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# Well-Posedness for a Generalized Nonlinear Derivative Schrödinger Equation

**Gleison do Nascimento Santos**

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*Dedicado à minha família*

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# Abstract

In this work we study the well-posedness for the initial value problem associated to a generalized derivative Schrödinger equation for small size initial data in weighted Sobolev space. The techniques used include parabolic regularization method combined with sharp linear estimates. An important point in our work is that the contraction principle is likely to fail but gives us inspiration to obtain certain uniform estimates that are crucial to obtain the main result. To prove such uniform estimates we assume smallness on the initial data in weighted Sobolev spaces.

# Resumo

Neste trabalho estudamos a boa colocação para o problema de valor inicial associado a equação de Schrödinger com derivada generalizada com dado inicial pequeno em espaço de Sobolev com peso. A técnica que empregamos para conseguir tal resultado é o método da regularização parabólica mais estimativas ótimas da solução do problema linear. Um ponto importante no nosso trabalho é que o argumento do princípio de contração tende a falhar mas ele nos da inspiração para obter certas estimativas uniformes que são cruciais na obtenção do resultado principal. Para estabelecer tais estimativas uniformes, nós assumimos dados iniciais pequenos em espaços com peso.

# Notations

- The notation $A \lesssim B$ means there is a constant $c > 0$ such that $A \leq cB$. And we say, $A \sim B$ when $A \lesssim B$ and $B \lesssim A$.
- For a real number $r$ we shall denote $r+$ instead of $r + \epsilon$, whenever $\epsilon$ is a positive number whose value is not important. Also we write $T^{+}$, to denote $T$ raised to some positive power.
- We denote $\langle x \rangle = (1 + x^2)^{1/2}$, called Japanese bracket.
- If $f \in L^1(\mathbb{R})$ then we denote $\hat{f}$ the Fourier transform of $f$, and $\check{f}$ denotes its inverse Fourier transform.
- For a $s \in \mathbb{R}$, $J^s$, and $D^s$ stand the Bessel and the Riesz potentials of order $s$, given via Fourier transform by the formulas

$$\widehat{J^s f} = \langle \xi \rangle^s \widehat{f}, \quad \text{and} \quad \widehat{D^s f} = |\xi|^s \widehat{f}.$$

- $H^s(\mathbb{R})$ denotes the Sobolev space of order $s$ defined by

$$H^s(\mathbb{R}) = \{f \in S'(\mathbb{R}); J^s f \in L^2(\mathbb{R})\}$$

  endowed with the norm

$$\|f\|_{H^s} = \|J^s f\|_{L^2};$$

- We also consider the Sobolev spaces

$$L^p_\alpha(\mathbb{R}) = \{f \in S'(\mathbb{R}); J^\alpha f \in L^p(\mathbb{R})\}$$

  for $p \geq 1$.

- For a function $f = f(x,t)$, with $(x,t) \in \mathbb{R} \times [0,T]$, we denote

$$\|f\|_{L^q_T L^p_x} = \left( \int_0^T \left( \int_{\mathbb{R}} |f(x,t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

  and

$$\|f\|_{L^q_T H^s_x} = \left( \int_0^T \|f(\cdot,t)\|^q_{H^s} dt \right)^{\frac{1}{q}}.$$

- Similarly we denote

$$\|f\|_{L^p_x L^q_T} = \left( \int_{\mathbb{R}} \left( \int_0^T |f(x,t)|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

- When $f(x,t)$ is defined for time $t$ in the whole line $\mathbb{R}$ we shall consider the notations $\|f\|_{L^q_t L^p_x}$, $\|f\|_{L^q_t H^s_x}$ and $\|f\|_{L^p_x L^q_t}$.

# Contents

# Introduction

We shall study the following initial value problem (IVP)

$$\begin{cases} i\partial_t u + \partial_x^2 u + i|u|^{1+a}\partial_x u &= 0 \\ u(\cdot, 0) &= u_0 \end{cases} \tag{1}$$

where $u$ is a complex valued function of $(x,t) \in \mathbb{R} \times \mathbb{R}$ and $0 < a < 1$.

The equation in (1) is a generalization of the derivative nonlinear Schrödinger equation, (DNLS)

$$i\partial_t u + \partial_x^2 u + i(|u|^2 u)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{2}$$

The DNLS equation appears in physics as a model that describes the propagation of Alfvén waves in plasma (see [22], [23], [25]). In mathematics, this equation has also been extensively studied in regard of well-posedness of its associated IVP ([6], [8], [11], [12], [13], [14], [30], [31]). Tsutsumi and Fukuda [32], using parabolic regularization, proved local well-posedness in Sobolev spaces $H^s(\mathbb{R})$, $s > 3/2$. Hayashi [12] proved well-posedness for initial data $u_0 \in H^1(\mathbb{R})$ satisfying the smallness condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}. \tag{3}$$

His idea was to use a gauge transformation to turn the DNLS equation into a system of nonlinear Schrödinger equations without derivative in the nonlinearity. This system, in turn, can be treated using Strichartz estimates. It is known that DNLS can be writen as a Hamiltonian system

$$\frac{d}{dt}u(t) = -iE'(u(t)) \tag{4}$$

where $E(u)$ the energy of $u$ defined by

$$E(u)(t) = \frac{1}{2}\int |\partial_x u|^2 dx + \frac{1}{4}\mathrm{Im}\int |u|^2 \bar{u}\partial_x u dx. \tag{5}$$

As a consequence of (4) it follows that $E$ is a conserved quantity. In particular, the result of Hayashi is global in time. Later on Hayashi and Ozawa [13] based on the same gauge transformation proved global well-posedness for initial condition in $H^m(\mathbb{R})$, $m \in \mathbb{N}$, also satisfying the smallness condition (3). The best result regarding local well-posedness was obtained by Takaoka in [30]. He proved the IVP associated to DNLS was well-posed in $H^s(\mathbb{R})$, for $s \geq 1/2$. He considered the gauge transformation

$$v(x,t) = u(x,t) \exp\left(\frac{i}{2}\int |u(y,t)|^2 dy\right).$$

to convert the DNLS into

$$i\partial_t v + \partial_x^2 v = -iv^2 \partial_x \bar{v} - \frac{1}{2}|v|^4 v \tag{6}$$

and used the Fourier restriction norm method introduced by Bourgain [3]. Biagioni and Linares [2] proved that the IVP associated to DNLS is not well-posed in $H^s(\mathbb{R})$ for $s < 1/2$, which implies that Takaoka's result is sharp. Using the I-method Colliander, Keel, Staffilani, Takaoka and Tao ([5], [6]) showed that the IVP associated to DNLS is global well-posed for $s > 1/2$.

The main difficulty to deal with DNLS is the presence of the derivative in the nonlinearity, which causes the so called loss of derivatives. This means that the standard way of proving existence of solution of

$$u(t) = U(t)u_0 - \int_0^t U(t-t')\partial_x(|u|^2 u)dt' \tag{7}$$

can not be accomplished only using the property of unitary group and Strichartz (Lemma 1.2.1) of the Schrödinger propagator $U(t) = e^{it\partial_x^2}$. In fact the right hand side of (7) has less derivative than the left hand side and Strichartz estimates do not provide us gain of derivatives. This is one point that makes the study of DNLS more difficulty than the corresponding cubic nonlinear Schrödinger equation (NLS), namely

$$i\partial_t u + \partial_x^2 u + \lambda |u|^2 u = 0.$$

The equation in (1) admits a family of solitary waves solutions given explicitily by

$$\psi(x,t) = \varphi_{\omega,c}(x-ct) \exp i \left\{\omega t + \frac{c}{2}(x-ct) - \frac{1}{a+3}\int_{-\infty}^{x-ct} \varphi_{\omega,c}^{1+a}(y)dy\right\},$$

where $\omega > c^2/4$ and

$$\varphi_{\omega,c}(y)^{1+a} = \frac{(3+a)(4\omega - c^2)}{4\sqrt{\omega}\left(\cosh(\frac{a+1}{2}\sqrt{4\omega - c^2}y) - \frac{c}{2\sqrt{\omega}}\right)}.$$

Liu, Simpson and Sulem in [21] studied the orbital stability of these solitary waves for the equation in (1). More precisely,

**Definition 0.0.1.** *The solitary wave $\psi_{\omega,c}$ is said to be orbitally stable for (1) if for each initial data $u_0 \in H^1(\mathbb{R})$ sufficiently close to $\psi_{\omega,c}(0)$ corresponds a unique $u$ global solution of (1) and*

$$\sup_{t\geq 0} \inf_{(\theta,y)\in\mathbb{R}\times\mathbb{R}} \|u(t) - e^{i\theta}\psi_{\omega,c}(t, \cdot - y)\|_{H^1}$$

*is sufficiently small. Otherwise $\psi_{\omega,c}$ is said to be orbitally unstable.*

Results of orbital stability or instability were obtained according with the value of $a$. There it was assumed the existence of solution of the IVP (1) for arbitrary $a > -1$, and initial data $u_0 \in H^1(\mathbb{R})$. However, besides the case $a = 1$ there is only a few well-posedness results for (1). For integer powers $a \geq 4$ Hao [10] proved local well-posedness in $H^{1/2}(\mathbb{R})$.

In the present work we carry out the case $a > 0$. The main result regards the most interesting case $0 < a < 1$.

**Theorem 0.0.2.** *Let $X = H^{3/2}(\mathbb{R}) \cap \{f \in S'(\mathbb{R}); xf \in H^{1/2}(\mathbb{R})\}$. If $u_0 \in X$ and*

$$\|u_0\|_X = \|u_0\|_{H^{3/2}} + \|xu_0\|_{H^{1/2}}$$

*is sufficiently small, then there exist $T = T(\|u_0\|_X) > 0$ and an unique*

$$u \in C([0,T]; H^{3/2-}) \cap L^\infty([0,T]; X)$$

*solution of the integral equation*

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')(|u|^{1+a}\partial_x u)(t')dt'. \tag{8}$$

Our strategy to prove this theorem will be based on the technique known as parabolic regularization (or viscosity argument) introduced by T. Kato [16]. It consists in the following: First we prove that for each positive parameter $\epsilon$ (viscosity) the initial value problem

$$\left\{\begin{array}{rcl} i\partial_t u_\epsilon + \partial_x^2 u_\epsilon + i|u_\epsilon|^{1+a}\partial_x u_\epsilon &=& i\epsilon\partial_x^2 u_\epsilon \\ u_\epsilon(\cdot,0) &=& u_0 \end{array}\right. \tag{9}$$

has solution in $[0, T_\epsilon]$. The second step is to prove that the solutions $u_\epsilon$ converge as $\epsilon$ goes to zero and the limit solves (1). The first step is quite simple due to the presence of the parabolic term $i\epsilon\partial_x^2 u$. As is usual we can obtain solution $u_\epsilon \in C([0,T_\epsilon];H^s(\mathbb{R}))$, with $T_\epsilon = \epsilon/\|u_0\|_{H^s}$, in some Sobolev space $H^s(\mathbb{R})$. The difficulty lies in the second step. But before considering the problem of convergence we have to argue that all the solutions can be extended to a same interval of time. This uniform extention will be possible if

$$\sup_{\epsilon>0} \|u_\epsilon\|_{L^\infty_{T_\epsilon} H^s_x} < \infty. \tag{10}$$

The estimate (10) permits to extend all the solutions to an interval of time $[0,T]$ independent of $\epsilon$ and the extension still satisfies $\sup_{\epsilon>0} \|u_\epsilon\|_{L^\infty_T H^s_x} < \infty$. Using compactness, for each $t \in [0,T]$ we have weak convergence $u_\epsilon(t)$ of some subsequence to an element $u(t)$ in $H^s(\mathbb{R})$. So, $u$ is the candidate to be a solution. Thus we have a great chance to be successful with the parabolic regularization if we can prove the uniform estimate (10). In [15] it is already presented how (10) can be obtained in $H^s(\mathbb{R})$, $s > 3/2$, for nonlinearities like $u^m\partial_x u$, $m \in \mathbb{N}$. There, the argument can also be adapted to the nonlinearity $|u|^{1+a}\partial_x u$ whenever we have some smoothness of the nonlinearity, for example $a \geq 1$, that allows us to prove

$$\||u|^{1+a}\|_{H^s} \leq c\|u\|^{1+a}_{H^s}. \tag{11}$$

However, (11) is no longer true for low powers $a$ since $|z|^{1+a}$ does not have enough regularity. In this work we obtain a uniform estimate like (10) in $H^{3/2}(\mathbb{R})$ when $0 < a < 1$ (See Theorem 3.20). Our argument was inspired trying to use the contraction mapping principle. Using sharp smoothing properties associated to the linear equation we define a subspace $E \subset C([0,T];H^s(\mathbb{R}))$ and prove that the associated integral operator

$$\Psi(u) = U(t)u_0 - \int_0^t U(t-t')(|u|^{1+a}\partial_x u)dt'$$

is well defined, i.e, $\Psi : E \to E$ for small data. However $\Psi$ is not a contraction. This is why we use parabolic regularization to study this problem.

This work is organized as follows:

In Chapter 1 we present a list of results that will be used along the work.

In Chapter 2 we consider the regularized equation (2.1). We prove the existence of solutions $u_\epsilon$ defined in an interval of time that depends on the parameter $\epsilon$. Aiming to extend the solution to an interval of time independent of the parameter $\epsilon$ we establish some uniform estimates for the linear solutions of the regularized equation.

We prove that the solutions $u_\epsilon$ of the regularized problem can be all extended to an interval $[0,T]$ where $T$ depends only on the size of the initial data. We also prove that the solutions $u_\epsilon$ are uniformly bounded in $H^{3/2}$ with respect to the parameter $\epsilon$. To achieve this uniform estimate we impose that the initial data belongs to a weighted Sobolev space and has small size on it.

In Chapter 4 the convergence of the solutions defined in $[0,T]$ will be studied. First it is proved that this sequence converges strongly in $L^\infty_T L^2_x$ to some function $u$. Then using the uniform estimate provided in the third chapter and compactness argument, it is shown that for each $t \in [0,T]$ there is some subsequence $\{u_{\epsilon'}(t)\}$ converging weakly in $H^{3/2}(\mathbb{R})$ and the weak limit is in fact $u(t)$. Finally, we prove that $u$ is solution of the integral equation (8) and it is the unique possible solution.

In Chapter 5 we study the case $a > 1$. In particular, we establish well-posedness for the IVP (1) in $H^s(\mathbb{R})$, $s > 1$, for small data via contraction argument.

Finally in Chapter 6 we add some remarks.

# Chapter 1

# Preliminaries

author

## 1.1 Basic results

In this section we list without proof some elementary inequalities and some commutator estimates useful in our analysis below.

**Lemma 1.1.1.** (Gronwall inequality) *Let $f$ be a nonnegative absolutely continuous function satisfying the differential inequality*

$$f'(t) \leq \varphi(t)f(t) + \psi(t)$$

*almost everywhere, with $\varphi$ and $\psi$ also nonegative. Then we have*

$$f(t) \leq e^{\int_0^t \varphi(s)ds} \left[ f(0) + \int_0^t \psi(s)ds \right].$$

*Proof.* See for example [26]. □

**Lemma 1.1.2.** *Given $s_0 < s < s_1$, we have*

$$\|f\|_{H^s} \leq \|f\|_{H^{s_0}}^{\theta} \|f\|_{H^{s_1}}^{1-\theta}$$

*where $\theta = \dfrac{s_1 - s}{s_1 - s_0}$.*

*Proof.* See for example [20]. □

**Lemma 1.1.3.** (Sobolev embeddings)
i) *For* $2 \leq p < \infty$*, there exists a constant* $c > 0$ *such that,*

$$\|f\|_{L^p} \leq c\|D^s f\|_{L^2}$$

*for all* $f \in H^s(\mathbb{R})$*, where* $s = 1/2 - 1/p$*;*
ii) *When* $p = \infty$*, there exists a constant* $c > 0$ *such that,*

$$\|f\|_{L^\infty} \leq c\|f\|_{H^{1/2+}}$$

*or all* $f \in H^{1/2+}(\mathbb{R})$*.*

*Proof.* See [20] Chapter 3. □

**Lemma 1.1.4.** (Leibniz rule for fractional derivatives)
(i) *Let* $0 < s < 1$*,* $s_1, s_2 \in [0, s]$ *with* $s = s_1 + s_2$ *and* $1 < p, p_1, p_2 < \infty$*, such that* $1/p = 1/p_1 + 1/p_2$*. Then, for some constant* $c >$ *we have*

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \leq c\|D^{s_1} f\|_{L^{p_1}}\|D^{s_2} g\|_{L^{p_1}}.$$

(ii) *For* $f = f(x,t), g = g(x,t)$ *we have*

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p_x L^q_T} \leq c\|D^{s_1} f\|_{L^{p_1}_x L^{q_1}_T}\|D^{s_2} g\|_{L^{p_1}_x L^{q_1}_T} \tag{1.1}$$

*for* $1 < p, p_1, p_2, q, q_1, q_2 < \infty$*, such that* $1/p = 1/p_1 + 1/p_2$ *and* $1/q = 1/q_1 + 1/q_2$*. Moreover, for* $s_1 = 0$ *the value* $q_1 = \infty$ *is allowed.*
(iii) *If* $p = 1$ *and* $q = 2$ *, there exists some constant* $c > 0$ *such that*

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^1_x L^2_T} \leq c\|D^{s_1} f\|_{L^{p_1}_x L^{q_1}_T}\|D^{s_2} g\|_{L^{p_1}_x L^{q_1}_T}$$

*for* $1 < p_1, p_2, q_1, q_2 < \infty$*, such that* $1 = 1/p_1 + 1/p_2$ *and* $1/2 = 1/q_1 + 1/q_2$*.*

**Lemma 1.1.5.** (Chain rule) *Let* $0 < s < 1$ *and* $F : \mathbb{C} \to \mathbb{C}$ *be given such that* $F$ *is* $C^1$ *(regarded as a function in* $\mathbb{R}^2$*).*
(i) *For all* $1 < p, p_1, p_2 < \infty$ *such that* $1/p = 1/p_1 + 1/p_2$*, there is a constant* $c > 0$ *such that*

$$\|D^s(F(f))\|_{L^p} \leq c\|F'(f)\|_{L^{p_1}}\|D^s f\|_{L^{p_2}}.$$

(ii) *In the case of functions* $f = f(x,t)$ *it holds*

$$\|D^s(F(f))\|_{L^p_x L^q_T} \leq c\|F'(f)\|_{L^{p_1}_x L^{q_1}_T}\|D^s f\|_{L^{p_2}_x L^{q_2}_T}$$

*for* $1 < p, p_1, p_2, q, q_1, q_2 < \infty$*, such that* $1/p = 1/p_1 + 1/p_2$ *and* $1/q = 1/q_1 + 1/q_2$*.*

The proof of Lemma 1.1.4 and Lemma 1.1.5 were established by Kenig, Ponce e Vega [17].

**Remark 1.** *Visan and Killip* [33] *established a chain rule when $F$ is only a Hölder continuous function of order $\alpha \in (0,1)$.*

## 1.2 The Schrödinger propagator

Consider the IVP associated to the free Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u &= 0, x \in \mathbb{R}, t > 0 \\ u(\cdot, 0) &= f \end{cases} \tag{1.2}$$

whose solutions is given by

$$u(x,t) = \{e^{-it\xi^2} \hat{f}\}^{\vee}(x)$$

and is denoted by $U(t)f$. The family $\{U(t)\}_{t\in\mathbb{R}}$ forms a unitary group in $H^s(\mathbb{R})$, for all $s \in \mathbb{R}$. In the following we will list some estimates satisfied for solutions of IVP (1.2). We begin by the so called $L^qL^p$ estimates or Strichartz estimates.

**Lemma 1.2.1.** (Strichartz estimates) *For all pair $(p,q)$ satisfying*

$$2 \le p \le \infty \quad \text{and} \quad 2/q = 1/2 - 1/p$$

*we have*

$$\|U(t)f\|_{L_T^q L_x^p} \le c\|f\|_{L^2} \tag{1.3}$$

*and*

$$\left\| \int_0^t U(t-t')F(x,t')dt' \right\|_{L_T^q L_x^p} \le \|F\|_{L_T^{q'} L_x^{p'}} \tag{1.4}$$

*where $1/p' + 1/p = 1/q' + 1/q = 1$, for some constant $c > 0$.*

*Proof.* See Ginibre and Velo [7] □

Next we have the smoothing effects estimates

**Lemma 1.2.2.** (homogeneous smoothing effect) *There exists a constant $c > 0$, such that*

$$\|D^{1/2}U(t)f\|_{L_x^\infty L_T^2} \le c\|f\|_{L^2}$$

*for all $f \in L^2(\mathbb{R})$.*

By duality it follows from Lemma 1.2.2:

**Lemma 1.2.3.** *There exists a constant $c > 0$ such that*

$$\left\| D^{1/2} \int_{\mathbb{R}} U(t')F(\cdot, t')dt' \right\|_{L^\infty_T L^2_x} \le c\|F\|_{L^1_x L^2_T}$$

*for all $F \in L^1_x L^2_T$.*

For the inhomogeneous problem

$$\begin{cases} i\partial_t u + \partial_x^2 u & = & F(x,t) \\ u(\cdot, 0) & = & 0 \end{cases} \tag{1.5}$$

whose solution is

$$u(x,t) = \int_0^t U(t - t')F(\cdot, t')dt' \tag{1.6}$$

we have:

**Lemma 1.2.4.** (Inhomogeneus smoothing effect) *For $u$ given as (1.6) we have*

$$\|\partial_x u\|_{L^2_T L^\infty_x} \lesssim \|F\|_{L^1_x L^2_T}$$

The estimates above were proved by Kenig, Ponce and Vega [17]. For a detailed proof of Lemmas 1.2.1 to 1.2.4 see [20] Chapter 4.

Finally, we present some maximal function estimates for the Schrödinger propagator. More precisely,

**Lemma 1.2.5.** (Maximal $L^2$ estimate) *Given $s > 1/2$ there exists a constant $c > 0$ such that*

$$\|U(t)f\|_{L^2_x L^\infty_T} \le c\|f\|_{H^s}$$

*for all $f \in H^s(\mathbb{R})$.*

*Proof.* See Kenig, Ponce and Vega in [18] . □

**Lemma 1.2.6.** (Maximal $L^4$ estimate) *There exists a constant $c > 0$ such that*

$$\|U(t)f\|_{L^4_x L^\infty_t} \le c\|D^{1/4}f\|_{L^2}$$

*for all $f \in H^{1/4}(\mathbb{R})$.*

*Proof.* See Kenig and Ruiz [19] . □

## 1.3 Two technical lemmas

In this section we shall prove the following lemmas.

**Lemma 1.3.1.** (Interpolation) *Let $\theta \in [0,1]$ be given. There exists some constant $c > 0$ such that,*

$$\|J^{1/2+\theta}(\langle x\rangle^{1-\theta} f)\|_{L^2} \leq c\|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2} f\|_{L^2}^{\theta}.$$

**Lemma 1.3.2.** *Given $0 < s < 1$ we have*

$$\|J^s(\langle x\rangle f)\|_{L^2} \leq c\|J^s(xf)\|_{L^2} + \|J^s f\|_{L^2}$$

*for some constant $c > 0$.*

Before going through the proof of these lemmas we shall present some facts that will help us in their proofs. To prove Lemma 1.3.1 we consider for $0 < \alpha < 2$ the following derivative

$$\mathcal{D}_\alpha f(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{f(x+y) - f(x)}{|y|^{1+\alpha}} dy. \tag{1.7}$$

**Theorem 1.3.3** (Characterization I)**.** *Let $0 < \alpha < 2$ and $1 < p < \infty$. Then $f \in L^p_\alpha(\mathbb{R})$ if, and only if, $f \in L^p(\mathbb{R})$ and the limit defined in* (1.7) *converges in $L^p$ norm. In this case*

$$\|J^\alpha f\|_{L^p} \approx \|f\|_{L^p} + \|\mathcal{D}_\alpha f\|_{L^p}.$$

*Proof.* See Stein [27] or [29]. □

**Lemma 1.3.4.** *Let $0 < \alpha < 1$ and $1 < p < \infty$. Then there exists a constant $c > 0$ such that*

$$(c(1+|t|))^{-1}\|J^\alpha(\langle\cdot\rangle^{it} f)\|_{L^p} \leq \|J^\alpha f\|_{L^p} \leq c(1+|t|)\|J^\alpha(\langle\cdot\rangle^{it} f)\|_{L^p}$$

*for all $t \in \mathbb{R}$.*

*Proof.* Denote $\varphi(x) = \log\langle x\rangle$.

$$\begin{aligned}
\mathcal{D}_\alpha(\langle\cdot\rangle^{it} f)(x) &= \mathcal{D}_\alpha(e^{it\varphi} f)(x) \\
&= \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{e^{it\varphi(x+y)} f(x+y) - e^{it\varphi(x)} f(x)}{|y|^{1+\alpha}} dy \\
&= e^{it\varphi(x)} \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{e^{it(\varphi(x+y)-\varphi(x))} - 1}{|y|^{1+\alpha}} f(x+y) dy \\
&\quad + e^{it\varphi(x)} \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{f(x+y) - f(x)}{|y|^{1+\alpha}} dy \\
&= e^{it\varphi(x)} \phi_{\alpha,t}(f)(x) + e^{it\varphi(x)} \mathcal{D}_\alpha f(x).
\end{aligned}$$

Thus

$$\|\mathcal{D}_\alpha(\langle\cdot\rangle^{it}f)(x)\|_{L^p} \le \|\phi_{\alpha,t}(f)\|_{L^p} + \|\mathcal{D}_\alpha f\|_{L^p}.$$

Let us estimate $\|\phi_{\alpha,t}(f)\|_{L^p}$.

$$\begin{aligned}\phi_{\alpha,t}(f)(x) &= \lim_{\epsilon\to 0}\int_{\epsilon<|y|<1}\frac{e^{it(\varphi(x+y)-\varphi(x))}-1}{|y|^{1+\alpha}}f(x+y)dy \\ &\quad + \int_{|y|>1}\frac{e^{it(\varphi(x+y)-\varphi(x))}-1}{|y|^{1+\alpha}}f(x+y)dy \\ &= I_0 + I_\infty.\end{aligned}$$

Since $|e^{i\theta}-1| \le 2$ we have

$$\begin{aligned}\|I_\infty\|_{L^p} &\le 2\int_{|y|>1}\frac{1}{|y|^{1+\alpha}}\|f(\cdot+y)\|_{L^p}dy \\ &= 2\|f\|_{L^p}\int_{|y|>1}\frac{1}{|y|^{1+\alpha}}dy \\ &= c_\alpha\|f\|_{L^p}.\end{aligned}$$

To take care of $I_0$ we first note that

$$|e^{i\theta}-1| = |e^{i\theta/2}-e^{-i\theta/2}| = 2|\sin(\theta/2)| \le |\theta|.$$

