1t_3 - \frac{1}{3}t_2^2 \end{cases}, \tag{0.2}$$

that Ramanujan worked on this system and found a solution of Eisenstein series for it.

After these works, H. Movasati [31, 33] considered a one parameter family of Calabi-Yau 3-folds, which is known as the family of mirror quintic 3-folds, and studied on it. If W is a mirror quintic 3-fold, then the Hodge filtration of $H^3_{\mathrm{dR}}(W)$ is given as follow,

$$\{0\} = F^4 \subset F^3 \subset \ldots \subset F^0 = H^3_{\mathrm{dR}}(W), \quad \dim_{\mathbb{C}}(F^i) = 4 - i,$$

and there is a holomorphic 3-form $\omega \in F^3$ such that $L\omega = 0$, where L is the Picard-Fuchs equation

$$L = \vartheta^5 - 5^5 z(\vartheta + \frac{1}{5})(\vartheta + \frac{2}{5})(\vartheta + \frac{3}{5})(\vartheta + \frac{4}{5}),$$

in which $\vartheta := \nabla_{z\frac{\partial}{\partial z}}$. Movasati treated on the moduli space T of the pair $(W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4])$, where $\alpha_i \in F^{4-i} \setminus F^{5-i}$, and

$$(\langle \omega_i, \omega_j \rangle)_{1 \leq i,j \leq 4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

He proved that T is a 7-dimensional space and there is a unique vector field H and a unique meromorphic function y on T such that,

$$\nabla_{\mathrm{H}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

Indeed he express H and y explicitly, and he show that y is the Yukawa coupling (for more details see Theorem 1.1).

After what we saw about the family of Calabi-Yau 1-folds and the family of mirror quintic 3-folds, it is natural to ask whether there exist such a moduli space T and such a vector field H in higher dimensions. In the present text we give a positive answer to this question. In fact, we prove it for a Calabi-Yau n-fold that satisfies some certain conditions. To do this, suppose that W is a Calabi-Yau n-fold that its complex deformation is given by the one parameter family $\pi : \mathcal{W} \to P$ of n-dimensional Calabi-Yau manifolds parameterized by z, such that $\dim H^n_{\mathrm{dR}}(\mathcal{W}/P) = n+1$, and the Picard-Fuchs equation L associated with the unique nowhere vanishing holomorphic n-form $\omega \in \mathcal{F}^n$, where $\mathcal{F}^\bullet$ is the Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$, is given by

$$L = \vartheta^{n+1} - a_n(z)\vartheta^n - \ldots - a_1(z)\vartheta - a_0(z), \tag{0.3}$$

in which $\vartheta := \nabla_{z\frac{\partial}{\partial z}}$ and $a_i(z) \in \mathbb{Q}(z)$, $i = 0, 1, \ldots, n$. Before stating the main theorem of this thesis, we fix the $(n+1) \times (n+1)$ matrix Φ as follow. If n is an *odd* integer, then let

$$\Phi := \begin{pmatrix} 0_{\frac{n+1}{2}} & J_{\frac{n+1}{2}} \\ -J_{\frac{n+1}{2}} & 0_{\frac{n+1}{2}} \end{pmatrix}, \tag{0.4}$$

where for a positive integer k, by 0_k we mean a $k \times k$ block of zeros, and J_k is the following

$k \times k$ block

$$J_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{0.5}$$

If n is an *even* integer, then $\Phi = J_{n+1}$.

Theorem 0.1. *Let W be the Calabi-Yau n-fold given above and T be the moduli of the pair $(W, [\alpha_1, \alpha_2, \dots, \alpha_{n+1}])$, where $\{\alpha_i\}_{i=1}^{n+1}$ is a basis of $H^n_{\mathrm{dR}}(W; \mathbb{C})$ satisfying*

$$\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}, \; i = 1, 2, \dots, n+1, \tag{0.6}$$

and the matrix of their intersection forms satisfies the following:

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le n+1} = \Phi. \tag{0.7}$$

Then there exist a unique vector field H and unique meromorphic functions y_i, $i = 1, 2, \dots, n-2$, on T such that the composition of Gauss-Manin connection ∇ with the vector field H satisfy:

$$\nabla_{\mathrm{H}} \alpha = Y \alpha, \tag{0.8}$$

in which

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \end{pmatrix}^{\mathsf{t}},$$

and

$$Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & y_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & y_{n-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{0.9}$$

And moreover,

$$\dim_{\mathbb{C}} \mathsf{T} = \begin{cases} \frac{(n+1)(n+3)}{4} + 1; & \text{if } n \text{ is odd}, \\ \frac{n(n+2)}{4} + 1; & \text{if } n \text{ is even}, \end{cases}.$$

We prove this theorem in Chapter 5.

The chapters of this thesis are organized as follows:

Chapter 1. In this chapter, we first explain the main problem in some special cases of low dimensions that have been done, and reformulate them in our language. Then in distinct sections we review the works of Darboux, Halphen and Ramanujan, that are in relationship with our main problem.

Chapter 2. In this chapter we briefly recall the basic facts related with Hodge theory. Firstly, the concept of integrable connections are stated. Secondly, the general theory of Hodge structure is provided. After introducing de Rham cohomology, as a nice example of Hodge structure, the Hodge decomposition of complexified de Rham cohomology and Hodge filtration is presented. Next the definitions and important results of families of complex manifolds are established. In sequence we present Gauss-Manin connection and Griffiths transversality that provide a family of complex manifolds with a Hodge variation. Finally the concept of intersection forms is given. In particular we verify the existence relationship between intersection forms and Hodge filtration in the context of de Rham cohomology.

Chapter 3. The composition of Gauss-Manin connection with a vector field yields a differential operator on the relative de Rham cohomology group of a family of complex manifolds. In this chapter we are going to study this operator and its generated linear differential equations. First, we briefly state some basic facts related with differential operators. We give an algorithm to find the relationships among coefficients of a self-dual linear differential operator of an arbitrary degree. Next, a special linear differential operator associated with a holomorphic n-form, which is called Picard-Fuchs equation, is presented. At the end of this chapter, after fixing some certain hypothesis on a family of complex manifolds, we prove that its relative de Rham cohomology group has a special type of frame, which we call it Yukawa frame. To do this we use the properties of coupling function that we prove them in Proposition 3.3 in context of de Rham cohomology. Also in Proposition 3.4, we give a relationship between the dimensions of the relative de Rham cohomology group and its Hodg filtration, which weakens our primary hypothesis.

Chapter 4. In this chapter we are going to recall fundamental definitions and facts related to Calabi-Yau manifolds. Here we first announce Calabi-Yau theorem and then we review the equivalence definitions of Calabi-Yau manifolds that are used in different contexts. once we fix the definition of Calabi-Yau manifold, some primary examples and classifications of low dimensional Calabi-Yau manifolds are given. Finally, we review the properties of the families of Calabi-Yau manifolds and also some more examples of families of Calabi-Yau manifolds are provided.

Chapter 5. In this chapter we state our main result about the encountering DHR vector field. We first fix some hypothesis on a Calabi-Yau n-fold, under which we are working in the whole of this chapter. We first give a special moduli space T of a fixed Calabi-Yau manifold. Then we prove that there exist a unique vector field, which we call it DHR vector field, and several unique meromorphic functions on T that satisfy certain properties. Next, after computing the matrix of intersection forms and finding the relationships among the coefficients of Picard-Fuchs equation, we express DHR vector field explicitly in dimension five and three.

Chapter 6. During my works on this thesis, I encountered with various natural problems that seem interesting. Hence it worth to organize them for more future researches. And since they are directly related to my thesis, I state them in this chapter in different sections.

Appendix A. Here we provide several tables, which include information about Picard-Fuchs equation of Calabi-Yau manifolds.

The reader who has bacgrounds of Hodge theory, Gauss-Manin connection, intersection form and Calabi-Yau manifolds, it is just enough to read chapters 3 and 5.

Chapter 1

Historical Backgrounds

In this thesis we are going to introduce a special vector field on a moduli space of a family of Calabi-Yau manifolds. This vector field is in a close relationships with, and in a sense it is an extension of, the vector fields introduced by Darboux, Halphen and Ramanujan that we will review them below. Because of this relationship, the vector field is called *Darboux-Halphen-Ramanujan*, abbreviatly DHR, *vector field*. In this chapter, in §1.1 we explain the main problem in some special cases of low dimensions that have been done, and reformulate them in our language. Then in distinct sections §1.2, §1.3 and §1.4, resp., we review the works of Darboux, Halphen and Ramanujan, resp., that are in relationship with our main problem.

1.1 Problem Statement

To have an idea of DHR vector field, we explain the issue on a family of 1-dimensional Calabi-Yau manifolds, which are elliptic curves. For more details one can see [32] and [34].

Let E be an elliptic curve over $\mathbb{C}$. We know that there are $t_1, t_2, t_3 \in \mathbb{C}^3$, such that $\Delta := t_2^3 - 27t_3^2 \neq 0$, and E in Weirestrass form is considered as a following projective curve in $\mathbb{P}^2$,

$$E = \{[x; y; z] \in \mathbb{P}^2 |\, F(x, y, z) := zy^2 - 4(x - t_1 z)^3 + t_2(x - t_1 z)z^2 + t_3 z^3 = 0\},$$

where $F(x, y, z)$ is the homogenization of the function

$$f(x, y) := y^2 - 4(x - t_1)^3 + t_2(x - t_1) + t_3,$$

that gives the following affine elliptic curve E_0

$$E_0 = \{[x; y; 1] \in \mathbb{P}^2 |\, f(x, y) = 0\}.$$

Indeed Δ is discriminant function of $P(x) := 4(x-t_1)^3 + t_2(x-t_1) + t_3$, and since $\Delta \neq 0$, P and P' do not vanish simultaneously. Also we can state

$$y^2 = P(x) \Rightarrow 2ydy = P'(x)dx \Rightarrow \frac{dx}{y} = 2\frac{dy}{P'(x)}.$$

So the 1-form $\frac{dx}{y}$ is holomorphic on E_0. On the other hand, we know that $O := [0;1;0] \in E$ is the only point at infinity and $O \notin E_0$. So the one form $\frac{xdx}{y}$ is as well holomorphic on E_0. And if we let $H^1_{\rm dR}(E_0/\mathbb{C})$ be the first algebraic de Rham cohomology of E_0, then one can see in [34, Proposition 2.2] that $H^1_{\rm dR}(E_0/\mathbb{C})$ is freely generated by $[\frac{dx}{y}]$ and $[\frac{xdx}{y}]$. It is obvious that $E = E_0 \setminus \{[0;1;0]\}$, so we have the inclusion $\iota : E_0 \to E$. In [34, Proposition 2.4] we can find that the inclusion ι induces the isomorphism

$$\iota^* : H^1_{\rm dR}(E/\mathbb{C}) \to H^1_{\rm dR}(E_0/\mathbb{C}), \tag{1.1}$$

hence there is a holomorphic 1-form $\alpha_1 \in H^1_{\rm dR}(E/\mathbb{C})$ and a differential 1-form $\alpha_2 \in H^1_{\rm dR}(E/\mathbb{C})$ such that $\iota^*(\alpha_1) = [\frac{dx}{y}]$ and $\iota^*(\alpha_2) = [\frac{xdx}{y}]$. Since $[\frac{dx}{y}]$ and $[\frac{xdx}{y}]$ generate $H^1_{\rm dR}(E_0/\mathbb{C})$, α_1 and α_2 as well generate $H^1_{\rm dR}(E/\mathbb{C})$, so $\alpha_1 \wedge \alpha_2 \neq 0$. We can repeat this history with another presentation of E where

$$F(x,y,z) := zy^2 - 4(x-zt_1)(x-zt_2)(x-zt_3),$$
$$f(x,y) = y^2 - 4(x-t_1)(x-t_2)(x-t_3),$$

with $t_1 \neq t_2 \neq t_3$ (for more details see [34, § 3.5]). Because of isomorphism (1.1), in the continue the necessary calculations attached to $H^1_{\rm dR}(E/\mathbb{C})$, such as computation of Gauss-Manin connection, will be done on the first de Rham cohomology $H^1_{\rm dR}(E_0/\mathbb{C})$ of affine curve E_0.

Let E be an affine elliptic curve over $\mathbb{C}$ and

$$\{0\} = F^2 \subset F^1 \subset F^0 = H^1_{\rm dR}(E), \quad \dim_{\mathbb{C}}(F^i) = 2-i,$$

be the Hodge filtration of $H^1_{dR}(E)$ (see §2.3). For any $\alpha_i \in F^i, i = 1, 2$, that satisfy the following intersection condition (see §2.6)

$$\frac{1}{2\pi i}\int_E \alpha_1 \wedge \alpha_2 = 1, \tag{1.2}$$

there exist a unique

$$h = (h_1, h_2, h_3) \in T_H := \{(t_1, t_2, t_3) \in \mathbb{C}^3 | t_1 \neq t_2 \neq t_3\}, \tag{1.3}$$

such that E is given by

$$E_{\rm H}: \; y^2 - 4(x-h_1)(x-h_2)(x-h_3) = 0 \tag{1.4}$$

in $\mathbb{P}^2$, and α_1, α_2, respectively, are given by $\frac{dx}{y}$, $\frac{xdx}{y}$, respectively. In fact $T_{\rm H}$ is the moduli of $(E, \alpha_1, \alpha_2, a_1, a_2, a_3)$, where the ordered triple (a_1, a_2, a_3) is the non-zero 2-torsion points of E, i.e. $2a_i = 0, \ i = 1, 2, 3$. There is a unique vector field H, that we call it DHR vector field, on $T_{\rm H}$ such that

$$\nabla_{\rm H}(\tfrac{dx}{y}) = -\tfrac{xdx}{y}, \ \nabla_{\rm H}(\tfrac{xdx}{y}) = 0, \tag{1.5}$$

in which, ∇ is the Gauss-Manin connection (see §2.5) defined on $T_{\rm H}$ given by

$$\nabla \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix} = A_{\rm H} \begin{pmatrix} \frac{dx}{y} \\ \frac{xdx}{y} \end{pmatrix} \tag{1.6}$$

with

$$\begin{aligned} A_{\rm H} &= \frac{dh_1}{2(h_1-h_2)(h_1-h_3)} \begin{pmatrix} -h_1 & 1 \\ h_2h_3 - h_1(h_2+h_3) & h_1 \end{pmatrix} \\ &+ \frac{dh_2}{2(h_2-h_1)(h_2-h_3)} \begin{pmatrix} -h_2 & 1 \\ h_1h_3 - h_2(h_1+h_3) & h_2 \end{pmatrix} \\ &+ \frac{dh_3}{2(h_3-h_1)(h_3-h_2)} \begin{pmatrix} -h_3 & 1 \\ h_1h_2 - h_3(h_1+h_2) & h_3 \end{pmatrix}. \end{aligned}$$

The vector field H is given by

$$\begin{aligned} {\rm H} &= (h_1(h_2+h_3) - h_2h_3)\frac{\partial}{\partial h_1} \\ &+ (h_2(h_1+h_3) - h_1h_3)\frac{\partial}{\partial h_2} \\ &+ (h_3(h_1+h_2) - h_1h_2)\frac{\partial}{\partial h_3}, \end{aligned}$$

that can be seen as the following ordinary differential equation in $\mathbb{C}^3$:

$${\rm H}: \begin{cases} \dot{h}_1 = h_1(h_2+h_3) - h_2h_3 \\ \dot{h}_2 = h_2(h_1+h_3) - h_1h_3 \\ \dot{h}_3 = h_3(h_1+h_2) - h_1h_2 \end{cases}. \tag{1.7}$$

So briefly, for the 1-forms $\alpha_1 = \frac{dx}{y}$ and $\alpha_2 = x\frac{dx}{y}$ that satisfy the following intersection condition

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le 2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.8}$$

the DHR vector field H given by (1.7) satisfies the following equation,

$$\nabla_{\mathrm{H}}\alpha = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \alpha, \tag{1.9}$$

in which $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}^{\mathtt{t}}$, where $\mathtt{t}$ refers to matrix transpose. The system of differential equation (1.7) appeared in the work of G. Darboux in 1978, and then G. Halphen (1881) and M. Brioschi (1881) contributed to the study of this differential equation system, that more details are presented in §1.2 and §1.3.

We can see the moduli of elliptic curves from another point of view and find another DHR vector field that is presented by Ramanujan's system of differential equations. To see this, consider the triple (E, α_1, α_2), where E is an elliptic curve with two 1-forms α_1 and α_2 satisfying (1.2). The moduli of (E, α_1, α_2)'s is given by

$$T_{\mathrm{R}} := \{(t_1, t_2, t_3) \in \mathbb{C}^3 | 27t_3^2 - t_2^3 = 0\}, \tag{1.10}$$

and more precisely for any (E, α_1, α_2) there exist a unique $(r_1, r_2, r_3) \in T_{\mathrm{R}}$, such that E is given by

$$E_{\mathrm{R}} : \; y^2 = 4(x - r_1)^2 - r_2(x - r_1) - r_3, \tag{1.11}$$

in $\mathbb{P}^2$, and $\alpha_1 = \frac{dx}{y}$, $\alpha_2 = \frac{xdx}{y}$. The Gauss-Manin connection of T_{R} is given by the matrix

$$A_{\mathrm{R}} = \frac{1}{\Delta} \begin{pmatrix} -\frac{3}{2}r_1\alpha - \frac{1}{12}d\Delta & \frac{3}{2}\alpha \\ \Delta r_1 - \frac{1}{16}r_1 d\Delta - (\frac{3}{2}r_1^2 + \frac{1}{8}r_2)\alpha & \frac{3}{2}r_1\alpha + \frac{1}{12}d\Delta \end{pmatrix},$$

$$\Delta = 27r_3^2 - r_2^3, \;\; \alpha = 3r_3dr_2 - 2r_2dr_3,$$

and the intersection form matrix is given by (1.8). It is seen that there exist a unique vector field R on T_{R}, which satisfies the following equation

$$\nabla_{\mathrm{R}}\alpha = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \alpha, \tag{1.12}$$

and R is given by the Ramanujan's system of differential equations

$$\mathrm{R} : \begin{cases} \dot{r}_1 = r_1^2 - \frac{1}{12}r_2 \\ \dot{r}_2 = 4r_1r_2 - 6r_3 \\ \dot{r}_3 = 6r_1r_3 - \frac{1}{3}r_2^2 \end{cases} . \tag{1.13}$$

It is natural to ask if there is any relationship between T_{H} and T_{R}, and the response is positive. The algebraic morphism $\phi : T_{\mathrm{H}} \to T_{\mathrm{R}}$ defined by

$$\phi : (h_1, h_2, h_3) \mapsto (T, 4\sum_{1\leq i<j\leq 3}(T - h_i)(T - h_j), 4(T - h_1)(T - h_2)(T - h_3)),$$

where

$$T := \frac{1}{3}(h_1 + h_2 + h_3),$$

connects two families. That is, if in E_{R} we replace r with $\phi(h)$ we obtain the family E_{H}, and ϕ maps all related concepts of E_{H}, to the corresponding concepts of E_{R}. In particular $\phi_*\mathrm{H} = \mathrm{R}$ and $\phi^* A_{\mathrm{R}} = A_{\mathrm{H}}$.

H. Movasati in [31, 33] worked on the family of mirror quintic 3-fold Calabi-Yau manifolds and he found DHR vector field on a moduli space constructed on the family of mirror quintic 3-folds. We know that the family of quintic 3-folds are hypersurfaces of $\mathbb{P}^4$ given by homogeneous polynomials of degree 5. It is seen in [8] that the mirror of the quintic 3-folds are given by the family of W_ψ's defined as the variety obtained by a resolution of singularities of the following quotient:

$$W_\psi := \{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \mid x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0x_1x_2x_3x_4 = 0\}/G, \quad (1.14)$$

where G is the group

$$G := \{(\zeta_1, \zeta_2, \cdots, \zeta_5) \mid \zeta_i^5 = 1,\ \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5 = 1\},$$

acting in a canonical way and $\psi^5 \neq 1$ (for more details see Example 4.2). H. Movasati started to work with a special moduli spaces consisting of a Calabi-Yau manifolds and a basis of de Rham cohomology. More precisely, Let W_1, W_2 be two mirror quintic 3-folds and $\{\alpha_1^i, \alpha_2^i, \alpha_3^i, \alpha_4^i\}$ be a basis of $H^n_{\mathrm{dR}}(W_i; \mathbb{C}), i = 1, 2$. Then we have the following equivalence relation,

$$(W_1, [\alpha_1^1, \alpha_2^1, \alpha_3^1, \alpha_4^1]) \sim (W_2, [\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2]) \quad (1.15)$$

if and only if there exist a biholomorphic function $f : W_1 \to W_2$ such that $f^*(\alpha_j^2) = \alpha_j^1,\ \ j = 1, 2, 3, 4$. He considered T to be the moduli of pairs $(W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4])$ under above equivalence relation (1.15), where

$$\alpha_i \in F^{4-i} \backslash F^{5-i}, \quad i = 1, 2, 3, 4,$$

in which $F^\bullet$ is the Hodge filtration of $H^3_{\mathrm{dR}}(W)$, satisfying the following intersection condition,

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le 4} = \Phi.$$

with

$$\Phi := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (1.16)$$

Movasati stated the following theorem to find DHR vector field.

Theorem 1.1. *Let T be the moduli of pairs $(W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4])$ defined above. Then there is a unique vector field* Ra *and a unique regular function y on T such that the Gauss-Manin connection of the universal family of mirror quintic Calabi-Yau varieties over T composed with the vector field* Ra*, namely ∇_{Ra}, satisfies*

$$\nabla_{\mathrm{Ra}}\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \alpha, \tag{1.17}$$

in which

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}^{\mathsf{t}}.$$

In fact,

$$T \cong \{(t_0, t_1, t_2, t_3, t_4, t_5, t_6) \in \mathbb{C}^7 \mid t_5 t_4 (t_4 - t_0^5) \neq 0\}, \tag{1.18}$$

and under this isomorphism the vector field Ra *as an ordinary differential equation is*

$$\mathrm{Ra}: \begin{cases} \dot t_0 = \frac{1}{t_5}(6 \cdot 5^4 t_0^5 + t_0 t_3 - 5^4 t_4) \\ \dot t_1 = \frac{1}{t_5}(-5^8 t_0^6 + 5^5 t_0^4 t_1 + 5^8 t_0 t_4 + t_1 t_3) \\ \dot t_2 = \frac{1}{t_5}(-3 \cdot 5^9 t_0^7 - 5^4 t_0^5 t_1 + 2 \cdot 5^5 t_0^4 t_2 + 3 \cdot 5^9 t_0^2 t_4 + 5^4 t_1 t_4 + 2 t_2 t_3) \\ \dot t_3 = \frac{1}{t_5}(-5^{10} t_0^8 - 5^4 t_0^5 t_2 + 3 \cdot 5^5 t_0^4 t_3 + 5^{10} t_0^3 t_4 + 5^4 t_2 t_4 + 3 t_3^2) \\ \dot t_4 = \frac{1}{t_5}(5^6 t_0^4 t_4 + 5 t_3 t_4) \\ \dot t_5 = \frac{1}{t_5}(-5^4 t_0^5 t_6 + 3 \cdot 5^5 t_0^4 t_5 + 2 t_3 t_5 + 5^4 t_4 t_6) \\ \dot t_6 = \frac{1}{t_5}(3 \cdot 5^5 t_0^4 t_6 - 5^5 t_0^3 t_5 - 2 t_2 t_5 + 3 t_3 t_6) \end{cases}, \tag{1.19}$$

and $y = \frac{5^8 (t_4 - t_0^5)^2}{t_5^3}$ is the Yukawa coupling.

1.2 Darboux

The system of differential equations

$$\begin{cases} \dot t_1 + \dot t_2 = t_1 t_2 \\ \dot t_2 + \dot t_3 = t_2 t_3 \\ \dot t_1 + \dot t_3 = t_1 t_3 \end{cases}, \tag{1.20}$$

first time appeared in 1878 in the works of Gaston Darboux (1842-1917) when he was studying the curvilinear coordinates and orthogonal systems in [12]. The problem that he was trying to prove is as follow: *Let A and B be two fixed surfaces in the 3-dimensional Euclidean space $\mathbb{R}^3$ and suppose that Σ is the family of surfaces which are the locus of the points that the sum of their distances from the surfaces A and B are constant; and Σ' is the family of surfaces which are the locus of the points that the difference of their distances* *from the surfaces A and B are constant. Is there a third family of surfaces that intersects Σ and Σ' orthogonally?* This problem is equivalent to the following problem: *Let A and B be as before and suppose that Σ is a family of surfaces parallel to A which is parameterized by v, and Σ' is a family of surfaces parallel to B that is parameterized by w. Is there a third family of surfaces parameterized by u such that intersects Σ and Σ' orthogonally?* Note that, two surfaces A_1 and A_2 are said to be *parallel*, if there exist a constant c such that for any $a_1 \in A_1$ and $a_2 \in A_2$, $d(a_1, A_2) = d(a_2, A_1) = c$, in which d refers to the Euclidean distance of $\mathbb{R}^3$; and we say that a family of surfaces is parameterized by $s = s(x, y, z)$, if any surface belonging to this family is given by $s(x, y, z) = constant$, in which x, y, z are the standard coordinates of $\mathbb{R}^3$. If for a function $s = s(x, y, z)$, we define

$$s_x = \frac{\partial s}{\partial x} \ , \qquad s_y = \frac{\partial s}{\partial y} \ , \qquad s_z = \frac{\partial s}{\partial z} \ ,$$

then in the latter problem, the condition of parallelism is given by

$$v_x^2 + v_y^2 + v_z^2 = 1 \qquad \& \qquad w_x^2 + w_y^2 + w_z^2 = 1,$$

and the condition of orthogonality is given by

$$u_x v_x + u_y v_y + u_z v_z = 0 \qquad \& \qquad u_x w_x + u_y w_y + u_z w_z = 0.$$

So th problem is equivalent to the following system of equations,

$$\begin{cases} v_x^2 + v_y^2 + v_z^2 = 1 \\ w_x^2 + w_y^2 + w_z^2 = 1 \\ u_x v_x + u_y v_y + u_z v_z = 0 \\ u_x w_x + u_y w_y + u_z w_z = 0 \end{cases} . \tag{1.21}$$

The more interesting case of this problem is when the family (u) is of second degree and Darboux proved that in this case this family is given by

$$\frac{x^2}{t_1(u)} + \frac{y^2}{t_2(u)} + \frac{z^2}{t_3(u)} = 1,$$

in which t_1, t_2, t_3 are functions of u given by the following equation,

$$t_3(\frac{dt_1}{du} + \frac{dt_2}{du}) = t_2(\frac{dt_1}{du} + \frac{dt_3}{du}) = t_1(\frac{dt_2}{du} + \frac{dt_3}{du}). \tag{1.22}$$

Hence the system of equations (1.20) is a particular case of the equation (1.22).

