Hyperopic Strict Topologies on l^∞

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Abstract

We explicitly define a family of seminorms on the space of all bounded real sequence l^∞. This family gives rise to a Hausdorff locally convex topology which is not equivalent to the usual ones: the weak topology $\sigma(l^\infty, l^1)$, the norm topology τ_∞, the Mackey topology $m(l^\infty, l^1)$ and the strict topology β. We show that this new topology, denoted by β_h, is weaker than the norm topology, τ_∞. Finally, we show that the dual of l^∞ with respect to β_h, called hyperopic strict dual, is not l^1 anymore but rather, is identified with the set of all purely finitely additive measures.

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1 Introduction

On l^∞, the set of bounded real-valued sequences, many topologies have been defined. Among the most well-known we find: the weak topology $\sigma(l^\infty, l^1)$, the norm topology τ_∞, the Mackey toplogy $m(l^\infty, l^1)$ and the strict topology β. All these topologies on l^∞, as well as their respective duals are important mathematical objects in their own right and have been studied by influential mathematicians like Mackey [8], Buck [1,2] , Conway [6], and Collins [5], among others.

Our concern in this paper is with the topologies defined on l^∞ which are defined by families of semi-norms indexed by a subset $\mathcal{A} \subset l^1$, the set of all summable sequences $a \in l^1$ such that $a_n \neq 0$ for all but finitely many n. Thus, $\sum_{n \geq N} |a_n|$ is always non zero. Any element of that family, $p_a : l^\infty \to R$, is defined by

$$p_a(x) = \lim \sup_{N \to \infty} \frac{\sum_{n \geq N} |a_n x_n|}{\sum_{n \geq N} |a_n|}$$

This topology will be called strict because it is obtained from a family of semi-norms indexed by $\mathcal{A} \subset l^1$ and because it also recalls Buck [2], who defined it for first time. In order to distinguish it from strict topology[1], introduced by Buck, we call it hyperopic. This name will be justified in Proposition 2 below.

Our topology is shown to not be equivalent to the topology introduced by Buck [2] since it has mathematical properties[2] completely opposed to the strict topology. For instance, the strict dual of l^∞ is l^1 which is well-known to be equivalent to the set of all countably additive measures. While the dual of l^∞

[1]It is defined by the family indexed by all sequences converging to zero: $p_a(x) = \sup_{k \geq 1} |a_k x_k|$ with $a_k \to 0$ as $k \to \infty$.

[2]For other non mathematical properties see, for instance, Lewis [7].

with respect to our topology is proven to be equivalent to the set of all purely finitely additive measures.

The paper is organized in the following way: Section 2 deals with the existence of hyperopic strict topologies. Section 3 defines hyperopic preferences and from them hyperopic topologies are defined and characterized in Section 4. In Section 5, we establish a representation theorem which allows us to prove that the hyperopic strict dual of l^∞ is the set of all bounded pure charges. Finally, in Section 6, we give a short conclusion.

2 Hyperopic strict topologies

Let $\Gamma = \{p_a : a \in \mathcal{A}\}$ be the family of semi-norms on l^∞ indexed by $\mathcal{A}$. Any $p_a \in \Gamma$ satisfies the following properties that define semi-norms

1. Positivity
$$p_a(x) \geq 0$$

2. Homogeneity
$$p_a(\alpha x) = \lim \sup_{N\to\infty} \frac{\sum_{n\geq N} |a_n(\alpha x_n|)}{\sum_{n\geq N} |a_n|} = |\alpha| p_a(x)$$

3. Triangle Inequality
$$p_a(x+y) = \lim \sup_{N\to\infty} \frac{\sum_{n\geq N} |a_n(x_n+y_n)|}{\sum_{n\geq N} |a_n|} \leq p_a(x) + p_a(y)$$

Definition 1. *The hyperopic strict topology β_h on l^∞ is the locally convex topology defined by the family Γ of semi-norms on l^∞.*

In what follows we will guarantee the existence of β_h, and will offer some of its properties.

2.1 Existence

Let $A \subset \mathcal{A}$ be a finite subset of $\mathcal{A}$ and $\epsilon > 0$ be a positive number real. For each $x \in l^\infty$, define $V_{A,\epsilon}(x)$ as

$$V_{A,\epsilon}(x) = \{y \in l^\infty : p_a(y - x) < \epsilon, a \in A\}$$

Define $\mathcal{V} : l^\infty \to 2^{l^\infty}$ so that for each $x \in l^\infty$, $\mathcal{V}(x)$ consists of all sets $V \subset l^\infty$ such that there exists $V_{A,\epsilon}(x) \subset V$. We have the following theorem.