So

$$|e^{it(\varphi(x+y)-\varphi(x))}-1| \le |t||\varphi(x+y)-\varphi(x)|.$$

Since $\varphi$ is Lipschitz we have

$$|e^{it(\varphi(x+y)-\varphi(x))}-1| \le |t||y|. \tag{1.8}$$

Then

$$\begin{aligned}|I_0| &\le \lim_{\epsilon\to 0}\int_{\epsilon<|y|<1}\frac{|t||y|}{|y|^{1+\alpha}}|f(x+y)|dy \\ &= |t|\lim_{\epsilon\to 0}\int_{\epsilon<|y|<1}\frac{1}{|y|^{\alpha}}|f(x+y)|dy. \qquad (1.9)\end{aligned}$$

Applying Minkowsky's inequality in the integral (1.9) we get

$$\begin{aligned}\|I_0\|_{L^p} &\le |t|\lim_{\epsilon\to 0}\int_{\epsilon<|y|<1}\frac{1}{|y|^{\alpha}}\|f(\cdot+y)\|_{L^p}dy \\ &= |t|\|f\|_{L^p}\lim_{\epsilon\to 0}\int_{\epsilon<|y|<1}\frac{1}{|y|^{\alpha}}dy \\ &= c_\alpha|t|\|f\|_{L^p}.\end{aligned}$$

We conclude that

$$\|\phi_{\alpha,t}(f)\|_{L^p} \lesssim (1+|t|)\|f\|_{L^p}.$$

Then applying Theorem 1.3.3 we obtain

$$\begin{aligned}\|\mathcal{D}_\alpha(\langle\cdot\rangle^{it}f)\|_{L^p} &\lesssim (1+|t|)(\|f\|_{L^p}+\|\mathcal{D}_\alpha f\|_{L^p})\\ &\lesssim (1+|t|)\|J^\alpha f\|_{L^p}.\end{aligned}$$

Therefore

$$\begin{aligned}\|J^\alpha(\langle\cdot\rangle^{it}f)\|_{L^p} &\lesssim \|\mathcal{D}_\alpha(\langle\cdot\rangle^{it}f)\|_{L^p}+\|f\|_{L^p}\\ &\lesssim (1+|t|)\|J^\alpha f\|_{L^p}. \qquad (1.10)\end{aligned}$$

The opposite inequality follows immediately by applying (1.10) to the function $\langle\cdot\rangle^{-it}f$ instead of $f$. $\square$

Now we are ready to prove Lemma 1.3.1

**Proof of Lemma 1.3.1 :**

Given $g \in L^2(\mathbb{R})$ such that $\|g\|_{L^2}=1$ define $F_g : S \longrightarrow \mathbb{C}$ by

$$F_g(z) = e^{z^2-1}\int_{\mathbb{R}} J^{1/2+z}(\langle x\rangle^{1-z}f)(x)\bar{g}(x)dx$$

where $S$ is the strip $S=\{z\in\mathbb{C}; 0\le \mathrm{Re}(z)\le 1\}$. Using the Cauchy-Schwarz inequality and Lemma 1.3.4

$$\begin{aligned}|F_g(iy)| &= e^{-(y^2+1)}\left|\int_{\mathbb{R}} J^{1/2+iy}(\langle x\rangle^{1-iy}f)(x)\bar{g}(x)dx\right|\\ &\le e^{-(y^2+1)}\|J^{1/2+iy}(\langle x\rangle^{1-iy}f)\|_{L^2}\|\bar{g}\|_{L^2}\\ &= e^{-(y^2+1)}\|J^{1/2}(\langle x\rangle^{1-iy}f)\|_{L^2}\\ &\lesssim (1+|y|)e^{-(y^2+1)}\|J^{1/2}(\langle x\rangle f)\|_{L^2}.\end{aligned}$$

Since the function $(1+|y|)e^{-(y^2+1)}$ is bounded it follows

$$|F_g(iy)| \lesssim \|J^{1/2}(\langle x\rangle f)\|_{L^2}.$$

Similarly, we have

$$\begin{aligned}|F_g(1+iy)| &= e^{-y^2}\left|\int_{\mathbb{R}} J^{3/2+iy}(\langle x\rangle^{-iy}f)(x)\bar{g}(x)dx\right|\\ &\le e^{-y^2}\|J^{3/2+iy}(\langle x\rangle^{-iy}f\|_{L^2}\|g\|_{L^2}\\ &= e^{-y^2}\|J^{3/2}(\langle x\rangle^{-iy}f)\|_{L^2} \qquad (1.11)\end{aligned}$$

Before applying Lemma 1.3.4 to (1.11) we make the following computation

$$\begin{aligned}
\|J^{3/2}(\langle x\rangle^{-iy}f)\|_{L^2} &\sim \|\langle x\rangle^{-iy}f\|_{L^2}+\|D^{3/2}(\langle x\rangle^{-iy}f)\|_{L^2}\\
&= \|f\|_{L^2}+\|D^{1/2}\partial_x(\langle x\rangle^{-iy}f)\|_{L^2}\\
&= \|f\|_{H^{3/2}}+|y|\left\|D^{1/2}\left(\langle x\rangle^{-iy}\frac{x}{\langle x\rangle^2}f\right)\right\|_{L^2}. \qquad (1.12)
\end{aligned}$$

Now we can use Lemma 1.3.4 to obtain

$$\left\|D^{1/2}\left(\langle x\rangle^{-iy}\frac{x}{\langle x\rangle^2}f\right)\right\|_{L^2}\lesssim(1+|y|)\left\|J^{1/2}\left(\frac{x}{\langle x\rangle^2}f\right)\right\|_{L^2}. \qquad (1.13)$$

It turns out we can bound (1.13) as

$$\begin{aligned}
\left\|J^{1/2}\left(\frac{x}{\langle x\rangle^2}f\right)\right\|_{L^2} &\le \left\|J^{1}\left(\frac{x}{\langle x\rangle^2}f\right)\right\|_{L^2}\\
&\sim \left\|\frac{x}{\langle x\rangle^2}f\right\|_{L^2}+\left\|\partial_x\left(\frac{x}{\langle x\rangle^2}f\right)\right\|_{L^2}\\
&\le \|f\|_{L^2}+\left\|\partial_x\left(\frac{x}{\langle x\rangle^2}\right)\right\|_{L^\infty}\|f\|_{L^2}+\left\|\frac{x}{\langle x\rangle^2}\right\|_{L^\infty}\|\partial_x f\|_{L^2}\\
&\lesssim \|f\|_{L^2}+\|\partial_x f\|_{L^2}. \qquad (1.14)
\end{aligned}$$

We conclude from the estimates (1.12), (1.13) and (1.14)

$$\|J^{3/2}(\langle x\rangle^{-iy}f)\|_{L^2}\lesssim(1+|y|)^2\|f\|_{H^{3/2}}$$

and then we finally get

$$|F_g(1+iy)|\lesssim\|f\|_{H^{3/2}}.$$

Therefore using the Three Lines Theorem (see [28]) we obtain

$$|F_g(\theta)|\lesssim\|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta} \qquad (1.15)$$

for all $\theta\in[0,1]$. Taking the supremum over all $g\in L^2(\mathbb{R})$ such that $\|g\|_{L^2}=1$ in (1.15) we obtain

$$e^{\theta^2-1}\|J^{1/2+\theta}(\langle x\rangle^{1-\theta}f)\|_{L^2}\lesssim\|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta}.$$

So we finally conclude

$$\|J^{1/2+\theta}(\langle x\rangle^{1-\theta}f))\|_{L^2}\lesssim\|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta}.$$

□

To prove Lemma 1.3.2 it will be more convenient to consider the characterization of Sobolev spaces in terms of the following derivative

$$\mathcal{D}^s f(x) = \left( \int_{\mathbb{R}} \frac{|f(x+y) - f(x)|^2}{|y|^{1+2s}} dy \right)^{1/2}. \tag{1.16}$$

We have the following characterization

**Theorem 1.3.5.** *Let $s \in (0,1)$ and $2/(1+2s) < p < \infty$. Then $f \in L^p_s(\mathbb{R})$ if, and only if $f$ and $\mathcal{D}^s f$ belong to $L^p(\mathbb{R})$. Moreover*

$$\|J^s f\|_{L^p} \sim \|f\|_{L^p} + \|\mathcal{D}^s f\|_{L^p}$$

*for all $f \in L^p_s(\mathbb{R})$.*

*Proof.* See Stein [27] or [29]. □

**Remark 2.** *When $p = 2$ we have the following product rule*

$$\|\mathcal{D}^s(fg)\|_{L^2} \leq \|f\mathcal{D}^s(g)\|_{L^2} + \|g\mathcal{D}^s f\|_{L^2}. \tag{1.17}$$

**Proof of Lemma 1.3.2**: Consider the cut-off function $\chi \in C_c^\infty(\mathbb{R})$, supported in $[-2,2]$ and identically 1 in $[-1,1]$. Before going any further notice the following

CLAIM 1: If $\varphi$ is Lipschitz and bounded then $\mathcal{D}^s\varphi$ is bounded as well.

In fact,

$$\begin{aligned} (\mathcal{D}^s\varphi(x))^2 &= \int_{\mathbb{R}} \frac{|\varphi(x+y) - \varphi(x)|^2}{|y|^{1+2s}} dy \\ &\lesssim \int_{|y|<1} \frac{|y|^2}{|y|^{1+2s}} dy + 2\|\varphi\|^2_{L^\infty} \int_{|y|>1} \frac{1}{|y|^{1+2s}} \\ &< \infty. \end{aligned}$$

To prove Lemma 1.3.2 it is enough to prove

$$\|D^s(\langle x \rangle f)\|_{L^2} \lesssim \|J^s(xf)\|_{L^2} + \|J^s f\|_{L^2}.$$

We have

$$\begin{aligned} \|D^s(\langle x \rangle f)\|_{L^2} &\leq \|D^s(\langle x \rangle f\chi(x)f)\|_{L^2} + \|D^s(\langle x \rangle f(1-\chi(x))f)\|_{L^2} \\ &\leq I + II. \end{aligned}$$

We are going to use the characterization in Theorem 1.3.5 to estimate $I$ and $II$. In fact, using Theorem 1.3.5, together with (1.17) and CLAIM 1 with $\varphi(x) = \langle x\rangle\chi(x)$ we have

$$\begin{aligned}
\|D^s(\langle x\rangle f\chi(x)f)\|_{L^2} &\lesssim \|\varphi f\|_{L^2} + \|\mathcal{D}^s(\varphi f)\|_{L^2} \\
&\lesssim \|\varphi f\|_{L^2} + \|\varphi\mathcal{D}^s f\|_{L^2} + \|f\mathcal{D}^s\varphi\|_{L^2} \\
&\lesssim \|\varphi\|_{L^\infty}\left(\|f\|_{L^2} + \|\mathcal{D}^s f\|_{L^2}\right) + \|\mathcal{D}^s\varphi\|_{L^\infty}\|f\|_{L^2} \\
&\lesssim \|f\|_{L^2} + \|\mathcal{D}^s f\|_{L^2}.
\end{aligned}$$

Then we conclude

$$I \lesssim \|J^s f\|_{L^2}.$$

Similarly, using Theorem 1.3.5, (1.17) and CLAIM 1 for the function $\varphi(x) = \frac{\langle x\rangle}{x}(1-\chi(x))$ we have

$$\begin{aligned}
\|D^s(\langle x\rangle(1-\chi(x))f)\|_{L^2} &= \|D^s(\varphi xf)\|_{L^2} \\
&\lesssim \|\varphi xf\|_{L^2} + \|\mathcal{D}^s(\varphi xf)\|_{L^2} \\
&\lesssim \|\varphi xf\|_{L^2} + \|xf\mathcal{D}^s\varphi\|_{L^2} + \|\varphi\mathcal{D}^s(xf)\|_{L^2} \\
&\lesssim \|\varphi\|_{L^\infty}\left(\|xf\|_{L^2} + \|\mathcal{D}^s(xf)\|_{L^2}\right) + \|\mathcal{D}^s\varphi\|_{L^\infty}\|xf\|_{L^2} \\
&\lesssim \|xf\|_{L^2} + \|\mathcal{D}^s(xf)\|_{L^2}.
\end{aligned}$$

Then

$$II \lesssim \|J^s(xf)\|_{L^2}.$$

□

# Chapter 2

# The viscosity argument

In this chapter we are going to perform the viscosity argument. That is, for every $\epsilon > 0$ we are going to solve the (IVP) problem

$$\begin{cases} i\partial_t u + \partial_x^2 u + i|u|^{1+a}\partial_x u &= i\epsilon\partial_x^2 u \\ u(\cdot, 0) &= u_0 \end{cases} \tag{2.1}$$

where $(x,t) \in \mathbb{R} \times [0,\infty)$ and $a \in (0,1)$. This problem is simpler to solve than (1) since the additional term $i\epsilon\partial_x^2 u$ provides us with the propagator $e^{it(1-i\epsilon)\partial_x^2}$ which has stronger decay properties than the original $e^{it\partial_x^2}$. These decay properties translate into gain of derivatives which are very convenient in handling the loss of derivative introduced by the nonlinear term. A crucial point here is to establish estimates for $e^{it(1-i\epsilon)\partial_x^2}$ uniformly in the parameter $\epsilon$. These uniform estimates, in turn, will allow us to pass to the limit as $\epsilon$ goes to zero. In the next chapter we will prove that the solutions $u_\epsilon$ can be uniformly estimated in $H^{3/2}(\mathbb{R})$. Therefore in this chapter we will solve the reguralized problem (2.1) for initial data $u_0$ in $H^{3/2}(\mathbb{R})$, even though we could do so for $u_0$ in a Sobolev space of lower regularity.

## 2.1 Solution of the $\epsilon$-nonlinear problem in $H^{3/2}$

In our work we are going to denote by $U_\epsilon(t)$ the linear propagator $e^{it(1-i\epsilon)\partial_x^2}$, which describes the solution of

$$\begin{cases} i\partial_t u + \partial_x^2 u &= i\epsilon\partial_x^2 u, \\ u(\cdot, 0) &= u_0, \end{cases} \tag{2.2}$$

defined via Fourier transform by

$$U_\epsilon(t)u_0 = \{e^{-(i+\epsilon)t\xi^2}\hat{u}_0\}^\vee(x)$$

for all $t \geq 0$. We will solve the integral equation

$$u(t) = U_\epsilon(t)u_0 - \int_0^t U_\epsilon(t-t')(|u|^{1+a}\partial_x u)(t')dt', \tag{2.3}$$

and later on we will justify that the solution of (2.3) is actually solution of the diffential equation (2.1). Denoting by $\Psi_\epsilon$ the right hand side of (2.3) we are looking for $u$ such that $\Psi_\epsilon(u) = u$. Following the contraction principle argument we have to prove in a first step that there exist a time $T_\epsilon > 0$ and a constant $A > 0$ such that the integral operator $\Psi_\epsilon$ maps the space

$$E_{A,T} = \{u \in C([0,T] : H^{3/2}(\mathbb{R})); \|u\|_{L_T^\infty H_x^{3/2}} \leq A\} \tag{2.4}$$

to itself for all $0 < T \leq T_\epsilon$, then we have to prove that $\Psi_\epsilon : E_{A,T} \longrightarrow E_{A,T}$ is a contraction, for all those $T$. Consequently the Banach Theorem for Contraction Mappings assures the existence and uniqueness of $u_\epsilon \in E_{A,T}$ satisfying $\Psi_\epsilon(u_\epsilon) = u_\epsilon$. In this task we will use the following lemma:

**Lemma 2.1.1.** *Let $s \geq 0$. Then,*
*i) There exists $c_s > 0$ such that*

$$\|U_\epsilon(t)f\|_{H_x^s} \leq c_s\left(1 + \frac{1}{|\epsilon t|^{s/2}}\right)\|f\|_{L^2}, \tag{2.5}$$

*for all $f \in L^2(\mathbb{R})$, and $t > 0$.*
*ii) for all $f \in L^2(\mathbb{R})$ the map*

$$t \in (0,\infty) \longmapsto U_\epsilon(t)f \in H^s(\mathbb{R}) \tag{2.6}$$

*is continuous. Moreover, if $f \in H^s(\mathbb{R})$ then the map in (2.6) is continuous to the right at $t = 0$.*

*Proof.* Indeed,

$$\begin{aligned}
\|D^s U_\epsilon(t)f\|_{L_x^2} &= \||\xi|^s e^{-(\epsilon+i)\xi^2 t}\widehat{f}\|_{L_\xi^2} \\
&= \||\xi|^s e^{-\epsilon\xi^2 t}\widehat{f}\|_{L_\xi^2} \\
&= \frac{1}{|\epsilon t|^{s/2}}\||\epsilon\xi^2 t|^{s/2} e^{-\epsilon\xi^2 t}\widehat{f}\|_{L_\xi^2}.
\end{aligned}$$

Since the function $y^{s/2}e^{-y}$ is bounded in $\{y \in \mathbb{R}; y > 0\}$ we obtain

$$\begin{aligned} \|D^s U_\epsilon(t) f\|_{L^2_x} &\leq \frac{1}{|\epsilon t|^{s/2}} \sup_{y>0}\{|y|^{s/2}e^{-y}\}\|\widehat{f}\|_{L^2} \\ &= \frac{1}{|\epsilon t|^{s/2}} \sup_{y>0}\{|y|^{s/2}e^{-y}\}\|f\|_{L^2}. \end{aligned} \tag{2.7}$$

Estimate (2.7) with $s = 0$ gives us

$$\|U_\epsilon(t) f\|_{L^2_x} \leq \|f\|_{L^2}. \tag{2.8}$$

Then, using (2.7) and (2.8) we obtain (2.5). Now we are going to prove the property *ii*). Let $t_0 > 0$ be given and $t > t_0$. We have from (2.5) that

$$\begin{aligned} \|U_\epsilon(t)f - U_\epsilon(t_0)f\|_{H^s_x} &= \|U_\epsilon(t_0)(U_\epsilon(t-t_0)f - f)\|_{H^s_x} \\ &\lesssim \left(1 + \frac{1}{|\epsilon t_0|^{s/2}}\right) \|U(t-t_0)f - f\|_{L^2_x}. \end{aligned}$$

Turns out

$$\|U(t-t_0)f - f\|_{L^2_x} = \|[e^{-(\epsilon+i)(t-t_0)\xi^2} - 1]\widehat{f}\|_{L^2_\xi} \tag{2.9}$$

Since for each $t_0 < t$ the function

$$\xi \longmapsto |(e^{-(\epsilon+i)(t-t_0)\xi^2} - 1)\widehat{f}(\xi)|^2$$

is integrable and bounded by the function $2|\widehat{f}(\xi)|^2$, which is integrable, it follows from the Dominated Convergence Theorem that

$$\lim_{t \searrow t_0} \|(e^{-(\epsilon+i)(t-t_0)\xi^2} - 1)\widehat{f}\|_{L^2_\xi} = 0.$$

Therefore we have continuity to right. To prove the continuity to the left we consider $0 < t < t_0$. Using (2.5) we have already proved we get

$$\|U_\epsilon(t)f - U_\epsilon(t_0)f\|_{H^s_x} = \|U_\epsilon(t)(U_\epsilon(t_0-t)f - f)\|_{H^s_x} \tag{2.10}$$

$$\lesssim \left(1 + \frac{1}{|\epsilon t|^{s/2}}\right) \|U(t_0-t)f - f\|_{L^2_x}. \tag{2.11}$$

Then we proceed using the Dominated Convergence Theorem to conclude

$$\lim_{t \nearrow t_0} \|U(t_0-t)f - f\|_{L^2_x} = 0.$$

Moreover, supposing $f \in H^s(\mathbb{R})$ we have that for each $t > 0$ the function

$$\xi \longmapsto |(e^{-(i+\epsilon)t\xi^2} - 1)\langle\xi\rangle^s \widehat{f}(\xi)|^2$$

is integrable and bounded by $2|\langle\xi\rangle^s\widehat{f}|^2$ which is integrable as well. Using one more time the Dominated Convergence Theorem it follows

$$\begin{aligned}\lim_{t\searrow 0}\|U_\epsilon(t)f-f\|_{H^s_x} &= \lim_{t\searrow 0}\|(e^{-(i+\epsilon)t\xi^2}-1)\langle\xi\rangle^s\widehat{f}\|_{L^2_\xi}\\ &= 0.\end{aligned}$$

□

We start the contraction principle argument by controlling $\Psi_\epsilon(u)$ in the norm $\|\cdot\|_{L^\infty_T H^{3/2}_x}$ for a function $u$ belonging to the class $E_{A,T}$ defined in (2.4). Indeed, Lemma 2.1.1 with $s=0$ implies

$$\begin{aligned}\|U_\epsilon(t)u_0\|_{H^{3/2}_x} &= \|U_\epsilon(t)(J^{3/2}u_0)\|_{L^2_x}\\ &\le \|J^{3/2}u_0\|_{L^2}.\end{aligned}$$

Then we have

$$\begin{aligned}\|\Psi_\epsilon(u)\|_{H^{3/2}_x} &\le \|U_\epsilon(t)u_0\|_{H^{3/2}_x}+\|\int_0^t U_\epsilon(t-t')(|u|^{1+a}\partial_x u)(t')dt'\|_{H^{3/2}_x}\\ &\le \|u_0\|_{H^{3/2}}+(NL).\end{aligned}$$

Lemma 2.1.1 also implies

$$\begin{aligned}(NL) &\le \int_0^t \|U_\epsilon(t-t')(|u|^{1+a}\partial_x u)(t')\|_{H^{3/2}_x}dt'\\ &\lesssim \int_0^t\left(1+\frac{1}{(\epsilon|t-t'|)^{3/4}}\right)\||u|^{1+a}\partial_x u(t')\|_{L^2_x}dt'\\ &\lesssim \left(T+\frac{4T^{1/4}}{\epsilon^{3/4}}\right)\||u|^{1+a}\partial_x u\|_{L^\infty_T L^2_x}.\end{aligned}$$

Using the Sobolev embedding $H^{1/2+}(\mathbb{R})\subset L^\infty(\mathbb{R})$ we have

$$\begin{aligned}\||u|^{1+a}\partial_x u\|_{L^\infty_T L^2_x} &\le \|u\|^{a+1}_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x}\\ &\lesssim \|u\|^{a+2}_{L^\infty_T H^{3/2}_x}.\end{aligned}$$

We conclude

$$\|\Psi_\epsilon(u)\|_{L^\infty_T H^{3/2}_x}\le \|u_0\|_{H^{3/2}}+c\left(T+\frac{4cT^{1/4}}{\epsilon^{3/4}}\right)\|u\|^{a+2}_{L^\infty_T H^{3/2}_x}, \tag{2.12}$$

for some constant $c$. If we take $T_\epsilon=T(\epsilon,\|u_0\|_{H^{3/2}})>0$ sufficiently small such that

$$T_\epsilon+\frac{4T_\epsilon^{1/4}}{\epsilon^{3/4}}\le\frac{1}{2c(2\|u_0\|_{H^{3/2}})^{1+a}}$$

and $A = 2\|u_0\|_{H^{3/2}}$ it follows from the estimate (2.12) that $\Psi_\epsilon(u) \in L^\infty([0,T]; H^{3/2}(\mathbb{R}))$ and

$$\|\Psi_\epsilon(u)\|_{L^\infty_T H^{3/2}_x} \leq A$$

for all $0 < T \leq T_\epsilon$. To conclude $\Psi_\epsilon(u) \in E_{A,T}$ it remains to prove $\Psi_\epsilon(u)$ is continuous with respect to $t$. We already know from Lemma 2.1.1 the map

$$t \in [0,T] \longmapsto U_\epsilon(t)u_0 \in H^{3/2}(\mathbb{R})$$

is continuous. It remains to prove the continuity of

$$v_\epsilon(t) = -\int_0^t U_\epsilon(t-t')(|u|^{1+a}\partial_x u)dt'.$$

Denote $F = |u|^{1+a}\partial_x u$ and let $t_0 \in [0,T]$. For each $t > t_0$ we have

$$\begin{aligned} v_\epsilon(t) - v_\epsilon(t_0) &= -\int_0^{t_0} U_\epsilon(t_0-t')(U_\epsilon(t-t_0)F(t') - F(t'))dt' \\ &\quad -\int_{t_0}^t U_\epsilon(t-t')F(t')dt' \\ &= I + II. \end{aligned}$$

Before estimating $I$ and $II$ notice that $F \in L^\infty([0,T]; L^2(\mathbb{R}))$. In fact, using Sobolev embedding

$$\begin{aligned} \|F\|_{L^\infty_T L^2_x} &\leq \|u\|^{1+a}_{L^\infty_T L^\infty_x} \|\partial_x u\|_{L^\infty_T L^2_x} \\ &\lesssim \|u\|^{1+a}_{L^\infty_T H^{1/2+}_x} \|u\|_{L^\infty_T H^1_x} \\ &\leq \|u\|^{2+a}_{L^\infty_T H^{3/2}_x}. \end{aligned}$$

Now using Lemma 2.1.1 we conclude

$$\begin{aligned} \|II\|_{H^{3/2}_x} &\leq \int_{t_0}^t \|U_\epsilon(t-t')F(t')\|_{H^{3/2}_x} dt' \\ &\lesssim \|F\|_{L^\infty_T L^2_x} \int_{t_0}^t \left(1 + \frac{1}{(\epsilon(t-t'))^{3/4}}\right) dt'. \end{aligned} \tag{2.13}$$

The integral in (4.25) goes to zero as t goes to $t_0$, because the function

$$t' \in [0,T] \longmapsto 1 + \frac{1}{(\epsilon(t-t'))^{3/4}}$$

is integrable. Next, we look at the term $I$. Also using Lemma 2.1.1 we have

$$\begin{aligned} \|I\|_{H^{3/2}} &\leq \int_0^{t_0} \|U_\epsilon(t_0-t')(U_\epsilon(t-t_0)F(t') - F(t'))\|_{H^{3/2}_x} dt' \\ &\lesssim \int_0^{t_0} \left(1 + \frac{1}{(\epsilon(t_0-t'))^{3/4}}\right) \|U_\epsilon(t-t_0)F(t') - F(t')\|_{L^2_x} dt'. \end{aligned} \tag{2.14}$$

Since

$$\|U_\epsilon(t-t_0)F(t') - F(t')\|_{L^2_x} \le 2\|F(t')\|_{L^2_x}$$

we can apply the Dominated Convergence Theorem to the integral in (2.14) and conclude $\|I\|_{H^{3/2}}$ goes to zero as $t$ goes to $t_0$. This concludes the proof that $v_\epsilon$ is continuous to the right. The continuity on the left is proved similarly. Therefore,

$$\Psi_\epsilon(E_{A,T}) \subset E_{A,T}.$$

Now let us prove that we can still choose $T_\epsilon > 0$ such that the map

$$\Psi_\epsilon : E_{A,T} \longrightarrow E_{A,T}$$

is a contraction with respect to the norm $\|\cdot\|_{L^\infty_T H^{3/2}_x}$ for all $0 < T < T_\epsilon$. Consider $u, v \in E_{A,T}$ and denote $G(u,v) = |u|^{1+a}\partial_x u - |v|^{1+a}\partial_x v$. We have

$$\Psi_\epsilon(u) - \Psi_\epsilon(v) = -\int_0^t U_\epsilon(t-t')G(u,v)(t')dt'.$$

Then using Lemma 2.1.1

$$\begin{aligned}
\|\Psi_\epsilon(u) - \Psi_\epsilon(v)\|_{H^{3/2}_x} &= \|\int_0^t U_\epsilon(t-t')G(u,v)(t')dt'\|_{L^\infty_T H^{3/2}_x} \\
&\lesssim \int_0^t \left(1 + \frac{1}{(\epsilon|t-t'|)^{3/2}}\right) \|G(u,v)(t')\|_{L^2_x} dt' \\
&\lesssim \left(T + \frac{4cT^{1/4}}{\epsilon^{3/4}}\right) \|G(u,v)\|_{L^\infty_T L^2_x}.
\end{aligned}$$

To estimate $\|G(u,v)\|_{L^\infty_T L^2_x}$ first note that $G(u,v)$ can be written as

$$G(u,v) = \left(|u|^{1+a} - |v|^{1+a}\right)\partial_x u + |v|^{1+a}\partial_x(u-v).$$

Then using

$$\left||u|^{1+a} - |v|^{1+a}\right| \lesssim \left(|u|^a + |v|^a\right)|u-v| \tag{2.15}$$

we have

$$\begin{aligned}
\|G(u,v)\|_{L^\infty_T L^2_x} &\le \|\left(|u|^{1+a} - |v|^{1+a}\right)\partial_x u\|_{L^\infty_T L^2_x} + \||v|^{1+a}\partial_x(u-v)\|_{L^\infty_T L^2_x} \\
&\le \||u|^{1+a} - |v|^{1+a}\|_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} + \|v\|^{1+a}_{L^\infty_T L^\infty_x}\|\partial_x(u-v)\|_{L^\infty_T L^2_x} \\
&\lesssim \left(\|u\|^a_{L^\infty_T L^\infty_x} + \|v\|^a_{L^\infty_T L^\infty_x}\right)\|u-v\|_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} \\
&+ \|v\|^{1+a}_{L^\infty_T L^\infty_x}\|\partial_x(u-v)\|_{L^\infty_T L^2_x}.
\end{aligned} \tag{2.16}$$

Applying Sobolev embedding in (2.16) we conclude

$$\begin{aligned}\|G(u,v)\|_{L^\infty_T L^2_x} &\lesssim \left(\|u\|_{L^\infty_T H^{3/2}_x} + \|v\|_{L^\infty_T H^{3/2}_x}\right)^{1+a} \|u-v\|_{L^\infty_T H^{3/2}_x} \\ &\lesssim \left(4\|u_0\|_{H^{3/2}}\right)^{1+a} \|u-v\|_{L^\infty_T H^{3/2}_x}.\end{aligned}$$

Therefore

$$\|\Psi_\epsilon(u) - \Psi_\epsilon(v)\|_{L^\infty_T H^{3/2}_x} \leq c\left(T + \frac{4cT^{1/4}}{\epsilon^{3/4}}\right)\left(4\|u_0\|_{H^{3/2}}\right)^{1+a} \|u-v\|_{L^\infty_T H^{3/2}_x}, \tag{2.17}$$

for some constant $c > 0$. Take $T_\epsilon$ such that

$$T_\epsilon + \frac{4cT_\epsilon^{1/4}}{\epsilon^{3/4}} \leq \frac{1}{2c(4\|u_0\|_{H^{3/2}})^{1+a}}.$$

Thus from (2.17) we end up with

$$\|\Psi_\epsilon(u) - \Psi_\epsilon(v)\|_{L^\infty_T H^{3/2}_x} \leq \frac{1}{2}\|u-v\|_{L^\infty_T H^{3/2}_x}.$$

**Remark 3.** *We can choose*

$$T_\epsilon = \frac{\epsilon^3}{c\|u_0\|^{1+a}_{H^{3/2}}}$$

*for some constant $c$ sufficiently large.*

We have proved the following

**Theorem 2.1.2.** *Let $\epsilon > 0$ and $u_0 \in H^{3/2}(\mathbb{R})$ be given. Then for all $0 < T \leq \dfrac{\epsilon^3}{c\|u_0\|^{1+a}_{H^{3/2}}}$ there exists one, and only one, $u_\epsilon \in C([0,T]; H^{3/2}(\mathbb{R}))$ satisfying the integral equation (2.3).*

Next we prove that the solution $u_\epsilon$ in Theorem 2.1.2 is in fact solution of the differential equation (2.1).