1.3 Halphen

In 1881, G. Halphen [23] studied the following system of differential equations

$$\begin{cases} \frac{dt_1}{dz} + \frac{dt_2}{dz} = t_1 t_2 \\ \frac{dt_2}{dz} + \frac{dt_3}{dz} = t_2 t_3 \\ \frac{dt_1}{dz} + \frac{dt_3}{dz} = t_1 t_3 \end{cases}, \tag{1.23}$$

in which t_1, t_2, t_3 are three unknown variables. Halphen proved that this system satisfies an important invariant property. To express this invariant property, for the constants a, b, a', b', let

$$w = \frac{az+b}{a'z+b'}, \tag{1.24}$$

$$t_i = -\frac{2a'}{a'z+b'} + \frac{ab'-ba'}{(a'z+b')^2} s_i, \quad i = 1, 2, 3. \tag{1.25}$$

Then by substituting (1.24) and (1.25) in the system (1.23), we have

$$\begin{cases} \frac{ds_1}{dw} + \frac{ds_2}{dw} = s_1 s_2 \\ \frac{ds_2}{dw} + \frac{ds_3}{dw} = s_2 s_3 \\ \frac{ds_1}{dw} + \frac{ds_3}{dw} = s_1 s_3 \end{cases}. \tag{1.26}$$

Hence, the system (1.23) is invariant under the change of variables (1.24) and (1.25). So to find a general solution of (1.23), it is enough to first find a particular solution of (1.26), and then applying (1.25). Halphen expressed a solution of the system (1.23) in terms of the logarithmic derivatives of the null theta functions; namely,

$$\begin{aligned} t_1 &= 2(\ln \theta_4(0|z))', \\ t_2 &= 2(\ln \theta_2(0|z))', \qquad ' = \frac{\partial}{\partial z} \\ t_3 &= 2(\ln \theta_3(0|z))'. \end{aligned}$$

where

$$\begin{cases} \theta_2(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(0|z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \quad q = e^{2\pi i z}, \ z \in \mathbb{H}.$$

From it, we can found following three particular solutions of the system (1.23),

$$\begin{aligned} T_1(z) &= 8\frac{e^z + 4e^{4z} + 9e^{9z} + 16e^{16z} + \dots}{1 + 2e^z + 2e^{4z} + 2e^{9z} + 2e^{16z} + \dots}, \\ T_2(z) &= 8\frac{-e^z + 4e^{4z} - 9e^{9z} + 16e^{16z} - \dots}{1 - 2e^z + 2e^{4z} - 2e^{9z} + 2e^{16z} - \dots}, \\ T_3(z) &= \frac{1 + 9e^{2z} + 25e^{6z} + 49e^{12z} + 81e^{20z} + \dots}{1 + e^{2z} + e^{6z} + e^{12z} + e^{20z} + \dots}. \end{aligned}$$

And the general solution of (1.23), is given as

$$t_i(z) = -\frac{2a'}{a'z+b'} + \frac{ab'-ba'}{(a'z+b')^2} T_i(\frac{az+b}{a'z+b'}), \quad i = 1,2,3.$$

In [5], Fr. Brioschi in 1881 studied the following extension of the system (1.23)

$$\begin{cases} \frac{dt_1}{dz} + \frac{dt_2}{dz} = t_1t_2 + \varphi(z) \\ \frac{dt_2}{dz} + \frac{dt_3}{dz} = t_2t_3 + \varphi(z) \\ \frac{dt_1}{dz} + \frac{dt_3}{dz} = t_1t_3 + \varphi(z) \end{cases}, \tag{1.27}$$

in which, $\varphi(z)$ is a function of z. Again Halphen in [21] introduced and studied a class of differential equations that the system (1.27) belongs to this class. In the case of three variables, he showed that this class is given by

$$\begin{cases} \frac{dt_1}{dz} = a_1t_1^2 + (\lambda - a_1)(t_1t_2 + t_1t_3 - t_2t_3) \\ \frac{dt_2}{dz} = a_2t_2^2 + (\lambda - a_2)(t_2t_3 + t_2t_1 - t_3t_1) \\ \frac{dt_3}{dz} = a_3t_3^2 + (\lambda - a_3)(t_3t_1 + t_3t_2 - t_1t_2) \end{cases}, \tag{1.28}$$

where, a_1, a_2, a_3, λ are constants. One can see that the system (1.28) is equivalent to the system (1.23), when $a_1 = a_2 = a_3 = 0$ and $\lambda = 1$. He proved that the system (1.28) also satisfies the invariant property and it is in a direct relationship with the second order linear differential equations and he gave a solution of the system (1.28) in terms of hypergeometric functions X, Y, Z which had been introduced in [22].

1.4 Ramanujan

In this section we are going to find the origin of the system of equations (1.13) which has been done by Ramanujan in [37]. To do this, let

$$\sigma_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \ldots + \sigma_r(n)\sigma_s(0),$$

in which, $\sigma_\nu(n) = \sum_{d|n} d^\nu$ and, by definition, $\sigma_s(0) = \frac{1}{2}\zeta(-s)$, where $\zeta(s)$ is the Riemann Zeta-function. Ramanujan proves that

$$\begin{aligned} \sigma_{r,s}(n) = & \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)} \frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)} \sigma_{r+s+1}(n) \\ & + \frac{\zeta(1-r)+\zeta(1-s)}{r+s} n\sigma_{r+s-1}(n) + O\{n^{2/3(r+s+1)}\}. \end{aligned}$$

To prove this equation Ramanujan encounters the following equations

$$P(q) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k},$$
$$Q(q) = 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k},$$
$$R(q) = 1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k},$$

that satisfy the following system of differential equations

$$\begin{cases} q\frac{dP}{dq} &= \frac{P^2(q)-Q(q)}{12} \\ q\frac{dQ}{dq} &= \frac{P(q)Q(q)-R(q)}{3} \\ q\frac{dR}{dq} &= \frac{P(q)R(q)-Q^2(q)}{2} \end{cases}. \tag{1.29}$$

Next we give the relationships between the equations introduced by Ramanujan and Eisenstein series. First note that, Eisenstein series G_{2j} of weight $2j$, for integers $j \geq 2$, are given as follow

$$G_{2j}(\tau) := \sum_{\substack{m_1,m_2\in\mathbb{Z} \\ (m_1,m_2)\neq(0,0)}} (m_1\tau + m_2)^{-2j},$$

and working with the q-expansion of the Eisenstein series, the alternate notations of Eisenstein series , for complex τ's with $\mathrm{Im}\tau > 0$, are defined

$$\begin{aligned} E_{2j}(\tau) := \frac{G_{2j}(\tau)}{2\zeta(2j)} &= 1 - \frac{4j}{B_{2j}}\sum_{k=1}^{\infty}\frac{k^{2j-1}e^{2\pi ik\tau}}{1-e^{2\pi ik\tau}} \\ &= 1 - \frac{4j}{B_{2j}}\sum_{r=1}^{\infty}\sigma_{2j-1}(r)q^r, \quad q = e^{2\pi i\tau}, \end{aligned} \tag{1.30}$$

in which B_k's are Bernoulli's numbers. So $E_4(\tau) = Q(q)$, $E_6(\tau) = R(q)$, and also $P(q) = E_2(\tau)$, where $E_2(\tau)$ is defined by (1.30). Moreovere, if we let

$$(r_1(\tau), r_2(\tau), r_3(\tau)) = (\frac{2\pi i}{12}E_2(\tau), 12(\frac{2\pi i}{12})^2E_4(\tau), 8(\frac{2\pi i}{12})^3E_6(\tau)),$$

then using the system (1.29), one easily finds that (r_1, r_2, r_3) stisfy the following system of differential equations

$$\begin{cases} \frac{dr_1}{d\tau} &= r_1^2 - \frac{1}{12}r_2 \\ \frac{dr_2}{d\tau} &= 4r_1r_2 - 6r_3 \\ \frac{dr_3}{d\tau} &= 6r_1r_3 - \frac{1}{3}r_2^2 \end{cases}, \tag{1.31}$$

which is the system (1.13).

Chapter 2

Hodge Theory and Families

In this chapter we briefly recall the basic facts related with Hodge theory. Firstly, in §2.1 the concept of integrable connections are stated. Secondly, in §2.2 the general theory of Hodge structure is provided. Then in §2.3 after introducing de Rham cohomology, as a nice example of Hodge structure, the Hodge decomposition of complexified de Rham cohomology and Hodge filtration is presented. Next in §2.4 the definitions and important results of families of complex manifolds are established. To provide a family of complex manifolds with a Hodge variation, one needs Gauss-Manin connection and Griffiths transversality that are stated in §2.5. Finally in §2.6 the concept of intersection forms is given. In particular we give a proof of the existence relationship between intersection forms and Hodge filtration in the context of de Rham cohomology.

In this chapter, a more detailed account as well as further information and omitted proofs to most of the results can be found in [13, 36, 41].

2.1 Local Systems and Integrable Connections

Definition 2.1. Let (X, o) be a pointed topological space and R be a commutative ring with a unit. A *local system of R-modules* on X is defined to be a collection $\mathbb{V} = \{V_x\}_{x\in X}$ of R-modules together with a collection of isomorphisms $\{\rho([\gamma])|V_x \xrightarrow{\sim} V_y\}_{x,y\in X \text{ and } [\gamma]\in\pi_1(X,(x,y))}$, in which $\pi_1(X,(x,y))$ is the homotopy classes of pathes from x to y, that satisfy followings

(i) for the class of constant path e_x at x, $\rho([e_x]) = \mathrm{id}_{V_x}$,

(ii) for any two classes $[\gamma] \in \pi_1(X,(x,y))$ and $[\gamma'] \in \pi_1(X,(y,z))$, $\rho([\gamma*\gamma']) = \rho([\gamma'])\circ\rho([\gamma])$, in which the composition $\gamma * \gamma'$ of pathes γ and γ' means first traverse γ and then γ', both with double speed.

Usually we denote this local system by $\mathbb{V}$, and moreover if R is a field and the fibers V_x are R-vector spaces, then the rank of $\mathbb{V}$, which is denoted by $\mathrm{rk}(\mathbb{V})$, by definition is $\mathrm{rk}(\mathbb{V}) := \dim_R V_x$. Also we denote a *constant system* with fibers V on X by $\underline{V}_X$.

Definition 2.2. Let (X, o) be a pointed path-connected topological space. Then the collection $\{\rho([\gamma])|\gamma \text{ is a loop at } o\}$ defines the associated *monodromy representation*

$$\rho : \pi_1(X, o) \to GL(V_o).$$

The Definition 2.1 of a local system looks a little long and to make it more applicable we present an equivalent statement of it. To do this, we define a *locally constant sheaf* $\mathcal{F}$ on X to be a sheaf with the property that for some open cover $\{U_i\}_{i\in\mathcal{I}}$ of X, the restrictions $\rho_{U_i,x} : \mathcal{F}(U_i)\tilde{\to}\mathcal{F}_x$, $x \in U_i$ are isomorphisms. Now suppose that X is path-connected and locally simply-connected with a covering $\{U_i\}_{i\in\mathcal{I}}$ of simply-connected open subsets of X. If $\mathbb{V}$ is a locally system on X, then the unique isomorphisms $f_{x,y} : V_x \xrightarrow{\sim} V_y$, $x, y \in U_i$, defined by any path connecting x to y within U_i, give a locally constant sheaf by trivializations

$$\phi_i : \mathbb{V}|_{U_i} \xrightarrow{\sim} \underline{V_{U_i}}.$$

Conversely, let $\mathcal{F}$ be a locally constant sheaf on X. To associate a local system $\mathbb{V}$ to $\mathcal{F}$, for any $x \in X$, let V_x be the stalk $\mathcal{F}_x$, and for any path $\gamma : [a, b] \to X$, in which $[a, b] \subset U_i$ for some $i \in \mathcal{I}$, define $\rho(\gamma) = \rho_{U_i,b} \circ \rho^{-1}_{U_i,a}$. In the case that $[a, b]$ is not a subset of U_i, we divide $[a, b]$ to the subdivisions that any of them is a subset of some U_i and then define $\rho(\gamma)$ as compositions of isomorphisms. Hence we have the following theorem:

Theorem 2.1. *Let X be a path connected and locally simply-connected topological space. Then there is an one to one correspondence between local systems of R-modules and locally constant sheaves of R-modules.*

From now on we substitute the topological space X with the complex manifold S and by a *complex local system on S* we mean a *locally constant sheaf of $\mathbb{C}$-vector space*. If we denote the category of complex local systems on S by $\mathrm{LocSys}_{\mathbb{C}}(S)$, next we see that it is equivalent to the category of holomorphic vector bundles on S with an integral connection.

Definition 2.3. Let S be an n-dimensional complex manifold and $\mathcal{V}$ be an $\mathcal{O}_S$-module on it. A *holomorphic connection* ∇ on $\mathcal{V}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{V} \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{V}$, such that it satisfies the *Leibniz rule*, i.e. for any local section f of $\mathcal{O}_S$ and any local section v of $\mathcal{V}$,

$$\nabla(fv) = df \otimes v + f\nabla(v). \tag{2.1}$$

A local section v of $\mathcal{V}$ is said to be *horizontal* if $\nabla(v) = 0$, and by notation $\mathcal{V}^\nabla$ denotes $\mathrm{Ker}(\nabla)$.

Observation 2.1. If $\mathcal{V}$ is locally free of finite rank m, then by considering the frame $\{v_j\}_{j=1}^m$ on an open subset U of S, we can write $\nabla(v_j) = \sum_{i=1}^m C_{ij}v_i$, and then define the *connection matrix* as matrix of holomorphic 1-forms on U given by $C_U = (C_{ij})_{1\leq i,j\leq m}$. So for a holomorphic section $v = \sum_{j=1}^m g_jv_j$, by applying the Leibniz' rule we have

$$\nabla(v) = \sum_{j=1}^m dg_j \otimes v_j + \sum_{i,j=1}^m g_jC_{ij} \otimes v_i,$$

hence we can abbreviate the connection locally as follow

$$\nabla_U = d + C_U. \tag{2.2}$$

By defining $\Omega_S^p(\mathcal{V}) := \Omega_S^p \otimes_{\mathcal{O}_S} \mathcal{V}$, the wedge product of differential forms induces

$$\begin{aligned} \nabla^{(p)} :&\Omega_S^p(\mathcal{V}) \to \Omega_S^{p+1}(\mathcal{V}) \\ &\omega \otimes v \mapsto d\omega \otimes v + (-1)^p \omega \wedge \nabla(v) \end{aligned} \tag{2.3}$$

Definition 2.4. The *curvature* F_∇ of connection ∇ by definition is the map

$$\begin{aligned} F_\nabla :&\mathcal{V} \to \Omega_S^2(\mathcal{V}), \\ F_\nabla(v) &= \nabla^{(1)} \circ \nabla(v). \end{aligned} \tag{2.4}$$

The connection ∇ is said to be *flat* or *integrable* if $F_\nabla = 0$.

Let $\mathcal{V}$ be a holomorphic vector bundle on S of finite rank m. Then by using (2.2) we see that $F_{\nabla_U} = dC_U - C_U \wedge C_U$, in which $(C_U \wedge C_U)_{ij} = \sum_{k=1}^m C_{ik} \wedge C_{kj}$. Thus the integrability of ∇ is equivalent to $C_U \wedge C_U = dC_U$. From now on, the pair $(\mathcal{V}, \nabla)$ refers to a holomorphic vector bundle $\mathcal{V}$ with integrable connection ∇ and we call it *integrable connection* on S. Integrable connections on S form a category that we denote it by $\mathrm{IntCon}(S)$.

If $\mathbb{V}$ is a complex local system on S, then

$$\mathcal{V} := \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_S \tag{2.5}$$

is a holomorphic vector bundle on S. By defining $\nabla(v \otimes f) = df \otimes v$, for any local sections f and v, resp., of $\mathcal{O}_S$ and $\mathbb{V}$, resp., we can see that ∇ is an integrable connection of $\mathcal{V}$, hence $(\mathcal{V}, \nabla)$ is an integrable connection. Conversely, let $(\mathcal{V}, \nabla)$ be an integrable connection. Then by defining $\mathbb{V} := \mathcal{V}^\nabla$, it is seen that $\mathbb{V}$ is a complex local system on S and $(\mathcal{V}, \nabla) \simeq \mathbb{V} \otimes_{\mathbb{C}} (\mathcal{O}_S, d)$ (for more details see [36, § 10.3]). So we have the following theorem:

Theorem 2.2. *Let S be a complex manifold. Then* $\mathrm{LocSys}_{\mathbb{C}}(S) \cong \mathrm{IntCon}(S)$.

2.2 Hodge Structure

Definition 2.5. Let V be a $\mathbb{Z}$-module of finite rank and $V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ its complexification. A *Hodge structure of weight k* on V is a direct sum decompositio

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}, \quad \textit{(Hodge decomposition)} \tag{2.6}$$

with $V^{p,q} = \overline{V^{q,p}}$. The numbers $h^{p,q}(V) := \dim V^{p,q}$ are called *Hodge numbers* of the Hodge structure.

The *Hodge filtration* of V associated to the Hodge structure (2.6) is given by

$$F^p(V) = \bigoplus_{r \geq p} V^{r,k-r}. \tag{2.7}$$

To simplify the notation, we write F^p instead of $F^p(V)$ when no confusion arise. So obviously we have the decreasing filtration

$$0 = F^{k+1} \subset F^k \subset \ldots \subset F^1 \subset F^0 = V_{\mathbb{C}}. \tag{2.8}$$

Also one can see $V^{p,q} = F^p \cap \overline{F^q}$ and $V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}$ (or equivalently $F^p \cap \overline{F^{k-p+1}} = 0$). Conversely, the decreasing filtration (2.8) of $V_{\mathbb{C}}$ with the property $V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}$ gives a Hodge filtration of weight k by defining $V^{p,q} = F^p \cap \overline{F^q}$.

Definition 2.6. Let S be a complex manifold. A *variation of Hodge structure of weight* k on S is a quadruple $(S, \mathbb{V}, \nabla, \mathcal{F}^\bullet)$, in which $\mathbb{V}$ is a local system of finitely generated abelian groups on S, $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_s} \Omega^1_S$ is an integrable connection and $\mathcal{F}^\bullet := \{\mathcal{F}^p\}$ is a finite decreasing filtration of the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic subbundles (the *Hodge filtration*), such that following conditions hold:

(i) for each $s \in S$ the filtration $\{\mathcal{F}^p(s)\}$ of $\mathcal{V}(s) \simeq \mathbb{V}_s \otimes_{\mathbb{Z}} \mathbb{C}$ defines a Hodge structure of weight k on the finitely generated abelian group $\mathbb{V}_s$,

(ii) the connection ∇, whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}}$, satisfies the *Griffiths' transversality condition*

$$\nabla(\mathcal{F}^p) \subset \Omega^1_S \otimes \mathcal{F}^{p-1}. \tag{2.9}$$

Note that in all the concepts of this section we can substitute the group of integers $\mathbb{Z}$ by the group of reals $\mathbb{R}$, and have the *real Hodge structure* or *real variation of Hodge structure*.

2.3 de Rham Cohomology and Hodge Filtration

Let X be an n-dimensional differentiable manifold, $\mathcal{T}X$ the tangent bundle of X and $\mathcal{T}^*X$ its cotangent bundle. By definition let

$$\mathcal{E}^m_X := \Gamma\left(X, \Lambda^m \mathcal{T}^* X\right), \ \ m = 1, 2, 3, \ldots$$

be the bundle of differential m-forms; note that by the space of 0-forms we mean the space of differentiable functions. Then by considering the long exact sequence

$$\mathcal{E}^\bullet_X : \ \ 0 \xrightarrow{d} \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}^{n-1}_X \xrightarrow{d} \mathcal{E}^n_X \xrightarrow{d} 0,$$

in which d is exterior derivative, we define the q-th *de Rham cohomology* $H^q_{\mathrm{dR}}(X)$ of X as follow,

$$H^q_{\mathrm{dR}}(X) := H^q(\mathcal{E}^\bullet_X, d) = \frac{\mathrm{Ker}\left(\mathcal{E}^q_X \xrightarrow{d} \mathcal{E}^{q+1}_X\right)}{\mathrm{Im}\left(\mathcal{E}^{q-1}_X \xrightarrow{d} \mathcal{E}^q_X\right)}. \tag{2.10}$$

We can also see de Rham cohomology in the point of view of Čech cohomology. To do this let $\Omega^q(X)$, $q = 1, 2, \dots$, be the sheaf of differential q-forms, and $\Omega^0(X) := \mathcal{O}(X)$ the sheaf of differential functions on X. Then we have the de Rham complex

$$\Omega^\bullet(X): \ \ 0 \to \underline{\mathbb{R}_X} \hookrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}(X) \xrightarrow{d} \Omega^n(X) \xrightarrow{d} 0,$$

in which $\underline{\mathbb{R}_X}$ is a constant sheaf on X. If X is compact, then we have the following isomorphism

$$H^q_{dR}(X) \cong \check{H}^q(X, \mathbb{R}).$$

Also in the context of singular cohomology, by *de Rham lemma* one has the following isomorphism

$$H^q_{dR}(X) \cong H^q(X, \mathbb{R}), \ (de \ \ Rham \ \ lemma). \tag{2.11}$$

Note that if X is connected, then for $q = 0$ we have

$$H^0_{dR}(X) \cong \{\text{locally constant } \mathbb{R}\text{-valued functions on } X\} \cong \mathbb{R}. \tag{2.12}$$

Next we suppose that X is a complex n-dimensional manifold, and hence $m = 2n$ dimensional differential manifold. Let J be the complex structure of X. Then for any $p \in X$, complexify $\mathcal{T}_pX$ to obtain $(\mathcal{T}_pX)_{\mathbb{C}} := \mathcal{T}_pX \otimes_{\mathbb{R}} \mathbb{C}$ which is a complex vector space isometric to $\mathbb{C}^{2n}$. The automorphism $J_p : \mathcal{T}_pX \to \mathcal{T}_pX$ extends naturally to a automorphism $J_p : (\mathcal{T}_pX)_{\mathbb{C}} \to (\mathcal{T}_pX)_{\mathbb{C}}$, which is linear over $\mathbb{C}$. Since $J_p^2 = -id$, the eigenvalues of J_p are $\pm i$, where $i = \sqrt{-1}$. Let $\mathcal{T}_p^{(1,0)}X$ and $\mathcal{T}_p^{(0,1)}X$, resp., be the eigenspaces of J_p corresponding to eigenvalues i and $-i$, resp. Then we see that $\mathcal{T}_p^{(1,0)}X \cong \mathbb{C}^n \cong \mathcal{T}_p^{(0,1)}X$ and that $(\mathcal{T}_pX)_{\mathbb{C}} = \mathcal{T}_p^{(1,0)}X \oplus \mathcal{T}_p^{(0,1)}X$. As this is valid for every point $p \in X$, we have defined two subbundles of $(\mathcal{T}X)_{\mathbb{C}} := \mathcal{T}X \otimes_{\mathbb{R}} \mathbb{C}$ such that $(\mathcal{T}X)_{\mathbb{C}} = \mathcal{T}^{(1,0)}X \oplus \mathcal{T}^{(0,1)}X$. By a similar argument we can see that the complexified cotangent bundle $(\mathcal{T}^*X)_{\mathbb{C}} := \mathcal{T}^*X \otimes_{\mathbb{R}} \mathbb{C}$ splits into pieces $(\mathcal{T}^*X)_{\mathbb{C}} = \mathcal{T}^{*(1,0)}X \oplus \mathcal{T}^{*(0,1)}X$. If we define $\Lambda^{p,q}X := \Lambda^p\mathcal{T}^{*(1,0)}X \otimes \Lambda^q\mathcal{T}^{*(0,1)}X$, then by using of splitting of $(\mathcal{T}^*X)_{\mathbb{C}}$ it follows that

$$\Lambda^k\mathcal{T}^*X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{j=0}^{k} \Lambda^{j,k-j}X.$$

This is decomposition of the exterior k-forms on X induced by the complex structure J. A section of $\Lambda^{p,q}X$ is called a (p,q)-form and by notation we write

$$\mathcal{E}_X^{p,q} := \Gamma(X, \Lambda^{p,q}X).$$

Now we define ∂ and $\bar{\partial}$ to be the components of exterior derivative d as follow,

$$\partial : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p+1,q} \quad \& \quad \bar{\partial} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}.$$

Then ∂ and $\bar{\partial}$ are first-order partial differential operators on complex k-forms which satisfy $d = \partial + \bar{\partial}$. The identity $d^2 = 0$ implies that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. So by using the differential operator $\bar{\partial}$ we have the following long exact sequence

$$\mathcal{E}_X^{p,\bullet}: \ \ 0 \to \mathcal{E}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{E}_X^{p,q} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{p,q+1} \xrightarrow{\bar{\partial}} \dots, \quad p = 0, 1, 2, \dots$$

and hence we define the (p,q)-th Dolbeault cohomology $H^{p,q}$ as follow

$$H^{p,q}(X) := H^q(\mathcal{E}^{p,\bullet}, \bar{\partial}) = \frac{\operatorname{Ker}\left(\mathcal{E}_X^{p,q} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{p,q+1}\right)}{\operatorname{Im}\left(\mathcal{E}_X^{p,q-1} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{p,q}\right)}.$$

If we denote the sheaf of holomorphic p-forms by Ω_X^p and the sheaf of (p,q)-forms by $\Omega_X^{p,q}$, then we consider the *Dolbeault complex* as follow

$$\Omega_X^{p,\bullet}: \ 0 \to \Omega_X^p \hookrightarrow \Omega_X^{p,0} \xrightarrow{\bar{\partial}} \Omega_X^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega_X^{p,q} \xrightarrow{\bar{\partial}} \Omega_X^{p,q+1} \xrightarrow{\bar{\partial}} \dots, \quad p = 0, 1, 2, \dots$$

which is a resolution of Ω_X^p, and by *Dolbeault's theorem* we have the following isomorphism

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p), \quad (\textit{Dolbeault's theorem}). \tag{2.13}$$

Next we are going to introduce a real Hodge structure of weight k on $H^k_{\mathrm{dR}}(X)$. To do this we need X to be a Kähler manifold. By definition, a *Kähler manifold* is a 4-tuple (X, ω, g, J), in which (X, ω) is a symplectic manifold, i.e. ω is a non-degenerate closed 2-form on X, with the complex structure J and the Riemannian metric g such that for any two vector fields v, w on X we have $g(v,w) = \omega(v, Jw)$. Hence one can easily see that

$$h := g + i\omega \tag{2.14}$$

is a *hermitian metric* on X. As we saw in (2.6), we also need to complexify $H^k_{\mathrm{dR}}(X)$ and we denote its complexification as follow

$$H^k_{\mathrm{dR}}(X; \mathbb{C}) := H^k_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathbb{C},$$

and by de Rham lemma (2.11) we have

$$H^k_{dR}(X; \mathbb{C}) \cong H^k(X, \mathbb{C}). \tag{2.15}$$

Following theorem gives the real Hodge structure that we desire(see [36, Theorem 1.8]).

Theorem 2.3. (Hodge Decomposition Theorem) *If X is a compact Kähler manifold, then we have*

$$H^k_{dR}(X; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{2.16}$$

and moreover, $H^{p,q}(X) = \overline{H^{q,p}(X)}$, in which "$\bar{.}$" refers to complex conjugation.

Hence the complex conjugation map implies the isomorphism

$$H^{p,q}(X) \cong H^{q,p}(X), \tag{2.17}$$

thus if we denote the Hodge numbers by $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$, then $h^{p,q}(X) = h^{q,p}(X)$. If there is no danger of confusion, we just denote the Hodge number by $h^{p,q}$. Also we have

$h^{p,q} = h^{n-q,n-p}$; to see this first we need to know a little about the Hodge $*$-operator. The Kähler metric g of X present an inner product on $\mathcal{T}_x X$, and hence it induces an inner product on $\mathcal{T}_x^* X$ and on the wedge product of $\mathcal{T}_x^* X$, i.e. the bundle of q-forms $\Lambda^q \mathcal{T}_x^* X$, which we denote it also by g. Using g we can define the positive definite hermitian form $g_x^{\mathbb{C}}$ on $(\mathcal{T}_x X)_{\mathbb{C}}$, for any $x \in X$, as follow

$$g_x^{\mathbb{C}}(v \otimes \lambda, w \otimes \mu) = (\lambda \bar{\mu}) g_x(v, w), \ \forall \lambda, \mu \in \mathbb{C} \text{ and } and v, w \in \mathcal{T}_x X. \tag{2.18}$$

Definition 2.7. The *Hodge $*$-operator*

$$* : \Lambda^q \mathcal{T}^* X \to \Lambda^{2n-q} \mathcal{T}^* X, \tag{2.19}$$

in any point $x \in X$ is defined by

$$\alpha \wedge *\beta = g_x(\alpha, \beta) \mathrm{vol}_g(x), \ \forall \alpha, \beta \in \Lambda^q \mathcal{T}_x^* X, \tag{2.20}$$

in which $\mathrm{vol}_g(x)$ refers to the volume form of X with respect to the metric g in x, of course with a fixed orientation on X.

