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Weighted Projective Planes and Irreducible Components of the Space of Foliations

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“La vida es una búsqueda de momentos de felicidad. ”

R.L. Monje

“ Valores não se trocam ... se conquistam! E a sorte só acompanha os audazes. Um fator determinante para o sucesso é a auto confiança. ... um fator determinante para a auto confiança é a preparação ”

D. Jesus

Resumo

Nosso trabalho será dividido em dois focos. O primeiro foco de estudo é a densidade de folheações sem soluções algébricas nos planos projetivos com pesos. Nós provamos que uma folheção genérica com grau do fibrado normal muito grande nos planos projetivos com pesos não possui solução algébrica.

É bem conhecido que as resoluçoes minimais de singularidades de um tipo "especial" de planos projetivos com pesos são as superfícies de Hirzebruch. Neste contexto, nós provamos que uma folheção genérica com bigrau do fibrado normal muito grande nas superfícies de Hirzebruch não possui solução algébrica.

O segundo foco de estudo é as componentes irredutíveis do espaço de folheações de codimensão um no espaço projetivo 3-dimensional. Nós construimos uma família de componentes irredutíveis associadas à uma álgebra de Lie afim.

Palavras chaves: Folheações nos planos projetivos com pesos. Superfícies de Hirzebruch. Componentes irredutíveis do espaço de folheações .

Abstract

Our work is divided into two focus. The first one is study the density of foliations without algebraic solutions in weighted projective planes. We prove that a generic foliation with very large degree of the normal bundle in weighted projective planes has no algebraic solution.

It is well known that the minimal resolutions of singularities of a “special” type of weighted projective planes are the Hirzebruch surfaces. In this context, we prove that a generic foliation with very large bidegree of normal bundle of Hirzebruch surfaces has no algebraic solution..

The second focus of study is the irreducible components of the space of holomorphic foliations of codimension one in 3-dimensional projective space. We construct a family of irreducible components associated with an affine Lie algebra.

Keywords: Foliations in weighted projective planes. Hirzebruch surfaces. Irreducible components of the space of foliations.

Contents

Introdução

O estudo das folheações holomorfas nos espaços projetivos complexos tem suas origens no século XIX com os trabalhos de G. Darboux, H. Poincaré e P. Painlevé. Muito tempo depois em 1970, J. P. Jouanolou reformolou e extendeu os trabalhos do Darboux [22] na linguagem da geometria algébrica fornecida por Grothendieck. Um dos trabalhos mais importantes sobre folheações nos espaços projetivos complexos encontra-se na célebre monografia de Jouanolou [29]. Tal trabalho desenvolveu duas linhas de pesquisa: densidade de folheações sem soluções algébricas e o problema das componentes irredutíveis do espaço de folheações. Nosso trabalho está focado nestas duas linhas de pesquisa. Vejamos um breve resumo histórico de cada uma delas.

Densidade das folheações sem soluções algébricas. Na teoria clássica de folheações holomorfas (ou equacões diferenciais) no plano projetivo complexo $\mathbb{P}^2$ introduz-se um invariante destas que é o chamado grau da folheação. O problema da densidade de folheações sem soluções algébricas em $\mathbb{P}^2$ foi originalmente tratado por Jouanolou em [29]. Neste trabalho foi mostrado que uma folheação genérica de grau pelo menos 2 em $\mathbb{P}^2$ não admite solução algébrica. Aqui genérico significa que o espaço das folheações que não tem curva algébrica invariante é o complemento de uma união contável de subconjuntos próprios algébricos fechados. A prova de Jouanolou é baseada na construção de exemplos, mais especificamente, ele mostra que as folheações de grau d induzidas pelos campos de vetores polinomiais

$$X = y^d \frac{\partial}{\partial x} + z^d \frac{\partial}{\partial y} + x^d \frac{\partial}{\partial z},$$

não tem solução algébrica se $d \geq 2$.

Uma generalização deste resultado foi obtida por Cerveau e Lins Neto em [12]. Eles mostraram que, para todo $d \geq 2$, existe um aberto e denso U no espaço das folheações de grau d, tal que toda folheação em U não tem solução algébrica.

Outras provas foram dadas por Zoladek [42], Ollagnier e Nowicki [34]. Para estes resultados existem versões em dimensão superior, por exemplo, veja os artigos de M. Soares [21], A. Lins-Neto e M. Soares [39], S. C. Coutinho e J. V. Pereira [17], Zoladek [41] e Bersntein-Lunts [7].

Neste trabalho serão abordadas as folheações nos planos projetivos complexos com pesos, as quais foram estudadas por Corrêa and Soares em [16]. Faremos uma generalização da definição do "grau" da folheação. Para tais folheações, o invariante é o chamado grau do normal da folheação, que é um inteiro canonicamente associado a esta. Primeiro, nosso estudo irá concentrar-se nas folheações nos planos projetivos complexos com pesos dos tipos (l_0, l_1, l_2), $1 \leq l_0 \leq l_1 \leq l_2$, l_i coprimos dois a dois. É possivel perguntar se o resultado de Jouanolou [29] ainda é válido para folheações nos planos projetivos com pesos destes tipos. A resposta é sim, e é um dos resultados principais desta tese.

Teorema 1. *Uma folheação genérica com $\mathbb{Q}$-fibrado normal de grau d em $\mathbb{P}^2_{(l_0,l_1,l_2)}$, l_i coprimos dois a dois, $1 \leq l_0 \leq l_1 \leq l_2$, não possui curva algébrica invariante se $d \gg 0$.*

Na verdade, a cota obtida é $d = l_0 l_1 l_2 + l_0 l_1 + 2l_2$. Os resultados não são ótimos no sentido que nós podemos achar exemplos onde o teorema acima ainda é verdadeiro para valores fora das hipóteses.

No caso particular de folheações nos planos projetivos com pesos tipos $(1, 1, l_2)$, $l_2 \geq 2$ temos um resultado mais preciso para estas perguntas.

Teorema 2. *Uma folheação genérica com $\mathbb{Q}$-fibrado normal de grau d em $\mathbb{P}^2_{(1,1,l_2)}$, $l_2 \geq 2$, não possui curva algébrica invariante se $d \geq 2l_2 + 1$.*

O cota acima é ótima no sentido que toda folheação com $\mathbb{Q}$-fibrado normal de grau menor do que $2l_2 + 1$ admite uma reta invariante.

O próximo objeto de estudo são as folheações nas superfícies de Hirzebruch. É bem conhecido que a resolução minimal do planos projetivos com pesos $\mathbb{P}^2_{(1,1,l_2)}$, $l_2 \geq 2$, são as superfícies de Hirzebruch $\mathbb{F}_{l_2} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$ (ver [36]). O invariante destas é o chamado bigrau, que é um par ordenado (a, b) de números inteiros canonicamente associado à folheação. Nosso próximo resultado é uma nova generalização do teorema de Jouanolou [29] para folheações nas superfícies de Hirzebruch.

Teorema 3. *Uma folheação holomorfa de bigrau (a, b) genérica em $\mathbb{F}_{l_2}$ não possui solução algébrica se $a \geq bl_2 + 2$ e $b \geq 3$.*

O resultado acima também é o melhor, pois toda folheação de bigrau (a, b) com $b < 3$ ou $a < bl_2+2$ admite uma curva invariante. Cabe mencionar, pelo dito anteriormente que a resolução minimal de $\mathbb{P}^2_{(1,1,l_2)}$ é $\mathbb{F}_{l_2}$, nós poderíamos pensar que o Teorema 2 implique o Teorema 3, e vice-versa. Isso não é verdade pelo seguinte fato: As folheações em $\mathbb{F}_{l_2}$ cuja seção excepcional de $\mathbb{F}_{l_2}$, (isto é, a curva com auto interseção $-l_2$.) é invariante traduz-se via o mapa de resolução como uma condição aberta nas folheações em $\mathbb{P}^2_{(1,1,l_2)}$, contrariamente as folheações em $\mathbb{F}_{l_2}$ que não são invariantes pela seção excepcional interpreta-se como uma condição fechada nas folheações em $\mathbb{P}^2_{(1,1,l_2)}$. Isto justifica que os dois problemas são totalmente diferentes.

Componentes irredutíveis do espaço de folheações. O segundo foco do nosso trabalho são as folheações holomorfas de codimensão um em espaços projetivos de dimensão maior ou igual a três. É bem conhecido que o espaço de folheações holomorfas de codimensão um e grau k em $\mathbb{P}^n$, $n \geq 3$, denotado por $\mathbb{F}ol(k, n)$ é um conjunto algébrico que tem decomposição em componentes irredutíveis. Em [29], Jouanolou mostra que $\mathbb{F}ol(0, n)$ tem uma só componente irredutível e $\mathbb{F}ol(1, n)$ tem duas componentes irredutíveis. Em 1996, D. Cerveau e Lins Neto [13], mostraram que $\mathbb{F}ol(2, n)$ tem seis componentes irredutveis. Outra nova prova desse resultado foi obtida por Loray, Touzet e Pereira em [33]. Para $k \geq 3$, ainda está aberto o problema das componentes. Existem construções de famílias de componentes irredutíveis, por exemplo, veja os artigos de Calvo-Andrade, Cerveau, Giraldo e Lins Neto [11], Cukierman e Pereira [20], Calvo-Andrade [10].

No trabalho de Calvo-Andrade, Cerveau, Giraldo e Lins Neto [11], eles constroem famílias de componentes irredutíveis associadas à uma álgebra de Lie afim. Eles introduziram o seguinte conceito:

Para $1 \leq r < p < q$ inteiros positivos com $\mathrm{mdc}(p, q, r) = 1$, considere o campo vetorial linear em $\mathbb{C}^3$

$$S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z},$$

e suponha que exista um outro campo vetorial polinomial X em $\mathbb{C}^3$ tal que $[S, X] = \lambda X$, para algum $\lambda \in \mathbb{Z}$. Denotemos por $\mathcal{F}(S, X)$ a folheação induzida pela forma $\omega = i_S i_X dx \wedge dy \wedge dz$. Eles definem

$$\mathbb{F}ol((p, q, r), \lambda, \nu) = \{\mathcal{F} \in \mathbb{F}ol(\nu, 3) |\ \mathcal{F} = \mathcal{F}(S, X) \text{ em alguma carta afim}\}.$$

E mostram o seguinte:

Teorema (Calvo et al., '04)**.** *Seja $d \geq 1$ um inteiro. Então*

$$\overline{\mathbb{F}ol((d^2 + d + 1, d + 1, 1), -1, d + 1)}$$

é uma componente irredutiveil de $\mathbb{F}ol(d + 1, 3)$.

O objetivo de nosso trabalho é construir novas famílias de componentes irredutíveis. O segundo resultado principal é o teorema a seguir:

Teorema 4. *Se $l_0 > l_1$, $\mathrm{mdc}(l_0, l_1) = 1$, $l_0 \geq 3$ e $q \geq 1$, então*

$$\overline{\mathbb{F}ol((l_0, l_1, l_1), l_1(ql_0 - 1), ql_0 + 1)}$$

é uma componente irredutível de $\mathbb{F}ol(ql_0 + 1, 3)$ e

$$\overline{\mathbb{F}ol((l_0, l_1, l_1), l_0 l_1 q, ql_0 + 2)}$$

é uma componente irredutível de $\mathbb{F}ol(ql_0 + 2, 3)$.

Esta tese está dividida em três capítulos. No Capítulo 1 nos concentraremos em definir os espaços projetivos com pesos e as folheações nestes espaços. Além disso, fixaremos as notações que serão utilizadas nos Capítulos seguintes.

O Capítulo 2 é voltado para a densidade das folheações sem soluções algébricas. Serão provados os Teoremas 1, 2 e 3.

O Capítulo 3 é dedicado ao problema das componentes irredutíveis do espaço das folheações. E probaremos o Teorema 4.

Introduction

The study of holomorphic foliations in complex projective spaces has its origins in the 19th century with the works of G. Darboux, H. Poincaré and P. Painlevé. In the late 1970's, J. P. Jouanolou reformulated and extended the work of Darboux [22] in the algebraic geometry framework provided by Grothendieck. One of the most important works about holomorphic foliations in complex projective spaces is found in Jouanolou's celebrated monograph [29]. Such a work developed two lines of research: the density of algebraic foliations without algebraic solutions and the problem concerning irreducible components of the space of holomorphic foliations. Our work is focused in this two lines of research. The reader can see a brief summary of each one of them bellow.

Density of algebraic foliations without algebraic solutions: In the classic theory of holomorphic foliations or differential equations in the complex projective plane $\mathbb{P}^2$, it is introduced an invariant which is called the degree of the foliation. The issue of the density of algebraic foliations without algebraic solutions in $\mathbb{P}^2$ was originally proved by Jouanolou in [29]. In this work it was proved that a generic foliation of degree at least 2 does not admit any algebraic solution. By generic we mean that the set of foliations that does not have any invariant curve is the complement of a countable union of algebraic closed proper subsets. The Jouanolou's proof is based in the construction of the examples, more specifically, he has showed that the foliations of degree d that are induced by the polynomial vector fields

$$X = y^d \frac{\partial}{\partial x} + z^d \frac{\partial}{\partial y} + x^d \frac{\partial}{\partial z},$$

have no algebraic solutions if $d \geq 2$.

A generalization of this result was obtained by Cerveau and Lins Neto in [12]. They have showed that, for all $d \geq 2$, there exists an open and dense subset U in the space of foliations of degree d, such a foliation in U has no algebraic solutions.

Other proofs have been given by Zoladek [42], Ollagnier and Nowicki [34].For this results there are versions in dimension greater than 2, for example, see the papers of M. Soares [21], A. Lins-Neto and M. Soares [39], S. C. Coutinho and J. V. Pereira [17], Zoladek [41], and Bersntein-Lunts [7].

In this work it will be discussed the foliations in the weighted projective planes. We will consider a generalization of the definition of "degree" of the foliation. In this case such an invariant is called the normal degree of the foliation, which is an integer canonically associated to the foliation. Our study will focus on foliations in the weighted projective planes of types (l_0, l_1, l_2), $1 \leq l_0 \leq l_1 \leq l_2$, l_i pairwise coprimes. Hence, one can ask naturally if the Jouanolou's result is still true for foliations in the weighted projective planes of these types. In this work we provide a positive answer to this question.

Theorem 1. *A generic foliation with normal $\mathbb{Q}$-bundle of degree d in $\mathbb{P}^2_{(l_0,l_1,l_2)}$, l_0, l_1, l_2 pairwise coprimes, $1 \leq l_0 \leq l_1 \leq l_2$, does not admit any invariant algebraic curve if $d \gg 0$.*

Actually, the bound is $d = l_0 l_1 l_2 + l_0 l_1 + 2l_2$. The results are not sharp in the sense that we can find where the above theorem is still true for values outside of the hypothesis.

In the particular case of foliations on the weighted projective planes of types $(1, 1, l_2)$, $l_2 \geq 2$, we have a more accurate answer to these questions.

Theorem 2. *A generic foliation with normal $\mathbb{Q}$-bundle of degree d in $\mathbb{P}^2_{(1,1,l_2)}$ with $l_2 \geq 2$ has no algebraic solutions if $d \geq 2l_2 + 1$.*

The above result is sharp in the sense that every foliation with $\mathbb{Q}$-bundle normal of degree less than $2l_2 + 1$ admits an invariant line.

A second issue of our study are the foliations on the Hirzebruch surfaces. It is well known that the minimal resolution of the weighted projective planes $\mathbb{P}^2_{(1,1,l_2)}$, $l_2 \geq 2$, are the Hirzebruch surfaces $\mathbb{F}_{l_2} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$ (see [36]). The invariant of the foliations on Hirzebruch surfaces is called the bidegree, which is an ordered pair (a, b) of integers numbers canonically associated to them. Our next result is a generalized version of Jouanolou's Theorem for foliations in Hirzebruch surfaces.

Theorem 3. *A generic foliation with normal bundle of bidegree (a, b) in $\mathbb{F}_{l_2}$ does not admit any invariant algebraic curve if $b \geq 3$ and $a \geq bl_2 + 2$.*

The above result is also sharp, because every foliation of bidegree (a, b) with $b < 3$ or $a < bl_2 + 2$ admits an invariant curve. It should be noted, by the previously discussion the minimal resolution of $\mathbb{P}^2_{(1,1,l_2)}$ is $\mathbb{F}_{l_2}$, as such we might think that Theorem 2 implies Theorem 3 and vice versa. This is not true by the following fact: The foliations in $\mathbb{F}_{l_2}$ in which the excepcional section of $\mathbb{F}_{l_2}$, (*i.e.*, the unique curve with selfintersection $-l_2$,) is invariant by the foliations, are translated via the resolution map as a open condition for the foliations in $\mathbb{P}^2_{(1,1,l_2)}$, conversely the foliations in which excepcional section of $\mathbb{F}_{l_2}$ is not invariant by them, are translated as a closed condition for the foliations in $\mathbb{P}^2_{(1,1,l_2)}$. This justifies why the two problems are different.

Irreducible components of the space of holomorphic foliations: The second focus are the holomorphic codimension one foliations on complex projective spaces of dimension greater or equal than 3. It is known that the space of holomorphic codimension one foliations of degree k on $\mathbb{P}^n$, $n \geq 3$, which is denoted by $\mathbb{F}ol(k, n)$, is an algebraic set. Therefore, it has an unique decomposition into irreducible components. The second problem is to determinate the irreducible components of the space of foliations. In [29], Jouanolou shows that $\mathbb{F}ol(0, n)$ has only one irreducible component and $\mathbb{F}ol(1, n)$ has two irreducible components. In 1996, D. Cervau and Lins Neto [13], showed that $\mathbb{F}ol(2, n)$ has six irreducible components. Another new proof of this result was obtained by Loray, Touzet e Pereira in [33]. For $k \geq 3$, the problem of the components is still open. There are constructions of families of irreducible components, for example, Calvo-Andrade, Cerveau, Giraldo and Lins Neto [11], Cukierman and Pereira [20], Calvo-Andrade [10]. In the work of Calvo-Andrade, Cerveau, Giraldo and Lins Neto [11], they have constructed families of the irreducible components associated to the affine Lie algebra. They have introduced the following definition. Let $1 \leq p < q < r$ be are positive integers with $\gcd(p, q, r) = 1$. Consider the linear vector field on $\mathbb{C}^3$

$$S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}.$$

Suppose that there is another polynomial vector field X on $\mathbb{C}^3$ such that $[S, X] = \lambda X$, for some $\lambda \in \mathbb{Z}$. Denoted by $\mathcal{F}(S, X)$, the foliation on $\mathbb{C}^3$ induced by the 1-form $\Omega = i_S i_X(dx \wedge dy \wedge dz)$, which is associated to a representation of the affine algebra of polynomial vector fields in $\mathbb{C}^3$. It can be extended to a foliation on $\mathbb{P}^3$ of certain degree ν. They define

$$\mathbb{F}ol((p, q, r), \lambda, \nu) := \{\mathcal{F} \in \mathbb{F}ol(\nu, 3) | \, \mathcal{F} = \mathcal{F}(S, X) \text{ in some affine chart}\},$$

And they prove the following:

Theorem (Calvo et al., '04). *Let $d \geq 1$ be an integer. There is an N-dimensional irreducible component*

$$\overline{\mathbb{F}ol((d^2 + d + 1, d + 1, 1), -1, d + 1)},$$

of the space $\mathbb{F}ol(d + 1, 3)$ whose general point corresponds to a GK Klein-Lie foliation with exactly one quasi-homogeneous singularity, where $N = 13$ if $d = 1$ and $N = 14$ if $d > 1$. Moreover, this component is the closure of a $\mathbb{P}GL(4, \mathbb{C})$ orbit on $\mathbb{F}ol(d + 1, 3)$.

The goal of our work is to construct new families of irreducible components. The third result is the theorem bellow.

Theorem 4. *If $l_0 > l_1$, $\gcd(l_0, l_1) = 1$, $l_0 \geq 3$ and $q \geq 1$, then*

$$\overline{\mathbb{F}ol((l_0, l_1, l_1), l_1(ql_0 - 1), ql_0 + 1)},$$

is an irreducible component of $\mathbb{F}ol(ql_0 + 1, 3)$ and

$$\overline{\mathbb{F}ol((l_0, l_1, l_1), l_0 l_1 q, ql_0 + 2)},$$

is an irreducible component of $\mathbb{F}ol(ql_0 + 2, 3)$.

This work is organized in three chapters.

In Chapter 1, we focus on the definition of the weighted projective spaces and the codimension one foliations on weighted projective spaces. In addition, we fix the notations that will be used in the following chapters.

Chapter 2 is devoted to the density of algebraic foliations without algebraic solutions on weighted projective spaces. Theorem 1, Theorem 2 and Theorem 3 are proved in this Chapter.

Chapter 3 is concerned with the problem of the irreducible components of holomorphic foliations. We prove Theorem 4.

Chapter 1

Weighted Projective Spaces

In this first chapter, we introduce the weighted projective spaces and present the basic definitions and properties we shall need in the sequel. See [2], [5], [19] and [23] for more details. Once we understand these spaces, we will talk about holomorphic foliations on weighted projective spaces and stablish some index theorems. The main reference for the last part is [9].

Theorem 1.0.1. *let*

1.1 Definition of weighted projective space

Let $\ell = (l_0, l_1, \ldots, l_n)$ be a vector of positive integers which is called *a weight vector*, and set

$$|\ell| = l_0 + \cdots + l_n.$$

Then there is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\backslash\{0\}$ given by

$$(x_0, \ldots, x_n) \mapsto (t^{l_0}x_0, \ldots, t^{l_n}x_n), \quad \text{for all } t \in \mathbb{C}^*.$$

The set of orbits $\frac{\mathbb{C}^{n+1}\backslash\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^n_\ell$ (or $\mathbb{P}^n(\ell)$) and is called the *weighted projective space* of type ℓ. It comes equipped with a natural quotient topology: a subset $U \subseteq \mathbb{P}^n_\ell$ is open iff $U = V/\mathbb{C}^*$ for some $\mathbb{C}^*$-invariant open subset of $\mathbb{C}^{n+1}\backslash\{0\}$.

The class of a nonzero element $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ is denoted by $[x_0 : \ldots : x_n]_\ell$ and the weight vector can be omitted if no ambiguity can arise.

For $x \in \mathbb{C}^{n+1}\backslash\{0\}$, the closure of $[x]_\ell$ in $\mathbb{C}^{n+1}$ is obtained by adding the origin and it is an algebraic curve.

Remark 1.1.1. When $\ell = (l_0, \ldots, l_n) = (1, \ldots, 1)$ one obtains the usual complex projective spaces and the weight vector will always be omitted.

1.2 Analytic structure

As in the classical case, weighted projective spaces can be endowed with an analytic structure. However, in general they contain cyclic quotient singularities. To understand this structure, consider the decomposition $\mathbb{P}^n_\ell = U_0 \cup \ldots \cup U_n$, where U_i is the open set consisting of the elements

$[x_0 : \ldots : x_i : \ldots : x_n]_\ell$ with $x_i \neq 0$. Let μ_k be the cyclic group of k-th roots of unity in $\mathbb{C}$. Consider the map

$$\begin{array}{rcl} \tilde{\psi}_i : \qquad \mathbb{C}^n & \to & U_i, \\ (y_1, \ldots, y_n) & \mapsto & [y_1 : \ldots : y_i : 1 : y_{i+1} : \ldots : y_n]_\ell \,. \end{array}$$

It is a surjective map but it is not a chart, since injectivity fails. In fact, $[y_1 : \ldots : y_i : 1 : y_{i+1} : 1 : y_n]_\ell = [y'_1 : \ldots : y'_i : 1 : y'_{i+1} : \ldots : y'_n]_\ell$ if and only if there exists $g_i \in \mu_{l_i}$ such that $y'_j = g_i^{l_{j-1}} y_j$, for all $j = 1, \ldots, i$ and $y'_j = g_i^{l_j} y_j$, for all $j = i+1, \ldots, n$. Hence the above map induces the bijection

$$\begin{array}{rcl} \psi_i : \mathbb{C}^n/\mu_{l_i} & \to & U_i, \\ [(y_1, \ldots, y_n)] & \mapsto & [y_1 : \ldots : y_i : 1 : y_{i+1} : \ldots : y_n]_\ell, \end{array}$$

where $\mathbb{C}^n/\mu_{l_i}$ is the quotient of $\mathbb{C}^n$ by the action

$$\begin{array}{rcl} \mu_{l_i} \times \mathbb{C}^n & \to & \mathbb{C}^n, \\ (g_i, (y_1, \ldots, y_n)) & \mapsto & (g_i^{l_0} y_1, \ldots, g_i^{l_{i-1}} y_i, g_i^{l_{i+1}} y_{i+1}, \ldots, g_i^{l_n} y_n). \end{array}$$

For $i < j$:

$$\begin{array}{rcl} \psi_j^{-1} \circ \psi_i : \psi_i^{-1}(U_i \cap U_j) \subseteq \mathbb{C}^n/\mu_{l_i} & \to & \psi_j^{-1}(U_i \cap U_j) \subseteq \mathbb{C}^n/\mu_{l_j}, \\ [(y_1, \ldots, y_n)] & \mapsto & \left[\left(\frac{y_1}{y_j^{l_1/l_j}}, \ldots, \frac{y_i}{y_j^{l_i/l_j}}, \frac{1}{y_j^{1/l_j}}, \frac{y_{i+1}}{y_j^{l_{i+1}/l_j}}, \ldots, \frac{y_{j-1}}{y_j^{l_{j-1}/l_j}}, \ldots, \frac{y_n}{x_j^{l_n/l_j}} \right) \right]. \end{array}$$

Since the *transition* maps are analytic, $\mathbb{P}^n_\ell$ is an analytic space with cyclic quotient singularities as claimed.

1.3 Interpretation

The weighted projective space can be seen as a quotient of $\mathbb{P}^n$ by a group acting on it. For $r \in \mathbb{N}$, let μ_r be the finite cyclic group of r-th roots of unity in $\mathbb{C}$, and $G_\ell = \mu_{l_0} \times \cdots \times \mu_{l_n}$ be the product of cyclic groups. Consider the action of the group G_ℓ on $\mathbb{P}^n$ given as follows

$$\begin{array}{rcl} G_\ell \times \mathbb{P}^n & \to & \mathbb{P}^n, \\ (g, [x_0 : \ldots : x_n]) & \mapsto & [g_0 x_0 : \ldots : g_n x_n], \end{array}$$

where $g = (g_0, \ldots, g_n) \in G_\ell$.

For every $g \in G_\ell$, the map $[x_0 : \ldots : x_n] \to [g_0 x_0 : \ldots : g_n x_n]$ is an automorphism of $\mathbb{P}^n$. We will also denote it by g. The set of all orbits $\mathbb{P}^n/G_\ell$ is isomorphic to the weighted projective space of type ℓ and the isomorphism is induced by a surjective natural morphism

$$\begin{array}{rcl} \varphi_\ell : \mathbb{P}^n & \to & \mathbb{P}^n_\ell, \\ [x_0 : \ldots : x_n] & \mapsto & [x_0^{l_0} : \ldots : x_n^{l_n}]_\ell \,, \end{array}$$

This is a branched covering, unramified over

$$\mathbb{P}^n_\ell \setminus \{[x_0 : \ldots : x_n]_\ell \mid \quad x_0 x_1 \ldots x_n = 0\},$$

and has $\bar{\ell} = lcm(l_0, \ldots, l_n)$ sheets. Moreover, the covering respects the coordinate axes.

Example 1.3.1. $\mathbb{P}^2_{(1,1,l_2)}$, $l_2 \geq 2$ is the cone over the rational curve of degree l_2 in $\mathbb{P}^{l_2}$ given by the following embedding

$$\begin{array}{rcl} \mathbb{P}^2_{(1,1,l_2)} & \to & \mathbb{P}^{l_2}, \\ [x_0 : x_1 : x_2]_{(1,1,l_2)} & \mapsto & [x_0^{l_2} : x_0^{l_2-1}x_1 : \ldots : x_1^{l_2} : x_2]. \end{array}$$

This surface is obtained by blowing down the exceptional section of the ruled surface $\mathbb{F}_{l_2} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$, a Hirzebruch surface, see [36, page 29]. In §1.8 we will check this.

Example 1.3.2. For $\ell = (1, l_1, \ldots, l_n)$ the space $\mathbb{P}^n_{(1,l_1,\ldots,l_n)}$ is a compactification of the affine space $\mathbb{C}^n$, that is, the open set U_0 is isomorphic to $\mathbb{C}^n$. Its complement coincides with the weighted projective space $\mathbb{P}^{n-1}_{(l_1,\ldots,l_n)}$.

Example 1.3.3. $\mathbb{P}^2_{(1,2,3)}$ is covered by 3 open sets $U_0 \simeq \mathbb{C}^2$, $U_1 \simeq \mathbb{C}^2/\mu_2$ induced by the action

$$\begin{array}{rcl} \mu_2 \times \mathbb{C}^2 & \to & \mathbb{C}^2, \\ (g_2, (x, y)) & \mapsto & (g_2 x, g_2 y), \end{array}$$

and $U_2 \simeq \mathbb{C}^2/\mu_3$ induced by the action

$$\begin{array}{rcl} \mu_3 \times \mathbb{C}^2 & \to & \mathbb{C}^2, \\ (g_3, (x, y)) & \mapsto & (g_3 x, g_3^2 y). \end{array}$$

As in the Example 1.3.1 we can find an embedding in $\mathbb{P}^6$. It suffices to take

$$\begin{array}{rcl} \mathbb{P}^2_{(1,2,3)} & \to & \mathbb{P}^6, \\ [x_0 : x_1 : x_2]_{(1,2,3)} & \mapsto & [x_0^6 : x_0^4 x_1 : x_0^3 x_2 : x_0^2 x_1^2 : x_0 x_1 x_2 : x_1^3 : x_2^2]. \end{array}$$

1.4 Quasi-homogeneous polynomials and divisors on $\mathbb{P}^n_\ell$

The main reference for this and the next section are [23] and [19, Chapter 4].

Definition 1.4.1. Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial and $\ell = (l_0, \ldots, l_n)$ be a weight vector.

1. The polynomial F is said to be a *quasi-homogeneous polynomial* of degree d if for all $t \in \mathbb{C}^*$ we have
$$F(t^{l_0} x_0, \ldots, t^{l_n} x_n) = t^d F(x_0, \ldots, x_n).$$
Note that if $l_0 = \ldots = l_n = 1$ then we are in the case of standard homogeneous polynomial.

2. A *rational function* on $\mathbb{P}^n_\ell$ is a quotient of quasi-homogeneous polynomials of the same degree, and the *field of rational functions* of $\mathbb{P}^n_\ell$ is denoted by $\mathbb{C}(\mathbb{P}^n_\ell)$.

3. An irreducible subvariety of codimension one on $\mathbb{P}^n_\ell$ is the set of zeros of an irreducible quasi-homogeneous polynomial.

Remark 1.4.2. The definition of the degree given above induces a natural grading of the coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$ and this ring, considered as a graded ring, is denoted by $S(\ell)$, that is

$$S(\ell) = \bigoplus_{d \geq 0} S(\ell)_d,$$

where $S(\ell)_d$ denotes the vector space of the quasi-homogeneous polynomials of degree d.

Definition 1.4.3. 1. $Div(\mathbb{P}^n_\ell)$ is the free abelian group generated by the irreducible subvarieties of codimension one on $\mathbb{P}^n_\ell$. A *Weil divisor* is an element of $Div(\mathbb{P}^n_\ell)$.

2. The divisor of $f \in \mathbb{C}(\mathbb{P}^n_\ell)^*$ is

$$div(f) = \sum_{D \in Div(\mathbb{P}^n_\ell)} ord_D(f)D,$$

where $ord_D : \mathbb{C}(\mathbb{P}^n_\ell)^* \to \mathbb{Z}$ is a discrete valuation, see [19, page 155].

3. $div(f)$ is called a *principal divisor*, and the set of all principal divisors is denoted by $Div_0(\mathbb{P}^n_\ell)$.