**Theorem 2.1.3.** *Let $u_\epsilon$ be the solution of (2.3) given in Theorem 2.1.2. Then,*

$$u_\epsilon \in C((0, T_\epsilon]; H^2(\mathbb{R})) \tag{2.18}$$

*and $\partial_t u_\epsilon$ exists in the topology of $H^{-1}(\mathbb{R})$ and satisfies*

$$\partial_t u_\epsilon = (i+\epsilon)\partial_x^2 u_\epsilon - |u_\epsilon|^{1+a}\partial_x u_\epsilon. \tag{2.19}$$

*Proof.* We already know from Lemma 2.1.1 that $U_\epsilon(t)u_0 \in C((0,T_\epsilon];H^2(\mathbb{R}))$. So we have just to investigate the continuity of

$$v_\epsilon(t) = -\int_0^t U_\epsilon(t-t')F_\epsilon(t')dt'$$

where $F_\epsilon = |u_\epsilon|^{1+a}\partial_x u_\epsilon$. From Lemma 2.1.1 we have,

$$\begin{aligned} \|v_\epsilon(t)\|_{H^2_x} &\leq \int_0^t \|U_\epsilon(t-t')F_\epsilon(t')\|_{H^2_x} dt' \\ &\lesssim \|F_\epsilon\|_{L^\infty_T H^{1/2}_x} \int_0^t \left(1 + \frac{1}{(\epsilon(t-t'))^{3/4}}\right) dt' \\ &\lesssim \|F_\epsilon\|_{L^\infty_T H^{1/2}_x}. \end{aligned}$$

So, if we assume

$$F_\epsilon \in L^\infty([0,T];H^{1/2}(\mathbb{R})) \tag{2.20}$$

we obtain

$$v_\epsilon \in L^\infty([0,T];H^2(\mathbb{R})). \tag{2.21}$$

Lets keep the assumption (2.20) and prove (2.18). We claim that (2.21) together with the continuity in $H^{3/2}$ gives us (2.18). Indeed, choose some $s \in (2,5/2)$. Using Lemma 1.1.2 we have

$$\|v_\epsilon(t) - v_\epsilon(t_0)\|_{H^2_x} \leq \|v_\epsilon(t) - v_\epsilon(t_0)\|^\theta_{H^{3/2}_x} \|v_\epsilon(t) - v_\epsilon(t_0)\|^{1-\theta}_{H^{s'}_x} \tag{2.22}$$

for $\theta = \dfrac{2s-4}{2s-3}$. Using (2.21) and (2.22) we obtain

$$\|v_\epsilon(t) - v_\epsilon(t_0)\|_{H^s_x} \lesssim \|v_\epsilon(t) - v_\epsilon(t_0)\|^\theta_{H^{3/2}_x}. \tag{2.23}$$

Since $v_\epsilon \in C([0,T_\epsilon];H^{3/2}(\mathbb{R}))$ it follows from (2.23) that $v_\epsilon \in C([0,T_\epsilon];H^2(\mathbb{R}))$. To finish the proof of (2.18) we need to justify (2.20). Indeed, using the Leibniz rule (Lemma 1.1.4) we have

$$\|D^{1/2}F_\epsilon\|_{L^2_x} \lesssim \|D^{1/2}(|u_\epsilon|^{1+a})\|_{L^p_x}\|\partial_x u_\epsilon\|_{L^q_x} + \||u_\epsilon|^{1+a}\|_{L^\infty_x}\|D^{1/2}\partial_x u_\epsilon\|_{L^2_x}, \tag{2.24}$$

for $2 < p,q < \infty$ satisfying $1/2 = 1/p + 1/q$. Using Sobolev embedding in (2.24) we obtain

$$\|D^{1/2}F_\epsilon\|_{L^2_x} \lesssim \||u_\epsilon|^{1+a}\|_{H^1_x}\|u_\epsilon\|_{H^{3/2}_x} + \|u_\epsilon\|^{2+a}_{H^{3/2}_x}. \tag{2.25}$$

Computing $\partial_x(|u_\epsilon|^{1+a})$ and using Sobolev embedding we obtain

$$\begin{aligned} \||u_\epsilon|^{1+a}\|_{H^1_x} &\lesssim \|u_\epsilon\|^{1+a}_{L^{2+2a}_x} + \||u_\epsilon|^a \partial_x u_\epsilon\|_{L^2_x} \\ &\lesssim \|u_\epsilon\|^{1+a}_{H^{3/2}_x} + \|u_\epsilon\|^a_{L^\infty_x}\|\partial_x u_\epsilon\|_{L^2_x} \\ &\lesssim \|u_\epsilon\|^{2+a}_{H^{3/2}_x}. \end{aligned} \tag{2.26}$$

Combining (2.26) and (2.25) we finally obtain (2.20). Now we are going to prove (2.19). Consider $t \in (0,T]$ and $h$ sufficiently small. Regarding the linear term $U_\epsilon(t)u_0$ we have

$$\frac{U_\epsilon(t+h)u_0 - U_\epsilon(t)u_0}{h} - (i+\epsilon)\partial_x^2 U_\epsilon(t)u_0 = \{g(t;h,\xi)\widehat{u_0}\}^\vee$$

where

$$g(t;h,\xi) = \frac{e^{-(i+\epsilon)\xi^2(t+h)} - e^{-(i+\epsilon)\xi^2 t}}{h} + (i+\epsilon)\xi^2 e^{-(i+\epsilon)\xi^2 t}.$$

Notice that

$$|g(t;h,\xi)| \leq 2(1+\epsilon)\xi^2.$$

Then we can use the Dominated Convergence Theorem and conclude that

$$\lim_{h\to 0} \|g(t;h,\xi)\langle\xi\rangle^{-1}\widehat{u_0}\|_{L^2_\xi} = 0,$$

which means

$$\lim_{h\to 0}\left\|\frac{U_\epsilon(t+h)u_0 - U_\epsilon(t)u_0}{h} - (i+\epsilon)\partial_x^2 U_\epsilon(t)u_0\right\|_{H_x^{-1}} = 0. \tag{2.27}$$

Next we analyse the nonlinear part $v_\epsilon$. For all $h>0$ we have,

$$\begin{aligned}
\frac{v_\epsilon(t+h) - v_\epsilon(t)}{h} &= -\frac{1}{h}\int_0^t \big[U_\epsilon(t+h-t') - U_\epsilon(t-t')\big]F_\epsilon(t')dt' \\
&\quad -\frac{1}{h}\int_t^{t+h} U_\epsilon(t+h-t')F_\epsilon(t')dt' \\
&= I_1(t,h) + I_2(t,h).
\end{aligned}$$

We have

$$\begin{aligned}
I_1(t,h) &= -\frac{1}{h}\int_0^t U_\epsilon(h)U_\epsilon(t-t')F_\epsilon(t')dt' \\
&\quad -\frac{1}{h}\int_0^t U_\epsilon(t-t')F_\epsilon(t')dt' \\
&= \frac{U_\epsilon(h)v_\epsilon(t) - v_\epsilon(t)}{h}.
\end{aligned} \tag{2.28}$$

Since $v_\epsilon(t) \in H^2(\mathbb{R})$, it follows from (2.28) that

$$\lim_{h\searrow 0} \|I_1(t,h) - (i+\epsilon)\partial_x^2 v_\epsilon(t)\|_{H_x^{-1}} = 0. \tag{2.29}$$

Regarding the term $I_2(t,h)$ we have

$$\begin{aligned}\|I_2(t,h)-F_\epsilon(t)\|_{H_x^{-1}} &\leq \frac{1}{h}\int_t^{t+h}\|U_\epsilon(t+h-t')(F_\epsilon(t')-F_\epsilon(t))\|_{H_x^{-1}}dt' \\ &\quad +\frac{1}{h}\int_t^{t+h}\|U_\epsilon(t+h-t')F_\epsilon(t)-F_\epsilon(t)\|_{H_x^{-1}}dt' \\ &\lesssim \frac{1}{h}\int_t^{t+h}\|F_\epsilon(t')-F_\epsilon(t)\|_{H_x^{-1}}dt' \\ &\quad +\frac{1}{h}\int_t^{t+h}\|U_\epsilon(t+h-t')F_\epsilon(t)-F_\epsilon(t)\|_{H_x^{-1}}dt'. \end{aligned} \tag{2.30}$$

Using that $F_\epsilon$ belongs to the class $C([0,T];H^{-1}(\mathbb{R}))$ as well as the map

$$\tau\in[0,T]\longmapsto U_\epsilon(\tau)F_\epsilon(t)$$

we conclude from (2.30) that

$$\lim_{h\searrow 0}\|I_2(t,h)-F_\epsilon(t)\|_{H_x^{-1}}=0. \tag{2.31}$$

Consequently

$$\lim_{h\searrow 0}\frac{u_\epsilon(t+h)-u_\epsilon(t)}{h}=(i+\epsilon)\partial_x^2u_\epsilon(t)-|u_\epsilon|^{1+a}\partial_xu_\epsilon(t)$$

in the $H^{-1}(\mathbb{R})$ topology. To conclude our proof, we study the limit

$$\lim_{h\nearrow 0}\frac{v_\epsilon(t+h)-v_\epsilon(t)}{h}.$$

Indeed, for each $h<0$ we write

$$\begin{aligned}\frac{v_\epsilon(t+h)-v_\epsilon(t)}{h} &= -\frac{1}{h}\int_0^{t+h}\big[U_\epsilon(t+h-t')-U_\epsilon(t-t')\big]F_\epsilon(t')dt' \\ &\quad +\frac{1}{h}\int_{t+h}^{t}U_\epsilon(t-t')F_\epsilon(t')dt' \\ &= \tilde{I}_1(t,h)+\tilde{I}_2(t,h).\end{aligned}$$

The argument we used to prove (2.31) can also be applied to the term $\tilde{I}_2(h,t)$. Thus we also obtain

$$\lim_{h\searrow 0}\|\tilde{I}_2(t,h)-F_\epsilon(t)\|_{H_x^{-1}}=0. \tag{2.32}$$

Finally, we consider $\tilde{I}_1(t,h)$. Using the definition of $v_\epsilon$ we can write

$$\tilde{I}_1(t,h)-(i+\epsilon)\partial_x^2v_\epsilon(t)=\tilde{I}_{11}(t,h)+\tilde{I}_{12}(t,h),$$

where

$$\tilde{I}_{11}(t,h) = -(i+\epsilon)\int_{t+h}^{t} \partial_x^2 U_\epsilon(t-t')F_\epsilon(t')dt',$$

and

$$\tilde{I}_{12}(t,h) = \int_0^{t+h} \left( \frac{U_\epsilon(t+h-t') - U_\epsilon(t-t')}{h} F_\epsilon(t') - (i+\epsilon)\partial_x^2 U_\epsilon(t-t')F_\epsilon(t') \right) dt'.$$

To prove both $\tilde{I}_{11}(t,h)$ and $\tilde{I}_{12}(t,h)$ converge to zero as $h$ goes to zero, notice that

$$F_\epsilon \in L^\infty([0,T];H^1(\mathbb{R})) \tag{2.33}$$

This follows from the fact $u_\epsilon \in L^\infty([0,T];H^2(\mathbb{R}))$. Using the boundness property of $U_\epsilon$ we have

$$\begin{aligned} \|\tilde{I}_{11}(t,h)\|_{H_x^{-1}} &\lesssim \int_{t+h}^{t} \|\partial_x^2 U_\epsilon(t-t')F_\epsilon(t')\|_{H_x^{-1}} dt' \\ &\lesssim \int_{t+h}^{t} \|F_\epsilon(t')\|_{H_x^1} dt' \\ &\lesssim -h\|F_\epsilon\|_{L_T^\infty H_x^1}. \end{aligned}$$

Hence

$$\lim_{h\nearrow 0} \|\tilde{I}_{11}(t,h)\|_{H_x^{-1}} = 0.$$

It is remaining to analyse $\tilde{I}_{12}(t,h)$. Notice we can write

$$\frac{U_\epsilon(t+h-t') - U_\epsilon(t-t')}{h} F_\epsilon(t') - (i+\epsilon)\partial_x^2 U_\epsilon(t-t')F_\epsilon(t') = \{q(h,t',\xi)\widehat{F_\epsilon(t')}\}^\vee,$$

where

$$q(h,t',\xi) = \frac{e^{-(i+\epsilon)(t+h-t')\xi^2} - e^{-(i+\epsilon)(t-t')\xi^2}}{h} + (i+\epsilon)\xi^2 e^{-(i+\epsilon)(t-t')\xi^2}.$$

Thus

$$\|\tilde{I}_{12}(t,h)\|_{H_x^{-1}} \le \int_0^{t+h} \|q(t,t',\xi)\widehat{F_\epsilon(t')}\langle\xi\rangle^{-1}\|_{L_\xi^2} dt'.$$

Since

$$|q(h,t',\xi)\chi_{[0,t+h]}(t')| \lesssim \xi^2, \tag{2.34}$$

it follows the function

$$t' \in [0,t] \longmapsto \|\langle\xi\rangle^{-1} q(h,t',\xi)\chi_{[0,t+h]}(t')\widehat{F_\epsilon(t')}\|_{L_\xi^2}$$

is bounded by

$$t' \longmapsto \|\langle\xi\rangle\widehat{F_\epsilon(t')}\|_{L_\xi^2},$$

which, in turn, is integrable because $F_\epsilon \in L^\infty([0,T]; H^1(\mathbb{R}))$. Consequently, we can apply the Dominated Convergence Theorem and get

$$\lim_{h \nearrow 0} \|\tilde{I}_{12}(t,h)\|_{H_x^{-1}} = \int_0^t \lim_{h \nearrow 0} \|\langle \xi \rangle^{-1} q(h,t',\xi)\chi_{[0,t+h]}(t')\widehat{F_\epsilon(t')}\|_{L_\xi^2} dt' \tag{2.35}$$

Using (2.34) again we can apply the Dominated Convergence Theorem to conclude that for each $t' \in [0,t]$

$$\begin{aligned} \lim_{h \nearrow 0} \|\langle \xi \rangle^{-1} q(h,t',\xi)\chi_{[0,t+h]}(t')\widehat{F_\epsilon(t')}\|_{L_\xi^2} &= \| \lim_{h \nearrow 0} \langle \xi \rangle^{-1} q(h,t',\xi)\chi_{[0,t+h]}(t')\widehat{F_\epsilon(t')}\|_{L_\xi^2} \\ &= 0. \end{aligned}$$

This finishes our proof.

□

**Remark 4.** *The argument we presented above in the proof of (2.19) implies that*

$$u_\epsilon \in C([0,T]; H^s(\mathbb{R}))$$

*whenever*

$$F_\epsilon \in L^\infty([0,T]; H_x^{(s-2)+}) \tag{2.36}$$

*Since we proved $F_\epsilon \in L^\infty([0,T]; H^1(\mathbb{R}))$, it is possible to prove $u_\epsilon \in C([0,T]; H^{3-}(\mathbb{R}))$. But we will not need this in our work. Unfortunately, we can not derivate the nonlinearity $F_\epsilon = |u_\epsilon|^{1+a}\partial_x u_\epsilon$ twice when $0 < a < 1$. This means we do not have (2.36) for $s \geq 4$. Consequently we can not guarantee $u_\epsilon$ is sufficiently regular.*

## 2.2 Uniform estimates for the $\epsilon$-linear problem

In this section we establish properties that are true for $U(t)$ and still hold for $U_\epsilon(t)$ uniformly in the parameter $\epsilon$. All these properties will be important to prove uniform estimates for the solution of the problem (2.1). We are looking for estimates independent of $\epsilon$ because we intend to take the limit when $\epsilon$ goes to zero later on.

**Proposition 2.2.1.** *There exists a constant $c > 0$ independent of $\epsilon$ such that*

$$\|U_\epsilon(t)f\|_{L_T^q L_x^p} \leq c\|f\|_{L^2}$$

*for all pairs $(p,q)$ satisfying*

$$2 \leq p \leq \infty, \quad \frac{1}{2} = \frac{2}{q} + \frac{1}{p}.$$

*Proof.* Notice that $U_\epsilon(t) = E_\epsilon(t)U(t)$ where

$$E_\epsilon(t)g = \{e^{-\epsilon t\xi^2}\hat{g}\}^\vee.$$

Let $\varphi(x) = e^{-\pi x^2}$ and denote $\varphi_\rho$ the function $\varphi_\rho(x) = \rho\varphi(\rho x)$. Since $\check{\varphi} = \varphi$, we have from the properties of the Fourier transform that

$$\begin{aligned}
\{e^{-\epsilon t\xi^2}\}^\vee(x) &= \Big\{\varphi(\sqrt{\frac{\epsilon t}{\pi}}\cdot)\Big\}^\vee(x) \\
&= \sqrt{\frac{\pi}{\epsilon t}}\check{\varphi}\Big(\sqrt{\frac{\pi}{\epsilon t}}x\Big) \\
&= \sqrt{\frac{\pi}{\epsilon t}}\varphi\Big(\sqrt{\frac{\pi}{\epsilon t}}x\Big) \\
&= \varphi_{\rho_{\epsilon,t}}(x)
\end{aligned}$$

where $\rho_{\epsilon,t} = \sqrt{\frac{\pi}{\epsilon t}}$. So we can write $E_\epsilon(t)$ as

$$E_\epsilon(t)g(\cdot,t) = \varphi_{\rho_{\epsilon,t}} * g. \tag{2.37}$$

Finally, we combine the Young inequality and Strichartz estimate for the Schrödinger group to conclude

$$\begin{aligned}
\|U_\epsilon(t)f\|_{L^q_T L^p_x} &= \|E_\epsilon(t)U(t)f\|_{L^q_T L^p_x} \\
&\lesssim \|\varphi_{\rho_{\epsilon,t}} * U(t)f\|_{L^q_T L^p_x} \\
&\lesssim \|\|\varphi_{\rho_{\epsilon,t}}\|_{L^1_x}\|U(t)f\|_{L^p_x}\|_{L^q_T}.
\end{aligned} \tag{2.38}$$

Using that

$$\int \varphi_{\rho_{\epsilon,t}}(x)dx = \int \varphi(x)dx$$

in (2.38) we conclude that

$$\begin{aligned}
\|U_\epsilon(t)f\|_{L^q_T L^p_x} &\lesssim \|U(t)f\|_{L^q_T L^p_x} \\
&\lesssim \|f\|_{L^2}.
\end{aligned}$$

□

Next we are concerned with the question whether or not the smoothing effects for the Schrödinger propagator are still true for $U_\epsilon(t)$. If we knew, for example, that

$$\|E_\epsilon(t)f\|_{L^\infty_x L^2_T} \le c\|f\|_{L^\infty_x L^2_T} \tag{2.39}$$

with $c$ independent of $\epsilon$ we would obtain

$$\sup_{\epsilon>0} \|D^{1/2}U_\epsilon(t)f\|_{L^\infty_x L^2_T} \leq c\|f\|_{L^2}.$$

Unfortunately we do not know if (2.39) is true. Looking for a replacement we introduce a norm somehow similar to $\|\cdot\|_{L^\infty_x L^2_T}$ but in which we are able to prove that the operators $E_\epsilon(t)$ are bounded uniformly in $\epsilon$. The norm is

$$\|\cdot\|_{l^\infty_j(L^2(Q_j))} = \sup_{j\in\mathbb{Z}} \|\cdot\|_{L^2(Q_j)}$$

where $Q_j$ is the rectangle $Q_j = [0,T] \times [j, j+1]$, $j \in \mathbb{Z}$. This norm is weaker than $\|\cdot\|_{L^\infty_x L^2_T}$, i.e.

$$\|\cdot\|_{l^\infty_j(L^2(Q_j))} \leq \|\cdot\|_{L^\infty_x L^2_T}. \tag{2.40}$$

In this weaker norm we are able to prove analogous of the Lemmas 1.2.2, 1.2.3 and 1.2.4 for $U_\epsilon(t)$ with bounds independents of $\epsilon$.

**Proposition 2.2.2.** *Let $T > 0$ and $\epsilon > 0$ satisfying $0 < \epsilon < \pi/T$. Then there exists $c > 0$ independent of $\epsilon$ such that*

$$\|D^{1/2}U_\epsilon(t)f\|_{l^\infty_j(L^2(Q_j))} \leq c\|f\|_{L^2}$$

*for all $f \in L^2(\mathbb{R})$.*

To prove this proposition we need to prove the following lemma.