Theorem 1. *Let $\mathcal{V}(x)$ be the collection of subsets defined above. Then the following holds:*

(1) The collection $\mathcal{V}(x)$ is a fundamental system of neighborhoods of $x \in l^\infty$.

(2) There is a unique topology β_h on l^∞ such that for each $x \in l^\infty$, $\mathcal{V}(x)$ is a fundamental system of neighborhoods of x. In addition, the members of β_h are characterized in the following way: $\mathcal{O}$ of l^∞ is open in β_h if and only if $\mathcal{O} = \cup_{arbitrary} V_{A,\epsilon}(x)$.

(3) Let $\{x^n\}$ be a sequence in l^∞. Then $x^n \to x$ (in β_h) if and only if $p_a(x^n - x) \to 0, \forall a \in \mathcal{A}$.

(4) (l^∞, β_h) is a topological vector space (TVS) which is locally convex.

(5) $\beta_h \subset \tau_\infty$.

Proof. We will prove these items separately.

Proof of (1). Clearly $\mathcal{V}(x)$ satisfies the following properties: a) $x \in V$, for all $V \in \mathcal{V}(x)$; b)If $V \in \mathcal{V}(x)$ and $V \subset W$, then $W \in \mathcal{V}(x)$; c) If $V, W \in \mathcal{V}(x)$, then $V \cap W \in \mathcal{V}(x)$ and d)

If $V \in \mathcal{V}(x)$, there exists $U \in \mathcal{V}(x)$ such that $y \in U$, then $V \in \mathcal{V}(y)$. These conditions define a fundamental system of neighborhoods of x.

Proof of (2). This item follows from Proposition 2, §1, $n^{\underline{o}}2$ of Bourbaki [4].

Proof of (3). For every $\mathcal{O} \in \beta_h$ containing x, there exists $n_o \in N$ such that $n > n_o$ implies that $x^n \in \mathcal{O}$. This in turn is equivalent to stating that $p_a(x^n - y) \to 0, \forall a \in \mathcal{A}$.

Proof of (4). The vector space operations are β_h-continuous. This follows from properties b) and c) of the definition of β_h. The local convexity of (l^∞, β_h) follows from the fact that each member, $V_{A,\epsilon}(x)$, of the generating family is convex.

Proof of (5). We will prove that any open set in β_h contains an open ball in τ_∞. It is sufficient to prove that any $V(x, \epsilon, A) = \{y \in l^\infty : \rho_a(y - x) < \epsilon, a \in A\} \supset B_\infty(x, \epsilon) = \{y \in l^\infty : \sup_k |y_k - x_k| < \frac{\epsilon}{2}\}$. Let $y \in B_\infty(x, \epsilon)$. This implies that $|y_k - x_k| < \frac{\epsilon}{2}, \forall k$. Computing $\rho_a(y - x)$ with $a \in A \subset \mathcal{A} \subset l^1$ with A finite, one has

$$\rho_a(y-x) = \lim \sup_{N \to \infty} \frac{\sum_{n \geq N} |a_n(y_n - x_n)|}{\sum_{n \geq N} |a_n|} \leq \frac{\epsilon}{2} \lim \sup_{N \to \infty} \frac{\sum_{n \geq N} |a_n|}{\sum_{n \geq N} |a_n|}$$

therefore $\rho_a(y - x) < \epsilon$. □

3 Hyperopic preferences

Myopia via preference relations on l^∞, which is nothing else but a complete transitive binary relation on l^∞, has been studied by Brown and Lewis [3]. By following the spirit of these authors, who studied myopic preferences, we define hyperopic preferences.

Definition 2. *A preference relation on l^∞ is a binary relation on l^∞ satisfying the following properties:*

1. *Complete:* $\forall x, y \in l^\infty,$ *either* $x \succeq y$ *or* $y \succeq x$

2. *Transitive:* $\forall x, y, z \in l^\infty, x \succeq y, y \succeq z \Rightarrow x \succeq z$

By $x \succ y$ we mean that $x \succeq y$ and $\neg(y \succeq x)$, and $x \sim y$ means $x \succeq y$ and $y \succeq x$.

For any $x \in l^\infty$, we define its $n-$head denoted by x_{hn} to be

$$x_{hn}(k) = \begin{cases} x_k, & 1 \leq k \leq n \\ 0, & k > n \end{cases}$$

and its $n-$tail as $x_n^t = x - x_{hn}$.