4. A Weil divisor D on $\mathbb{P}^n_\ell$ is *Cartier* if it is locally principal, meaning that $\mathbb{P}^n_\ell$ has an open cover $\{U_i\}_{i\in I}$ such that $D|_{U_i}$ is principal in U_i for every $i \in I$. It follows that the Cartier divisors on $\mathbb{P}^n_\ell$ form a group $CDiv(\mathbb{P}^n_\ell)$ satisfying

$$Div_0(\mathbb{P}^n_\ell) \subseteq CDiv(\mathbb{P}^n_\ell) \subseteq Div(\mathbb{P}^n_\ell).$$

5. The *class group* of $\mathbb{P}^n_\ell$ is

$$Cl(\mathbb{P}^n_\ell) = Div(\mathbb{P}^n_\ell)/Div_0(\mathbb{P}^n_\ell),$$

and the *Picard group* of $\mathbb{P}^n_\ell$ is

$$Pic(\mathbb{P}^n_\ell) = CDiv(\mathbb{P}^n_\ell)/Div_0(\mathbb{P}^n_\ell).$$

Theorem 1.4.4. *If* $\gcd(l_0, l_1, \ldots, l_n) = 1$. *Then the following assertions hold true.*

1. *The natural map*

$$\begin{aligned} \deg : Cl(\mathbb{P}^n_\ell) &\to \mathbb{Z}, \\ [\{F = 0\}] &\mapsto \deg(F), \end{aligned}$$

is an isomorphism. Furthermore the natural inclusion $Pic(\mathbb{P}^n_\ell) \subset Cl(\mathbb{P}^n_\ell)$ *induces the following isomorphism*

$$deg|_{Pic(\mathbb{P}^n_\ell)} : Pic(\mathbb{P}^n_\ell) \to m\mathbb{Z},$$

where $m = lcm(l_0, \ldots, l_n)$.

2. *The linear map*

$$Pic(\mathbb{P}^n_\ell) \otimes_{\mathbb{Z}} \mathbb{Q} \to Cl(\mathbb{P}^n_\ell) \otimes_{\mathbb{Z}} \mathbb{Q},$$

is an isomorphism of $\mathbb{Q}$*-vector spaces. In particular, for a given Weil divisor* D *on* $\mathbb{P}^n_\ell$ *there always exists* $k \in \mathbb{Z}$ *such that* $kD \in Pic(\mathbb{P}^n_\ell)$. *In this case it is said that* D *is a* $\mathbb{Q}$*-Cartier divisor or* $\mathbb{Q}$*-bundle.*

Proof. See [19, page 76 and page 188] and [1, Theorem 4.12]. □

Definition 1.4.5. Let $D \in Div(\mathbb{P}^n_\ell)$. We define the *sheaf* $\mathcal{O}_{\mathbb{P}^n_\ell}(D)$ *of a Weil divisor* D as

$$U \mapsto \mathcal{O}_{\mathbb{P}^n_\ell}(D)(U) = \{f \in \mathbb{C}(\mathbb{P}^n_\ell)^* | \, (div(f) + D)|_U \geq 0\} \cup \{0\},$$

where U is any open subset of $\mathbb{P}^n_\ell$.

Note that $\mathcal{O}_{\mathbb{P}^n_\ell}(D) \simeq \mathcal{O}_{\mathbb{P}^n_\ell}(D')$ if $\deg(D) = \deg(D')$. Denote $\mathcal{O}_{\mathbb{P}^n_\ell}(\deg(D)) := \mathcal{O}_{\mathbb{P}^n_\ell}(D)$.

The following theorem relates the global sections of a Weil divisor with the quasi-homogeneous polynomials. One can find it in [23, page 39].

Theorem 1.4.6. *For every $d \in \mathbb{N}$ the following assertions hold true.*

1. $H^0(\mathbb{P}^n_\ell, \mathcal{O}_{\mathbb{P}^n_\ell}(d)) \simeq S(\ell)_d$.
2. $H^i(\mathbb{P}^n_\ell, \mathcal{O}_{\mathbb{P}^n_\ell}(d)) = 0$ *for* $i \neq 0, n$.

Thus we can identify $H^0(\mathbb{P}^n_\ell, \mathcal{O}_{\mathbb{P}^n_\ell}(d))$ as the vector space of the quasi-homogeneous polynomials of degree d. It will convenient to set

$$h^0(n, d, \ell) := \dim_{\mathbb{C}} H^0(\mathbb{P}^n_\ell, \mathcal{O}_{\mathbb{P}^n_\ell}(d)).$$

Consider the map

$$\begin{array}{rcl} \tilde{\varphi}_\ell : \mathbb{C}^{n+1} & \to & \mathbb{C}^{n+1}, \\ (x_0, \dots, x_n) & \mapsto & (x_0^{l_0}, \dots, x_n^{l_n}). \end{array}$$

It induces a branched covering

$$\begin{array}{rcl} \varphi_\ell : \mathbb{P}^n & \to & \mathbb{P}^n_\ell, \\ [x_0 : \dots : x_n] & \mapsto & [x_0^{l_0} : \dots : x_n^{l_n}]_\ell. \end{array}$$

If $F \in S(\ell)_d$, then

1. $\varphi_\ell^*(F) := F \circ \tilde{\varphi}_\ell$ is a homogeneous polynomial of degree d.
2. $g^*(F \circ \tilde{\varphi}_\ell) = F \circ \tilde{\varphi}_\ell, \quad \forall\, g \in G_\ell$.

Therefore, it is natural to give the following definition.

Definition 1.4.7. Let $\tilde{F} \in \mathbb{C}[x_0, \dots, x_n]$ be a polynomial and $\ell = (l_0, \dots, l_n)$ be a weight vector.

1. The polynomial $\tilde{F}$ is said to be *G_ℓ-invariant homogeneous polynomial* of degree d if it is homogeneous of degree d and $g^*\tilde{F} = \tilde{F}$, $\forall\, g \in G_\ell$.
2. The vector space of the G_ℓ-invariant homogeneous polynomials of degree d is denoted by $S_d^{G_\ell}$.

Lemma 1.4.8. *The natural homomorphism*

$$\begin{array}{rcl} \varphi_\ell^* : S(\ell)_d & \to & S_d^{G_\ell}, \\ F & \mapsto & F \circ \varphi_\ell\,, \end{array}$$

is a linear isomorphism $\forall\, d \geq 0$, $\forall\, n \geq 1$. In particular we have the natural isomorphisms

$$H^0(\mathbb{P}^n_\ell, \mathcal{O}_{\mathbb{P}^n_\ell}(d)) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^{G_\ell},$$

where $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^{G_\ell}$ refers to the image of $S_d^{G_\ell}$ by the isomorphism given in the Theorem 1.4.6 item 1 with $\ell = (1, \dots, 1)$.

Proof. The application φ_ℓ^* is clearly linear and injective. To see that it is surjective it is enough to prove it for monomials. Let $F = x_0^{a_0} \dots x_n^{a_n}$ be a monomial such that $g^*F = F, \quad \forall\, g \in G_\ell$, that is

$$g_0^{a_0} \dots g_n^{a_n} x_0^{a_0} \dots x_n^{a_n} = x_0^{a_0} \dots x_n^{a_n},$$

so $g_0^{a_0} \dots g_n^{a_n} = 1, \quad \forall\, g \in G_\ell$, and then $a_i = m_i l_i$. Hence we can define $\tilde{F} = y_0^{m_0} \dots y_n^{m_n}$ such that $\varphi^*\tilde{F} = F$. This concludes the proof of the lemma. $\square$

1.4.1 Well-formed weighted projective spaces

For different weight vectors ℓ and ℓ' the corresponding weighted projective spaces $\mathbb{P}^n_\ell$ and $\mathbb{P}^n_{\ell'}$ can be isomorphic. Let

$$\begin{aligned} d_i &= \gcd(l_0,\dots,l_{i-1},l_{i+1},\dots,l_n),\\ a_i &= lcm(d_0,\dots,d_{i-1},d_{i+1},\dots,d_n),\\ a &= lcm(d_0,\dots,d_n). \end{aligned}$$

Note that $a_i|a$, $\gcd(a_i,d_i)=1$, $\gcd(d_i,d_j)=1$ for $i\neq j$ and $a_id_i=a$.

Proposition 1.4.9. *Let $\ell'=(l'_0,\dots,l'_n)=(\frac{l_0}{a_0},\dots,\frac{l_n}{a_n})$. The natural morphism*

$$\begin{array}{cccc} \varphi: & \mathbb{P}^n_\ell & \to & \mathbb{P}^n_{\ell'},\\ & [x_0:\ldots:x_n]_\ell & \mapsto & [x_0^{d_0}:\ldots:x_n^{d_n}]_{\ell'}, \end{array} \tag{1.1}$$

is an algebraic isomorphism.

Proof. See [23, page 37] or [1, Proposition 2.3]. □

Remark 1.4.10. The isomorphism (1.1) induces isomorphisms

$$\begin{array}{rcl} \varphi^*: H^0(\mathbb{P}^n_{\ell'},\mathcal{O}_{\mathbb{P}^n_{\ell'}}(d)) & \to & H^0(\mathbb{P}^n_\ell,\mathcal{O}_{\mathbb{P}^n_\ell}(ad)),\\ F & \mapsto & F\circ\varphi, \end{array}$$

for all $d\geq 0$.

Remark 1.4.11. In the case $n=2$, if $\gcd(l_0,l_1,l_2)=1$, then l'_0,l'_1,l'_2 are two-by-two coprimes obtained from Proposition 1.4.9.

1.5 Quasi-homogeneous k-forms on $\mathbb{P}^n_\ell$

To define foliations on weighted projective spaces we will need quasi-homogeneous 1-forms. Let us start to study in a more general way the quasi-homogeneous k-forms. As we shall see there is a natural identification between quasi-homogeneous k-forms and homogeneous k-forms invariant under a group. This construction was motived by [23, Chapter 2].

Definition 1.5.1. Let η be a polynomial k-form on $\mathbb{C}^{n+1}$ and $\ell=(l_0,\dots,l_n)$ be a weight vector. The k-form η is said to be *quasi-homogeneous k-form* of degree d if

$$\psi_t^*\eta=t^d\eta,\quad \forall t\in\mathbb{C}^*,$$

where $\psi_t(x_0,\dots,x_n)=(t^{l_0}x_0,\dots,t^{l_n}x_n)$.

A polynomial vector field X on $\mathbb{C}^{n+1}$ is called *quasi-homogeneous vector field* of degree r if

$$(\psi_t)_*X=t^{|\ell|-r}X\circ\psi_t.$$

Remark 1.5.2. It follows from the definition that $\deg(dx_{i_1}\wedge\ldots\wedge dx_{i_k})=l_{i_1}+\cdots+l_{i_k}$. Moreover if

$$\eta=\sum_{0\leq i_1<\cdots<i_k\leq n}F_{i_1,\dots,i_k}(x_0,\dots,x_n)dx_{i_1}\wedge\ldots\wedge dx_{i_k},$$

is a quasi-homogeneous k-form of degree d, then

$$\psi_t^*\eta = \sum_{0\leq i_1<\cdots<i_k\leq n} F_{i_1,\ldots,i_k}(t^{l_0}x_0,\ldots,t^{l_n}x_n)t^{l_{i_1}+\ldots l_{i_k}}dx_{i_1}\wedge\ldots\wedge dx_{i_k} = t^d\eta,$$

therefore,

$$F_{i_1,\ldots,i_k}(t^{l_0}x_0,\ldots,t^{l_n}x_n) = t^{d-(l_{i_1}+\cdots+l_{i_k})}F_{i_1,\ldots,i_k}(x_0,\ldots,x_n), \quad \forall t\in\mathbb{C}^*.$$

Briefly, $\deg(\eta)=d$ if only if the polynomials $F_{i_1,\ldots,i_k}$ are quasi-homogeneous polynomials of degree $d-(l_{i_1}+\cdots+l_{i_k})$.

The definition of the degree given above induces a natural $S(\ell)$-grading in the free module of polynomials k-forms and this $S(\ell)$-module considered as graded $S(\ell)$-module is denoted by $\Omega^k_{S(\ell)}$, that is

$$\Omega^k_{S(\ell)} = \bigoplus_{d\geq 0}\Omega^k_{S(\ell)}(d),$$

where $\Omega^k_{S(\ell)}(d)$ denotes the vector space of quasi-homogeneous polynomial k-forms of degree d.

According to our convention $\deg(\frac{\partial}{\partial x_i}) = \sum_{j\neq i} l_j$. Furthermore, if

$$X = \sum_{i=0}^n A_i\frac{\partial}{\partial x_i},$$

is a quasi-homogeneous vector field of degree r, hence

$$(\psi_t)_*X = \sum_{i=0} t^{l_i}A_i(x_0,\ldots,x_n)\frac{\partial}{\partial y_i} = t^{|\ell|-r}\sum_{i=0}^n A_i(t^{l_0}x_0,\ldots,t^{l_n}x_n)\frac{\partial}{\partial y_i},$$

thus,

$$A_i(t^{l_0}x_0,\ldots,t^{l_n}x_n) = t^{r-\sum_{j\neq i}l_j}A_i(x_0,\ldots,x_n),\ \forall t\in\mathbb{C}^*.$$

That is, $\deg(X)=r$ if and only if the polynomials A_i are quasi-homogeneous polynomials of degree $r-\sum_{j\neq i}l_j$.

Exterior derivative. Let

$$\begin{aligned} d: S(\ell) &\longrightarrow \Omega^1_{S(\ell)}, \\ F &\longmapsto \sum_{i=0}^n \frac{\partial F}{\partial x_i}dx_j. \end{aligned}$$

be the canonical universal differentiation. The k-linear map d extends to the exterior differentiation

$$d:\Omega^k_S\to\Omega^{k+1}_S,$$

determined uniquely by the condition

$$d(\eta\wedge\eta') \quad = \quad d\eta\wedge\eta' + (-1)^k\eta\wedge d\eta', \quad \eta\in\Omega^k_{S(\ell)}, \quad \eta'\in\Omega^j_{S(\ell)},$$

We also have an analogue of *Euler's formula*:

$$mF = \sum_{i=0}^n \frac{\partial F}{\partial x_i}l_ix_j, \quad \forall F\in S(\ell)_m.$$

Using the linearity of both sides of this identity we need to verify this formula only in the case when F is a monomial $x_0^{i_0}\dots x_n^{i_n}$. But in this case it can be done without any difficulties.

Interior product is the homomorphism of graded $S(\ell)$-modules defined as follows. If $X = \sum_{j=0}^{n} A_j \frac{\partial}{\partial x_j}$ is a quasi-homogeneous vector field of degree r, interior product with X is

$$i_X : \Omega^k_{S(\ell)} \to \Omega^{k-1}_{S(\ell)},\ k \geq 1,$$

defined by the formula

$$i_X(dx_{j_1} \wedge \ldots \wedge dx_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1} A_{j_i} dx_{j_1} \wedge \ldots \wedge \hat{dx_{j_i}} \wedge \ldots \wedge dx_{j_k},$$

In the case $R = \sum_{i=0}^{n} l_i x_i \frac{\partial}{\partial x_i}$ we have the following properties:

1. $i_R(dF) = mF$, $F \in S(\ell)_m$,

2. $i_R(d\eta) + d(i_R(\eta)) = m\eta$, $\eta \in \Omega^k_{S(\ell)}(m)$.

Using the branched covering map

$$\begin{array}{rcl} \varphi_\ell : \mathbb{P}^n & \to & \mathbb{P}^n_\ell, \\ [x_0 : \cdots : x_n] & \mapsto & [x_0^{l_0} : \cdots : x_n^{l_n}]_\ell. \end{array}$$

induced by the map

$$\begin{array}{rcl} \tilde{\varphi}_\ell : \mathbb{C}^{n+1} & \to & \mathbb{C}^{n+1}, \\ (x_0, \ldots, x_n) & \mapsto & (x_0^{l_0}, \ldots, x_n^{l_n}), \end{array}$$

we see that if $\eta \in \Omega^k_{S(\ell)}(d)$ then

1. $\varphi_\ell^*(\eta) := \tilde{\varphi}_\ell^*(\eta)$ is a homogeneous polynomial k-form of degree d on $\mathbb{C}^{n+1}$.

2. $g^*(\varphi_\ell^*\eta) = \varphi_\ell^*\eta, \quad \forall\, g \in G_\ell = \mu_{l_0} \times \ldots \times \mu_{l_n}$.

Therefore, it is natural to give the following definition.

Definition 1.5.3. Let ω be a homogeneous k-form ω of degree d on $\mathbb{C}^{n+1}$, i.e., $\omega \in \Omega^k_S(d)$.

1. The 1-form ω is called *G_ℓ-invariant* if $g^*\omega = \omega$, for all $g \in G_\ell$.

2. The vector space of the G_ℓ-invariant homogeneous k-forms of degree d is denoted by $\Omega^k_S(d)^{G_\ell}$.

Lemma 1.5.4. *Under above conditions. The natural homomorphism*

$$\begin{array}{rcl} \varphi_\ell^* : \Omega^k_{S(\ell)}(d) & \to & \Omega^k_S(d)^{G_\ell}, \\ \eta & \mapsto & \varphi_\ell^*\eta, \end{array}$$

is a linear isomorphism $\forall\, d \geq 0$, $\forall\, k \geq 0$, $\forall\, n \geq 1$.

Proof. φ_ℓ^* is clearly linear and injective. To see that it is surjective, by linearity of φ_ℓ^*, we can assume

$$\omega = (\prod_{i\neq 0,\ldots,k} x_{j_i}^{a_i})x_{j_0}^{a_0-1}\ldots x_{j_k}^{a_k-1}dx_{j_0}\wedge\ldots\wedge dx_{j_k},$$

such that $g^*\omega=\omega, \quad \forall\, g\in G_\ell$. This implies that

$$\prod_i g_{j_i}^{a_i}(\prod_{i\neq 0,\ldots,k} x_{j_i}^{a_i})x_{j_0}^{a_0-1}\ldots x_{j_k}^{a_k-1}dx_{j_0}\wedge\ldots\wedge dx_{j_k}=\omega,$$

therefore, $\prod_{i=1\ldots n} g_{j_i}^{a_i}=1$, and hence $a_i=m_i l_{j_i}$, $\forall\, i$. Thus we define

$$\eta = (\prod_{i=0,\ldots,k} y_{j_i}^{m_i})y_{j_0}^{m_0-1}\ldots y_{j_k}^{m_k-1}\frac{dy_{j_0}}{l_{j_0}}\wedge\ldots\wedge\frac{dy_{j_k}}{l_{j_k}},$$

and we can conclude that $\varphi^*\eta=\omega$. □

Notice that the following sequence

$$0\longrightarrow\Omega^{n+1}_{S(\ell)}\xrightarrow{i_R}\Omega^n_{S(\ell)}\xrightarrow{i_R}\cdots\xrightarrow{i_R}\Omega^0_{S(\ell)}\longrightarrow 0\ ,$$

is exact, see [23, page 44].

Definition 1.5.5. Let $d\geq 0$ be an integer. The *twisted sheaf of Zariski k-forms* $\Omega^k_{\mathbb{P}^n_\ell}(d)$ is defined as follow

$$\Omega^k_{\mathbb{P}^n_\ell}(d)(U_i)=\left\{\frac{\eta}{x_i^j}\mid \eta\in\ker\{i_R:\Omega^k_{S(\ell)}(d+jl_i))\to\Omega^{k-1}_{S(\ell)}(d+jl_i)\}\right\}.$$

If $k=n$ and $d=0$, the sheaf $\Omega^n_{\mathbb{P}^n_\ell}$ which is called *canonical sheaf* of $\mathbb{P}^n_\ell$.

Proposition 1.5.6. *Under the above conditions we have*

$$H^0(\mathbb{P}^n_\ell,\Omega^k_{\mathbb{P}^n_\ell}(d))\simeq\ker\{i_R:\Omega^k_{S(\ell)}(d)\to\Omega^{k-1}_{S(\ell)}(d)\}. \tag{1.2}$$

Furthermore, $j_*(\Omega^k_{\mathbb{P}^n_\ell\backslash Sing(\mathbb{P}^n_\ell)})=\Omega^k_{\mathbb{P}^n_\ell}$*, where* $j:\mathbb{P}^n_\ell\backslash Sing(\mathbb{P}^n_\ell)\to\mathbb{P}^n_\ell$ *is the natural inclusion.*

Proof. See [23, 2.1.5, page 44 and 2.2.4, page 47]. □

By Lemma 1.5.4 and the commutativity of the following diagram

$$\begin{array}{ccc}\Omega^k_{S(\ell)}(d) & \xrightarrow{i_R} & \Omega^{k-1}_{S(\ell)}(d)\\ \downarrow{\scriptstyle\varphi_\ell^*} & & \downarrow{\scriptstyle\varphi_\ell^*}\\ \Omega^k_S(d) & \xrightarrow{i_{R_0}} & \Omega^{k-1}_S(d)\end{array}$$

in which $R=\sum_{i=0}^n l_i y_i\frac{\partial}{\partial y_i}$, $R_0=\sum_{i=0}^n x_i\frac{\partial}{\partial x_i}$, we have the following isomorphism

$$\ker\{i_R:\Omega^k_{S(\ell)}(d)\to\Omega^{k-1}_{S(\ell)}(d)\}\xrightarrow{\varphi_\ell^*}\ker\{i_{R_0}:\Omega^k_S(d)^{G_\ell}\to\Omega^{k-1}_S(d)\}. \tag{1.3}$$

Furthermore

$$H^0(\mathbb{P}^n_\ell,\Omega^k_{\mathbb{P}^n_\ell}(d))\simeq H^0(\mathbb{P}^n,\Omega^k_{\mathbb{P}^n}(d))^{G_\ell},$$

where $H^0(\mathbb{P}^n,\Omega^k_{\mathbb{P}^n}(d))^{G_\ell}\subset H^0(\mathbb{P}^n,\Omega^k_{\mathbb{P}^n}(d))$ refers to the image of $\ker\{i_{R_0}:\Omega^k_S(d)^{G_\ell}\to\Omega^{k-1}_S(d)\}$ given by the isomorphism (1.2).

Remark 1.5.7. In [4, Theorem 12.1., page 38], Batyrev and Cox constructs an exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n_\ell} \longrightarrow \sum_{i=0}^n \mathcal{O}_{\mathbb{P}^n_\ell}(-l_i) \xrightarrow{i_R} \mathcal{O}_{\mathbb{P}^n_\ell} \longrightarrow 0 \ ,$$

called the *generalized Euler exact sequence of* $\mathbb{P}^n_\ell$.

Proposition 1.5.8. *If* $\gcd(l_0,\ldots,l_n) = 1$, *then* $\Omega^n_{\mathbb{P}^n_\ell}$, *the canonical sheaf of* $\mathbb{P}^n_\ell$, *is isomorphic to* $\mathcal{O}_{\mathbb{P}^n_\ell}(-|\ell|)$. *That is, the canonical sheaf is the sheaf associated to a Weil divisor of degree* $-|\ell|$ *on* $\mathbb{P}^n_\ell$.

Proof. See [19, Theorem 8.2.3] □

1.6 Foliations on Weighted Projective Spaces

The purpose of this section is to extend the definition of codimension one foliations on usual projective spaces to the weighted projective spaces. It is worth noting that M. Corrêa and M.G. Soares already studied the subject in [16].

Definition 1.6.1. A *codimension one foliation* $\mathcal{F}$ on $\mathbb{P}^n_\ell$ is given by $\eta \in H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(d))$ such that

1. $\eta \wedge d\eta = 0$ (*Integrability condition*),
2. $\operatorname{codim}(Sing(\eta)) \geq 2$, where $Sing(\eta) = \{p \in \mathbb{P}^n_\ell \,|\, \eta(p) = 0\}$.

Remark 1.6.2. If η_1 and η_2 satisfies condition 2 and define $\mathcal{F}$ then $\eta_1 = \lambda\eta_2$, for some $\lambda \in \mathbb{C}^*$.

Remark 1.6.3. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n_\ell$ given by the quasi-homogeneous 1-form of degree d

$$\eta = \sum_{i=0}^n A_i(x_0,\ldots,x_n)dx_i,$$

in which $\sum_{i=0}^n l_i x_i A_i = 0$.

In the open set $U_0 \simeq \mathbb{C}^n/\mu_{l_0}$, using the map $\tilde{\psi}_0(y_1,\ldots,y_n) = (1 : y_1 : \cdots : y_n)$ we lift $\mathcal{F}|_{U_0}$ to $\mathbb{C}^n$ where it is given by

$$\eta_0 = \tilde{\psi}_0^*\eta = \sum_{i=1}^n A_i(1,y_1,\ldots,y_n)dy_i.$$

Note that

$$\lambda_{g_0}^*\eta_0 = \lambda(g_0)\eta_0, \ \forall g_0 \in \mu_{l_0},$$

where $\lambda_{g_0}(y_1,\ldots,y_n) = (g_0^{l_1}y_1,\ldots,g_0^{l_n}y_n)$ and $\lambda(g_0) = g_0^d$. In particular

$$\lambda(g_0) = g_0^\alpha, \quad d \equiv \alpha \,(\mod l_0).$$

Reciprocally, if η is a polynomial 1-form on $\mathbb{C}[y_1,\ldots,y_n]$ such that

$$\lambda_{g_0}^*\eta = g_0^\alpha\eta, \ \alpha \in \mathbb{Z}/l_0\mathbb{Z}. \tag{1.4}$$

The 1-form polynomial η can be write as

$$\eta = \eta_k + \eta_{k+1} + \cdots + \eta_d,$$

in which η_j are quasi-homogeneous 1-forms of degree j with respect to the weight vector $(l_1, \dots, l_n)$ and $\eta_k \neq 0$, $\eta_d \neq 0$. By condition (1.4) we have that

$$\eta = \eta_k + \eta_{k+l_0} + \cdots + \eta_d,$$

where $k \equiv \alpha(\mod l_0)$. Hence the foliation on $\mathbb{C}^n$ induced by η extends to a foliation on $\mathbb{P}^n_\ell$ induced by a quasi-homogeneous 1-form of degree

$$\begin{cases} d & \text{, if } i_S\eta_d = 0, \\ d + l_0 & \text{, if } i_S\eta_d \neq 0, \end{cases}$$

where $S = l_1 y_1 \frac{\partial}{\partial y_1} + \cdots + l_n y_n \frac{\partial}{\partial y_n}$. This remark will be used in the next chapter.

Using the change of coordinates we have that

$$\eta_j = \frac{x_0^{d/l_0}}{x_j^{d/l_j}} \eta_0, \text{ in } U_j \cap U_0,\ j > 0,$$

where $g_{j0} = \left(\frac{x_0^{1/l_0}}{x_j^{1/l_j}} \right)^d$ and g_{j0} is multiplicative cocycle that defines the following $\mathbb{Q}$-bundle

$$N\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_\ell}(d),$$

called *the normal* $\mathbb{Q}$*-bundle* of the foliation $\mathcal{F}$.

The space of codimension one foliations with normal $\mathbb{Q}$-bundle of degree d on $\mathbb{P}^n_\ell$ is denoted by $\mathbb{F}ol(d, n, \ell) \subseteq \mathbb{P}H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(d))$.

Note that in the case $\ell = (1, \dots, 1)$ the $d = \deg(\eta)$ denotes the degree of the normal bundle of the foliation $\mathcal{F}$, *i.e.*, $\deg(N\mathcal{F}) = d$, and we will denote $\mathbb{F}ol(d, n) := \mathbb{F}ol(d, n, (1, \dots, 1))$. The degree of normal bundle should not be confused with the degree of the foliation on $\mathbb{P}^n$ (number of tangencies with general line). For codimension one foliations on $\mathbb{P}^n$ we have

$$\deg(N\mathcal{F}) = \deg(\mathcal{F}) + 2.$$

Remark 1.6.4. For the case of $n = 2$. Let $\mathcal{F}$ be a codimension one foliation with normal $\mathbb{Q}$-bundle of degree d on $\mathbb{P}^2_\ell$. Since we have the 3-form $dx_0 \wedge dx_1 \wedge dx_2$ in $\mathbb{C}^3$, then the foliation $\mathcal{F}$ is given by a quasi-homogeneous vector field X which is not a multiple of $R = l_0 x_0 \frac{\partial}{\partial x_0} + l_1 x_1 \frac{\partial}{\partial x_1} + l_2 x_2 \frac{\partial}{\partial x_2}$,

$$X = B_0 \frac{\partial}{\partial x_0} + B_1 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2},$$

where B_0, B_1, B_2 are quasi-homogeneous polynomials of degree $d - l_1 - l_2$, $d - l_0 - l_2$, $d - l_0 - l_1$ respectively. The foliation $\mathcal{F}$ is also induced by quasi-homogeneous 1-form of degree d

$$\eta = i_X i_R (dx_0 \wedge dx_1 \wedge dx_2) = (l_1 x_1 B_2 - l_2 x_2 B_1) dx_0 + (l_2 x_2 B_0 - l_0 x_0 B_2) dx_1 + (l_0 x_0 B_1 - l_1 x_1 B_0) dx_2.$$

As we saw in the case of forms, it is not difficult to see $(\psi_0^{-1})_*(X|_{U_0}) = x_0^{\frac{d-|\ell|}{l_0}} X_0$, where

$$X_0 = (B_1(1, y_1, y_2) - \frac{l_1}{l_0} y_1 B_0(1, y_1, y_2)) \frac{\partial}{\partial y_1} + (B_2(1, y_1, y_2) - \frac{l_2}{l_0} y_2 B_0(1, y_1, y_2)) \frac{\partial}{\partial y_2},$$

and

$$X_j = \frac{x_0^{\frac{d-|\ell|}{l_0}}}{x_j^{\frac{d-|\ell|}{l_j}}} X_0, \quad j > 0 \text{ in } U_{j0},$$

where $f_{j0} = \left(\frac{x_0^{1/l_0}}{x_j^{1/l_j}} \right)^{d-|\ell|}$ and f_{j0} is multiplicative cocycle that defines the following $\mathbb{Q}$-bundle

$$K\mathcal{F} = \mathcal{O}_{\mathbb{P}^2_\ell}(d - |\ell|),$$

called *the cotangent or canonical* $\mathbb{Q}$*-bundle* of the foliation $\mathcal{F}$.

Note that canonical and normal $\mathbb{Q}$-bundle of $\mathcal{F}$ are related by the formula

$$K_{\mathbb{P}^2_\ell} = K\mathcal{F} \otimes N^*\mathcal{F},$$

where $N^*\mathcal{F}$ is dual sheaf of $N\mathcal{F}$ called the *conormal* $\mathbb{Q}$*-bundle* of $\mathcal{F}$.

Remark 1.6.5. In [16], Corrêa and Soares introduce the notion of degree of a foliation $\mathcal{F}$ on weighted projective plane which verifies the following relation:

$$\deg(N\mathcal{F}) = l_0 l_1 l_2 \deg(\mathcal{F}) + |l| - 1.$$

Example 1.6.6. Consider the foliation $\mathcal{F}$ with normal $\mathbb{Q}$-bundle of degree 7 on $\mathbb{P}^2_{(1,2,3)}$ given by

$$\eta = 6x_2(x_2 - x_0x_1)dx_0 + 3x_2(x_0^2 - x_1)dx_1 + 2(x_1^2 - x_0x_2)dx_2.$$

The singular set of $\mathcal{F}$ is

$$sing(\mathcal{F}) = \{[1:0:0], [1:1:1]\}.$$

In the open set $U_0 \simeq \mathbb{C}^2$, the foliation $\mathcal{F}|_{U_0}$ is given by

$$\eta_0 = 3y_2(1 - y_1)dy_1 + 2(y_1^2 - y_2)dy_2.$$

In the open set $U_1 \simeq \mathbb{C}^2/\mu_2$, using the map $\tilde{\psi}_1(u_1, u_2) = [u_1 : 1 : u_2]$ we lift $\mathcal{F}|_{U_1}$ to $\mathbb{C}^2$ which is given by

$$\eta_1 = 6u_2(u_2 - u_1)du_1 + 2(1 - u_1u_2)du_2.$$

In the open set $U_2 \simeq \mathbb{C}^2/\mu_3$, using the map $\tilde{\psi}_2(v_1, v_2) = [v_1 : v_2 : 1]$ we lift $\mathcal{F}|_{U_2}$ to $\mathbb{C}^2$ which is given by

$$\eta_2 = 6(1 - v_1v_2)dv_1 + 3(v_1^2 - v_2)dv_2.$$

Note that $\{x_2 = 0\}$ is $\mathcal{F}$-invariant and using the

$$\begin{array}{rcl} \varphi_{(1,2,3)} : \mathbb{P}^2 & \to & \mathbb{P}^2_{(1,2,3)} \\ {[x:y:z]} & \mapsto & [x : y^2 : z^3], \end{array}$$

we have

$$\varphi^*_{(1,2,3)}\eta = z^2(6z(z^3 - xy^2)dx + 6yz^3(x^2 - y^2)dy + 6(y^4 - xz^3)dz).$$

As we can see $codim\, Sing(\varphi^*_{(1,2,3)}\eta) = 1$, then $\varphi^*_{(1,2,3)}\eta$ induces a foliation on $\mathbb{P}^2$ with normal degree five as we need to divide $\varphi^*\eta$ by z^2 in order to obtain a 1-form with singular set of codimension 2.