The Hodge $*$-operator (2.19) $\mathbb{C}$-linearly extends to the operator

$$* : \Lambda^q (\mathcal{T}^* X)_{\mathbb{C}} \to \Lambda^{2n-q} (\mathcal{T}^* X)_{\mathbb{C}}, \tag{2.21}$$

that is induced by

$$\alpha \wedge *\bar{\beta} = g_x^{\mathbb{C}}(\alpha, \beta) \mathrm{vol}_g(x), \ \forall \alpha, \beta \in \Lambda^q \mathcal{T}_x^* X, \tag{2.22}$$

in which $g_x^{\mathbb{C}}(.,.)$ is given by (2.18). The operator given in (2.21) is also called Hodge $*$-operator and it is not difficult to see that it sends $\Lambda^{p,q} X$ to $\Lambda^{n-q,n-p} X$ (for more details one refers to [24]). And finally the equality $h^{p,q} = h^{n-q,n-p}$, follows from following theorem which is proved in [24, Corollary 3.3.14].

Theorem 2.4. *The Hodge $*$-operator on a compact Kähler manifold X, induces the following isomorphism,*

$$H^{p,q}(X) \cong H^{n-q,n-p}(X).$$

Notation 2.1. For a compact Kähler manifold X, we denote by $b_k(X) := \dim_{\mathbb{C}} H^k_{\mathrm{dR}}(X; \mathbb{C})$ which is known as k-th Betti number of X. And if there is no ambiguity about the manifold X, we just denote it by b_k.

We can summarize above facts in the following lemma.

Lemma 2.1. *Let X be a compact Kähler manifold of complex dimension n. Then the followings hold:*

(i) $b_k = \sum_{p+q=k} h^{p,q}, \ 0 \le k \le 2n.$

(ii) $h^{p,q} = h^{q,p}, \ 0 \le p, q \le n.$

(iii) $h^{p,q} = h^{n-q,n-p}$, $0 \leq p, q \leq n$.

By using Lemma 2.1, we consider the *Hodge diamond* of an n-dimensional compact Kähler manifold X as follow

$$
\begin{array}{ccccccccccc}
 & & & & \overset{\text{Lemma 2.1(ii)}}{\longleftrightarrow} & & & & & & \\
 & & & & & h^{0,0} & & & & & \\
 & & & & h^{1,0} & & h^{1,0} & & & & \\
 & & & h^{2,0} & & h^{1,1} & & h^{2,0} & & & \\
 & & \iddots & & & & & & \ddots & & \\
 & h^{n-1,0} & & h^{n-2,1} & & \dots & & h^{n-2,1} & & h^{n-1,0} & \\
\xrightarrow{\text{Lemma 2.1(i)}} h^{n,0} & & h^{n-1,1} & & & \dots & & & h^{n-1,1} & & h^{n,0} \ \updownarrow \text{Lemma 2.1 (iii)} \\
 & h^{n-1,0} & & h^{n-2,1} & & \dots & & h^{n-2,1} & & h^{n-1,0} & \\
 & & \ddots & & & & & & \iddots & & \\
 & & & h^{2,0} & & h^{1,1} & & h^{2,0} & & & \\
 & & & & h^{1,0} & & h^{1,0} & & & & \\
 & & & & & h^{0,0} & & & & &
\end{array}
\tag{2.23}
$$

Now, as we saw in (2.7), we present the Hodge filtration of $H^k_{\mathrm{dR}}(X)$, $0 \leq k \leq n$, as follow

$$F^p(X) = \bigoplus_{p \leq r \leq k} H^{r,k-r}(X), \ 0 \leq p \leq k, \tag{2.24}$$

and hence we have the following decreasing filtration

$$F^\bullet : \ 0 = F^{k+1} \subset F^k \subset \ldots \subset F^1 \subset F^0 = H^k_{\mathrm{dR}}(X; \mathbb{C}). \tag{2.25}$$

One can easily see the following lemma.

Lemma 2.2. *Following hold for the Hodge filtration $F^\bullet$ of $H^k_{\mathrm{dR}}(X)$:*

(i) $H^{p,q} = F^p \cap \overline{F^q}$, $0 \leq p, q \leq k$.

(ii) $H^k_{\mathrm{dR}}(X; \mathbb{C}) = F^p \oplus \overline{F^{k-p+1}}$, $0 \leq p \leq k$.

2.4 Families and Complex Deformations

Definition 2.8. By a *family of complex manifolds* we mean a holomorphic proper submersion $\pi : \mathcal{X} \to S$, in which $\mathcal{X}$ and S are complex manifolds. The manifold S sometimes is called the *base manifold* and in the case that there is no ambiguity about the base manifold, we simply denote the family of complex manifolds by $\mathcal{X}$.
For any $s \in S$, let X_s be the fiber of π over the point s, i.e. $X_s := \pi^{-1}(s)$. If S is connected and $o \in S$ is a base point, then we say that $\mathcal{X}$ is a *family of deformations* of the fiber X_o, and any fiber X_s, $s \in S$, is said a *complex deformation* of X_o. An *infinitesimal complex deformation* of X is a deformation with base space $S := \mathrm{Spec}(\mathbb{C}[\epsilon])$. Following theorem gives a representation of infinitesimal deformation of X (see [18, Theorem 22.1]).

Theorem 2.5. *The isomorphism classes of infinitesimal deformations of a compact complex manifold X are parametrized by elements in $H^1(X, \mathcal{T}_X)$.*

Remark 2.1. Let X be an n-dimensional compact complex manifold. The space of complex deformations of X sometimes is called *complex moduli space* of X which is denoted by $\mathcal{M}_{cmplx}(X)$. If E is a holomorphic vector bundle on X, then *Serre duality* states that

$$H^q(X, E) \cong H^{n-q}(X, K_X \otimes E^*)^*,$$

in which $K_X = \Lambda^{n,0}X$ is the canonical bundle of X. So by Theorem 2.5, we have

$$\mathcal{M}_{cmplx}(X) \cong H^{n-1}(X, K_X \otimes \mathcal{T}^*X)^*. \tag{2.26}$$

Next we state the well known Ehresmann Lemma, and then by applying it to the family of complex manifolds we conclude a trivialization of them.

Theorem 2.6. (Ehresmann Lemma) *Let $\pi : \mathcal{X} \to S$ be a proper smooth submersion of differentiable manifolds over a contractible pointed base (S, o). Then there exists a diffeomorphism $T = (T_1, T_2) : \mathcal{X} \xrightarrow{\sim} X_o \times S$, such that $T_2 = \pi$.*

Hence the first component of trivialization T, i.e. $T_1 : \mathcal{X} \to X_o$, induces a diffeomorphism $X_s \cong X_o$, for any $s \in S$.
Now in [41, Proposition 9.5] we have the following theorem.

Theorem 2.7. *Let $\pi : \mathcal{X} \to S$ be a family of complex manifolds over a pointed base (S, o). Then up to replacing S by a neighborhood of the base point o, there exists a C^∞ trivialization $T = (T_1, \pi) : \mathcal{X} \to X_o \times S$, such that the fibers of T_1 are complex submanifolds of $\mathcal{X}$.*

Note that in the complex case the trivialization T, in general, is not holomorphic, but the diffeomorphismes $T_1 : X_s \xrightarrow{\sim} X_o$, $s \in S$, enable us to verify that the family of complex structures on X_o parameterized by S, i.e. X_s's, varies holomorphically with $s \in S$.
In the continue, we announce some results about a family of complex manifolds $\mathcal{X}$ in which X_o is a Kähler manifold. For more details one refers to [41, § 9.3].

Proposition 2.1. [41, Propositions 9.20 and 9.21] *Let $\pi : \mathcal{X} \to S$ be a family of complex manifolds that X_o is a Kähler manifold. Then for any $s \in S$, close enough to o, following hold:*

(i) $h^{p,q}(X_s) = h^{p,q}(X_o)$, *for suitable p and q's.*

(ii) $H^k_{\mathrm{dR}}(X_s; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_s)$.

Theorem 2.8. [41, Theorem 9.23] *Let $\pi : \mathcal{X} \to S$ be a family of complex manifolds. If X_o is a Kähler manifold, then for any $s \in S$, close enough to o, X_s is also a Kähler manifold.*

Definition 2.9. By a *family of Kähler manifolds* we mean a family of complex manifolds $\mathcal{X}$ over the pointed base (S, o) that any fiber X_s, $s \in S$, is a Kähler manifold.

Remark 2.2. (i) Considering Theorem 2.8, in the family of complex manifolds $\mathcal{X}$, up to a neighborhood of base point o, to have a family of Kähler manifolds it is enough that the fiber X_o be a Kähler manifold.

(ii) Note that in the family of complex manifolds $\mathcal{X}$ any fiber X_s is a compact complex manifold, and to further emphasize sometimes we use the *family of compact complex* (*Kähler*) *manifolds* instead of the family of complex (Kähler) manifolds.

(iii) Under hypothesis of Proposition 2.1, for any $s \in S$ near to o, $b_k(X_s) = b_k(X_o)$, and (2.24) gives the following decreasing Hodge filtration of $H^k_{\mathrm{dR}}(X_s; \mathbb{C})$

$$0 = F^{k+1}(X_s) \subset F^k(X_s) \subset \ldots \subset F^1(X_s) \subset F^0(X_s) = H^k_{\mathrm{dR}}(X_s; \mathbb{C}). \tag{2.27}$$

And also $\dim_{\mathbb{C}} F^j(X_s) = \dim_{\mathbb{C}} F^j(X_o)$, $0 \leq j \leq k+1$.

2.5 Gauss-Manin Connection and Griffiths Transversality

Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds. In this section first we are going to introduce a locally constant sheaf on S and then after presenting an integrable connection, which is known as *Gauss-Manin connection*, we conclude a real variation of Hodge structure on S.

Let $\underline{V_{\mathcal{X}}}$ be a constant sheaf on $\mathcal{X}$ for some group V, for example $V = \mathbb{C}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{Z}$. Then consider the sheaf $R^k\pi_*\underline{V_{\mathcal{X}}}$ on S, in which $R^k\pi_*$ refers to k-th derived functor of the pushforward (to know more about the derived functor one can see [36, § A.2.4]). By [13, Corollary 2.25] we have the following more understandable presentation of the stalks $(R^k\pi_*\underline{V_{\mathcal{X}}})_s$:

Theorem 2.9. *For any $s \in S$, one has the following isomorphism*

$$(R^k\pi_*\underline{V_{\mathcal{X}}})_s \simeq H^k(X_s, V).$$

In particular, in the case that $V = \mathbb{C}$, we have

$$(R^k \pi_* \underline{\mathbb{C}_{\mathcal{X}}})_s \simeq H^k(X_s, \mathbb{C}) \overset{(2.15)}{\simeq} H^k_{\mathrm{dR}}(X_s; \mathbb{C}),$$

hence $R^k \pi_* \underline{\mathbb{C}_{\mathcal{X}}}$ is a locally constant sheaf on S. Formally speaking, $R^k \pi_* \underline{\mathbb{C}_{\mathcal{X}}}$ is the sheaf associated to the presheaf $U \mapsto H^k(\pi^{-1}(U), \mathbb{C})$. In fact, for a contractible open subset $U \subset S$, by Ehresmann Lemma $\pi^{-1}(U) \cong U \times X_s$ for some $s \in S$, so $H^k(\pi^{-1}(U), \mathbb{C}) \simeq H^k(X_s, \mathbb{C})$. Thus $(R^k \pi_* \underline{\mathbb{C}_{\mathcal{X}}})|_U$ is just the constant sheaf with group $H^k_{\mathrm{dR}}(X_s; \mathbb{C})$. Hence, by Theorem 2.1, equivalently we can consider $R^k \pi_* \underline{\mathbb{C}_{\mathcal{X}}}$ as a local system on S. Next by following the process of constructing holomorphic vector bundle (2.5) from a local system, we define

$$H^k_{\mathrm{dR}}(\mathcal{X}/S) := R^k \pi_* \underline{\mathbb{C}_{\mathcal{X}}} \otimes_{\mathbb{C}} \mathcal{O}_S, \tag{2.28}$$

which is a holomorphic vector bundle on S, so by Theorem 2.2 there exist a unique integrable connection on $H^k_{\mathrm{dR}}(\mathcal{X}/S)$ such that the space of its horizontal local section is $R^k \pi_* \underline{\mathbb{C}_{\mathcal{X}}}$. And also, with a little neglect, we have

$$H^k_{\mathrm{dR}}(\mathcal{X}/S)_s \cong H^k_{\mathrm{dR}}(X_s; \mathbb{C}). \tag{2.29}$$

For more details one refers to [36, Prop-Def 10.24].

Definition 2.10. The holomorphic vector bundle $H^k_{\mathrm{dR}}(\mathcal{X}/S)$ defined in (2.28) is called *k-th relative de Rham cohomology group* and the unique integrable connection

$$\nabla^{\mathsf{GM}} : H^k_{\mathrm{dR}}(\mathcal{X}/S) \to \Omega^1_S \otimes_{\mathcal{O}_S} H^k_{\mathrm{dR}}(\mathcal{X}/S),$$

is said *Gauss-Manin connection*.

For a vector field v on S, consider the map

$$v \otimes \mathrm{Id} : \Omega^1_S \otimes_{\mathcal{O}_S} H^k_{\mathrm{dR}}(\mathcal{X}/S) \to H^k_{\mathrm{dR}}(\mathcal{X}/S),$$

then by composing the Gauss-Manin connection ∇^{GM} with $v \otimes \mathrm{Id}$ we define

$$\begin{aligned} \nabla^{\mathsf{GM}}_v :& H^k_{\mathrm{dR}}(\mathcal{X}/S) \to H^k_{\mathrm{dR}}(\mathcal{X}/S) \\ \nabla^{\mathsf{GM}}_v &= (v \otimes \mathrm{Id}) \circ \nabla^{\mathsf{GM}}. \end{aligned} \tag{2.30}$$

From now on, if there is no danger of confusion we denote the Gauss-Manin connection by ∇ instead of ∇^{GM}.

Observation 2.2. Proposition 2.1 implies that the k-th relative de Rham cohomology group $H^k_{\mathrm{dR}}(\mathcal{X}/S)$ is local free of finite rank, say m. By considering the locally frame $\varpi := \{\omega_j\}_{j=1}^m$, the same as Observation 2.1, we have the *Gauss-Manin matrix connection* which is denoted by GM_ϖ. More precisely if we consider the matrix presentation of ϖ, which we denote it also by ϖ, as follow

$$\varpi := \begin{pmatrix} \omega_1 & \omega_2 & \dots & \omega_m \end{pmatrix}^{\mathsf{t}},$$

in which t refers to the matrix transpose, then

$$\nabla\varpi := (\nabla\omega_1 \quad \nabla\omega_2 \quad \ldots \quad \nabla\omega_m)^{\mathsf{t}} = \mathsf{GM}_{\varpi} \otimes \varpi.$$

Note that for any $s \in S$ and any $j \in \{1, 2, \ldots, m\}$, $\omega_j(s) \in H^k_{\mathrm{dR}}(X_s; \mathbb{C})$ and we can present it by a k-form in X_s that we denote it also by $\omega_j(s)$.

Considering Remark 2.2**(iii)**, each fiber of $H^k_{\mathrm{dR}}(\mathcal{X}/S)$ has a Hodge filtration, and this yields a decreasing filtration of $H^k_{\mathrm{dR}}(\mathcal{X}/S)$ by holomorphic subbundles

$$\mathcal{F}^{\bullet}: \ 0 = \mathcal{F}^{k+1} \subset \mathcal{F}^k \subset \ldots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = H^k_{\mathrm{dR}}(\mathcal{X}/S), \tag{2.31}$$

such that for any $s \in S$ and any $p \in \{0, 1, 2, \ldots, k\}$,

$$\mathcal{F}^p_s \cong F^p(X_s) = \bigoplus_{p \le r \le k} H^{r,k-r}(X_s).$$

By definition, the filtration $\mathcal{F}^{\bullet}$ given in (2.31) is called *Hodge filtration of* $H^k_{\mathrm{dR}}(\mathcal{X}/S)$. However, one can still define the bundles $\mathcal{H}^{p,n-p} = \mathcal{F}^p/\mathcal{F}^{p+1}$ such that for any $s \in S$, $\mathcal{H}^{p,n-p}_s \cong H^{p,n-p}(X_s)$. And now to complete the real Hodge variation of S that we are looking for, it is enough to show the Griffiths' transversality which is given in the following theorem (see [18, Theorem 16.4]).

Theorem 2.10. (Griffiths' transversality) *Let $\pi : \mathcal{X} \to S$ be a family of compact Kähler manifolds and $H^k_{\mathrm{dR}}(\mathcal{X}/S)$ be its k-th relative de Rham cohomology group. If ∇ is the Gauss-Manin connection and $\mathcal{F}^{\bullet}$ is the Hodge filtration of $H^k_{\mathrm{dR}}(\mathcal{X}/S)$ given in* (2.31)*, then*

$$\nabla\mathcal{F}^p \subset \Omega^1_S \otimes \mathcal{F}^{p-1}, \ \ p = 1, 2, \ldots k.$$

2.6 Intersection Forms

In this section, we suppose that $\mathcal{X}$ is the family $\pi : \mathcal{X} \to S$ of n-dimensional compact Kähler manifolds. For any $\alpha, \omega \in H^n_{\mathrm{dR}}(\mathcal{X}/S)$, the *intersection form* of α and ω by definition is

$$\langle\alpha, \omega\rangle(s) := \mathrm{Tr}(\alpha(s) \smile \omega(s)), \ \forall s \in S,$$

in which "$\smile$" refers to the cup product. But in de Rham cohomology, the cup product of differential forms is induced by the wedge product, hence in the family $\mathcal{X}$ and its relative de Rham cohomology group defined by differential forms, the intersection form is defined as

$$\langle\alpha, \omega\rangle(s) = \int_{X_s} \alpha(s) \wedge \omega(s). \tag{2.32}$$

For more details one can see [14].

For a fixed $s \in S$, we know that X_s is an n-dimensional compact Kähler manifold. It is obvious that if $\eta, \nu \in H^n_{\mathrm{dR}}(X_s;\mathbb{C})$, then $\eta \wedge \nu = (-1)^n(\nu \wedge \eta)$, thus $\langle \eta, \nu\rangle = (-1)^n\langle \nu, \eta\rangle$. Also it is easy to see that

$$\eta \wedge \nu = 0, \text{ for any } \eta \in F^i(X_s)\ ,\ \nu \in F^j(X_s) \text{ with } i+j \geq n+1, \tag{2.33}$$

in which $F^\bullet(X_s)$ is the Hodge filtration of $H^n_{\mathrm{dR}}(X_s;\mathbb{C})$. To show this, first suppose that $(x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is a local chart of X_s, so since $F^i(X_s) = \bigoplus_{i\leq r\leq n} H^{r,n-r}(X_s)$, η is presented in this chart as follow

$$\eta = \sum_{\substack{\sharp I_\eta \geq i \\ \sharp J_\eta \leq n-i}} f_{I_\eta J_\eta} dx_{I_\eta} d\bar{x}_{J_\eta},$$

and similarly $\nu \in F^j(X_s)$ is locally presented as follow

$$\nu = \sum_{\substack{\sharp I_\nu \geq j \\ \sharp J_\nu \leq n-j}} g_{I_\nu J_\nu} dx_{I_\nu} d\bar{x}_{J_\nu},$$

thus,

$$\eta \wedge \nu = \sum_{\substack{\sharp I_\eta \geq i,\, \sharp J_\eta \leq n-i \\ \sharp I_\nu \geq j,\, \sharp J_\nu \leq n-j}} h_{I_\eta J_\eta I_\nu J_\nu} dx_{I_\eta} dx_{I_\nu} d\bar{x}_{J_\eta} d\bar{x}_{J_\nu}.$$

Now we know that $\sharp I_\eta + \sharp I_\nu \geq i+j \geq n+1$, hence $dx_{I_\eta} dx_{I_\nu} = 0$, and this completes the proof of (2.33). So we can state the following lemma.

Lemma 2.3. *Let $\mathcal{X}$ be a family of n-dimensional compact Kähler manifolds. Then following hold:*

(i) $\langle \alpha, \omega\rangle = (-1)^n\langle \omega, \alpha\rangle$, *for any* $\alpha, \omega \in H^n_{\mathrm{dR}}(\mathcal{X}/S)$.

(ii) *If $\mathcal{F}^\bullet$ is the Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{X}/S)$, then*

$$\langle \mathcal{F}^i, \mathcal{F}^j\rangle = 0, \ \textit{for} \ \ i+j \geq n+1. \tag{2.34}$$

Notation 2.2. Let $\mathcal{X}$ be the family of n-dimensional compact Kähler manifolds over the pointed base (S, o). By $\dim H^j_{\mathrm{dR}}(\mathcal{X}/S) = l$ we mean that $\dim_{\mathbb{C}} H^j_{\mathrm{dR}}(X_o;\mathbb{C}) = l$. Proposition 2.1 guaranties that, up to a neighborhood of o, for any $s \in S$, $\dim_{\mathbb{C}} H^j_{\mathrm{dR}}(X_s;\mathbb{C}) = l$. Also by $\dim \mathcal{F}^j = k$ we mean that $\dim_{\mathbb{C}} F^j(X_s) = k$, for any $s \in S$.

Remark 2.3. We are going to consider a special case with respect to the dimension of de Rham cohomology. Let $\mathcal{X}$ be the family of n-dimensional compact Kähler manifolds such that $\dim H^n_{\mathrm{dR}}(\mathcal{X}/S) = n+1$. Also suppose that $\dim \mathcal{F}^i = (n+1) - i,\ \ i = 0, 1, \ldots, n$, where $\mathcal{F}^\bullet$ is the Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{X}/S)$. Then it is an easy consequence that for any $s \in S$,

$\dim H^{p,q}(X_s)=1$ if $p+q=n$. So if we consider a local frame $\{\omega_i\}_{i=1}^{n+1}$ of $H^n_{\mathrm{dR}}(\mathcal{X}/S)$ with $\omega_i\in\mathcal{F}^{(n+1)-i}$, then we can define the *matrix of intersection forms* as follow

$$\Omega=(\Omega_{ij})_{1\leq i,j\leq n+1}:=(\langle\omega_i,\omega_j\rangle)_{1\leq i,j\leq n+1}\,.$$

By using of Lemma 2.3, in the case that n is an odd integer Ω is given as

$$\Omega=\begin{pmatrix} 0 & 0 & \cdots & 0 & \Omega_{1(n+1)} \\ 0 & 0 & \cdots & \Omega_{2n} & \Omega_{2(n+1)} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -\Omega_{2n} & \cdots & 0 & \Omega_{n(n+1)} \\ -\Omega_{1(n+1)} & -\Omega_{2(n+1)} & \cdots & -\Omega_{n(n+1)} & 0 \end{pmatrix}, \tag{2.35}$$

and in the case that n is an even integer Ω is presented as follow

$$\Omega=\begin{pmatrix} 0 & 0 & & \cdots & & 0 & \Omega_{1(n+1)} \\ 0 & 0 & & \cdots & & \Omega_{2n} & \Omega_{2(n+1)} \\ & & & & \iddots & & \\ \vdots & \vdots & & \Omega_{ll} & & \vdots & \vdots \\ & & \iddots & & & & \\ 0 & \Omega_{2n} & & \cdots & & \Omega_{nn} & \Omega_{n(n+1)} \\ \Omega_{1(n+1)} & \Omega_{2(n+1)} & & \cdots & & \Omega_{n(n+1)} & \Omega_{(n+1)(n+1)} \end{pmatrix}, \tag{2.36}$$

in which $l=\frac{n}{2}+1$.

Chapter 3

Picard-Fuchs Equation as a Self-Dual Linear Differential Equation

In the pervious chapter we introduced the Gauss-Manin connection. We saw that the composition of Gauss-Manin connection with a vector field yields a differential operator on the relative de Rham cohomology group of a family of complex manifolds. In this chapter we are going to study this operator and its generated linear differential equations. First in §3.1, we briefly state some basic facts related with differential operators. We give an algorithm to find the relationships among coefficients of a self-dual linear differential operator of an arbitrary degree. Next in §3.2, a special linear differential operator associated with a holomorphic n-form, which is called Picard-Fuchs equation, is presented. At the end of this chapter, in §3.3, after fixing some certain hypothesis on a family of complex manifolds, we prove that its relative de Rham cohomology group has a special type of frame, which we call it Yukawa frame. To do this we use the properties of coupling function that we prove them in Proposition 3.3 in context of de Rham cohomology. Also in Proposition 3.4, we give a relationship between the dimensions of the relative de Rham cohomology group and its Hodg filtration, which weakens our primary hypothesis.

The basic concepts of this chapter are discussed more detailed in [2, 3].

3.1 Differential Operators

In this section by R we mean a simple commutative differential ring, with quotient field k and derivative $(.)'$, and $R[\partial]$ is the ring of differential operators. We assume that the ring of constants C is an algebraically closed field of characteristic zero, such that k also has the same field of constants C and $k \neq C$.

In this section, for more details and proofs, one refers to [3].

Definition 3.1. A *differential R-module* is a pair (M, ∂), where M is a finitely generated R-module and $\partial : M \to M$ is a map satisfying

(i) $\partial(m+n) = \partial(m) + \partial(n)$ for every $m, n \in M$,

(ii) $\partial(fm) = f'm + f\partial(m)$ for every $f \in R$ and every $m \in M$.

A differential morphism between differential R-modules (M, ∂) and (N, ∂) is a morphism of R-modules $\psi : M \to N$ satisfying

$$\partial\psi(m) = \psi(\partial m)$$

for every $m \in M$.

Definition 3.2. Let (M, ∂) be a differential k-module. Then for each $m \in M$ we define the *evaluation map* $ev_m : k[\partial] \to M$ by

$$\sum_{i=0}^{n} a_i \partial^i \mapsto \sum_{i=0}^{n} a_i \partial^i m.$$

The monic generator of the kernel of ev_m as a left ideal is called the *minimal operator* of m over $k[\partial]$. Furthermore, we call m a *cyclic vector* of M if the degree of its minimal operator equals the k-dimension of M, i.e. the set $\{m, \partial m, ..., \partial^{\dim_k(M)-1} m\}$ is a k-basis of M. We call a pair (M, e) consisting of a differential module M and a cyclic vector $e \in M$ a *marked differential module.*

By a result due to N. Katz (see [39]), there is a one to one correspondence between monic differential operators $L \in k[\partial]$ and marked differential modules (M, e).

Proposition 3.1. *If the field of constants C of k is algebraically closed, then each differential k-module M has a cyclic vector. In particular, there is a differential operator $L \in k[\partial]$ such that M is isomorphic to $k[\partial]/k[\partial]L$.*

In the continue of this section, we suppose that $k = \mathbb{C}(z)$, the operator ∂ is the usual derivation $\frac{\partial}{\partial z}$ or logarithmic derivation $z\frac{\partial}{\partial z}$. Note that it is seen $k[\frac{\partial}{\partial z}]$ and $k[z\frac{\partial}{\partial z}]$ are isomorphic, so we can freely switch between these two differential rings. Also we assume that

$$L = \partial^{n+1} + \sum_{i=0}^{n} a_i \partial^i \in \mathbb{Q}(z)[\partial],$$

be an irreducible monic differential operator and

$$(M_L, e) \cong (\mathbb{C}(z)[\partial]/\mathbb{C}(z)[\partial]L, [1]), \tag{3.1}$$

be its corresponding marked differential $\mathbb{C}(z)$-module.

Definition 3.3. It is said that L satisfies *property* (P), if there is a non-degenerate form $\langle ., . \rangle : M_L \times M_L \to \mathbb{C}(z)$ such that

(i) $\langle ., . \rangle$ is a $(-1)^n$-symmetric form, i.e. $\langle ., . \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]}(\mathrm{Sym}^2 M_L, \mathbb{C}(z))$ if n is even, and $\langle ., . \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]}(\bigwedge^2 M_L, \mathbb{C}(z))$ if n is odd.

(ii) $\langle e, \partial^i e\rangle = 0$ for $i = 0, 1, \ldots, n-1$.

Following we state a proposition that gives an equivalence condition for property (P) and for proof see [3]. Note that for $\psi \in \mathbb{C}(z)$, the operator $\partial\psi$ is given as follow,

$$\partial\psi = \partial(\psi) + \psi\partial,$$

and for convenient we denote $\psi' = \partial(\psi)$, so $\psi^{(i)} = \underbrace{\partial(\partial(\ldots(\partial}_{i-\text{times}}(\psi))\ldots))$. Before stating the proposition, we give the following definition.