Now we are ready to define hyperopic preferences.

Definition 3. *The preference relation $\succeq$ on l^∞ is said to be hyperopic if and only if it satisfies the following condition.*

$$\forall x, y, z \in l_+^\infty, \text{ if } x \succ y \text{ then } x \succ y + z_{hn}, \forall n. \tag{1}$$

Definition 4. *Let $\succeq$ be a preference relation on l_+^∞.*

1. *$\succeq$ is said to be weakly hyperopic $\Leftrightarrow$ for all $x, y \in l_+^\infty$ and c a constant sequence in l_+^∞, if $x \succ y$ then there exists N such that $x \succ y + c_{hn}$ for all $n \geq N$.*

2. *$\succeq$ is said to be strongly hyperopic $\Leftrightarrow$ for all $x, y, z \in l_+^\infty$ if $x \succ y$ then there exists N such that $x \succ y + z_{hn}$ for all $n \geq N$.*

3. *$\succeq$ is said to be monotonically hyperopic $\Leftrightarrow$ for all $x, y \in l_+^\infty$, if $x \succ y$ then there exists N such that $x_n^t \succ y_n^t$ for all $n \geq N$, where $x_n^t = x - x_{hn}$.*

For any two $x, y \in l^\infty$ let us define $x \geq y$ if and only if $x_n \geq y_n, \forall n$. Thus, $\succeq$ is said to be monotonic in the classic sense if $x \geq y$ implies $x \succeq y$.

The following results follow directly from the definition.

Proposition 1.

1. *Hyperopia implies strong hyperopia which in turn implies weak hyperopia.*

2. *Under classic monotonicity, strong hyperopia and hyperopia are equivalent.*

Proof.

1. Item 1 directly follows from the definition.

2. Strong hyperopia implies that there exists N such that $x \succ y + z_{hn}, \forall n \geq N$. This fact together with the monotonicity, imply that for all $n \leq N, x \succ y + z_{hN} \succeq y + z_{hn}$.

□

The following definition is usual in the literature

Definition 5. *Let τ be a topology on l^∞. A preference relation $\succeq$ is said to be $\tau-$continuous if for all $x \in l^\infty_+$ the sets $\{y \in l^\infty_+ : y \succeq x\}$ and $\{y \in l^\infty_+ : x \succeq y\}$ are closed in the topology τ.*

Remark 1: If we only have that for all $x \in l^\infty_+$ the set $\{y \in l^\infty_+ : y \succeq x\}(\{y \in l^\infty_+ : x \succeq y\})$, then $\succeq$ would $\tau-$ upper semi continuous (lower semi continuous).

Example 1. *We consider preferences represented[3] by the utility function $u : l^\infty_+ \to R$ defined by*

$$u(x) = \limsup_{n\to\infty} x_n, \forall x \in l^\infty_+.$$

[3] $\succeq$ on X is represented by $u : X \to R$ if and only if $x \succeq y \Leftrightarrow u(x) \geq u(y)$.

We claim that this preference is β_h- upper semi continuous.

Let $x^n \to_{\beta_h} y$. Then, $p_a(x^n - y) \to 0, \forall a \in \mathcal{A}$. This implies that there exists N_o such that for all $N \geq N_o$ one has

$$\frac{\sum_{k\geq N} |a_k(x_k^n - y_k)|}{\sum_{k\geq N} |a_k|} \leq \epsilon, \forall n \geq n_o$$

Therefore,

$$|x_k^n - y_k| \leq \frac{||a||_1 \epsilon}{\inf\{a_k : a_k \neq 0\}}, \forall n \geq n_o$$

On the other hand, one has

$$u(x^n) - u(y) = \limsup_{k\to\infty} x_k^n - \limsup_{k\to\infty} y_k \leq \limsup_{k\to\infty} (x_k^n - y_k).$$

Then there exists a subsequence $(x_{k_m}^n - y_{k_m})$ of $(x_k^n - y_k)$ for which we have that

$$u(x^n) - u(y) \leq (x_{k_m}^n - y_{k_m}) < 2|x_{k_m}^n - y_{k_m}|, \forall m$$

Choose m_o such that $k_m \geq \max\{N_o, k_{m_o}\}$. Then,

$$2|x_{k_m}^n - y_{k_m}| \leq \frac{2||a||_1 \epsilon}{\inf\{a_{k_m} : a_{k_m} \neq 0\}}, \forall n \geq n_o$$

and therefore

$$u(x^n) - u(y) < \frac{2||a||_1 \epsilon}{\inf\{a_{k_m} : a_{k_m} \neq 0\}}, \forall n \geq n_o$$

Since ϵ is small enough, we have that

$$\limsup_{n\to\infty} u(x^n) \leq u(y)$$

and therefore the preference represented by u is β_h- upper semi continuous.