1.6.1 Foliations on $\mathbb{P}^n_\ell$ and G_ℓ-invariant foliations on $\mathbb{P}^n$

As we saw in the previous section there is a correspondence between quasi-homogeneous 1-forms and G_l-invariant homogeneous 1-forms. In a similar way there is one such correspondence for foliations. It is important to see that a G_ℓ-invariant homogeneous 1-form ω has in general a singular set of codimension one, see Example 1.6.6. Note that $\varphi_\ell : \mathbb{P}^n_\ell \to \mathbb{P}^n$ is unramified covering over $\mathbb{P}^n \backslash \{x_0 \dots x_n = 0\}$, therefore if $codimSing(\omega) = 1$, then

$$\omega = \frac{\omega'}{x_0^{i_0} \dots x_n^{i_n}},$$

where $codim\, Sing(\omega') \geq 2$. Because of that we have the following definitions.

Definition 1.6.7. Fix $\ell = (l_0, \dots, l_n)$, let $I_n = \{1, \dots, n\}$ and denote by $I_n' = \{i \in I_n \mid \quad l_i > 1\}$.

1. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n_\ell$ given by η, for $i = 0, \dots, n$, the hypersurface $\{x_i = 0\}$ is $\mathcal{F}$-invariant if there exists a quasi-homogeneous 2-form θ_i such that

$$\eta \wedge dx_i = x_i \theta_i.$$

2. A foliation $\mathcal{G}$ on $\mathbb{P}^n$ given by ω with $\{x_{i_0} \dots x_{i_k} = 0\}$ invariant by $\mathcal{G}$ is called G_ℓ-*invariant* if

$$g^*\mathcal{G} = \mathcal{G},\ \forall\, g \in G_\ell = \mu_{l_0} \times \dots \times \mu_{l_n},\ \textit{i.e.},$$

$$g^*\omega = \lambda(g)\omega,\ \forall\, g \in G_\ell,$$

where $\lambda(g) = g_{i_0} \dots g_{i_k}$.

3. We define

$$\mathbb{F}ol_{i_0,\cdots,i_k}(d,n,\ell) = \Big\{\ \mathcal{F} \in \mathbb{F}ol(d,n,\ell) \mid \{x_{i_0} \cdots x_{i_k} = 0\} \text{ is } \mathcal{F}\text{-invariant}\ \Big\},$$

$$\mathbb{F}ol'_{i_0,\cdots,i_k}(d,n,\ell) = \left\{ \begin{array}{ll} \mathcal{F} \in \mathbb{F}ol(d,n,\ell) \mid & \{x_{i_0} \cdots x_{i_k} = 0\} \text{ is } \mathcal{F}\text{-invariant}, \\ & \{x_{i_j} = 0\} \text{ is not } \mathcal{F}\text{-invariant}, \\ & \forall\, i_j \in I_n' - \{i_0, \cdots, i_k\} \end{array} \right\}.$$

In the case of $\ell = (1, \dots, 1)$, we recall that

$$\mathbb{F}ol(d,n) = \mathbb{F}ol(d,n,\ell),$$

$$\mathbb{F}ol_{i_0,\cdots,i_k}(d,n) = \mathbb{F}ol_{i_0,\cdots,i_k}(d,n,\ell),$$

$$\mathbb{F}ol'_{i_0,\cdots,i_k}(d,n) = \mathbb{F}ol'_{i_0,\cdots,i_k}(d,n,\ell).$$

4. For foliations on $\mathbb{P}^n$

$$\mathbb{F}ol_{i_0,\cdots,i_k}(d,n)^{G_\ell} = \Big\{\ \mathcal{G} \in \mathbb{F}ol(d,n) \mid \{x_{i_0} \cdots x_{i_k} = 0\} \text{ is } \mathcal{G}\text{-invariant and } \mathcal{G} \text{ is } G_\ell\text{-invariant.}\ \Big\},$$

$$\mathbb{F}ol'_{i_0,\cdots,i_k}(d,n)^{G_\ell} = \mathbb{F}ol'_{i_0,\cdots,i_k}(d,n) \cap \mathbb{F}ol_{i_0,\cdots,i_k}(d,n)^{G_\ell}.$$

Remark 1.6.8. The natural map

$$\begin{array}{rcl}\varphi_\ell : \mathbb{P}^n & \to & \mathbb{P}^n_\ell \\ [x_0 : \cdots : x_n] & \mapsto & [y_0 : \cdots : y_n]_\ell = [x_0^{l_0} : \cdots : x_n^{l_n}]_\ell,\end{array}$$

induces the natural isomorphism

$$\varphi_\ell^* : H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(d)) \to H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))^{G_\ell}. \tag{1.5}$$

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^n_\ell$ induced by η. Observe that if $\{y_i = 0\}$ is $\mathcal{F}$-invariant then we can write

$$\eta = A_i dy_i + y_i(\sum_{j\neq i} l_i A_j dy_j),$$

in which $A_i = -\sum_{j\neq i} l_j y_j A_j$. Therefore

$$\begin{array}{rcl}\varphi_\ell^*\eta & = & x_i^{l_i-1} A_i \circ \varphi_\ell dx_i + x_i^{l_i}(\sum_{j\neq i} l_i x_j^{l_j-1} A_j \circ \varphi_\ell dx_j), \\ & = & x_i^{l_i-1}(A_i \circ \varphi_\ell dx_i + x_i(\sum_{j\neq i} l_i x_j^{l_j-1} A_j \circ \varphi_\ell dx_j)).\end{array}$$

Reciprocally if $\{y_i = 0\}$ is not $\mathcal{F}$-invariant, then $\{x_i = 0\}$ is not invariant by the foliation induced by $\varphi^*\eta$ on $\mathbb{P}^n$. This last and the isomorphism (1.5) induce the natural morphism

$$\begin{array}{rcl}\varphi_\ell^* : \mathbb{F}ol'_{i_0,\cdots,i_k}(d,n,\ell) & \to & \mathbb{F}ol'_{i_0,\cdots,i_k}(d+k+1-l_{i_0}-\cdots-l_{i_k},n)^{G_\ell} \\ [\eta] & \mapsto & \left[\dfrac{\varphi_\ell^*\eta}{x_{i_0}^{l_{i_0}-1}\cdots x_{i_k}^{l_{i_k}-1}}\right]\end{array} \tag{1.6}$$

Proposition 1.6.9. *The induced natural morphism (1.6) is an isomorphism. In particular, if* $I'_n - \{i_0,\cdots,i_k\} = \emptyset$*, then the induced natural morphism*

$$\begin{array}{rcl}\varphi_\ell^* : \mathbb{F}ol_{i_0,\cdots,i_k}(d,n,\ell) & \to & \mathbb{F}ol_{i_0,\cdots,i_k}(d+k+1-l_{i_0}-\cdots-l_{i_k},n)^{G_\ell}, \\ [\eta] & \mapsto & \left[\dfrac{\varphi_\ell^*\eta}{x_{i_0}^{l_{i_0}-1}\cdots x_{i_k}^{l_{i_k}-1}}\right],\end{array}$$

is an isomorphism.

Proof. The application φ_ℓ^* is clearly injective. To see its surjectivity, let us take

$$\mathcal{G} \in \mathbb{F}ol'_{i_0,\cdots,i_k}(d+k+1-l_{i_0}-\cdots-l_{i_k},n)^{G_\ell},$$

given by ω, so we define $\tilde{\omega} = x_{i_0}^{l_{i_0}-1}\cdots x_{i_k}^{l_{i_k}-1}\omega$ that verify $g^*\tilde{\omega} = \tilde{\omega} \quad \forall g \in G_\ell$. Then by the isomorphism (1.3) there exist $\eta \in H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(d))$ such that

$$\omega = \frac{\varphi_\ell^*\eta}{x_{i_0}^{l_{i_0}-1}\cdots x_{i_k}^{l_{i_k}-1}}.$$

It follows that η induces a foliation $\mathcal{F} \in \mathbb{F}ol'_{i_0,\cdots,i_k}(d,n,\ell)$. This conclude the proof. □

Example 1.6.10. Let $n = 2$ and $\ell = (1,1,l_2)$, $l_2 \geq 2$. Then we have two important isomorphism:

The first isomorphism,

$$\begin{array}{rcl}\varphi_l^* : \mathbb{F}ol_2(d,2,(1,1,l_2)) & \to & \mathbb{F}ol_2(d-l_2+1,2)^{G_\ell}, \\ [\eta] & \mapsto & \left[\dfrac{\varphi_\ell^*\eta}{x_2^{l_2-1}}\right],\end{array}$$

for any $d \geq \ l_2 + 1$.

The second isomorphism,

$$\begin{array}{rcl}\varphi_\ell^* : \mathbb{F}ol'(d,2,(1,1,l_2)) & \to & \mathbb{F}ol'(d,2)^{G_\ell}, \\ [\eta] & \mapsto & [\varphi_\ell^* \eta],\end{array}$$

for any $d \geq \ 2$.

Proposition 1.6.11. *Let $\ell = (l_0, l_1, \dots, l_n)$ a weighted vector with $1 \leq l_0 \leq l_1 \leq l_2 \leq \dots \leq l_n$, $\gcd(l_0, \dots, l_n) = 1$ and $n \geq 2$. Then the lowest possible normal $\mathbb{Q}$-bundle degree of the codimension one holomorphic foliations on $\mathbb{P}^n_\ell$ is $d = l_0 + l_1$ and the folations are pencil of hypersurfaces on $\mathbb{P}^n_\ell$.*

Proof. We consider two cases:

1. If $l_0 = l_1 = \cdots = l_k < l_{k+1} \leq \cdots \leq l_n$ and $1 \leq k \leq n$, then

$$H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(l_0 + l_1)) = \bigoplus_{0 \leq i < j \leq k} \mathbb{C}\, l_0(x_j dx_i - x_i dx_j).$$

Thus $\dim_{\mathbb{C}} H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(l_0 + l_1)) = \frac{k(k+1)}{2}$ and $\dim_{\mathbb{C}} \mathbb{F}ol(l_0 + l_1, n, \ell) = \frac{k(k+1)}{2} - 1$. Let $\mathcal{F} \in \mathbb{F}ol(l_0 + l_1, n, \ell)$ given by

$$\eta = l_0 (\sum_{0 \leq i < j \leq k}^{k} a_{ij}(x_j dx_i - x_i dx_j), \text{ for some } a_{ij} \neq 0.$$

Hence $\mathcal{F}$ can be thought of as a foliation on $\mathbb{P}^k$ of degree 0. So there exists automorphism on $\mathbb{P}^k$ such that a foliation is given by

$$y_1 dy_0 - y_0 dy_1.$$

2. If $l_0 < l_1 = l_2 = \cdots = l_k < l_{k+1} \leq \cdots \leq l_n$ and $1 \leq k \leq n$, then

$$H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(l_0 + l_1)) = \bigoplus_{j=1}^{k} \mathbb{C}(l_1 x_j dx_0 - l_0 x_0 dx_j).$$

Therefore $\dim_{\mathbb{C}} H^0(\mathbb{P}^n_\ell, \Omega^1_{\mathbb{P}^n_\ell}(l_0 + l_1)) = k$ and $\dim_{\mathbb{C}} \mathbb{F}ol(l_0 + l_1, n, \ell) = k - 1$. Let $\mathcal{F} \in Fol(l_0 + l_1, n, \ell)$ given by

$$\begin{array}{rcl}\eta & = & l_1(\sum_{i=1}^k a_i) x_i dx_0 - l_0 x_0 \sum_{i=1}^k a_i dx_i, \\ \eta & = & l_1(\sum_{i=1}^k a_i) x_i dx_0 - l_0 x_0 d(\sum_{i=1}^k a_i x_i).\end{array}$$

Using the automorphism $\phi(x_0, \dots, x_n) = (x_0, \sum_{i=1}^k a_i x_i, x_2, \cdots, x_n) = (y_0, \dots, y_n)$, we have

$$\phi^* \eta = l_1 y_1 dy_0 - l_0 y_0 dy_1.$$

We conclude the proposition. □

From now on we restrict us to $n = 2$.

1.7 Intersection formulas for foliations on singular surface

Before specifying the intersection formulas for foliations on weighted projective spaces, we will present the formulas for more general surfaces. The intersection formulas on singular surfaces has already studied and results about this topic can be found in [1] and [2]. The focus of this section is in the intersection formulas for foliations on singular surfaces. The general reference here is [9].

Definition 1.7.1. Let μ_r be the cyclic group of r-roots of unity and $(a,b) \in \mathbb{Z}^2$ be a vector of weights.

1. Consider the action
$$\begin{array}{rcl} \mu_r \times \mathbb{C}^2 & \to & \mathbb{C}^2, \\ (\xi, (x,y)) & \mapsto & (\xi^a x, \xi^b y). \end{array}$$
The set of all orbits $\mathbb{C}^2/\mu_d$ is called a (cyclic) quotient space of type $(r; a, b)$ and it is denoted by $X(r,(a,b))$.

2. The space $X(r,(a,b))$ is written in a normalized form if $\gcd(r,a) = \gcd(r,b) = 1$. It is possible to convert the general types $X(r,(a,b))$ into normalized form, see [1, Lemma 1.8, page 4].

3. A *singular surface* M *with only abelian quotient singularities* is an analytic surface such that for all $p \in Sing(M)$ there is a neighborhood U of p isomorphic to $X(r,(a,b))$.

Note that for every $\xi \in \mu_r$, the map $\lambda_\xi(x,y) = (\xi^a x, \xi^b y)$ is an automorphism of $\mathbb{C}^2$ induced by the action of μ_r.

Example 1.7.2. If $\ell = (l_0, l_1, l_2)$, where l_0, l_1, l_2 are two-by-two coprimes then $\mathbb{P}^2_\ell$, $X(l_0,(l_1,l_2))$, $X(l_1,(l_0,l_2))$, $X(l_2,(l_0,l_1))$ are surfaces with abelian quotient singularities.

Definition 1.7.3. Let M be a surface with only abelian quotient singularities and C be a curve on M.

1. A foliation $\mathcal{F}$ on M is a foliation on $M \backslash Sing(M)$ that extends to M. That is, if $p \in Sing(M)$ and $U \simeq X(r,(a,b))$ is a neighborhood of p, the foliation can be defined by a 1-form ω in $\mathbb{C}^2$ such that $\lambda_\xi^* \omega = \lambda(\xi)\omega$, $\forall\, \xi \in \mu_r$ where $\lambda : \mu_r \to \mathbb{C}^*$ is a group homomorphism which is known as a *character* of μ_r.

2. A curve C is called $\mathcal{F}$-invariant if the curve $C \cap (M \backslash Sing(M))$ is $\mathcal{F}$-invariant.

Remark 1.7.4. Given a foliation $\mathcal{F}$ on a surface M, we can still define its normal sheaf $N\mathcal{F}$ in the following way: the foliation $\mathcal{F}|_{M\backslash Sing(M)}$ will be denoted by $\mathcal{F}_0$, $N\mathcal{F}_0$ is the normal bundle of $\mathcal{F}_0$ and $j : M\backslash Sing(M) \to M$ is the inclusion, then

$$N\mathcal{F} = j_* N\mathcal{F}_0,$$

in which j_* is the direct image functor.

Similar considerations also hold for $T\mathcal{F}$, $K\mathcal{F}$, $N\mathcal{F}^*$ and of course K_M. We still have the equality

$$K_M = K\mathcal{F} \otimes N\mathcal{F}^*,$$

as sheaves or $\mathbb{Q}$-bundles.

1.7.1 Intersection multiplicity

As before, let M be a surface with only abelian quotient singularities and let $\mathcal{F}$ be a foliation on M. For every $p \in Sing(\mathcal{F})$ we can define an index $m(\mathcal{F}, p)$ which is called *the multiplicity of* $\mathcal{F}$ *at* p.

Definition 1.7.5. To define $m(\mathcal{F}, p)$ we consider the following two cases:

1. If $p \notin Sing(M)$, we take a local holomorphic 1-form $\eta = A(x,y)\,dx + B(x,y)\,dy$ that generates $\mathcal{F}$ around p, and define
$$m(\mathcal{F},p) = dim_{\mathbb{C}} \frac{\mathcal{O}_p}{\langle A,B\rangle},$$
where $\mathcal{O}_p$ is the local algebra of M at p (germ of holomorphic functions) and $\langle A,B\rangle$ is the ideal generated by A, B as elements of $\mathcal{O}_p$.

2. If $p \in Sing(M)$, we take a neighborhood $U \simeq X(r,(a,b))$ in normalized form, and lift $\mathcal{F}|_U$ to $\mathbb{C}^2$ which is denoted by $\tilde{\mathcal{F}}$ and then set
$$m(\mathcal{F},p) = \frac{m(\tilde{\mathcal{F}},(0,0))}{r}.$$

Now we can define, if X is compact:
$$m(\mathcal{F}) = \sum_{p\in M} m(\mathcal{F},p).$$

For foliations on $\mathbb{P}^2_\ell$, we have the following proposition.

Proposition 1.7.6. *Let $\ell = (l_0,l_1,l_2)$ be a weighted vector, l_i pairwise coprimes and $\mathcal{F}$ be a foliation with normal degree d on $\mathbb{P}^2_\ell$. Then*
$$l_0l_1l_2m(\mathcal{F}) = d^2 - |\ell|d + l_0l_1 + l_0l_2 + l_1l_2 = d(d-|\ell|) + l_0l_1 + l_0l_2 + l_1l_2.$$

Proof. The foliation $\mathcal{F}$ is induced by
$$\eta = A_0(x_0,x_1,x_2)dx_0 + A_1(x_0,x_1,x_2)dx_1 + A_2(x_0,x_1,x_2)dx_2,$$
such that
$$l_0x_0A_0 + l_1x_1A_1 + l_2x_2A_2 = 0. \tag{1.7}$$
First, we can suppose $\mathcal{F} \in \mathbb{F}ol'(d,2,\ell)$, *i.e.*, $\{x_0=0\}$, $\{x_1=0\}$ and $\{x_2=0\}$ are not $\mathcal{F}$-invariant. By Proposition 1.6.9 we have a foliation $\tilde{\mathcal{F}}$ of normal degree d on $\mathbb{P}^2$ given by
$$\omega = \varphi_l^*\eta = \tilde{A}_0dy_0 + \tilde{A}_1dy_1 + \tilde{A}_2dy_2,$$
where $\tilde{A}_i(y_0,y_1,y_2) = l_iy_i^{l_i-1}A_i\circ\varphi_\ell$. In the case of $\mathbb{P}^2$, we have
$$m(\tilde{\mathcal{F}}) = (d-2)^2 + (d-2) + 1. \tag{1.8}$$
Let us take a local coordinate $U_0 \simeq X(l_0,(l_1,l_2))$. Then lift $\mathcal{F}|_{U_0}$, $\varphi_\ell|_{\varphi_\ell^{-1}(U_0)}$ to $\mathbb{C}^2$ and denote them by $\mathcal{F}_0$, φ_0 which are given by
$$\eta_0 = A_1(1,u,v)du + A_2(1,u,v)dv,$$
and
$$\begin{array}{rccc} \varphi_0 : & \mathbb{C}^2 & \to & \mathbb{C}^2, \\ & (y,z) & \mapsto & (u,v), \end{array}$$
where $(u,v) = (y^{l_1}, z^{l_2})$, respectively and $\tilde{\mathcal{F}}|_{\varphi_\ell^{-1}(U_0)}$ given by
$$\omega_0 = \varphi_0^*\eta_0 = \tilde{A}_1(1,y,z)dy + \tilde{A}_2(1,y,z)dz.$$

If we denote by $(\tilde{A}_1, \tilde{A}_2)_{\tilde{q}} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\tilde{q}}}{\langle \tilde{A}_1, \tilde{A}_2 \rangle}$, then we have

$$(\tilde{A}_1, \tilde{A}_2)_{\tilde{q}} = (A_1 \circ \varphi_0, A_2 \circ \varphi_0)_{\tilde{q}} + (\frac{l_1 - 1}{l_1})(u \circ \varphi_0, A_2 \circ \varphi_0)_{\tilde{q}} + (\frac{l_2 - 1}{l_2})(v \circ \varphi_0, A_1 \circ \varphi_0)_{\tilde{q}} + \frac{(l_1 - 1)(l_2 - 1)}{l_1 l_2}(u \circ \varphi_0, v \circ \varphi_0)_{\tilde{q}}.$$

Taking average over $l_1 l_2$,

$$\frac{1}{l_1 l_2} \sum_{\tilde{q} \in \varphi_0^{-1}(q)} \underbrace{(\tilde{A}_1, \tilde{A}_2)_{\tilde{q}}}_{m(\tilde{\mathcal{F}}, \tilde{q})} = \underbrace{\frac{1}{l_1 l_2} \sum_{\tilde{q} \in \varphi_0^{-1}(q)} (A_1 \circ \varphi, A_2 \circ \varphi)_{\tilde{q}}}_{(A_1, A_2)_q} + \frac{l_1 - 1}{l_1}(u, A_2)_q + \frac{l_2 - 1}{l_2}(v, A_1)_q + \frac{(l_1 - 1)(l_2 - 1)}{l_1 l_2}(u, v)_q,$$

it is equivalent to

$$\frac{1}{l_1 l_2} \sum_{\tilde{q} \in \varphi_0^{-1}(q)} m(\tilde{\mathcal{F}}, \tilde{q}) = m(\mathcal{F}_0, q) + \frac{l_1 - 1}{l_1}(u, A_2)_q + \frac{l_2 - 1}{l_2}(v, A_1)_q + \frac{(l_1 - 1)(l_2 - 1)}{l_1 l_2}(u, v)_q.$$

Taking average over l_0 and using the definition of multiplicity at p, we have

$$\frac{1}{l_0 l_1 l_2} \sum_{q \in \varphi_\ell^{-1}(p)} m(\tilde{\mathcal{F}}, q) = m(\mathcal{F}, p) + \frac{l_1 - 1}{l_1}(x_1, A_2)_p + \frac{l_2 - 1}{l_2}(x_2, A_1)_p + \frac{(l_1 - 1)(l_2 - 1)}{l_1 l_2}(x_1, x_2)_p, \ \forall p \in U_0. \tag{1.9}$$

Analogously, we have

$$\frac{1}{l_0 l_1 l_2} \sum_{q \in \varphi_\ell^{-1}(p)} m(\tilde{\mathcal{F}}, q) = m(\mathcal{F}, p) + \frac{l_0 - 1}{l_0}(x_0, A_2)_p + \frac{l_2 - 1}{l_2}(x_2, A_0)_p + \frac{(l_0 - 1)(l_2 - 1)}{l_0 l_2}(x_0, x_2)_p, \ \forall p \in U_1. \tag{1.10}$$

$$\frac{1}{l_0 l_1 l_2} \sum_{q \in \varphi_\ell^{-1}(p)} m(\tilde{\mathcal{F}}, q) = m(\mathcal{F}, p) + \frac{l_0 - 1}{l_0}(x_0, A_1)_p + \frac{l_1 - 1}{l_1}(x_1, A_0)_p + \frac{(l_0 - 1)(l_1 - 1)}{l_0 l_1}(x_0, x_1)_p, \ \forall p \in U_2. \tag{1.11}$$

Using the equality (1.7), we have the following equalities

$$(x_1, A_2)_p = -(x_1, x_2)_p + (x_1, x_0)_p + (x_1, A_0)_p, \tag{1.12}$$

$$(x_2, A_2)_p = -(x_1, x_2)_p + (x_0, x_2)_p + (x_2, A_0)_p, \tag{1.13}$$

$$(x_0, A_2)_p = -(x_0, x_2)_p + (x_0, x_1)_p + (x_0, A_1)_p. \tag{1.14}$$

Adding the equalities (1.9), (1.10), (1.11) and using the equalities (1.12), (1.13), (1.14) and Bézout's Theorem for weighted projective planes see [2, Proposition 8.2, page 23], we have

$$\begin{aligned}\frac{1}{l_0 l_1 l_2} m(\tilde{\mathcal{F}}) &= m(\mathcal{F}) + \frac{l_1 - 1}{\bar{\ell}}(d - l_0 - l_2) + \frac{l_2 - 1}{\bar{\ell}}(d - l_0 - l_1) + \frac{l_0 - 1}{\bar{\ell}}(d - l_1 - l_2) \\ &\quad - \frac{(l_1 - 1)(l_2 - 1)}{\bar{\ell}} - \frac{(l_0 - 1)(l_1 - 1)}{\bar{\ell}} - \frac{(l_0 - 1)(l_2 - 1)}{\bar{\ell}}.\end{aligned}$$

Finally, using the equality (1.8) then

$$l_0 l_1 l_2 m(\mathcal{F}) = d^2 - |\ell| d + l_0 l_1 + l_0 l_2 + l_1 l_2 = d(d - |\ell|) + l_0 l_1 + l_0 l_2 + l_1 l_2. \tag{1.15}$$

The other cases are similar. □

1.7.2 Tangency Index

As before, let $\mathcal{F}$ be a foliation on the surface M, not necessarily compact, and C be a compact connected curve, possibly singular, and suppose that each irreducible component of C is not invariant by $\mathcal{F}$. For every $p \in C$ we can define an index $Tang(\mathcal{F}, C, p)$ which measure *the tangency order of $\mathcal{F}$ with C at p* (and thus it is 0 for a generic $p \in C$, where we have transversality).

Definition 1.7.7. The definition of tangency index $Tang(\mathcal{F}, C, p)$ is given in the following cases:

1. If $p \notin Sing(M)$ take a local equation f of C at p, and a local holomorphic vector field v generating $\mathcal{F}$ around p. Then we define

$$Tang(\mathcal{F}, C, p) = dim_{\mathbb{C}} \frac{\mathcal{O}_p}{\langle f, v(f) \rangle},$$

where $\mathcal{O}_p$ is the local algebra of M at p (germ of holomorphic functions), $v(f)$ is the Lie derivative of f along v, and $\langle f, v(f) \rangle$ is the ideal generated by f, $v(f)$ as elements of $\mathcal{O}_p$. This index is finite, because C is not $\mathcal{F}$-invariant; it is 0 iff $p \notin Sing(\mathcal{F})$ and $\mathcal{F}$ is transverse to C at p.

2. If $p \in Sing(M)$, we take a neighborhood $U \simeq X(r, (a, b))$ in normalized form, lift $\mathcal{F}|_U$ and $C \cap U$ to $\mathbb{C}^2$ and denote them by $\tilde{\mathcal{F}}$ and $\tilde{C}$ respectively. Then we define

$$Tang(\mathcal{F}, C, p) = \frac{Tang(\tilde{\mathcal{F}}, \tilde{C}, (0, 0))}{r}.$$

Note that $Tang(\mathcal{F}, C, p)$ is a nonnegative rational number, and it is 0 if and only if $\mathcal{F}$ is transverse to C at p (in the sense that $\tilde{\mathcal{F}}$ is transverse $\tilde{C}$ at 0.)

Now we can set:

$$Tang(\mathcal{F}, C) = \sum_{p \in C} Tang(\mathcal{F}, C, p).$$

We also introduce the orbifold-arithmetic Euler characteristic $\chi_{orb}(C)$ of C via the adjunction formula:

$$\chi_{orb}(C) = -K_M.C - C.C$$

Proposition 1.7.8. *Let M be a surface with only abelian quotient singularities, $\mathcal{F}$ be a foliation on M and $C \subset M$ be a compact curve, each component of which is not invariant by $\mathcal{F}$. Then*

$$\begin{aligned} N\mathcal{F}.C &= Tang(\mathcal{F}, C) + \chi_{orb}(C), \\ K\mathcal{F}.C &= Tang(\mathcal{F}, C) - C.C. \end{aligned}$$

Proof. See [9, page 4]. □

1.7.3 Vanishing and Camacho-Sad Index

Let us consider now the case where each irreducible component of C is invariant by $\mathcal{F}$. For every $p \in C$, we can define *the vanishing and Camacho-Sad indices* at p denoted by $Z(\mathcal{F}, C, p)$ and $CS(\mathcal{F}, C, p)$ respectively.

Definition 1.7.9. The definition of $Z(\mathcal{F}, C, p)$ and $CS(\mathcal{F}, C, p)$ is given in two cases:

1. If $p \notin Sing(M)$ take a local equation f of C at p, and a local holomorphic 1-form ω generating $\mathcal{F}$ around p. We can factorize around p (see [40, Chapter V])

$$g\omega = hdf + f\eta,$$

where η is a holomorphic 1-form, g and h are holomorphic functions, and g and h do not vanish identically on each local branch of C at p. Then we define

$$Z(\mathcal{F}, C, p) = \text{vanishing order of } \tfrac{h}{g}|_C \text{ at } p,$$

and

$$CS(\mathcal{F}, C, p) = -\frac{1}{2\pi i}\int_\gamma \frac{\eta}{h},$$

where $\gamma \subset C$ is a union of small circles around p, one for each local irreducible component C_i of C, oriented as the boundary of a small disc contained in C_i and containing p.

2. If $p \in Sing(M)$, we take a neighborhood $U \simeq X(r,(a,b))$ in normalized form, lift $\mathcal{F}|_U$ and $C \cap U$ to $\mathbb{C}^2$ and denote them by $\tilde{\mathcal{F}}$ and $\tilde{C}$ respectively. Then we define

$$Z(\mathcal{F}, C, p) = \frac{Z(\tilde{\mathcal{F}}, \tilde{C}, (0,0))}{r},$$

and

$$CS(\mathcal{F}, C, p) = \frac{CS(\tilde{\mathcal{F}}, \tilde{C}, (0,0))}{r}.$$

Note that $Z(\mathcal{F}, C, p)$ can be a negative rational number. Now we can set:

$$Z(\mathcal{F}, C) = \sum_{p \in C} Z(\mathcal{F}, C, p),$$

$$CS(\mathcal{F}, C) = \sum_{p \in C} CS(\mathcal{F}, C, p).$$

Proposition 1.7.10. *Let M be a surface with only abelian quotient singularities, $\mathcal{F}$ be a foliation on M and $C \subset M$ be a compact curve, each component of which is invariant by $\mathcal{F}$. Then*

$$\begin{array}{rcl} N\mathcal{F}.C &=& Z(\mathcal{F}, C) + C.C, \\ K\mathcal{F}.C &=& Z(\mathcal{F}, C) - \chi_{orb}(C), \\ C.C &=& CS(\mathcal{F}, C). \end{array}$$

Proof. See [9, page 5]. □

Example 1.7.11. Consider the foliation $\mathcal{F}$ of with normal degree 4 on $\mathbb{P}^2_{(2,1,1)}$ given by

$$\eta = -\frac{1}{2}(x_1^2 + x_2^2)dx_0 + x_0x_1dx_1 + x_0x_2dx_2.$$

Then we have

$$Sing(\mathcal{F}) = \{[0 : i : 1]_{(2,1,1)}, [0 : -i : 1]_{(2,1,1)}, [1 : 0 : 0]_{2,1,1}\},$$

and $m(\mathcal{F}, [0 : i : 1]_{(2,1,1)}) = m(\mathcal{F}, [0 : -i : 1]_{(2,1,1)}) = 1$, $m(\mathcal{F}, [1 : 0 : 0]_{2,1,1}) = \frac{1}{2}$. Hence

$$m(\mathcal{F}) = 1 + 1 + \frac{1}{2} = \frac{5}{2},$$

that coincides with multiplicity formula.