Definition 3.4. Given an n-th order linear differential operator

$$L = \sum_{i=0}^{n} a_i(z)\partial^i \in k[\partial], \tag{3.2}$$

its *dual operator* $\check{L}$ is given by

$$\check{L} = \sum_{i=0}^{n} (-1)^{n-i}\partial^i a_i, \tag{3.3}$$

in which, $\partial a_i = a_i' + a_i\partial,\ i = 0, 1, \ldots, n$.

Proposition 3.2. *The operator L satisfies the property* (P) *if and only if L is self-dual, i.e. there is an $0 \neq \psi \in \mathbb{C}(z)$, such that*

$$L\psi = \psi\check{L}. \tag{3.4}$$

Using Proposition 3.2, we prove the following lemma that gives existence relationships among coefficients of L.

Lemma 3.1. *Let $L = \sum_{i=0}^{n+1} a_i\partial^i$, with $a_{n+1} = 1$, be a linear differential operator satisfying property* (P). *Then the following hold:*

(i) *If $n = 2$, then*

$$a_0 = \frac{1}{3}a_1a_2 - \frac{1}{3}a_2a_2' - \frac{2}{27}a_2^3 + \frac{1}{2}a_1' - \frac{1}{6}a_2''. \tag{3.5}$$

(ii) *If $n = 3$, then*

$$a_1 = -\frac{3}{4}a_3a_3' + a_2' - \frac{1}{2}a_3'' - \frac{1}{8}a_3^3 + \frac{1}{2}a_2a_3. \tag{3.6}$$

(iii) *If $n = 5$, then*

$$a_3 = \frac{2}{3}a_4a_5 - \frac{5}{3}a_5a_5' - \frac{5}{27}a_5^3 - \frac{5}{3}a_5'' + 2a_4', \tag{3.7}$$

$$\begin{aligned} a_1 = {} & a_2' - a_4''' + a_5^{(4)} - a_4^{(2)}a_5 - a_4'a_5' + \frac{5}{3}a_5(a_5')^2 + \frac{1}{3}a_2a_5 \\ & - \frac{1}{27}a_4a_5^3 + \frac{10}{27}a_5^3a_5' + \frac{1}{81}a_5^5 - \frac{1}{3}a_4'a_5^2 - \frac{1}{3}a_4a_5a_5' + \frac{10}{9}a_5^2a_5'' \\ & + \frac{10}{3}a_5'a_5'' - \frac{1}{3}a_4a_5'' + \frac{5}{3}a_5a_5'''. \end{aligned} \tag{3.8}$$

Proof. We prove **(iii)**, and the proofs of **(i)** and **(ii)** are given similarly. First we compute $L\psi$, for $\psi \in \mathbb{C}(z) \setminus \{0\}$, as follow

$$L\psi = \sum_{i=0}^{6} \left(\sum_{j=i}^{6} \begin{pmatrix} j \\ i \end{pmatrix} a_j \psi^{(j-i)} \right) \partial^i. \tag{3.9}$$

Next we find $\check{L}$,

$$\check{L} = \sum_{i=0}^{6} \left(\sum_{j=i}^{6} (-1)^{6-j} \begin{pmatrix} j \\ i \end{pmatrix} a_j^{(j-i)} \right) \partial^i, \tag{3.10}$$

hence,

$$\psi\check{L} = \sum_{i=0}^{6} \psi \left(\sum_{j=i}^{6} (-1)^{6-j} \begin{pmatrix} j \\ i \end{pmatrix} a_j^{(j-i)} \right) \partial^i. \tag{3.11}$$

Since L satisfies the property (P), by Proposition 3.2

$$L\psi = \psi\check{L}. \tag{3.12}$$

By comparing coefficients of ∂^5 in (3.9) and (3.11) we find

$$\psi' = -\frac{1}{3} a_5 \psi, \tag{3.13}$$

and then using (3.13), we compute $\psi^{(i)}$'s, i=2,3,. . . ,6, in terms of ψ and a_5, and substitute them in (3.9). Finally (3.7) and (3.8), resp., follow from comparing the coefficients of ∂^3 and ∂, resp., in (3.12). ■

Next we give the general theory of property (P) based on an algorithm to find the existence relationships among coefficients. Let $L = \sum_{i=0}^{n+1} a_i \partial^i$, with $a_{n+1} = 1$, be a linear differential operator satisfying property (P). Suppose that $n = 2m$ or $2m+1$, for a positive integer m. Then m coefficients $a_{n-2}, a_{n-4}, \ldots, a_{n-2m}$ depend to the rest of coefficients and their derivations. First one by induction can easily verify that for $\psi \in \mathbb{C}(z)$,

$$\partial^j \psi = \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} \psi^{(j-i)} \partial^i.$$

So it follows that

$$\check{L} = \sum_{i=0}^{n+1} \left(\sum_{j=i}^{n+1} (-1)^{n+1-j} \begin{pmatrix} j \\ i \end{pmatrix} a_j^{(j-i)} \right) \partial^i,$$

and

$$L\psi = \sum_{i=0}^{n+1} \left(\sum_{j=i}^{n+1} \begin{pmatrix} j \\ i \end{pmatrix} a_j \psi^{(j-i)} \right) \partial^i.$$

Hence if we substitute them in $L\psi = \psi\check{L}$, then we have:

$$\sum_{i=0}^{n+1} \left(\sum_{j=i}^{n+1} \begin{pmatrix} j \\ i \end{pmatrix} a_j \psi^{(j-i)} \right) \partial^i = \sum_{i=0}^{n+1} \psi \left(\sum_{j=i}^{n+1} (-1)^{n+1-j} \begin{pmatrix} j \\ i \end{pmatrix} a_j^{(j-i)} \right) \partial^i. \tag{3.14}$$

Now by comparing the coefficient of ∂^n in (3.14) we find ψ' and its derivations in terms of ψ and a_5 and derivations of a_n as follows

$$\begin{aligned}
\psi' &= -\frac{2}{n+1} a_n \psi, \\
\psi'' &= \left((-\frac{2}{n+1})^2 a_n^2 - \frac{2}{n+1} a_n' \right) \psi, \\
\psi''' &= \left((-\frac{2}{n+1})^3 a_n^3 + 3(-\frac{2}{n+1})^2 a_n a_n' - \frac{2}{n+1} a_n'' \right) \psi, \\
&\vdots
\end{aligned}$$

and substitute them in the left side of (3.14). Therefore to express a_{n-2k}, $k = 1, 2, \ldots, m$, as equation of $a_n, a_{n-1}, a_{n-3}, \ldots, a_{n-(2k-1)}$ and their derivations it is enough to compare the coefficient of ∂^{n-2k} of both sides of (3.14), i.e.,

$$\sum_{j=n-2k}^{n+1} \begin{pmatrix} j \\ n-2k \end{pmatrix} a_j \psi^{(j-(n-2k))} = \left(\sum_{j=n-2k}^{n+1} (-1)^{n+1-j} \begin{pmatrix} j \\ n-2k \end{pmatrix} a_j^{(j-(n-2k))} \right) \psi,$$

thus,

$$\begin{aligned}
2a_{n-2k} = \sum_{j=n-2k+1}^{n} (-1)^{n+1-j} \begin{pmatrix} j \\ n-2k \end{pmatrix} a_j^{(j-(n-2k))} \\
- \sum_{j=n-2k+1}^{n+1} \begin{pmatrix} j \\ n-2k \end{pmatrix} a_j (\psi^{(j-(n-2k))}/\psi).
\end{aligned}$$

For example when $k = 1$, then a_{n-2} is given as follow,

$$a_{n-2} = \frac{n-1}{n+1} a_{n-1} a_n - \frac{n(n-1)}{2(n+1)} a_n a_n' - \frac{n(n-1)}{3(n+1)^2} a_n^3 + \frac{(n-1)}{2} a_{n-1}' - \frac{1}{12} n(n-1) a_n'',$$

and one can see the truth of Lemma 3.1(i),(ii) by using of this equation.

3.2 Picard-Fuchs Equation

In this section $\mathcal{X}$ refers to a family of n-dimensional compact Kähler manifolds on a pointed base (S, o). Fix the local section $\omega \in \mathcal{F}^n$ at o, in which $\mathcal{F}^\bullet$ is the Hodge filtration of $H^n(\mathcal{X}/S)$. In fact for any $s \in S$, $\omega(s) \in H^{n,0}(X_s)$ is a holomorphic n-form. Let $\mathcal{D}$ be the ring of linear differential operators on S. If $\dim S = r$ and $z_1, z_2, \ldots, z_r$ is a local coordinate of (S, o), then we have

$$\mathcal{D} = \mathbb{C}\{z_1, z_2, \ldots, z_r\}[\partial_1, \partial_2, \ldots, \partial_r], \tag{3.15}$$

where $\mathbb{C}\{z_1, z_2, \ldots, z_r\}$ is the ring of convergent power series of $z_1, z_2, \ldots, z_r$ and $\partial_i = \frac{\partial}{\partial z_i}$. Now by using the composition of Gauss-Manin connection ∇ with a vector field on S given in (2.30), we define the $\mathcal{O}_S$-homomorphism $\Psi : \mathcal{D} \to H^n_{\mathrm{dR}}(\mathcal{X}/S)$ which for vector fields $v_1, v_2, \ldots, v_k$ is determined by

$$\Psi(v_1 v_2 \ldots v_k) = \nabla_{v_1} \nabla_{v_2} \ldots \nabla_{v_k} \omega.$$

By this definition, Ψ gives the structure of a $\mathcal{D}$-module to $H^n_{\mathrm{dR}}(\mathcal{X}/S)$.

Definition 3.5. The ideal $\mathcal{I} = \ker \Psi$, consist of the differential operators that annihilate ω, by definition is called *Picard-Fuchs ideal* and any $L \in \mathcal{I}$ is called a *Picard-Fuchs equation.*

Definition 3.6. We say that $\mathcal{X}$ is a *one parameter family* of n-dimensional compact Kähler manifolds if $\dim S = 1$.

Now suppose that $\mathcal{X}$ is a one parameter family of n-dimensional compact Kähler manifolds. Let z be a coordinate of S and define the differential operator $\partial := \nabla_{\frac{\partial}{\partial z}}$. Then $(H^n_{\mathrm{dR}}(\mathcal{X}/S), \partial)$ is a differential $\mathbb{C}(z)$-module. Considering the terminologies introduced in §3.1, we present the following definition of Picard-Fuchs equation.

Definition 3.7. Let $\mathcal{X}$ be a one parameter family of n-dimensional compact Kähler manifolds and $\omega \in H^n_{\mathrm{dR}}(\mathcal{X}/S)$ be a fixed non-zero element. Then the minimal operator of ω is called the *Picard-Fuchs equation* of ω.

The Proposition 3.1 guaranties the existence of a cyclic vector in $H^n_{\mathrm{dR}}(\mathcal{X}/S)$, i.e. there exist an $\omega \in H^n_{\mathrm{dR}}(\mathcal{X}/S)$ such that

$$\{\omega, \partial\omega, \ldots, \partial^{l-1}\omega\}, \tag{3.16}$$

construct a frame for $H^n_{\mathrm{dR}}(\mathcal{X}/S)$ where $l = \dim H^n_{\mathrm{dR}}(\mathcal{X}/S)$ and

$$\partial^i \omega = \underbrace{\nabla_{\frac{\partial}{\partial z}} \nabla_{\frac{\partial}{\partial z}} \ldots \nabla_{\frac{\partial}{\partial z}}}_{i-\text{times}} \omega. \tag{3.17}$$

Observation 3.1. In the one parameter family of n-dimensional compact Kähler manifolds $\mathcal{X}$, if we define the differential operator $\vartheta := \nabla_{z\frac{\partial}{\partial z}}$, then $\vartheta = z\partial$. What we stated above about the operator ∂, is valid for the operator ϑ. In particular, there exist

$\omega \in H^n_{\mathrm{dR}}(\mathcal{X}/S)$ such that $\{\omega, \vartheta\omega, \ldots, \vartheta^n\omega\}$ construct a frame for $H^n_{\mathrm{dR}}(\mathcal{X}/S)$. Considering $\vartheta^{n+1}\omega \in H^n_{\mathrm{dR}}(\mathcal{X}/S)$, there are rational holomorphic functions $a_i(z)$'s, $i = 0, 1, \ldots, n$, such that

$$\vartheta^{n+1}\omega = a_0(z)\omega + a_1(z)\vartheta\omega + \ldots + a_n(z)\vartheta^n\omega.$$

Thus if we define

$$L = \vartheta^{n+1} - a_0(z) - a_1(z)\vartheta + \ldots - a_n(z)\vartheta^n,$$

then $L\omega = 0$. In the other word, L is the Picard-Fuchs equation of ω.

What we are interesting more is that $\omega \in H^n_{\mathrm{dR}}(\mathcal{X}/S) \cap \mathcal{F}^n$, and we will study it in the next section.

3.3 Self-Duality

In this section we follow some terminologies and results given in [2, § 4.5] and we state some new results that we will use them future. First we fix the following assumptions.

Assumption 3.1. During the whole of this section we are working with the family $\mathcal{X}$ satisfying followings:

(i) $\pi : \mathcal{X} \to S$ is a one parameter family of n-dimensional compact Kähler manifolds, i.e. $\dim S = 1$.

(ii) z is a local coordinate of S at the point $o \in S$, i.e $z(o) = 0$, and by notation ϑ refers either to the differential operator $\nabla_{z\frac{\partial}{\partial z}}$ if it operate on the elements of $H^n_{\mathrm{dR}}(\mathcal{X}/S)$, or to $z\frac{\partial}{\partial z}$ if it acts on the elements of $\mathbb{C}(z)$. Also by definition we consider $\vartheta^0 = 1$ to be the identity differential operator.

(iii) For $i \in \{0, 1, \ldots, n\}$, $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1$, where $\mathcal{F}^\bullet$ is the Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{X}/S)$, or equivalently $\dim \mathcal{F}^i = (n+1) - i$. In the other word, $h^{i,j}(X_s) = 1$ for any $s \in S$ and any non-negative integers i, j with $i + j = n$. So $\dim H^n_{\mathrm{dR}}(\mathcal{X}/S) = n + 1$.

(iv) $\omega \in \mathcal{F}^n$ is a nowhere vanishing holomorphic n-form satisfying $L\omega = 0$, in which L is the following Picard-Fuchs equation

$$L = \vartheta^{n+1} + a_n(z)\vartheta^n + \ldots + a_1(z)\vartheta + a_0(z), \tag{3.18}$$

where $a_i(z) \in \mathbb{Q}(z)$, $i = 0, 1, \ldots, n$.

Remark 3.1. Under hypothesis **(i), (ii), (iii)** of Assumption 3.1, one can see in [2] that if the family $\mathcal{X}$ has maximal unipotent monodromy at $z = 0$, then **(iv)** of Assumption 3.1 holds.

Notation 3.1. By notation, for $i = 1, \ldots, n + 1$, we define ω_i as follow

$$\omega_i := \vartheta^{i-1}\omega. \tag{3.19}$$

The aim of this section is to prove that $\{\omega_1, \omega_2, \dots, \omega_{n+1}\}$ construct a frame for $H^n_{\mathrm{dR}}(\mathcal{X}/S)$ and for any $i \in \{1, 2, \dots, n+1\}$,

$$\omega_i \in \mathcal{F}^{(n+1)-i} \setminus \mathcal{F}^{(n+2)-i}. \tag{3.20}$$

Note that by Griffiths' transversality we know

$$\omega_i \in \mathcal{F}^{(n+1)-i}. \tag{3.21}$$

Definition 3.8. For any non-negative integers k, l, we define the *coupling functions* $\mathcal{X}_{k,l}(z)$ as follow

$$\mathcal{X}_{k,l}(z) = \int_{X_z} \vartheta^k\omega(z) \wedge \vartheta^l\omega(z) = \int_{X_z} \omega_{k+1}(z) \wedge \omega_{l+1}(z).$$

The coupling function $\mathcal{X}_{n,0}$ is called *n-point Yukawa function.*

In fact, the coupling function $\mathcal{X}_{k,l}$ is the intersection form of ω_{k+1} and ω_{l+1}, i.e.

$$\mathcal{X}_{k,l}(z) = \langle \omega_{k+1}, \omega_{l+1} \rangle(z).$$

Proposition 3.3. *Following properties hold for coupling functions $\mathcal{X}_{k,l}(z)$:*

(i) $\mathcal{X}_{k,l}(z) = (-1)^n \mathcal{X}_{l,k}(z)$.

(ii) $\mathcal{X}_{k,l}(z) = 0$ *for* $k + l < n$.

(iii) $\vartheta\mathcal{X}_{k,l}(z) = \mathcal{X}_{k+1,l}(z) + \mathcal{X}_{k,l+1}(z)$.

(iv) $\mathcal{X}_{n,0}(z) = (-1)^i \mathcal{X}_{n-i,i}(z),\ i = 0, 1, 2, \dots, n$.

(v) $\mathcal{X}_{n+k+1,0}(z) + a_n(z)\mathcal{X}_{n+k,0}(z) + \cdots + a_0(z)\mathcal{X}_{k,0}(z) = 0$.

Proof.

(i),(ii) Directly follow from Lemma 2.3 and (3.21).

(iv) By **(ii)**, $\mathcal{X}_{n-i-1,i}(z) = 0$, so $\vartheta\mathcal{X}_{n-i-1,i}(z) = 0$, hence**(iii)** implies

$$\mathcal{X}_{n-j,j}(z) + \mathcal{X}_{n-j-1,j+1}(z) = 0 \ \text{ for } j = 0, 1, \dots, n, \tag{3.22}$$

thus,

$$\mathcal{X}_{n,0} \overset{j=0}{=} (-1)^1\mathcal{X}_{n-1,1} \overset{j=1}{=} (-1)^2\mathcal{X}_{n-2,2} = \dots \overset{j=i-1}{=} (-1)^i\mathcal{X}_{n-i,i}.$$

(v) By denoting $A = \mathcal{X}_{n+k+1,0}(z) + a_n(z)\mathcal{X}_{n+k,0}(z) + \cdots + a_0(z)\mathcal{X}_{k,0}(z)$, we have

$$\begin{aligned} A &= \int_{X_z} \vartheta^k(\vartheta^{n+1}\omega + a_n(z)\vartheta^n\omega + \ldots + a_0(z)\omega)(z) \wedge \omega(z) \\ &= \int_{X_z} \vartheta^k(L\omega)(z) \wedge \omega(z) \overset{L\omega=0}{=} 0. \end{aligned}$$

■

Theorem 3.1. *The n-point Yukawa function $\mathcal{X}_{n,0}(z)$ satisfies the following first order linear differential equation*

$$\vartheta\mathcal{X}_{n,0}(z) + \frac{2}{n+1}a_n(z)\mathcal{X}_{n,0}(z) = 0. \tag{3.23}$$

Proof. By Proposition 3.3**(iv)** we have

$$\mathcal{X}_{n,0}(z) = (-1)^i\mathcal{X}_{n-i,i}, \tag{3.24}$$

and Proposition 3.3**(iii)** implies

$$\vartheta\mathcal{X}_{n-i,i} = \mathcal{X}_{n-i+1,i}(z) + \mathcal{X}_{n-i,i+1}(z),\ i = 0, 1, \ldots, n. \tag{3.25}$$

It follows from (3.22), (3.24) and (3.25) that for $k = 1, 2, \ldots, n+1$

$$k\vartheta\mathcal{X}_{n,0}(z) = \sum_{i=0}^{k-1}(-1)^i\vartheta\mathcal{X}_{n-i,i}(z) = \mathcal{X}_{n+1,0}(z) + (-1)^{k-1}\mathcal{X}_{n-k+1,k}(z). \tag{3.26}$$

Depending to n, we prove the theorem in the following two cases:

Case 1. n is odd. By Proposition 3.3**(i)**, $\mathcal{X}_{\frac{n+1}{2},\frac{n+1}{2}}(z) = 0$. Then using (3.26) for $k = (n+1)/2$, we obtain

$$\frac{(n+1)}{2}\vartheta\mathcal{X}_{n,0}(z) = \mathcal{X}_{n+1,0}(z). \tag{3.27}$$

In Proposition 3.3, **(ii)** and **(v)** implies

$$\mathcal{X}_{n+1,0}(z) = -a_n(z)\mathcal{X}_{n,0}(z) \tag{3.28}$$

Substituting $\mathcal{X}_{n+1,0}(z)$ from (3.28) in (3.27) completes the proof of (3.23).

Case 2. n is even. By Proposition 3.3**(i)**, $\mathcal{X}_{n+1,0}(z) = \mathcal{X}_{0,n+1}(z)$. So using (3.26) for $k = n+1$, we obtain

$$(n+1)\vartheta\mathcal{X}_{n,0}(z) = 2\mathcal{X}_{n+1,0}(z).$$

Now we are in the same situation (3.27) of **Case 1**, and analogously the proof is complete. ■

Corollary 3.1. *The n-point Yukawa function $\mathcal{X}_{n,0}(z)$, explicitly is expressed as follow*

$$\mathcal{X}_{n,0}(z) = c_0\exp\left(-\frac{2}{n+1}\int_0^z a_n(v)\frac{dv}{v}\right),$$

for some nonzero constant $c_0 = \mathcal{X}_{n,0}(0)$.

Proof. The proof follows directly from differential equation (3.23). ■

Example 3.1. Assume that $L = \vartheta^4 - z(\vartheta+1/5)(\vartheta+2/5)(\vartheta+3/5)(\vartheta+4/5)$ be the Picard-Fuchs differential equation of the family of mirror quintic 3-folds in $\mathbb{P}^4$ (see Example 4.2). In the other word, we can rewrite L as follow

$$L = \vartheta^4 - \frac{2z}{1-z}\vartheta^3 - \frac{5}{7}\frac{z}{1-z}\vartheta^2 - \frac{2}{5}\frac{z}{1-z}\vartheta - \frac{24}{625}\frac{z}{1-z}.$$

Then the Yukawa 3-point function $\mathcal{X}_{3,0}(z)$ is given as

$$\mathcal{X}_{3,0}(z) = c_0\frac{1}{1-z}$$

and in [8] c_0 is computed

$$c_0 = \frac{1}{5^7}.$$

Corollary 3.2. *If we denote by* $\tilde{a}(z) = c_0 \exp\left(-\frac{2}{n+1}\int_0^z a_n(v)\frac{dv}{v}\right)$*, then for any* $i \in \{1,\dots,n\}$

$$\langle \omega_i, \omega_{n+2-i}\rangle = (-1)^{i-1}\tilde{a}. \tag{3.29}$$

Proof. Proposition 3.3**(iv)** and Corollary 3.1 directly verify (3.29). ■

Remark 3.2. Following the notation of Corollary 3.2, since $\tilde{a}(z) \neq 0$ for any z,

$$\langle \omega_i, \omega_{n+2-i}\rangle(z) \neq 0, \ \forall i \in \{0,1,\dots,n\}.$$

Theorem 3.2. *The set* $\{\omega_1, \omega_2, \dots, \omega_{n+1}\}$ *construct a frame for* $H^n_{\rm dR}(\mathcal{X}/S)$ *and for any* $i \in \{1,2,\dots,n+1\}$,

$$\omega_i \in \mathcal{F}^{(n+1)-i} \setminus \mathcal{F}^{(n+2)-i}. \tag{3.30}$$

Proof. We know that $\dim H^n_{\rm dR}(\mathcal{X}/S) = n+1$, so it is enough to show that for any z, the set $\{\omega_1(z), \omega_2(z), \dots, \omega_{n+1}(z)\}$ is linearly independent. To do this suppose that there are meromorphic functions $b_1, b_2, \dots, b_{n+1}$ such that

$$b_1(z)\omega_1(z) + b_2(z)\omega_2(z) + \ldots + b_{n+1}(z)\omega_{n+1}(z) = 0, \tag{3.31}$$

and let

$$k = \max\{i \,|\, b_i(z) \neq 0,\, i = 1,2,\dots,n+1\}.$$

Then we can write

$$\omega_k(z) = c_1(z)\omega_1(z) + c_2(z)\omega_2(z) + \ldots + c_{k-1}(z)\omega_{k-1}(z),$$

in which $c_i(z) = \frac{b_i}{b_k}(z)$. By intersecting ω_k with ω_{n+2-k} and using Proposition 3.3**(ii)** we have

$$\langle \omega_k, \omega_{n+2-k}\rangle(z) = c_1(z)\langle \omega_1, \omega_{n+2-k}\rangle(z) + \ldots + c_{k-1}(z)\langle \omega_{k-1}, \omega_{n+2-k}\rangle(z) = 0.$$

But by Remark 3.2 we know that $\langle\omega_k, \omega_{n+2-k}\rangle(z) \neq 0$, which is an contradiction. Therefore for any z and any $i \in \{1, 2, \ldots, n+1\}$, $b_i(z) = 0$, that says $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ is a linearly independent subset.
To prove (3.30), first note that Griffiths' transversality implies that $\omega_i \in \mathcal{F}^{(n+1)-i}$, $i = 1, 2, \ldots, n+1$. On the other hand, since $\dim \mathcal{F}^{(n+2)-i} = i-1$, and $\{\omega_1, \omega_2, \ldots, \omega_i\}$ is an independent subset of $H^n_{\rm dR}(\mathcal{X}/S)$, $\omega_i \notin \mathcal{F}^{(n+2)-i}$. ■

Definition 3.9. The frame $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ is called *Yukawa frame* of $H^n_{\rm dR}(\mathcal{X}/S)$.

Considering the terminologies introduced in §3.1, $(H^n_{\rm dR}(\mathcal{X}/S), \vartheta)$ is a differential $\mathbb{C}(z)$-module and $(H^n_{\rm dR}(\mathcal{X}/S), \omega)$ is a marked differential module and the equation L given in (3.18) is the minimal operator of ω.

Theorem 3.3. *The picard-Fuchs equation L satisfies the property* (P)*, or equivalently L is self dual, i.e., there is a non-zero $\psi \in \mathbb{C}(z)$ such that,*

$$L\psi = \psi\check{L}, \tag{3.32}$$

where $\check{L}$ is the dual operator of L.

Proof. Consider the intersection form defined as follow,

$$\langle ., . \rangle : H^n_{\rm dR}(\mathcal{X}/S) \times H^n_{\rm dR}(\mathcal{X}/S) \to \mathbb{C}(z).$$

Remark 3.2 implies that $\langle ., . \rangle$ is non-degenerate, and the Lemma 2.3**(i)** verifies that $\langle ., . \rangle$ is a $(-1)^n$-symmetric form. Also Lemma 2.3**(ii)** guaranties that in Yukawa frame $\{\omega, \vartheta\omega, \ldots, \vartheta^n\omega\}$ we have,

$$\langle \omega, \vartheta^i\omega \rangle = 0, \text{ for } i = 0, 1, \ldots, n-1.$$

Hence by Definition 3.3, L satisfies the property (P). And finally (3.32) follows directly from Proposition 3.2. ■

In the following proposition we give an equivalent statement of Assumption 3.1**(iii)**.

Proposition 3.4. *Let $\pi : \mathcal{X} \to S$ be a one parameter family of n-dimensional compact Kähler manifolds, z be a local coordinate of S at the point $o \in S$, and $\omega \in \mathcal{F}^n$ be a nowhere vanishing holomorphic n-form satisfying $L\omega = 0$, where L is the following Picard-Fuchs equation*

$$L = \vartheta^{n+1} + a_n(z)\vartheta^n + \ldots + a_1(z)\vartheta + a_0(z), \tag{3.33}$$

in which $a_i(z) \in \mathbb{Q}(z)$, $i = 0, 1, \ldots, n$. Then $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1$ for any $i \in \{0, 1, \ldots, n\}$, if and only if $\dim H^n_{\rm dR}(\mathcal{X}/S) = n+1$.