**Lemma 2.2.3.** *Let $T > 0$ and $\epsilon > 0$ satisfying $0 < \epsilon < \pi/T$. Then there exists $c > 0$ independent of $\epsilon$ such that*

$$\|E_\epsilon(t)g\|_{l^\infty_j(L^2(Q_j))} \leq c\|g\|_{l^\infty_j(L^2(Q_j))}$$

*for all $g \in l^\infty_j(L^2(Q_j))$.*

*Proof.* Writing $E_\epsilon(t)$ as the convolution in (2.37) we have

$$\begin{aligned} E_\epsilon(t)g(\cdot,t)(x) &= \int_{\mathbb{R}} \varphi_{\rho_{\epsilon,t}}(y)g(x-y,t)dy \\ &= \int_{|y|<1} \varphi_{\rho_{\epsilon,t}}(y)g(x-y,t)dy + \int_{|y|>1} \varphi_{\rho_{\epsilon,t}}(y)g(x-y,t)dy \\ &= I_0(x,t) + I_\infty(x,t). \end{aligned}$$

By using Minkowsky inequality, a change of variables and $[j-y, j+1-y] \subset [j-1, j+2]$ for all $|y| < 1$, we have that

$$\begin{aligned}
\|I_0(\cdot,t)\|_{L^2([j,j+1])} &\leq \int_{|y|<1} \varphi_{\rho_{\epsilon,t}}(y)\|g(\cdot - y,t)\|_{L^2([j,j+1])}dy \\
&= \int_{|y|<1} \varphi_{\rho_{\epsilon,t}}(y)\|g(\cdot,t)\|_{L^2([j-y,j-y+1])}dy \\
&= \int_{|y|<1} \varphi_{\rho_{\epsilon,t}}(y)\|g(\cdot,t)\|_{L^2([j-1,j+2])}dy \\
&= \|g(\cdot,t)\|_{L^2([j-1,j+2])} \int_{\mathbb{R}} \varphi(y)dy \\
&= \|g(\cdot,t)\|_{L^2([j-1,j+2])}
\end{aligned}$$

for all $j \in \mathbb{Z}$. So we obtain

$$\begin{aligned}
\|I_0\|_{l_j^\infty(L^2(Q_j))} &= \sup_{j\in\mathbb{Z}} \|\|I_0\|_{L^2([j,j+1])}\|_{L^2([0,T])} \\
&= \sup_{j\in\mathbb{Z}} \|g\|_{L^2([j-1,j+2]\times[0,T])} \\
&\leq 3\|g\|_{l_j^\infty(L^2(Q_j))}.
\end{aligned}$$

Now let us estimate $\|I_\infty\|_{l_j^\infty(L^2(Q_j))}$. In order to do so notice that $\rho_{\epsilon,t} \leq \rho_{\epsilon,t}^2$ whenever $\rho_{\epsilon,t} > 1$, i.e. $\epsilon < \frac{\pi}{T}$. Then

$$\begin{aligned}
\varphi_{\rho_{\epsilon,t}}(y) &= \rho_{\epsilon,t}e^{-\pi|\rho_{\epsilon,t}y|^2} \\
&\leq \rho_{\epsilon,t}^2 e^{-\pi|\rho_{\epsilon,t}y|^2}.
\end{aligned}$$

This allows us to conclude

$$\begin{aligned}
|I_\infty(x,t)| &\leq \int_{|y|>1} \varphi_{\rho_{\epsilon,t}}(y)|g(x-y,t)|dy \\
&\leq \int_{|y|>1} \rho_{\epsilon,t}^2 e^{-|\rho_{\epsilon,t}y|^2}|g(x-y,t)|dy \\
&= \int_{|y|>1} \frac{1}{y^2}|y\rho_{\epsilon,t}|^2 e^{-|\rho_{\epsilon,t}y|^2}|g(x-y,t)|dy \\
&\leq \sup\{\delta e^{-\delta} : \delta > 0\} \int_{|y|>1} \frac{1}{y^2}|g(x-y,t)|dy.
\end{aligned}$$

Using change of variables we have

$$\begin{aligned}
\|I_\infty\|_{L^2(Q_j)} &\lesssim \int_{|y|>1} \frac{1}{y^2}\|g(\cdot - y,\cdot)\|_{L^2(Q_j)}dy \\
&\lesssim \int_{|y|>1} \frac{1}{y^2}\|g\|_{L^2([j-y,j+1-y]\times[0,T])}dy \\
&\lesssim \int_{|y|>1} \frac{1}{y^2}\|g\|_{L^2([m_{j,y},m_{j,y}+2]\times[0,T])}dy
\end{aligned}$$

where $m_{j,y}$ denotes the integer number such that $m_{j,y} \le j - y < m_{j,y} + 1$. Then we conclude

$$\begin{aligned} \|I_\infty\|_{L^2(Q_j)} &\lesssim \sup_{m\in\mathbb{Z}} \|g\|_{L^2([m,m+1]\times[0,T])} \int_{|y|>1} \frac{1}{y^2} dy \\ &\lesssim \|g\|_{l^\infty_j(L^2(Q_j))}. \end{aligned}$$

Therefore

$$\|I_\infty\|_{l^\infty_j(L^2(Q_j))} \lesssim \|g\|_{l^\infty_j(L^2(Q_j))}.$$

□

**Proof of Proposition 2.2.2 :** Since

$$\begin{aligned} D^{1/2} U_\epsilon(t) f &= D^{1/2} E_\epsilon(t) U(t) f \\ &= E_\epsilon(t) D^{1/2} U(t) f \end{aligned}$$

we have from Lemma 2.2.3 that

$$\|D^{1/2} U_\epsilon(t) f\|_{l^\infty_j(L^2(Q_j))} \le c \|D^{1/2} U(t) f\|_{l^\infty_j(L^2(Q_j))}.$$

To complete the proof we recall (2.40) as well as the smoothing effect Lemma 1.2.2 to conclude

$$\begin{aligned} \|D^{1/2} U_\epsilon(t) f\|_{l^\infty_j(L^2(Q_j))} &\le c \|D^{1/2} U(t) f\|_{L^\infty_x L^2_T} \\ &\le c \|f\|_{L^2}. \end{aligned}$$

Next we establish the smoothing effect for the inhomogeneous problem associated to (2.2).

**Proposition 2.2.4.** *Let $T > 0$ and $\epsilon > 0$ , such that $0 < \epsilon\pi/T$. Then there exists $c > 0$ independent of $\epsilon$ such that*

$$\|D^{1/2} \int_0^t U_\epsilon(t - t') F(\cdot, t') dt'\|_{L^\infty_T L^2_x} \le c T^{1/2} \|F\|_{l^1_j(L^2(Q_j))}.$$

*Proof.* By duality it is equivalent to prove

$$\sup_G \left| \int_{\mathbb{R}} \left( D^{1/2} \int_0^t U_\epsilon(t - t') F(\cdot, t')(x) dt' \right) \overline{G}(x) dx \right| \le c \|F\|_{l^1_j(L^2(Q_j))}$$

where the supremum is taken over all $G \in L^2(\mathbb{R})$, such that $\|G\|_{L^2} = 1$. Using Fubini's theorem, Parserval's identity and the definition of $U_\epsilon(t)$ it follows

$$\begin{aligned}
&\int_{\mathbb{R}}\left(D^{1/2}\int_0^t U_\epsilon(t-t')F(\cdot,t')(x)dt'\right)\overline{G}(x)dx\\
=&\int_{\mathbb{R}}\left(\int_0^t D^{1/2}U_\epsilon(t-t')F(\cdot,t')(x)dt'\right)\overline{G}(x)dx\\
=&\int_0^t\int_{\mathbb{R}}\left(D^{1/2}U_\epsilon(t-t')F(\cdot,t')(x)\overline{G}(x)dx\right)dt'\\
=&\int_0^t\int_{\mathbb{R}}|\xi|^{1/2}e^{-(t-t')(i-\epsilon)\xi^2}\widehat{F(\cdot,t')}(\xi)\widehat{\overline{G}}(\xi)d\xi dt'\\
=&\int_0^t\int_{\mathbb{R}}F(x,t')\overline{D^{1/2}E_\epsilon(t-t')U(t'-t)G(x)}dxdt'. \qquad (2.41)
\end{aligned}$$

Splitting the integral with respect to the variable $x$ in (2.41) into a sum of integrals over the intervals $[j,j+1]$ we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\left(D^{1/2}\int_0^t U_\epsilon(t-t')F(\cdot,t')(x)dt'\right)\overline{G}(x)dx\right|\\
\leq&\sum_{j\in\mathbb{Z}}\int_j^{j+1}\int_0^t|F(x,t')||D^{1/2}E_\epsilon(t-t')U(t'-t)\overline{G}(x)|dt'dx. \qquad (2.42)
\end{aligned}$$

Using Hölder's inequality we have that (2.42) is bounded by

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\|F\|_{L^2([j,j+1]\times[0,t])}\|D^{1/2}E_\epsilon(t-\cdot)U(\cdot-t)\overline{G}\|_{L^2([j,j+1]\times[0,t])}\\
\leq&\sup_{j\in\mathbb{Z}}\|D^{1/2}E_\epsilon(t-\cdot)U(\cdot-t)\overline{G}\|_{L^2([j,j+1]\times[0,t])}\sum_{j\in\mathbb{Z}}\|F\|_{L^2([j,j+1]\times[0,t])}.
\end{aligned}$$

Notice that by the change of variables $s\longmapsto t-t'$ and Lemma 2.2.3

$$\begin{aligned}
\|D^{1/2}E_\epsilon(t-\cdot)U(\cdot-t)\overline{G}\|_{L^2([j,j+1]\times[0,t])} &= \|D^{1/2}E_\epsilon(s)U(-s)\overline{G}\|_{L^2([j,j+1]\times[0,t])}\\
&= \|E_\epsilon(s)D^{1/2}U(-s)\overline{G}\|_{L^2([j,j+1]\times[0,t])}\\
&\lesssim \|D^{1/2}U(-s)\overline{G}\|_{l^\infty_j(L^2(Q_j))}\\
&= \sup_{k\in\mathbb{Z}}\|D^{1/2}U(-s)\overline{G}\|_{L^2([k,k+1]\times[0,T])}\\
&\leq \|D^{1/2}U(-s)\overline{G}\|_{L^\infty_x L^2_T}\\
&\lesssim \|G\|_{L^2}.
\end{aligned}$$

Then we conclude that

$$\left|\int_{\mathbb{R}}\left(\int_0^t U_\epsilon(t-t')F(\cdot,t')(x)dt'\right)\overline{G}(x)dx\right|\lesssim\|F\|_{l^1_j(L^2(Q_j))}$$

for all $G\in L^2(\mathbb{R})$ with $\|G\|_{L^2}=1$. □

**Proposition 2.2.5.** *Let $T > 0$ and $\epsilon > 0$ , such that $0 < \epsilon\pi/T$. Then there exists $c > 0$ independent of $\epsilon$ such that*

$$\|\partial_x \int_0^t U_\epsilon(t-t')F(\cdot,t')dt'\|_{l_j^\infty(L^2(Q_j))} \leq c\|F\|_{l_j^1(L(Q_j))}.$$

To prove this proposition we basically follow the ideas presented by Kenig, Ponce e Vega [17]. Our additional effort consists basically in proving the following:

**Lemma 2.2.6.** *For each $w \in \mathbb{C}$ define the function $f_w(x) = \dfrac{x}{x^2+w^2}$ . Then there exists a constant $c > 0$ such that*

$$\|\widehat{f_w}\|_{L^\infty} \leq c$$

*for all $w \in \mathbb{C}$.*

*Proof.* We consider three cases

**Case 1:** $\mathrm{Re}(w) > 0$.

Consider for each $\xi \in \mathbb{R}$ the complex function

$$F_+(z) = \frac{e^{i|\xi|z}}{z^2+w^2}$$

and $\Gamma_R^+$ the boundary of

$$D_R^+ = \{z \in \mathbb{C}; |z| < R, \ \ \mathrm{Im}(z) > 0\}.$$

Since $F_+$ is holomorphic in $D_R^+ \setminus \{iw\}$ , we have from the Residue Formula

$$\begin{aligned}
\int_{\Gamma_R^+} F_+(z)dz &= 2\pi i \mathrm{Res}(F_+, iw) \\
&= 2\pi i \frac{e^{-w|\xi|}}{2iw} \\
&= \frac{\pi}{w} e^{-w|\xi|}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\Gamma_R^+} F_+(z)dz &= \int_{-R}^{R} \frac{e^{i\xi x}}{x^2+w^2}dx + \int_0^\pi F_+(\gamma_R(\theta))\gamma_R'(\theta)d\theta \\
&= I_R^+ + II_R^+
\end{aligned}$$

where $\gamma_R$ is the curve $\theta \in [0,\pi] \longmapsto Re^{i\theta}$. Let us estimate the integral $II_R^+$.

$$\begin{aligned}
\left|II_R^+\right| &= \left| \int_0^\pi \frac{e^{i|\xi|R(\cos\theta + i\sin\theta)} iRe^{i\theta}}{(Re^{i\theta})^2+w^2} d\theta \right| \\
&\leq \int_0^\pi \frac{e^{-|\xi|R\sin\theta}R}{|(Re^{i\theta})^2+w^2|}d\theta \\
&\leq \int_0^\pi \frac{e^{-|\xi|R\sin\theta}R}{|R^2-|w|^2|}d\theta. \qquad (2.43)
\end{aligned}$$

Taking $R > \sqrt{2}|w|$ we have that the integral (2.43) is less than

$$\frac{1}{2R^2}\int_0^\pi e^{-|\xi|R\sin\theta} R d\theta.$$

Since $\sin\theta$ is positive in the range $0 < \theta < \pi$ we conclude

$$|II_R^+| \leq \frac{\pi}{2R}$$

and then $II_R^+ \longrightarrow 0$ as $R \longrightarrow +\infty$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}} \frac{e^{ix|\xi|}}{x^2+w^2}dx &= \lim_{R\to+\infty} I_R^+ + II_R^+ \\
&= \lim_{R\to+\infty} \int_{\Gamma_R^+} F_+(z)dz \\
&= \frac{\pi}{w}e^{-w|\xi|}
\end{aligned}$$

To finish this case notice that

$$\int_{\mathbb{R}} \frac{\cos(\xi x)}{x^2+w^2}dx = \int_{\mathbb{R}} \frac{\cos(|\xi|x)}{x^2+w^2}dx$$

because the function $\dfrac{\cos(\xi x)}{x^2+w^2}$ is even and

$$\int_{\mathbb{R}} \frac{\sin(\xi x)}{x^2+w^2}dx = 0 = \int_{\mathbb{R}} \frac{\sin(|\xi|x)}{x^2+w^2}dx$$

because the function $\dfrac{\sin(\xi x)}{x^2+w^2}$ is odd. Then

$$\begin{aligned}
\int_{\mathbb{R}} \frac{e^{-i\xi x}}{x^2+w^2}dx &= \int_{\mathbb{R}} \frac{\cos(\xi x)}{x^2+w^2}dx - i\int_{\mathbb{R}} \frac{\sin(\xi x)}{x^2+w^2}dx \\
&= \int_{\mathbb{R}} \frac{\cos(|\xi| x)}{x^2+w^2}dx + i\int_{\mathbb{R}} \frac{\sin(|\xi| x)}{x^2+w^2}dx \\
&= \int_{\mathbb{R}} \frac{e^{i|\xi|x}}{x^2+w^2}dx \\
&= \frac{\pi}{w}e^{-w|\xi|}.
\end{aligned}$$

We have just proven that

$$\left\{\frac{1}{x^2+w^2}\right\}^\wedge(\xi) = \frac{\pi}{w}e^{-w|\xi|}.$$

Then using the relation between Fourier transform and differentiation we finally obtain

$$\begin{aligned}
\widehat{f_w}(\xi) &= \left(\frac{x}{x^2+w^2}\right)^\wedge(\xi) \\
&= i\frac{d}{d\xi}\left(\frac{1}{x^2+w^2}\right)^\wedge(\xi) \\
&= \frac{i\pi}{w}\frac{d}{d\xi}\left(e^{-w|\xi|}\right) \\
&= -i\pi\mathrm{sgn}(\xi)e^{-w|\xi|}.
\end{aligned}$$

**Case 2:** $\mathrm{Re}(w) < 0$.

Applying the previous case to $-w$ and noticing that $f_w = f_{-w}$ we get

$$\widehat{f_w}(\xi) = -i\pi \mathrm{sgn}(\xi) e^{w|\xi|}$$

**Case 3:** $\mathrm{Re}(w) = 0$.

In this case we have

$$f_w(x) = \frac{x}{x^2 - a^2}, \quad a = \mathrm{Im}(w).$$

We are considering the Fourier transform of $f_w$ as

$$\lim_{\delta \to 0} \int_{\delta < |x^2 - a^2| < 1/\delta} \frac{x}{x^2 - a^2} e^{-ix\xi} dx.$$

Note that

$$\frac{x}{x^2 - a^2} = \frac{1}{2}\left[\frac{1}{x-a} + \frac{1}{x+a}\right].$$

Then

$$\begin{aligned}
\widehat{f_w}(\xi) &= \frac{1}{2} \lim_{\delta \to 0} \int_{\delta < |x^2 - a^2| < 1/\delta} \frac{1}{x-a} e^{-ix\xi} dx \\
&\quad + \frac{1}{2} \lim_{\delta \to 0} \int_{\delta < |x^2 - a^2| < 1/\delta} \frac{1}{x+a} e^{-ix\xi} dx \\
&= I + II.
\end{aligned}$$

In the first limit we can replace the region $\{\delta < |x^2 - a^2| < 1/\delta\}$ by the region $\{\delta < |x-a| < 1/\delta\}$ since $\dfrac{1}{x-a}$ is not singular at $x = -a$. So

$$\begin{aligned}
I &= \frac{1}{2} \lim_{\delta \to 0} \int_{\delta < |x-a| < 1/\delta} \frac{1}{x-a} e^{-ix\xi} dx \\
&= \frac{1}{2} \lim_{\delta \to 0} \int_{\delta < |x| < 1/\delta} \frac{e^{-i(x+a)\xi}}{x} dx \\
&= \frac{e^{-ia\xi}}{2} \lim_{\delta \to 0} \int_{\delta < |x| < 1/\delta} \frac{e^{-ix\xi}}{x} dx.
\end{aligned}$$

Similarly, since $\dfrac{1}{x+a}$ is not singular at the point $x = a$ we replace the region $\{\delta < |x^2 - a^2| < 1/\delta\}$ by the region $\{\delta < |x+a| < 1/\delta\}$ in the limit to obtain

$$\begin{aligned}
II &= \frac{1}{2} \lim_{\delta \to 0} \int_{\delta < |x+a| < 1/\delta} \frac{1}{x+a} e^{-ix\xi} dx \\
&= \frac{1}{2} \lim_{\delta \to 0} \int_{\delta < |x| < 1/\delta} \frac{e^{-i(x-a)\xi}}{x} dx \\
&= \frac{e^{ia\xi}}{2} \lim_{\delta \to 0} \int_{\delta < |x| < 1/\delta} \frac{e^{-ix\xi}}{x} dx.
\end{aligned}$$

Using that the function $x \longmapsto \dfrac{\cos(x\xi)}{x}$ is odd, so it has integral zero over any symmetric interval, we obtain

$$\begin{aligned} \int_{\delta<|x|<1/\delta} \frac{e^{-ix\xi}}{x} dx &= \int_{\delta<|x|<1/\delta} \frac{\cos(x\xi) - i\sin(x\xi)}{x} dx \\ &= -i\mathrm{sgn}(\xi) \int_{\delta<|x|<1/\delta} \frac{\sin(x|\xi|)}{x} dx \\ &= -i\mathrm{sgn}(\xi) \int_{\delta|\xi|<|y|<|\xi|/\delta} \frac{\sin(y)}{y} dy \end{aligned} \tag{2.44}$$

Taking the limite $\delta \to 0$ in both sides of (2.44) we have

$$\lim_{\delta\to 0} \int_{\delta<|x|<1/\delta} \frac{e^{-ix\xi}}{x} dx = -i\mathrm{sgn}(\xi) \int_{\mathbb{R}} \frac{\sin(y)}{y} dy \tag{2.45}$$

$$= -i\pi\mathrm{sgn}(\xi). \tag{2.46}$$

Adding $I$ and $II$ we finally obtain

$$\begin{aligned} \widehat{f_w}(\xi) &= -i\pi \frac{e^{-ia\xi} + e^{ia\xi}}{2} \mathrm{sgn}(\xi) \\ &= -i\pi \cos(a\xi)\mathrm{sgn}(\xi). \end{aligned}$$

In all three cases we have $|\widehat{f_w}(\xi)| \leq \pi$, for all $\xi \in \mathbb{R}$. □

**Proof of Proposition 2.2.5 :** Denote by $v$ the following function

$$v(x,t) = \int_0^t U_\epsilon(t-t')F(\cdot,t')(x)dt'.$$

The function $v$ satisfies the inhomogeneous problem

$$\begin{cases} i\partial_t v + \partial_x^2 v &= i\epsilon\partial_x^2 v + F(x,t), \quad x \in \mathbb{R}, \quad t > 0, \\ v(\cdot,0) &= 0. \end{cases} \tag{2.47}$$

CLAIM : It holds the following formula

$$v(x,t) = -\int\int \frac{e^{it\tau} - e^{-it(1-i\epsilon)\xi^2}}{\tau + (1-i\epsilon)\xi^2} e^{ix\xi} \widetilde{F}(\xi,\tau) d\xi d\tau, \tag{2.48}$$

where $\tilde{F}$ denotes the Fourier transform of $F$ with respect to both variables $x$ and $t$. Let us assume this claim is true for a while and go on in our proof. First of all, we differentiate the formula (2.48) with respect to the variable $x$ to obtain

$$\begin{aligned}
\partial_x v(x,t) &= -\int\int \frac{i\xi}{\tau+(1-i\epsilon)\xi^2}(e^{it\tau}-e^{-it(1-i\epsilon)\xi^2})e^{ix\xi}\widetilde{F}(\xi,\tau)d\xi d\tau \\
&= -\int\int \frac{i\xi e^{it\tau}}{\tau+(1-i\epsilon)\xi^2}e^{ix\xi}\widetilde{F}(\xi,\tau)d\xi d\tau \\
&\quad +\int\int \frac{i\xi e^{-it(1-i\epsilon)\xi^2}}{\tau+(1-i\epsilon)\xi^2}e^{ix\xi}\widetilde{F}(\xi,\tau)d\xi d\tau \\
&= \partial_x v_1(x,t)+\partial_x v_2(x,t).
\end{aligned}$$

Write

$$\partial_x v_1(x,t) = -\frac{1}{1-i\epsilon}\int\int \frac{i\xi}{w^2+\xi^2}\widetilde{F}(\xi,\tau)e^{ix\xi}e^{it\tau}d\xi d\tau \tag{2.49}$$

where $w = w_{\epsilon,\tau}$ is a complex number such that $w^2 = \dfrac{\tau}{1-i\epsilon}$. Denoting

$$K_w(x) = \left\{\frac{i\xi}{w^2+\xi^2}\right\}^{\vee}(x)$$

and using the properties of convolution in the integral (2.49) we have

$$\begin{aligned}
\partial_x v_1(x,t) &= -\frac{1}{1-i\epsilon}\int\left(\int \widehat{K_w}(\xi)\widetilde{F}(\xi,\tau)e^{ix\xi}d\xi\right)e^{it\tau}d\tau \\
&= -\frac{1}{1-i\epsilon}\int\left(\int \{K_w * \widehat{F}^{(t)}(\tau)\}^{\wedge}(\xi)e^{ix\xi}d\xi\right)e^{it\tau}d\tau \\
&= -\frac{1}{1-i\epsilon}\int \{K_w * \widehat{F}^{(t)}(\tau)\}(x)e^{it\tau}d\tau. 
\end{aligned}\tag{2.50}$$

Using Plancherel's Theorem with respect to the variable $t$ in the integral (2.50)

$$\|\partial_x v_1(x,\cdot)\|_{L^2_t} = \frac{1}{\sqrt{1+\epsilon^2}}\|\{K_{w_{\tau,\epsilon}} * \widehat{F}^{(t)}(\tau)\}(x)\|_{L^2_\tau}.$$

Then applying Minkowsky's inequality for integrals and finally Lemma 2.2.6 we obtain

$$\begin{aligned}
\|\partial_x v_1(x,\cdot)\|_{L^2_t} &\le \|\int K_{w_{\tau,\epsilon}}(x-y)\widehat{F(y,\cdot)}^{(t)}(\tau)dy\|_{L^2_\tau} \\
&\le \int \|K_{w_{\tau,\epsilon}}(x-y)\widehat{F(y,\cdot)}^{(t)}(\tau)\|_{L^2_\tau}dy \\
&\le \sup_{\tau,\epsilon}\|K_{w_{\tau,\epsilon}}\|_{L^\infty}\int \|\widehat{F(y,\cdot)}^{(t)}\|_{L^2_\tau}dy \\
&= c\int \|F(y,\cdot)\|_{L^2_t}dy \\
&= c\|F\|_{L^1_xL^2_t}.
\end{aligned}$$

From (2.40) we finally conclude

$$\begin{aligned}
\|\partial_x v_1\|_{l^\infty_j(L^2(Q_j))} &\lesssim \|F\|_{L^1_xL^2_t} \\
&\lesssim \|F\|_{l^1_j(L^2(Q_j))}.
\end{aligned}$$

Now we turn our attention to $\partial_x v_2$. First of all note that this term can be written as

$$\begin{aligned}\partial_x v_2(x,t) &= \int |\xi|^{1/2} e^{-it(1-i\epsilon)\xi^2}\left(\int \frac{i\mathrm{sgn}(\xi)|\xi|^{1/2}\widetilde{F}(\xi,\tau)}{\tau+(1-i\epsilon)\xi^2}d\tau\right)e^{ix\xi}d\xi \\ &= D^{1/2}U_\epsilon(t)G(x) \qquad (2.51)\end{aligned}$$

where $G$ is defined via Fourier Transform by

$$\widehat{G}(\xi) = \int \frac{i\mathrm{sgn}(\xi)|\xi|^{1/2}\widetilde{F}(\xi,\tau)}{\tau+(1-i\epsilon)\xi^2}d\tau.$$

Next notice that

$$\left\{\text{p.v.}\frac{1}{\tau+(1-i\epsilon)\xi^2}\right\}^{\vee}(t) = c\ \ \mathrm{sgn}(t)e^{-it(1-i\epsilon)\xi^2}. \qquad (2.52)$$

So using Parseval's identity and (2.52) we have

$$\begin{aligned}\widehat{G}(\xi) &= i\mathrm{sgn}(\xi)|\xi|^{1/2}\int \frac{\widetilde{F}(\xi,\tau)}{\tau+(1-i\epsilon)\xi^2}dt \\ &= i\mathrm{sgn}(\xi)|\xi|^{1/2}\int \widehat{F(\cdot,t)}(\xi)\left\{\text{p.v.}\frac{1}{\tau+(1-i\epsilon)\xi^2}\right\}^{\vee}(t)d\tau \\ &= i\mathrm{sgn}(\xi)|\xi|^{1/2}\int \widehat{F(\cdot,t)}(\xi)\mathrm{sgn}(t)e^{-it(1-i\epsilon)\xi^2}dt \\ &= \left\{\mathcal{H}D^{1/2}\int U_\epsilon(t)F(\cdot,t)\mathrm{sgn}(t)dt\right\}^{\wedge}(\xi) \qquad (2.53)\end{aligned}$$

where $\mathcal{H}$ is the Hilbert Transform. Finally using (2.51), Proposition 2.2.2, Proposition 2.2.4 , identity (2.53) and the fact the Hilbert transform is an isometry in $L^2(\mathbb{R})$ it follows that

$$\begin{aligned}\|\partial_x v_2\|_{l_j^\infty(L^2(Q_j))} &= \|D^{1/2}U_\epsilon(t)G\|_{l_j^\infty(L^2(Q_j))} \\ &\lesssim |G\|_{L^2} \\ &= \|\mathcal{H}D^{1/2}\int U_\epsilon(t)F(\cdot,t)\mathrm{sgn}(t)dt\|_{L^2} \\ &= \|D^{1/2}\int U_\epsilon(t)F(\cdot,t)\mathrm{sgn}(t)dt\|_{L^2} \\ &\leq \|F\|_{l_j^1(L^2(Q_j))}.\end{aligned}$$

To complete the proof it remains to prove the formula (2.48). In fact, from the definition of $U_\epsilon(t)$, it follows that

$$U_\epsilon(t-t')F(\cdot,t')(x) = \{e^{-(i+\epsilon)(t-t')\xi^2}\widehat{F(\cdot,t')}(\xi)\}^{\vee(\xi)}(x). \qquad (2.54)$$

Using the property

$$\{f(\cdot+h)\}^{\vee}(t') = e^{-iht'}\check{f}(t')$$

we have

$$e^{(i+\epsilon)t'\xi^2}\widehat{F(\cdot,t')}(\xi) = \{\widetilde{F}(\xi,\tau' - (1-i\epsilon)\xi^2)\}^{\vee(\tau')}(t'). \tag{2.55}$$

Replacing the relation (2.55) into (2.54) it follows

$$U_\epsilon(t-t')F(\cdot,t')(x) = \int_{\mathbb{R}} e^{-(i+\epsilon)t\xi^2}\left[\int_{\mathbb{R}} \widetilde{F}(\xi,\tau' - (1-i\epsilon)\xi^2)e^{i\tau' t'}d\tau'\right]e^{ix\xi}d\xi. \tag{2.56}$$

Integrating (2.56) over the interval $[0,t]$ and using Fubini's theorem

$$\begin{aligned}\int_0^t U_\epsilon(t-t')F(\cdot,t')(x)dt' &= \int_{\mathbb{R}} e^{-(i+\epsilon)t\xi^2}\left[\int_{\mathbb{R}} \widetilde{F}(\xi,\tau' - (1-i\epsilon)\xi^2)\left(\int_0^t e^{i\tau' t'}d\tau'\right)d\tau'\right]e^{ix\xi}d\xi\\ &= \int_{\mathbb{R}} e^{-(i+\epsilon)t\xi^2}\left[\int_{\mathbb{R}} \widetilde{F}(\xi,\tau' - (1-i\epsilon)\xi^2)\frac{e^{i\tau' t}-1}{i\tau'}d\tau'\right]e^{ix\xi}d\xi \end{aligned} \tag{2.57}$$

Performing the change of variables $\tau = \tau' - (1-i\epsilon)\xi^2$ in the integral (2.57) we obtain

$$\begin{aligned}\int_0^t U_\epsilon(t-t')F(\cdot,t')(x)dt' &= \int_{\mathbb{R}} e^{-(i+\epsilon)t\xi^2}\left[\int_{\mathbb{R}} \widetilde{F}(\xi,\tau)\frac{e^{i(\tau+(1-i\epsilon)\xi^2)t}-1}{\tau+(1-i\epsilon\xi^2)}d\tau\right]e^{ix\xi}d\xi\\ &= \int_{\mathbb{R}^2} \widetilde{F}(\xi,\tau)\frac{e^{it\tau}-e^{-it(1-i\epsilon)\xi^2}}{\tau+(1-i\epsilon\xi^2)}e^{ix\xi}d\xi d\tau.\end{aligned}$$

This finishes the proof. $\square$

Since we will work in weighted Sobolev we need to know how to handle the commutator of multiplication by $x$ and $U_\epsilon(t)$.