The curve $C_1 = \{x_1 = 0\}$ is not $\mathcal{F}$-invariant, it is easy to see that $Tang(\mathcal{F}, C_1, [1:0:0]) = 1/2$, so

$$Tang(\mathcal{F}, C_1) = 1/2.$$

The curve $C_0 = \{x_0 = 0\}$ is $\mathcal{F}$-invariant, so we have

$$Z(\mathcal{F}, C_0, [0:i:1]) = Z(\mathcal{F}, C_0, [0:-i:1]) = CS(\mathcal{F}, C_0, [0:i:1]) = CS(\mathcal{F}, \mathbb{C}_0, [0:-i:1]) = 1,$$

thus

$$Z(\mathcal{F}, C_0) = CS(\mathcal{F}, C_0) = 2.$$

1.8 Intersection Numbers and Weighted Blow-ups

In this section we want to study what is happening with intersection formula of foliations by weighted blow-up. A comprehensive reference for weighted blow-up is [2].

Weighted (l_0, l_1)*-blow-up of* $\mathbb{C}^2$. Let $\ell = (l_0, l_1)$ be a weight vector with coprime entries. We consider the space

$$\hat{\mathbb{C}}^2_\ell := \{((x,y),[t,s]_\ell) \in \mathbb{C}^2 \times \mathbb{P}^1_\ell |\ (x,y) \in \overline{[t,s]_\ell}\},$$

$\hat{\mathbb{C}}^2_\ell$ is covered by $\{U_0, U_1\}$ and the charts are given by

$$\begin{array}{rcl} X(l_0,(-1,l_1)) & \to & U_0, \\ {[(x,y)]} & \mapsto & ((x^{l_0}, x^{l_1}y),[1:y]_\ell). \end{array}$$

$$\begin{array}{rcl} X(l_1,(l_0,-1)) & \to & U_1, \\ {[(x,y)]} & \mapsto & ((xy^{l_0}, y^{l_1}),[x:1]). \end{array}$$

We denote by $\pi_\ell : \hat{\mathbb{C}}^2_\ell \to \mathbb{C}^2$ the natural projection. The exceptional divisor $E = \pi_\ell^{-1}(0)$ is isomorphic to $\mathbb{P}^1_\ell$.

Blow-up of $X(l_2,(l_0,l_1))$ *with respect to* $\ell = (l_0, l_1)$. Let $X(l_2,(l_0,l_1))$ be a surface with be a weight vector with $\gcd(l_2, l_0) = 1$ and $\gcd(l_2, l_1) = 1$. The action μ_{l_2} on $\mathbb{C}^2$ extends naturally to an action on $\hat{\mathbb{C}}^2_\ell$ as follows,

$$g_2.((x,y),[t,s]_\ell) \xrightarrow{\mu_{l_2}} ((g_2^{l_0}x, g_2^{l_1}y),[t:s]_\ell)$$

Let $\hat{X}(l_2,(l_0,l_1)) := \hat{\mathbb{C}}^2_\ell/\mu_{l_2}$ denote the quotient space of this action. Then the induced projection

$$\pi : \hat{X}(l_2,(l_0,l_1)) \to X(l_2,(l_0,l_1))$$

is an isomorphism over $\hat{X}(l_2,(l_0,l_1)) \backslash \pi^{-1}([0])$ and the exceptional divisor $E := \pi^{-1}([0])$ is identified with $\mathbb{P}^1_\ell$. We cover $\hat{X}(l_2,(l_0,l_1)) = \hat{U}_0 \cup \hat{U}_1$ and the charts are given by

$$\begin{array}{rcl} X(l_0,(-l_2,l_1)) & \to & \hat{U}_0, \\ {[(x^{l_2},y)]} & \mapsto & ((x^{l_0}, x^{l_1}y),[1:y]_\ell). \end{array} \tag{1.16}$$

$$\begin{array}{rcl} X(l_1,(l_0,-l_2)) & \to & \hat{U}_1, \\ {[(x,y^{l_2})]} & \mapsto & ((xy^{l_0}, y^{l_1}),[x:1]). \end{array} \tag{1.17}$$

Proposition 1.8.1. *Let M be a surface with abelian quotient singularities. Let $\pi : \hat{M} \to M$ be the weighted blow-up at a point p of type $(l_2; l_0, l_1)$ with respect to (l_0, l_1). Assume $(l_2, l_1) = (l_2, l_0) = 1$. Consider C and D two $\mathbb{Q}$-divisors on M, denote by E the exceptional divisor of π, and by $\hat{C}$ (resp. $\hat{D}$) the strict transform of C (resp. D). Let μ and ν be the (l_0, l_1)-multiplicities of C and D at p. i.e., x (resp. y) has (l_0, l_1)-multiplicity l_0 (resp. l_1). Then there are the following equalities:*

1. $\pi^*(C) = \hat{C} + \frac{\mu}{l_2} E$,
2. $E^2 = -\frac{l_2}{l_0 l_1}$,
3. $E.\hat{C} = \frac{\mu}{l_0 l_1}$,
4. $\hat{C}.\hat{D} = C.D - \frac{\mu\nu}{l_0 l_1 l_2}$.

In addition, if C has compact support then $\hat{C}^2 = C^2 - \frac{\mu^2}{l_0 l_1 l_2}$.

Proof. See [2, Proposition 7.3, page 19]. □

1.8.1 Foliations on Weighted Blow-ups

Let us again consider a foliation $\mathcal{F}$ on a surface $U = X(l_2, (l_0, l_1))$. Let $[(0,0)] \in U$ be a singular point of $\mathcal{F}$, and let $\pi : \hat{X}(l_2, (l_0, l_1)) \to X(l_2, (l_0, l_1))$ be a blow-up of $X(l_2, (l_0, l_1))$ with respect to (l_0, l_1), with exceptional divisor $E = \pi^{-1}([(0,0)]) \simeq \mathbb{P}^1_{(l_0, l_1)}$. Let us explain how can define a foliation $\hat{\mathcal{F}}$ on $\hat{X}(l_2, (l_0, l_1))$. We lift $\mathcal{F}$ to $\mathbb{C}^2$ and denote it by $\tilde{\mathcal{F}}$ generated by a 1-form η such that

$$\psi_{g_2}^* \eta = \lambda(g_2)\eta, \tag{1.18}$$

where λ is a character of μ_{l_2} and $\psi_{g_2}(x, y) = (g_2^{l_0} x, g_2^{l_1} y)$.

The 1-form η can be write as

$$\eta = \eta_k + \eta_{k+1} + \cdots ,$$

in which η_j are quasi-homogeneous 1-forms of degree j with respect to the weight vector (l_0, l_1) and $\eta_k \neq 0$. By condition (1.18) we have that

$$\eta = \eta_k + \eta_{k+l_2} + \cdots ,$$

and

$$\lambda(g_2) = g_2^k.$$

Hence we can write

$$\eta_{k+jl_2} = A_{k+jl_2-l_0}(x, y)dx + B_{k+jl_2-l_1}(x, y)dy,$$

where $A_{k+jl_2-l_0}(x, y)$, $B_{k+jl_2-l_1}(x, y)$ are quasi-homogeneous polynomials of degree $k + jl_2 - l_0$ and $k + jl_2 - l_1$ respectively. Under these conditions we say that k is called the (l_0, l_1)-*multiplicity or algebraic multiplicity of* $\mathcal{F}$ *at* p, which is denoted by

$$\text{multalg}_p(\mathcal{F}).$$

In the first chart $\tilde{U}_0 = X(l_0,(-l_2,l_1))$ we denote by $\pi_0 : \mathbb{C}^2 \to X(l_0,(-l_2,l_1))$ the natural projection. To define a foliation on $X(l_0,(-l_2,l_1))$, the natural idea is to define a foliation $\mathcal{F}_0$ on $\mathbb{C}^2$. Using the following change of coordinates

$$\begin{cases} x = u^{\frac{l_0}{l_2}}, \\ y = u^{\frac{l_1}{l_2}} v. \end{cases}$$

we have that

$$\pi_0^*(\eta) = u^{\frac{k}{l_2}-1}\left(\frac{l_0}{l_2}A_{k-l_0}(1,v) + \frac{l_1}{l_2}B_{k-l_1}(1,v)\right)du + u^{\frac{k}{l_2}}B_{k-l_1}(1,v)dv + \cdots .$$

From this last equality, we get

$$\pi_0^*(\eta) = u^{\frac{e}{l_0}}\tilde{\eta},$$

where

$$e = \begin{cases} k - l_2 & \text{, if } l_0xA_{k-l_0} + l_1yB_{k-l_1} \neq 0, \\ k & \text{, if } l_0xA_{k-l_0} + l_1yB_{k-l_1} = 0. \end{cases}$$

in addition $\psi_{g_0}^*(\tilde{\eta}) = g_0^{-e}\tilde{\eta}$, this means that $\tilde{\eta}$ induces a foliation $\hat{\mathcal{F}}$ on $X(l_0,(-l_2,l_1))$. In the second chart similarly can be treated and so we can define a foliation $\hat{\mathcal{F}}$ on $\hat{X}(l_2,(l_0,l_1))$. Note that

$$e = \begin{cases} k - l_2 & \text{, if } E \text{ is } \hat{\mathcal{F}}\text{-invariant} \\ k & \text{, if } E \text{ is not } \hat{\mathcal{F}}\text{-invariant} \end{cases}$$

So, we have the following proposition.

Proposition 1.8.2. *Let M be a surface with abelian quotient singularities. Let $\pi : \hat{M} \to M$ be the weighted blow-up at a point p of type $(l_2; l_0, l_1)$ with respect to (l_0, l_1). Assume* $\gcd(l_2,l_1) = \gcd(l_2,l_0) = 1$*. Consider $\mathcal{F}$ a foliation on M and C a $\mathcal{F}$-invariant compact curve, denote by E the exceptional divisor of π, by $\hat{\mathcal{F}} = \pi^*(\mathcal{F})$ the foliation induced on $\hat{M}$ and by $\hat{C}$ the strict transform of C. Let μ and k be the (l_0,l_1)-multiplicities of C and $\mathcal{F}$ at p. i.e., x (resp. y) has (l_0,l_1)-multiplicity l_0 (resp. l_1). Then the following equalities hold:*

1. $\pi^*(N\mathcal{F}) = N\hat{\mathcal{F}} + \frac{e}{l_2}E$,
2. $\pi^*(K\mathcal{F}) = K\hat{\mathcal{F}} + \frac{e-l_0-l_1+l_2}{l_2}E$,

where

$$e = \begin{cases} k - l_2 & \text{, if } E \text{ is } \hat{\mathcal{F}}\text{-invariant,} \\ k & \text{, if } E \text{ is not } \hat{\mathcal{F}}\text{-invariant.} \end{cases}$$

Proof. 1. Follow from the construction of $\tilde{\mathcal{F}}$.
2. Notice that the foliation $\tilde{\mathcal{F}}$ in $\mathbb{C}^2$ is also induced by the vector field

$$v = (B_{k-l_1} + \cdots + B_{k+jl_2-l_1}\cdots)\frac{\partial}{\partial x} - (A_{k-l_0} + \cdots + A_{k+jl_2-l_0} + \cdots)\frac{\partial}{\partial y}.$$

Using the following change of coordinates

$$\begin{cases} x = u^{\frac{l_0}{l_2}}, \\ y = u^{\frac{l_1}{l_2}} v. \end{cases}$$

We get $(\pi_0^{-1})_*(v) = u^{\frac{e-l_0-l_1+l_2}{l_2}}\tilde{v}$, in which the foliation $\hat{\mathcal{F}}$ is induced by $\tilde{v}$. From this follows the item 2.

□

Proposition 1.8.3. *Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{P}^2_\ell$. Let $\pi : \hat{\mathbb{P}^2_\ell} \to \mathbb{P}^2_\ell$ be the weighted blow-up at the point $p = [0:0:1]_\ell$ of type $(l_2; l_0, l_1)$ with respect to (l_0, l_1). Assume $\gcd(l_2, l_1) = \gcd(l_2, l_0) = 1$. Denote by E the exceptional divisor of π, by $\hat{\mathcal{F}} = \pi^*(\mathcal{F})$. Let k be the (l_0, l_1)-multiplicity of $\mathcal{F}$ at p. Then $\hat{\mathcal{F}}$ is a Ricatti foliation with respect to the natural fibration if and only if*

1. *$k = d - l_2$, if E is $\hat{\mathcal{F}}$-invariant,*
2. *$k = d - 2l_2$, if E is not $\hat{\mathcal{F}}$-invariant.*

Proof. Let C be a algebraic curve of degree $l_0 l_1$ passing by p. Then

$$\pi^*(C) = \hat{C} + \frac{l_0 l_1}{l_2} E,$$

and for the foliation we have

$$\pi^*(K\mathcal{F}) = K\hat{\mathcal{F}} + \frac{e - |l| + 2l_2}{l_2} E,$$

where

$$e = \begin{cases} k - l_2 & \text{, if } E \text{ is } \hat{\mathcal{F}}\text{-invariant,} \\ k & \text{, if } E \text{ is not } \hat{\mathcal{F}}\text{-invariant.} \end{cases}$$

Since $K\hat{\mathcal{F}}.\hat{C} = K\mathcal{F}.C - \frac{l_0 l_1 (e - |l| + 2l_2)}{l_0 l_1 l_2}$ and $K\mathcal{F} = \mathcal{O}_{\mathbb{P}^2_l}(d - |l|)$, thus

$$K\hat{\mathcal{F}}.\hat{C} = \frac{d - e - 2l_2}{l_2}.$$

If $\hat{\mathcal{F}}$ is a Ricatti foliation, that is, $\hat{\mathcal{F}}$ is transversal with respect to natural fibration, then we have that $K\hat{\mathcal{F}}.\hat{C} = 0$, therefore $e = d - 2l_2$, this completes the proof. □

Remark 1.8.4. If $\hat{\mathcal{F}}$ is a Ricatti foliation and E is $\hat{\mathcal{F}}$-invariant, then there exist a curve $\hat{C}$ invariant by $\hat{\mathcal{F}}$. In fact, since E is $\hat{\mathcal{F}}$-invariant and $E.E = \frac{-l_2}{l_0 l_1} \neq 0$, then there exist $\hat{p} \in sing(\hat{\mathcal{F}}) \cap E$ and a unique fiber $\hat{C}$ passing by $\hat{p}$. It is not difficult to see that $\hat{C}$ is $\hat{\mathcal{F}}$-invariant. In particular, $\mathcal{F}$ have algebraic curves invariant by $\mathcal{F}$.

Example 1.8.5. Consider the weighted projective planes $\mathbb{P}^2_{(1,1,l_2)}$, $l_2 \geq 2$. The only singular point of $\mathbb{P}^2_{(1,1,l_2)}$ is $p = [0:0:1] \in X(l_2, (1,1))$. Let $\pi : \hat{\mathbb{P}^2_\ell} \to \mathbb{P}^2_l$ be the weighted blow-up at p of type $(l_2, (1,1))$ with respect to $(1,1)$ and let E be the exceptional divisor. Since $\hat{X}(l_2, (1,1)) = X(1, (-l_2, 1)) \cup X(1, (1, -l_2))$, $\hat{\mathbb{P}^2_\ell}$ is smooth and $E.E = -l_2$. This surface is a Hirzebruch surface $\mathbb{F}_{l_2} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$ which has a structure of $\mathbb{P}^1$-bundle. Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{P}^2_{(1,1,l_2)}$ and $\hat{\mathcal{F}} = \pi^*(\mathcal{F})$ be a foliation on $\hat{\mathbb{P}}^2_\ell$ induces by $\mathcal{F}$. Let $L \subset \mathbb{P}^2_{(1,1,l_2)}$ be a line which pass through p and $\hat{L}$ be the strict transform of L. Let $k = k(p)$ and $e = e(p)$ be defined in the usual way. Then

1. $L = \mathcal{O}_{\mathbb{P}^2_l}(1)$,
2. $N\mathcal{F} = \mathcal{O}_{\mathbb{P}^2_l}(d)$,
3. $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^2_l}(d - |l|)$,

4. $\pi^*(L) = \hat{L} + \frac{1}{l_2}E$,

5. $N\hat{\mathcal{F}} = d\hat{L} + \frac{d-e}{l_2}E$,

6. $K\hat{\mathcal{F}} = (d - |l|)\hat{L} + \frac{(d-e-2l_2)}{l_2}E$.

Note that $d - e \equiv 0 (\mod l_2)$.

Chapter 2

Density of Foliations Without Algebraic Solutions

One the most important results of Jouanolou's celebrated monograph [29] states that the set of holomorphic foliations on the complex projective plane $\mathbb{P}^2$ of degree at least 2 which do not have an algebraic solution, is dense in the space of foliations. This result for dimension one holomorphic foliations on $\mathbb{P}^n$ was proved by Lins Neto - Soares in [31]. In [17] the authors prove a generalization of Jouanolou's result for one dimensional foliations over any smooth projective variety. On the other hand, in [35] the author gives a different proof of Jouanolou's theorem following the ideas of [17] and restricting to $\mathbb{P}^2$. In [24], one can find versions of Jouanolou's Theorem for second order differential equations on $\mathbb{P}^2$, for k-webs (first order differential equations) on $\mathbb{P}^2$ and for webs with sufficiently ample normal bundle on arbitrary projective surfaces.

The main theorem of this chapter provides a version of Jouanolou's Theorem for foliations in the weighted projective planes.

Theorem 1. *Let $\ell = (l_0, l_1, l_2)$ be a weighted vector, with l_0, l_1, l_2 pairwise coprimes and $1 \leq l_0 \leq l_1 \leq l_2$. A generic foliation with normal $\mathbb{Q}$-bundle of degree d in $\mathbb{P}^2_\ell$ does not admit any invariant algebraic curve if $d \geq l_0 l_1 l_2 + l_0 l_1 + 2l_2$.*

The bound above is not sharp. When $\ell = (1, 1, l_2)$, $l_2 > 1$, we have a more precise version of above statement which is sharp.

Theorem 2. *A generic foliation with normal $\mathbb{Q}$-bundle of degree d in $\mathbb{P}^2_{(1,1,l_2)}$ with $l_2 \geq 2$ does not admit any invariant algebraic curve if $d \geq 2l_2 + 1$. Moreover, if $d < 2l_2 + 1$ any foliation with normal $\mathbb{Q}$-bundle of degree d in $\mathbb{P}^2_{(1,1,l_2)}$ admits some invariant algebraic curve.*

In both statements, by generic we mean that the set of foliations that does not have any invariant curve is the complement of a countable union of algebraic closed proper subsets.

2.1 Holomorphic foliations on $\mathbb{P}^2_\ell$

In $\mathbb{P}^2_\ell$, we know that $\mathbb{F}ol(d, \ell) = \mathbb{P}(H^0(\mathbb{P}^2_\ell, \Omega^1_{\mathbb{P}^2_\ell}(d)))$ is the space of holomorphic foliations with normal $\mathbb{Q}$-bundle of the degree d. When the singular set is finite we say that the foliation $\mathcal{F}$ is saturated. If we denote by $S_k \subset \mathbb{C}[x_0, x_1, x_2]$ the complex vector space formed by the quasi-homogeneous polynomial of degree k, we see that for each $1 \leq k \leq d$ there is a natural application

$$\begin{array}{rcl} \phi_k : \mathbb{P}(S_k) \times \mathbb{F}ol(d-k, \ell) & \to & \mathbb{F}ol(d, \ell), \\ ([F], [\eta]) & \mapsto & [F\eta]. \end{array}$$

The set of the unsaturated foliations with normal $\mathbb{Q}$-bundle of degree d is equal to

$$\bigcup_{1 \leq k \leq d} \phi_k(\mathbb{P}(S_k) \times \mathbb{F}ol(d-k, \ell)).$$

In particular, the set of the saturated foliations with the normal $\mathbb{Q}$-bundle of degree d is an open set on $\mathbb{F}ol(d, \ell)$ in the Zariski topology. Remark however that this open set can be empty, as illustrated by the following examples.

Example 2.1.1. In the case $\ell = (1, 1, l_2)$, $l_2 \geq 2$, we have that

$$\mathbb{F}ol(d, (1, 1, l_2)) = \phi_{d-2}(\mathbb{P}(S_{d-2}) \times \mathbb{F}ol(2, \ell)) \quad \text{for all } 2 < d < l_2 + 1.$$

In fact, let $\mathcal{F} \in \mathbb{F}ol(d, (1, 1, l_2))$ be a foliation on $\mathbb{P}^2_{(1,1,l_2)}$ induced by η a quasi-homogeneous d-form. Since $2 < d < l_2 + 1$, it follows that

$$\eta = A_{d-1}(x_0, x_1)dx_0 + B_{d-1}(x_o, x_1)dx_1 + Cdx_2,$$

where A_{d-1}, B_{d-1} are quasi-homogeneous polynomials of degree $d-1$. From this last equality and the equality $x_0 A_{d-1} + x_1 B_{d-1} + l_2 x_2 C = 0$, we conclude that

$$\eta = F_{d-2}(x_0, x_1)(l_0 x_0 dx_1 - l_1 x_1 dx_0),$$

where F_{d-2} is a quasi-homogeneous polynomials of degree $d-2$.

Example 2.1.2. In the case $\ell = (3, 5, 11)$, we have that

$$\begin{array}{rcl} \mathbb{F}ol(d, (3, 5, 11)) & = & \emptyset, \quad \text{for all } d \in \{9, 10, 12\}, \\ \mathbb{F}ol(11, (3, 5, 11)) & = & \phi_3(\mathbb{P}(S_3) \times \mathbb{F}ol(8, \ell)), \\ \mathbb{F}ol(13, (3, 5, 11)) & = & \phi_5(\mathbb{P}(S_5) \times \mathbb{F}ol(8, \ell)). \end{array}$$

For simplicity, here and hereafter we will use $\mathbb{F}ol(d)$ to represent $\mathbb{F}ol(d, \ell)$.

2.1.1 Invariant algebraic curves

Let $\mathcal{F}$ be a foliation with normal $\mathbb{Q}$-bundle of degree d in $\mathbb{P}^2_\ell$ defined by η and $C \subset \mathbb{P}^2_\ell$ an irreducible algebraic curve. As in the case of $\mathbb{P}^2$, we say that C is $\mathcal{F}$-invariant if $i^*\eta \equiv 0$, where i is the inclusion of the smooth part of C into $\mathbb{P}^2_\ell \backslash Sing(\mathbb{P}^2_\ell)$.

Let us assume that C is given by the irreducible quasi-homogeneous polynomial F of degree k. Then C is $\mathcal{F}$-invariant if and only if there exists a quasi-homogeneous 2-form Θ_F of degree d such that

$$\eta \wedge dF - F\Theta_F = 0. \tag{2.1}$$

Remark 2.1.3. An important fact about equation (2.1) is that it still works for reducible curves, *i.e.*, if the decomposition of the curve is $F = F_1^{n_1} \dots F_r^{n_r}$, then the equation (2.1) holds if and only if each irreducible factor F_j defines a $\mathcal{F}$-invariant curve.

Definition 2.1.4. Let $p_0 := [1:0:0]$, $p_1 := [0:1:0]$, and $p_2 := [0:0:1]$ and consider the following sets

$$\begin{aligned} \mathcal{C}_k(d) &:= \{\mathcal{F} \in \mathbb{F}ol(d) | \quad \text{there exists an } \mathcal{F}\text{-invariant algebraic curve of degree } k\}, \\ \mathcal{D}_k(d) &:= \{(x,\mathcal{F}) \in \mathbb{P}^2_\ell \times \mathbb{F}ol(d) | \quad x \in C \text{ for some } \mathcal{F}\text{-invariant algebraic curve of degree } k\}, \end{aligned}$$

and for each $i \in \{0,1,2\}$

$$\mathcal{C}_k^{p_i}(d) := \{\mathcal{F} \in \mathbb{F}ol(d) | \quad p_i \in C \text{ for some } \mathcal{F}\text{-invariant algebraic curve of degree } k\}.$$

The following lemma will be used in §2.3 and §2.4.

Lemma 2.1.5. *The sets $\mathcal{C}_k(d)$, $\mathcal{D}_k(d)$ and $\mathcal{C}_k^{p_i}(d)$ are closed sets for every k and i.*

Proof. We consider the following set

$$\mathcal{Z}_k(d) \subset \mathbb{P}^2_\ell \times \mathbb{P}(H^0(\mathbb{P}^2_\ell, \Omega^1_{\mathbb{P}^2_\ell}(d)) \times H^0(\mathbb{P}^2_\ell, \Omega^2_{\mathbb{P}^2_\ell}(d))) \times \mathbb{P}(H^0(\mathbb{P}^2_\ell, \mathcal{O}_{\mathbb{P}^2_\ell}(k))),$$

$$\mathcal{Z}_k(d) = \{(x, [(\eta,\Theta)], [F]) | \quad \eta \wedge dF - F\Theta = 0 \text{ and } F(x) = 0\},$$

and the application

$$\begin{aligned} \pi : \mathbb{P}^2_\ell \times \mathbb{P}(H^0(\mathbb{P}^2_\ell, \Omega^1_{\mathbb{P}^2_\ell}(d)) \times H^0(\mathbb{P}^2_\ell, \Omega^2_{\mathbb{P}^2_\ell}(d))) \times \mathbb{P}(H^0(\mathbb{P}^2_\ell, \mathcal{O}_{\mathbb{P}^2_\ell}(k))) &\dashrightarrow \mathbb{P}^2_\ell \times \mathbb{F}ol(d) \times \mathbb{P}(H^0(\mathbb{P}^2_\ell, \mathcal{O}_{\mathbb{P}^2_\ell}(k))) \\ (x, [(\eta,\Theta)], [F]) &\mapsto (x, [\eta], [F]). \end{aligned}$$

The restriction of π to $\mathcal{Z}_k(d)$ is regular. Since $\mathcal{Z}_k(d)$ is a closed algebraic set then $\pi(\mathcal{Z}_k(d))$ is a closed algebraic set too. Since $\mathcal{C}_k(d)$ is the image of $\pi(\mathcal{Z}_k(d))$ by the projection

$$\mathbb{P}^2_\ell \times \mathbb{F}ol(d) \times \mathbb{P}(H^0(\mathbb{P}^2_\ell, \mathcal{O}_{\mathbb{P}^2_\ell}(k))) \to \mathbb{F}ol(d),$$

we have that $\mathcal{C}_k(d)$ is closed algebraic set. Similarly, the image of $\pi(\mathcal{Z}_k(d))$ by the projection

$$\mathbb{P}^2_\ell \times \mathbb{F}ol(d) \times \mathbb{P}(H^0(\mathbb{P}^2_\ell, \mathcal{O}_{\mathbb{P}^2_\ell}(k))) \to \mathbb{P}^2_\ell \times \mathbb{F}ol(d),$$

is $\mathcal{D}_k(d)$ and therefore it is closed set. Let

$$\begin{array}{ccc} \mathbb{P}^2_\ell \times \mathbb{F}ol(d) & \xrightarrow{\pi_1} & \mathbb{P}^2_\ell \\ \downarrow{\scriptstyle \pi_2} & & \\ \mathbb{F}ol(d) & & \end{array}$$

be the canonical projections. It follows that

$$\mathcal{C}_k^{p_i}(d) = \pi_2(\pi_1^{-1}(\{p_i\}) \cap \mathcal{D}_k(d)),$$

is closed set in $\mathbb{F}ol(d)$, this completes the proof of the lemma. □

2.2 Existence of algebraic leaves

The following result has been proved (in much more general context) by Bogomolov and McQuillan [6]. We will give another proof for foliations on the weighted projective planes, which tell us that the low degree foliations on $\mathbb{P}^2_{(l_0,l_1,l_2)}$ have infinitely many algebraic leaves. By technical questions that we will explain in §2.3.2, from now on we make the assumption: l_0, l_1, l_2 pairwise coprimes and $l_0 < l_1 < l_2$.

Proposition 2.2.1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_\ell$. If $\deg(K\mathcal{F}) < 0$, then $\mathcal{F}$ is a rational fibration.*

Proof. Let $d = \deg(N\mathcal{F})$, using Remark 1.6.4 and $\deg(K\mathcal{F}) < 0$, we see that

$$d = \deg(N\mathcal{F}) < |l| = l_0 + l_1 + l_2.$$

We have the following cases:

1. If $d = l_0 + l_1$, then $\mathcal{F}$ is induced by $\eta = l_1x_1dx_0 - l_0x_0dx_1$.

2. If $l_0 + l_1 < d < l_0 + l_2$, then $\mathcal{F}$ is an unsaturated foliation, that is, $\mathcal{F}$ is induced by a quasi-homegeneous 1-form $\eta = F_{d-l_o-l_1}(x_0, x_1)(l_1x_1dx_0 - l_0x_0dx_1)$.

3. If $l_0 + l_2 \leq d < l_1 + l_2$ or $l_1 + l_2 < d < |l|$, then the foliation $\mathcal{F}$ is given by

$$\eta = b_1x_0^{k-1}(l_2x_2dx_0 - l_0x_0dx_2) + A_{d-l_0-l_1}(x_0, x_1)(l_1x_1dx_0 - l_0x_0dx_1),$$

   where $l_0k + l_2 = d$, $k \geq 1$ and $A_{d-l_0-l_1}$ is a quasi-homogeneous polynomial of degree $d - l_0 - l_1$ in $\mathbb{C}[x_0, x_1]$. We can write $A_{d-l_0-l_1}(x_0, x_1) = \sum_{I=(i,j)} a_Ix_0^ix_1^j$. In open set $U_0 = \mathbb{C}^2/\mu_{l_0}$, we lift $\mathcal{F}|_{U_0}$ to $\mathbb{C}^2$ which is given by

$$\begin{aligned} \eta_0 &= -l_0b_1dx_2 - l_0A_{d-l_0-l_1}(1, x_1)dx_1, \\ &= d(-l_0b_1x_2 - l_0G_{d-l_0}(1, x_1)), \end{aligned}$$

   where $G_{d-l_0}(x_0, x_1) = \sum_I \frac{a_I}{j+1}x_0^ix_1^{j+1}$. Thus,

$$\eta = \frac{(-b_1x_0^{k-1}x_2 - G_{d-l_0})^{l_0-1}}{x_0^{d-l_0+1}}d\left(\frac{(-b_1x_0^{k-1}x_2 - G_{d-l_0})^{l_0}}{x_0^{d-l_0}}\right).$$

4. if $d = l_1 + l_2$, then the foliation $\mathcal{F}$ is given by

$$\eta = \begin{cases} A_{l_2-l_0}(x_0, x_1)(l_1x_1dx_0 - l_0x_0dx_1) + b_1(l_2x_2dx_1 - l_1x_1dx_2) & \text{, if } l_0 > 1, \\ A_{l_2-l_0}(x_0, x_1)(l_1x_1dx_0 - l_0x_0dx_1) + b_1(l_2x_2dx_1 - l_1x_1dx_2) & \text{, if } l_0 = 1. \\ +b_2x_0^{l_1-1}(l_2x_2dx_0 - x_0dx_2) & \end{cases}$$

   where $A_{l_2-l_0}$ is a quasi-homogeneous polynomial of degree $l_2 - l_0$ in $\mathbb{C}[x_0, x_1]$. By Proposition 1.7.6 we see that $m(\mathcal{F}) = \frac{1}{l_0}$, then we have two cases: If $l_0 > 1$, then $Sing(\mathcal{F}) = \{[1 : 0 : 0]\}$, and if $l_0 = 1$, then either $Sing(\mathcal{F}) \cap \{x_0 = 0\} = \emptyset$ or $Sing(\mathcal{F}) = \{[0 : 0 : 1]\}$. In the case $l_0 = 1$ and $Sing(\mathcal{F}) \cap \{x_0 = 0\} = \emptyset$, using a automorphism of $\mathbb{P}^2_\ell$, we can suppose that

$Sing(\mathcal{F}) = \{[1:0:0]\}$. Hence in the case $l_0 \geq 1$ and $Sing(\mathcal{F}) = \{[1:0:0]\}$, the foliation $\mathcal{F}$ is induced by

$$\eta = A_{l_2-l_0}(x_0,x_1)(l_1x_1dx_0 - l_0x_0dx_1) + (l_2x_2dx_1 - l_1x_1dx_2).$$

We can write $A_{l_2-l_0}(x_0,x_1) = \sum_{I=(i,j)} a_I x_0^i x_1^j$. Taking the following automorphism

$$\begin{array}{rcl} \psi : \mathbb{P}^2_{(l_0,l_1,l_2)} & \to & \mathbb{P}^2_{(l_0,l_1,l_2)} \\ [x_0 : x_1 : x_2] & \mapsto & [y_0 : y_1 : y_2] = [x_0 : x_1 : F(x_0,x_1,x_2)], \end{array}$$

where $F(x_0,x_1,x_2) = x_2 + \sum_I -(i+1)a_I x_0^{i+1}x_1^j$, thus we have that

$$\tilde{\eta} = (\psi^{-1})^*\eta = l_2y_2dy_1 - l_1y_1dy_2.$$

In other case $l_0 = 1$ and $Sing(\mathcal{F}) = \{[0:0:1]\}$, therefore the foliation $\mathcal{F}$ is induced by

$$\eta = A_{l_2-1}(x_0,x_1)(l_1x_1dx_0 - l_0x_0dx_1) + b_2x_0^{l_1-1}(l_2x_2dx_0 - x_0dx_2).$$

In open set $U_0 = \mathbb{C}^2/\mu_{l_0}$, we lift $\mathcal{F}|_{U_0}$ to $\mathbb{C}^2$ which is given by

$$\begin{array}{rcl} \eta_0 & = & -b_2dx_2 - l_0A_{l_2-1}(1,x_1)dx_1, \\ & = & d(-b_2x_2 - l_0G_{l_2+l_1-1}(1,x_1)), \end{array}$$

where $G_{l_2+l_1-1}(x_0,x_1) = \sum_I \frac{a_I}{j+1} x_0^i x_1^{j+1}$. Thus,

$$\eta = \frac{1}{x_0^{l_2+l_1}} d\left(\frac{-b_2x_0^{l_1-1}x_2 - G_{l_2+l_1-1}(x_0,x_1)}{x_0^{l_2+l_1-1}} \right).$$

This finishes the proof of the proposition. □

The following proposition characterizes the low degree foliations on $\mathbb{P}^2_{(1,1,l_2)}$ with some algebraic solution.