Proof. If $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1$, then it is obvious that $\dim H^n_{\rm dR}(\mathcal{X}/S) = n+1$. Conversely suppose that $\dim H^n_{\rm dR}(\mathcal{X}/S) = n+1$. Then it is enough to prove that $\dim \mathcal{F}^i/\mathcal{F}^{i+1} \neq 0$. The same as before, for $i = 1, \ldots, n+1$ define $\omega_i := \vartheta^{i-1}\omega$ and Griffiths' transversality gives $\omega_i \in \mathcal{F}^{(n+1)-i}$. One can repeat the proof of Theorem 3.23 to see that $\langle \omega_1, \omega_{n+1} \rangle \neq 0$, hence Proposition 3.3**(ii)** implies that

$$\omega_{n+1} \in \mathcal{F}^0 \setminus \mathcal{F}^1. \tag{3.34}$$

To prove $\dim \mathcal{F}^i/\mathcal{F}^{i+1} \neq 0$, by contradiction suppose that there is a $j \in \{1, 2, 3, \ldots, n-1\}$ such that $\dim \mathcal{F}^j/\mathcal{F}^{j+1} = 0$, and hence $\mathcal{F}^{j+1} = \mathcal{F}^j$. We know $\omega_{(n+1)-j} \in \mathcal{F}^j$, so by Griffiths' transversality $\omega_{(n+1)-j+1} = \vartheta\omega_{(n+1)-j} \in \mathcal{F}^{j+1} = \mathcal{F}^j$. Again using Griffiths' transversality we see that $\omega_{(n+1)-j+2} \in \mathcal{F}^j$, and by continuing this process it follows that $\omega_{n+1} \in \mathcal{F}^j$, which contradicts (3.34). Therefore $\dim \mathcal{F}^i/\mathcal{F}^{i+1} \neq 0$. ■

Chapter 4

Calabi-Yau Manifolds

As one can see in the title of this thesis, the space that we are working on is a Calabi-Yau manifold. In this chapter we are going to recall fundamental definitions and facts related to Calabi-Yau manifolds. Here we first announce Calabi-Yau theorem and then we review the equivalence definitions of Calabi-Yau manifolds that are used in different contexts. In §4.1, once we fix the definition of Calabi-Yau manifold, some primary examples and classifications of low dimensional Calabi-Yau manifolds are given. Finally in §4.2 we review the properties of the families of Calabi-Yau manifolds and also some more examples of families of Calabi-Yau manifolds are provided.

The name of *Calabi-Yau manifold* comes from the Calabi-Yau Theorem. In 1954 first E. Calabi [6, 7] proposed his conjecture, and could prove a part of it, then in 1976 S. T. Yau [42] completed the proof of Calabi's conjecture. After that in 1985 these manifolds were named "Calabi-Yau" by Candelas et al [9]. Before stating the Calabi-Yau Theorem, we need to know some preliminary concepts that for more details one refers to [25].

Let (W, ω, g, J) be an n-dimensional compact Kähler manifold. From Riemannian geometry we know the Levi-Civita connection and Riemann curvature tensor R with respect to the Riemannian metric g. In index notation, we show the curvature tensor by R^a_{bcd}, and by definition $R_{abcd} := g_{ae}R^e_{bcd}$. The *Ricci curvature* tensor R_{ab} of g by definition is $R_{ab} := R^c_{acb}$. It is seen that $R_{ab} = R_{ba}$. Then the *Ricci form* ρ is defined to be $\rho_{ac} := J^b_a R_{bc}$ and it is seen that $\rho_{ac} = -\rho_{ca}$, and so that ρ is a 2-form. In fact, ρ is a closed real $(1,1)$-form. It is said that W is *Ricci-flat* if $\rho \equiv 0$.

Next we are going to introduce the first Chern class of W denoted by $c_1(W)$. To do this, note that the exact sequence of sheaves

$$0 \to \mathcal{Z} \overset{2\pi i}{\to} \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0$$

gives a boundary map on cohomology $H^1(W, \mathcal{O}^*) \overset{\delta}{\to} H^2(W, \mathcal{Z})$, where $\mathcal{Z}, \mathcal{O}$ and $\mathcal{O}^*$, resp., are the sheaves of constant $\mathbb{Z}$-valued functions, holomorphic functions and nonzero holomorphic functions, resp. For any line bundle $E \in Pic(W) := H^1(W, \mathcal{O}^*)$, the *first Chern* *class* of E is defined to be $\delta(E) \in H^2(W, \mathcal{Z})$ and is denoted by $c_1(E)$. By de Rham theorem, we can consider $c_1(E) \in H^2_{\mathrm{dR}}(W)$.

Definition 4.1. The canonical bundle of W is defined by $K_W := \Lambda^{n,0}W$, i.e., the bundle of $(n, 0)$-forms on W.

Since $\dim W = n$, it is easily seen that K_W is a holomorphic line bundle over W. By definition the *first Chern class* $c_1(W)$ of W is defined to be the first Chern class $c_1(K_W)$ of canonical bundle K_W. In [25], one can see that

$$[\rho] = 2\pi c_1(W), \tag{4.1}$$

where $[\rho]$ is the class of $(1, 1)$-form ρ in $H^2_{\mathrm{dR}}(W)$.

Observation 4.1. It is well known that for a line bundle E on W, $c_1(E) = 0$ if and only E is trivial, i.e., $E \cong W \times \mathbb{C}$. So $c_1(W) = 0$ if and only if K_W is trivial (see [4, § 20]).

In the following theorem we present the Calabi-Yau Theorem and one can find its proof in [25, Chapter 5].

Theorem 4.1. (Calabi-Yau Theorem) *Let (W, ω, g, J) be a compact Kähler manifold. Suppose that ρ' is a real closed $(1, 1)$-form on W with $[\rho'] = 2\pi c_1(W)$. Then there exists a unique Kähler metric g' on W with Kähler form ω', such that $[\omega] = [\omega'] \in H^2(W, \mathbb{R})$, and the Ricci form of g' is ρ'.*

In particular if $c_1(W) = 0$, then W accepts a Kähler metric that the corresponding Ricci form $\rho \equiv 0$. Conversely if W is Ricci-flat, then (4.1) implies that $c_1(W) = 0$. Hence we can provide th following corollary.

Corollary 4.1. $c_1(W) = 0$ *if and only if W is Ricci-flat.*

We have another important result that follows from Calabi-Yau Theorem. Before giving this result, we introduce the Kähler cone of W. As we know, the Kähler form ω is a real closed $(1, 1)$-form, hence $[\omega] \in H^{1,1}(W) \cap H^2_{\mathrm{dR}}(W)$. Note that (W, J) may admit various Kähler metrics, so we present the following definition.

Definition 4.2. The *Kähler cone* $\mathcal{K}_W$ of W is defined to be the set of Kähler classes $[\omega] \in H^{1,1}(W) \cap H^2_{\mathrm{dR}}(W)$ of Kähler metrics.

It is easily seen that $\mathcal{K}_W$ is a *convex cone*, i.e., if $\alpha_1, \alpha_2 \in \mathcal{K}_W$ and $t_1, t_2 > 0$, then $t_1\alpha_1 + t_2\alpha_2 \in \mathcal{K}_W$. So we can present the desired result as a following theorem (e.g. see [25, Theorem 6.2.1]).

Theorem 4.2. *If $c_1(W) = 0$, then for any Kähler classes $[\omega] \in H^{1,1}(W) \cap H^2_{\mathrm{dR}}(W)$, there exist a unique Ricci-flat Kähler metric on W. Moreover, the Ricci-flat metrics on W form a smooth family of dimension $h^{1,1}(W)$, isomorphic to the Kähler cone $\mathcal{K}_W$ of W.*

Remark 4.1. Considering Theorem 4.2, it follows that the family of Ricci-flat metrics on the fixed complex manifold (W, J) construct a real manifold of dimension $h^{1,1}(W)$ that is isomorphic to the Kähler cone $\mathcal{K}_W$. We define *complex Kähler moduli space* $\mathcal{M}_{Kah}(W)$ of W as follow,

$$\mathcal{M}_{Kah}(W) = \frac{H^2(W, \mathbb{R}) + i\mathcal{K}_W}{H^2(W, \mathbb{Z})},$$

and since $H^{2,0}(W) = H^{0,2}(W) = 0$, $\dim_{\mathbb{R}} H^2(W, \mathbb{R}) = h^{1,1}$. So $\dim \mathcal{M}_{Kah}(W) = h^{1,1}$ (see [18]).

Now we can give a primary definition of Calabi-Yau manifolds. A compact Kähler manifold W is called a *Calabi-Yau manifold* if its canonical bundle is trivial, or equivalently it is Ricci-flat. But depending on the contexts, authors give different equivalent definitions of Calabi-Yau manifolds that we state the most important of them in Theorem 4.3. Before doing that, we need to know the holonomy group.

Another concept that is in interested, is the holonomy group of W denoted by $\mathrm{Hol}(W)$. Let E be a k-dimensional holomorphic bundle on W and ∇^E be a connection on E. For any continues and piecewise-smooth curve $\gamma : [0, 1] \to W$,

$$P_\gamma : E_x \to E_y$$

is the parallel transport map along γ, where $\gamma(0) = x$ and $\gamma(1) = y$. More detailed, for any $e \in E_x$ there exist a unique continues and piecewise-smooth section ζ of γ^*E that $\zeta(0) = e$, and satisfies $\nabla^E_{\dot\gamma(t)}\zeta(t) = 0$ for any $t \in [0, 1]$. Then the paarallel transport of e is defined $P_\gamma(e) = \zeta(1)$. It is obvious that $P_\gamma \in \mathrm{GL}_k(\mathbb{C})$.

Definition 4.3. Let E be a holomorphic bundle on W and ∇^E be a connection on E. For any $x \in W$, the *holonomy group* $\mathrm{Hol_x(\nabla^E)}$ of ∇^E based at x is defined as follow,

$$\mathrm{Hol}_x(\nabla^E) := \{P_\gamma | \ \gamma \text{ is a loop based at } x\} \subset \mathrm{GL}_k(\mathbb{C}).$$

Moreover, since W is connected, $\mathrm{Hol}_x(\nabla^E)$ does not depend to x, and simply we show it by $\mathrm{Hol}(\nabla^E)$. If we denot by ∇^{LC} the Levi-Civita connection of the metric g, then the *holonomy group* $\mathrm{Hol}(W)$ *of* W by definition is $\mathrm{Hol}(\nabla^{LC})$.

So we are in the position where can state the following theorem that gives the equivalent definitions of Calabi-Yau manifolds.

Theorem 4.3. *Let W be an n-dimensional compact Kähler manifold. Then followings are equivalent:*

(i) *The canonical bundle of W is trivial.*

(ii) *The first Chern class of W is zero.*

(iii) *W accept a Ricci-flat Kähler metric.*

(iv) *There exist a, up to multiplication by a constant, unique holomorphic nowhere vanishing $(n, 0)$-form on W.*

(v) *The holonomy group of W is a subgroup of* $\mathrm{SU}(n)$.

Proof.

((i)⇔(ii)) See Observation 4.1.

((ii)⇔(iii)) See Corollary 4.1.

((ii)⇔(iv)) If $c_1(W) = 0$, then by Observation 4.1 the canonical bundle of W is trivial, i.e., $K_W \cong W \times \mathbb{C}$, hence the unit section η of K_W is a globally non-vanishing $(n, 0)$-form. For uniqueness, since $\dim W = n$, for any other globally non-vanishing $(n, 0)$-form $\hat{\eta}$ there exist a holomorphic function f on W such that $\hat{\eta} = f\eta$. But because of compactness of W, f is constant. For proof of the converse, i.e., **(iv)⇒(ii)** see [40, Page 26].

((i)⇔(v)) For **(i)⇒(v)** see [25, Corollary 6.2.5]. And for **(v)⇒(i)** see [25, Page 122].

■

In the following proposition we see that what happen for Hodge numbers of W when $\mathrm{Hol}(W) = \mathrm{SU}(n)$, and for a proof see [25, Proposition 6.2.6].

Proposition 4.1. *If* $\mathrm{Hol}(W) = \mathrm{SU}(n)$, *then* $h^{0,0} = h^{n,0} = 1$ *and* $h^{p,0} = 0$ *for* $0 < p < n$.

4.1 Definitions and Properties

As we saw there are different equivalent definitions of Calabi-Yau manifolds, and in particular if $\mathrm{Hol}(W) = \mathrm{SU}(n)$, then $h^{p,0} = 0$ for $0 < p < n$, which is more interested for us. So in this thesis we are considering the following definition.

Definition 4.4. An n-dimensional *Calabi-Yau manifold* is a compact Kähler manifold W of complex dimension n with a trivial canonical bundle such that the Hodge numbers $h^{k,0}$ vanish for any $0 < k < n$. Sometimes *Calabi-Yau n-fold* is used instead of n-dimensional Calabi-Yau manifold.

So if W is a Calabi-Yau n-fold, then by (2.23), its Hodge diamond is as follow

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & \iddots & & & & & & \ddots & \\
 & 0 & & h^{n-2,1} & \dots & h^{n-2,1} & & 0 & \\
1 & & h^{n-1,1} & & \dots & & h^{n-1,1} & & 1 \\
 & 0 & & h^{n-2,1} & \dots & h^{n-2,1} & & 0 & \\
 & \ddots & & & & & & \iddots & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
\tag{4.2}
$$

Before providing some important properties of Calabi-Yau manifolds, we present some introductory examples of Calabi-Yau manifolds. To do this we need the following observation.

Observation 4.2. By using the "adjunction formula" from algebraic geometry, e.g. [17], one finds that given a polynomial equation $P = 0$ of degree d inside $\mathbb{P}^{k-1}$, the resulting hypersurface $W := P^{-1}(0)$ has $c_1(W) = (d-k)c_1(\mathbb{P}^{k-1})$.

Example 4.1. Consider the complex projective space $\mathbb{P}^n$. We know that $\mathbb{P}^n$ with Fubini-Study 2-form ω_{FS} is a Kähler manifold (e.g., see [10]). Let P be a homogeneous polynomial of degree $n+1$ in $\mathbb{P}^n$ without singularity, which is true for generic P. Then the hypersurface $W := P^{-1}(0)$ is a compact submanifold of $\mathbb{P}^n$ that inherits the Kähler structure generated by ω_{FS} on $\mathbb{P}^n$. Observation 4.2 guaranties that $c_1(W) = 0$, hence W is an $(n-1)$-dimensional compact Kähler manifold with trivial canonical bundle. Theorem 4.2 implies that W has a family of Ricci-flat Kähler metrics, and in fact these metrics have holonomy $\mathrm{SU}(n-1)$, and W is a Calabi-Yau $(n-1)$-fold. So for any $n \geq 2$ the family of homogeneous polynomials of degree $n+1$, gives a family of $n-1$ dimensional Calabi-Yau manifolds. In particular we study a little more when $n = 2, 3, 4$ in followings:

n=2. In this case, the given family is the family of elliptic curves. In fact the family of 1-dimensional Calabi-Yau manifolds are classified by the family of elliptic curves. We know that elliptic curves are diffeomorphic to the torus, and the Ricci-flat metric on a torus is actually the flat metric of the page. Thus the Hodge diamond of elliptic curves is as follow,

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}.$$

n=3. The resulted family is the family of complex K3 surfaces. By definition, a *complex* K3 *surface* W is a compact complex surface (W, J) with $h^{1,0}(X) = 0$ and trivial canonical bundle. So all Calabi-Yau 2-folds are classified as K3 surfaces. We know that the Euler characteristic of K3 surface W is $\chi(W) = 24$, on the other hand since

$$\chi(W) = \sum_{i=0}^{4} (-1)^i b_i(W),$$

where $b_i(W)$'s are Betti numbers of W, $h^{0,0} = h^{4,4} = h^{2,0} = h^{0,2} = 1$ and $h^{1,0} = h^{0,1} = h^{3,0} = h^{0,3} = 0$, it follows that $h^{1,1} = 20$. Hence the Hodge diamond of a complex K3 surface is as follow

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

n=4. In this case we have the family of hypersurfaces of degree 5 in $\mathbb{P}^4$ which are known as the *family of generic quintic threefolds* or simply *quintic threefolds*. In spite of the previous two cases, here the family of quintic threefolds does not classify the Calabi-Yau threefolds and we will see in the future that there are too many examples of Calabi-Yau threefolds. Indeed, the classification of Calabi-Yau threefolds is an open problem, even though Yau suspects that there is a finite number of threefolds families. The Hodge diamond of quintic threefolds is given as follow

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1. \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{4.3}$$

The importance of Calabi-Yau manifolds was discovered by physicists with introducing the concept of *mirror symmetry* for Calabi-Yau manifolds. In fact, mirror symmetry is a conjecture that says there exist mirror pair of Calabi-Yau manifolds W and $\check{W}$ that have a certain relationships, which are quite complicated and long to write all and also we doesn't need them in this thesis. We just concentrate on one of mathematical significance of mirror symmetry which says the pair of Calabi-Yau manifolds W and $\check{W}$ are mirror symmetry if locally we have following isomorphisms,

$$\mathcal{M}_{cmplx}(W) \cong \mathcal{M}_{Kah}(\check{W}) \text{ and } \mathcal{M}_{Kah}(W) \cong \mathcal{M}_{cmplx}(\check{W}), \tag{4.4}$$

that for $\mathcal{M}_{cmplx}(W)$ and $\mathcal{M}_{Kah}(W)$ see Remarks 2.1 and 4.1.

Lemma 4.1. *Let W be a Calabi-Yau n-fold. Then* $\dim \mathcal{M}_{cmplx}(W) = h^{n-1,1}(W)$.

Proof. Since the canonical bundle K_W is trivial, (2.26) of Remark 2.1 implies that

$$\mathcal{M}_{cmplx}(W) \cong H^{n-1}(W, \Omega^1_W)^*,$$

and then by Dolbeault's theorem we have

$$\mathcal{M}_{cmplx}(W) \cong H^{1,n-1}(W)^*.$$

Thus $\dim \mathcal{M}_{cmplx}(W) = h^{1,n-1}(W) = h^{n-1,1}(W)$. ■

Remark 4.2. Considering Remark 4.1, Lemma 4.1 and (4.4), It follows that two necessary conditions that under which the pair of Calabi-Yau n-folds W and $\check{W}$ construct a mirror symmetry pair, are $h^{n-1,1}(W) = h^{1,1}(\check{W})$ and $h^{1,1}(W) = h^{n-1,1}(\check{W})$. For example from Hodge diamond (4.3) of the family of quintic 3-folds we know that $h^{1,1} = 1$, i.e., this family accepts one Kähler structure which is the one comes from Fubini-Study Kähler form. And since $h^{2,1} = 101$, it accepts 101 nonholomorphic complex structure. In Example 4.2 we will know the mirror symmetry of this family that accepts 101 Kähler structure and one complex structure.

In the following example we introduce the mirror family of quintic 3-folds, which is known as the *family of mirror quintic 3-folds*. This family, for the first time in 1990 was presented by physicists Greene and Plesser [16], and then in 1991 Candelas et al [8] studied it in more details. As we saw in Theorem 1.1, Movasati [31, 33] worked on a moduli space of this family and presented the DHR vector field explicitly.

Example 4.2. (Mirror quintic 3-fold) For any $\psi \in \mathbb{C}$, let

$$X_\psi = \{(x_0, x_1, x_2, x_3, x_4)|\ f_\psi(x_0, x_1, x_2, x_3, x_4) = 0\},$$

in which

$$f_\psi(x_0, x_1, x_2, x_3, x_4) = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4.$$

If we define $X_\infty := \{(x_0, x_1, x_2, x_3, x_4) | \prod_{i=0}^{4} x_i = 0\}$, then we can see X_ψ as the family $\pi : \mathcal{X} \to \mathbb{P}^1$, where $\mathcal{X} \subset \mathbb{P}^4 \times \mathbb{C}$. One can easily see that the singular points of this family are $\psi^5 = 1, \infty$. Now consider the following group

$$G := \{(\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4) \mid \zeta_i^5 = 1,\ \zeta_0\zeta_1\zeta_2\zeta_3\zeta_4 = 1\},$$

that acts on X_ψ as follow

$$(\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4).(x_0, x_1, x_2, x_3, x_4) = (\zeta_0 x_0, \zeta_1 x_1, \zeta_2 x_2, \zeta_3 x_3, \zeta_4 x_4).$$

It is easy to see that this action is well defined. Now let $Y_\psi = X_\psi/G$ be the quotient space of this action, which is quite singular. Indeed Y_ψ is singular in any $x \in X_\psi$ which its stabilizer in G is nontrivial. The way of solving the singularities of Y_ψ are not important in this thesis, and we just state the final result that says, for $\psi \neq 1, \infty$ there exist a resolution of singularities $W_\psi \to Y_\psi$ such that W_ψ is a Calabi-Yau 3-fold with

$$h^{1,1}(W_\psi) = 101 \ \text{ and } \ h^{2,1}(W_\psi) = 1.$$

For details one can see above mentioned original references or [18, Theorem 18.1]. Note that we construct a one parameter family W_ψ that its complex structure depends to ψ which suggests why $h^{2,1} = 1$. So the Hodge diamond of this family is as follow,

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{4.5}$$

which satisfies the necessary conditions, mentioned in Remark 4.2, for being the mirror symmetry of the family of quintic 3-folds.
Another coordinate is defined on the family of mirror quintic 3-folds as $z = (5\psi)^{-5}$. So in this coordinate the singularities are $z = 0, 5^{-5}$. We denote the family of mirror quintic 3-folds by $\mathcal{W}$ on the base space $S = \mathbb{P} \backslash \{0, 5^{-5}\}$. If $\mathcal{F}^\bullet$ is the Hodge filtration of $H^3_{\mathrm{dR}}(\mathcal{W}/S)$, then $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1,\ i = 0, 1, 2, 3$. There is a global holomorphic 3-form $\omega \in H^{3,0}(\mathcal{W})$ that its Picard-Fuchs equation is as follow,

$$L = \vartheta^5 - 5^5 z(\vartheta + \frac{1}{5})(\vartheta + \frac{2}{5})(\vartheta + \frac{3}{5})(\vartheta + \frac{4}{5}).$$

in which $\vartheta = \frac{d}{dz}$. So the family $\mathcal{W}$ satisfies the items of Assumption 3.1. In fact we are interested in such families that satisfies the Assumption 3.1 and in the future we will know more examples of this type.

4.2 Families and More Examples

Here we are going to follow some terminologies introduced in §2.4. For more emphasize, if $\pi : \mathcal{W} \to S$ is a family of complex manifolds, then we denote by $W_s := \pi^{-1}(s)$ the fibers of π. In the following proposition we see that an infinitesimal deformation of complex structure of a Calabi-Yau manifold leaves it again a Calabi-Yau manifold.

Proposition 4.2. *Let $\pi : \mathcal{W} \to S$ be a family of complex manifolds. If W_o is a Calabi-Yau manifold, then for any $s \in S$, close enough to o, W_s is also a Calabi-Yau manifold.*

Proof. It is a straightforward result of Proposition 2.1 and Theorem 2.8. ■

Definition 4.5. By a *family of n-dimensional Calabi-Yau manifolds* we mean a family of complex manifolds $\mathcal{W}$ over the pointed base (S, o) that any fiber W_s, $s \in S$, is an n-dimensional Calabi-Yau manifold.

Observation 4.3. Let $\mathcal{W}$ be a family of n-dimensional Calabi-Yau manifolds. We know that the Calabi-Yau structure of $\mathcal{W}$ depends to its complex structure and its Kähler structure. So if we denote the moduli of Calabi-Yau structures of $\mathcal{W}$ by $\mathcal{M}_{\mathrm{CY}}(\mathcal{W})$, then $\mathcal{M}_{\mathrm{CY}}(\mathcal{W})$ is a real manifold of dimension

$$\begin{aligned} \dim_{\mathbb{R}} \mathcal{M}_{\mathrm{CY}}(\mathcal{W}) &= \dim_{\mathbb{R}} \mathcal{K}_{\mathcal{W}} + \dim_{\mathbb{R}} \mathcal{M}_{cmplx}(\mathcal{W}) \\ &= h^{1,1}(\mathcal{W}) + 2h^{n-1,1}(\mathcal{W}). \end{aligned} \tag{4.6}$$

As we saw in Example 4.1, one dimensional Calabi-Yau manifolds are elliptic curves that are algebraic varieties. Calabi-Yau 2-folds are classified by K3 surfaces. In general K3 surfaces are classified as quartic hypersurfaces in $\mathbb{P}^3$ (Example 4.1 for n=3) or Kummer surfaces. Considering the Hodge diamond of K3 surfaces, we obtain that the moduli space $\mathcal{M}_{\mathrm{K3}}$ of K3 surfaces is a connected 20-dimensional singular complex manifold. Some K3 surfaces are algebraic, that is, they can be embedded as complex submanifolds in $\mathbb{P}^N$ for some N, and the others are not. So by (4.6) the family of Calabi-Yau 2-folds is a 60-dimensional real manifold. And if $n \geq 3$, then for Calabi-Yau n-folds we have the following theorem that for proof one refers to [18, § 5.4].

Theorem 4.4. *Let W be a Calabi-Yau n-fold and $n \geq 3$. Then W is projective. That is, M is isomorphic as a complex manifold to a complex submanifold of $\mathbb{P}^N$, for some N, and is an algebraic variety.*

This shows that CalabiYau manifolds or at least, their underlying complex manifolds, can be studied using complex algebraic geometry.

In the same reference we can also find the following theorem.

Theorem 4.5. *Any Calabi-Yau $2n$-fold is simply connected.*

Next we will give more examples of Calabi-Yau 3-folds.

Example 4.3. First we are going to explain another way of constructing Calabi-Yau manifolds, which is called complete intersection Calabi-Yau manifolds in projective spaces. Let $X = X(d_1, d_2, \dots, d_r)$ be the complete intersection of r hypersurfaces X_i, $i = 1, 2, \dots, r$, of degree d_i in $\mathbb{P}^n$, with $r \leq n$. That is, for any $i \in \{1, 2, \dots, r\}$ there is a polynomial f_i of degree d_i such that $X_i = \{f_i = 0\}$, and hence $X = \{f_1 = f_2 = \dots, f_r = 0\} \subset \mathbb{P}^n$. For a generic choice of f_i's, X is a smooth manifold of dimension $n - r$, and of course a compact Kähler submanifold of $\mathbb{P}^n$. By using of adjunction formula, one can see that the canonical bundle K_X of X is trivial if and only if $\sum_{i=1}^{r} d_i = n + 1$. So $X(d_1, d_2, \dots, d_r) \subset \mathbb{P}^n$ is a Calabi-Yau $(n - r)$-fold if and only if $\sum_{i=1}^{r} d_i = n + 1$.

There are other constructions for Calabi-Yau manifolds, such as hypersurfaces in weighted projective spaces and complete intersection in weighted projective spaces, which the way of these constructions are not so important in this thesis. We just need the existence of some examples of Calabi-Yau manifolds that satisfy some certain properties and these constructions provide us with them (see [1, 26, 27]). As we know the normal projective space $\mathbb{P}^n$ is the weighted projective space $\mathbb{P}(\underbrace{1, 1, \dots, 1}_{(n+1)-\mathrm{th}})$. So all other constructions of Calabi-Yau manifolds, i.e., hypersurfaces in projective spaces, complete intersections in projective spaces and hypersurfaces in weighted projective spaces, are spacial cases of complete intersection Calabi-Yau manifolds in weighted projective spaces.

In [11] there is a large list of Calabi-Yau 3-folds where among them there are 14 cases that we are more interested in them and are listed in Table A.1. For these 14 families of Calabi-Yau 3-folds $h^{1,1} = 1$, and hence for their mirror families $h^{2,1} = 1$. So the mirror families of these 14 families of Calabi-Yau 3-folds are one parameter families and they satisfies the following Picard-Fuchs differential equations which are hypergeometric equations:

$$L = \vartheta^5 - cz(\vartheta + r_1)(\vartheta + r_2)(\vartheta + 1 - r_2)(\vartheta + 1 - r_1), \tag{4.7}$$

in which r_1, r_2, c are given in Table A.1. Note that $\sharp 1$ is the family of quintic 3-folds that we discussed it and its mirror family in Example 4.2. $\sharp 2, 3, 4$ are hypersurfaces in weighted projective spaces, $\sharp 5, 6, 7, 8$ are complete intersection Calabi-Yau 3-folds in projective spaces, and finally $\sharp 9, 10, 11, 12, 13, 14$ are complete intersection Calabi-Yau 3-folds in weighted projective spaces. For more details one can see the references given in Table A.1.