**Proposition 2.2.7.** *If $f$ is differentiable then for all $x$, $t \in \mathbb{R}$ we have*

$$xU_\epsilon(t)f(x) = U_\epsilon(t)(xf)(x) - 2(i+\epsilon)tU_\epsilon(t)(\partial_x f)(x).$$

*Proof.* By using properties of the Fourier transform and the definition of $U_\epsilon(t)$ we have

$$\begin{aligned} xU_\epsilon(t)f(x) &= x\{e^{-(\epsilon+i)t\xi^2}\hat{f}\}^\vee(x)\\ &= i\{\partial_\xi(e^{-(\epsilon+i)t\xi^2}\hat{f})\}^\vee(x)\\ &= -2(i+\epsilon)t\{e^{-(\epsilon+i)t\xi^2}i\xi\hat{f}\}^\vee(x) + \{e^{-(\epsilon+i)t\xi^2}\partial_\xi\hat{f}\}^\vee(x)\\ &= -2(i+\epsilon)tU_\epsilon(t)(\partial_x f)(x) + U_\epsilon(t)(xf)(x).\end{aligned}$$

$\square$

# Chapter 3

# Uniform estimate for the solutions $u_\epsilon$

Throughout this chapter we consider the solutions $u_\epsilon$ obtained in the previous chapter. We are going to prove that such solutions are bounded with respect to the norm

$$\begin{aligned}\Omega_T(u) &\equiv \|u\|_{L^\infty_T H^{3/2}_x} + \|xu\|_{L^\infty_T H^{1/2}_x} + \|\partial_x^2 u\|_{l^\infty_j(L^2(Q_j))} + \|\partial_x(xu)\|_{l^\infty_j(L^2(Q_j))} \\ &\equiv \Omega_1(u) + \Omega_2(u) + \Omega_3(u) + \Omega_4(u)\end{aligned}$$

whenever the initial data $u_0$ belongs to the weighted Sobolev space

$$X = H^{3/2}(\mathbb{R}) \cap \{f \in \mathcal{S}'(\mathbb{R}); xf \in H^{1/2}(\mathbb{R})\}$$

and

$$\|u_0\|_X = \|u_0\|_{H^{3/2}} + \|xu_0\|_{H^{1/2}}$$

is sufficiently small. Since the time of existence of the solutions $u_\epsilon$ depends on $\|u_\epsilon\|_{H^{3/2}}$ we will be able to extend each solution $u_\epsilon$ to an interval of time that depends only on the size of the initial data.

## 3.1 Estimate for the norms $\Omega_1$ and $\Omega_3$

We start using Propositions 2.2.2, 2.2.4 and 2.2.5 to obtain

$$\begin{aligned}\Omega_1(u_\epsilon) + \Omega_3(u_\epsilon) &\lesssim \|U_\epsilon(t)u_0\|_{L^\infty_T H^{3/2}_x} + \|D^{1/2}\int_0^t U_\epsilon(t-t')(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon))(t')dt'\|_{L^\infty_T L^2_x} \\ &\quad + \|\partial_x^2 U_\epsilon(t)u_0\|_{l^\infty_j L^2(Q_j)} + \|\partial_x \int_0^t U_\epsilon(t-t')(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon))(t')dt'\|_{l^\infty_j(L^2(Q_j))} \\ &\lesssim \|u_0\|_{H^{3/2}} + \|\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))} \\ &\lesssim \|u_0\|_{H^{3/2}} + \||u_\epsilon|^{1+a}\partial_x^2 u_\epsilon\|_{l^1_j(L^2(Q_j))} + (1+a)\||u_\epsilon|^a(\partial_x u_\epsilon)^2\|_{l^1_j(L^2(Q_j))}.\end{aligned} \tag{3.1}$$

Let us take care of each term in (3.1) separately. First we use Hölder's inequality to obtain

$$\begin{aligned}\||u_\epsilon|^{1+a}\partial_x^2 u_\epsilon\|_{l_j^1(L^2(Q_j))} &\leq \|u_\epsilon\|^{1+a}_{l_j^{1+a}(L^\infty(Q_j))}\|\partial_x^2 u_\epsilon\|_{l_j^\infty(L^2(Q_j))}\\ &\leq \|u_\epsilon\|^{1+a}_{l_j^{1+a}(L^\infty(Q_j))}\Omega_3(u_\epsilon).\end{aligned}$$

In order to estimate the norm $\|u_\epsilon\|_{l_j^{1+a}(L^\infty(Q_j))}$ we are going to introduce the weight $\langle x\rangle = (1+x^2)^{1/2}$ so we can pass from the norm $\|u_\epsilon\|_{l_j^{1+a}(L^\infty(Q_j))}$ to another one easier to handle. In fact, using Hölder's inequality and the fact that $\langle j\rangle \leq 2\langle x\rangle$ for $x\in[j,j+1]$, we have

$$\begin{aligned}\|u_\epsilon\|_{l_j^{1+a}(L^\infty(Q_j))} &\leq \|\langle j\rangle^{1-}u_\epsilon\|_{l_j^\infty(L^\infty(Q_j))}\left\|\frac{1}{\langle j\rangle^{1-}}\right\|_{l_j^{1+a}}\\ &\lesssim \|\langle x\rangle^{1-}u_\epsilon\|_{l_j^\infty(L^\infty(Q_j))}\\ &= \|\langle x\rangle^{1-}u_\epsilon\|_{L_T^\infty L_x^\infty}. \qquad (3.2)\end{aligned}$$

Using Sobolev embedding in (3.2) we obtain

$$\|u_\epsilon\|_{l_j^{1+a}(L^\infty(Q_j))} \lesssim \|J^{1/2+}\big(\langle x\rangle^{1-}u_\epsilon\big)\|_{L_T^\infty L_x^2}.$$

Combining Lemmas 1.3.1 and 1.3.2 it follows that

$$\begin{aligned}\|u_\epsilon\|_{l_j^{1+}(L^\infty(Q_j))} &\lesssim \|J^{1/2+}(\langle x\rangle^{1-}u_\epsilon)\|_{L_T^\infty L_x^2}\\ &\lesssim \|J^{1/2}(\langle x\rangle u_\epsilon)\|^{1-\theta}_{L_T^\infty L_x^2}\|J^{3/2}u_\epsilon\|^{\theta}_{L_T^\infty L_x^2}\\ &\lesssim \Omega(u_\epsilon), \qquad (3.3)\end{aligned}$$

for some $\theta\in[0,1]$. We conclude

$$\||u_\epsilon|^{1+a}\partial_x^2 u_\epsilon\|_{l_j^1(L^2(Q_j))} \lesssim \Omega(u_\epsilon)^{2+a}. \qquad (3.4)$$

Now we turn our attention to $\||u_\epsilon|^a(\partial_x u_\epsilon)^2\|_{l_j^1(L^2(Q_j))}$. First we use Hölder's inequality and then estimate (3.3) to obtain

$$\begin{aligned}\||u_\epsilon|^a(\partial_x u_\epsilon)^2\|_{l_j^1(L^2(Q_j))} &\lesssim \||u_\epsilon|^a\|_{l_j^{\frac{1}{a}+}(L^\infty(Q_j))}\|\partial_x u_\epsilon\|^2_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))}\\ &\lesssim \|u_\epsilon\|^a_{l_j^{1+}(L^\infty(Q_j))}\|\partial_x u_\epsilon\|^2_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))}\\ &\lesssim \Omega(u_\epsilon)^a\|\partial_x u_\epsilon\|^2_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))}. \qquad (3.5)\end{aligned}$$

To examine (3.5) we consider two cases:

**Case 1:** $1/2 < a \leq 1$.

In this case we have $\dfrac{2}{1-a} > 4$ so

$$\|\cdot\|_{L^4(Q_j)} \leq T^+\|\cdot\|_{L^{\frac{2}{1-a}-}(Q_j)},$$

and then

$$\begin{aligned}
\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))} &\leq T^+\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{1-a}-}(L^{\frac{2}{1-a}-}(Q_j))}\\
&= T^+\left(\sum_{j\in\mathbb{Z}}\int_0^T\int_j^{j+1}|\partial_x u_\epsilon(x,t)|^{\frac{2}{1-a}-}dxdt\right)^{\frac{1-a}{2}+}\\
&\leq T^+\|\partial_x u_\epsilon\|_{L_T^{\frac{2}{1-a}-}L_x^{\frac{2}{1-a}-}}.
\end{aligned}$$

Since from Sobolev embedding we have

$$\|\partial_x u_\epsilon\|_{L_x^{\frac{2}{1-a}-}} \lesssim \|u_\epsilon\|_{H_x^{3/2}}$$

we conclude that

$$\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))} \lesssim T^+\Omega_1(u_\epsilon).$$

**Case 2:** $0 < a \leq 1/2$

First let us prove the following

CLAIM: Consider $\rho = \left(\dfrac{1-2a}{4}\right)+$. Then

$$\|D^{1/4}(\langle x\rangle^\rho\partial_x f)\|_{L^2} \lesssim \|J^{1/2}(xf)\|_{L^2} + \|J^{3/2}f\|_{L^2}.$$

In fact, since

$$D^{1/4}(\langle x\rangle^\rho\partial_x f) = \partial_x D^{1/4}(\langle x\rangle^\rho f) - D^{1/4}\left(f\partial_x(\langle x\rangle^\rho)\right)$$

we have

$$\|D^{1/4}(\langle x\rangle^\rho\partial_x f)\|_{L^2} \leq \|D^{5/4}(\langle x\rangle^\rho f)\|_{L^2} + \|D^{1/4}\left(f\partial_x(\langle x\rangle^\rho)\right)\|_{L^2}. \tag{3.6}$$

Since $\partial_x(\langle x\rangle^\rho)$ and $\partial_x^2(\langle x\rangle^\rho)$ are bounded we have

$$\begin{aligned}
\|D^{1/4}\left(f\partial_x(\langle x\rangle^\rho)\right)\|_{L^2} &\leq \|J^1\left(f\partial_x(\langle x\rangle^\rho)\right)\|_{L^2}\\
&\sim \|f\partial_x(\langle x\rangle^\rho)\|_{L^2} + \|\partial_x\left(f\partial_x(\langle x\rangle^\rho)\right)\|_{L^2}\\
&\leq \|f\|_{L^2}\|\partial_x(\langle x\rangle^\rho)\|_{L^\infty} + \|\partial_x f\|_{L^2}\|\partial_x(\langle x\rangle^\rho)\|_{L^\infty} + \|f\|_{L^2}\|\partial_x^2(\langle x\rangle^\rho)\|_{L^\infty}\\
&\lesssim \|f\|_{H^1}. \qquad (3.7)
\end{aligned}$$

Now using Lemma 1.3.1 with $\theta = 1-\rho$ (notice that $0 \leq 1-\rho \leq 1$) we obtain

$$\begin{aligned}
\|D^{5/4}(\langle x\rangle^\rho f)\|_{L^2} &\leq \|J^{1/2+\theta}(\langle x\rangle^{1-\theta})\|_{L^2}\\
&\lesssim \|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2}f\|_{L^2}^\theta. \qquad (3.8)
\end{aligned}$$

Applying Lemma 1.3.2 in (3.8) we get

$$\begin{aligned} \|D^{5/4}(\langle x\rangle^\rho f)\|_{L^2} &\lesssim \left(\|J^{1/2}(xf)\|_{L^2}+\|J^{3/2}f\|_{L^2}\right)^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta} \\ &\lesssim \|J^{1/2}(xf)\|_{L^2}+\|J^{3/2}f\|_{L^2}. \end{aligned} \tag{3.9}$$

Replacing the estimates (3.7) and (3.9) in (3.6) we finish the proof of our claim.

Going back to the proof of the case 2 we consider

$$(p,q)=\left(4,\frac{4}{1-2a}-\right),$$

but such that $q\rho>1$. So using Hölder's inequality we have

$$\begin{aligned} \|\partial_x u_\epsilon\|_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))} &\leq \left\|\frac{1}{\langle j\rangle^\rho}\right\|_{l_j^q}\|\langle j\rangle^\rho\partial_x u_\epsilon\|_{l_j^4(L^4(Q_j))} \\ &\lesssim \|\langle j\rangle^\rho\partial_x u_\epsilon\|_{l_j^4(L^4(Q_j))} \\ &\lesssim \|\langle x\rangle^\rho\partial_x u_\epsilon\|_{L_T^4L_x^4}. \end{aligned} \tag{3.10}$$

Using Sobolev embedding in (3.10)

$$\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))}\lesssim T^{1/4}\|D^{1/4}(\langle x\rangle^\rho\partial_x u_\epsilon)\|_{L_T^\infty L_x^2}. \tag{3.11}$$

Then, using the claim we have just proved we conclude that in the case $0<a\leq 1/2$ we also have

$$\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{1-a}-}(L^4(Q_j))}\lesssim T^+\Omega(u_\epsilon). \tag{3.12}$$

Thus, from (3.5) and the claim we conclude

$$\||u_\epsilon|^a(\partial_x u_\epsilon)^2\|_{l_j^1(L^2(Q_j))}\lesssim T^+\Omega(u_\epsilon)^{2+a}. \tag{3.13}$$

Plugging the estimates (3.4) and (3.13) in (3.1) we finally obtain

$$\Omega_1(u_\epsilon)+\Omega_3(u_\epsilon)\lesssim\|u_0\|_{H^{3/2}}+(1+T^+)\Omega(u_\epsilon)^{2+a}. \tag{3.14}$$

## 3.2 Estimates for the norms $\Omega_2$ and $\Omega_4$

To estimate $\Omega_2(u_\epsilon)$ and $\Omega_4(u_\epsilon)$ we start applying Proposition 2.2.7 to obtain

$$\begin{aligned} xu_\epsilon(x,t) &= U_\epsilon(t)(xu_0)-2(i+\epsilon)tU_\epsilon(t)(\partial_x u_0) \\ &\quad -i\int_0^t U_\epsilon(t-t')(x|u_\epsilon|^{1+a}\partial_x u_\epsilon)(t')dt' \\ &\quad -2(1-i\epsilon)\int_0^t(t-t')U_\epsilon(t-t')\big(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt' \\ &= L+NL_1+NL_2. \end{aligned}$$

Then we have

$$\begin{aligned}\Omega_2(u_\epsilon)+\Omega_4(u_\epsilon) &\leq \|L\|_{L^\infty_T H^{1/2}_x}+\|\partial_x L\|_{l^\infty_j(L^2(Q_j))}\\ &\quad +\|NL_1\|_{L^\infty_T H^{1/2}_x}+\|\partial_x NL_1\|_{l^\infty_j(L^2(Q_j))}\\ &\quad +\|NL_2\|_{L^\infty_T H^{1/2}_x}+\|\partial_x NL_2\|_{l^\infty_j(L^2(Q_j))}.\end{aligned}$$

### 3.2.1 Linear terms

The linear terms can be estimated very easily by using Proposition 2.2.2. Indeed,

$$\begin{aligned}\|L\|_{L^\infty_T H^{1/2}_x}+\|\partial_x L\|_{l^\infty_j(L^2(Q_j))} &= \|U_\epsilon(t)(xu_0)-2(i+\epsilon)tU_\epsilon(t)(\partial_x u_0)\|_{L^\infty_T H^{1/2}_x}\\ &\quad +\|\partial_x U_\epsilon(t)(xu_0)-2(i+\epsilon)t\partial_x U_\epsilon(t)(\partial_x u_0)\|_{l^\infty_j(L^2(Q_j))}\\ &\lesssim \|U_\epsilon(t)(xu_0)\|_{L^\infty_T H^{1/2}_x}+T\|U_\epsilon(t)(\partial_x u_0)\|_{L^\infty_T H^{1/2}_x}\\ &\quad +\|\partial_x U_\epsilon(t)(xu_0)\|_{l^\infty_j(L^2(Q_j))}+T\|\partial_x U_\epsilon(t)(\partial_x u_0)\|_{l^\infty_j(L^2(Q_j))}\\ &\lesssim \|xu_0\|_{H^{1/2}_x}+T\|u_0\|_{H^{3/2}}\\ &\lesssim \|u_0\|_X.\end{aligned}$$

### 3.2.2 Nonlinear terms - Part I

Here we focus on the nonlinear term $NL_1$. In fact, Proposition 2.2.4 implies for each $t\in[0,T]$

$$\begin{aligned}\|NL_1(t)\|_{H^{1/2}_x} &= \|\int_0^t U_\epsilon(t-t')(x|u_\epsilon|^{1+a}\partial_x u_\epsilon)(t')dt'\|_{L^\infty_T H^{1/2}_x}\\ &\leq \|\int_0^t U_\epsilon(t-t')(x|u_\epsilon|^{1+a}\partial_x u_\epsilon)(t')dt'\|_{L^2_x}\\ &\quad +\|D^{1/2}\int_0^t U_\epsilon(t-t')(x|u_\epsilon|^{1+a}\partial_x u_\epsilon)(t')dt'\|_{L^2_x}\\ &\lesssim \int_0^t \|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{L^2_x}dt'+\|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{l^1_j(L^2(Q_j))}\\ &\lesssim T^{1/2}\|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{L^2_T L^2_x}+\|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{l^1_j(L^2(Q_j))}.\end{aligned}$$

Using Hölder's inequality and the estimate for the norm $\|\cdot\|_{l^{1+}_j(L^\infty(Q_j))}$ in (3.3) we obtain

$$\begin{aligned}\|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{l^1_j(L^2(Q_j))} &\leq \||u_\epsilon|^{1+a}\|_{l^1_j(L^\infty(Q_j))}\|x\partial_x u_\epsilon\|_{l^\infty_j(L^2(Q_j))}\\ &\leq \|u_\epsilon\|^{1+a}_{l^{1+a}_j(L^\infty(Q_j))}\|x\partial_x u_\epsilon\|_{l^\infty_j(L^2(Q_j))}\\ &\leq \Omega(u_\epsilon)^{1+a}\|x\partial_x u_\epsilon\|_{l^\infty_j(L^2(Q_j))}.\end{aligned}$$

Since

$$\begin{aligned}
\|x\partial_x u_\epsilon\|_{l_j^\infty(L^2(Q_j))} &= \|\partial_x(xu_\epsilon) - u_\epsilon\|_{l_j^\infty(L^2(Q_j))} \\
&\leq \|\partial_x(xu_\epsilon)\|_{l_j^\infty(L^2(Q_j))} + \|u_\epsilon\|_{l_j^\infty(L^2(Q_j))} \\
&\leq \Omega_4(u_\epsilon) + \|u_\epsilon\|_{L_T^2 L_x^2} \\
&\leq \Omega_4(u_\epsilon) + T^{1/2}\|u_\epsilon\|_{L_T^\infty L_x^2} \\
&\leq \Omega_4(u_\epsilon) + T^{1/2}\Omega_1(u_\epsilon) \\
&\leq (1+T^{1/2})\Omega(u_\epsilon)
\end{aligned}$$

we conclude

$$\|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{l_j^1(L^2(Q_j))} \lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}. \tag{3.15}$$

Since

$$\|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{L_T^2 L_x^2} = \|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{l_j^2(L^2(Q_j))}$$

it follows from a similar argument that,

$$\|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{l_j^2(L^2(Q_j))} \lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}, \tag{3.16}$$

and so

$$\|NL_1\|_{L_T^\infty H_x^{1/2}} \lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}. \tag{3.17}$$

Finally, Proposition 2.2.5 and estimate (3.15) imply

$$\begin{aligned}
\|\partial_x NL_1\|_{l_j^\infty(L^2(Q_j))} &\leq \|\partial_x \int_0^t U_\epsilon(t-t')(x|u_\epsilon|^{1+a}\partial_x u_\epsilon)(t')dt'\|_{l_j^\infty(L^2(Q_j))} \\
&\leq \|x|u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{l_j^1(L^2(Q_j))} \\
&\lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}.
\end{aligned}$$

### 3.2.3 Nonlinear terms - Part II

Now we shall analyse the nonlinear term $NL_2$. First, we bound its $H^{1/2}$ norm as follows.

$$\begin{aligned}
\|NL_2\|_{H_x^{1/2}} &\leq 2(1+\epsilon)\|\int_0^t (t-t')U_\epsilon(t-t')\big(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt'\|_{H_x^{1/2}} \\
&\leq 2(1+\epsilon)\|\int_0^t (t-t')U_\epsilon(t-t')\big(|u_\epsilon|^{1+a}\partial_x u_\epsilon\big)(t')dt'\|_{H_x^{3/2}} \\
&\lesssim 2(1+\epsilon)\|\int_0^t (t-t')U_\epsilon(t-t')\big(|u_\epsilon|^{1+a}\partial_x u_\epsilon\big)(t')dt'\|_{L_x^2} \\
&\quad +2(1+\epsilon)\|D^{1/2}\int_0^t (t-t')U_\epsilon(t-t')\big(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt'\|_{L_x^2}.
\end{aligned}$$

After that we apply Minkowski's inequality and the smoothing effect in Proposition 2.2.4 to obtain

$$\begin{aligned}
\|NL_2(t)\|_{H^{1/2}} &\leq 2(1+\epsilon)t\int_0^t \||u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{L^2_x}dt' \\
&\quad +2(1+\epsilon)t\|D^{1/2}\int_0^t U_\epsilon(t-t')\big(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt'\|_{L^2_x} \\
&\quad +2(1+\epsilon)\|D^{1/2}\int_0^t U_\epsilon(t-t')\big(-t'\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt'\|_{L^2_x} \\
&\lesssim 2(1+\epsilon)T^2\||u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{L^\infty_T L^2_x} \\
&\quad +2(1+\epsilon)T\|\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))}.
\end{aligned}$$

Using the estimates (3.4) and (3.13) we have

$$\begin{aligned}
\|\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))} &\lesssim \||u_\epsilon|^{1+a}\partial_x^2 u_\epsilon\|_{l^1_j(L^2(Q_j))} + \||u_\epsilon|^a(\partial_x u_\epsilon)^2\|_{l^1_j(L^2(Q_j))} \\
&\lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}, \qquad (3.18)
\end{aligned}$$

and using Sobolev embedding

$$\begin{aligned}
\||u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{L^\infty_T L^2_x} &\leq \|u_\epsilon\|^{1+a}_{L^\infty_T L^\infty_x}\|\partial_x u_\epsilon\|_{L^\infty_T L^2_x} \\
&\lesssim \|u_\epsilon\|^{1+a}_{L^\infty_T H^{1/2+}_x}\|u_\epsilon\|_{L^\infty_T H^1_x} \\
&\lesssim \Omega(u_\epsilon)^{2+a}.
\end{aligned}$$

Then we conclude

$$\|NL_2\|_{L^\infty_T H^{1/2}_x} \lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}.$$

Finally, to bound $\partial_x NL_2$ with respect to the norm $l^\infty_j(L^2(Q_j))$ we employ Proposition 2.2.5 to obtain

$$\begin{aligned}
\|\partial_x NL_2\|_{l^\infty_j(L^2(Q_j))} &\leq 2(1+\epsilon)\|\partial_x\int_0^t (t-t')U_\epsilon(t-t')\big(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt'\|_{l^\infty_j(L^2(Q_j))} \\
&\leq 2(1+\epsilon)T\|\partial_x\int_0^t U_\epsilon(t-t')\big(\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt'\|_{l^\infty_j(L^2(Q_j))} \\
&\quad +2(1+\epsilon)\|\partial_x\int_0^t U_\epsilon(t-t')\big(\partial_x(t'|u_\epsilon|^{1+a}\partial_x u_\epsilon)\big)(t')dt'\|_{l^\infty_j(L^2(Q_j))} \\
&\leq 2(1+\epsilon)T\|\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))} \\
&\quad +2(1+\epsilon)\|\partial_x(t'|u_\epsilon|^{1+a}\partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))} \\
&\leq 4(1+\epsilon)T\|\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))}.
\end{aligned}$$

Using (3.18) it follows

$$\|\partial_x NL_2\|_{l^1_j(L^2(Q_j))} \lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}.$$

Therefore we obtain

$$\Omega_2(u_\epsilon) + \Omega_4(u_\epsilon) \lesssim \|u_0\|_X + (1+T^+)\Omega(u_\epsilon)^{2+a}. \tag{3.19}$$

## 3.3 Uniform time of definition and uniform estimate

Gathering the estimates (3.14) and (3.19) we conclude that there exist $\beta > 0$ and $c \geq 1$, constants independent of $\epsilon$ such that

$$\Omega_T(u_\epsilon) \leq c\|u_0\|_X + c(1+T^\beta)\Omega_T(u_\epsilon)^{2+a} \tag{3.20}$$

for all $\epsilon > 0$ whenever we have $u_\epsilon$ solution of (2.3) defined in $[0,T]$. A consequence of that is the following

**Theorem 3.3.1.** *Let $u_0 \in X$, with $\|u_0\|_X < 1/4c$. Considering the constants $c$ and $\beta$ in (3.20) we define*

$$T_* = \left(\frac{1-4c\|u_0\|_X}{4c\|u_0\|_X}\right)^{1/\beta}.$$

*If $u_\epsilon$ is a solution of (2.3) defined in $[0,T]$ with $T < T_*$ we have*

$$\Omega_T(u_\epsilon) \leq \frac{1-\sqrt{1-4c(1+T^\beta)\|u_0\|_X}}{2c}. \tag{3.21}$$

*Proof.* For $0 \leq \tilde{T} \leq T$ consider the polynomial $p(x) = c_{\tilde{T}}x^2 - x + c\|u_0\|_X$, $c_{\tilde{T}} = c(1+\tilde{T}^\beta)$, and note that $p(x)$ has two roots

$$r_0 = \frac{1-\sqrt{1-4c_{\tilde{T}}\|u_0\|_X}}{2c_{\tilde{T}}} \quad \text{and} \quad r_1 = \frac{1+\sqrt{1-4c_{\tilde{T}}\|u_0\|_X}}{2c_{\tilde{T}}}$$

that, in its turn, belong to $(0,1)$. Then since $c_{\tilde{T}}x^{2+a} - x + c\|u_0\|_X \leq p(x)$ in $(0,1)$ and $p(x)$ is negative in $(r_0, r_1)$ we have

$$c_{\tilde{T}}x^{2+a} - x + c\|u_0\|_X < 0$$

for all $x \in (r_0, r_1)$. So, if we have $x \leq c\|u_0\|_X + c_{\tilde{T}}x^{2+a}$ then either $x \leq r_0$ or $x \geq r_1$. On the other hand we have from (3.20) that

$$\Omega_{\tilde{T}}(u_\epsilon) \leq c\|u_0\|_X + c_{\tilde{T}}\Omega_{\tilde{T}}(u_\epsilon)^{2+a}.$$

Thus $\Omega_{\tilde{T}}(u_\epsilon) \in (0, r_0] \cup [r_1, +\infty)$. Since

$$\tilde{T} \in [0,T] \longmapsto \Omega_{\tilde{T}}(u_\epsilon)$$

is continuous it follows that either

$$\Omega_{\tilde{T}}(u_\epsilon) \leq r_0, \quad \text{for all} \quad 0 \leq \tilde{T} \leq T \tag{3.22}$$

or

$$\Omega_{\tilde{T}}(u_\epsilon) \geq r_1, \quad \text{for all} \quad 0 \leq \tilde{T} \leq T \tag{3.23}$$

If the second possibility (3.23) happened we would have

$$\lim_{\tilde{T}\to 0} \Omega_{\tilde{T}}(u_\epsilon) \geq r_1.$$

This is not possible because

$$\lim_{\tilde{T}\to 0} \Omega_{\tilde{T}}(u_\epsilon) = \|u_0\|_X$$

and $\|u_0\|_X \leq r_0$. Therefore

$$\Omega_{\tilde{T}}(u_\epsilon) \leq r_0, \quad \text{for all} \quad 0 \leq \tilde{T} \leq T.$$

$\square$

We finish this chapter by proving that all the solution $u_\epsilon$ can be defined in the same interval of time.