Remark 2: An argument similar to the one above shows that the preference relation represented by $u : l_+^\infty \to R$ defined by

$$u(x) = \liminf_{n \to \infty} x_n, \forall x \in l_+^\infty$$

is β_h-lower semi-continuous. That is, if $x^n \to_{\beta_h} y$, then $\liminf_{n\to\infty} u(x^n) \geq u(y)$.

4 Hyperopic topologies

Once the hyperopic preferences have been defined we are ready to define hyperopic topologies. We offer three degrees of hyperopia:

Definition 6. *A topology τ on l^∞ is said to be weakly (strongly) hyperopic if and only if every preference relation, $\succeq$, which is $\tau-$continuous, is weakly (strongly) hyperopic.*

The following proposition shows that β_h, the one defined by Γ, is strongly hyperopic.

Proposition 2. *The topology β_h is strongly hyperopic in the sense of Item 2 of Definition 4. This property justifies its name. Namely, hyperopic strict topology.*

Proof. We pick out any β_h-continuous $\succeq$ on l^∞ . Let $x, y, z \in l^\infty$. Suppose that $x \succ y$.

Computing $p_a(z_{hn})$ we have that $p_a(z_{hn}) = 0, \forall n$. This implies that

$$p_a(y + z_{hn} - y) = 0, \forall n, \tag{2}$$

and therefore

$$y + z_{hn} \to_{\beta_h} y \tag{3}$$

From the continuity of $\succeq$ one has that (3) implies that $x \succ y + z^n$, for any large enough n, implying strongly hyperopic $\succeq$. $\square$

Example 2. *The preference defined in Example 1 is strongly hyperopic. This follows directly from Proposition 2 and from the* β_h- *upper semi continuity of* $u(x) = \limsup_{n\to\infty} x_n, \forall x \in l_+^\infty$.

Remark 3: We can show that $u(x) = \limsup_{n\to\infty} x_n$ is strongly hyperopic without using the fact that it is β_h- upper semicontinuous. This follows from the following fact: for all $y, z \in l_+^\infty$ and for all n, one has $\limsup_{k\to\infty}(y_k + z_{hn}(k)) = \limsup_{k\to\infty} y_k$. That is, $u(y + z_{hn}) = u(y), \forall n$. Therefore, $u(x) > u(y)$ implies $u(x) > u(y + z_{hn}), \forall n$. Thus, from Item 2 of Proposition 1 the strong hyperopia of u follows since u is clearly monotonic.

Brown and Lewis [3] have characterized myopic topologies via convergence to zero of the sequence of the tails of any bounded sequence. Here, we obtain a characterization for hyperopic topologies, but instead of using the sequence of tails we use the sequence of the heads of any bounded sequence.

Theorem 2. *If* τ *is a Hausdorff locally convex topology on* l^∞ *then* τ *is strongly hyperopic if and only if for all* $z \in l^\infty$, $z_{hn} \to_\tau 0$

Proof. In a general way the proof of this theorem follows the proof of an theorem in Brown and Lewis [3] and may carry over to the present context with little changes. $\square$

Corollary 1. *Let* τ *be a Hausdorff locally convex topology on* l^∞ *which is strongly hyperopic. Then* τ *is monotonically hyperopic.*

Proof. Suppose that $x \succ y$. Strong hyperopia implies that $x_{hn} \to_\tau 0$ and $y_{hn} \to_\tau 0$. It then follows that $x_n^t = x - x_{hn} \to_\tau x$ and $y_n^t = y - y_{hn} \to_\tau y$. Hence, there exists n_o, such that $x_n^t \succ y_n^t$ for all $n \geq n_o$. □

5 The hyperopic strict dual of l^∞

We begin by characterizing the β_h-continuous linear functionals. That is, $(l^\infty, \beta_h)'$.

Lemma 1. *A linear functional $f : (l^\infty, \beta_h) \to R$ is β_h-continuous if and only if $f(x_{hn}) \to 0$.*

Proof. Lemma 1 readily follows from the fact that β_h is strongly hyperopic, see Theorem 2, and from the β_h- continuity of the linear functionals. □

A linear functional $f : l^\infty \to R$ is said to be countably additive if $f(x) = \sum_{k=1}^{\infty} b_k x_k$, for some $b = \{b_k\}_{K=1}^{\infty} \in l^1$.