Example 4.4. For higher dimensions, H. Movasati and K. M. Shokri in [35] claim that there are 40 numbers of one parameter Calabi-Yau 5-fold families, for which the Picard-Fuchs differential equations are hypergeometric of the form

$$L = \vartheta^6 - z(\vartheta + r_1)(\vartheta + r_2)(\vartheta + r_3)(\vartheta + r_4)(\vartheta + r_5)(\vartheta + r_6) \tag{4.8}$$

such that, $r_i = 1 - r_{i-3}$, $i = 4, 5, 6$, and r_i's are given in Table A.2.

Chapter 5

Darboux-Halphen-Ramanujan Vector Field

In this chapter we state our main result about the encountering DHR vector field. We first fix some hypothesis on a Calabi-Yau n-fold, under which we are working in the whole of this chapter. In §5.1 and §5.2 we give a special moduli space T of a fixed Calabi-Yau manifold. Then we prove that there exist a unique vector field, which we call it DHR vector field, and several unique meromorphic functions on T that satisfy certain properties. Next in §5.3 and §5.4, after computing the matrix of intersection forms and finding the relationships among the coefficients of Picard-Fuchs equation, we express DHR vector field explicitly in dimension five and three.

We first fix the following notations.

Notation 5.1. Let $\mathcal{W}$ be a one parameter family of Calabi-Yau n-folds parameterized by z on the base space P. Then during this chapter we have followings:

- $\mathcal{F}^\bullet$ is the Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$.
- ∇ is the Gauss-Manin connection.
- The operator ϑ refers either to the differential operator $\nabla_{z\frac{\partial}{\partial z}}$ if it operates on the elements of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$, or to $z\frac{\partial}{\partial z}$ if it acts on the elements of $\mathbb{C}(z)$. And also by definition we consider $\vartheta^0 = 1$ to be the identity differential operator.

To avoid repeating the hypothesis, we consider the following assumption.

Assumption 5.1. During the whole of this chapter W is a Calabi-Yau n-fold that its complex deformation is given by the family $\mathcal{W}$ satisfying followings:

(i) $\pi : \mathcal{W} \to P$ is a one parameter family of n-dimensional Calabi-Yau manifolds parameterized by z. Indeed, $W_0 := \pi^{-1}(0) = W$.

(ii) $\dim H^n_{\mathrm{dR}}(\mathcal{W}/P) = n + 1$.

(iii) The unique nowhere vanishing holomorphic n-form $\omega \in \mathcal{F}^n$ satisfies $L\omega = 0$, where L is the following Picard-Fuchs equation of minimum degree

$$L = \vartheta^{n+1} - a_n(z)\vartheta^n - \ldots - a_1(z)\vartheta - a_0(z), \tag{5.1}$$

in which $a_i(z) \in \mathbb{Q}(z)$, $i = 0, 1, \ldots, n$.

Remark 5.1. Proposition 3.4, guaranties that $\dim \mathcal{F}^i/\mathcal{F}^{i+1} = 1$, $i = 0, 1, \ldots, n$, or in the other word for any z, $\dim H^{i,j}(W_z) = 1$ when $i + j = n$.

For more convenient, we fix one more notation in the following.

Notation 5.2. For the fixed holomorphic n-form $\omega \in \mathcal{F}^n$, by notation we define

$$\omega_i = \vartheta^{i-1}\omega, \ \ i = 1, 2, \ldots, n+1.$$

So Griffiths' transversality implies that $\omega_i \in \mathcal{F}^{(n+1)-i}$.

Because of different behaviors of intersection form for odd or even integer n, we separate the cases for odd and even integers and first we state the results in the odd case. Also for $n = 3, 5$ we give an explicit computation of results.

5.1 Odd Case

In the whole of this section n is considered to be an odd positive integer. We know that Hodge filtration of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$ is as follow,

$$\mathcal{F}^\bullet: \ \{0\} = \mathcal{F}^{n+1} \subset \mathcal{F}^n \subset \ldots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = H^n_{\mathrm{dR}}(\mathcal{W}/P), \ \ \dim(\mathcal{F}^i) = (n+1) - i, \tag{5.2}$$

and by Lemma 2.3 we have the following relationship between Hodge filtration and the intersection form:

$$\langle \mathcal{F}^i, \mathcal{F}^j \rangle = 0, \ \ i + j \geq n+1. \tag{5.3}$$

As we saw in Theorem 3.2, the set $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ construct Yukawa frame of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$, such that

$$\omega_i \in \mathcal{F}^{(n+1)-i} \setminus \mathcal{F}^{(n+2)-i}, \tag{5.4}$$

and by using of Picard-Fuchs equation (5.1) we have

$$\vartheta^{n+1}\omega = \vartheta\omega_{n+1} = a_0\omega_1 + a_1\omega_2 + \ldots + a_n\omega_{n+1}. \tag{5.5}$$

Hence, considering Observation 2.2, if we apply the Gauss-Manin connection to the column of n-forms

$$\varpi = \begin{pmatrix} \omega_1 & \omega_2 & \ldots & \omega_{n+1} \end{pmatrix}^{\mathsf{t}}, \tag{5.6}$$

then

$$\nabla\varpi = \mathsf{GM}_\varpi \otimes \varpi, \tag{5.7}$$

where

$$\mathsf{GM}_{\varpi} = \frac{1}{z}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix} dz. \tag{5.8}$$

To see this, for $j = 1, 2, \dots, n$, we have

$$z\nabla_{\frac{\partial}{\partial z}}\omega_i = \nabla_{z\frac{\partial}{\partial z}}\omega_i = \vartheta\omega_i = \omega_{i+1} \Longrightarrow \nabla\omega_i = \frac{1}{z}dz \otimes \omega_{i+1},$$

and similarly, by using (5.5) for ω_{n+1} we get

$$\nabla\omega_{n+1} = \frac{1}{z}\sum_{i=0}^{n} a_i dz \otimes \omega_{i+1}.$$

In continue we are going to talk about intersection forms. So, if we define the matrix of intersection forms as

$$\Omega = (\Omega_{ij})_{1\leq i,j\leq n+1} := (\langle\omega_i, \omega_j\rangle)_{1\leq i,j\leq n+1}, \tag{5.9}$$

then we find some entries of Ω in the following lemma.

Lemma 5.1. *Followings hold for the matrix Ω:*

(i) *If $i + j \leq n + 1$, then $\Omega_{ij} = \langle\omega_i, \omega_j\rangle = 0$.*

(ii) $\Omega^{\mathrm{t}} = -\Omega$.

(iii) *For any $i = 1, 2, \dots, n + 1$, $\langle\omega_i, \omega_{n+2-i}\rangle = (-1)^{i+1}\tilde{a}$, where*

$$\tilde{a}(z) = c_0 \exp\left(\frac{2}{n+1}\int_0^z a_n(v)\frac{dv}{v}\right), \tag{5.10}$$

with nonzero constant c_0.

Proof. By using Lemma 2.3, Remark 2.3 and Corollary 3.2 we can directly complete the proof. ■

So Lemma 5.1 gives the matrix Ω as follow,

$$\Omega = \begin{pmatrix} 0 & 0 & \dots & 0 & \tilde{a} \\ 0 & 0 & \dots & -\tilde{a} & \Omega_{2(n+1)} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \tilde{a} & \dots & 0 & \Omega_{n(n+1)} \\ -\tilde{a} & -\Omega_{2(n+1)} & \dots & -\Omega_{n(n+1)} & 0 \end{pmatrix}. \tag{5.11}$$

Next, we would like to introduce a special frame for $H^n_{\rm dR}(\mathcal{W}/P)$. To do this, we first fix the $(n+1)\times(n+1)$ matrix Φ as follow

$$\Phi := \begin{pmatrix} 0_{\frac{n+1}{2}} & J_{\frac{n+1}{2}} \\ -J_{\frac{n+1}{2}} & 0_{\frac{n+1}{2}} \end{pmatrix}, \tag{5.12}$$

where for a positive integer k, by 0_k we mean a $k\times k$ block of zeros, and J_k is the following $k\times k$ block

$$J_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{5.13}$$

Definition 5.1. We say that the frame $\{\alpha_1,\alpha_2,\dots,\alpha_{n+1}\}$ of $H^n_{\rm dR}(\mathcal{W}/P)$ is *compatible with Hodge filtration* $\mathcal{F}^\bullet$ of $H^n_{\rm dR}(\mathcal{W}/P)$, if for any $i\in\{1,2,\dots,n+1\}$,

$$\alpha_i \in \mathcal{F}^{n+1-i}\setminus\mathcal{F}^{n+2-i}, \tag{5.14}$$

and it called *standard frame* if moreover the matrix of their intersection forms satisfies the following:

$$(\langle\alpha_i,\alpha_j\rangle)_{1\le i,j\le n+1} = \Phi. \tag{5.15}$$

Note that if $\{\alpha_1,\alpha_2,\dots,\alpha_{n+1}\}$ is a basis of W, we have the same definitions for *compatible with Hodge filtration* and *standard basis*.

In the continue we introduce a special moduli space of Calabi-Yau manifold W that we will work on it. To do this, we first provide an equivalence relation.

Definition 5.2. Let W_1, W_2 be two Calabi-Yau n-folds and $\{\alpha^i_1,\alpha^i_2,\dots,\alpha^i_{n+1}\}$ be a basis of $H^n_{\rm dR}(W_i;\mathbb{C}), i=1,2$, compatible with its Hodge filtration. Then we write

$$(W_1,[\alpha^1_1,\alpha^1_2,\dots,\alpha^1_{n+1}]) \sim (W_2,[\alpha^2_1,\alpha^2_2,\dots,\alpha^2_{n+1}]) \tag{5.16}$$

if and only if there exist a biholomorphism $\varphi: W_1\to W_2$ such that $\varphi^*(\alpha^2_j)=\alpha^1_j,\ \ j=1,2,\dots,n+1$. It is obvious that "$\sim$" is an equivalence relation. For Calabi-Yau n-fold W, and a basis $\{\alpha_i\}_{i=1}^{n+1}$ of $H^n_{\rm dR}(W;\mathbb{C})$ compatible with its Hodge filtration, the *moduli space* $\tilde{\mathsf{T}}$ of pair $(W,[\alpha_1,\alpha_2,\dots,\alpha_{n+1}])$ is defined under the above equivalence relation (5.16).

Remark 5.2. We know that the family $\pi:\mathcal{W}\to P$ is the complex deformation of W. So for any different $z_1,z_2\in P$, W_{z_1} and W_{z_2} are not biholomorph, thus we have different members $(W_{z_1},[\omega_1(z_1),\omega_2(z_1),\dots\omega_{n+1}(z_1)])$ and $(W_{z_2},[\omega_1(z_2),\omega_2(z_2),\dots\omega_{n+1}(z_2)])$ of moduli space $\tilde{\mathsf{T}}$. Also suppose that $\{\mu_i\}_{i=1}^{n+1}$ and $\{\nu_i\}_{i=1}^{n+1}$ are two bases of $H^n_{\rm dR}(W;\mathbb{C})$ compatible with its Hodge filtration. If for any

$$\varphi\in \mathrm{Aut}(W) := \{f: W\to W \,|\, f \text{ is a biholomorphism}\},$$

it does not preserve the bases, i.e., there exist a $j \in \{1, 2, \dots, n+1\}$ such that $\varphi^*\nu_j \neq \mu_j$, then $(W, [\mu_1, \mu_2, \dots, \mu_{n+1}])$ and $(W, [\nu_1, \nu_2, \dots, \nu_{n+1}])$ give two different elements of moduli space $\tilde{\mathsf{T}}$.

As we fixed in the beginning of this chapter, for any z, $\{\omega_1(z), \omega_2(z), \dots, \omega_{n+1}(z)\}$ construct a basis for $H^n_{\mathrm{dR}}(W_z, \mathbb{C})$ that is compatible with its Hodge filtration, and by abuse of notation, we remove the letter z from this basis and denote it by $\{\omega_1, \omega_2, \dots, \omega_{n+1}\}$, so $(W_z, [\omega_1, \omega_2, \dots \omega_{n+1}]) \in \tilde{\mathsf{T}}$. Let S be the change of basis matrix $\alpha = S\varpi$, where $\{\alpha_i\}_{i=1}^{n+1}$ is a basis of $H^n_{\mathrm{dR}}(W_z; \mathbb{C})$ and

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \end{pmatrix}^{\mathsf{t}}. \tag{5.17}$$

Then (5.4) and (5.14) imply that S is a lower triangular matrix which we consider it as follow,

$$S = \begin{pmatrix} s_{11} & 0 & 0 & \dots & 0 \\ s_{21} & s_{22} & 0 & \dots & 0 \\ s_{31} & s_{32} & s_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{(n+1)1} & s_{(n+1)2} & s_{(n+1)3} & \dots & s_{(n+1)(n+1)} \end{pmatrix}. \tag{5.18}$$

So the entries of the change of basis matrix S, present coordinates of a chart of $\tilde{\mathsf{T}}$ that we use it in the proof of next proposition. Before stating the proposition, we give the following lemma.

Lemma 5.2. *Let $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ be a frame of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$ compatible with its Hodge filtration.*

(i) *If we define $\Psi := (\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq n+1}$, then*

$$\Psi = S\Omega S^{\mathsf{t}}. \tag{5.19}$$

(ii) *If we let $\nabla\alpha = \mathsf{GM}_\alpha \otimes \alpha$, then*

$$\mathsf{GM}_\alpha = (dS + S.\mathsf{GM}_\varpi)S^{-1}, \tag{5.20}$$

where

$$dS = \begin{pmatrix} ds_{11} & 0 & 0 & \dots & 0 \\ ds_{21} & ds_{22} & 0 & \dots & 0 \\ ds_{31} & ds_{32} & ds_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ds_{(n+1)1} & ds_{(n+1)2} & ds_{(n+1)3} & \dots & ds_{(n+1)(n+1)} \end{pmatrix}. \tag{5.21}$$

Proof.

(i) By using the $\alpha = S\varpi$, verifying (5.19) is an easy exercise of linear algebra.

(ii) If we apply the Gauss-Manin connection to the equation $\alpha = S\varpi$, then by using (5.7) we have

$$\begin{aligned}\nabla\alpha &= dS \otimes \varpi + S\nabla\varpi = (dS + S.\mathsf{GM}_{\varpi}) \otimes \varpi \\ &= (dS + S.\mathsf{GM}_{\varpi})S^{-1} \otimes \alpha,\end{aligned}$$

which completes the proof. ■

Following proposition give the more important part of the proof of main theorem.

Proposition 5.1. *Let $\tilde{\mathsf{T}}$ be the moduli of $(W, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, where W is the Calabi-Yau n-fold satisfying Assumption 5.1 and*

$$\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}, \;\; i = 1, 2, \ldots, n+1. \tag{5.22}$$

Then there exist a unique vector field $\tilde{\mathrm{H}}$ and unique meromorphic functions y_i, $i = 1, 2, \ldots, n-2$, on $\tilde{\mathsf{T}}$ such that

$$\nabla_{\tilde{\mathrm{H}}}\alpha = Y\alpha, \tag{5.23}$$

in which $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{pmatrix}^{\mathsf{t}}$, and

$$Y = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & y_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & y_{n-2} & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{5.24}$$

Proof. The idea of the proof is to present the vector field $\tilde{\mathrm{H}}$ explicitly in a chart of $\tilde{\mathsf{T}}$. It is easily seen that the dimension of $\tilde{\mathsf{T}}$ is $k+1$, where $k = \frac{(n+1)(n+2)}{2}$. For any $(W_z, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \in \tilde{\mathsf{T}}$, let S be the change of basis matrix $\alpha = S\varpi$ given in (5.18). We consider the chart $t = (t_0, t_1, \ldots, t_k)$ on $\tilde{\mathsf{T}}$, for which the coordinates are defined as $t_0 = z, t_1 = s_{11}, t_2 = s_{12}, \ldots, t_k = s_{(n+1)(n+1)}$. We suppose that the vector field $\tilde{\mathrm{H}}$ is given as follow:

$$\tilde{\mathrm{H}} = \sum_{i=0}^{k} \tilde{\mathrm{H}}_i(t)\frac{\partial}{\partial t_i},$$

where $\tilde{\mathrm{H}}_i$'s, $i = 0, 1, \ldots, k$, are meromorphic functions on $\tilde{\mathsf{T}}$. Since $\tilde{\mathrm{H}}$ satisfies $\nabla_{\tilde{\mathrm{H}}}\alpha = Y\alpha$, Lemma 5.2**(ii)** implies that

$$(dS + S.\mathsf{GM}_{\varpi})S^{-1}(\tilde{\mathrm{H}}) = Y. \tag{5.25}$$

We have $S.\mathsf{GM}_{\varpi}(\tilde{\mathrm{H}}) = \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$, where $\dot{z}(t) := \tilde{\mathrm{H}}_0(t)$ and

$$\widehat{\mathsf{GM}}_{\varpi} := \frac{1}{z}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}. \tag{5.26}$$

Also if we define $\dot{s}_{11}(t) := \tilde{\mathrm{H}}_1(t)$, $\dot{s}_{21}(t) := \tilde{\mathrm{H}}_2(t)$, $\dots$, $\dot{s}_{(n+1)(n+1)}(t) := \tilde{\mathrm{H}}_k(t)$, then we have $dS(\tilde{\mathrm{H}}) = \dot{S}$, where

$$\dot{S} = \begin{pmatrix} \dot{s}_{11} & 0 & 0 & \dots & 0 \\ \dot{s}_{21} & \dot{s}_{22} & 0 & \dots & 0 \\ \dot{s}_{31} & \dot{s}_{32} & \dot{s}_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dot{s}_{(n+1)1} & \dot{s}_{(n+1)2} & \dot{s}_{(n+1)3} & \dots & \dot{s}_{(n+1)(n+1)} \end{pmatrix}. \tag{5.27}$$

Therefore (5.25) gives

$$(\dot{S} + \dot{z}S.\widehat{\mathsf{GM}}_{\varpi})S^{-1} = Y. \tag{5.28}$$

Thus if we let

$$\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}, \tag{5.29}$$

then we can find $\dot{z}$ (or $\tilde{\mathrm{H}}_0$) and y_i's (that we state them in Lemma 5.3). Also by defining $\dot{S}$ as (5.29), we find $\tilde{\mathrm{H}}_i$'s, $i = 1, 2, \dots, k$, and hence the existence of vector field $\tilde{\mathrm{H}}$ that satisfies (5.23) is verified. The uniqueness of $\tilde{\mathrm{H}}$ and y_i's follow from Lemma 5.3**(i)**,**(ii)**. ■

The proof of Proposition 5.1, implies more results about entries of Y that we express them in the following lemma. Before of stating the lemma, we provide the following fact as a remark.

Remark 5.3. Since the matrix S is the change of basis matrix, it is invertible, thus for any $1 \leq i \leq n+1$, $s_{ii} \neq 0$.

Lemma 5.3. *Let the matrices $\widehat{\mathsf{GM}}_{\varpi}$, Ω, Φ, S, Y and $\dot{S}$, respectively, be the same as given in* (5.26), (5.11), (5.12), (5.18), (5.24) *and* (5.27), *respectively. Then the equation*

$$\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}, \tag{5.30}$$

implies that

(i) $\dot{z} = \frac{zs_{22}}{s_{11}} = \frac{zs_{(n+1)(n+1)}}{s_{nn}}$.

(ii) $y_{i-1} = \frac{s_{22}s_{ii}}{s_{11}s_{(i+1)(i+1)}}$, *for all* $i = 2, 3, \ldots, n-1$.

(iii) *Moreover, if* $S\Omega S^{\mathsf{t}} = \Phi$, *then* $y_{i-1} = -y_{n-i}$, *for* $i \neq \frac{n+1}{2}$, *and*

$$y_{\frac{n-1}{2}} = (-1)^{\frac{n+3}{2}} \frac{\tilde{a} s_{22} s^2_{\frac{n+1}{2}\frac{n+1}{2}}}{s_{11}}.$$

In the other word

$$Y\Phi = -\Phi Y^{\mathsf{t}}. \tag{5.31}$$

Proof. Let's define the matrix B as follow

$$B = (b_{ij})_{1\leq i,j\leq n+1} := YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}.$$

(i) The equation (5.30) implies that

$$b_{12} = s_{22} - \frac{\dot{z}}{z}s_{11} = 0 \text{ and } b_{n(n+1)} = s_{(n+1)(n+1)} - \frac{\dot{z}}{z}s_{nn} = 0,$$

which prove **(i)**.

(ii) The proof of **(ii)** follows from **(i)** and

$$b_{i(i+1)} = y_{i-1}s_{(i+1)(i+1)} - \frac{\dot{z}}{z}s_{ii} = 0, \; i = 2, 3, \ldots, n-1.$$

(iii) Let's define the matrix C as follow

$$C = (c_{ij})_{1\leq i,j\leq n+1} := S\Omega S^{\mathsf{t}}.$$

Then from equation $C = \Phi$ it follows that,

$$c_{i(n+2-i)} = (-1)^{i+1}\tilde{a}s_{ii}s_{(n+2-i)(n+2-i)} = 1, \; i = 1, 2, \ldots, \frac{n+1}{2},$$

hence,

$$s_{(n+2-i)(n+2-i)} = (-1)^{i+1}\frac{1}{\tilde{a}s_{ii}}, \; i = 1, 2, \ldots, \frac{n+1}{2}, \tag{5.32}$$

thus,

$$\frac{s_{ii}}{s_{(i+1)(i+1)}} = -\frac{s_{(n+1-i)(n+1-i)}}{s_{(n+2-i)(n+2-i)}}, \; i = 1, 2, \ldots, \frac{n-1}{2},$$

therefore by using **(ii)**, the proof of **(iii)** is complete. ∎

Lemma 5.4. *Let the matrices* Ω *and* $\widehat{\mathsf{GM}}_{\varpi}$, *respectively, be the same as given in* (5.11) *and* (5.26), *respectively. Then following equation holds:*

$$\vartheta\Omega = A\Omega + \Omega A^{\mathsf{t}}, \tag{5.33}$$

where $A := z\widehat{\mathsf{GM}}_{\varpi}$.

Proof. By using of Proposition 3.3**(iii)** and Picard-Fuchs equation (5.1), the proof is an easy exercise of linear algebra. ■

Finally, we state the main result of this thesis in the following theorem.

Theorem 5.1. *Let W be the Calabi-Yau n-fold satisfying Assumption 5.1, and T be the moduli of $(W, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, where*

$$\alpha_i \in F^{n+1-i} \setminus F^{n+2-i},\ i = 1, 2, \ldots, n+1, \tag{5.34}$$

and

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le n+1} = \Phi. \tag{5.35}$$

Then there exist a unique vector field H *and unique meromorphic functions y_i, $i = 1, 2, \ldots, n-2$, on T such that*

$$\nabla_{\mathrm{H}} \alpha = Y \alpha, \tag{5.36}$$

in which

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{pmatrix}^{\mathsf{t}},$$

and

$$Y = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & y_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & y_{n-2} & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{5.37}$$

Furthermore,

$$\dim \mathsf{T} = \frac{(n+1)(n+3)}{4} + 1.$$

Proof. Let $\tilde{\mathsf{T}}$ be the moduli space introduced in Proposition 5.1, and suppose that $(W_z, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \in \tilde{\mathsf{T}}$ be arbitrary. As we saw in the proof of Proposition 5.1, there exist the matrix S such that

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le n+1} = S \Omega S^{\mathsf{t}}.$$

Define the vector subspace $M \subset \mathrm{Mat}_{n+1}(\mathbb{C})$ to be

$$M = \{B = (b_{ij})_{1 \le i,j \le n+1} \in \mathrm{Mat}_{n+1}(\mathbb{C}) |\ b_{ij} = 0,\ if\ i \le n+1-j\}.$$

If we define the function f as follow

$$f : \tilde{\mathsf{T}} \to M$$
$$f(\ W_z, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) = S \Omega S^{\mathsf{t}},$$

then $\mathsf{T} = f^{-1}(\Phi)$, because $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ construct a standard frame of $H^n_{\mathrm{dR}}(\mathcal{W}/P)$ and $\langle \alpha_i, \alpha_j \rangle_{1 \leq i,j \leq n+1} = \Phi$. Hence T is a subspace of $\tilde{\mathsf{T}}$ and to prove the existence of the vector field H on T , it is enough to show that the vector field $\tilde{\mathrm{H}}$, which was introduced in Proposition 5.1, is tangent to T and set $\mathrm{H} := \tilde{\mathrm{H}}|_{\mathsf{T}}$. To demonstrate the tangency of $\tilde{\mathrm{H}}$ to T , it suffices to prove that $df|_{\mathsf{T}}(\tilde{\mathrm{H}}) = 0$, or equivalently verify that

$$(\dot{S}\Omega S^{\mathsf{t}} + S\dot{\Omega}S^{\mathsf{t}} + S\Omega\dot{S}^{\mathsf{t}})|_{\mathsf{T}} = 0, \tag{5.38}$$

in which $\dot{\Omega} = d\Omega(\tilde{\mathrm{H}})$. We have $\dot{\Omega} = \dot{z}\frac{\partial}{\partial z}\Omega$, and by using of Lemma 5.4, it is seen that

$$\dot{\Omega} = \frac{\dot{z}}{z}(A\Omega + \Omega A^{\mathsf{t}}) = \dot{z}(\widehat{\mathsf{GM}}_{\varpi}\Omega + \Omega\widehat{\mathsf{GM}}^{\mathsf{t}}_{\varpi}).$$

On the other hand as we saw in (5.29), $\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$, so

$$\dot{S}\Omega S^{\mathsf{t}} + S\dot{\Omega}S^{\mathsf{t}} + S\Omega\dot{S}^{\mathsf{t}} = YS\Omega S^{\mathsf{t}} + S\Omega S^{\mathsf{t}}Y^{\mathsf{t}}.$$

Since $S\Omega S^{\mathsf{t}}|_{\mathsf{T}} = \Phi$, by using Lemma 5.3**(iii)** we have

$$(\dot{S}\Omega S^{\mathsf{t}} + S\dot{\Omega}S^{\mathsf{t}} + S\Omega\dot{S}^{\mathsf{t}})|_{\mathsf{T}} = (YS\Omega S^{\mathsf{t}} + S\Omega S^{\mathsf{t}}Y^{\mathsf{t}})|_{\mathsf{T}} = Y\Phi + \Phi Y^{\mathsf{t}} = 0,$$

and the proof of existence of H is complete.