**Theorem 3.3.2.** *Let $u_0 \in X$, with $\|u_0\|_X \leq 1/4c$. Consider $T_*$ as in Theorem 3.3.1. Then all the solutions $u_\epsilon$ of (2.3) can be extended to the interval of time $[0, T_*]$.*

*Proof.* We know that there is a solution $u_\epsilon$ defined in $[0, T_\epsilon]$ where

$$T_\epsilon = \frac{\epsilon^3}{c\|u_0\|_{H^{3/2}}^{1+a}}.$$

We assume $T_\epsilon < T_*$, otherwise we do not have anything to prove. Since $\|u_0\|_X < 1/4c$ we have in particular that $u_\epsilon$ is defined in $[0, \epsilon^3]$. Consider the problem (2.3) with the initial condition $v_0 = u_\epsilon(\cdot, \epsilon^3)$ instead of $u_0$. Theorem 2.1.2 assures that there exists a solution $v_\epsilon$ defined in $[0, \tilde{T}_\epsilon]$ where

$$\tilde{T}_\epsilon = \frac{\epsilon^3}{c\|v_0\|_{H^{3/2}}^{1+a}}.$$

By Theorem 3.3.1 we have

$$\|v_0\|_{H^{3/2}} \leq \Omega(u_\epsilon) \leq \frac{1}{2c}.$$

Thus $v_\epsilon$ is defined in $[0, \epsilon^3]$. Consider then $u_\epsilon^1$ the function defined in $[0, 2\epsilon^3]$ by

$$u_\epsilon^1(\cdot, t) = \begin{cases} u_\epsilon(\cdot, t), & \text{if} \quad t \in [0, \epsilon^3] \\ v_\epsilon(\cdot, t - \epsilon^3), & \text{if} \quad t \in [\epsilon^3, 2\epsilon^3] \end{cases}$$

We have $u_\epsilon^1$ is a solution of (2.3) defined in $[0, 2\epsilon^3]$. Assume $2\epsilon^3 < T_*$, otherwise there is nothing to prove. Consider the problem (2.3) with initial data $u_\epsilon^1(2\epsilon^3)$ instead of $u_0$. By theorem 2.1.2 we have that there exists a solution $v_\epsilon$ of this initial value problem defined in $[0, \widetilde{T}_\epsilon^1]$, where

$$\widetilde{T}_\epsilon^1 = \frac{\epsilon^3}{c\|u_\epsilon^1(2\epsilon^3)\|_{H_x^{3/2}}^{1+a}}.$$

From Theorem 3.3.1 we have $\|u_\epsilon^1(2\epsilon^3)\|_{H_x^{3/2}} \leq 1/2c$, then $\widetilde{T}_\epsilon^1 > \epsilon^3$. Define

$$u_\epsilon^2(\cdot, t) = \begin{cases} u_\epsilon^1(\cdot, t), & \text{if} \quad t \in [0, 2\epsilon^3], \\ v_\epsilon^1(\cdot, t - \epsilon^3), & \text{if} \quad t \in [2\epsilon^3, 3\epsilon^3]. \end{cases}$$

We have $u_\epsilon^2$ is a solution of (2.3) defined in $[0, 3\epsilon^3]$. We can repeat this argument to obtain extensions $u_\epsilon^1$, $u_\epsilon^2$, ... , $u_\epsilon^{k-1}$ of $u_\epsilon$ to the intervals $[0, 2\epsilon^3]$, $[0, 3\epsilon^3]$, ... , $[0, k\epsilon^3]$ as long as we have $k\epsilon^3 < T_*$. So this argument finishes for some $k_*$ and then $u_\epsilon^{k_*}$ will be an extension of $u_\epsilon$ to $[0, T_*]$. $\square$

# Chapter 4

# Sending $\epsilon$ to zero

In the previous chapter we proved that all solutions $u_\epsilon$ of (2.1) can be defined in the same interval of time $[0,T]$. Here we intend to let $\epsilon$ go to zero, and then to prove the limit is in fact the solution of the integral equation (8).

## 4.1 Convergence in $L^2$

We will first prove that the sequence $\{u_\epsilon\}$ converges in $L^2$. To do so it will be important the following lemma

**Lemma 4.1.1.** *Let $u_0 \in X$ satisfying the smallness assumption of Theorem 3.3.1. Let $u_\epsilon$ be the corresponding sequence of solutions of (2.1). Then,*

$$\sup_{\epsilon>0} \|\partial_x u_\epsilon\|_{L^4_T L^\infty_x} < \infty$$

*Proof.* First we differentiate the integral equation (2.3) to obtain

$$\begin{aligned}
\partial_x u_\epsilon(t) &= U_\epsilon(t)\partial_x u_0 - i\int_0^t U_\epsilon(t-t')\partial_x(|u_\epsilon|^{1+a}\partial_x u_\epsilon)(t')dt' \\
&= U_\epsilon(t)\partial_x u_0 - i(1+a)\int_0^t U_\epsilon(t-t')(|u_\epsilon|^{a-1}\mathrm{Re}(\bar{u}_\epsilon\partial_x u_\epsilon)\partial_x u_\epsilon)(t')dt' \\
&\quad -i\int_0^t U_\epsilon(t-t')(|u_\epsilon|^{a+1}\partial_x^2 u_\epsilon)(t')dt'.
\end{aligned}$$

By applying Lemma 2.2.1 to the pair $(4,\infty)$ we obtain

$$
\begin{aligned}
\|\partial_x u_\epsilon\|_{L^4_T L^\infty_x} &\lesssim \|\partial_x u_0\|_{L^2} + (1+a)\int_0^T \||u_\epsilon|^a(\partial_x u_\epsilon)^2\|_{L^2_x} dt' \\
&\quad + \int_0^T \||u_\epsilon|^{1+a}\partial_x^2 u_\epsilon\|_{L^2_x} dt' \\
&\lesssim \|\partial_x u_0\|_{L^2} + (1+a)T\||u_\epsilon|^a(\partial_x u_\epsilon)^2\|_{L^1_T L^2_x} + T^{1/2}\||u_\epsilon|^{a+1}\partial_x^2 u_\epsilon\|_{L^2_T L^2_x} \\
&\lesssim \|\partial_x u_0\|_{L^2} + (1+a)T\|u_\epsilon\|^a_{L^\infty_T L^\infty_x}\|\partial_x u_\epsilon\|^2_{L^\infty_T L^4_x} + T^{1/2}\||u_\epsilon|^{a+1}\partial_x^2 u_\epsilon\|_{L^2_x L^2_T} \\
&\lesssim \|\partial_x u_0\|_{L^2} + (1+a)T\|u_\epsilon\|^{a+2}_{L^\infty_T H^{3/2}_x} + T^{1/2}\||u_\epsilon|^{a+1}\partial_x^2 u_\epsilon\|_{L^2_T L^2_x}.
\end{aligned} \tag{4.1}
$$

Using the computation (3.3) we have

$$
\begin{aligned}
\||u_\epsilon|^{a+1}\partial_x^2 u_\epsilon\|_{L^2_T L^2_x} &= \||u_\epsilon|^{a+1}\partial_x^2 u_\epsilon\|_{l^2_j(L^2(Q_j))} \\
&\le \|u_\epsilon\|^{1+a}_{l_j^{2+2a}(L^\infty(Q_j))}\|\partial_x^2 u_\epsilon\|_{l_j^\infty(L^2(Q_j))} \\
&= \|u_\epsilon\|^{1+a}_{l_j^{1+}(L^\infty(Q_j))}\Omega_3(u_\epsilon) \\
&\lesssim \Omega(u_\epsilon)^{2+a}.
\end{aligned} \tag{4.2}
$$

Then substituting (4.2) in (4.1) we conclude

$$
\|\partial_x u_\epsilon\|_{L^4_T L^\infty_x} \lesssim \|u_0\|_{H^1} + T^+\Omega(u_\epsilon)^{2+a}
$$

□

In what follows we fix $u_\epsilon$, $u_{\epsilon'}$ and we consider $w = w_{\epsilon,\epsilon'} = u_\epsilon - u_{\epsilon'}$. Taking the difference between the equations

$$
i\partial_t u_\epsilon + (1-i\epsilon)\partial_x^2 u_\epsilon + i|u_\epsilon|^{1+a}\partial_x u_\epsilon = 0 \tag{4.3}
$$

and

$$
i\partial_t u_{\epsilon'} + (1-i\epsilon')\partial_x^2 u_{\epsilon'} + i|u_{\epsilon'}|^{1+a}\partial_x u_{\epsilon'} = 0. \tag{4.4}
$$

we get that $w$ satisfies

$$
i\partial_t w + (1-i\epsilon)\partial_x^2 w + i(\epsilon'-\epsilon)\partial_x^2 u_{\epsilon'} + i(|u_\epsilon|^{1+a} - |u_{\epsilon'}|^{1+a})\partial_x u_\epsilon + i|u_{\epsilon'}|^{1+a}\partial_x^2 w = 0. \tag{4.5}
$$

Considering the conjugate of (4.5) we have that $\bar{w}$ satisfies

$$
-i\partial_t \bar{w} + (1+i\epsilon)\partial_x^2 \bar{w} - i(\epsilon'-\epsilon)\partial_x^2 \bar{u}_{\epsilon'} - i(|u_\epsilon|^{1+a} - |u_{\epsilon'}|^{1+a})\partial_x \bar{u}_\epsilon - i|u_{\epsilon'}|^{1+a}\partial_x^2 \bar{w} = 0. \tag{4.6}
$$

Multiplying (4.5) by $\bar{w}$ and (4.6) by $w$ and integrating their difference we obtain

$$\begin{aligned}
& i\frac{d}{dt}\int |w|^2dx - (1-i\epsilon)\int |\partial_x w|^2dx - (1+i\epsilon)\int |\partial_x w|^2dx \\
= \;& 2i(\epsilon-\epsilon')\mathrm{Re}\int \bar{w}\partial_x^2 u_{\epsilon'}dx \\
& +2i\mathrm{Re}\int (|u_\epsilon|^{1+a}-|u_{\epsilon'}|^{1+a})\bar{w}\partial_x u_\epsilon dx \\
& -i\int |u_{\epsilon'}|^{1+a}\bar{w}\partial_x w dx - i\int |u_{\epsilon'}|^{1+a} w\partial_x \bar{w}dx. \qquad (4.7)
\end{aligned}$$

Using integration by parts in (4.7) and the fact the functions involved vanish at infinity we have

$$\begin{aligned}
& -i\int |u_{\epsilon'}|^{1+a}\bar{w}\partial_x w dx - i\int |u_{\epsilon'}|^{1+a} w\partial_x \bar{w}dx \\
= \;& -i\int |u_{\epsilon'}|^{1+a}\bar{w}\partial_x w dx + i\int \partial_x(|u_{\epsilon'}|^{1+a}w)\bar{w}dx \\
= \;& -i\int |u_{\epsilon'}|^{1+a}\bar{w}\partial_x w dx + i\int \partial_x(|u_{\epsilon'}|^{1+a})|w|^2dx + i\int |u_{\epsilon'}|^{1+a}\bar{w}\partial_x w dx \\
= \;& i\int \partial_x(|u_{\epsilon'}|^{1+a})|w|^2dx.
\end{aligned}$$

Then

$$\begin{aligned}
i\frac{d}{dt}\|w(\cdot,t)\|_{L^2_x}^2 + 2i\epsilon\|\partial_x w(\cdot,t)\|_{L^2_x}^2 &= 2i(\epsilon-\epsilon')\mathrm{Re}\int \bar{w}\partial_x^2 u_{\epsilon'}dx \\
&\quad +2i\mathrm{Re}\int (|u_{\epsilon'}|^{1+a}-|u_\epsilon|^{1+a})\bar{w}\partial_x u_\epsilon dx \\
&\quad +i\int \partial_x(|u_{\epsilon'}|^{1+a})|w|^2dx \\
&= I + II + III.
\end{aligned}$$

So

$$\frac{d}{dt}\|w(\cdot,t)\|_{L^2_x}^2 \le |I| + |II| + |III|. \qquad (4.8)$$

Next we are going to estimate each of the terms $I$, $II$ and $III$. First we take care of $I$. In fact, using integration by parts and Hölder's inequality

$$\begin{aligned}
\left|\mathrm{Re}\int \bar{w}\partial_x^2 u_{\epsilon'}(x,t)dx\right| &= \left|\mathrm{Re}\int \partial_x\bar{w}\partial_x u_{\epsilon'}(x,t)dx\right| \\
&\le \|\partial_x\bar{w}\|_{L^2_x}\|\partial_x u_{\epsilon'}\|_{L^2_x} \\
&\le (\|u_\epsilon\|_{H^1_x} + \|u_{\epsilon'}\|_{H^1_x})\|u_{\epsilon'}\|_{H^1_x} \\
&\le 2\sup_{\varepsilon>0}\Omega(u_\varepsilon)^2.
\end{aligned}$$

Then

$$|I| \leq c|\epsilon - \epsilon'| \tag{4.9}$$

where $c$ is a constant independent of $\epsilon$ and $\epsilon'$. Using (2.15) we have

$$\begin{aligned}
&\left|\mathrm{Re}\int(|u_{\epsilon'}|^{1+a} - |u_\epsilon|^{1+a})\bar{w}\partial_x u_\epsilon(x,t)dx\right| \\
\lesssim\ & \int(|u_{\epsilon'}|^a + |u_\epsilon|^a)|u_\epsilon - u_{\epsilon'}||w||\partial_x u_\epsilon|dx \\
\lesssim\ & (\|u_{\epsilon'}\|^a_{L^\infty_x} + \|u_\epsilon\|^a_{L^\infty_x})\|\partial_x u_\epsilon\|_{L^\infty_x}\|w\|^2_{L^2_x} \\
\lesssim\ & \sup_{\varepsilon>0}\Omega(u_\varepsilon)^a\|\partial_x u_\epsilon\|_{L^\infty_x}\|w\|^2_{L^2_x}.
\end{aligned} \tag{4.10}$$

Also,

$$\begin{aligned}
\left|\int\partial_x(|u_{\epsilon'}|^{1+a})|w|^2dx\right| &\leq \|\partial_x(|u_{\epsilon'}|^{1+a})\|_{L^\infty_x}\|w\|^2_{L^2_x} \\
&= \|(1+a)|u_{\epsilon'}|^{a-1}\mathrm{Re}(\bar{u}_{\epsilon'}\partial_x u_{\epsilon'})\|_{L^\infty_x}\|w\|^2_{L^2_x} \\
&\leq (1+a)\||u_{\epsilon'}|^a\partial_x u_{\epsilon'}\|_{L^\infty_x}\|w\|^2_{L^2_x} \\
&\leq (1+a)\|u_{\epsilon'}\|^a_{L^\infty_x}\|\partial_x u_{\epsilon'}\|_{L^\infty_x}\|w\|^2_{L^2_x} \\
&\lesssim \sup_{\varepsilon>0}\Omega(u_\varepsilon)^a\|\partial_x u_{\epsilon'}\|_{L^\infty_x}\|w\|^2_{L^2_x}.
\end{aligned} \tag{4.11}$$

Therefore

$$|II| + |III| \lesssim (\|\partial_x u_\epsilon\|_{L^\infty_x} + \|\partial_x u_{\epsilon'}\|_{L^\infty_x})\|w\|^2_{L^2_x}. \tag{4.12}$$

Gathering the estimates (4.9) and (4.12) in (4.8) we obtain

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2_x} \lesssim |\epsilon - \epsilon'| + \|\partial_x u_{\epsilon'}(\cdot,t)\|_{L^\infty_x}\|w(\cdot,t)\|^2_{L^2_x}. \tag{4.13}$$

Finally, applying Lemma 1.1.1 we obtain from (4.13)

$$\|w(\cdot,t)\|^2_{L^2_x} \leq \exp\left(c\int_0^t\|\partial_x u_{\epsilon'}(\cdot,s)\|_{L^\infty_x}ds\right)\int_0^t c|\epsilon - \epsilon'|ds.$$

So, by Hölder inequality

$$\|w(\cdot,t)\|^2_{L^2_x} \lesssim 2|\epsilon - \epsilon'|T\exp\left(cT^{3/4}\|\partial_x u_{\epsilon'}\|_{L^4_T L^\infty_x}\right).$$

Since the quantity $\|\partial_x u_{\epsilon'}\|_{L^4_T L^\infty_x}$ is uniformly bounded in $\epsilon'$ we conclude that

$$\lim_{\epsilon,\epsilon'\to 0}\|u_\epsilon - u_{\epsilon'}\|_{L^\infty_T L^2_x} = 0$$

that is, $\{u_\epsilon\}_{\epsilon>0}$ is a Cauchy sequence in $L^\infty_T L^2_x$ and then there exists $u \in L^\infty_T L^2_x$ such that

$$\lim_{\epsilon\to 0}\|u_\epsilon - u\|_{L^\infty_T L^2_x} = 0.$$

## 4.2 Existence of solution

We have constructed a family $\{u_\epsilon\}_{\epsilon>0}$ where each $u_\epsilon$ is the unique solution of

$$\begin{cases} i\partial_t u + \partial_x^2 u + i|u|^{1+a}\partial_x u &= i\epsilon\partial_x^2 u \\ u(\cdot,0) &= u_0 \end{cases}$$

in $C([0,T];H^{3/2}(\mathbb{R}))$. There the solutions also satisfy

i ) $\{u_\epsilon\}_{\epsilon>0}$ is bounded in $L_T^\infty H_x^{3/2}$;

ii) There exists a function $u \in C([0,T];L^2(\mathbb{R}))$ such that $u_\epsilon$ converges to $u$ in $L_T^\infty L_x^2$ as $\epsilon$ goes to zero. Furthermore, using property (i) and the interpolation

$$\|\cdot\|_{H^s} \leq \|\cdot\|_{L^2}^{1-\theta}\|\cdot\|_{H^{3/2}}^{\theta}, \quad \theta = \frac{3/2-s}{3/2}$$

we conclude

iii) For all $0 \leq s < 3/2$, $u \in C([0,T];H^s(\mathbb{R}))$ and $u_\epsilon$ converges to $u$ in $L_T^\infty H_x^s$ as $\epsilon$ goes to zero.

We are going to use these properties to show that $u$ is a solution of the integral equation (8). First we look at the linear part. Using Plancherel's theorem we have

$$\begin{aligned} \|U_\epsilon(t)u_0 - U(t)u_0\|_{L_x^2}^2 &= \int |e^{-(i+\epsilon)t\xi^2} - e^{-it\xi^2}|^2|\hat{u}_0(\xi)|^2 d\xi \\ &\leq \int |e^{-\epsilon t\xi^2} - 1|^2|\hat{u}_0(\xi)|^2 d\xi. \end{aligned}$$

Now we use the Dominated Convergence Theorem to conclude that the last integral goes to zero as $\epsilon$ goes to zero. Then for each $t \in [0,T]$ we have

$$\lim_{\epsilon\to 0}\|U_\epsilon(t)u_0 - U(t)u_0\|_{L_x^2} = 0. \tag{4.14}$$

To investigate the convergence of the nonlinear part denote

$$v(x,t) = -\int_0^t U(t-t')F(t')dt'$$

where $F(x,t) = |u|^{1+a}\partial_x u$ and similarly

$$v_\epsilon(x,t) = -\int_0^t U_\epsilon(t-t')F_\epsilon(t')dt'$$

where $F_\epsilon(x,t) = |u_\epsilon|^{1+a}\partial_x u_\epsilon$. We have

$$\begin{aligned}
\|v - v_\epsilon\|_{L^2_x} &\leq \|\int_0^t U(t-t')(F-F_\epsilon)(t')dt'\|_{L^2_x} \\
&\quad +\|\int_0^t \left[U(t-t') - U_\epsilon(t-t')\right](F_\epsilon)(t')dt'\|_{L^2_x} \\
&\leq \int_0^T \|F(t') - F_\epsilon(t')\|_{L^2_x} dt' \\
&\quad + \int_0^t \|\left(e^{-i(t-t')\xi^2} - e^{-(i+\epsilon)(t-t')\xi^2}\right)\widehat{F_\epsilon(t')}\|_{L^2_\xi} dt' \\
&\leq T\|F - F_\epsilon\|_{L^\infty_T L^2_x} \\
&\quad + \int_0^t \|\left(1 - e^{-\epsilon(t-t')\xi^2}\right)(\widehat{F_\epsilon(t')} - \widehat{F(t')})\|_{L^2_\xi} dt' \\
&\quad + \int_0^t \|\left(1 - e^{-\epsilon(t-t')\xi^2}\right)\widehat{F(t')}\|_{L^2_\xi} dt' \\
&\leq 2T\|F - F_\epsilon\|_{L^\infty_T L^2_x} + \int_0^t \|(1 - e^{-\epsilon(t-t')\xi^2})\widehat{F(t')}\|_{L^2_\xi} dt'. \qquad (4.15)
\end{aligned}$$

Since $|(1 - e^{-\epsilon(t-t')\xi^2})\widehat{F(t')}|$ is bounded by $2|F(t')|$ and

$$\begin{aligned}
\|\widehat{F(t')}\|_{L^1_T L^2_\xi} &= \|\widehat{F(t')}\|_{L^1_T L^2_x} \\
&= \||u|^{1+a}\partial_x u\|_{L^1_T L^2_x} \\
&\leq T\|u\|^{1+a}_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} \\
&\leq T\|u\|^{a+2}_{L^\infty_T H^1_x}
\end{aligned}$$

we can apply the Dominated Convergence Theorem to conclude

$$\lim_{\epsilon\to 0}\int_0^t \|(1 - e^{-\epsilon(t-t')\xi^2})\widehat{F(t')}\|_{L^2_\xi} dt' = 0.$$

Finally, we take care of $\|F - F_\epsilon\|_{L^\infty_T L^2_x}$. Adding and subtracting $|u_\epsilon|^{1+a}\partial_x u$ and using triangle inequality

$$\begin{aligned}
\|F - F_\epsilon\|_{L^\infty_T L^2_x} &= \||u|^{1+a}\partial_x u - |u_\epsilon|^{1+a}\partial_x u_\epsilon\|_{L^\infty_T L^2_x} \\
&\leq \|\left(|u|^{1+a} - |u_\epsilon|^{1+a}\right)\partial_x u\|_{L^\infty_T L^2_x} \\
&\quad +\||u_\epsilon|^{1+a}\partial_x(u - u_\epsilon)\|_{L^\infty_T L^2_x} \\
&\leq \||u|^{1+a} - |u_\epsilon|^{1+a}\|_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} \\
&\quad +\||u_\epsilon|^{1+a}\|_{L^\infty_T L^\infty_x}\|\partial_x(u - u_\epsilon)\|_{L^\infty_T L^2_x}.
\end{aligned}$$

Using (2.15) and the Sobolev embedding, we have

$$\begin{aligned}
\|F - F_\epsilon\|_{L^\infty_T L^2_x} &\leq \|(|u|^a + |u_\epsilon|^a)|u - u_\epsilon|\|_{L^\infty_T L^\infty_x} \|\partial_x u\|_{L^\infty_T L^2_x} \\
&\quad + \|u_\epsilon\|^{1+a}_{L^\infty_T L^\infty_x} \|\partial_x (u - u_\epsilon)\|_{L^\infty_T L^2_x} \\
&\leq (\|u\|^a_{L^\infty_T H^{1/2+}_x} + \|u_\epsilon\|^a_{L^\infty_T H^{1/2+}_x}) \|\partial_x u_\epsilon\|_{L^\infty_T L^2_x} \|u - u_\epsilon\|_{L^\infty_T H^{1/2+}_x} \\
&+ \|u_\epsilon\|^{1+a}_{L^\infty_T H^{1/2+}_x} \|\partial_x (u_\epsilon - u)\|_{L^\infty_T L^2_x}.
\end{aligned}$$

As $u_\epsilon$ is bounded in $L^\infty_T H^{3/2}_x$ we conclude

$$\|F - F_\epsilon\|_{L^\infty_T L^2_x} \lesssim \|u - u_\epsilon\|_{L^\infty_T H^1_x}.$$

Then we conclude

$$\lim_{\epsilon \to 0} \|v_\epsilon(t) - v(t)\|_{L^2_x} = 0 \tag{4.16}$$

for each $t \in [0, T]$. It follows from (4.14) and (4.16) that for every $t \in [0, T]$

$$u(t) = \lim_{\epsilon \to 0} u_\epsilon(t) = U(t)u_0 - \int_0^t U(t - t')(|u|^{1+a} \partial_x u)(t') dt'$$

in $L^2(\mathbb{R})$.