Lemma 2. *A countably additive linear functional on l^∞ is β_h-continuous if and only if it is the zero functional.*

Proof. Let x be any element of l^∞ and f be a countably additive linear functional on l^∞ which is β_h- continuous. Thus,

$$f(x) = \sum_{k\geq 1} a_k x_k \text{ for some } a \in l^1.$$

For all n one has that

$$|f(x - x_{hn})| = |\sum_{k\geq n+1} a_k x_k| \leq \sum_{k\geq n+1} |a_k||x_k| \leq ||x||_\infty \sum_{k\geq n+1} |a_k|.$$

Since f is β_h- continuous and $a \in l^1$, then Lemma 1 implies that $f(x) = 0$. Because x is arbitrary, the functional f must be the zero functional. Thus, Lemma 2 follows.

□

5.1 A representation theorem

Any pure charge (positive) $\mu \in ba(N)$ induces a β_h- continuous linear functional (positive). So, let $\mu \in pa(N)$ be a pure charge. Define the following linear functional $H_\mu : (l^\infty, \beta_h) \to R$ to be

$$H_\mu(x) = D\int_N x(n)d\mu, \forall x \in l^\infty$$

where the integral on the right hand is known as the Dunford-Schwartz integral.

For any $x \in l^\infty$ the sequence of $n-$heads of x is defined to be

$$x_{hn}(m) = x(m), 1 \leq m \leq n, \text{ and } x_{hn}(m) = 0, m > n.$$

Thus

$$H_\mu(x_{hn}) = D\int_{N_n} x(n)d\mu \leq ||x||_\infty \mu(N_n), \text{ where } N_n = \{1, \ldots, n\}$$

From Theorem 3.2 and Theorem 5.7 (pages: 189 and 193 respectively) in Olubummo [9] , it follows that $\mu(N_n) = 0$. Therefore $H_\mu(x_{hn}) = 0$ since μ is positive. Therefore, by using Lemma 1 above, it follows that H_μ is β_h-continuous.

Hence, from the Jordan Decomposition theorem, see for instance Theorem 2.2.2(1) in Rao and Rao [10], we can then state that any $\mu \in pa(N)$ induces a β_h- continuous linear functional.

Theorem 3. *For all $F \in (l^\infty, \beta_h)'$, there exists a unique bounded pure charge, μ, such that*

$$F(x) = D\int_N x(n)d\mu$$

Proof. From Item 5 of Theorem 1 it follows that $F \in (l^\infty, \tau_\infty)' = ba(N)$. From Theorem 4.7.4 in Rao and Rao [10], one has that there exists a unique bounded charge $\mu \in ba(N)$ such that:

$$F(x) = D\int_N x(n)d\mu(n), \forall x \in l^\infty.$$

where the integral on the right hand is known as the Dunford-Schwartz integral.

From Theorem 1.23 in Yosida and Hewitt [12] there exists a unique decomposition for $\mu = \mu_c + \mu_p$ with μ_c countably additive and μ_p purely finitely additive. Thus

$$F(x) = D\int_N x(n)d\mu_c(n) + D\int_N x(n)d\mu_p(n).$$

Since $ca(N) \equiv l^1$, we have that there exists $a \in l^1$ defined by $a_n = \mu_c(n), \forall n$, such that the first integral can be written as a series. Thus F can be rewritten as:

$$F(x) = \sum_n^{\infty} x_n a_n + D\int_N x d\mu_p(n)$$

Lemma 2 implies that the only continuous linear functional $F_{\mu_c}(x) = \sum_n^{\infty} x_n a_n$ with respect to topology β_h is zero. Thus, $\mu_c = 0$. So,

$$F(x) = D\int_N x(n)d\mu_p(n)$$

Thus, we have shown that for any F defined on l^∞ which is linear and β_h-continuous, there is a unique pure charge μ which represents F. □

Now, the following corollary is straightforward.

Corollary 2.

$$(l^\infty, \beta_h)' = \{\mu \in ba(N) : \mu \textit{ is a bounded pure charge}\}$$

6 Concluding remarks

We have defined a new topology on l^∞ and have also characterized its dual. All results in this paper have been obtained without invoking any interpretation of any kind. Naturally, our results can be applied to capture behaviors of economic agents opposed to those obtained by Brown and Lewis [3] and Raut [11]. Interpretations, applications and extensions, for more general spaces, of our results will be subjects of other papers.

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