To prove the uniqueness, first notice that Lemma 5.3(ii) guaranties the uniqueness of y_i's. So we just need to prove that vector field H is unique. Suppose that there are two vector fields H_1 and H_2 such that $\nabla_{\mathrm{H}_i}\alpha = Y\alpha$, $i = 1, 2$. If we let $\mathrm{R} = \mathrm{H}_1 - \mathrm{H}_2$, then

$$\nabla_{\mathrm{R}}\alpha = 0. \tag{5.39}$$

We need to prove that $\mathrm{R} = 0$, and to do this it is enough to verify that any integral curve of R is a constant point. So assume that γ is an integral curve of R which is given by

$$\begin{aligned} \gamma : (\mathbb{C}, 0) &\to \mathsf{T} \\ x &\mapsto \gamma(x). \end{aligned}$$

Let's denote the trajectory $\gamma(\mathbb{C}, 0)$ of R by $\mathcal{C}$, i.e., $\mathcal{C} = \gamma(\mathbb{C}, 0) \subset \mathsf{T}$. We know that the members of T are in the form of the pairs $(\widehat{W}, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, where $\widehat{W}$ is a Calabi-Yau manifold of the family and $\{\alpha_i\}_{i=1}^{n+1}$ form a basis of $H^n_{\mathrm{dR}}(\widehat{W}; \mathbb{C})$ that is compatible with its Hodge filtration and has constant intersection matrix Φ. So for any $x \in (\mathbb{C}, 0)$, we have $\gamma(x) = (\widehat{W}(x), [\alpha_1(x), \alpha_2(x), \ldots, \alpha_{n+1}(x)])$, and as well the vector field R on $\mathcal{C}$ is reduced to $\frac{\partial}{\partial x}$. We know that $\widehat{W}(x)$ depends only on the parameter z and so x holomorphically depends to z, i.e., there exist a holomorphic function f such that $x = f(z)$. Now we prove that f is constant. Otherwise, by contradiction suppose that $f' \neq 0$, then we have

$$\nabla_{\frac{\partial}{\partial x}}\alpha_1 = \frac{\partial z}{\partial x}\nabla_{\frac{\partial}{\partial z}}\alpha_1. \tag{5.40}$$

Equation (5.39) gives that $\nabla_{\frac{\partial}{\partial x}}\alpha_1 = 0$, but the right hand side of (5.40) is not zero, because $\alpha_1 = s_{11}\omega_1$, which is a contradiction. Thus f is constant and $\widehat{W}(x)$ does not depend on the parameter x. Since $\widehat{W}(x) = \widehat{W}$ does not depend on x, we can write the Taylor series of $\alpha_i(x),\ \ i = 2, 3, \ldots, n+1$, in x at some point x_0 as

$$\alpha_i(x) = \sum_j (x - x_0)^j \alpha_{i,j}$$

where $\alpha_{i,j}$'s are elements in $H^n_{\mathrm{dR}}(\widehat{W};\mathbb{C})$ independent of x. In this way the action of $\nabla_{\frac{\partial}{\partial x}}$ on α_i is just the usual derivation $\frac{\partial}{\partial x}$. Again using (5.39) gives us $\nabla_{\frac{\partial}{\partial x}}\alpha_i = 0$, and we conclude that α_i's also do not depend on x and so the image of γ is a point and the proof of uniqueness is complete.

To prove that

$$\dim \mathsf{T} = \frac{(n+1)(n+3)}{4} + 1,$$

it is enough to observe that $S\Omega S^{\mathsf{t}} = \Phi$ gives $\frac{(n+1)(n+3)}{4}$ independent equations and that $\mathcal{W}$ is a one parameter family. ■

As we saw in §1.1, the vector field H has a relationship with the works of Darboux, Halphen and Ramanujan, so we give the following definition.

Definition 5.3. The vector field H founded in Theorem 5.1, is called *Darboux-Halphen-Ramanujan*, abbreviatly DHR , *vector field.*

Note that by using of equation $S\Omega S^{\mathsf{t}} = \Phi$, we can write $\frac{(n+1)^2}{4}$ entries of S in terms of the rest $\frac{(n+1)(n+3)}{4}$ independent entries, and we call them *dependent* entries of S. Let s_{ij} be a dependent entry of S. Then we can compute $\dot{s}_{ij}$ in two ways; the first one is using the relationship following from $S\Omega S^{\mathsf{t}} = \Phi$ and writing it in terms of $\dot{s}_{rs}$ of independent entries and then using the equation $\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$; the second way is using directly the equation $\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$. We say that *the equations* $S\Omega S^{\mathsf{t}} = \Phi$ *and* $\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ *are compatible* if the results of computing $\dot{s}_{ij}$, for all dependent entries s_{ij}, from above introduced two ways are the same.

Corollary 5.1. *The equations* $S\Omega S^{\mathsf{t}} = \Phi$ *and* $\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ *are compatible on* T.

Proof. It follows from uniqueness of DHR vector field. ■

As well, one can find that the compatibility of equations $S\Omega S^{\mathsf{t}} = \Phi$ and $\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}$ implies the uniqueness of DHR vector field. We see this clearly in §5.3 and §5.4, where we compute DHR vector field explicitly.

5.2 Even Case

During this section n refers to an even positive integer. As we mentioned before, the deference of even case with the odd case is just the symmetry of intersection form. Lemma 2.3 implies that in the odd case the matrix of intersection forms is anti-symmetric, but in the even case it is symmetric. So in this section we follow all the notations and definitions of §5.1, except the related concept with intersection form. In particular the matrix $\Omega = (\Omega_{ij})_{1\le i,j\le n+1} := (\langle\omega_i,\omega_j\rangle)_{1\le i,j\le n+1}$, is given as follow,

$$\Omega = \begin{pmatrix} 0 & 0 & & \dots & & 0 & a \\ 0 & 0 & & \dots & & -a & \Omega_{2(n+1)} \\ & & & & \iddots & & \\ \vdots & \vdots & & \Omega_{ll} = (-1)^{\frac{n}{2}} a & & \vdots & \vdots \\ & & \iddots & & & & \\ 0 & -a & & \dots & & \Omega_{nn} & \Omega_{n(n+1)} \\ a & \Omega_{2(n+1)} & & \dots & & \Omega_{n(n+1)} & \Omega_{(n+1)(n+1)} \end{pmatrix}, \tag{5.41}$$

in which $l = \frac{n}{2}+1$. And as well, the constant matrix Φ is defined as follow,

$$\Phi = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{5.42}$$

In this section matrices ω, α, GM_ω, $\widehat{\mathsf{GM}}_\omega$, GM_α, S, dS, $\dot{S}$ and Y are the sam as given in §5.1. Almost all results of odd case are valid in even case. More precisely, we can repeat Lemma 5.2, Proposition 5.1 and Lemma 5.4 exactly the same. But Lemma 5.3 is valid with some changes that we rewrite it as follow.

Lemma 5.5. *The equation*

$$\dot{S} = YS - \dot{z}S.\mathsf{GM}_{\varpi}, \tag{5.43}$$

implies that,

(i) $\dot{z} = \frac{zs_{22}}{s_{11}} = -\frac{zs_{(n+1)(n+1)}}{s_{nn}}$.

(ii) $y_{i-1} = \frac{s_{22}s_{ii}}{s_{11}s_{(i+1)(i+1)}}$, *for all* $i = 2,3,\dots,n-1$.

(iii) *Moreover, if* $S\Omega S^{\mathsf{t}} = \Phi$, *then* $y_{i-1} = -y_{n-i}$, *for* $i = 2,3,\dots,\frac{n}{2}$. *In the other word*

$$Y\Phi = -\Phi Y^{\mathsf{t}}. \tag{5.44}$$

Proof. The proof is given similar to the proof of Lemma 5.3. ■

Now we state the main theorem, and it is similar to the Theorem 5.1 and just the dimension of moduli space T is changed.

Theorem 5.2. *Let W be the Calabi-Yau n-fold satisfying Assumption 5.1, and T be the moduli of $(W, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, where*

$$\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}, \; i = 1, 2, \ldots, n+1, \tag{5.45}$$

and

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \le i,j \le n+1} = \Phi. \tag{5.46}$$

Then there exist a unique vector field H *and unique meromorphic functions y_i, $i = 1, 2, \ldots, n-2$, on T such that*

$$\nabla_{\mathrm{H}} \alpha = Y\alpha, \tag{5.47}$$

in which

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{pmatrix}^{\mathsf{t}},$$

and

$$Y = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & y_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & y_{n-2} & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{5.48}$$

Furthermore,

$$\dim \mathsf{T} = \frac{n(n+2)}{4} + 1.$$

Proof. Considering the proof of Theorem 5.1, analogously we can proof this theorem. ■

Remark 5.4. If we are more exact on the dimension of moduli space T in the even case and odd case, then we find a nice relationship between them. Let T be the moduli space of a Calabi-Yau n-fold, where n is even, and T' be the moduli space of a Calabi-Yau $(n-1)$-fold. Then we have

$$\dim \mathsf{T} = \frac{n(n+2)}{4} + 1 = \frac{((n-1)+1)((n-1)+3)}{4} + 1 = \dim \mathsf{T}'.$$

So if we can know better the algebraic and geometrical structure of the moduli space, then we may find more relationships between structures of T and T'.

5.3 Five-Dimensional Case

In this section we give an explicit presentation of DHR vector field H, and in particular we verify its uniqueness by using of the existence relationships among coefficients of Picard-Fuchs equation. Here we are following the notations and terminologies of §5.1 for $n = 5$. To review them briefly, $\mathcal{W}$ is a one parameter family of Calabi-Yau 5-folds satisfying Assumption 5.1. In particular the Hodge filtration of $H^5_{\mathrm{dR}}(\mathcal{W}/P)$ is as follow,

$$\{0\} = \mathcal{F}^6 \subset \mathcal{F}^5 \subset \ldots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = H^5_{\mathrm{dR}}(\mathcal{W}/P), \quad \dim(\mathcal{F}^i) = (6) - i, \tag{5.49}$$

and the Picard-Fuchs equation (5.1), for the fixed holomorphic 5-form $\omega \in \mathcal{F}^5$ reduces to

$$\vartheta^6 = a_0(z) + a_1(z)\vartheta + a_2(z)\vartheta^2 + a_3(z)\vartheta^3 + a_4(z)\vartheta^4 + a_5(z)\vartheta^5. \tag{5.50}$$

and since this equation satisfies property (P), we have the following lemma to find the relationships among the coefficients a_i's.

Lemma 5.6. *The coefficients a_i's given in equation* (5.50) *satisfy following equations,*

$$\begin{aligned} a_3 &= -\tfrac{2}{3}a_4a_5 + \tfrac{5}{3}a_5\vartheta a_5 - \tfrac{5}{27}a_5^3 - \tfrac{5}{3}\vartheta^2 a_5 + 2\vartheta a_4, \\ a_1 &= \vartheta a_2 - \vartheta^3 a_4 + \vartheta^4 a_5 + \vartheta^2 a_4 a_5 + \vartheta a_4 \vartheta a_5 + \tfrac{5}{3}a_5(\vartheta a_5)^2 - \tfrac{1}{3}a_2a_5 \\ &\quad + \tfrac{1}{27}a_4a_5^3 - \tfrac{10}{27}a_5^3\vartheta a_5 + \tfrac{1}{81}a_5^5 - \tfrac{1}{3}\vartheta a_4 a_5^2 - \tfrac{1}{3}a_4a_5\vartheta a_5 + \tfrac{10}{9}a_5^2\vartheta^2 a_5 \\ &\quad - \tfrac{10}{3}\vartheta a_5 \vartheta^2 a_5 + \tfrac{1}{3}a_4\vartheta^2 a_5 - \tfrac{5}{3}a_5\vartheta^3 a_5. \end{aligned} \tag{5.51}$$

Proof. By Theorem 3.3 the Picard-Fuchs equation (5.50) satisfies the property (P), and the proof follows from Lemma 3.1 **(iii)** . ■

We know that if the Picard-Fuchs equation is hypergeometric, then it is of the form

$$\vartheta^6 - z(\vartheta + r_1)(\vartheta + r_2)(\vartheta + r_3)(\vartheta + r_4)(\vartheta + r_5)(\vartheta + r_6) = 0, \tag{5.52}$$

in which r_i's are constants, and by rewriting it in the form of equation (5.50), we have

$$\vartheta^6 = \frac{c_1 z}{1-z}\vartheta^5 + \frac{c_2 z}{1-z}\vartheta^4 + \frac{c_3 z}{1-z}\vartheta^3 + \frac{c_4 z}{1-z}\vartheta^2 + \frac{c_5 z}{1-z}\vartheta + \frac{c_6 z}{1-z}. \tag{5.53}$$

In the following Lemma we find the relationships among c_i's.

Lemma 5.7. *Following relationships hold for constants c_i's,*

$$\begin{aligned} c_3 &= -\tfrac{5}{27}c_1^3 + \tfrac{2}{3}c_1c_2, \\ c_3 &= -\tfrac{5}{6}c_1^2 + \tfrac{2}{6}c_1c_2 + \tfrac{5}{6}c_1 + c_2, \\ c_3 &= -\tfrac{5}{3}c_1 + 2c_2, \\ c_5 &= \tfrac{1}{81}c_1^5 - \tfrac{1}{27}c_1^3c_2 + \tfrac{1}{3}c_1c_4, \\ c_5 &= \tfrac{10}{108}c_1^4 - \tfrac{1}{108}c_1^3c_2 - \tfrac{10}{36}c_1^3 - \tfrac{1}{6}c_1^2c_2 + \tfrac{5}{12}c_1^2 + \tfrac{1}{3}c_1c_2 + \tfrac{1}{4}c_1c_4 - \tfrac{1}{4}c_1 - \tfrac{1}{4}c_2 + \tfrac{1}{4}c_4, \\ c_5 &= \tfrac{25}{54}c_1^3 - \tfrac{1}{9}c_1^2c_2 - \tfrac{10}{6}c_1^2 - \tfrac{1}{6}c_1c_2 + \tfrac{1}{6}c_1c_4 + \tfrac{11}{6}c_1 + \tfrac{3}{6}c_2 + \tfrac{3}{6}c_4, \\ c_5 &= \tfrac{5}{4}c_1^2 - \tfrac{7}{12}c_1c_2 + \tfrac{1}{12}c_1c_4 - \tfrac{11}{4}c_1 + \tfrac{3}{4}c_2 + \tfrac{3}{4}c_4, \\ c_5 &= c_1 - c_2 + c_4. \end{aligned} \tag{5.54}$$

In particular if $c_1 = 3$*, then the equations of* (5.54) *reduce to the following two equations:*

$$\begin{aligned} c_3 &= 2c_2 - 5, \\ c_5 &= c_4 - c_2 + 3. \end{aligned} \tag{5.55}$$

Proof. By Lemma 5.6, we have

$$\begin{aligned} c_3 &= \frac{(-5c_1^3 + 18c_1c_2)z^2 + (45c_1^2 - 18c_1c_2 - 45c_1 - 54c_2)z + (-45c_1 + 54c_2)}{27z^2 - 54z + 27}, \\ c_5 &= \frac{A_4z^4 + A_3z^3 + A_2z^2 + A_1z + A_0}{81z^4 - 324z^3 + 486z^2 - 324z + 81}, \end{aligned} \tag{5.56}$$

for any z, where

$$\begin{aligned} A_0 &= 81c_1 - 81c_2 + 81c_4, \\ A_1 &= -405c_1^2 + 189c_1c_2 - 27c_1c_4 + 891c_1 - 243c_2 - 243c_4, \\ A_2 &= 225c_1^3 - 54c_1^2c_2 - 810c_1^2 - 81c_1c_2 + 81c_1c_4 + 891c_1 + 243c_2 + 243c_4, \\ A_3 &= -30c_1^4 + 3c_1^3c_2 + 90c_1^3 + 54c_1^2c_2 - 135c_1^2 - 108c_1c_2 - 81c_1c_4 + 81c_1 + 81c_2 - 81c_4, \\ A_4 &= c_1^5 - 3c_1^3c_2 + 27c_1c_4. \end{aligned}$$

So by comparing the coefficients of z^i in (5.56) for $i = 0, 1, 2, 3, 4$, results follow. ■

Example 5.1. As we saw in Example 4.4 there are 40 families of Calabi-Yau 5-folds that each of them satisfies Assumption 5.1, and their Picard-Fuchs differential equations are hypergeometric of the form (5.52), such that $r_i = 1 - r_{i-3}$, $i = 4, 5, 6$, and r_i's are given in Table A.2. If we write their Picard-Fuchs equations in the form (5.53), then for all cases $c_1 = 3$, and other c_i's are given in Table A.3, for which the equations given in (5.55) hold.

In Lemma 5.1 we computed some enteries of the intersection matrix $\Omega = (\langle \omega_i, \omega_j \rangle)_{1 \leq i,j \leq 6}$ given in (5.11). In the following proposition we compute all entries of the intersection matrix in the case $n = 5$.

Proposition 5.2. *The matrix of intersection forms,* $\Omega = (\langle \omega_i, \omega_j \rangle)_{1 \leq i,j \leq 6}$*, is given by*

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \tilde{a} \\ 0 & 0 & 0 & 0 & -\tilde{a} & \Omega_{26} \\ 0 & 0 & 0 & \tilde{a} & \Omega_{35} & \Omega_{36} \\ 0 & 0 & -\tilde{a} & 0 & \Omega_{45} & \Omega_{46} \\ 0 & \tilde{a} & -\Omega_{35} & -\Omega_{45} & 0 & \Omega_{56} \\ -\tilde{a} & -\Omega_{26} & -\Omega_{36} & -\Omega_{46} & -\Omega_{56} & 0 \end{pmatrix}, \tag{5.57}$$

where

$$\tilde{a} = c_0 \exp\left(\frac{1}{3} \int_0^z a_5(v) \frac{dv}{v}\right), \tag{5.58}$$

for some nonzero constant c_0, and

$$\Omega_{26} = -\tfrac{2}{3}\tilde{a}a_5,$$

$$\Omega_{35} = \tfrac{1}{3}\tilde{a}a_5,$$

$$\Omega_{36} = \tilde{a}a_4 + \tfrac{4}{9}\tilde{a}a_5^2 - \tfrac{2}{3}\tilde{a}\vartheta a_5,$$

$$\Omega_{45} = -\tilde{a}a_4 - \tfrac{1}{3}\tilde{a}a_5^2 + \tilde{a}\vartheta a_5,$$

$$\Omega_{46} = -\tilde{a}a_3 - \tilde{a}a_4a_5 - \tfrac{8}{27}\tilde{a}a_5^3 + \tilde{a}\vartheta a_4 + \tfrac{4}{3}\tilde{a}a_5\vartheta a_5 - \tfrac{2}{3}\tilde{a}\vartheta^2 a_5,$$

$$\begin{aligned}\Omega_{56} = {}& \tilde{a}a_2 + \tfrac{2}{3}\tilde{a}a_3a_5 + \tilde{a}a_4^2 + \tilde{a}a_4a_5^2 + \tfrac{16}{81}\tilde{a}a_5^4 - \tilde{a}\vartheta a_3 - \tfrac{5}{3}\tilde{a}a_5\vartheta a_4 - \tfrac{16}{9}\tilde{a}a_5^2\vartheta a_5 \\ & -2\tilde{a}a_4\vartheta a_5 + \tfrac{4}{3}\tilde{a}(\vartheta a_5)^2 + \tilde{a}\vartheta^2 a_4 + \tfrac{16}{9}\tilde{a}a_5\vartheta^2 a_5 - \tfrac{2}{3}\tilde{a}\vartheta^3 a_5.\end{aligned}$$

Proof. By Lemma 5.1 we see that when $n = 5$, the matrix Ω is given by (5.57), and we just need to find the entries $\Omega_{26}, \Omega_{35}, \Omega_{36}, \Omega_{45}, \Omega_{46}, \Omega_{56}$. First note that from (5.58) it follows that,

$$\vartheta\tilde{a} = \frac{1}{3}\tilde{a}a_5. \tag{5.59}$$

By Picard-Fuchs equation (5.50) we know that

$$\vartheta\omega_6 = \vartheta^6\omega = a_0\omega_1 + a_1\omega_2 + a_2\omega_3 + a_3\omega_4 + a_4\omega_5 + a_5\omega_6. \tag{5.60}$$

Since $\langle\omega_1, \omega_6\rangle = \tilde{a}$, by considering (5.60) and the fact that $\langle\omega_1, \omega_i\rangle = 0$, for $i = 1, 2, \ldots, 5$, we find Ω_{26} as follow

$$\begin{aligned}&\vartheta\langle\omega_1, \omega_6\rangle = \langle\omega_2, \omega_6\rangle + \langle\omega_1, \vartheta\omega_6\rangle \\ &\qquad \Rightarrow \Omega_{26} = \vartheta\tilde{a} - \tilde{a}a_5 \Rightarrow \Omega_{26} = -\frac{2}{3}\tilde{a}a_5.\end{aligned}$$

To find the rest of entries, similarly we have

$$\begin{aligned}&\vartheta\langle\omega_2, \omega_5\rangle = \langle\omega_3, \omega_5\rangle + \langle\omega_2, \omega_6\rangle \\ &\qquad \Rightarrow \Omega_{35} = -(2\vartheta\tilde{a} - \tilde{a}a_5) \Rightarrow \Omega_{35} = \frac{1}{3}\tilde{a}a_5.\end{aligned}$$

$$\begin{aligned}\Omega_{36} &= \vartheta\Omega_{26} - \langle\omega_2, \vartheta\omega_6\rangle = \vartheta\Omega_{26} - a_4\Omega_{25} - a_5\Omega_{26} \\ &= \tilde{a}a_4 + \frac{4}{9}\tilde{a}a_5^2 - \frac{2}{3}\tilde{a}\vartheta a_5.\end{aligned}$$

$$\Omega_{45} = \vartheta\Omega_{35} - \Omega_{36} = \tilde{a}a_4 + \frac{4}{9}\tilde{a}a_5^2 - \frac{2}{3}\tilde{a}\vartheta a_5.$$

$$\Omega_{46} = \vartheta\Omega_{36} - \langle\omega_3, \vartheta\omega_6\rangle = \vartheta\Omega_{36} - a_3\Omega_{34} - a_4\Omega_{35} - a_5\Omega_{36}$$
$$= -\tilde{a}a_3 - \tilde{a}a_4a_5 - \frac{8}{27}\tilde{a}a_5^3 + \tilde{a}\vartheta a_4 + \frac{4}{3}\tilde{a}a_5\vartheta a_5 - \frac{2}{3}\tilde{a}\vartheta^2 a_5.$$

$$\Omega_{56} = \vartheta\Omega_{46} - \langle\omega_4, \vartheta\omega_6\rangle = \vartheta\Omega_{46} - a_2\Omega_{43} - a_4\Omega_{45} - a_5\Omega_{46}$$
$$= \tilde{a}a_2 + \frac{2}{3}\tilde{a}a_3a_5 + \tilde{a}a_4^2 + \tilde{a}a_4a_5^2 + \frac{16}{81}\tilde{a}a_5^4 - \tilde{a}\vartheta a_3 - \frac{5}{3}\tilde{a}a_5\vartheta a_4 - \frac{16}{9}\tilde{a}a_5^2\vartheta a_5$$
$$- 2\tilde{a}a_4\vartheta a_5 + \frac{4}{3}\tilde{a}(\vartheta a_5)^2 + \tilde{a}\vartheta^2 a_4 + \frac{16}{9}\tilde{a}a_5\vartheta^2 a_5 - \frac{2}{3}\tilde{a}\vartheta^3 a_5.$$

Thus the proof is complete. ■

Let T be the moduli space presented in Theorem 5.1, for $n = 5$. We are going to use the chart which was presented in the proof of Proposition 5.1, and find a chart for T , for which we can compute the DHR vector field H explicitly.

Theorem 5.3. *Let T be the moduli space introduced in Theorem 5.1, for $n = 5$. Then there is a chart $(t_0, t_1, \dots, t_{12})$ for T such that in this chart,*

$$y_1 = \frac{t_3^2}{t_1 t_6}, \qquad \& \qquad y_2 = \frac{\tilde{a}t_3t_6^2}{t_1},$$

and DHR *vector field* H *is given by*

$$\mathrm{H} = \sum_{i=0}^{12} \mathrm{H}_i \frac{\partial}{\partial t_i}$$

where,

$$\mathrm{H}_0 = \frac{t_0t_3}{t_1}, \qquad \mathrm{H}_1 = t_2, \qquad \mathrm{H}_2 = \frac{t_3^2 t_4}{t_1 t_6},$$
$$\mathrm{H}_3 = \frac{-t_2t_3t_6 + t_3^2 t_5}{t_1t_6} \qquad \mathrm{H}_4 = \frac{\tilde{a}t_3t_6^2t_7}{t_1}, \qquad \mathrm{H}_5 = \frac{-t_3t_4 + \tilde{a}t_3t_6^2t_8}{t_1},$$
$$\mathrm{H}_6 = \frac{-t_3t_5 + \tilde{a}t_3t_6^2t_9}{t_1}, \qquad \mathrm{H}_7 = \frac{-t_3^2t_{10}}{t_1t_6}, \qquad \mathrm{H}_8 = \frac{-t_3^2t_{11} - t_3t_6t_7}{t_1t_6},$$
$$\mathrm{H}_9 = \frac{3\tilde{a}t_3t_5t_9 - 6\tilde{a}t_3t_6t_8 - 3t_3a_4 - t_3a_5^2 + 3t_3\vartheta a_5}{3\tilde{a}t_1t_6}, \qquad \mathrm{H}_{10} = -t_{12},$$
$$\mathrm{H}_{11} = \frac{27\tilde{a}t_2t_{11} - 54\tilde{a}t_3t_{10} + 27\tilde{a}t_4t_8 - 27\tilde{a}t_5t_7 + 27a_2 + 27a_3a_5 - 27\vartheta a_3 + 15a_4a_5^2}{27\tilde{a}t_1}$$
$$+ \frac{-9a_4\vartheta a_5 - 54a_5\vartheta a_4 + 27\vartheta^2 a_4 + 4a_5^4 - 42a_5^2\vartheta a_5 + 54a_5\vartheta^2 a_5 + 18(\vartheta a_5)^2 - 18\vartheta^3 a_5}{27\tilde{a}t_1},$$
$$\mathrm{H}_{12} = \frac{-t_3a_0}{\tilde{a}t_1^2}.$$

Proof. By Theorem 5.1 we know that T is 13-dimensional. Using the equation

$$\Phi = S\Omega S^{\mathsf{t}}, \tag{5.61}$$

and considering $s_{11}, s_{21}, s_{22}, s_{31}, s_{32}, s_{33}, s_{41}, s_{42}, s_{43}, s_{51}, s_{52}, s_{61}$, as independent entries of S, then dependent entries are given in the following equations,

$$\begin{aligned}
s_{44} &= \tfrac{1}{\tilde{a}s_{33}}, \\
s_{53} &= \tfrac{-3\tilde{a}s_{32}s_{43}+3\tilde{a}s_{33}s_{42}+3a_4+a_5^2-3\vartheta a_5}{3\tilde{a}s_{22}}, \\
s_{54} &= \tfrac{-3s_{32}+s_{33}a_5}{3\tilde{a}s_{22}s_{33}}, \\
s_{55} &= -\tfrac{1}{\tilde{a}s_{22}}, \\
s_{62} &= \tfrac{-27\tilde{a}s_{21}s_{52}+27\tilde{a}s_{22}s_{51}-27\tilde{a}s_{31}s_{42}+27\tilde{a}s_{32}s_{41}-27a_2-27a_3a_5+27\vartheta a_3-15a_4a_5^2}{27\tilde{a}s_{11}} \\
&\quad + \tfrac{9a_4\vartheta a_5+54a_5\vartheta a_4-27\vartheta^2a_4-4a_5^4+42a_5^2\vartheta a_5-54a_5\vartheta^2a_5-18(\vartheta a_5)^2+18\vartheta^3a_5}{27\tilde{a}s_{11}},
\end{aligned} \tag{5.62}$$

$$\begin{aligned}
s_{63} &= \tfrac{27\tilde{a}s_{21}s_{32}s_{43}-27\tilde{a}s_{21}s_{33}s_{42}-27\tilde{a}s_{22}s_{31}s_{43}+27\tilde{a}s_{22}s_{33}s_{41}-27s_{21}a_4-9s_{21}a_5^2}{27\tilde{a}s_{11}s_{22}} \\
&\quad + \tfrac{27s_{21}\vartheta a_5-27s_{22}a_3-9s_{22}a_4a_5+27s_{22}\vartheta a_4-2s_{22}a_5^3+18s_{22}a_5\vartheta a_5-18s_{22}\vartheta^2a_5}{27\tilde{a}s_{11}s_{22}}, \\
s_{64} &= \tfrac{9s_{21}s_{32}-3s_{21}s_{33}a_5-9s_{22}s_{31}-9s_{22}s_{33}a_4-2s_{22}s_{33}a_5^2+6s_{22}s_{33}\vartheta a_5}{9\tilde{a}s_{11}s_{22}s_{33}}, \\
s_{65} &= \tfrac{3s_{21}-2s_{22}a_5}{3\tilde{a}s_{11}s_{22}}, \\
s_{66} &= \tfrac{1}{\tilde{a}s_{11}}.
\end{aligned} \tag{5.63}$$

We know that $\mathcal{W}$ is a family of one parameter 5-dimensional Calabi-Yau manifolds parameterized by z. So we present a chart of T given by

$$t = (t_0, t_1, \ldots, t_{12})$$

where $t_0 = z,\ t_1 = s_{11},\ t_2 = s_{21},\ t_3 = s_{22},\ t_4 = s_{31},\ t_5 = s_{32},\ t_6 = s_{33},\ t_7 = s_{41},\ t_8 = s_{42},\ t_9 = s_{43},\ t_{10} = s_{51},\ t_{11} = s_{52},\ t_{12} = s_{61}$.