Proposition 2.2.2. *Any foliation on $\mathbb{P}^2_{(1,1,l_2)}$ with normal $\mathbb{Q}$-bundle having degree d satisfying $2 \leq d \leq 2l_2$ admits some invariant algebraic curve. Furthermore*

1. *$\mathbb{F}ol(2) = \{x_0dx_1 - x_1dx_0\}$, every foliation in $\mathbb{F}ol(2)$ admits rational first integral and in particular admits infinite number of algebraic solutions.*

2. *$\mathbb{F}ol(d)$, $2 < d \leq l_2$, has no saturated foliations.*

3. *$\mathbb{F}ol(l_2+1)$: every foliation in $\mathbb{F}ol(l_2+1)$ admits rational first integral.*

4. *$\mathbb{F}ol(l_2+2)$: the generic element is defined by a logarithmic 1-form with poles on two curves of degree one and a curve of degree l_2.*

5. *$\mathbb{F}ol(d)$, if $l_2+3 \leq d \leq 2l_2$: every foliation in $\mathbb{F}ol(d)$ is a transversaly projective foliation (Ricatti).*

Proof. If $2 \leq d \leq l_2$, then $\mathcal{F} \in \mathbb{F}ol(d)$ is given by

$$\eta = A_{d-2}(x_0, x_1)(x_1 dx_0 - x_0 dx_1),$$

where A_{d-2} is a homogeneous polynomial of degree $d-2$ in $\mathbb{C}[x_0, x_1]$. Therefore $\{x_0 x_1 = 0\}$ is $\mathcal{F}$-invariant $\forall \mathcal{F} \in \mathbb{F}ol(d)$.

If $l_2 < d \leq 2l_2$, let $\mathcal{F} \in \mathbb{F}ol(d)$ be a saturated foliation given by

$$\eta = A_{d-l_2-1}(x_0,x_1)(l_2 x_2 dx_0 - x_0 dx_2) + B_{d-l_2-1}(x_0,x_1)(l_2 x_2 dx_1 - x_1 dx_2) + C_{d-2}(x_0,x_1)(x_1 dx_0 - x_0 dx_1),$$

where $A_{d-l_2-1}, B_{d-l_2-1}, C_{d-2}$ are homogeneous polynomials of degree $d-l_2-1$, $d-l_2-1$ and $d-2$ respectively. In the open set $U_2 \simeq \mathbb{C}^2/\mu_{l_2}$, we lift $\mathcal{F}|_{U_2}$ to $\mathbb{C}^2$ which is given by

$$\eta_2 = l_2 A_{d-l_2-1} dx_0 + l_2 B_{d-l_2-1} dx_1 + C_{d-2}(x_1 dx_0 - x_0 dx_1).$$

Then we have that $\text{multalg}_{[0:0:1]}(\mathcal{F}) = d - l_2$ and $x_0 A_{d-l_2-1} + x_1 B_{d-l_2-1} \neq 0$. Therefore by Proposition 1.8.3 we have that $\mathcal{F}$ is Ricatti foliation. Hence by Remark 1.8.4 it admits an invariant curve.

1. A foliation $\mathcal{F} \in \mathbb{F}ol(2)$ is induced by $\lambda(x_0 dx_1 - x_1 dx_0)$, for some $\lambda \in \mathbb{C}^*$.
2. Since $\mathcal{F} \in \mathbb{F}ol(d)$ is given by

$$\eta = A_{d-2}(x_0, x_1)(x_1 dx_0 - x_0 dx_1),$$

where A_{d-2} is a homogeneous polynomial of degree $d-2$. We conclude that $\mathcal{F}$ is unsaturated.

3. Let $\mathcal{F} \in \mathbb{F}ol(l_2+1)$ be a saturated foliation. We can see that $\mathcal{F}$ is induced by

$$\eta = (l_2 x_2 A_0(x_0,x_1) + x_1 A_{l_2-1}(x_0,x_1))dx_0 + (l_2 x_2 B_0 - x_0 A_{l_2-1}(x_0,x_1))dx_1 - (x_0 A_0 + x_1 B_0)dx_2, \quad (2.2)$$

where A_0, B_0 and A_{l_2-1} are homogeneous polynomials of degree 0, 0 and $l_2 - 1$ respectively. By the multiplicity formula Proposition 1.7.6, we have that $m(\mathcal{F}) = 1$, then there exists a point $p \in Sing(\mathcal{F})$. By the equation (2.2) and $\mathcal{F}$ is a saturated foliation we conclude that $p \in Sing(\mathcal{F}) \cap (\mathbb{P}^2_{(1,1,l_2)} \setminus [0:0:1])$. Take the line L_p that through the points p and p_2. Suppose that L_p is not invariant by $\mathcal{F}$. Using the Tangency Formula for $\mathcal{F}$ and L_p we have that

$$0 < Tang(\mathcal{F}, L_p) = 0,$$

this is contradiction, then L_p is $\mathcal{F}$-invariant. For a suitable choice of coordinates we can assume that the point $p = p_0$ and $\mathcal{F}$ is induced by the 1-form

$$\eta = -x_1^2 A_{l_2-2}(x_0,x_1)dx_0 + (-l_2 x_2 + x_0 x_1 A_{l_2-2}(x_0,x_1))dx_1 + x_1 dx_2,$$

where $A_{l_2-2}(x_0,x_1) = a_{l_2-1} x_1^{l_2-2} + \cdots + a_1 x_0^{l_2-2}$. Taking the following automorphism

$$\begin{array}{rcl} \psi : \mathbb{P}^2_{(1,1,l_2)} & \rightarrow & \mathbb{P}^2_{(1,1,l_2)} \\ [x_0 : x_1 : x_2] & \mapsto & [y_0 : y_1 : y_2] = [x_0 : x_1 : F(x_0,x_1,x_2)], \end{array}$$

where $F(x_0,x_1,x_2) = x_2 - a_{l_2-1} x_0 x_1^{l_2-1} - \frac{a_{l_2-2}}{2} x_0^2 x_1^{l_2-2} - \cdots - \frac{a_1}{l_2-1} x_0^{l_2-1} x_1$. Therefore we have that

$$\tilde{\eta} = (\psi^{-1})^* \eta = -l_2 y_2 dy_1 + y_1 dy_2.$$

4. If $\mathcal{F} \in \mathbb{F}ol(l_2+2)$ is a saturated foliation, then by the multiplicity formula Proposition 1.7.6, we have that $m(\mathcal{F}) = 2 + \frac{1}{l_2}$, then $p_2 \in Sing(\mathcal{F})$. Take a nonempty open U_1 in $\mathbb{F}ol(l_2+2)$ such that if $\mathcal{F} \in U_1$ thus, $Sing(\mathcal{F})$ consists of three different points and one of them is p_2. Let $\mathcal{F} \in U_1$, then for a suitable choice of coordinates we can assume that $Sing(\mathcal{F}) = \{p_0, p_1, p_2\}$ and $\mathcal{F}$ is induced by the 1-form

$$\eta = -(l_2 a x_1 x_2 + x_0 x_1^2 A_{l_2-2}(x_0,x_1))dx_0 + (-l_2 b x_0 x_2 + x_0^2 x_1 A_{l_2-2}(x_0,x_1))dx_1 + (a+b)x_0 x_1 dx_2$$

where $A_{l_2-2}(x_0,x_1) = a_{l_2-1}x_1^{l_2-2} + \cdots + a_1 x_0^{l_2-2}$. We define the nonempty open

$$U = \{[\eta] \in U_1 |\, (l_2-i)a - ib \neq 0,\ \forall\, i = 1,\ldots,l_2-1\}.$$

If $\mathcal{F} \in U$, take the following automorphism

$$\begin{array}{rcl} \psi : \mathbb{P}^2_{(1,1,l_2)} & \to & \mathbb{P}^2_{(1,1,l_2)} \\ {[x_0 : x_1 : x_2]} & \mapsto & [y_0 : y_1 : y_2] = [x_0 : x_1 : F(x_0,x_1,x_2)], \end{array}$$

where $F(x_0,x_1,x_2) = x_2 + \frac{a_{l_2-1}}{(l_2-1)a-b}x_0 x_1^{l_2-1} + \frac{a_{l_2-2}}{a(l_2-2)-2b}x_0^2 x_1^{l_2-2} + \cdots + \frac{a_1}{a+(1-l_2)b}x_0^{l_2-1}x_1$, therefore we have that

$$\tilde{\eta} = (\psi^{-1})^*\eta = -l_2 a y_1 y_2 dy_0 - l_2 b y_0 y_2 dy_1 + (a+b)y_0 y_1 dy_2,$$

$$\tilde{\eta} = y_0 y_1 y_2 \left(-a\frac{dy_0}{y_0} - b\frac{dy_1}{y_1} + (a+b)\frac{dy_2}{l_2 y_2} \right)$$

□

2.3 Foliations without algebraic leaves on $\mathbb{P}_{(l_0,l_1,l_2)}$

For any positive integers a and b with $\gcd(a,b) = 1$, define $g(a,b)$ to be the greatest positive integer N for which the equation

$$ax_1 + bx_2 = N, \tag{2.3}$$

is not solvable in nonnegative integers.

Lemma 2.3.1. *(Sylvester, 1894 [3]) Let a and b be positive integers with* $\gcd(a,b) = 1$. *Then*

$$g(a,b) = ab - a - b.$$

Proof. Suppose that $N > ab - a - b$. Note that if $(x_1,x_2) = (y_1,y_2)$ is a particular solution to (2.3), then every integer solution is of the form $(x_1,x_2) = (y_1+bt, y_2-at)$, $t \in \mathbb{Z}$. Let t be an integer such that $0 \leq y_2 - at \leq a-1$. Then

$$(y_1+bt)a = N - (y_2-at)b > ab - a - b - (a-1)b = -a,$$

which implies $y_1 + bt > -1$, *i.e.*, $y_1 + bt \geq 0$. It follows that in this case the equation $ax_1 + bx_2 = N$ is solvable in nonnegative integers. Thus

$$g(a,b) \leq ab - a - b.$$

Now we need only to show that the equation $ax_1 + bx_2 = ab - a - b$ is not solvable in nonnegative integers. Otherwise, we have

$$ab = a(x_1 + 1) + b(x_2 + 1).$$

Since $\gcd(a, b) = 1$, we see that $a|(x_2 + 1)$ and $b|(x_1 + 1)$, which implies $x_2 + 1 \geq a$ and $x_1 + 1 \geq b$. Hence

$$ab = a(x_1 + 1) + b(x_2 + 1) \geq 2ab,$$

and this contradiction shows that

$$g(a, b) \geq ab - a - b.$$

Therefore $g(a, b) = ab - a - b$. □

Definition 2.3.2. Let $\ell = (l_0, l_1, l_2)$ be a weight vector. Assume that its entries are ordered ($l_0 \leq l_1 \leq l_2$) and are pairwise coprime. Let $X = \mathbb{P}^2_\ell \backslash Sing(\mathbb{P}^2_\ell)$. Recall that $p_0 = [1 : 0 : 0]$, $p_1 = [0 : 1 : 0]$, $p_2 = [0 : 0 : 1]$. We define the following sets

$$\begin{aligned}
\mathcal{S}(d) &:= \{(x, \mathcal{F}) \in \mathbb{P}^2_\ell \times \mathbb{F}ol(d)) \,|\, x \in sing(\mathcal{F})\},\\
\mathcal{S}_X(d) &:= \{(x, \mathcal{F}) \in X \times \mathbb{F}ol(d) |\, x \in sing(\mathcal{F})\},\\
\mathcal{S}_{p_i}(d) &:= \{(p_i, \mathcal{F}) \in \mathcal{S}(d)\},\ i = 0, 1, 2.
\end{aligned}$$

Proposition 2.3.3. *For all $d > l_1l_2$, $\mathcal{S}_X(d)$ is an irreducible subvariety and has codimension two on $X \times \mathbb{F}ol(d)$.*

Proof. Consider the projection $\pi_1 : \mathcal{S}(d) \to \mathbb{P}^2_\ell$. For every $x \in \mathbb{P}^2_\ell$, the fiber $\pi_1^{-1}(x)$ is a subvariety of $\{x\} \times \mathbb{F}ol(d)$ contained in $\mathcal{S}(d)$ and isomorphic to a projective space, *i.e.*, if two 1-forms η and η' vanish in x thus the same is true for a linear combination of them.

The automorphism group of $\mathbb{P}^2_\ell$ has four distinct orbits on $\mathbb{P}^2_\ell$. In order to proof the proposition it suffices to exhibit for each of these orbits, two 1-forms which are linearly independent at a point of them.

Since $d > l_1l_2$ we can apply Lemma 2.3.1 to obtain positive integers i_{12} and j_{12} such that $i_{12}l_1 + j_{12}l_2 = d$. Hence

$$\alpha = x_1^{i_{12}-1} x_2^{j_{12}-1}(l_2x_2dx_1 - l_1x_1dx_2)$$

belongs $H^0(\mathbb{P}^2_\ell, \Omega^1_{\mathbb{P}^2_\ell}(d))$. Similarly, since $d > l_0l_2$ and $d > l_0, l_1$ we obtain integers $i_{02}, i_{01}, j_{01}, j_{02}$ such that $i_{01}l_0 + j_{01}l_1 = i_{02}l_0 + j_{02}l_2 = d$, and consequently

$$\beta = x_0^{i_{01}-1} x_1^{j_{01}-1}(l_1x_1dx_0 - l_0x_0dx_1) \text{ and } \gamma = x_0^{i_{02}-1} x_2^{j_{02}-1}(l_2x_2dx_0 - l_0x_0dx_2)$$

also belong to $H^0(\mathbb{P}^2_\ell, \Omega^1_{\mathbb{P}^2_\ell}(d))$.

If $p = (1, 1, 1)$ then $V_p =< \alpha(p), \beta(p), \gamma(p) >$, the vector space generated by the evaluation of α, β, and γ at p has dimension 2.

At the point $p_0 = (0, 1, 1)$, the analogue vector space has dimension one. But we can apply Lemma 2.3.1 to write $d - l_0 = i_0l_1 + j_0l_2$ with i_0 and j_0 positive integers, and see that

$$\delta_0 = x_1^{i_0-1} x_2^{j_0}(l_0x_0dx_1 - l_1x_1dx_0)$$

belongs to $H^0(\mathbb{P}^2_\ell, \Omega^1_{\mathbb{P}^2_\ell}(d))$. Now considering the evaluation of δ_0 and α at p_0 we see that they are $\mathbb{C}$-linearly independent. Applying the same argument to the others points $p_1 = (1, 0, 1)$ and $p_2 = (1, 1, 0)$, we conclude that $\pi_1^{-1}(x) \subset X \times \mathbb{F}ol(d)$ always have codimension two. It follows that $\mathcal{S}_X(d)$ is projective bundle over X, and therefore is smooth and irreducible. □

Proposition 2.3.4. *Let $d > l_1 l_2$ and suppose that $\mathcal{C}_k(d) = \mathbb{F}ol(d)$ for some $k > 0$.*

1. *If $\dim \mathcal{S}_{p_i}(d) = \dim \mathbb{F}ol(d)$ and $\mathcal{C}_k^{p_i}(d) = \mathcal{C}_k(d)$ for some i, then $\mathcal{S}_{p_i}(d) \cap \mathcal{D}_k(d) = \mathcal{S}_{p_i}(d)$.*
2. *If $\dim \mathcal{S}_{p_i}(d) < \dim \mathbb{F}ol(d)$ or $\mathcal{C}_k^{p_i}(d) \subsetneq \mathcal{C}_k(d)$ for every i, then $\mathcal{S}_X(d) \cap \mathcal{D}_k(d) = \mathcal{S}_X(d)$.*

Proof. We denote by π the restriction of the natural projection to $\mathcal{S}_X(d)$

$$\pi : \mathcal{S}_X(d) \to \mathbb{F}ol(d).$$

For item 1, the condition $\dim \mathcal{S}_{p_i}(d) = \dim \mathbb{F}ol(d)$ implies that $\mathcal{S}_{p_i}(d) = \{p_i\} \times \mathbb{F}ol(d)$. Hence $\mathcal{S}_{p_i}(d) \cap \mathcal{D}_k(d) = \mathcal{S}_{p_i}(d)$. For item 2, since $\mathcal{C}_k(d) = \mathbb{F}ol(d)$ and $\mathcal{C}_k^{p_i}(d) \subsetneq \mathcal{C}_k(d)$ for all i, we define the nonempty open set

$$U_k = \mathbb{F}ol(d) \backslash \cup_{i=0}^{2} (\mathcal{C}_k^{p_i}(d)).$$

Let $\mathcal{F} \in U_k$ be a foliation saturated. Then there exists a curve C of degree k invariant by $\mathcal{F}$. Using Camacho-Sad Theorem in weighted projective planes $C^2 = \sum_{p \in X \cap C} CS(\mathcal{F}, C, p)$. This implies that there exists a $p \in Sing(\mathcal{F}) \cap X$. Therefore

$$\pi(\pi^{-1}(U_k) \cap \mathcal{D}_k(d)) = U_k.$$

By Proposition 2.3.3 we have that $\pi^{-1}(U_k)$ is an irreducible open set and

$$\dim \pi^{-1}(U_k) = \dim \mathbb{F}ol(d),$$

and since $\dim \pi(\pi^{-1}(U_k) \cap \mathcal{D}_k(d) = \dim U_k = \dim \mathbb{F}ol(d)$ implies that

$$\dim \pi^{-1}(U_k) \cap \mathcal{D}_k(d) = \dim \pi^{-1}(U_k),$$

and hence

$$\pi^{-1}(U_k) \cap \mathcal{D}_k(d)) = \pi^{-1}(U_k).$$

Taking closure on $\mathbb{P}^2_\ell \times \mathbb{F}ol(d)$, we get

$$\overline{\mathcal{S}_X(d)} \cap \mathcal{D}_k(d) = \overline{\mathcal{S}_X(d)}.$$

In particular we have

$$\mathcal{S}_X(d) \cap \mathcal{D}_k(d) = \mathcal{S}_X(d).$$

□

In order to prove our result we will construct examples contradicting the above proposition for $d \gg 0$ in the next subsection.

2.3.1 Existence of singularities without algebraic separatrix

First, we construct a family examples that contradict item 2 of Proposition 2.3.4 for $d \gg 0$. The following example is an adaptation of an example of J. V. Pereira, see [35, page 5].

Let $\ell = (l_0, l_1, l_2)$, $l_0 \le l_1 \le l_2$, and $\mathcal{F}_0$ be a foliation in $\mathbb{P}^2_{(l_0,l_1,l_2)}$ induced by the following 1-form

$$\eta_0 = x_0x_1x_2(x_0^{l_1l_2}+x_1^{l_0l_2}+x_2^{l_0l_1}) \left(\lambda l_1 l_2 \frac{dx_0}{x_0} + \mu l_0 l_2 \frac{dx_1}{x_1} + + \gamma l_0 l_2 \frac{dx_2}{x_2} - (\lambda+\mu+\gamma) \frac{d(x_0^{l_1l_2} + x_1^{l_0l_2} + x_2^{l_0l_2})}{x_0^{l_1l_2} + x_1^{l_0l_2} + x_2^{l_0l_1}} \right).$$

Then $\deg(N\mathcal{F}_0) = l_0l_1l_2 + l_0 + l_1 + l_2$ and

$$Sing(\mathcal{F}_0) = \left\{ \begin{array}{c} [1:0:0], [0:1:0], [0:0:1], [0:1:a^{1/l_1}] \,|\, a^{l_0} = -1, \\ [1:0:b^{1/l_0}] \,|\, b^{l_1} = -1, \\ [1:c^{1/l_0}:0] \,|\, c^{l_2} = -1, \\ [x_0:x_1:x_2] \,|\, x_0^{l_1l_2} = \lambda,\, x_1^{l_0l_2} = \mu,\, x_2^{l_0l_1} = \gamma \end{array} \right\}.$$

Notice that the foliation $\mathcal{F}_0$ is induced by η admits a multivalued first integral, that is,

$$F(x_0, x_1, x_2) = \frac{x_0^{\lambda} x_1^{\mu} x_2^{\gamma}}{(x_0^{l_1l_2} + x_1^{l_0l_2} + x_2^{l_0l_1})^{\lambda+\mu+\gamma}}.$$

Proposition 2.3.5. *If λ,μ and γ are $\mathbb{Z}$-linearly independent, then the previous foliation $\mathcal{F}_0$ in $\mathbb{P}^2_\ell$ which does not have a $\mathcal{F}_0$-invariant algebraic curve passing through the points*

$$Sing(\mathcal{F}_0) \cap \mathbb{P}^2_\ell \backslash \{x_0x_1x_2 = 0\}.$$

Proof. Let $p \in Sing(\mathcal{F}_0) \cap \mathbb{P}^2_\ell \backslash \{x_0x_1x_2 = 0\}$, since λ,μ and γ are $\mathbb{Z}$-linearly independent we have that p does not belong none of the curves $\{x_0 = 0\}$, $\{x_1 = 0\}$, $\{x_2 = 0\}$, $\{x_0^{l_1l_2} + x_1^{l_0l_2} + x_2^{l_0l_1} = 0\}$. Suppose that there exists a curve C invariant by $\mathcal{F}_0$ passing through p. By Bézout's Theorem for weighted projective planes we have that C intersects to $\{x_0 = 0\}$. It is clear that the points of intersection between C and $\{x_0 = 0\}$ are contained in the singularities of $\mathcal{F}$. Let q one of these points of intersection between C and $\{x_0 = 0\}$. Without loss of generality we can assume $q = [0:0:1]$. The others case are similar. In a neighborhood of q, we can write the first integral F as

$$F(x_0, x_1, 1) = u(x_0, x_1) x_0^{\lambda} x_1^{\mu},$$

where u is a function non zero and without ramification in a neighborhood of $q = (0, 0)$. Again, because λ,μ and γ are $\mathbb{Z}$-linearly independent we have that $\lambda/\mu \notin \mathbb{Q}$. This implies that the unique separatrices passing through q are $\{x_0 = 0\}$ and $\{x_1 = 0\}$. This completes the proof. □

Corollary 2.3.6. *For all $d > l_0l_1l_2 + l_0l_1 + l_2$, there exists a foliation $\mathcal{F}$ with normal $\mathbb{Q}$-bundle of degree d in $\mathbb{P}^2_\ell$ that does not have a $\mathcal{F}$-invariant algebraic curve passing through the points $Sing(\mathcal{F}) \cap \mathbb{P}^2_\ell \backslash \{x_0x_1x_2 = 0\} \neq \emptyset$.*

Proof. Let $\mathcal{F}_0$ be the foliation on $\mathbb{P}^2_\ell$ of Lemma 2.3.5 induced by η_0. Note that the system $il_0 + jl_1 = d_0$ always has solution solution in $\mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}$ if $d_0 > l_0l_1 - l_0 - l_1$, according Lemma 2.3.1. Therefore, we define a foliation $\mathcal{F}$ on $\mathbb{P}^2_\ell$ induced by

$$\eta = x_0^i x_1^j \eta_0,$$

in which $\deg(N\mathcal{F}) > l_0l_1l_2 + l_0l_1 + l_2$. This completes the proof of the corollary. □

Now, we are going to construct a family examples that contradict item 2 of Proposition 2.3.4 for $d \gg 0$.

For every $j_0 = 1, \ldots, l_2$, let j_1 be the unique integer satisfying $1 \leq j_1 \leq l_2$ and $l_0j_0 \equiv l_1j_1 \mod l_2$. Consider the foliation $\mathcal{F}$ in $\mathbb{C}^2$, induced by the 1-form

$$\eta = (x_1^{l_2} - 1)x_0^{j_0-1} dx_0 - a(x_0^{l_2} - 1)x_1^{j_1-1} dx_1,$$

in which $a \in \mathbb{C}\backslash\mathbb{R}$.

We have to consider two cases:

1. First case: $j_0 = j_1 = l_2$ (nondicrital case).

Lemma 2.3.7. *The foliation $\mathcal{F}$ does not have a $\mathcal{F}$-invariant algebraic curve passing through the point $(0,0)$.*

Proof. We extend $\mathcal{F}$ to a foliation on $\mathbb{P}^2$ that is denoted by $\mathcal{G}$. The foliation $\mathcal{G}$ is induced by

$$\omega = (x_1^{l_2} - x_2^{l_2})x_0^{l_2-1}x_2dx_0 - a(x_0^{l_2} - x_2^{l_2})x_1^{l_2-1}x_2dx_1 + (a(x_0^{l_2} - x_2^{l_2})x_1^{l_2} - (x_2^{l_2} - x_2^{l_2})x_0^{l_2})dx_2\,.$$

Thus $\deg(\mathcal{G}) = 2l_2 - 1$, and $\{x_2 = 0\}$, $\{x_1^{l_2} - x_2^{l_2} = 0\}$, $\{x_0^{l_2} - x_2^{l_2} = 0\}$ are $\mathcal{G}$-invariant.

Notice that the singularities of $\mathcal{G}$ on $\{x_1^{l_2} - x_2^{l_2} = 0\} \cap \{x_0^{l_2} - x_2^{l_2} = 0\}$ are reduced. Also over each of these lines $\mathcal{G}$ has only one extra singularity corresponding to the intersection of the line with $\{x_2 = 0\}$.

Suppose there exists an algebraic curve C invariant by $\mathcal{G}$ passing by $[0:0:1]$. Bézout's Theorem implies that C must intersect the line $\{x_1 - x_2 = 0\}$. Since the singularities of $\mathcal{G}$ on this line outside $\{x_2 = 0\}$ are all reduced and contain two separatrices which do not pass through $[0:0:1]$ we conclude the C intersect this line only at the point $[1:0:0]$. We proceed to make a blow-up at the point $[1:0:0]$. Let $\pi : M \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the point $[1:0:0]$, E be the exceptional divisor, $\tilde{C}$ be the strict transform of C, L_2 be the strict transform of $\{x_2 = 0\}$, F be the strict transform of $\{x_1^{l_2} - x_2^{l_2} = 0\}$, and $\tilde{\mathcal{G}} = \pi^*(\mathcal{G})$ be the foliation on M.

We claim that the singularity at the point $[1:0:0]$ is nondicritical, the singularities of $\tilde{\mathcal{G}}$ on E are all reduced and are contained in the intersection of E with $L_2 \cup F$. In fact, the open set $\pi^{-1}(U_0)$ is covered by $V_0 \cup V_1$ and the first chart is given by

$$\begin{array}{rccl} \pi|_{V_0}: & V_0 & \to & U_0 \\ & (u,v) & \mapsto & (x_1,x_2) = (u,uv). \end{array}$$

The foliation $\tilde{\mathcal{G}}$ on V_0 is induced by

$$\tilde{\eta} = -v(1 - v^{l_2})du + u(a(1 - u^{l_2}v^{l_2}) - (1 - v^{l_2}))dv.$$

We see that E is $\tilde{\mathcal{G}}$-invariant, this implies that the point $[1:0:0]$ is nondicritical. Also, $Sing(\tilde{\mathcal{G}}) \cap E \cap V_0 = \{(0,0), (0,\xi) |\ \xi^{l_2} = 1\}$, and the quotients of eigenvalues of $\tilde{\mathcal{G}}$ at the points $(0,0)$ and $(0,\xi)$ are $1-a$ and $\frac{1-a}{l_2}$. Since $a \in \mathbb{C}\backslash\mathbb{R}$ we conclude that these singularities are all reduced. Similarly to the second chart. This claim contradicts the fact that $\tilde{C} \cap E$ is contained in $Sing(\mathcal{G})$. Therefore we conclude that there is no such curve C invariant by $\mathcal{G}$ passing by $[0:0:1]$. □

2. Second case: $l_2 > j_0, j_1$ (dicritical case).

Lemma 2.3.8. *The foliation $\mathcal{F}$ does not have a $\mathcal{F}$-invariant algebraic curve passing through the point $(0,0)$.*

Proof. Again, we extend $\mathcal{F}$ to a foliation on $\mathbb{P}^2$ that is denoted by $\mathcal{G}$. Assume, without loss of generality, that $j_0 \geq j_1$. In this case the foliation $\mathcal{G}$ is induced by

$$\omega = (x_1^{l_2} - x_2^{l_2})x_0^{j_0-1}x_2dx_0 - a(x_0^{l_2} - x_2^{l_2})x_1^{j_1-1}x_2^{j_0-j_1+1}dx_1 + (a(x_0^{l_2} - x_2^{l_2})x_1^{j_1}x_2^{j_0-j_1} - (x_1^{l_2} - x_2^{l_2}x_0^{j_0}))dx_2\,.$$

Thus $\deg(\mathcal{G}) = l_2 + j_0 - 1$, and $\{x_2 = 0\}$, $\{x_1^{l_2} - x_2^{l_2} = 0\}$, $\{x_0^{l_2} - x_2^{l_2} = 0\}$ are $\mathcal{G}$-invariant.

Notice that the singularities of $\mathcal{G}$ on $\{x_1^{l_2} - x_2^{l_2} = 0\} \cap \{x_0^{l_2} - x_2^{l_2} = 0\}$ are reduced. Also over each of these lines $\mathcal{G}$ has only one extra singularity corresponding to the intersection of the line with $\{x_2 = 0\}$.

Suppose there exists an algebraic curve C invariant by $\mathcal{G}$ passing by $[0 : 0 : 1]$. By the same procedure as above, we conclude the C intersect at points $[1 : 0 : 0]$ and $[0 : 1 : 0]$. We proceed to make a blow-up at the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$. Let $\pi : M \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points $[0 : 1 : 0]$ and $[1 : 0 : 0]$, $E = E_1 \cup E_2$ be the exceptional divisor, $\tilde{C}$ be the strict transform of C, L_2 be the strict transform of $\{x_2 = 0\}$, L_0 be the strict transform of $\{x_1 - x_2 = 0\}$, L_1 be the strict transform of $\{x_0 - x_2 = 0\}$, and $\tilde{\mathcal{G}} = \pi^*(\mathcal{G})$ the foliation on M.