Now we prove that $u \in L^\infty([0, T]; X) \cap C([0, T]; H^s(\mathbb{R}))$ for all $s < 3/2$. To do so we first note that since $\{u_\epsilon(t)\}_{\epsilon > 0}$ is bounded in $H^{3/2}(\mathbb{R})$ and $\{x u_\epsilon(t)\}_{\epsilon > 0}$ is bounded in $H^{1/2}(\mathbb{R})$ there exist a subsequence $\{u_{\epsilon_j}(t)\}_{j=1}^\infty$, and functions $v(t) \in H^{3/2}(\mathbb{R})$ and $w(t) \in H^{1/2}(\mathbb{R})$ such that

$$u_{\epsilon_j}(t) \rightharpoonup v(t), \quad \text{in} \quad H^{3/2}(\mathbb{R}) \tag{4.17}$$

and

$$x u_{\epsilon_j}(t) \rightharpoonup w(t), \quad \text{in} \quad H^{1/2}(\mathbb{R}). \tag{4.18}$$

We are going to argue that $u(t) = v(t)$ and $xu(t) = w(t)$ and then conclude $u(t) \in X$. The convergence (4.17) means

$$\lim_{j \to \infty} \int \widehat{u_{\epsilon_j}(t)}(\xi) \bar{\hat{\varphi}}(\xi) (1 + \xi^2)^3 d\xi = \int \widehat{v(t)}(\xi) \bar{\hat{\varphi}}(\xi) (1 + \xi^2)^3 d\xi$$

for all $\varphi \in H^{3/2}(\mathbb{R})$. So, considering

$$\varphi = \frac{\phi}{(1 + \xi^2)^3},$$

with $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$\lim_{j \to \infty} \int \widehat{u_{\epsilon_j}(t)}(\xi) \bar{\hat{\phi}}(\xi) d\xi = \int \widehat{v(t)}(\xi) \bar{\hat{\phi}}(\xi) d\xi$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. After using Parserval Identity we obtain

$$\lim_{j\to\infty}\int u_{\epsilon_j}(x,t)\phi(x)dx = \int v(x,t)\phi(x)dx \tag{4.19}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. On the other hand, recalling that $u_{\epsilon_j}(t) \longrightarrow u(t)$ in $L^2(\mathbb{R})$ we also have

$$\lim_{j\to\infty}\int u_{\epsilon_j}(x,t)\phi(x)dx = \int u(x,t)\phi(x)dx \tag{4.20}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Then from (4.19) and (4.20) we have

$$\int v(x,t)\phi(x)dx = \int u(x,t)\phi(x)dx$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Then we conclude $v(x,t) = u(x,t)$ for almost every $x \in \mathbb{R}$. Now we are going stablish the second identity $xu(t) = w(t)$. Using the same argument we conclude from the convergence (4.18) that

$$\lim_{j\to\infty}\int xu_{\epsilon_j}(x,t)\phi(x)dx = \int w(x,t)\phi(x)dx \tag{4.21}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Again, using that $\{u_{\epsilon_j}(t)\}_{j=1}^{\infty}$ converges to $u(t)$ in $L^2(\mathbb{R})$ we obtain

$$\lim_{j\to\infty}\int xu_{\epsilon_j}(x,t)\phi(x)dx = \int xu(x,t)\phi(x)dx. \tag{4.22}$$

Then from (4.21) and (4.22) we obtain

$$\int w(x,t)\phi(x)dx = \int xu(x,t)\phi(x)dx$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Then $w(x,t) = xu(x,t)$ for almost every $x \in \mathbb{R}$.

We summarize what we have proved above in the following theorem

**Theorem 4.2.1.** (Existence) *Let $u_0 \in X$, such that $\|u_0\|_X$ is sufficiently small. Then there exist $T = T(\|u_0\|_X) > 0$ and $u : [0,T] \to X$, $u \in L^\infty([0,T];X) \cap C([0,T];H^{3/2-}(\mathbb{R}))$, solution of*

$$u(t) = U(t)u_0 - \int_0^t U(t-t')(|u|^{1+a}\partial_x u)(t')dt'.$$

## 4.3 Uniqueness

Here we are going to prove that the function $u$ is the unique solution of the integral equation (8) in the class $L^\infty([0,T];X) \cap C([0,T];H^{3/2-}(\mathbb{R}))$.

**Lemma 4.3.1.** *If $v \in L^\infty([0,T];X) \cap C([0,T];H^{3/2-}(\mathbb{R}))$ is a solution of the integral equation (8) then $\partial_x v \in L^\infty_x L^2_T$.*

*Proof.* In fact, using Lemma 1.2.2 and Lemma 1.2.4 we have

$$\begin{aligned} \|\partial_x v\|_{L^\infty_x L^2_T} &\lesssim \|D^{1/2}u_0\|_{L^2} + \||v|^{1+a}\partial_x v\|_{L^1_x L^2_T} \\ &\lesssim \|u_0\|_{H^{3/2}} + \||v|^{1+a}\|_{L^2_x L^\infty_T}\|\partial_x v\|_{L^2_x L^2_T} \\ &\lesssim \|u_0\|_{H^{3/2}} + T^{1/2}\||v|^{1+a}\|_{L^2_x L^\infty_T}\|\partial_x v\|_{L^\infty_T L^2_x}. \end{aligned} \tag{4.23}$$

Using Hölder's inequality and Sobolev embedding we have

$$\begin{aligned} \|v\|_{L^{2+2a}_x L^\infty_T} &\leq \|\langle\cdot\rangle^{1-}v\|_{L^\infty_x L^\infty_T}\left\|\frac{1}{\langle\cdot\rangle^{1-}}\right\|_{L^{2+2a}} \\ &\lesssim \|J^{1/2+}(\langle\cdot\rangle^{1-}v)\|_{L^\infty_T L^2_x}. \end{aligned} \tag{4.24}$$

Using Lemma 1.3.1 we have

$$\|J^{1/2+}(\langle\cdot\rangle^{1-}v)\|_{L^\infty_T L^2_x} \leq \|v\|_{L^\infty([0,T];X)}.$$

Then we obtain $\partial_x v \in L^\infty_x L^2_T$ and

$$\|\partial_x v\|_{L^\infty_x L^2_T} \lesssim \|u_0\|_{H^{3/2}} + T^{1/2}\|v\|^{2+a}_{L^\infty([0,T];X)}.$$

□

Let $\tilde{u}$ be another solution of (8) in $L^\infty([0,T];X) \cap C([0,T];H^{3/2-}(\mathbb{R}))$. We will prove that $\tilde{u}(t) = u(t)$ for all $t$ sufficiently small. To help us on this task we consider the norm

$$|||v|||_T = \|v\|_{L^\infty_T H^{1/2}_x} + \|\partial_x v\|_{L^\infty_x L^2_T}$$

which allows us to compute the difference $u - \tilde{u}$ without differentiate the nonlinearity. Indeed, using the smoothing effects we have

$$\begin{aligned} |||u-\tilde{u}|||_T &\lesssim \|\int_0^t U(t-t')(|u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u})(t')dt'\|_{L^\infty_T L^2_x} \\ &\quad + \|D^{1/2}\int_0^t U(t-t')(|u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u})(t')dt'\|_{L^\infty_T L^2_x} \\ &\quad + \|\partial_x\int_0^t U(t-t')(|u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u})(t')dt'\|_{L^\infty_x L^2_T} \\ &\lesssim \||u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L^1_T L^2_x} + \||u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L^1_x L^2_T}. \end{aligned} \tag{4.25}$$

Note that for (2.15)

$$\begin{aligned} ||u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x \tilde{u}| &= ||u|^{1+a} - |\tilde{u}|^{1+a}||\partial_x \tilde{u}| + ||u|^{1+a}\partial_x(\tilde{u}-u)| \\ &\lesssim (|\tilde{u}|^a + |u|^a)|\tilde{u}-u||\partial_x\tilde{u}| + |u|^{1+a}|\partial_x(\tilde{u}-u)|. \end{aligned} \tag{4.26}$$

Then using Hölder's inequality with $\frac{1}{2} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ and $q \geq \frac{2}{a}$ and then Sobolev's embedding

$$\begin{aligned} \||u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L^1_T L^2_x} &\lesssim cT\||\tilde{u}|^a + |u|^a\|_{L^\infty_T L^q_x}\|\tilde{u}-u\|_{L^\infty_T L^p_x}\|\partial_x\tilde{u}\|_{L^\infty_T L^r_x} \\ &\quad + T^{1/2}\||u|^{1+a}\partial_x(\tilde{u}-u)\|_{L^2_T L^2_x} \\ &\lesssim T(\|\tilde{u}\|_{L^\infty_T H^{3/2}_x} + \|u\|_{L^\infty_T H^{3/2}_x})^{1+a}\|\tilde{u}-u\|_{L^\infty_T H^{1/2}_x} \\ &+ T^{1/2}\||u|^{1+a}\|_{L^2_x L^\infty_T}\|\partial_x(\tilde{u}-u)\|\|_{L^\infty_x L^2_T}. \end{aligned}$$

Arguing as we did in (4.24) we get

$$\||u|^{1+a}\|_{L^2_x L^\infty_T} \lesssim \|u\|^{1+a}_{L^\infty([0,T];X)}.$$

Thus

$$\begin{aligned} \||u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L^1_T L^2_x} &\lesssim T(\|\tilde{u}\|_{L^\infty_T H^{3/2}_x} + \|u\|_{L^\infty_T H^{3/2}_x})^{1+a}\|\tilde{u}-u\|_{L^\infty_T H^{1/2}_x} \\ &\quad + \|u\|_{L^\infty([0,T];X)}\|\tilde{u}-u\|. \end{aligned} \tag{4.27}$$

Regarding the norm $L^1_x L^2_T$ we have from (4.26) that

$$\begin{aligned} \||u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L^1_x L^2_T} &\lesssim \||\tilde{u}|^a + |u|^a\|_{L^{\frac{1}{a}+}_x L^\infty_T}\||\tilde{u}-u|\partial_x\tilde{u}\|_{L^{\frac{1}{1-a}-}_x L^2_T} \\ &\quad + \|u\|^{1+a}_{L^{1+a}_x L^\infty_T}\|\partial_x(\tilde{u}-u)\|_{L^\infty_x L^2_T} \\ &\lesssim (\|u\|^a_{L^{1+}_x L^\infty_T} + \|\tilde{u}\|^a_{L^{1+}_x L^\infty_T})\||\tilde{u}-u|\partial_x\tilde{u}\|_{L^{\frac{1}{1-a}-}_x L^2_T} \\ &\quad + \|u\|^{1+a}_{L^{1+a}_x L^\infty_T}\|\tilde{u}-u\|_T. \end{aligned}$$

Now we choose $n$ sufficiently large and use Hölder's inequality with

$$\frac{1}{\frac{1}{1-a}-} = \frac{1}{\frac{1}{1-a}} + \frac{1}{n}, \quad \text{and} \quad \frac{1}{2} = \frac{1}{2+} + \frac{1}{n}.$$

Applying Sobolev embedding we obtain that

$$\begin{aligned} \||\tilde{u}-u|\partial_x\tilde{u}\|_{L^{\frac{1}{1-a}-}_x L^2_T} &\leq \|\partial_x\tilde{u}\|_{L^{\frac{1}{1-a}}_x L^{2+}_T}\|\tilde{u}-u\|_{L^n_x L^n_T} \\ &\lesssim \|\partial_x\tilde{u}\|_{L^{\frac{1}{1-a}}_x L^{2+}_T}\|\tilde{u}-u\|_{L^\infty_T H^{1/2}_x} \end{aligned}$$

Using one more time that

$$\|\cdot\|_{L^{1+}_x L^\infty_T} \lesssim \|\cdot\|_{L^\infty([0,T];X)}$$

we obtain

$$\begin{aligned}\||u|^{1+a}\partial_x u - |\tilde{u}|^{1+a}\partial_x \tilde{u}\|_{L^1_x L^2_T} &\lesssim \left(\|\tilde{u}\|^a_{L^\infty([0,T];X)} + \|u\|^a_{L^\infty([0,T];X)}\right)\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{1-a}} L_T^{2+}} \|\tilde{u}-u\| \\ &+\|u\|^{1+a}_{L^\infty([0,T];X)} \|\tilde{u}-u\|_T.\end{aligned}$$

It remains to estimate $\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{1-a}} L_T^{2+}}$.

Case $1:\ a > 1/2$.

In this case we can take $2+ = \frac{1}{1-a}$ and then interchange $\|\cdot\|_{L_x^{\frac{1}{1-a}} L_T^{2+}} = \|\cdot\|_{L_T^{2+} L_x^{\frac{1}{1-a}}}$ and after using Sobolev embedding we have

$$\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{1-a}} L_T^{2+}} \lesssim T^+ \|\tilde{u}\|_{L_T^\infty H_x^{3/2}}.$$

Case $2:\ 0 < a \leq 1/2$.

In this case we consider $\rho > \dfrac{1-2a}{2}$. Then using Hölder's inequality with

$$\frac{1}{\frac{1}{1-a}} = \frac{1}{2+} + \frac{2}{1-2a}$$

and that the function $\langle\cdot\rangle^{-\rho} \in L^{\frac{2}{1-2a}-}(\mathbb{R})$, since $\rho > \dfrac{1-2a}{2}$, we obtain

$$\begin{aligned}\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{1-a}} L_T^{2+}} &\leq \|\langle\cdot\rangle^\rho \partial_x \tilde{u}\|_{L^{2+}_x L^{2+}_T} \|\langle\cdot\rangle^{-\rho}\|_{L^{\frac{2}{1-2a}-}} \\ &\lesssim T^+ \|\langle\cdot\rangle^\rho \partial_x \tilde{u}\|_{L^\infty_T L^{2+}_x} \\ &\lesssim T^+ \|D^\epsilon(\langle\cdot\rangle^\rho \partial_x \tilde{u})\|_{L^\infty_T L^2_x},\end{aligned} \tag{4.28}$$

where $\epsilon = \frac{1}{2} - \frac{1}{2+}$ is sufficiently small. To finish this case we claim that

$$\|D^\epsilon(\langle x\rangle^\rho \partial_x f)\|_{L^2} \lesssim \|J^{1/2}(xf)\|_{L^2} + \|J^{3/2} f\|_{L^2}. \tag{4.29}$$

In fact, first notice that

$$\begin{aligned}\|D^\epsilon(\langle x\rangle^\rho \partial_x f)\|_{L^2} &\leq \|D^{\epsilon+1}(\langle x\rangle^\rho f)\|_{L^2} + \|D^\epsilon(f\partial_x(\langle x\rangle^\rho)\|_{L^2} \\ &\lesssim \|D^{\epsilon+1}(\langle x\rangle^\rho f)\|_{L^2} + \|f\partial_x\langle x\rangle^\rho\|_{L^2} + \|\partial_x(f\partial_x\langle x\rangle^\rho)\|_{L^2} \\ &\lesssim \|D^{\epsilon+1}(\langle x\rangle^\rho f)\|_{L^2} + \left(\|\partial_x\langle x\rangle^\rho\|_{L^\infty} + \|\partial_x^2\langle x\rangle^\rho\|_{L^\infty}\right)\|f\|_{H^1}.\end{aligned}$$

Next consider $\theta = 1-\rho$. Notice $0 \leq 1-\rho \leq 1$ and $1+\epsilon \leq 1/2+\theta$ for small $\epsilon$. So by Lemma 1.3.1 we obtain

$$\|D^{\epsilon+1}(\langle x\rangle^\rho f)\|_{L^2} \lesssim \|J^{1/2}(\langle x\rangle f)\|^{1-\theta}_{L^2} \|J^{3/2} f\|^\theta_{L^2}.$$

Finally, we use Lemma 1.3.2 to conclude

$$\begin{aligned}\|D^{\epsilon+1}(\langle x\rangle^\rho f)\|_{L^2} &\lesssim \|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta}\\ &\lesssim \left(\|J^{1/2}(xf)\|_{L^2}+\|J^{3/2}f\|_{L^2}\right)^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta}\\ &\lesssim \|J^{1/2}(xf)\|_{L^2}+\|J^{3/2}f\|_{L^2}.\end{aligned}$$

Then using (4.29) in (4.28) we deduce that

$$\|\partial_x\tilde{u}\|_{L_x^{\frac{1}{1-a}}L_T^{2+}} \lesssim T^+\|\tilde{u}\|_{L^\infty([0,T];X)}.$$

Then

$$\begin{aligned}\||u|^{1+a}\partial_x u-|\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L_x^1L_T^2} &\lesssim T^+\left(\|\tilde{u}\|_{L^\infty([0,T];X)}+\|u\|_{L^\infty([0,T];X)}\right)^{1+a}\|u-\tilde{u}\|_T \quad (4.30)\\ &\quad +\|u\|_{L^\infty([0,T];X)}^{1+a}\|\tilde{u}-u\|_T.\end{aligned}$$

Therefore

$$\|u-\tilde{u}\|_T \lesssim \left[T^+\left(\|\tilde{u}\|_{L^\infty([0,T];X)}+\|u\|_{L^\infty([0,T];X)}\right)^{1+a}+\|u\|_{L^\infty([0,T];X)}^{1+a}\right]\|u-\tilde{u}\|. \qquad (4.31)$$

Taking $T$ sufficiently small and recalling that $\|u\|_{L^\infty([0,T];X)}$ is small we obtain from (4.31) that

$$\|u-\tilde{u}\|_T = 0.$$

Hence $\tilde{u}(t)=u(t)$ for all $t$ sufficiently small. To prove uniqueness in the whole $[0,T_*]$ consider

$$J=\{T\in[0,T_*]; u(t)=\tilde{u}(t), \quad \text{for all} \quad t\in[0,T]\}$$

and let $T_*'$ be its supremum . We would like to prove $T_*'=T_*$. Suppose $T_*'<T_*$, and consider $T\in(T_*',T_*)$. Since $u$ and $\tilde{u}$ are equals in the interval $[0,T_*']$ we have

$$\|u-\tilde{u}\|_T \lesssim \||u|^{1+a}\partial_x u-|\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L^1([T_*',T];L^2(\mathbb{R})}+\||u|^{1+a}\partial_x u-|\tilde{u}|^{1+a}\partial_x\tilde{u}\|_{L^1(\mathbb{R};L^2([T_*',T])}. \quad (4.32)$$

Using in (4.32) the same argument applied to (4.25) to obtain (4.31) we get

$$\begin{aligned}\|u-\tilde{u}\|_T &\lesssim (T-T_*')^+(\|\tilde{u}\|_{L^\infty([T_*',T];X)}+\|u\|_{L^\infty([T_*',T];X)})^{1+a}\|u-\tilde{u}\|_T\\ &\quad +\|u\|_{L^\infty([T_*',T];X)}^{1+a}\|u-\tilde{u}\|_T.\end{aligned}$$

We conclude that $\|u-\tilde{u}\|_T=0$, i.e. $T\in J$, for all $T$ sufficiently close to $T_*'$ on the right. That is a contradiction because $T_*'$ is the supremum of $J$. Therefore $u(t)=\tilde{u}(t)$ for all $t\in[0,T_*]$.

**Theorem 4.3.2** (Existence and Uniqueness)**.** *There exists $\delta > 0$ such that for each $u_0 \in X$, with*

$$\|u_0\|_X \leq \delta$$

*there exist $T = T(\delta)$ and a unique $u \in L^\infty([0,T]; X) \cap C([0,T]; H^{3/2-}(\mathbb{R}))$ solution of*

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')(|u|^{1+a}\partial_x u)(t')dt'.$$

# Chapter 5

# Further results

In this chapter we regard the high power case $a > 1$. The case $a = 1$ was already mentioned in the introduction. A great difficulty we had in the low power case $0 < a < 1$ was to control the term $\|u\|_{L_x^{1+a} L_T^\infty}$. Since we do not have maximal estimates for $\|U(t)f\|_{L_x^p L_T^\infty}$ when $1 \leq p < 2$, we were obligated to introduce weights and many other difficulties arose from that. But now, for $a \geq 1$ we are in a more comfortable situation, because we do have maximal estimate for exponents $p \geq 2$. These maximal estimates will be our main ingredient in our approach here. Another facility of the case $a > 1$ is the fact we have that the function $|z|^a$ is Lipschitz, so we will be able to perform the contraction principal argument. So we will obtain a stronger theorem of existence. Indeed, we have

**Theorem 5.0.3.** *Let $a > 1$ and $T > 0$ be given. There exists $\delta > 0$ such that for all initial data $u_0 \in H^{1+}(\mathbb{R})$ with $\|u_0\|_{H^{1+}} \leq \delta$ there exists one, and only one $u \in C([0,T]; H^{1+}(\mathbb{R}))$ solution of*

$$u(t) = U(t)u_0 - \int_0^t U(t-t')(|u|^{1+a}\partial_x u)(t')dt' \tag{5.1}$$

*satisfying*

$$\|D^{1/2+}\partial_x u\|_{L_x^\infty L_T^2}, \quad \|u\|_{L_x^2 L_T^\infty} < \infty.$$

*Proof.* Consider the integral operator $\Psi = \Psi_{u_0}$ defined by the right hand side of the equality in (5.1), the norm

$$\Omega(u) = \|u\|_{L_T^\infty H_x^{1+}} + \|D^{1/2+}\partial_x u\|_{L_x^\infty L_T^2} + \|u\|_{L_x^2 L_T^\infty}$$

and the space

$$E_{A,T} = \{u \in C([0,T]; H^{1+}(\mathbb{R})); \Omega(u) \leq A\}.$$

Next we consider $u \in E_{A,T}$ and initial data $\|u_0\|_{H^{1+}} \leq \delta$. We shall prove $\Psi(E_{A,T}) \subset E_{A,T}$ for small values $\delta$. Applying the smoothing effects Lemma 1.2.3 and Lemma 1.2.4 in the nonlinear of $\Psi(u)$ we have

$$\|\Psi(u)\|_{L_T^\infty H_x^{1+}} + \|D^{1/2+}\partial_x \Psi(u)\|_{L_x^\infty L_T^2} \lesssim \delta + \|D^{1/2+}(|u|^{1+a}\partial_u)\|_{L_x^1 L_T^2}.$$

Now we use the Leibniz rule for fractional derivatives Lemma 1.1.4 to obtain

$$\|D^{1/2+}(|u|^{1+a}\partial_x u)\|_{L_x^1 L_T^2} \lesssim \|D^{1/2+}(|u|^{1+a})\|_{L_x^{2-} L_T^M}\|\partial_x u\|_{L_x^{2+} L_T^{2+}} + \||u|^{1+a}\|_{L_x^1 L_T^\infty}\|D^{1/2+}\partial_x u\|_{L_x^\infty L_T^2}$$

where $M$ is a sufficiently large number. Sobolev embedding Lemma 1.1.3 implies

$$\|\partial_x u\|_{L_x^{2+}} \lesssim \|u\|_{H_x^{1+}}$$

and from the fact $a \geq 1$

$$\||u|^{1+a}\|_{L_x^1 L_T^\infty} \leq \|u\|^2_{L_x^2 L_T^\infty}\|u\|^{a-1}_{L_x^\infty L_T^\infty}.$$

Notice $L_x^\infty L_T^\infty = L_T^\infty L_x^\infty$ and using again Sobolev embedding we obtain

$$\||u|^{1+a}\|_{L_x^1 L_T^\infty} \lesssim \Omega(u)^{1+a}. \tag{5.2}$$

Thus

$$\|D^{1/2+}(|u|^{1+a}\partial_x u)\|_{L_x^1 L_T^2} \lesssim T^+\Omega(u)\|D^{1/2+}(|u|^{1+a})\|_{L_x^{2-} L_T^M} + \Omega(u)^{2+a}.$$

Since the $F(z) = |z|^{a+1}$ is $C^1$ (as a function in $\mathbb{R}^2$), the chain rule Lemma 1.1.5 implies

$$\|D^{1/2+}(|u|^{a+1})\|_{L_x^{2-} L_T^M} \lesssim \||u|^a\|_{L_x^2 L_T^{M+}}\|D^{1/2+}u\|_{L_x^{M+} L_T^{M+}}.$$

Using Sobolev embedding

$$\begin{aligned}
\||u|^a\|_{L_x^2 L_T^{M+}} &\leq T^+\||u|^a\|_{L_x^2 L_T^\infty} \\
&\leq \|u\|_{L_x^2 L_T^\infty}\|u\|^{a-1}_{L_x^\infty L_T^\infty} \\
&\lesssim \|u\|_{L_x^2 L_T^\infty}\|u\|^{a-1}_{L_T^\infty H_x^{1+}}
\end{aligned}$$

and

$$\begin{aligned}
\|D^{1/2+}u\|_{L_x^{M+} L_T^{M+}} &= \|D^{1/2+}u\|_{L_T^{M+} L_x^{M+}} \\
&\lesssim T^+\|u\|_{L_T^\infty H_x^{1+}}.
\end{aligned}$$

We obtain

$$\|D^{1/2+}(|u|^{1+a})\|_{L_x^{2-}L_T^M} \lesssim T^+\Omega(u)^{1+a}.$$

We conclude

$$\|\Psi(u)\|_{L_T^\infty H_x^{1+}} + \|D^{1/2+}\partial_x\Psi(u)\|_{L_x^\infty L_T^2} \lesssim \delta + (1+T^+)\Omega(u)^{2+a}. \tag{5.3}$$

Next we estimate $\|\Psi(u)\|_{L_x^2 L_T^\infty}$. Indeed, using the maximal function estimate Lemma 1.2.5 we have

$$\begin{aligned}\|\Psi(u)\|_{L_x^2L_T^\infty} &\leq \|u_0\|_{H^{1/2+}} + \int_0^T \|U(t)\big(U(-t')(|u|^{1+a}\partial_x u)\big)\|_{L_x^2L_T^\infty}dt' \\ &\lesssim \delta + \||u|^{1+a}\partial_x u\|_{L_T^1 H_x^{1/2+}}.\end{aligned}$$

Note

$$\||u|^{1+a}\partial_x u\|_{L_T^1 H_x^{1/2+}} \simeq \||u|^{1+a}\partial_x u\|_{L_T^1 L_x^2} + \|D^{1/2+}\big(|u|^{1+a}\partial_x u\big)\|_{L_T^1 L_x^2}.$$

It turns out, by Sobolev embedding

$$\begin{aligned}\||u|^{1+a}\partial_x u\|_{L_T^1L_x^2} &\leq T\|u\|_{L_T^\infty L_x^\infty}^{1+a}\|\partial_x u\|_{L_T^\infty L_x^2} \\ &\lesssim T\|u\|_{L_T^\infty H_x^{1/2+}}^{1+a}\|u\|_{L_T^\infty H_x^1} \\ &\leq T\Omega(u)^{2+a}.\end{aligned}$$

Using the Leibniz rule Lemma 1.1.4

$$\begin{aligned}\|D^{1/2+}\big(|u|^{1+a}\partial_x u\big)\|_{L_T^1L_x^2} &\leq T^{1/2}\|D^{1/2+}\big(|u|^{1+a}\partial_x u\big)\|_{L_x^2L_T^2} \\ &\lesssim \|D^{1/2+}(|u|^{1+a})\|_{L_x^ML_T^M}\|\partial_x u\|_{L_x^{2+}L_T^{2+}} \\ &\quad +\||u|^{1+a}\|_{L_x^2L_T^\infty}\|D^{1/2+}\partial_x u\|_{L_x^2L_T^\infty}\end{aligned} \tag{5.4}$$

where $M$ is sufficiently large. Now using Sobolev embedding in (5.4) it follows

$$\begin{aligned}\|D^{1/2+}(|u|^{1+a})\|_{L_x^ML_T^M} &= \|D^{1/2+}(|u|^{1+a})\|_{L_T^ML_x^M} \\ &\leq T^+\|D^{1/2+}(|u|^{1+a})\|_{L_T^\infty L_x^M} \\ &\lesssim T^+\||u|^{1+a}\|_{L_T^\infty H_x^1} \\ &\lesssim T^+\||u|^{1+a}\|_{L_T^\infty L_x^2} + T^+\|\partial_x(|u|^{1+a})\|_{L_T^\infty L_x^2} \\ &\lesssim T^+\|u\|_{L_T^\infty L_x^{2+2a}}^{1+a} + T^+\||u|^a\partial_x u\|_{L_T^\infty L_x^2} \\ &\lesssim T^+\|u\|_{L_T^\infty H_x^{1+}}^{1+a} + T^+\|u\|_{L_T^\infty L_x^\infty}^a\|\partial_x u\|_{L_T^\infty L_x^2} \\ &\leq T^+\Omega(u)^{1+a}.\end{aligned} \tag{5.5}$$

Keeping using Sobolev embedding to the other terms of (5.4) we also have

$$\begin{aligned} \|\partial_x u\|_{L^{2+}_x L^{2+}_T} &= T^+ \|\partial_x u\|_{L^\infty_T L^{2+}_x} \\ &\lesssim T^+ \|u\|_{L^\infty_T H^{1+}_x}, \end{aligned}$$

and

$$\begin{aligned} \||u|^{1+a}\|_{L^2_x L^\infty_T} &\le \|u\|_{L^2_x L^\infty_T} \|u\|^a_{L^\infty_x L^\infty_T} \\ &= \|u\|_{L^2_x L^\infty_T} \|u\|^a_{L^\infty_T L^\infty_x} \\ &\lesssim \|u\|_{L^2_x L^\infty_T} \|u\|^a_{L^\infty_T H^{1+}_x} \\ &\le \Omega(u)^{1+a}. \end{aligned}$$

We conclude

$$\|\Psi(u)\|_{L^2_x L^\infty_T} \lesssim T^+ \Omega(u)^{2+a}. \tag{5.6}$$

and then gathering with the estimate (5.3) we finally conclude

$$\Omega(\Psi(u)) \lesssim \delta + (1+T^+)\Omega(u)^{2+a}. \tag{5.7}$$

If $\delta \le A/2$ and $A$ satisfies

$$c(1+T^+)A^{2+a} \le \frac{A}{2}$$

we have $\Omega(u) \le A$. Then if we take

$$A = \frac{1}{(2c(1+T^+))^{\frac{1}{1+a}}}$$

and initial data with size $\delta \le A/2$ we will have $\Psi(E_{A,T}) \subset E_{A,T}$.