If we consider $\nabla\alpha = \mathsf{GM}_\alpha \otimes \alpha$, then we saw in (5.20) that

$$\mathsf{GM}_\alpha = (dS + S.\mathsf{GM}_\varpi)S^{-1}. \tag{5.64}$$

We want to have $\nabla_{\mathrm{H}}\alpha = Y\alpha$, so the same as the proof of Proposition 5.1, define H as follow:

$$\mathrm{H} = \sum_{i=0}^{12} \mathrm{H}_i(t)\frac{\partial}{\partial t_i},$$

and as well set $\dot{t}_i := \mathrm{H}_i(t)$. Then H_0 follows from Lemma 5.3**(i)**, and we can find the rest of H_i's from the equation

$$\dot{S} = YS - \dot{z}S.\widehat{\mathsf{GM}}_{\varpi}, \tag{5.65}$$

after using the equations given in (5.62) and (5.63) and also substituting a_1 and a_3 from Lemma 5.6. What remains to verify is the compatibility of (5.61) and (5.65). After doing calculations, it directly follows that (5.62), (5.63) and (5.65) give the same $\dot{s}_{44}, \dot{s}_{54}, \dot{s}_{55}, \dot{s}_{64}, \dot{s}_{65}, \dot{s}_{66}$. By using of Lemma 5.6, we find the same $\dot{s}_{53}, \dot{s}_{62}, \dot{s}_{63}$, from (5.62), (5.63) and (5.65), and the proof is complete. ■

5.4 Three-Dimensional Case

In this section also we are working under Assumption 5.1 and the notations of §5.1, for $n = 3$. The statements of this section will be similar to the ones in §5.3.

For the one parameter family $\mathcal{W}$ of Calabi-Yau 3-folds, the Picard-Fuchs equation of the fixed non-vanishing holomorphic 3-form ω is given as

$$\vartheta^4 = a_0(z) + a_1(z)\vartheta + a_2(z)\vartheta^2 + a_3(z)\vartheta^3. \tag{5.66}$$

Lemma 5.8. *The coefficients a_i's given in equation* (5.66) *satisfy the following relationship,*

$$a_1 = \frac{3}{4}a_3\vartheta a_3 + \vartheta a_2 - \frac{1}{2}\vartheta^2 a_3 - \frac{1}{8}a_3^3 - \frac{1}{2}a_2 a_3. \tag{5.67}$$

Proof. We know that the equation (5.66) satisfies the property (P), and the proof follows from Lemma 3.1**(ii)**. ■

In the following proposition we provide all the entries of the intersection matrix (5.11), for $n = 3$.

Proposition 5.3. *Let $\Omega = (\langle\omega_i, \omega_j\rangle)_{1\le i,j\le 4}$ be the matrix of intersection forms. Then*

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & \tilde{a} \\ 0 & 0 & -\tilde{a} & \Omega_{24} \\ 0 & \tilde{a} & 0 & \Omega_{34} \\ -\tilde{a} & -\Omega_{24} & -\Omega_{34} & 0 \end{pmatrix}, \tag{5.68}$$

in which

$$\tilde{a} = c_0 \exp\left(\frac{1}{2}\int_0^z a_3(v)\frac{dv}{v}\right), \tag{5.69}$$

for a constant $c_0 \neq 0$, *and*

$$\Omega_{24} = -\tfrac{1}{2}\tilde{a}a_3,$$

$$\Omega_{34} = \tfrac{1}{4}\tilde{a}a_3^2 + \tilde{a}a_2 - \tfrac{1}{2}\tilde{a}\vartheta a_3.$$

Proof. The proof is similar to the proof of Proposition 5.2. ■

The following theorem is an analogous version of Theorem 5.3, for $n = 3$, to compute the DHR vector field H explicitly.

Theorem 5.4. *Let* T *be the moduli space introduced in Theorem 5.1, for* $n = 3$. *Then there is a chart* $(t_0, t_1, \dots, t_6)$ *for* T *such that in this chart,*

$$y_1 = -\frac{\tilde{a}t_3^3}{t_1},$$

and the DHR *vector field* H *is given by*

$$\mathrm{H} = \sum_{i=0}^{6} \mathrm{H}_i \frac{\partial}{\partial t_i}$$

in which,

$$\mathrm{H}_0 = \frac{t_0 t_3}{t_1}, \qquad \mathrm{H}_1 = t_2,$$

$$\mathrm{H}_2 = -\frac{\tilde{a}t_3^3 t_4}{t_1}, \qquad \mathrm{H}_3 = -\frac{t_2 t_3 + \tilde{a}t_3^3 t_5}{t_1}$$

$$\mathrm{H}_4 = -t_6, \qquad \mathrm{H}_5 = \frac{4\tilde{a}t_2 t_5 - 8\tilde{a}t_3 t_4 + a_3^2 + 4a_2 - 2\vartheta a_3}{4\tilde{a}t_1},$$

$$\mathrm{H}_6 = -\frac{t_3 a_0}{\tilde{a}t_1^2}.$$

Proof. The same as the proof of Theorem 5.3, analogously we can work for $n = 3$. First note that T is 7-dimensional and by considering $s_{11}, s_{21}, s_{22}, s_{31}, s_{32}, s_{41}$ as independent variables, the equation

$$\Phi = S\Omega S^{\mathsf{t}}, \tag{5.70}$$

implies the following

$$\begin{aligned} s_{33} &= -\tfrac{1}{\tilde{a}s_{22}}, \\ s_{42} &= \tfrac{4\tilde{a}s_{22}s_{31} - 4\tilde{a}s_{21}s_{32} - a_3^2 - 4a_2 + 2\vartheta a_3}{4\tilde{a}s_{11}}, \\ s_{43} &= \tfrac{2s_{21} - s_{22}a_3}{2\tilde{a}s_{11}s_{22}}, \\ s_{44} &= \tfrac{1}{\tilde{a}s_{11}}. \end{aligned} \tag{5.71}$$

Next we present a chart of T given by $t=(t_0,t_1,\ldots,t_6)$, where $t_0=z,\ t_1=s_{11},\ t_2=s_{21},\ t_3=s_{22},\ t_4=s_{31},\ t_5=s_{32},\ t_6=s_{41}$, and then H_i's follow from Lemma 5.3**(i)** and equation

$$\dot{S}=YS-\dot{z}S.\widehat{\mathsf{GM}}_{\varpi}. \tag{5.72}$$

Finally the Lemma 5.8, guaranties the compatibility of (5.70) and (5.72). ∎

Example 5.2. As we saw in Table A.1, there are 14 families of one-parameter Calabi-Yau 3-folds that satisfy Assumption 3.1, and Theorem 5.4 holds for them. In particular, H. Movasati in [31, 33] worked on the family $\sharp 1$ with more details, that we announced it in Theorem 1.1.

Chapter 6

Related Problems

During my works on this thesis, I encountered with various natural problems that seem interesting. Hence it worth to organize them for more future researches. And since they are directly related to my thesis, I state them in this chapter in different sections.

6.1 The Moduli Space

Let T be the moduli space given in Theorem 5.1. By now, the structure of moduli space T is not well known to us, and it needs a deep study in both geometrical and algebraical structures. *So one of our goal is to work on such moduli spaces in a general context, which is predicted to be a quasi-affine space.*

6.2 Yukawa Coupling

Let $\pi : \mathcal{W} \to P$ be a one parameter family of n-dimensional Calabi-Yau manifolds satisfying Assumption 5.1.

In the case of 3-folds, i.e. $n = 3$, D. Morrison in [29] defines the *Yukawa coupling* the first non-zero l-point Yukawa function which is given by

$$\mathcal{W}_{0,l}(z) = \int_{W_z} \omega(z) \wedge \vartheta^l \omega(z).$$

By Proposition 3.3 we know that $\mathcal{W}_{0,0} = \mathcal{W}_{0,1} = \mathcal{W}_{0,2} = 0$, so for Calabi-Yau 3-folds Yukawa coupling is the function $\mathcal{W}_{0,3}$. If we apply the replacement $\omega(z) \mapsto f(z)\omega(z)$, then $\mathcal{W}_{0,l}$ transforms as

$$\mathcal{W}_{0,l} \mapsto f(z) \sum_{j=0}^{l} \binom{l}{j} \frac{d^j f(z)}{dz^j} \mathcal{W}_{0,l-j}.$$

Since $\mathcal{W}_{0,0} = \mathcal{W}_{0,1} = \mathcal{W}_{0,2} = 0$, the change in the Yukawa coupling is simply $\mathcal{W}_{0,3} \mapsto f(z)^2 \mathcal{W}_{0,3}$.

For the first time, by using the mirror symmetry, Candelas et al. in [8] have computed the coefficients of the q-expansion of the Yukawa coupling for quintic 3-folds in $\mathbb{P}^4$. More detailed, the coefficients $n_d d^3$ of the q-expansion of the normalized Yukawa coupling

$$\mathcal{W}_{0,3} = 5 + \sum_{d=1}^{\infty} (n_d d^3) \frac{q^d}{1-q^d}, \; q = e^{2\pi i z},$$

are conjectured to be the Gromov-Witten invariants of rational curves of degree d on a quintic 3-fold in $\mathbb{P}^4$. The integers n_d predict numbers of rational curves of degree d on quintic 3-folds. And several of these numbers have been confirmed by other researchers in mathematics and mathematical physic. And then, Morrison in [29] as well computed the coefficients of q-expansions of the normalized Yukawa coupling of the families $\sharp$ 2, 3, 4 given in Table A.1, that they predict the numbers of rational curves on the weighted projective hypersurfaces. For more details one refers to [8, 29, 30].

Then in [31, 33] H. Movasati changed his point of view and worked on a special moduli space of mirror quintic 3-folds. As we saw in Theorem 1.1, he found a unique vector field Ra on the moduli space T such that satisfies the equation $\nabla_{\mathrm{Ra}}\alpha = \alpha Y$, given in (1.17). In the chart that he provided for T , he proved that the function $y = \frac{5^8(t_4-t_0^5)^2}{t_5^3}$, which appears as an entry of Y, is the Yukawa coupling computed by Candelas et al. And in [31] he computed the same n_d's, which appear in the coefficients of q-expansion of normalized Yukawa coupling, in the new chart, and also he computed some coefficients of q-expansions of the chart functions and, up to multiplying by a constant, all these coefficients are integers.

Now we back to our DHR vector field H that we introduced it in Theorem 5.1. Here we are faced with a situation the same as H. Movasati's work. That is, we have a special moduli space T of Calabi-Yau n-folds and H is a unique vector field on T that satisfies $\nabla_{\mathrm{H}}\alpha = Y\alpha$, given in (5.47). We know that $n-2$ numbers of entries of the matrix Y are functions $y_1, y_2, \ldots, y_{n-2}$. But in Lemma 5.3(iii) we saw that $y_{i-1} = -y_{n-i}$, for $i \neq \frac{n+1}{2}$, hence we have $\frac{n-1}{2}$ numbers of functions $y_1, y_2, \ldots, y_{\frac{n-1}{2}}$, that we call them *Yukawa couplings of the family $\mathcal{W}$. And the question is that if we consider the q-expansion of these functions, what kind of information they will give us?* If we discuss in the context of Morrison, then just the Yukawa coupling

$$y_{\frac{n-1}{2}} = (-1)^{\frac{n+3}{2}} \frac{\tilde{a} s_{22} s^2_{\frac{n+1}{2}\frac{n+1}{2}}}{s_{11}}$$

is in relationship with the n-point Yukawa function $\mathcal{W}_{0,n} = \langle \omega, \vartheta^n \omega \rangle$. I think that $y_{\frac{n-1}{2}}$ plays a more important role.

6.3 Geometric Structure of Calabi-Yau 5-Folds

H. Movasati and K. M. Shokri in [35] worked on the generalized hypergeometric differential equation

$$\vartheta^n - z(\vartheta + r_1)(\vartheta + r_2)\ldots(\vartheta + r_n) = 0, \tag{6.1}$$

in which, $\vartheta = z\frac{\partial}{\partial z}$ and r_i's are rational numbers with $0 < r_i < 1$. A holomorphic solution of (6.1) is given by

$$F(r|z) := \ {}_nF_{n-1}(r_1, \ldots, r_n; 1, 1, \ldots, 1|z) = \sum_{k=0}^{\infty} \frac{(r_1)_k \ldots (r_n)_k}{k!^n} z^k, \quad |z| < 1$$

where, $r = (r_1, r_2, \ldots, r_n)$ and $(r_i)_k = r_i(r_i + 1)(r_i + 2)...(r_i + k - 1)$, $(r_i)_0 = 1$, is the Pochhammer symbol. As well, the first logarithmic solution in the Frobenius basis around $z = 0$ has the form $G(r|z) + F(r|z)\log z$, where

$$G(r|z) = \sum_{k=1}^{\infty} \frac{(r_1)_k \cdots (r_n)_k}{(k!)^n} \Big[\sum_{j=1}^{n} \sum_{i=0}^{k-1} (\frac{1}{r_j + i} - \frac{1}{1+i})\Big] z^k. \tag{6.2}$$

The mirror map

$$q(r|z) =: z\exp(\frac{G(r|z)}{F(r|z)}),$$

is a natural generalization of the Schwarz function. It is said that $q(r|z)$ is N-integral, if there is a natural number N such that $q(r|Nz)$ has integral coefficients. For even integer $n \geq 4$ with an N-integral mirror map $q(r|z)$, the set $\{r_1, r_2, \ldots, r_n\}$ is conjecturally invariant under $x \mapsto 1 - x$, and so we may identify $r = (r_1, r_2, \ldots, r_n)$ with its $\frac{n}{2}$ elements in the interval $[0, \frac{1}{2}]$. In the case $n = 4$, the Authors of [35] found that $q(a|z)$ is N-integral if and only if (r_1, r_2, r_3, r_4) belongs to the well-known 14 hypergeometric cases of Calabi-Yau equations, such that the first two elements (r_1, r_2) are given in Table A.1 and $r_3 = 1 - r_2,\ r_4 = 1 - r_1$. As we see in Table A.1, for any of these 14 hypergeometric differential equations L, there is a one parameter family of Calabi-Yau 3-folds that L is the Picard-Fuchs equation of a 3-form on this family, and in the same table the geometric structure of Calabi-Yau 3-folds and the references are given.

For $n = 6$ they found 40 examples of N-integral mirror maps. In this case (r_1, r_2, r_3) are given in Table A.2. As well as the case $n = 4$, we expect that in this case also there are 40 numbers of one parameter families of Calabi-Yau 5-folds. In [2], Batyrev and Starten provide some geometrical structures of complete intersections in weighted projective spaces. I think that by following these structures we can find and explain the topological, geometrical and algebraical structures of these 40 examples.

6.4 Semi-complete Vector Fields

For the first time J. C. Rebelo [38] in 1996 introduced the concept of semi-complete vector fields, and then Adolfo Guillot [19] in his thesis worked on these vector fields and gave some examples, including Halphen type vector fields, in $\mathbb{C}^3$. In fact he called these vector fields *uniformizable* instead of semi-complete. We first recall the following definition of semi-complete vector fields from [20]:

Definition 6.1. A holomorphic vector field H on a complex manifold W is *semi-complete* if for every $p \in W$ there exists a connected domain $0 \in U_p \subset \mathbb{C}$ and a map $\phi_p : U_p \to W$ such that:

(i) $\phi_p(0) = p$,

(ii) $\frac{d\phi_p(t)}{dt}|_{t=t_0} = \mathrm{H}(\phi_p(t_0))$,

(iii) for every sequence $\{t_i\} \subset U_p$ such that $\lim\limits_{i\to\infty} t_i \in \partial U_p$, the sequence $\{\phi_p(t_i)\}$ escapes from every compact subset of W.

A meromorphic vector field R on W is semi-complete if it is semi-complete in restriction to the open set where R is holomorphic.

If we look better to Definition 6.1, the first two conditions say that ϕ_p is a solution of vector field H with initial condition $\phi_p(0) = p$, and the third condition imply that the domain U_p is maximal, i.e., the domain of the solution ϕ_p can not be extended to a larger set. Of course every complete vector field is semi-complete, and hence this definition weakens the definition of completeness of a vector field. Indeed, this definition formalizes the notion of differential equations having only single-valued solutions. An interesting examples of semi-complete vector fields in $\mathbb{C}^3$ belong to the family of the Halphen's equation given by

$$\begin{cases} \frac{dt_1}{dz} = a_1 t_1^2 + (1-a_1)(t_1t_2 + t_1t_3 - t_2t_3) \\ \frac{dt_2}{dz} = a_2 t_2^2 + (1-a_2)(t_2t_3 + t_2t_1 - t_3t_1) \\ \frac{dt_3}{dz} = a_3 t_3^2 + (1-a_3)(t_3t_1 + t_3t_2 - t_1t_2) \end{cases}, \tag{6.3}$$

where the classification of semi-complete vector fields, which has single valued solutions, within this family is done by Halphen in [21]. One of this vector fields is when $a_1 = a_2 = a_3 = 0$ and $\lambda = 1$, i.e., following vector field

$$\mathrm{H} : \begin{cases} \frac{dt_1}{dz} = t_1t_2 + t_1t_3 - t_2t_3 \\ \frac{dt_2}{dz} = t_2t_3 + t_2t_1 - t_3t_1 \\ \frac{dt_3}{dz} = t_3t_1 + t_3t_2 - t_1t_2 \end{cases}. \tag{6.4}$$

Let $E = \sum_{j=1}^3 t_j \frac{\partial}{\partial t_j}$ and $Z = \sum_{j=1}^3 \frac{\partial}{\partial t_j}$. Then one can easily verify that

$$[E, \mathrm{H}] = \mathrm{H}, \qquad [E, Z] = -Z, \qquad [Z, \mathrm{H}] = 2E, \tag{6.5}$$

where $[.,.]$ refers to Lie bracket. Considering these equations gives the idea of following definition which is given by Guillot.

Definition 6.2. A quadratic homogeneous vector field H in $\mathbb{C}^3$ is called *Halphen tyepe* if there exist a rational vector field Z, homogeneous of degree zero, such that $[Z, \mathrm{H}] = 2E$.

And one can see that the homogeneity of the vector fields given in this definition implies als $[E,\mathrm{H}] = \mathrm{H}$, $[E,Z] = -Z$, so the vector fields H, Z and E construct a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Guillot also proves that the local solutions of a Halphen type vector field is invariant under the change of variables (1.24) and (1.25). Then he studies the Halphen type vector fields and give a criterion for semi-completeness of this vector fields. As well as Halphen vector field, Guillot and Rebelo in [20] study *Ramanujan* vector field

$$\mathrm{R}:\begin{cases} \frac{dr_1}{d\tau} = r_1^2 - \frac{1}{12}r_2 \\ \frac{dr_2}{d\tau} = 4r_1r_2 - 6r_3 \\ \frac{dr_3}{d\tau} = 6r_1r_3 - \frac{1}{3}r_2^2 \end{cases}, \tag{6.6}$$

and prove that it is meromorphic semi-complete.

Thus considering the above facts, the questions are that: *What we can say about* DHR *vector field* H*? Does it satisfies some conditions similar to what given in* (6.5)*? What we can say about the semi-completeness or meromorphic semi-completeness of* DHR *vector field? Does its solutions satisfy some invariant properties?*

Appendix A

Tables

Here we provide several tables, which include information about Picard-Fuchs equation of Calabi-Yau manifolds.

$\sharp$	r_1	r_2	c	Description	References
1	1/5	2/5	5^5	$X(5) \subset \mathbb{P}^4$	[8, 16]
2	1/6	2/6	$2^5 3^6$	$X(6) \subset \mathbb{P}^4(2,1,1,1,1)$	[29]
3	1/8	3/8	2^{18}	$X(8) \subset \mathbb{P}^4(4,1,1,1,1)$	[29]
4	1/10	3/10	$2^9 5^6$	$X(10) \subset \mathbb{P}^4(5,2,1,1,1)$	[29]
5	1/3	1/3	3^6	$X(3,3) \subset \mathbb{P}^5$	[28]
6	1/4	2/4	2^{10}	$X(2,4) \subset \mathbb{P}^5$	[28]
7	1/3	1/2	$2^4 3^3$	$X(2,2,3) \subset \mathbb{P}^6$	[28]
8	1/2	1/2	2^8	$X(2,2,2,2) \subset \mathbb{P}^7$	[28]
9	1/4	1/4	2^{12}	$X(4,4) \subset \mathbb{P}^5(2,2,1,1,1,1)$	[27]
10	1/6	1/6	$2^8 3^6$	$X(6,6) \subset \mathbb{P}^5(3,3,2,2,1,1)$	[27]
11	1/4	1/3	$2^6 3^3$	$X(3,4) \subset \mathbb{P}^5(2,1,1,1,1,1)$	[27]
12	1/6	3/6	$2^8 3^3$	$X(2,6) \subset \mathbb{P}^5(3,1,1,1,1,1)$	[27]
13	1/6	1/4	$2^{10} 3^3$	$X(4,6) \subset \mathbb{P}^5(3,2,2,1,1,1)$	[27]
14	1/12	5/12	12^6	$X(2,12) \subset \mathbb{P}^5(6,4,1,1,1,1)$	[15]

Table A.1: Calabi-Yau 3-folds with $h^{1,1} = 1$. Here $X(d_1, d_2, \ldots, d_r) \subset \mathbb{P}^s(l_0, l_1, \ldots, l_s)$ refers to the complete intersection of r hypersurfaces of degree $d_1, d_2, \ldots, d_r$, in weighted projective space $\mathbb{P}^s(l_0, l_1, \ldots, l_s)$ with $r \leq s$, such that $\sum_{i=1}^r d_i = \sum_{j=0}^s l_j$

♯	r_1	r_2	r_3	r_4	r_5	r_6
1	1/2	1/2	1/2	1/2	1/2	1/2
2	2/3	2/3	2/3	1/3	1/3	1/3
3	3/4	1/2	1/2	1/2	1/2	1/4
4	3/4	3/4	1/2	1/2	1/4	1/4
5	3/4	3/4	3/4	1/4	1/4	1/4
6	2/3	1/2	1/2	1/2	1/2	1/3
7	2/3	2/3	1/2	1/2	1/3	1/3
8	5/6	1/2	1/2	1/2	1/2	1/6
9	5/6	2/3	1/2	1/2	1/3	1/6
10	5/6	2/3	2/3	1/3	1/3	1/3
11	5/6	5/6	1/2	1/2	1/6	1/6
12	5/6	5/6	2/3	1/3	1/6	1/6
13	5/6	5/6	5/6	1/6	1/6	1/6
14	6/7	5/7	4/7	3/7	2/7	1/7
15	7/8	5/8	1/2	1/2	3/8	1/8
16	7/8	3/4	5/8	3/8	1/4	1/8
17	8/9	7/9	5/9	4/9	2/9	1/9
18	4/5	3/5	1/2	1/2	2/5	1/5
19	9/10	7/10	1/2	1/2	3/10	1/10
20	3/4	2/3	1/2	1/2	1/3	1/4
21	3/4	2/3	2/3	1/3	1/3	1/4
22	3/4	3/4	2/3	1/3	1/4	1/4
23	5/6	3/4	1/2	1/2	1/4	1/6
24	5/6	3/4	2/3	1/3	1/4	1/6
25	5/6	3/4	3/4	1/4	1/4	1/4
26	5/6	5/6	3/4	1/4	1/6	1/6
27	11/12	7/12	1/2	1/2	5/12	1/12
28	11/12	2/3	7/12	5/12	1/3	1/12
29	11/12	3/4	7/12	5/12	1/4	1/12
30	11/12	5/6	7/12	5/12	1/6	1/12
31	13/14	11/14	9/14	5/14	3/14	1/14
32	4/5	2/3	3/5	2/5	1/3	1/5
33	17/18	13/18	11/18	7/18	5/18	1/18
34	4/5	3/4	3/5	2/5	1/4	1/5
35	9/10	3/4	7/10	3/10	1/4	1/10
36	7/8	2/3	5/8	3/8	1/3	1/8
37	7/8	5/6	5/8	3/8	1/6	1/8
38	5/6	4/5	3/5	2/5	1/5	1/6
39	9/10	7/10	2/3	1/3	3/10	1/10
40	9/10	5/6	7/10	3/10	1/6	1/10

Table A.2: Constants r_i's of Picard-Fuchs equation $L=\vartheta^6-z(\vartheta+r_1)(\vartheta+r_2)(\vartheta+r_3)(\vartheta+r_4)(\vartheta+r_5)(\vartheta+r_6)$ and it is predicted that in any case, L is a Picard-Fuchs equation of a one parameter family of Calabi-Yau 5-folds.

♯	c_1	c_2	c_3	c_4	c_5	c_6
1	3	15/4	5/2	15/16	3/16	1/64
2	3	11/3	7/3	22/27	4/27	8/729
3	3	59/16	19/8	27/32	5/32	3/256
4	3	29/8	9/4	193/256	33/256	9/1024
5	3	57/16	17/8	171/256	27/256	27/4096
6	3	67/18	22/9	43/48	25/144	1/72
7	3	133/36	43/18	277/324	13/81	1/81
8	3	131/36	41/18	37/48	19/144	5/576
9	3	65/18	20/9	949/1296	157/1296	5/648
10	3	43/12	13/6	25/36	1/9	5/729
11	3	127/36	37/18	799/1296	115/1296	25/5184
12	3	7/2	2	251/432	35/432	25/5832
13	3	41/12	11/6	205/432	25/432	125/46656
14	3	25/7	15/7	232/343	36/343	720/117649
15	3	115/32	35/16	2889/4096	457/4096	105/16384
16	3	113/32	33/16	2545/4096	369/4096	315/65536
17	3	95/27	55/27	1318/2187	184/2187	2240/531441
18	3	73/20	23/10	1971/2500	173/1250	6/625
19	3	71/20	21/10	6439/10000	939/10000	189/40000
20	3	527/144	167/72	463/576	83/576	1/96
21	3	523/144	163/72	991/1296	43/324	1/108
22	3	259/72	79/36	1649/2304	91/768	1/128
23	3	515/144	155/72	197/288	31/288	5/768
24	3	511/144	151/72	3355/5184	511/5184	5/864
25	3	253/72	73/36	1385/2304	67/768	5/1024
26	3	499/144	139/72	1391/2592	185/2592	25/6912
27	3	257/72	77/36	13849/20736	2041/20736	385/82944
28	3	85/24	25/12	4363/6912	619/6912	385/93312
29	3	505/144	145/72	12139/20736	1627/20736	385/110592
30	3	83/24	23/12	1201/2304	145/2304	1925/746496
31	3	97/28	27/14	415/784	51/784	19305/7529536
32	3	163/45	101/45	4216/5625	716/5625	16/1875
33	3	377/108	107/54	19645/34992	2473/34992	85085/34012224
34	3	287/80	87/40	7009/10000	567/5000	9/1250
35	3	279/80	79/40	11253/20000	1503/20000	567/160000
36	3	1027/288	307/144	24625/36864	3761/36864	35/6144
37	3	1003/288	283/144	20497/36864	2705/36864	175/49152
38	3	637/180	187/90	14239/22500	1057/11250	2/375
39	3	317/90	92/45	54701/90000	2567/30000	21/5000
40	3	619/180	169/90	44951/90000	1817/30000	21/8000

Table A.3: Constants c_i's of Picard-Fuchs equation $L = \vartheta^6 - (\frac{c_1 z}{1-z}\vartheta^5 + \frac{c_2 z}{1-z}\vartheta^4 + \frac{c_3 z}{1-z}\vartheta^3 + \frac{c_4 z}{1-z}\vartheta^2 + \frac{c_5 z}{1-z}\vartheta + \frac{c_6 z}{1-z})$ and it is predicted that in any case, L is a Picard-Fuchs equation of a one parameter family of Calabi-Yau 5-folds.

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