We claim that the singularities at the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ are dicritical, the singularities of $\tilde{\mathcal{G}}$ on E do not belong to the lines L_1 and L_2. In fact, the open set $\pi^{-1}(U_0)$ is covered by $V_0 \cup V_1$ and the first chart is given by

$$\begin{array}{rcl} \pi|_{V_0} : \ V_0 & \to & U_0 \\ (u, v) & \mapsto & (x_1, x_2) = (u, uv). \end{array}$$

The foliation $\tilde{\mathcal{G}}$ on V_0 is induced by

$$\tilde{\eta} = -vu^{l_2 - j_0 - 1}(1 - v^{l_2})du + (a(1 - u^{l_2}v^{l_2})v^{j_0 - j_1} - u^{l_2 - j_0}(1 - v^{l_2}))dv.$$

We see that E_1 is not $\tilde{\mathcal{G}}$-invariant, this implies that $[1 : 0 : 0]$ is dicritical singularity. Also, $L_1 \cap V_0 = \{1 - v = 0\}$ and $Sing(\tilde{\mathcal{G}}) \cap E \cap V_0 = \{(0, 0)\}$. Therefore the singularities of $\tilde{\mathcal{G}}$ on E do not belong to L_1. The same argument show that the singularities of $\tilde{\mathcal{G}}$ on E do not belong to L_0.

Now we define the following map

$$\begin{array}{rcl} \phi : Pic(M) & \to & \mathbb{Z}^2 \\ D & \mapsto & (D.L_0, D.L_1) \end{array}$$

Observe that L_0 and L_1 generate the image of ϕ. From this and $L_2 \in \ker(\phi)$, it follows that

$$\ker(\phi) = \mathbb{Z}L_2.$$

By the last claim, we conclude that $\tilde{C} \in \ker(\psi)$. We can write $\tilde{C} = bL_2$ in $Pic(M)$ for some $b \in \mathbb{Z}$. Then

$$\begin{array}{rcl} \tilde{C}.E & > & 0, \\ bL_2.E & > & 0. \end{array}$$

Hence $b > 0$. On the other hand $\tilde{C}.L_2 \geq 0$ and $L_2.L_2 = -1$, thus $b \leq 0$. This is a contradiction. This implies that there is no such curve C invariant by $\mathcal{G}$ passing by $[0 : 0 : 1]$. $\square$

Corollary 2.3.9. *For all $d \geq l_2 l_1 + l_2 l_0 + l_2$, for all $k \in \mathbb{N}$ and for every $i = 0, 1, 2$, we have*

$$\mathcal{C}_k^{p_i}(d) \neq \mathbb{F}ol(d).$$

If $l_0 = l_1 = 1$ and $l_2 \geq 2$, then $\mathcal{C}_k^{p_2}(d) \neq \mathbb{F}ol(d)$, for all $d \geq 2l_2 + 1$, for all $k \in \mathbb{N}$.

Proof. We show that $\mathcal{C}_k^{p_2}(d) \neq \mathbb{F}ol(d)$. Same arguments can be used to other cases.

Take $d \geq l_2 l_1 + l_2 l_0 + l_2$, let j_0 j_1 be the unique integers satisfying $1 \leq j_0, j_1 \leq l_2$ and $d \equiv l_0 j_0 \equiv l_1 j_1$ mod l_2. By Lemma 2.3.7 and Lemma 2.3.8 the foliation $\mathcal{F}$ on $\mathbb{C}^2$ given by

$$\eta = (x_1^{l_2} - 1)x_0^{j_0 - 1}dx_0 - a(x_0^{l_2} - 1)x_1^{j_1 - 1}dx_1,$$

does not have a $\mathcal{F}$-invariant algebraic curve passing through the point $(0,0)$.

By Remark 1.6.3 we extend $\mathcal{F}$ to a foliation $\hat{\mathcal{F}}$ on $\mathbb{P}^2_\ell$, which is induced by θ and

$$\deg(N\hat{\mathcal{F}}) = \begin{cases} l_0 j_0 + l_1 l_2 + l_2 & , \text{if } l_0 j_0 \geq l_1 j_1, \\ l_1 j_1 + l_0 l_2 + l_2 & , \text{if } l_1 j_1 > l_0 j_0. \end{cases}$$

Since $d \equiv \deg(N\hat{\mathcal{F}}) \mod l_2$ and $d \geq l_2 l_1 + l_2 l_0 + l_2$, we can multiply the 1-form θ by an adequate power of x_2 and construct a foliation $\mathcal{H}$ on $\mathbb{P}^2_\ell$ with normal Q-bundle of degree d. The foliation $\mathcal{H}$ does not have a $\mathcal{H}$-invariant algebraic curve passing through the point $[0:0:1]$. Then $\mathcal{H} \notin \mathcal{C}_k^{p_2}(d)$.

If $l_0 = l_1 = 1$ and $d \geq 2l_2 + 1$, then $d \equiv j(\mod l_2)$, for an unique integer $1 \leq j \leq l_2$. We take $j = j_0 = j_1$ and the foliation $\hat{\mathcal{F}}$ constructed above satisfies $\deg(N\hat{\mathcal{F}}) = 2l_2 + j$. By the same procedure we can construct a foliation $\mathcal{H} \notin \mathcal{C}_k^{p_2}(d)$ for all $k \in \mathbb{N}$. □

Proof of Theorem 1. By Lemma 2.1.5 $\mathcal{C}_k(d)$ is an algebraic closed subset of $\mathbb{F}ol(d)$ for all $k \in \mathbb{N}$. We claim that if $d \geq l_0 l_1 l_2 + l_0 l_1 + 2l_2$, then $\mathcal{C}_k(d) \neq \mathbb{F}ol(d)$, for all $k \in \mathbb{N}$. In fact, if we had $\mathcal{C}_k(d) = \mathbb{F}ol(d)$ for some $k \in \mathbb{N}$, then we could apply Proposition 2.3.4 and this would contradict Corollary 2.3.9 and Corollary 2.3.6. Hence the complement of $\mathcal{C}_k(d)$ in $\mathbb{F}ol(d)$ is an dense open set if $d \geq l_0 l_1 l_2 + l_0 l_1 + 2l_2$. We conclude the proof of Jouanolou's Theorem on $\mathbb{P}^2_\ell$ applying Baire's Theorem. □

2.3.2 Reduction of problem

In this subsection, we only consider saturated foliations and the notations as in Chapter 1. Our goal is to reduce from the case of foliations on $\mathbb{P}^2_\ell$, $\ell = (l_0, l_1, l_2)$ with $\gcd(l_0, l_1, l_2) = 1$ to the case of foliations on $\mathbb{P}^2_{\ell'}$, $\ell' = (l'_0, l'_1, l'_2)$, with l'_0, l'_1, l'_2 pairwise coprimes.

Let $\ell = (l_0, l_1, l_2)$ be a weighted vector with $\gcd(l_0, l_1, l_2) = 1$. We have

$$l_0 = r_1 r_2 l'_0, \;\; l_1 = r_0 r_2 l'_1, \;\; l_2 = r_0 r_1 l'_2,$$

where

$$r_0 = \gcd(l_1, l_2), \;\; r_1 = \gcd(l_0, l_2), \;\; r_2 = \gcd(l_0, l_1).$$

Therefore, we have $l_0 l_1 l_2 = r^2 l'_0 l'_1 l'_2$, where $r = r_0 r_1 r_2$, $\ell' = (l'_0, l'_1, l'_2)$ and also the natural isomorphism

$$\begin{array}{rcl} \varphi^* : H^0(\mathbb{P}^2_{\ell'}, \Omega^k_{\mathbb{P}^2_{\ell'}}(d)) & \to & H^0(\mathbb{P}^2_\ell, \Omega^k_{\mathbb{P}^2_\ell}(rd)) \\ \eta & \mapsto & \varphi^*\eta, \end{array}$$

which is induced by the following isomorphism

$$\begin{array}{rcl} \varphi : \mathbb{P}^2_\ell & \to & \mathbb{P}^2_{\ell'} \\ {[x_0 : x_1 : x_2]_\ell} & \mapsto & [x_0^{r_0} : x_1^{r_1} : x_2^{r_2}]_{\ell'} = [y_0 : y_1 : y_2]_{\ell'}. \end{array}$$

Note that, in the case $k = 1$, the isomorphism is valid for any $d \geq \min\{l'_0 + l'_1, l'_0 + l'_2, l'_1 + l'_2\}$.

Remark 2.3.10. *Let $\mathcal{F} \in \mathbb{F}ol(d,\ell')$ given by η, and $\mathcal{G} = \varphi^*\mathcal{F}$ be the foliation on $\mathbb{P}^2_l$ induced by ω. Then*

$$\omega = \begin{cases} \varphi^*\eta & \text{, if } \{y_0=0\},\ \{y_1=0\} \text{ and } \{y_2=0\} \text{ are not } \mathcal{F}\text{-invariant,} \\[2ex] \dfrac{\varphi^*\eta}{x_0^{r_0-1}} & \text{, if } \{y_0=0\} \text{ is } \mathcal{F}\text{-invariant and } \{y_1=0\},\ \{y_2=0\} \text{ are not } \mathcal{F}\text{-invariant,} \\[2ex] \dfrac{\varphi^*\eta}{x_0^{r_0-1}x_1^{r_1-1}} & \text{, if } \{y_0y_1=0\} \text{ is } \mathcal{F}\text{-invariant and } \{y_2=0\} \text{ is not } \mathcal{F}\text{-invariant,} \\[2ex] \dfrac{\varphi^*\eta}{x_0^{r_0-1}x_1^{r_1-1}x_2^{r_2-1}} & \text{, if } \{y_0y_1y_2=0\} \text{ is } \mathcal{F}\text{-invariant.} \end{cases}$$

Recall that

$\mathbb{F}ol'(d,\ell') = \{\mathcal{F} \in \mathbb{F}ol(d,\ell')|\,\{y_0=0\},\{y_1=0\},\{y_2=0\}$ are not $\mathcal{F}$-invariant$\}$,

$\mathbb{F}ol'_i(d,\ell') = \{\mathcal{F} \in \mathbb{F}ol(d,\ell')|\,\{y_i=0\}$ is $\mathcal{F}$-invariant and $\{y_j=0\},\{y_k=0\}$ are not $\mathcal{F}$-invariant, $j \neq k\}$,

$\mathbb{F}ol'_{i,j}(d,\ell') = \{\mathcal{F} \in \mathbb{F}ol(d,\ell')|\,\{y_iy_j=0\}$ is $\mathcal{F}$-invariant and $\{y_k=0\}$ is not $\mathcal{F}$-invariant$\}$,

$\mathbb{F}ol_{0,1,2}(d,\ell') = \{\mathcal{F} \in \mathbb{F}ol(d,\ell')|\,\{y_0y_1y_2=0\}$ is $\mathcal{F}$-invariant$\}$.

Proposition 2.3.11. *Under the conditions above, we have the following isomorphisms:*

1. $H^0(\mathbb{P}^2_{\ell'},\Omega^k_{\mathbb{P}^2_{\ell'}}(d)) \xrightarrow{\varphi^*} H^0(\mathbb{P}^2_{\ell},\Omega^k_{\mathbb{P}^2_{\ell}}(rd)) \quad \forall\, d,\ \forall\, k \geq 0.$
2. $\mathbb{F}ol'(d,\ell') \xrightarrow{\varphi^*} \mathbb{F}ol'(rd,\ell) \quad \forall\, d.$
3. $\mathbb{F}ol'_i(d,\ell') \xrightarrow{\varphi^*} \mathbb{F}ol'_i(rd - l_i(r_i-1),\ell) \quad \forall\, d.$ *If* $l_i > 1$.
4. $\mathbb{F}ol'_{i,j}(d,\ell') \xrightarrow{\varphi^*} \mathbb{F}ol'_{i,j}(rd - l_i(r_i-1) - l_j(r_j-1),\ell) \quad \forall\, d.$ *If* $l_i,\, l_j > 1$ *and* $i \neq j$.
5. $\mathbb{F}ol_{0,1,2}(d,\ell') \xrightarrow{\varphi^*} \mathbb{F}ol_{0,1,2}(rd - l_0(r_0-1) - l_1(r_1-1) - l_2(r_2-1),l)$ *If* $l_0,\, l_1,\, l_2 > 1$.

In conclusion, for all $\mathcal{G} \in \mathbb{F}ol(\tilde{d},\ell)$, $\tilde{d} \geq \min\{l_0+l_1, l_0+l_2, l_1+l_2\}$, $\mathcal{G}$ satisfies one of the above items 2, 3, 4, or 5.

Proof. 1. Follows from the fact that φ is an isomorphism. To prove items 2 and 3, it is sufficient to prove surjectivity.

2. Let $\mathcal{G} \in \mathbb{F}ol'(rd,\ell)$ given by ω. Then by Remark 2.3.10 there exists $\eta \in H^0(\mathbb{P}^2_{\ell'},\Omega^1_{\mathbb{P}^2_{\ell'}}(d))$ such that $\varphi^*\eta = \omega$. We only need to show that $codim\,Sing(\eta) = 0$. To do this, suppose that $\eta = F\eta'$, then $(\varphi^*F)(\varphi^*\eta') = \varphi^*\eta = \omega$, and since $codim\,Sing(\omega) = 0$, we thus get φ^*F is a constant, hence F is a constant. By Remark 2.3.10, we concluded that η induces the foliation $\mathcal{F} \in \mathbb{F}ol'(d,\ell')$.

3. Suppose $i = 0$ and $l_0 > 1$. Let $\mathcal{G} \in \mathbb{F}ol'_0(rd - l_0(r_0-1),\ell)$ given by ω, we define

$$\tilde{\omega} = x_0^{r_0-1}\omega \in H^0(\mathbb{P}^2_{\ell},\Omega^1_{\mathbb{P}^2_{\ell}}(rd)).$$

Then by Remark 2.3.10 there exists $\eta \in H^0(\mathbb{P}^2_{\ell'}, \Omega^1_{\mathbb{P}^2_{\ell'}}(d))$ such that $\varphi^*\eta = \tilde{\omega}$. It suffices to prove that $codim\, Sing(\eta) = 0$. To see this, suppose that $\eta = F\eta'$, hence $(\varphi^* F)(\varphi^*\eta') = \varphi^*\eta = \tilde{\omega} = x_0^{r_0-1}\omega$, thus $x_0^{r_0-1}$ divides $\varphi^* F$, then $F = y_0^k$, and hence $k = 0$. Since $\mathcal{G} \in \mathbb{F}ol_0'(rd - l_0(r_0-1), \ell)$, Remark 2.3.10 implies that η induces the foliation $\mathcal{F} \in \mathbb{F}ol_0'(d, \ell')$.

Items 4 and 5 follow by analogous arguments used in 2 and 3, applying following definitions:

5. For example, for the case $i = 0$ and $j = 1$, define $\tilde{\omega} = x_0^{r_0-1}x_1^{r_1-1}\omega$.

6. Define $\tilde{\omega} = x_0^{r_0-1}x_1^{r_1-1}x_2^{r_2-1}\omega$.

Now let $\mathcal{G} \in \mathbb{F}ol(\tilde{d}, \ell)$ given by

$$\omega = Adx_0 + Bdx_1 + Cdx_2.$$

Let us prove that $\mathcal{G}$ satisfies one of the items 2, 3, 4 or 5. We will prove two cases, the other cases are analogous.

First case: Assume that $\{x_0 = 0\}$, $\{x_1 = 0\}$, and $\{x_2 = 0\}$ are not $\mathcal{G}$-invariant. We can always write

$$B = x_0^{q_1} B_{q_2}(x_1, x_2) + \ldots + B_{\tilde{d}-l_1}(x_1, x_2),$$

where $\tilde{d} - l_1 = q_1 l_0 + q_2$. Since $\{x_0 = 0\}$ is not $\mathcal{G}$-invariant, there exist $(i, j) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}$ such that $\tilde{d} - l_1 = il_1 + jl_2$. That implies $\gcd(l_1, l_2)|\tilde{d} - l_1$, so $r_0|\tilde{d}$. Analogously for the quasi-homogeneous polynomials A, B and using the fact that $\{x_1 = 0\}$, $\{x_2 = 0\}$ are not $\mathcal{G}$-invariant respectively, we get that $r_1|\tilde{d}$ and $r_2|\tilde{d}$. Therefore, we can conclude that $r_0r_1r_2|\tilde{d}$,*i.e.*, $\tilde{d} = rd$. Hence $\mathcal{G} \in \mathbb{F}ol'(rd, \ell)$, this finishes the proof of the first case.

Second case: Suppose that $\{x_0 = 0\}$ is $\mathcal{G}$-invariant and $\{x_1 = 0\}$, $\{x_2 = 0\}$ are not $\mathcal{G}$-invariant. Define

$$\tilde{\tilde{d}} = \tilde{d} + l_0(r_0 - 1).$$

Because $\{x_1 = 0\}$, $\{x_2 = 0\}$ are not $\mathcal{G}$-invariant, we can apply the first case, then $r_1r_2|\tilde{d}$. Moreover, $r_1r_2|l_0$, thus $r_0r_1|\tilde{\tilde{d}}$. We can write

$$A = x_0^{q_1} A_{q_2}(x_1, x_2) + \ldots + A_{\tilde{d}-l_0}(x_1, x_2),$$

since $\mathcal{G} \in \mathbb{F}ol(\tilde{d}, l)$ and $\{x_0 = 0\}$ is $\mathcal{G}$-invariant, then there exists $(i, j) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}$ such that $\tilde{d} - l_0 = il_1 + jl_2$, therefore $\gcd(l_1, l_2)|\tilde{d} - l_0$. Moreover $r_0|\tilde{d} - l_0$ and $\tilde{\tilde{d}} = (\tilde{d} - l_0) + l_0r_0$, hence $r_0|\tilde{\tilde{d}}$, we can conclude that $\tilde{d} = rd - l_0(r_0 - 1)$. Thus $\mathcal{G} \in \mathbb{F}ol_0'(rd - l_0(r_0 - 1), l)$, the proof of second case is complete. $\square$

Corollary 2.3.12. *Let $\ell = (l_0, l_1, l_2)$ be a weighted vector with $\gcd(l_0, l_1, l_2) = 1$ and $\mathcal{F}$ be a foliation on $\mathbb{P}^2_\ell$. If $\deg(K\mathcal{F}) < 0$, then $\mathcal{F}$ is a rational fibration.*

Proof. Since $\deg(K\mathcal{F}) < 0$ we have that $\deg(N\mathcal{F}) < |l| = l_0 + l_1 + l_2$. Let $\ell' = (l_0', l_1', l_2')$ and $\varphi : \mathbb{P}^2_\ell \to \mathbb{P}^2_{\ell'}$ as Proposition 2.3.11. We claim that if $\mathcal{F} = \varphi^*\mathcal{G}$ and $\deg(N\mathcal{F}) < l_0 + l_1 + l_2$, then $\deg(N\mathcal{G}) < l_0' + l_1' + l_2'$. In fact, by Proposition 2.3.11 we have

$$r\deg(N\mathcal{G}) - l_0(r_0 - 1) - l_1(r_1 - 1) - l_2(r_2 - 1) \leq \deg(N\varphi^*\mathcal{G}).$$

Using $\deg(N\varphi^*\mathcal{G}) < l_0 + l_1 + l_2$, we see that $r\deg(N\mathcal{G}) < r_0l_0 + l_1r_1 + l_2r_2$. Since $l_0 = l_0'r_1r_2$, $l_1 = l_1'r_0r_2$ and $l_2 = l_2'r_0r_1$ we conclude that $\deg(N\mathcal{G}) < l_0' + l_1' + l_2'$. The corollary follows from Proposition 2.2.1. $\square$

Corollary 2.3.13. *Under the conditions and notations of Proposition 2.3.11, a generic foliation with normal* $\mathbb{Q}$*-bundle of degree* rd *in* $\mathbb{P}^2_{(l_0,l_1,l_2)}$ *does not have any invariant algebraic curves if*

$$d \geq l_0'l_1'l_2' + l_0'l_1' + l_0'l_2' + l_1'l_2' + l_0' + l_1' + l_2'.$$

Proof. Using the isomorphism $\varphi^* : \mathbb{F}ol'(d,\ell') \to \mathbb{F}ol'(rd,\ell)$ of Proposition 2.3.11 and Theorem 1, the result follows. □

2.4 Foliations without algebraic leaves on $\mathbb{P}_{(1,1,l_2)}$, $l_2 > 1$

We first recall of Definition 2.3.2 that $X = \mathbb{P}^2_{(1,1,l_2)} \backslash \{p_2\}$ and the following algebraic sets

$$\begin{aligned} \mathcal{S}(d) &= \{(x,\mathcal{F}) \in \mathbb{P}^2_{(1,1,l_2)} \times \mathbb{F}ol(d)) \,|\, x \in sing(\mathcal{F})\}, \\ \mathcal{S}_X(d) &= \{(x,\mathcal{F}) \in X \times \mathbb{F}ol(d) |\, x \in sing(\mathcal{F})\}, \\ \mathcal{S}_{p_2}(d) &= \{(p_2,\mathcal{F}) \in \mathcal{S}(d)\}. \end{aligned}$$

The following proposition allows to determine when the algebraic variety $\mathcal{S}(d)$ is not irreducible.

Proposition 2.4.1. *Let* $\ell = (1,1,l_2)$ *be a weighted vector with* $l_2 \geq 2$ *and* $d > l_2$ *be an integer. The following assertions hold true:*

1. *If* $d \equiv 1(\mod l_2)$*, then* $\mathcal{S}(d)$ *is an irreducible variety of codimension two in* $\mathbb{P}^2_\ell \times \mathbb{F}ol(d)$*.*
2. *If* $d \not\equiv 1(\mod l_2)$*, then* $\mathcal{S}(d)$ *is not irreducible variety with decomposition in irreducible components equal to*
$$\overline{\mathcal{S}_X(d)} \cup \mathcal{S}_{p_2}(d).$$
Furthermore $\dim \mathcal{S}(d) = \dim \mathcal{S}_X(d) = \dim \mathcal{S}_{p_2}(d) = \dim \mathbb{F}ol(d)$.

Proof. 1. Consider the projection $\pi_1 : \mathcal{S}(d) \to \mathbb{P}^2_\ell$. For every $x \in \mathbb{P}^2_{(1,1,l_2)}$, the fiber $\pi_1^{-1}(x)$ is a subvariety of $\{x\} \times \mathbb{F}ol(d)$ contained in $\mathcal{S}(d)$ isomorphic to a projective space, *i.e.*, if two 1-forms η and η' vanish in x thus the same is true for a linear combination of them. Since the automorphism group of $\mathbb{P}^2_{(1,1,l_2)}$ acts transitively on X (see [18, section 4]), it follows that every fiber of π_1 restricted to $\mathcal{S}_X(d) \subset X \times \mathbb{F}ol(d)$ is smooth, irreducible and biholomorphic to a projective space of dimension N. It is clear that $\pi_1^{-1}(p_2) = \mathcal{S}_{p_2}(d)$ is smooth and irreducible, but in general it is not true that $\dim_{\mathbb{C}} \mathcal{S}_{p_2}(d) = \dim_{\mathbb{C}} \pi_1^{-1}(p_0)$, the condition $d \equiv 1(\mod l_2)$ is necessary. We claim that under this condition $\mathcal{S}_{p_2}(d)$ has codimension two on $\{p_2\} \times \mathbb{F}ol(d)$ and $\pi_1^{-1}(p_0)$ has codimension two on $\{p_0\} \times \mathbb{F}ol(d)$. The condition $d \equiv 1(\mod l_2)$ implies that there exists a quasi-homogeneous polynomials F of degree $d - l_2 - 1$ such that $F(p_2) \neq 0$, thus the 1-forms $\eta_1 = F(l_2x_2dx_0 - x_0dx_2)$ and $\eta_2 = F(l_2x_2dx_1 - x_1dx_2)$ are degree d, and they are such that $\eta_1(p_0)$ and $\eta_2(p_0)$ are $\mathbb{C}$-linearly independent. Thus any 1-form η of degree d such that $\eta(p_2) \neq 0$ can be written as a $\mathbb{C}$-linear combination of η_1 and η_2. Similarly for the fiber $\pi_1^{-1}(p_0)$. Since all the fibers $\pi_1^{-1}(x)$ for $x \in \mathbb{P}^2_{(1,1,l_2)}$ are irreducible and of the same dimension, then $\mathcal{S}(d)$ is irreducible, see [38, Theorem 8, page 77].

2. Notice that the condition $d \not\equiv 1(\mod l_2)$ implies that $\mathcal{S}_{p_2}(d) = \{p_2\} \times \mathbb{F}ol(d)$. Consider the following exact sequence

$$0 \longrightarrow \ker\psi \xrightarrow{\ i\ } H^0(\mathbb{P}^2_\ell, \Omega^1_{\mathbb{P}^2_\ell}(d)) \otimes \mathcal{O}_X \xrightarrow{\ \psi\ } \Omega^1_{\mathbb{P}^2_\ell}(d)|_X \ ,$$

where $\psi(x,\eta)=\eta(x)$. We claim that $\ker\psi$ is a vector bundle. From the proof of item 1, we have that all fibers $\pi_1^{-1}(x)$ for $x\in X$ are smooth and of dimension $dim_{\mathbb{C}}\mathbb{F}ol(d)-2$. This shows that $\dim_{\mathbb{C}}\ker\psi_x$ is constant as a function of $x\in X$. Hence $\ker\psi$ is a vector bundle and in particular $\mathbb{P}(\ker\psi)=\mathcal{S}_X(d)$ is a irreducible variety of codimension two on $X\times\mathbb{F}ol(d)$. Therefore, we have that $\overline{\mathcal{S}_X(d)}$ and $\mathcal{S}_{p_2}(d)$ are irreducible varieties of dimension equal to $\dim\mathbb{F}ol(d)$ and $\mathcal{S}(d)=\overline{\mathcal{S}_X(d)}\cup\mathcal{S}_{p_2}(d)$. We claim that the varieties $\overline{\mathcal{S}_X(d)}$ and $\mathcal{S}_{p_2}(d)$ are the irreducible components of $\mathcal{S}(d)$. By [28, Proposition 1.5, page 5] we only require that $\overline{\mathcal{S}_X(d)}\not\supset\mathcal{S}_{p_2}(d)$. Suppose that $\mathcal{S}_{p_2}(d)\subset\overline{\mathcal{S}_X(d)}$. Since $\dim\mathcal{S}_{p_2}(d)=\dim\overline{\mathcal{S}_X(d)}$ we get that $\mathcal{S}_{p_2}(d)=\overline{\mathcal{S}_X(d)}$. Hence $\mathcal{S}_{p_2}(d)\supset\mathcal{S}_X(d)$. This contradicts the fact that $\mathcal{S}_{p_2}(d)\cap\mathcal{S}_X(d)=\emptyset$. □

Recall that

$$\mathcal{C}_k^{p_2}(d)=\{\mathcal{F}\in\mathbb{F}ol(d)\,|\quad p_2\in C\text{ for some }\mathcal{F}\text{-invariant algebraic curve of degree }k\},$$

is a closed set.

Proposition 2.4.2. *Let $d>l_2$ and suppose that $\mathcal{C}_k(d)=\mathbb{F}ol(d)$ for some $k>0$. Then*

1. *If $d\equiv 1(\mod l_2)$, then $\mathcal{S}(d)\cap\mathcal{D}_k(d)=\mathcal{S}(d)$.*
2. *If $d\not\equiv 1(\mod l_2)$ and $\mathcal{C}_k^{p_2}(d)=\mathcal{C}_k(d)$, then $\mathcal{S}_{p_2}(d)\cap\mathcal{D}_k(d)=\mathcal{S}_{p_2}(d)$.*
3. *If $d\not\equiv 1(\mod l_2)$ and $\mathcal{C}_k^{p_2}(d)\subsetneq\mathcal{C}_k(d)$, then $\mathcal{S}_X(d)\cap\mathcal{D}_k(d)=\mathcal{S}_X$.*

Proof. Let $\pi_2:\mathbb{P}^2_\ell\times\mathbb{F}ol(d)\to\mathbb{F}ol(d)$ be the projection on the second coordinate and $\pi:\mathcal{S}_X(d)\to\mathbb{F}ol(d)$ be the restriction of π_2 to $\mathcal{S}_X(d)$.

1. If $\mathcal{C}_k(d)=\mathbb{F}ol(d)$, then all foliations with normal $\mathbb{Q}$-bundle of degree d admits a invariant algebraic curve of degree k. Using Camacho-Sad Theorem in weighted projective plane and the projection π_2, we get that

$$\pi_2(\mathcal{S}(d)\cap\mathcal{D}_k(d))=\mathcal{S}(d). \tag{2.4}$$

By Proposition 2.4.1 we have that $\mathcal{S}(d)$ is an irreducible variety and $\dim\mathcal{S}(d)=\dim\mathbb{F}ol(d)$. From (2.4) it follows that

$$\mathcal{S}(d)\cap\mathcal{D}_k(d)=\mathcal{S}(d).$$

2. Since $d\not\equiv 1(\mod l_2)$ implies that $\mathcal{S}_{p_2}(d)=\{p_2\}\times\mathbb{F}ol(d)$ and by the condition $\mathcal{C}_k^{p_2}(d)=\mathcal{C}_k(d)$, we thus get $\mathcal{S}_{p_2}(d)\cap\mathcal{D}_k(d)=\mathcal{S}_{p_2}(d)$.

3. We have that $\mathcal{C}_k(d)=\mathbb{F}ol(d)$ and $\mathcal{C}_k^{p_2}(d)\subsetneq\mathcal{C}_k(d)$ is a closed set by Lemma 2.1.5, then we define the nonempty open set

$$U_k=\mathbb{F}ol(d)\backslash\mathcal{C}_k^{p_2}(d).$$

Using Camacho-Sad Theorem in weighted projective plane, we get that

$$\pi(\pi^{-1}(U_k)\cap\mathcal{D}_k(d))=U_k.$$

By Proposition 2.4.1 we have that $\pi^{-1}(U_k)$ is an irreducible open set of $\mathcal{S}_X(d)$ and

$$\dim\pi^{-1}(U_k)=\dim\mathbb{F}ol(d).$$

Since $\dim\pi(\pi^{-1}(U_k)\cap\mathcal{D}_k(d))=\dim U_k=\dim\mathbb{F}ol(d)$ implies that

$$\dim\pi^{-1}(U_k)\cap\mathcal{D}_k(d)=\dim\pi^{-1}(U_k),$$

and hence

$$\pi^{-1}(U_k) \cap \mathcal{D}_k(d) = \pi^{-1}(U_k).$$

Taking closure on $\mathbb{P}^2_{(1,1,l_2)} \times \mathbb{F}ol(d)$, we get

$$\overline{\mathcal{S}_X(d)} \cap \mathcal{D}_k(d) = \overline{\mathcal{S}_X(d)}.$$

In particular we have

$$\mathcal{S}_X(d) \cap \mathcal{D}_k(d) = \mathcal{S}_X(d).$$

□

The next subsection, we will construct examples to prove Theorem 2.

2.4.1 Existence of singularities without algebraic separatrix

The following family of examples allow us to obtain the bound of Theorem 2.

Let $\mathcal{F}_1$ be a foliation in $\mathbb{P}^2_{(1,1,l_2)}$, $l_2 > 1$ induced by the following 1-form

$$\eta = -l_2 x_2(x_2 - x_0 x_1^{l_2-1})dx_0 + l_2 x_0 x_2(x_1^{l_2-1} - x_0 x_1^{l_2-2})dx_1 + x_0(x_2 - x_1^{l_2})dx_2.$$

Notice that $\deg(N\mathcal{F}_1) = 2l_2 + 1$, $Sing(\mathcal{F}_1) = \{[0:1:0], [1:0:0], [1:1:1]\}$ and $\{x_0 = 0\} \cap Sing(\mathcal{F}_1) = \{[0:1:0]\}$.