Next we will prove that the map $\Psi$ is a contraction in $E_{A,T}$ with respect to the norm $\Omega$ and then there will be a unique $u \in E_{A,T}$ satisfying $\Psi(u) = u$. We consider two elements $u, v$ of $E_{A,T}$. We have

$$\Psi(u) - \Psi(v) = -i \int_0^t U(t-t')(|u|^{1+a}\partial_x u - |v|^{1+a}\partial_x v)dt'. \tag{5.8}$$

Then using Lemma 1.2.3 and 1.2.4 we have

$$\begin{aligned} \|\Psi(u)-\Psi(v)\|_{L^\infty_T H^{1+}_x} &+ \|D^{1/2+}\partial_x(\Psi(u)-\Psi(v))\|_{L^\infty_x L^2_T} \\ &\lesssim \|D^{1/2+}(|u|^{1+a}\partial_x u - |v|^{1+a}\partial_x v)\|_{L^1_x L^2_T} \\ &\le \|D^{1/2+}((|u|^{1+a} - |v|^{1+a})\partial_x u)\|_{L^1_x L^2_T} + \|D^{1/2+}(|v|^{1+a}\partial_x(u-v))\|_{L^1_x L^2_T}. \end{aligned}$$

Similar to our argument presented in the first part we have by the Leibniz rule

$$\begin{aligned}
\|D^{1/2+}(|v|^{1+a}\partial_x(u-v))\|_{L_x^1L_T^2} &\lesssim \|D^{1/2+}(|v|^{1+a})\|_{L_x^{2-}L_T^M}\|\partial_x(u-v)\|_{L_x^{2+}L_T^{2+}} \\
&\quad +\||v|^{1+a}\|_{L_x^1L_T^\infty}\|D^{1/2+}\partial_x(u-v)\|_{L_x^\infty L_T^2} \\
&\lesssim T^+\|D^{1/2+}(|v|^{1+a})\|_{L_x^{2-}L_T^M}\Omega(u-v) \\
&\quad +\|v\|_{L_x^2L_T^\infty}^2\|v\|_{L_x^\infty L_T^\infty}\Omega(u-v) \\
&\lesssim (1+T^+)\Omega(u)^{1+a}\Omega(u-v).
\end{aligned}$$

Now, in order to estimate $\|D^{1/2+}((|u|^{1+a}-|v|^{1+a})\partial_x u)\|_{L_x^1L_T^2}$ we first notice that

$$\begin{aligned}
|u|^{1+a}-|v|^{1+a} &= \int_0^1 \frac{d}{d\theta}(|w(\theta)|^{1+a})d\theta \\
&= (1+a)\mathrm{Re}\left[\int_0^1 |w(\theta)|^{a-1}\bar{w}(\theta)d\theta(u-v)\right]
\end{aligned} \tag{5.9}$$

where we are denoting $w(\theta)=\theta u+(1-\theta)v$. Using the Leibniz rule Lemma 1.1.4

$$\begin{aligned}
\|D^{1/2+}((|u|^{1+a}-|v|^{1+a})\partial_x u)\|_{L_x^1L_T^2} &\leq \|D^{1/2+}(|u|^{1+a}-|v|^{1+a})\|_{L_x^{2-}L_T^M}\|\partial_x u\|_{L_x^{2+}L_T^{2+}} \\
&\quad +\||u|^{1+a}-|v|^{1+a}\|_{L_x^1L_T^\infty}\|D^{1/2+}\partial_x u\|_{L_x^\infty L_T^2}
\end{aligned}$$

where $M$ is taking sufficiently. Using Sobolev embedding it follows that

$$\|\partial_x u\|_{L_x^{2+}L_T^{2+}} \lesssim T^+\|u\|_{L_T^\infty H_x^{1+}}.$$

So we have

$$\begin{aligned}
\|D^{1/2+}((|u|^{1+a}-|v|^{1+a})\partial_x u)\|_{L_x^1L_T^2} &\lesssim T^+\Omega(u)\|D^{1/2+}(|u|^{1+a}-|v|^{1+a})\|_{L_x^1L_T^2} \\
&\quad +\Omega(u)\||u|^{1+a}-|v|^{1+a}\|_{L_x^1L_T^\infty}.
\end{aligned} \tag{5.10}$$

Using (5.9)

$$\begin{aligned}
\||u|^{1+a}-|v|^{1+a}\|_{L_x^1L_T^\infty} &\lesssim \int_0^1 \||w(\theta)|^a(u-v)\|_{L_x^1L_T^\infty}d\theta \\
&\lesssim \|u-v\|_{L_x^2L_T^\infty}\int_0^1 \||w(\theta)|^a\|_{L_x^2L_T^\infty}d\theta.
\end{aligned}$$

But

$$\begin{aligned}
\||w(\theta)|^a\|_{L_x^2L_T^\infty} &\leq \|w(\theta)\|_{L_x^2L_T^\infty}\|w(\theta)\|_{L_x^\infty L_T^\infty}^{a-1} \\
&\lesssim \|w(\theta)\|_{L_x^2L_T^\infty}\|w(\theta)\|_{L_T^\infty H_x^{1/2+}}^{a-1} \\
&\leq \Omega(w(\theta))^a \\
&\lesssim (\Omega(u)+\Omega(v))^a.
\end{aligned}$$

Therefore

$$\||u|^{1+a}-|v|^{1+a}\|_{L^1_xL^2_T}\lesssim(1+T^+)(\Omega(u)+\Omega(v))^a\Omega(u-v). \tag{5.11}$$

Now lets take care of

$$\|D^{1/2+}(|u|^{1+a}-|v|^{1+a})\|_{L^{2-}_xL^M_T}.$$

We use formula (5.9)

$$\|D^{1/2+}(|u|^{1+a}-|v|^{1+a})\|_{L^{2-}_xL^M_T}\lesssim\int_0^1\|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))(u-v)\|_{L^{2-}_xL^M_T}d\theta. \tag{5.12}$$

Applying the Leibniz rule with $1/M=1/M_1+1/M_2$ and then using Sobolev embedding we have

$$\begin{aligned}\|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))(u-v)\|_{L^{2-}_xL^M_T} &\lesssim \|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))\|_{L^M_xL^M_T}\|u-v\|_{L^2_xL^\infty_T}\\ &\quad+\||w(\theta)|^{a-1}\bar{w}\|_{L^2_xL^{M_1}_T}\|D^{1/2+}(u-v)\|_{L^{M_2}_xL^{M_2}_T}\\ &\lesssim \|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))\|_{L^M_xL^M_T}\Omega(u-v)\\ &\quad+T^+\|w(\theta)\|^{a-1}_{L^\infty_xL^\infty_T}\|w(\theta)\|_{L^2_xL^\infty_T}\|D^{1/2+}(u-v)\|_{L^\infty_TL^{M_2}_x}\\ &\lesssim \|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))\|_{L^M_xL^M_T}\Omega(u-v)\\ &\quad+T^+\|w(\theta)\|^{a-1}_{L^\infty_TH^{1/2+}_x}\|w(\theta)\|_{L^2_xL^\infty_T}\|u-v\|_{L^\infty_TH^{1+}_x}\\ &\lesssim \|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))\|_{L^M_xL^M_T}\Omega(u-v)\\ &\quad+T^+\Omega(w(\theta))^a\Omega(u-v).\end{aligned}$$

Finally, since $a>1$ the function $F(z)=|z|^{a-1}\bar{z}$ is $C^1$ and then we can apply the chain rule (Lemma 1.1.5) and obtain

$$\|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))\|_{L^M_x}\lesssim\||w(\theta)|^{a-1}\|_{L^{M_1}_x}\|D^{1/2+}w(\theta)\|_{L^{M_2}_x}$$

where $1/M=1/M_1+1/M_2$. For $M$ sufficiently large such that $(a-1)M_1\geq 2$ we have from the Sobolev embedding

$$\|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))\|_{L^M_x}\lesssim\|w(\theta)\|^a_{H^{1+}_x}.$$

Hence

$$\|D^{1/2+}(|w(\theta)|^{a-1}\bar{w}(\theta))(u-v)\|_{L^{2-}_xL^M_T}\lesssim T^+\Omega(\theta)^a\Omega(u-v). \tag{5.13}$$

Replacing (5.13) in the integral (5.12) and using that $\Omega(w(\theta))\leq\Omega(u)+\Omega(v)$ we get

$$\|D^{1/2+}(|u|^{1+a}-|v|^{1+a})\|_{L^1_xL^2_T}\lesssim T^+(\Omega(u)+\Omega(v))^a\Omega(u-v). \tag{5.14}$$

Replacing the estimates (5.12) and (5.14) in (5.10) we obtain

$$\|D^{1/2+}((|u|^{1+a}-|v|^{1+a})\partial_x u)\|_{L^1_x L^2_T} \lesssim (1+T^+)(\Omega(u)+\Omega(v))^{1+a}\Omega(u-v). \tag{5.15}$$

Therefore

$$\|\Psi(u)-\Psi(v)\|_{L^\infty_T H^{1+}_x}+\|D^{1/2+}\partial_x(\Psi(u)-\Psi(v))\|_{L^\infty_x L^2_T} \lesssim (1+T^+)(\Omega(u)+\Omega(v))^{1+a}\Omega(u-v). \tag{5.16}$$

To finish the proof it remains take care of $\|\Psi(u)-\Psi(v)\|_{L^2_x L^\infty_T}$. Indeed, applying the maximal estimate Lemma 1.2.5 we have

$$\begin{aligned}
\|\Psi(u)-\Psi(v)\|_{L^2_x L^\infty_T} &= \left\|\int_0^t U(t-t')(|u|^{1+a}\partial_x u-|v|^{1+a}\partial_x v)dt'\right\|_{L^2_x L^\infty_T} \\
&\leq \int_0^t \|U(t)U(-t')(|u|^{1+a}\partial_x u-|v|^{1+a}\partial_x v)\|_{L^2_x L^\infty_T} dt' \\
&\lesssim \||u|^{1+a}\partial_x u-|v|^{1+a}\partial_x v\|_{L^1_T H^{1/2+}_x}.
\end{aligned} \tag{5.17}$$

To estimate (5.17) we separate it in two partes

$$\||u|^{1+a}\partial_x u-|v|^{1+a}\partial_x v\|_{L^1_T L^2_x} \tag{5.18}$$

and

$$\|D^{1/2+}(|u|^{1+a}\partial_x u-|v|^{1+a}\partial_x v)\|_{L^1_T L^2_x}. \tag{5.19}$$

We use (2.15) combined with Sobolev embedding to obtain that (5.18) can be bounded by

$$\begin{aligned}
& T\|(|u|^{1+a}-|v|^{1+a})\partial_x u\|_{L^\infty_T L^2_x}+\||v|^{1+a}\partial_x(u-v)\|_{L^\infty_T L^2_x} \\
\leq\ & T\||u|^{1+a}-|v|^{1+a}\|_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} \\
& +\||v|^{1+a}\|_{L^\infty_T L^\infty_x}\|\partial_x(u-v)\|_{L^\infty_T L^2_x} \\
\lesssim\ & T\||u|^a+|v|^a\|_{L^\infty_T L^\infty_x}\|u-v\|_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} \\
& +\|v\|^{1+a}_{L^\infty_T L^\infty_x}\|\partial_x(u-v)\|_{L^\infty_T L^2_x} \\
\lesssim\ & T(\Omega(u)+\Omega(v))^{1+a}\Omega(u-v).
\end{aligned}$$

In (5.19), since we can not control $3/2+$ derivatives in the $L^2_x$, we first pass from $L^1_T L^2_x$ to the $L^2_x L^2_T$. So we first bound (5.19) by

$$T^{1/2}\|D^{1/2+}(|u|^{1+a}\partial_x u-|v|^{1+a}\partial_x u)\|_{L^2_x L^2_T}. \tag{5.20}$$

Then adding and subtracting $|v|^{1+a}\partial_x u$ we can bound (5.20) by

$$T^{1/2}\|D^{1/2+}((|u|^{1+a}-|v|^{1+a})\partial_x u)\|_{L^2_x L^2_T}+T^{1/2}\|D^{1/2+}(|v|^{1+a}\partial_x(u-v))\|_{L^2_x L^2_T} \tag{5.21}$$

Now we apply Leibniz rule keeping in mind we can only control $3/2+$ derivatives in $L_x^\infty L_T^2$. So in the second parcel of (5.21) we apply the Leibniz rule in the following way

$$\begin{aligned}
\|D^{1/2+}(|v|^{1+a}\partial_x(u-v))\|_{L_x^2L_T^2} &\lesssim \|D^{1/2+}(|v|^{1+a})\|_{L_x^ML_T^M}\|\partial_x(u-v)\|_{L_x^{2+}L_T^{2+}} \\
&\quad +\||v|^{1+a}\|_{L_x^2L_T^\infty}\|D^{1/2+}\partial_x(u-v)\|_{L_x^\infty L_T^2} \\
&\lesssim T^{1/2}\|D^{1/2+}(|v|^{1+a})\|_{L_T^\infty L_x^M}\Omega(u-v)+\||v|^{1+a}\|_{L_x^2L_T^\infty}\Omega(u-v)
\end{aligned}$$

Arguing as we did in (5.5) and in (5.2) we obtain

$$\|D^{1/2+}(|v|^{1+a}\partial_x(u-v))\|_{L_x^2L_T^2} \lesssim (1+T^+)\Omega(v)^{1+a}\Omega(u-v). \tag{5.22}$$

Now in the first parcel of (5.21) we use the Leibniz rule as we just did in the second parcel and then we obtain

$$\begin{aligned}
\|D^{1/2+}((|u|^{1+a}-|v|^{1+a})\partial_x u)\|_{L_x^2L_T^2} &\lesssim T^{1/2}\|D^{1/2+}(|u|^{1+a}-|v|^{1+a})\|_{L_T^\infty L_x^M}\Omega(u) \\
&\quad +\||u|^{1+a}-|v|^{1+a}\|_{L_x^2L_T^\infty}\Omega(u).
\end{aligned} \tag{5.23}$$

Using Sobolev embedding we have, for sufficiently large $M$,

$$\|D^{1/2+}(|u|^{1+a}-|v|^{1+a})\|_{L_T^\infty L_x^M} \lesssim \|\partial_x(|u|^{1+a}-|v|^{1+a})\|_{L_T^\infty L_x^2}.$$

Notice since $a\ge 1$ we have, $z\longmapsto |z|^a$ is Lipschtz, i.e

$$||u|^a-|v|^a| \lesssim (|u|^{a-1}+|v|^{a-1})|u-v|. \tag{5.24}$$

Using (5.24) and Sobolev embedding

$$\begin{aligned}
\|\partial_x(|u|^{1+a}-|v|^{1+a})\|_{L_T^\infty L_x^2} &\lesssim \||u|^a\partial_x u-|v|^a\partial_x v\|_{L_T^\infty L_x^2} \\
&\le \||u|^a-|v|^a\|_{L_T^\infty L_x^\infty}\|\partial_x u\|_{L_T^\infty L_x^2}+\|v\|_{L_T^\infty L_x^\infty}^a\|\partial_x(u-v)\|_{L_T^\infty L_x^2} \\
&\lesssim \||u|^{a-1}+|v|^{a-1}\|_{L_T^\infty L_x^\infty}\|u-v\|_{L_T^\infty L_x^\infty}\|\partial_x u\|_{L_T^\infty L_x^2} \\
&\quad +\|v\|_{L_T^\infty L_x^\infty}^a\|\partial_x(u-v)\|_{L_T^\infty L_x^2} \\
&\lesssim (\|v\|_{L_T^\infty H_x^{1/2+}}^{a-1}+\|u\|_{L_T^\infty H_x^{1/2+}}^{a-1})\|u-v\|_{L_T^\infty H_x^{1/2+}}\|u\|_{L_T^\infty H_x^1} \\
&\quad +\|v\|_{L_T^\infty H_x^{1/2+}}^a\|u-v\|_{L_T^\infty L_x^2} \\
&\lesssim (\Omega(u)+\Omega(v))^a\Omega(u-v).
\end{aligned} \tag{5.25}$$

Finally, still using the Lipschitz property (5.24) together with Sobolev embedding we obtain

$$\begin{aligned} \||u|^{1+a} - |v|^{1+a}\|_{L^2_x L^\infty_T} &\lesssim \||u|^a + |v|^a\|_{L^2_x L^\infty_T} \|u - v\|_{L^2_x L^\infty_T} \\ &\lesssim (\|u\|^a_{L^\infty_T H^{1/2+}_x} + \|v\|^a_{L^\infty_T H^{1/2+}_x}) \|u - v\|_{L^2_x L^\infty_T} \\ &\leq (\Omega(u) + \Omega(v))^a \Omega(u - v). \end{aligned}$$

Therefore we have the bound

$$\|D^{1/2+}((|u|^{1+a} - |v|^{1+a})\partial_x u)\|_{L^2_x L^2_T} \lesssim (1 + T^+)(\Omega(u) - \Omega(v))^a \Omega(u - v). \tag{5.26}$$

Collecting all these estimates we conclude finally obtain

$$\|\Psi(u) - \Psi(v)\|_{L^2_x L^\infty_T} \lesssim (1 + T^+)(\Omega(u) + \Omega(v))^a \Omega(u - v), \tag{5.27}$$

which combined with (5.16) provide us

$$\Omega(\Psi(u) - \Psi(v)) \lesssim (1 + T^+)(\Omega(u) + \Omega(v))^a \Omega(u - v).$$

We conclude

$$\Omega(\Psi(u) - \Psi(v)) \leq \frac{1}{2}\Omega(u - v) \tag{5.28}$$

for all $u, v \in E_{A,T}$, provided we choose $A$ convinient small. □

**Remark 5.** *Note that, if* $0 < a < 1$*, we would have*

$$||u|^a - |v|^a| \lesssim |u - v|^a.$$

*Then we would obtain* $\|\partial_x(|u|^{1+a} - |v|^{1+a})\|_{L^\infty_T L^2_x} \lesssim \Omega(u - v)^a$ *instead of (5.25) and so the contraction principle argument would provide*

$$\Omega(\Psi(u) - \Psi(v)) \leq \frac{1}{2}\Omega(u - v)^a$$

*instead of (5.16) which does not mean* $\Psi$ *is a contraction.*

# Chapter 6

# Additional Remarks

It is a well-known fact that the generalized derivative Schrödinger equation has conserved energy

$$E(u) = \frac{1}{2}\int |\partial_x u|^2 dx + \frac{1}{a+3}\mathrm{Im}\int |u|^{1+a}\bar{u}\partial_x u dx. \tag{6.1}$$

Other two conseved quantities are the mass

$$M(u) = \frac{1}{2}\int |u|^2 dx, \tag{6.2}$$

and the momentum

$$P(u) = -\frac{1}{2}\int \bar{u}\partial_x u dx. \tag{6.3}$$

Because the energy is conserved, the first question we would like to address is the existence of global solution for this equation for the case $0 < a < 5$.

In the following we would like to mention some results in the periodic context. Herr [14] adjusted the argument in the work of Takaoka [30] to obatin that the IVP associated to DNLS is local well-posed in $H^s(\mathbb{T})$ for $s \geq 1/2$. Later on Herr along with Grünrock [8] extended this result for initial data $u_0 \in \widehat{H^s_r}(\mathbb{T})$, where

$$\|u_0\|_{\widehat{H^s_r}(\mathbb{T})} = \|\langle\xi\rangle^s \widehat{u}_0\|_{l^{r'}_\xi},$$

and $s \geq 1/2$, $2 > r > 4/3$ and $1 = 1/r + 1/r'$.

Recently Simpson and Ambrose [1] studied the general case $a \geq 1$ in the periodic setting. It was proved existence of weak solution

$$u \in L^\infty([0,T]; H^1(\mathbb{T})) \cap C([0,T]; H^s(\mathbb{T})),$$

$s < 1$, for periodic initial data $u_0 \in H^1(\mathbb{T})$. They proved $u$ is a strong solution $\in C([0,T];H^1(\mathbb{T}))$ if $u_0 \in H^2(\mathbb{T})$. However it is still unknown if we have the existence of solutions $u \in C([0,T];H^1(\mathbb{T}))$ for initial data in $H^1(\mathbb{T})$.

Finally, we observe that the parabolic regularization method gives us, in the case a $\geq 1$, a local strong result in $H^s$ for $s > 3/2$. The argument can be applied in both periodic and nonperiodic case [32].

# Bibliography


[1] D. M. Ambrose and G. Simpson, *Local existence theory for derivative nonlinear Schrödinger equations with non-integer power nonlinearities*

[2] H. Biagioni and F. Linares, *Ill- posedness for the derivative schrödinger and generalized benjamin-Ono equations*, Trans. Amer. Math. Soc.,**353** (2001), 3649-3659.

[3] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Schrödinger equation, part II: The KdV-equation*, GAFA. 3 (1993), 107-156, 209-262.

[4] M. Colin, M. Ohta, *Stability of solitary waves for derivative nonlinear Schrödinger equation*, Ann. I. H. Poincaré - AN, **23** (2006), 753-764.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global well-posedness for the Schrödinger equations with derivative*, SIAM J. Math. Anal., **33** (2001), 649-669.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *A refined global well-posedness for the Schrödinger equations with derivative*, SIAM J. Math. Anal., **34** (2002), 64-86.

[7] J. Ginibre and G. Velo, *On the class of nonlinear Schrödinger equations*, J. Funct. Anal. **32** (1979), 1-32, 33-72.

[8] A. Grünrock and S. Herr, *Low regularity local well-posedness of the derivative nonlinear Schrödiger equation with periodic initial data* SIAM J. on Math. Anal., **39** (2008), 1890-1920.

[9] B. Guo, Y. Wu, *Orbital stability of solitary waves for the nonlunear derivative Schrödinger equation*, J. Diff. Eqs., **123** (1995), 35-55.

[10] C. Hao, *Well-posedness for one-dimensional derivative nonlinear Schrödinger equations*, Comm. Pure Appl. Anal. **6** (2007), 997-1021.

[11] N. Hayashi, *Global existence of small analytic solutions to nonlinear Schrödinger equations*, Duke Math. J., **60** (1990) 717-727.

[12] N. Hayashi, *The initial value problem for the derivative nonlinear Schrödinger equation in the energy space*, pre print (1991).

[13] N. Hayashi and T. Ozawa, *On the derivative nonlinear Schrödinger equation*, Phys D, **55** (1992), 14-36.

[14] S. Herr, *On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition*, International Mathematics Reseach Notices, (2006) 96763-33.

[15] R. Iorio, V. Iorio, *Fourier analysis and partial differential equations*, Cambridge studies in advanced mathmatics.

[16] T. Kato, *Nonstationary flows of viscous and ideal fluids*, J. Func. Anal., Vol. 9, No. 3, (1972), 296-305.

[17] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results fotr the generalized Korteweg-de Vries equation via contraction principle*, Comm. pure App. Math. **46** (1993), 527-620.

[18] C. E. Kenig, G. Ponce, and L. Vega, *Oscilatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), pp. 33-69.

[19] C. E. Kenig, and A. Ruiz, *A strong type (2,2) estimate for the maximal function associated to the Schrödinger equation*, Trans. Amer. Math. Soc. **230**, (1983), pp. 239-246.

[20] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*, Springer (2009).

[21] X. Liu, G. Simpson and C. Sulem, *Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation*, (2012). J. Nonlinear Science, 23(**4**), (2013) 557-583.

[22] K. Mio, T. Ogino, K. Minami and S. Takeda, *Modified nonlinear Schrödinger equation for Alfvén Waves propagating along magnectic field in cold plasma*, J. Phys. Soc. **41** (1976), 265-271.

[23] E. Mjolhus, *On the modulational instability of hydromagnetic waves parallel to the magnetic field*, J. Plasma Phys., **16** (1976), 321-334.

[24] T. Ozawa, *On the nonlinear Schrödinger equations of derivative type*, Indiana Univ. Math. J. **45** (1996), 137-163.

[25] T. Passot and P. L. Sulem, *On multidimensional modulation of Alfvén waves*, Phys. Rev. E, **48** (1993), 2966-2974.

[26] I. G. Petrovski, *Ordinary differential equations*, Rev. English ed., Englewood Cliffs, N. J.: Prentice-Hall (1966).

[27] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press (1970).

[28] E. M. Stein, R. Shakarchi, *Complex analysis*, Princeton University Press, 2003. 379 p.

[29] R. S. Strichartz, *Multipliers on Fractional Sobolev Spaces*, J. Math. Mech. **16** (1967) 1031-1060.

[30] H. Takaoka, *Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity*, Adv. Differential Equations, **4** (1999), 561-580.

[31] H. Takaoka, *Global well-posedness for the Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces*, J. Diff. Eqs., **2001**(2001) pp. 1-23.

[32] M. Tsutsumi and I. Fukuda, *On solutions of the derivatives nonlinear Schrödinger equation. Existence and uniqueness theorem*, Funkcialaj Ekvacioj, **23** (1980), 259-277.

[33] M. Visan, *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions*, Duke Math. J. **138** (2007), 281-374.