Lemma 2.4.3. *The foliation $\mathcal{F}_1$ does not have a $\mathcal{F}_1$-invariant algebraic curve passing through the point* $[1:1:1]$.

Proof. Observe that the lines $\{x_0 = 0\}$ and $\{x_2 = 0\}$ are $\mathcal{F}_1$-invariant. Suppose there exists an algebraic curve C invariant by $\mathcal{F}_1$ passing by $[1:1:1]$. Since $\{x_0 = 0\} \cap Sing(\mathcal{F}_1) = \{[0:1:0]\}$ and Bézout's Theorem for weighted projective planes we conclude the $\{x_0 = 0\}$ only intersects to C at the point $[0:1:0]$. In the open set $U_1 \simeq \mathbb{C}^2$ the foliation $\mathcal{F}_1|_{U_1}$ is induced by

$$\eta_1 = -l_2 x_2(x_2 - x_0)dx_0 + x_0(x_2 - 1)dx_2.$$

Note that $(0,0)$ is a saddle-node singularity with only two separatrices, then there no exists the curve C passing through the point $[0:1:0]$. □

2.4.2 Proof of Theorem 2

The proof is quite similar to the one presented on Theorem 1, but in contrast to that, in this case we obtain the best possible bound $d \geq 2l_2 + 1$. For this, it is enough to show that $\mathcal{C}_k(d) \neq \mathbb{F}ol(d)$, holds for all $k \in \mathbb{N}$ and all $d \geq 2l_2 + 1$. Since the complement of $\mathcal{C}_k(d)$ in $\mathbb{F}ol(d)$ is an dense open set if $d \geq 2l_2 + 1$. Reasoning by contradiction, suppose this does not hold. Then Proposition 2.4.2 and Corollary 2.3.9 would imply $\mathcal{S}_X(d) \cap \mathcal{D}_k(d) = \mathcal{S}_X$, that contradicts Corollary 2.3.6, for $d > 2l_2 + 1$, and Lemma 2.4.3, for $d = 2l_2 + 1$. □

2.5 Holomorphic foliations on Hirzebruch surfaces

As in Example 1.8.5, let $\mathbb{F}_{l_2} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(l_2))$ be a Hirzebruch surface, $\pi : \mathbb{F}_{l_2} \to \mathbb{P}^2_{(1,1,l_2)}$ be the weighted blow-up at $p = [0 : 0 : 1]$ of type $(l_2, (1, 1))$, E be the exceptional divisor, L be a line passing through p and F be the strict transform of L. Observe that

$$Pic(\mathbb{F}_{l_2}) = \mathbb{Z}F + \mathbb{Z}E,$$

in which $E.E = -l_2$, $F.F = 0$ and $F.E = 1$

We then denote by $\mathcal{R}(a, b) = \mathbb{P}H^0(\mathbb{F}_{l_2}, \Omega^1_{\mathbb{F}_{l_2}} \otimes \mathcal{O}_{\mathbb{F}_{l_2}}(aF + bE))$ the space of holomorphic foliations with normal bundle of bidegree (a, b) in $\mathbb{F}_{l_2}$.

Remark 2.5.1. Let $\mathcal{F}$ be a foliation with normal $\mathbb{Q}$-bundle d on $\mathbb{P}^2_{(1,1,l_2)}$ and $\mathcal{G} = \pi^*\mathcal{F}$ be the foliation on $\mathbb{F}_{l_2}$. We denote by $k = k(p) = \text{multalg}_p(\mathcal{F})$ and

$$e = \begin{cases} k - l_2 & \text{, if } E \text{ is } \mathcal{G}\text{-invariant,} \\ k & \text{, if } E \text{ is not } \mathcal{G}\text{-invariant.} \end{cases}$$

$$N\mathcal{G} = \mathcal{O}_{\mathbb{F}_{l_2}}\left(dF + \frac{d-e}{l_2}E\right).$$

Note that $d - e \equiv 0(\mod l_2)$.

The following proposition characterizes some foliations on $\mathbb{F}_{l_2}$.

Proposition 2.5.2. *The following assertions hold true:*

1. *If $b = 0$, then $a = 2$ and $\mathcal{G} \in \mathcal{R}(2, 0)$ is a rational fibration.*
2. *If $b = 2$, then $\mathcal{G} \in \mathcal{R}(a, b)$ is a Ricatti foliation.*

Proof. See [8, Proposition 1, page 51]. □

Remark 2.5.3. Let $\mathcal{F} \in \mathcal{R}(a, b)$ and suppose that E is not $\mathcal{F}$-invariant. Then by tangency formula

$$0 \leq Tang(\mathcal{G}, E) = -bl_2 + a - 2.$$

Hence

$$a \geq bl_2 + 2. \tag{2.5}$$

After we will see that this is one of the conditions for a generalization of Jouanolou's result for Hirzebruch surfaces.

The following proposition characterizes foliations on $\mathbb{F}_{l_2}$ with some algebraic solution.

Proposition 2.5.4. *If $a < bl_2 + 2$ or $b < 3$ then any foliation $\mathcal{G} \in \mathcal{R}(a, b)$ admits some invariant algebraic curve.*

Proof. If $a < bl_2 + 2$, then by Remark 2.5.3 we have that E is $\mathcal{G}$-invariant. We can assume that $a - bl_2 \geq 2$ and $0 < b \leq 2$. Baum-Bott formula implies that $\sum BB(N\mathcal{G}, p) = N\mathcal{G}^2 = (aF + bE)^2 = b(a + a - bl_2) > 0$, therefore there exists a point $p \in Sing(\mathcal{G})$. Let F be the fiber passing through the point p. We claim that F is $\mathcal{G}$-invariant. If we suppose that F is not $\mathcal{G}$-invariant, then by Tangency formula we see that

$$0 < Tang(\mathcal{G}, F) = N\mathcal{G}.F - \chi(F) = b - 2 \leq 0,$$

this is a contradiction. $\square$

Let $\chi \in \mathbb{Q}[t]$. Define two subsets of $\mathbb{F}_{l_2} \times \mathcal{R}(a,b)$ by

$$\mathcal{S}(a,b) = \{(x, \mathcal{G}) \in \mathbb{F}_{l_2} \times \mathcal{R}(a,b) |\, x \in Sing(\mathcal{G})\},$$

and

$$\mathcal{D}_\chi(a,b) = \{(x, \mathcal{G}) \in \mathbb{F}_{l_2} \times \mathcal{R}(a,b) |\ x \text{ is in subscheme, invariant by } \mathcal{G}, \text{ of Hilbert polynomial } \chi\}.$$

Proposition 2.5.5. *The following statements are true.*

1. *If $b \geq 2$ and $a \geq bl_2 + 2$, then $\mathcal{S}(a,b)$ is a closed irreducible variety of $\mathbb{F}_{l_2} \times \mathcal{R}(a,b)$ and*

$$\dim_{\mathbb{C}} \mathcal{S}(a,b) = \dim_{\mathbb{C}} \mathcal{R}(a,b).$$

2. *$\mathcal{D}_\chi(a,b)$ is a closed subset of $\mathbb{F}_{l_2} \times \mathcal{R}(a,b)$.*

Proof. 1. Let us denote by $\Sigma(a,b)$ the line bundle $\mathcal{O}_{\mathbb{F}_{l_2}}(aF + bE)$ on $\mathbb{F}_{l_2}$. Consider the following exact sequence

$$0 \longrightarrow \ker\psi \xrightarrow{\ i\ } H^0(\mathbb{F}_{l_2}, \Omega^1_{\mathbb{F}_{l_2}} \otimes \Sigma(a,b)) \otimes \mathcal{O}_{\mathbb{F}_{l_2}} \xrightarrow{\ \psi\ } \Omega^1_{\mathbb{F}_{l_2}} \otimes \Sigma(a,b)\ ,$$

where $\psi(x,\eta) = \eta(x)$. We claim that $\ker\psi$ is a vector bundle. In fact, since

$$\pi_{|_{\mathbb{F}_{l_2}\backslash E}} : \mathbb{F}_{l_2} \backslash E \to \mathbb{P}^2_{(1,1,l_2)} \backslash \{[0:0:1]\},$$

is an isomorphism and the automorphism group of $\mathbb{P}^2_{(1,1,l_2)}$ acts transitively on $\mathbb{P}^2_{(1,1,l_2)} \backslash \{[0:0:1]\}$, then the automorphism group of $\mathbb{P}^2_{(1,1,l_2)}$ acts transitively on $\mathbb{F}_{l_2} \backslash E$. Therefore $\ker\psi_x$ has same dimension for all $x \in \mathbb{F}_{l_2} \backslash E$. We claim that for all $x \in E$, $\ker\psi_x$ has dimension $\dim_{\mathbb{C}} \mathcal{R}(a,b) - 2$. In fact, we take two foliations $\mathcal{F}_1, \mathcal{F}_2$ on $\mathbb{P}^2_{(1,1,l_2)}$ with normal $\mathbb{Q}$-bundle of degree a induced by the 1-forms $\eta_1 = x_2^b A_{a-bl_2-2}(x_0,x_1)(x_0,x_1)(x_1dx_0 - x_0dx_1)$ and $\eta_2 = x_2^{b-2} B_{a-bl_2+l_2-1}(x_0,x_1)(l_2x_2dx_0 - x_0dx_2)$ respectively, where A_{a-bl_2-2} and $B_{a-bl_2+l_2-1}$ are homogeneous polynomials of degree $a - bl_2 - 2$ and $a - bl_2 + l_2 - 1$ respectively in $\mathbb{C}[x_0,x_1]$. Let $\mathcal{G}_1 = \pi^*(\mathcal{F}_1)$ and $\mathcal{G}_2 = \pi^*(\mathcal{F}_2)$ be the foliations on $\mathbb{F}_{l_2}$. Observe that $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{R}(a,b)$. In the chat $U_2 = X(l_2,(1,1)) \simeq \mathbb{C}^2/\mu_{l_2}$ we lift $\mathcal{F}_1|_{U_2}$ and $\mathcal{F}_2|_{U_2}$ to $\mathbb{C}^2$ given by

$$\eta_1|_{U_2} = A_{a-bl_2-2}(x_0,x_1)(x_0,x_1)(x_1dx_0 - x_0dx_1),$$

and

$$\eta_2|_{U_2} = l_2 B_{a-bl_2+l_2-1}(x_0,x_1)dx_0,$$

respectively. We have that $\pi^{-1}(U_2)$ is covered by $V_0 \cup V_1$. The first chart on $V_0 = X(1,(-l_2,1)) \simeq \mathbb{C}^2$ is given by

$$\begin{cases} x_0 = u^{1/l_2}, \\ x_1 = u^{1/l_2}v. \end{cases}$$

Therefore the foliations $\mathcal{G}_1|_{V_0}$ and $\mathcal{G}_2|_{V_0}$ are induced by the 1-forms $\tilde{\eta}_1 = A_{a-bl_2-2}(1,v)dv$ and $\tilde{\eta}_2 = B_{a-bl_2+l_2-1}(1,v)du$ respectively. Then for every $x \in E$, we can take $\tilde{\eta}_1(x)$ and $\tilde{\eta}_2(x)$ $\mathbb{C}$-linearly independent. Thus any 1-form $\eta \in \mathcal{R}(a,b)$ such that $\eta(x) \neq 0$ can be written as a $\mathbb{C}$-linear combination of η_1 and η_2. For $x \in \mathbb{F}_{l_2} \backslash E$, it sufficient to consider the point $\pi(x) = [1:0:0] \in \mathbb{P}^2_{(1,1,l_2)}$ and take the two foliations $\mathcal{F}'_1$ and $\mathcal{F}'_2$ with normal $\mathbb{Q}$-bundle of degree a induced by the 1-forms

$$\omega_1 = x_0^{a-l_2}(x_1dx_0 - x_0dx_1) + x_2^b A_{a-bl_2-2}(x_0,x_1)(x_0,x_1)(x_1dx_0 - x_0dx_1),$$

and

$$\omega_1 = x_0^{a-(l_2+1)}(l_2x_2dx_0 - x_0dx_1) + x_2^{b-2} B_{a-bl_2+l_2-1}(x_0,x_1)(l_2x_2dx_0 - x_0dx_2),$$

respectively. This shows that $\dim_{\mathbb{C}} \ker \psi_x$ is constant as a function of $x \in X$. Hence $\ker \psi$ is a vector bundle and in particular $\mathbb{P}(\ker \psi) = \mathcal{S}(a,b)$ is a irreducible variety of codimension two on $\mathbb{F}_{l_2} \times \mathcal{R}(a,b)$.

2. Follows directly from [17, Lemma 5.1, page 9].

□

Lemma 2.5.6. *Let $\mathcal{G} \in \mathcal{R}(a,b)$ be a foliation on $\mathbb{F}_{l_2}$ and C be an algebraic curve invariant by $\mathcal{G}$. If $b \geq 3$ and $a \geq bl_2 + 2$, then*

$$C \cap Sing(\mathcal{G}) \neq \emptyset.$$

Proof. We have two cases:

1. If $C = E$, then by Camacho-Sad formula $C^2 = -l_2$, this implies that $C \cap Sing(\mathcal{G}) \neq \emptyset$.

2. If $C \neq E$, then $C = mF + nE$ in $Pic(\mathbb{F}_{l_2})$, with $m > 0$, $n \geq 0$. Let us suppose that $C \cap Sing(\mathcal{G}) = \emptyset$, then by Camacho-Sad formula $C^2 = 0$ and by Vanishing formula

$$N\mathcal{G}.C = C^2 + Z(\mathcal{G}, C) = 0,$$

but $N\mathcal{G}.C = n(a - bl_2) + bm > 0$, this is a contradiction.

□

We obtain a generalization of Jouanolou's Theorem for Hirzebruch surfaces.

Theorem 3. *A generic foliation with normal bundle of bidegree (a,b) in $\mathbb{F}_{l_2}$ does not admit any invariant algebraic curve if $b \geq 3$ and $a \geq bl_2 + 2$.*

Proof. Consider the second projection $\pi_2 : \mathcal{S}(a,b) \to \mathcal{R}(a,b)$ and fix now a polynomial $\chi \in \mathbb{Q}[t]$ of degree one. Suppose that $\pi_2(\mathcal{D}_\chi(a,b)) = \mathcal{R}(a,b)$; that is, every foliation of bidegree (a,b) has an algebraic invariant curve with Hilbert polynomial χ. By Lemma 2.5.6 we have that

$$\pi_2(\mathcal{D}_\chi(a,b) \cap \mathcal{S}(a,b)) = \pi_2(\mathcal{S}(a,b)).$$

Since $\mathcal{S}(a,b)$ is an irreducible variety and $\dim_{\mathbb{C}} \mathcal{S}(a,b) = \dim_{\mathbb{C}} \mathcal{R}(a,b)$ by Proposition 2.5.5, we get $\mathcal{S}(a,b) \cap \mathcal{D}_\chi(a,b) = \mathcal{S}(a,b)$.

To conclude the theorem we take the foliation $\mathcal{F}_1$ on $\mathbb{P}^2_{(1,1,l_2)}$ of Lemma 2.4.3 of degree $2l_2+1$ induced by η_1. Let $\mathcal{F}$ be the foliation on $\mathbb{P}^2_{(1,1,l_2)}$ with normal $\mathbb{Q}$-bunlde of degree a induced by

$$\eta = x_0^{a-bl_2+l_2-1} x_2^{b-3} \eta_1.$$

Then we note that the foliation $\mathcal{G} = \pi^*(\mathcal{F})$ on $\mathbb{F}_{l_2}$ has bidegree (a,b) and through $\pi^{-1}([1:1:1])$, which is singularity of $\mathcal{G}$ we do not have any invariant curve, which is a contradiction. Since there are only countable many Hilbert polynomials, we conclude the theorem. □

Chapter 3

Components on $\mathbb{P}^3$

The problem of classifying of holomorphic foliations in projective spaces of dimension greater or equal than to 3 in the algebraic setting is given through the study of its irreducible components. This approach was initiated by Jouanolou in [29] who proved that the space of holomorphic codimension one foliations of degree one on $\mathbb{P}^n$, $n \geq 3$, has two irreducible components. Continuing with this work Cerveau and Lins-Neto in [13] have proved that the space of holomorphic codimension one foliations of degree two on $\mathbb{P}^n$, $n \geq 3$, has six irreducible components. In [11] Calvo Andrade, Cerveau, Giraldo and Lins Neto give a explicit construction of certain components of the space of holomorphic foliations of codimension one associated to some affine Lie algebra.

The main theorem of the thesis is to construct a family of components irreducible of the holomorphic foliations of codimension one associated to the affine Lie algebra on $\mathbb{P}^3$. The affine Lie algebra are induced by the vector fields $l_0 x\frac{\partial}{\partial x} + l_1 y\frac{\partial}{\partial y} + l_1 z\frac{\partial}{\partial z}$, when $l_0 > l_1$. Using these vector fields reduce the problem to the study of foliations holomorphic on weighted projective spaces.

3.1 Irreducible components of the space of foliations associated to the affine algebra Lie

In this section we will talk about the results that are known of irreducible components of the space of foliations which are associated to the affine algebra Lie, see [11] for more details. Let $1 \leq l_0 < l_1 < l_2$ be are positive integers with $\gcd(l_0, l_1, l_2) = 1$. Consider the linear vector field on $\mathbb{C}^3$

$$S = l_0 x\frac{\partial}{\partial x} + l_1 y\frac{\partial}{\partial y} + l_2 z\frac{\partial}{\partial z}.$$

Suppose that there is another polynomial vector field X on $\mathbb{C}^3$ such that $[S, X] = \lambda X$, for some $\lambda \in \mathbb{Z}$. Then the algebraic foliation $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(S, X)$ on $\mathbb{C}^3$ defined by the 1-form $\Omega = i_S i_X(dx \wedge dy \wedge dz)$ is associated to a representation of the affine algebra of polynomial vector fields in $\mathbb{C}^3$, it can be extended to a foliation on $\mathbb{P}^3$ of certain degree ν. The following definition is due to Calvo-Andrade, Cervau, Giraldo, Lins Neto, see [11].

$$\mathbb{F}ol((l_0, l_1, l_2), \lambda, \nu) := \{\tilde{\mathcal{F}} \in \mathbb{F}ol(\nu, 3) |\, \tilde{\mathcal{F}} = \mathcal{F}(S, X) \;\text{ in some affine chart}\},$$

They show that they are irreducible subvarieties of $\mathbb{F}ol(\nu,3)$, but not necessarily irreducible components.

Definition 3.1.1. Let Ω be an integrable 1-form defined in a neighborhood of $p \in \mathbb{C}^3$. We say that p is a *generalized Kupka* (GK) of Ω if $\Omega(p)=0$ and either $d\Omega(p)\neq 0$ or p is an isolated zero of $d\Omega$. A codimension one holomorphic foliation $\mathcal{F}$ in a complex three manifold M is *GK* if all the singularities of $\mathcal{F}$ are GK.

We have the following theorem.

Theorem 3.1.2 (Calvo and et al.)**.** *Suppose that $\mathbb{F}ol((l_0,l_1,l_2),\lambda,\nu)$ contains some GK foliation. Then*

$$\overline{\mathbb{F}ol((l_0,l_1,l_2),\lambda,\nu)},$$

is an irreducible component of $\mathbb{F}ol(\nu,3)$.

Corollary 3.1.3. *Let $d\geq 1$ be an integer. There is an N-dimensional irreducible component*

$$\overline{\mathbb{F}ol((d^2+d+1,d+1,1),-1,d+1)},$$

of the space $\mathbb{F}ol(d+1,3)$ whose general point corresponds to a GK Klein-Lie foliation with exactly one quasi-homogeneous singularity, where $N=13$ if $d=1$ and $N=14$ if $d>1$. Moreover, this component is the closure of a $\mathbb{P}GL(4,\mathbb{C})$ orbit on $\mathbb{F}ol(d+1,3)$.

The proof of can be found in [11, Theorem 1 and Corollary 3].

3.2 Foliations on $\mathbb{P}^3$ tangent to the fields $S=l_0x\frac{\partial}{\partial x}+l_1y\frac{\partial}{\partial y}+l_1z\frac{\partial}{\partial z}$ with $l_0>l_1$ and $\gcd(l_0,l_1)=1$

First, we study foliations tangent to homogeneous vector field $S=l_0x\frac{\partial}{\partial x}+l_1y\frac{\partial}{\partial y}+l_1z\frac{\partial}{\partial z}$. In this case, the vector field induces a natural rational map with generic fibers equal to orbits of S. In the affine neighborhood $w=1$, we can take composition of the natural quotient map

$$\begin{array}{rccc}\varphi: & \mathbb{C}^3 & \to & \mathbb{P}^2_{(l_0,l_1,l_1)}\\ & (x,y,z) & \mapsto & [x:y:z]_{(l_0,l_1,l_1)},\end{array}$$

with the natural isomorphism

$$\begin{array}{ccc}\mathbb{P}^2_{(l_0,l_1,l_1)} & \to & \mathbb{P}^2_{(l_0,1,1)}\\ [y_0:y_1:y_2]_{(l_0,l_1,l_1)} & \mapsto & [y_0^{l_1}:y_1:y_2]_{(l_0,1,1)}.\end{array}$$

We have the following map

$$\begin{array}{rccc}\varphi: & \mathbb{C}^3 & \to & \mathbb{P}^2_{(l_0,1,1)}\\ & (x,y,z) & \mapsto & [x^{l_1}:y:z]_{(l_0,1,1)},\end{array}$$

extending this map to $\mathbb{P}^3$, we get the following rational map

$$\begin{array}{rcl}\varphi:\mathbb{P}^3 & \dashrightarrow & \mathbb{P}^2_{(l_0,1,1)}\\ [x:y:z:w] & \mapsto & [x^{l_1}w^{l_0-l_1}:y:z]_{(l_0,1,1)}=[x_0:x_1:x_2]_{(l_0,1,1)}.\end{array}$$

Notice that

1. The indeterminacy locus of φ is composed of two points $[1:0:0:0]$ and $[0:0:0:1]$.
2. The generic fiber is irreducible.
3. The pre-image of the singular point $[1:0:0]_{(l_0,1,1)}$ is the line $\{x=y=0\}$.
4. The divisorial components of its critical locus are $\{z=0\}$(only when $l_1>1$) and $\{w=0\}$ (only when $l_0-l_1>1$). Both components are mapped to the curve $\{x_0=0\}$.

The following lemma characterizes the holomorphic foliations on $\mathbb{P}^3$ tangent to the S via the map φ with the holomorphic foliations on $\mathbb{P}^2_{(l_0,1,1)}$.

Lemma 3.2.1. *Let $\mathcal{F}$ be a foliation of degree normal $\mathbb{Q}$-bundle d on $\mathbb{P}^2_{(l_0,1,1)}$ given by η and $\tilde{\mathcal{F}}=\varphi^*\mathcal{F}$ be a foliation on $\mathbb{P}^3$ given by ω. Then*

$$\omega=\begin{cases}\varphi^*\eta & \text{, if } \{x_0=0\} \text{ is not } \mathcal{F}\text{-invariant,}\\ \dfrac{\varphi^*\eta}{x^{l_1-1}w^{l_0-l_1-1}} & \text{, if } \{x_0=0\} \text{ is } \mathcal{F}\text{-invariant.}\end{cases}$$

and has degree equal to

$$\deg(\tilde{\mathcal{F}})=\begin{cases}d-2 & \text{, if } \{x_0=0\} \text{ is not } \mathcal{F}\text{-invariant,}\\ d-l_0 & \text{, if } \{x_0=0\} \text{ is } \mathcal{F}\text{-invariant.}\end{cases}$$

Furthermore the following statements are equivalent

1. *$\{x=0\}$ is $\tilde{\mathcal{F}}$-invariant.*
2. *$\{w=0\}$ is $\tilde{\mathcal{F}}$-invariant.*
3. *$\{x_0=0\}$ is $\mathcal{F}$-invariant.*

Proof. We have two cases: In the first case, if $\{x_0=0\}$ is not $\mathcal{F}$-invariant, then

$$\eta=-\frac{1}{l_0}Adx_0+Bdx_1+Cdx_2$$

with $x_0A=x_1B+x_2C$ and $x_0\nmid B$, so we have that

$$\omega=\varphi^*\eta=-\frac{l_1}{l_0}x^{l_0-1}w^{l_0-l_1}A\circ\varphi dx+B\circ\varphi dy+C\circ\varphi dz-\frac{(l_0-l_1)}{l_0}x^{l_1}w^{l_0-l_1-1}A\circ\varphi dw.$$

Therefore $x\nmid B(x^{l_1}w^{l_0-l_1},y,z)$ and $w\nmid B(x^{l_1}w^{l_0-l_1},y,z)$. This implies that $\{x=0\}$ and $\{w=0\}$ are not invariant by $\tilde{\mathcal{F}}$. Then the degree of $\tilde{\mathcal{F}}$ is d.

In the other case, if $\{x_0=0\}$ is $\mathcal{F}$-invariant then

$$\eta=-\frac{A}{l_0}dx_0+x_0Bdx_1+x_0Cdx_2,$$

with $A=x_1B+x_2C$, so

$$\omega=\frac{\varphi^*\eta}{x^{l_1-1}w^{l_0-l_1-1}}=-\frac{l_1}{l_0}wA\circ\varphi dx+xwB\circ\varphi dy+xwC\circ\varphi dz-\frac{(l_0-l_1)}{l_0}xA\circ\varphi dw,$$

induces the foliation $\tilde{\mathcal{F}}$ of degree $d-l_0$ such that $\{x=0\}$ is $\tilde{\mathcal{F}}$-invariant and $\{w=0\}$ is $\tilde{\mathcal{F}}$-invariant. Note that

$$\deg(\omega)=\deg(N\tilde{\mathcal{F}})=\begin{cases}d & \text{, if } \{x_0=0\} \text{ is not } \mathcal{F}\text{-invariant,}\\ d-l_0+2 & \text{, if } \{x_0=0\} \text{ is } \mathcal{F}\text{-invariant.}\end{cases}$$

□

3.3 Foliations with split tangent bundle

We saw that holomorphic foliations on $\mathbb{P}^3$ tangent to the vector field S are the same as holomorphic foliations on $\mathbb{P}^2_{(l_0,1,1)}$ via the application φ. We will now investigate when the tangent bundle of the foliation on $\mathbb{P}^3$ tangent to S split as a sum of two line-bundles. For that we need some definitions.

Definition 3.3.1. Let $\tilde{\mathcal{F}}$ be a codimension one holomorphic foliation on $\mathbb{P}^3$, induced by a 1-form ω.

1. The *tangent sheaf* of $\tilde{\mathcal{F}}$, denoted by $T\tilde{\mathcal{F}}$, is a coherent subsheaf of $T\mathbb{P}^3$ generated by the germs of vector fields annihilating ω, *i.e.*, for every open set $U \subset \mathbb{P}^3$, we have that

$$T\tilde{\mathcal{F}}(U) = \{v \in T\mathbb{P}^3(U) |\, i_v\omega = 0\}.$$

2. We say that the tangent sheaf of $\tilde{\mathcal{F}}$ *splits* if

$$T\tilde{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^3}(e_1) \oplus \mathcal{O}_{\mathbb{P}^3}(e_2),$$

   for some integers e_i.

Remark 3.3.2. In general $T\tilde{\mathcal{F}}$ is not locally free, an example can be found in [20].

Note that, when the tangent sheaf of $\tilde{\mathcal{F}}$ splits, the inclusion of $T\tilde{\mathcal{F}}$ in $T\mathbb{P}^3$ induces sections $X_i \in H^0(\mathbb{P}^3, T\mathbb{P}^3(-e_i))$ for $i = 1, 2$. It follows from the Euler sequence that these sections are defined by homogeneous vector fields of degree $1 - e_i \geq 0$ on $\mathbb{C}^4$, which we still denote by X_i. The foliation $\tilde{\mathcal{F}}$ is induced by the homogeneous 1-form on $\mathbb{C}^4$

$$\omega = i_{X_1} i_{X_2} i_R \, dx \wedge dy \wedge dz \wedge dw.$$

Let us look at some previous lemmas.

Lemma 3.3.3. *Let $\tilde{\mathcal{F}}$ be a foliation on $\mathbb{P}^3$ with split tangent bundle i.e., $T\tilde{\mathcal{F}} = T\mathcal{G} \oplus T\mathcal{H}$, and let C be an irreducible hypersurfarce. Then*

1. *If C is $\mathcal{G}$-invariant and $\mathcal{H}$-invariant then C is $\tilde{\mathcal{F}}$-invariant.*

2. *If C is $\tilde{\mathcal{F}}$-invariant and $\mathcal{G}$-invariant then C is $\mathcal{H}$-invariant.*

3. *If C is $\tilde{\mathcal{F}}$-invariant and $\mathcal{H}$-invariant then C is $\mathcal{G}$-invariant.*

Proof. In an affine chart $\mathbb{C}^3$, we suppose that $\tilde{\mathcal{F}}$ is given by ω, $\mathcal{G}$ is given X_1, $\mathcal{H}$ is given by X_2 and C is given by f such that $\omega = i_{X_1} i_{X_2} dx \wedge dy \wedge dz$. We see that

$$\omega \wedge df = X_2(f) i_{X_1} dx \wedge dy \wedge dz - X_1(f) i_{X_2} dx \wedge dy \wedge dz. \tag{3.1}$$

From the last equality we conclude the lemma. □

Example 3.3.4. Let $S = l_0 x\frac{\partial}{\partial x} + l_1 y\frac{\partial}{\partial y} + l_1 z\frac{\partial}{\partial z}$ be a homogeneous vector field with $l_0 > l_1$ and $\mathcal{G}$ the induced foliation. Let $\tilde{\mathcal{F}}$ be a foliation on $\mathbb{P}^3$ tangent to S with split tangent sheaf. Then there exists a X homogeneous vector field such that $\tilde{\mathcal{F}}$ is given by

$$\omega = i_R i_S i_X dx \wedge dy \wedge dz \wedge dw,$$

where R is radial vector field in $\mathbb{C}^4$. Denote by $\mathcal{H}$ the induced foliation by X. Applying Lemma 3.3.3 we have that $\{x = 0\}$(or $\{w = 0\}$) is $\tilde{\mathcal{F}}$-invariant if only if $\{x = 0\}$(or $\{w = 0\}$) is $\mathcal{H}$-invariant.

Division lemma for the Koszul complex. Let v a singular vector field on $\mathbb{C}^3$ we denote by $\Omega^k(\mathbb{C}^3)$ the space of k-forms in $\mathbb{C}^3$. The interior product i_v by the vector field v of a k-form define a linear application $i_v : \Omega^k(\mathbb{C}^3) \to \Omega^{k-1}(\mathbb{C}^3)$. The *Koszul complex* of the vector field v is the complex

$$K(v):\ 0 \longrightarrow \Omega^3(\mathbb{C}^3) \xrightarrow{i_v} \Omega^2(\mathbb{C}^3) \xrightarrow{i_v} \Omega^1(\mathbb{C}^3) \xrightarrow{i_v} \Omega^0(\mathbb{C}^3) = \mathcal{O}(\mathbb{C}^3) \longrightarrow 0\ .$$

The homology of this complex is the obstruction to the following property: Let η be a k-form in $\mathbb{C}^3$ such that $i_v\eta = 0 \in \Omega^{k-1}(\mathbb{C}^3)$, then $\eta = i_v\theta$ for some $k+1$-form θ if only if the class $[\eta] \in H_k(K(v))$ is zero.

The vector field v have the following expression

$$v = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z},$$

and the singular set of v is $Z = \{(x, y, z) \in \mathbb{C}^3 |\ v_1(x, y, z) = v_2(x, y, z) = v_3(x, y, z) = 0\}$. Now, we are going to state the Division lemmas where the first one is for Koszul complex.

Lemma 3.3.5. *If Z have dimension 0, then*

$$H_1(K(v)) = 0.$$

Proof. In fact, let η be a 1-form in $\mathbb{C}^3$ such that $i_v\eta = 0$. Since the vector field v is not identically null in $U = \mathbb{C}^3\backslash Z$, then there is a covering $\mathcal{U} = \{U_j\}_j$ of U by open sets U_j such that $\eta = i_v\theta_j$ for some $\theta_j \in \Omega^2(U_j)$. Thus $i_v(\theta_j - \theta_k) = 0$ in $U_j \cap U_k$ and therefore there are $\nu_{jk} \in \Omega^3(U_j \cap U_k)$ such that $\theta_j - \theta_k = i_v\nu_{jk}$. We can write $\nu_{jk} = f_{jk}dx \wedge dy \wedge dz$, $f_{jk} \in \mathcal{O}(U_j \cap U_k)$ and $\{f_{jk}\} \in H^1(\mathcal{U}, \mathcal{O})$. Finally the hypothesis that Z have dimension zero is to apply [27, Theorem 5, page 160] in order to obtain $H^1(\mathcal{U}, \mathcal{O}) = 0$. Hence there are $f_j \in \mathcal{O}(U_j)$ such that $f_j - f_k = f_{jk}$ in $U_j \cap U_k$. Therefore we can define θ a 2-form in $\mathbb{C}^3$ such that $\eta = i_v\theta$ and $\theta|_{U_j} = \theta_j - i_v(f_jdx \wedge dy \wedge dz)$. □

With this Division lemma we can prove the following lemma that it will be used in Proposition 3.3.9.

Lemma 3.3.6. *Let $\mathcal{G}$ be a one-dimension foliation on $\mathbb{P}^3$ of degree one, with a invariant hyperplane H and isolated singularities on $\mathbb{P}^3\backslash H$ and $\tilde{\mathcal{F}}$ a codimension one foliation of degree d containing $\mathcal{G}$. If H is $\tilde{\mathcal{F}}$-invariant then $T\tilde{\mathcal{F}} \cong T\mathcal{G} \bigoplus T\mathcal{H}$, for a suitable one-dimensional foliation $\mathcal{H}$.*

Proof. By a change of coordinates we can assume that $H = \{w = 0\}$, and in the affine chart $\mathbb{C}^3 \simeq \mathbb{P}^3\backslash H$, we have that $\tilde{\mathcal{F}}$ and $\mathcal{G}$ are given ω and S respectively. Since H is $\tilde{\mathcal{F}}$-invariant we have that

$$\omega = \omega_0 + \omega_1 + \cdots + \omega_d,\ \ i_{R_0}\omega_d \neq 0, \tag{3.2}$$

where ω_j are homogeneous 1-form and $R_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Since $i_S\omega = 0$ and S has isolated singularities in $\mathbb{C}^3$ applying the Division Lemma 3.3.5, there exists a polynomial vector field X such that

$$\omega = i_S i_X(dx \wedge dy \wedge dz).$$

We write $X = X_0 + \cdots + X_{d-1}$ and $S = S_0 + S_1$ where X_j, S_j are homogeneous vector fields. Using the equality (3.2) we have that $\omega_d = i_{S_1}i_{X_{d-1}}dx\wedge dy\wedge dz$. Furthermore $i_{R_0}i_{S_1}i_{X_{d-1}}dx\wedge dy\wedge dz = i_{R_0}\omega_d \neq 0$. This implies that X_{d-1} is not multiple of R_0, i.e., $X_{d-1} \wedge R_0 \neq 0$. So, X induces to a foliation $\mathcal{H}$ of degree $d-1$ in $\mathbb{P}^3$ such that $\{w = 0\}$ is $\mathcal{H}$-invariant. Since $\{w = 0\}$ is $\mathcal{H}$-invariant and $\tilde{\mathcal{F}}$ we have that $T\tilde{\mathcal{F}} \simeq T\mathcal{G} \oplus T\mathcal{H}$. This concludes the proof. □

Remark 3.3.7. Note that if $\tilde{\mathcal{F}}$ is a holomorphic codimension one foliation of degree d, $\mathcal{G}$ and $\mathcal{H}$ are one-dimension foliations of degree one and $d-1$ on $\mathbb{P}^3$ given by ω, S and X respectively, then the following statements are equivalent:

1. $T\tilde{\mathcal{F}} \simeq T\mathcal{G} \oplus T\mathcal{H}$ is splits.

2. $\omega = i_R i_S i_X(dx \wedge dy \wedge dz \wedge dw)$, where R is radial vector field in $\mathbb{C}^4$.

Note also that any of these equivalent statements implies that there exist a generator Y of the foliation $\mathcal{H}$ and an integer λ such that $[S, Y] = \lambda Y$.

Lemma 3.3.8. *Under the notations of Remark 3.3.7, any of the equivalent statements 1 and 2 implies that there exist a generator Y of the foliation $\mathcal{H}$ and an integer λ such that $[S, Y] = \lambda Y$.*

Proof. By the integrability of ω one deduces that

$$[S, X] = \lambda X + F_1 S + F_2 R,$$

for some $\lambda \in \mathbb{C}$ and $F_1, F_2 \in S_{d-1}$. Recall that S_{d-1} denotes the space of homogeneous polynomials of degree $d-1$. Since, for arbitrary $\alpha \in \mathbb{C}$,

$$i_R i_S i_X(dx \wedge dy \wedge dz \wedge dw) = i_R i_{S+\alpha R} i_X(dx \wedge dy \wedge dz \wedge dw),$$

we can suppose that the linear map

$$\begin{array}{rcl} \psi : S_{d-1} & \to & S_{d-1} \\ G & \to & \lambda G - S(G) \end{array}$$

is invertible.

If we set $Y = X + \psi^{-1}(F_1)S + \psi^{-1}(F_2)R$, then

$$\begin{array}{rcl} [S, Y] & = & [S, X] + [S, \psi^{-1}(F_1)S] + [S, \psi^{-1}(F_2))R], \\ & = & \lambda X + F_1 S + F_2 R + S(\psi^{-1}(F_1))S + S(\psi^{-1}(F_2))R, \\ & = & \lambda X + (F_1 + S(\psi^{-1}(F_1)))S + (F_2 + S(\psi^{-1}(F_2)))R, \\ & = & \lambda Y. \end{array}$$

Notice that $i_R i_S i_X(dx \wedge dy \wedge dz \wedge dw) = i_R i_S i_Y(dx \wedge dy \wedge dz \wedge dw)$ to conclude the proof of the lemma. □

The following proposition characterizes codimension one holomorphic foliations with splits tangent sheaf.

Proposition 3.3.9. *Assume the notations and conditions of Lemma 3.2.1. Suppose that* $\deg(\tilde{\mathcal{F}}) > 1$ *and $\mathcal{G}$ is the induced foliation by S on $\mathbb{P}^3$. Then the following statements are equivalent:*

1. *There exists $\mathcal{H}$ a one-dimension foliation on $\mathbb{P}^3$ such that*

$$T\tilde{\mathcal{F}} \simeq T\mathcal{G} \oplus T\mathcal{H}.$$

2. *$\{x = 0\}$ (or $\{w = 0\}$) is $\tilde{\mathcal{F}}$-invariant.*

3. *$\{x_0 = 0\}$ is $\mathcal{F}$-invariant*

Proof. 1) $\Rightarrow$ 2) By Remark 3.3.7 we have that $\omega = i_R i_S i_X(dx \wedge dy \wedge dz \wedge dw)$ and $[S, X] = \lambda X$ for some $\lambda \in \mathbb{Z}$, where $\mathcal{H}$ is the induced one-dimension foliation by X. Suppose that $\{x = 0\}$ is not $\tilde{\mathcal{F}}$-invariant then by the Example 3.3.4 we have that $\{x = 0\}$ is not $\mathcal{H}$-invariant. We write

$$X = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + D\frac{\partial}{\partial w},$$

where $A = A_e(y, z, w) + \cdots + A_0 x^e$, $D = D_e(x, y, w) + \cdots + D_0 w^e$. Since $\{x = 0\}$ is not $\mathcal{H}$-invariant, then $A_e \neq 0$, *i.e.*, there exists non zero monomial $a_{i_0 j_0 k_0} y^{i_0} z^{j_0} w^{k_0}$, such that $i_0 + j_0 + k_0 = e$. Since $[S, X] = \lambda X$, then:

$$\begin{aligned} [S, a_{i_0 j_0 k_0} y^{i_0} z^{j_0} w^{k_0} \frac{\partial}{\partial x}] &= \lambda a_{i_0 j_0 k_0} y^{i_0} z^{j_0} w^{k_0} \frac{\partial}{\partial x}, \\ (l_1(i_0 + j_0) - l_0) a_{i_0 j_0 k_0} y^{i_0} z^{j_0} w^{k_0} \frac{\partial}{\partial x} &= \lambda a_{i_0 j_0 k_0} y^{i_0} z^{j_0} w^{k_0} \frac{\partial}{\partial x}, \end{aligned}$$

therefore

$$\lambda = l_1(i_0 + j_0) - l_0 = l_1(e - k_0) - l_0. \tag{3.3}$$

Note that $D_e = \sum_{i+j+k=e} d_{ijk} i x^i y^j z^k$, again using $[S, X] = \lambda X$ we have

$$\begin{aligned} [S, d_{ijk} x^i y^j z^k \tfrac{\partial}{\partial w}] &= \lambda d_{ijk} x^i y^j z^k \tfrac{\partial}{\partial w}, \\ (i l_0 + (j + k) l_1) d_{ijk} x^i y^j z^k \tfrac{\partial}{\partial w} &= \lambda d_{ijk} x^i y^j z^k \tfrac{\partial}{\partial w}, \end{aligned}$$

therefore

$$d_{ijk}(i l_0 + (j + k) l_1 - \lambda) = 0. \tag{3.4}$$

Using (3.3) and $i + j + k = e$ we have that

$$i l_0 + (j + k) l_1 - \lambda = i(l_0 - l_1) + l_0 + k_0 l_1 > 0. \tag{3.5}$$

Using (3.5) in the equality (3.4) we conclude that $d_{ijk} = 0$, $\forall\, i + j + k = e$, that is $D_e = 0$ therefore, $\{w = 0\}$ is $\mathcal{H}$-invariant then by Example 3.3.4 we have that $\{w = 0\}$ is $\tilde{\mathcal{F}}$-invariant and by Lemma 3.2.1 we have that $\{x = 0\}$ is $\tilde{\mathcal{F}}$-invariant. This is a contradiction.

2) $\Rightarrow$ 1) It is immediate using Lemma 3.3.6.

2) $\Leftrightarrow$ 3) Follows from Lemma 3.2.1. □

Now we will only be interested in holomorphic foliations with split tangent sheaf. The main idea to construct components is to use stability theorems for splits tangent sheaf following bellow.

Theorem 3.3.10. *Let $n \geq 3$, $d \geq 0$ and $\mathcal{F} \in \mathbb{F}ol(d, n)$ be a singular holomorphic foliation on $\mathbb{P}^n$ given by ω. If codim $Sing(d\omega) \geq 3$ and*

$$T\mathcal{F} \cong \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(e_i), \quad e_i \in \mathbb{Z},$$

then there exist a Zariski-open neighborhood $\mathcal{U} \subset \mathbb{F}ol(d, n)$ of $\mathcal{F}$ such that $T\mathcal{F}' \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(e_i)$ for every $\mathcal{F}' \in \mathcal{U}$.

The proof can be found in [20].

Observe that for applying the above theorem it is necessary a condition in the codimension of the singular set of $d\omega$. Therefore, we are going to study the codimension of the singular set of $d\omega$.

3.4 Codimension of the singular set of $d\omega$

Through the following lemma we can characterize the singular set $d\omega$.

Lemma 3.4.1. *Let $\mathcal{F}$ be a foliation with normal $\mathbb{Q}$-bundle of degree $d > l_0 + 1$ on $\mathbb{P}^2_{(l_0,1,1)}$ given by η and let $\tilde{\mathcal{F}} = \varphi^*\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^3$ with split tangent bundle given by ω. Then*

1. *If $(l_0, l_1) = (2, 1)$ then $\{x = w = 0\} \subset Sing(\omega) \cap Sing(d\omega)$.*
2. *If $\text{multialg}_{[1:0:0]}(\mathcal{F}) \geq 3$ then $\{y = z = 0\} \subset Sing(\omega) \cap Sing(d\omega)$.*

Proof. By Proposition 3.3.9 we have that $\{x_0 = 0\}$ is $\mathcal{F}$-invariant. Therefore, we can write

$$\eta = -\frac{A}{l_0}dx_0 + x_0Bdx_1 + x_0Cdx_2,$$

such that $A = x_1B + x_2C$. Since $\deg(B) = \deg(C) = d - l_0 - 1 = ql_0 + r,\ \ 0 \leq r < l_0$, we can express B and C as

$$\begin{array}{rcl} B &=& x_0^{q-i}B_{il_0+r}(x_1,x_2) + x_0^{q-(i+1)}B_{(i+1)l_0+r}(x_1,x_2) + \cdots + x_0B_{(q-1)l_0+r}(x_1,x_2) + B_{ql_0+r}(x_1,x_2),\\ C &=& x_0^{q-i}C_{il_0+r}(x_1,x_2) + x_0^{q-(i+1)}C_{(i+1)l_0+r}(x_1,x_2) + \cdots + x_0C_{(q-1)l_0+r}(x_1,x_2) + B_{ql_0+r}(x_1,x_2), \end{array}$$

where B_j, C_j are homogeneous polynomial of degree j and

$$\text{multialg}_{[1:0:0]}(\mathcal{F}) = il_0 + r + 1.$$

Applying Lemma 3.2.1 we have that

$$\omega = \frac{\varphi^*\eta}{x^{l_2-1}w^{l_0-l_1-1}} = -\frac{l_1}{l_0}wA\circ\varphi dx + xwB\circ\varphi dy + xwC\circ\varphi dz - \frac{(l_0-l_1)}{l_0}xA\circ\varphi dw,$$

where

$$\begin{array}{rcl} A\circ\varphi &=& x^{l_1(q-i)}w^{(l_0-l_1)(q-i)}A_{il_0+r}(y,z) + \cdots + A_{ql_0+r}(y,z),\\ B\circ\varphi &=& x^{l_1(q-i)}w^{(l_0-l_1)(q-i)}B_{il_0+r}(y,z) + \cdots + B_{ql_0+r}(y,z),\\ C\circ\varphi &=& x^{l_1(q-i)}w^{(l_0-l_1)(q-i)}C_{il_0+r}(y,z) + \cdots + C_{ql_0+r}(y,z). \end{array}$$

Then $\{x = w = 0\} \subset Sing(\omega)$. If $il_0 + r \geq 2$ we have that $\{y = z = 0\} \subset Sing(\omega)$.

$$\begin{array}{rcl} d\omega &=& \left(\frac{l_1}{l_0}w(A\circ\varphi)_y + wB\circ\varphi + xw(B\circ\varphi)_x\right)dx\wedge dy\\ && + \left(\frac{l_1}{l_0}w(A\circ\varphi)_z + wC\circ\varphi + xw(C\circ\varphi)_x\right)dx\wedge dz\\ && + \left(\frac{2l_1-l_0}{l_0}A\circ\varphi + \frac{l_1}{l_0}w(A\circ\varphi)_w - \frac{(l_0-l_1)}{l_0}x(A\circ\varphi)_x\right)dx\wedge dw\\ && + xw\left((C\circ\varphi)_y - (B\circ\varphi)_z\right)dy\wedge dz\\ && - x\left(B\circ\varphi + w(B\circ\varphi)_w + \frac{l_0-l_1}{l_0}(A\circ\varphi)_y\right)dy\wedge dw\\ && - x\left(C\circ\varphi + w(C\circ\varphi)_w + \frac{l_0-l_1}{l_1}(A\circ\varphi)_z\right)dz\wedge dw \end{array}$$

Lastly, note that if $(l_0, l_1) = (2, 1)$ we see that $\{x = w = 0\} \subset Sing(d\omega)$. If $il_0 + r \geq 2$ we have that $\{y = z = 0\} \subset Sing(d\omega)$. This concludes the lemma. □

In the case $\text{multialg}_{[1:0:0]}(\mathcal{F}) \leq 2$, it is possible to find examples of foliations induced by ω such that $d\omega$ has isolated singularities, as we see below.

Example 3.4.2. In the case $\text{multialg}_{[1:0:0]}(\mathcal{F}) = 1$. Let $\mathcal{F}$ be the foliation on $\mathbb{P}^2_{(l_0,1,1)}$ given by

$$\eta = -(x_0^q x_1 - 2x_0^q x_2 + x_1 x_2^{ql_0} - x_1^{ql_0} x_2)dx_0 + l_0 x_0(x_0^q + x_2^{ql_0})dx_1 + l_0 x_0(-2x_0^q - x_1^{ql_0})dx_2,$$

where $\deg(N\mathcal{F}) = (q+1)l_0 + 1$, $q \geq 1$, $l_0 > l_1 \geq 1$ and $l_0 \geq 3$.

Let $\tilde{\mathcal{F}} = \varphi^*\mathcal{F}$ be the foliation on $\mathbb{P}^3$ given by

$$\begin{aligned}
\omega &= \frac{\varphi^*\eta}{x^{l_1-1}w^{l_0-l_1-1}},\\
\omega &= -l_1 w(x^{ql_1}yw^{q(l_0-l_1)} - 2x^{ql_1}zw^{q(l_0-l_1)} + yz^{ql_0} - y^{ql_0}z)dx + l_0xw(x^{ql_1}w^{q(l_0-l_1)} + z^{ql_0})dy\\
&\quad + l_0xw(-2x^{ql_1}w^{q(l_0-l_1)} - y^{ql_0})dz - (l_0 - l_1)x(x^{ql_1}yw^{q(l_0-l_1)} - 2x^{ql_1}zw^{q(l_0-l_1)} + yz^{ql_0} - y^{ql_0}z)dw,
\end{aligned}$$

where $\deg(N\tilde{\mathcal{F}}) = ql_0 + 3$ and $\deg(\tilde{\mathcal{F}}) = ql_0 + 1$.

We calculate

$$\begin{aligned}
d\omega &= w\left((l_1 + (q+1)l_0)x^{ql_1}w^{q(l_0-l_1)} + (l_0 + l_1)z^{ql_0} - l_1 y^{ql_0-1}z\right) dx \wedge dy\\
&\quad + w\left(-2(l_1 + l_0(ql_1 + 1))x^{ql_1}w^{q(l_0-l_1)} + ql_0l_1yz^{ql_0-1} - (l_0 + l_1)y^{ql_0}\right) dx \wedge dz\\
&\quad + (2l_1 - l_0)\left(x^{ql_1}yw^{q(l_0-l_1)} - 2x^{ql_1}zw^{q(l_0-l_1)} + yz^{ql_0} - y^{ql_0}z\right) dx \wedge dw\\
&\quad - ql_0^2xw(z^{ql_0-1} - y^{ql_0-1})dy \wedge dz\\
&\quad + x\left(-(2l_0 - l_1)z^{ql_0} + ql_0y^{ql_0-1}z - ((l_0 - l_1)(ql_0 + 1) + l_0)x^{ql_0}w^{q(l_0-l_1)}\right) dy \wedge dw\\
&\quad + x\left(2(l_0(q(l_0 - l_1) + 1) + l_0 - l_1)x^{ql_1}w^{q(l_0-l_1)} + (2l_0 - l_1)y^{ql_0} - ql_0(l_0 - l_1)yz^{ql_0-1}\right) dz \wedge dw,
\end{aligned}$$

and

$$Sing(d\omega) = \{[0:0:0:1], [1:0:0:0], [0:\xi:1:0] \,|\, \xi^{ql_0-1} = 1\}.$$

Furthermore by Proposition 3.3.9 we have that

$$\omega = i_R i_S i_X dx \wedge dy \wedge dz \wedge dw,$$

where $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}$, $S = l_0x\frac{\partial}{\partial x} + l_1y\frac{\partial}{\partial y} + l_1z\frac{\partial}{\partial z}$, and

$$X = -(2x^{ql_1}w^{q(l_0-l_1)} - y^{ql_0})\frac{\partial}{\partial y} - (x^{ql_1}w^{q(l_0-l_1)} - z^{ql_0})\frac{\partial}{\partial z}.$$

It verifies that

$$[S, X] = l_1(ql_0 - 1)X.$$

Example 3.4.3. Now the case $\text{multialg}_{[1:0:0]}(\mathcal{F}) = 2$. Let $\mathcal{F}$ be the foliation on $\mathbb{P}^2_{(l_0,1,1)}$ given by

$$\eta = -\frac{(x_1^{e+1} + x_2^{e+1})}{l_0}dx_0 + x_0(-x_0^q x_2 + x_1^e)dx_1 + x_0(x_0^q x_1 + x_2^e)dx_2,$$

where $\deg(N\mathcal{F}) = (q+1)l_0 + 2$, $e = ql_0 + 1$, $q \geq 1$, $l_0 > l_1 \geq 1$ and $l_0 \geq 3$.

Let $\tilde{\mathcal{F}} = \varphi^*\mathcal{F}$ be the foliation on $\mathbb{P}^3$ given by

$$\begin{aligned}
\omega &= \frac{\varphi^*\eta}{x^{l_1-1}w^{l_0-l_1-1}},\\
\omega &= -\frac{l_1}{l_0}w(y^{e+1} + z^{e+1})dx + xw(-x^{l_1q}w^{q(l_0-l_1)}z + y^e)dy\\
&\quad + xw(x^{l_1q}w^{q(l_0-l_1)}y + z^e)dz + -\tfrac{(l_0-l_1)}{l_0}x(y^{e+1} + z^{e+1})dw.
\end{aligned}$$

where $\deg(N\tilde{\mathcal{F}}) = ql_0 + 4$ and $\deg(\tilde{\mathcal{F}}) = ql_0 + 2$.

Hence we calculate

$$\begin{array}{rcl} d\omega & = & w\left(\frac{l_1}{l_0}(e+1)y^e + y^e - (l_1q+1)x^{l_1q}w^{q(l_0-l_1)z}\right)dx\wedge dy \\ & & +w\left(\frac{l_1}{l_0}(e+1)z^e + z^e + (l_1q+1)x^{l_1q}w^{q(l_0l_1)}y\right)dx\wedge dz \\ & & +2x^{l_1q+1}w^{q(l_0-l_1)+1}dy\wedge dz \\ & & -x\left((\frac{(l_0-l_1)}{l_0}(e+1)+1)y^e - (q(l_0-l_1)+1)x^{l_1q}w^{q(l_0-l_1)}z\right)dy\wedge dw \\ & & -x\left((\frac{(l_0-l_1)}{l_0}(e+1)+1)z^e + (q(l_0-l_1)+1)x^{l_1q}w^{q(l_0-l_1)}y\right)dz\wedge dw \\ & & +\frac{(2l_1-l_0)}{l_0}(y^{e+1}+z^{e+1})dx\wedge dw, \end{array}$$

and

$$Sing(d\omega) = \{[0:0:0:1],[1:0:0:0],[0:-\xi:1:0]\,|\,\xi^{e+1}=1\}.$$

Also by Proposition 3.3.9 we have that

$$\omega = i_R i_S i_X dx\wedge dy\wedge dz\wedge dw,$$

where $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial}$, $S = l_0x\frac{\partial}{\partial x} + l_1y\frac{\partial}{\partial y} + l_1z\frac{\partial}{\partial z}$, and

$$X = -\frac{x^{ql_1+1}w^{q(l_0-l_1)}}{l_1}\frac{\partial}{\partial x} + \frac{z^{l_1}}{l_0}\frac{\partial}{\partial y} - \frac{y^e}{l_0}\frac{\partial}{\partial z},$$

and verify that

$$[S,X] = l_0l_1qX.$$

3.5 Automorphism of a foliation

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^3$ given by ω. The automorphism group of $\mathcal{F}$, $Aut(\mathcal{F})$, is the subgroup of $Aut(\mathbb{P}^3) = \mathbb{P}(GL(\mathbb{C},4))$ formed by automorphisms of $\mathbb{P}^3$ which send $\mathcal{F}$ to itself. In other words

$$Aut(\mathcal{F}) = \{\phi\in Aut(\mathbb{P}^3)|\,\phi^*\omega\wedge\omega = 0\}.$$

$Aut(\mathcal{F})$ is clearly a closed subgroup of $Aut(\mathbb{P}^3)$, and therefore the connected component of the identity is a finite dimensional connected Lie group. We will denote by $\mathfrak{aut}(\mathcal{F})$ its Lie algebra, which can be identified with a subalgebra of $\mathfrak{aut}(\mathbb{P}^3) = \mathfrak{sl}(4)$, more specifically,

$$\mathfrak{aut}(\mathcal{F}) = \{v\in\mathfrak{aut}(\mathbb{P}^3)|\,L_v\omega\wedge\omega = 0\},$$

where L is the Lie derivative. We define the $\mathfrak{fix}(\mathcal{F})$ as the subalgebra of $\mathfrak{aut}(\mathcal{F})$ annihilating ω, *i.e.*,

$$\mathfrak{fix}(\mathcal{F}) = \{v\in\mathfrak{aut}(\mathcal{F})|\,i_v\omega = 0\}.$$

Notice that $\mathfrak{fix}(\mathcal{F})$ is nothing more than $H^0(\mathbb{P}^3, T\mathcal{F})$. We also point out that $\mathfrak{fix}(\mathcal{F})$ is an ideal of $\mathfrak{aut}(\mathcal{F})$, and the subgroup $Fix(\mathcal{F})\subset Aut(\mathcal{F})$ generated by $\mathfrak{fix}(\mathcal{F})$ is not necessarily closed.

We have now the following lemma. The proof is an adaption of an argument of Cerveau and Mattei, see [14, page 35-36].

Lemma 3.5.1. *The following assertions hold true:*

1. *If $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$ then $\mathcal{F}$ is tangent to an algebraic action.*

2. *If $\mathfrak{fix}(\mathcal{F}) \neq \mathfrak{aut}(\mathcal{F})$ then $\mathcal{F}$ is generated by a closed rational 1-form without divisorial components in its zero set.*

Proof. The connected component of the identity of $Aut(\mathcal{F})$ is closed. If $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$ then $Fix(\mathcal{F})$ is also closed and therefore correspond to an algebraic subgroup of $Aut(\mathbb{P}^3)$. Item 1 follows. To prove Item 2, let v be a vector field in $\mathfrak{aut}(\mathcal{F}) - \mathfrak{fix}(\mathcal{F})$. If $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(d+2))$, where d is the degree of the foliation $\mathcal{F}$, then $\mathcal{F}$ is defined by the closed meromorphic 1-form over $\mathbb{P}^3$

$$\tilde{\omega} = \frac{\omega}{i_v\omega}.$$

It is sufficient to show that $\tilde{\omega}$ is closed 1-form. In fact

$$d\tilde{\omega} = \frac{\omega \wedge d\omega(v) - \omega(v)d\omega}{(\omega(v))^2}. \tag{3.6}$$

Since $v \in \mathfrak{aut}(\mathcal{F})$ we have that

$$L_v(\omega) \wedge \omega = 0,$$

where $L_v = di_v + i_v d$ is the Lie derivative. Therefore

$$d\omega(v) \wedge d\omega + i_v(d\omega) \wedge \omega = 0.$$

By the integrability of ω we obtain

$$\omega(v)d\omega + (i_v d\omega)\omega = 0.$$

From this last equality we derive that

$$\omega \wedge d\omega(v) - \omega(v)d\omega = 0. \tag{3.7}$$

Replacing (3.7) by (3.6) we conclude that $d\tilde{\omega} = 0$. □

3.6 Irreducible components on $\mathbb{P}^3$ tangent to $S = l_0 x\frac{\partial}{\partial x} + l_1 y\frac{\partial}{\partial y} + l_1 z\frac{\partial}{\partial z}$

For the foliations of higher degree, we have the following theorem.

Theorem 4. *If $l_0 > l_1$, $\gcd(l_0, l_1) = 1$, $l_0 \geq 3$ and $q \geq 1$, then*

$$\overline{\mathbb{F}ol((l_0, l_1, l_1), l_1(ql_0 - 1), ql_0 + 1)},$$

is an irreducible component of $\mathbb{F}ol(ql_0 + 1, 3)$ and

$$\overline{\mathbb{F}ol((l_0, l_1, l_1), l_0 l_1 q, ql_0 + 2)},$$

is an irreducible component of $\mathbb{F}ol(ql_0 + 2, 3)$.

Proof. Let $\tilde{\mathcal{F}}$ be the foliation of the Example 3.4.2 or 3.4.3 given by ω which is generated by two one-dimensional foliations on $\mathbb{P}^3$, say $\mathcal{G}$ and $\mathcal{H}$, the foliations defined by the homogeneous vector fields S and X respectively, furthermore its tangent bundle $T\tilde{\mathcal{F}}$ splits as the sum of two line bundles $T\tilde{\mathcal{F}} \cong T\mathcal{G} \oplus T\mathcal{H}$.

Now, let $\{\mathcal{F}_t\}_{t\in\Sigma}$, $0 \in \Sigma \subset \mathbb{C}$ be a holomorphic family of foliations such that $\tilde{\mathcal{F}} = \mathcal{F}_0$. Since $d\omega$ has isolated singularities, applying the Theorem 3.3.10 we have that for small $|t|$,

$$T\mathcal{F}_t = T\mathcal{G}_t \oplus T\mathcal{H}_t.$$

Then $\mathcal{F}_t$ is generated by two one-dimension foliations $\mathcal{G}_t$ and $\mathcal{H}_t$. As as consequence, $\mathcal{G}_t$ is generated by a global vector field S_t on $\mathbb{P}^3$ with zeros of codimension at least two. Notice that $\mathbb{C}S_t \subset \mathfrak{fix}(\mathcal{F}_t)$.

Suppose $\mathfrak{fix}(\mathcal{F}_t) \neq \mathfrak{aut}(\mathcal{F}_t)$. Lemma 3.5.1 implies that $\mathcal{F}_t$ is given by a closed meromorphic 1-form with zero set of codimension at least two, then by [33, Lemma 5.4] implies that $\mathcal{F}_t$ can be deformed to a foliation defined by a logarithmic 1-form. Thus $\mathcal{F}_t$ belongs to irreducible components of type rational or logarithmic. On the other hand [21, Theorem 3] that says the generic element of the logarithmic foliation of degree greater than or equal 3 on $\mathbb{P}^3$ has isolated singularity and thus its tangent sheaf is not split, this is a contradiction.

If we assume $\mathfrak{fix}(\mathcal{F}_t) = \mathfrak{aut}(\mathcal{F}_t)$ with $\dim \mathfrak{fix}(\mathcal{F}_t) > 1$ then, as S_t has no divisorial components in its zero set, any two elements in it will generate $T\mathcal{F}_t$. Thus $T\mathcal{F}_t \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}$, this is a contradiction with $\deg(\mathcal{F}_t) = \deg(\tilde{\mathcal{F}}) > 3$.

Finally, we have that $\mathfrak{fix}(\mathcal{F}_t) = \mathfrak{aut}(\mathcal{F}_t) = \mathbb{C}S_t$. Lemma 3.5.1 $\mathcal{F}_t$ is tangent to action of one-dimension Lie group. We can see S_t as a deformation of S by automorphisms of $\mathbb{P}^3$. In open set $U_3 = \mathbb{C}^3$ we can write

$$S_t = (\lambda_1(t)x + a_1(t)y + a_2(t)z)\frac{\partial}{\partial x} + (\lambda_2(t)y + a_3(t)z)\frac{\partial}{\partial y} + \lambda_3(t)z\frac{\partial}{\partial z},$$

such that $S_0 = S = l_0\frac{\partial}{\partial x} + l_1\frac{\partial}{\partial y} + l_1\frac{\partial}{\partial z}$. Notice that we can take S_t sufficiently close to S and similarly λ_1, λ_2 and λ_3 sufficiently close to l_0, l_1 and l_1 respectively. Hence $\lambda_1\lambda_2\lambda_3 \neq 0$. Since the solutions of foliation induced by S_t are algebraic, we have two possibilities: Either $a_1 = a_2 = a_3 = 0$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$, or $\lambda_1 = \lambda_2 = \lambda_3 = 0$. We conclude that $a_1 = a_2 = a_3 = 0$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$. Using the conditions $\lambda_1(0) = l_0$, $\lambda_2(0) = l_1$, $\lambda_3(0) = l_1$ it follows that $\lambda_1 = l_0$, $\lambda_2 = l_1$, $\lambda_3 = l_1$. This proves the theorem. □

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