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RIGIDITY CONJECTURE FOR C^3 CRITICAL CIRCLE MAPS

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... à Yuri.

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Abstract

We prove that any two C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism, for some universal $\alpha > 0$. The proof is based on the existence of a C^ω-compact set of real-analytic critical commuting pairs with the following property: given a C^3 critical circle map f with any irrational rotation number there exists a sequence $\{f_n\}$, contained in that compact set, such that $\mathcal{R}^n(f)$ is C^0-exponentially close to f_n at a universal rate, and both have the same rotation number. Here $\mathcal{R}$ denotes the renormalization operator for critical commuting pairs.

Keywords: Critical circle maps, rigidity, renormalization, commuting pairs.

Resumo

Provamos que quaisquer dois mapas críticos do círculo de classe C^3 com o mesmo número de rotação irracional do tipo limitado e a mesma ordem no ponto crítico são conjugados por um difeomorfismo de classe $C^{1+\alpha}$, para um $\alpha > 0$ universal. A prova está baseada na existência de um conjunto C^{ω}-compacto de pares críticos comutantes reais-analíticos com a seguinte propriedade: dado um mapa crítico do círculo f de classe C^3 com qualquer número de rotação irracional existe uma sequência $\{f_n\}$, contida nesse conjunto compacto, tal que $\mathcal{R}^n(f)$ e f_n estão C^0-exponencialmente perto, e têm o mesmo número de rotação. Aqui $\mathcal{R}$ denota o operador de renormalização para pares críticos comutantes.

Palavras-chave: Mapas críticos do círculo, rigidez, renormalização, pares comutantes.

Contents

Introduction

0.1 Critical Circle Maps

0.1.1 Rigidity in dynamics

In the theory of real one-dimensional dynamics there exist many levels of equivalence between two systems: combinatorial, topological, quasi-symmetric and smooth equivalence are major examples.

In the circle case, a classical result of Poincaré [45, Chapter 1, Theorem 1.1] states that circle homeomorphisms with the same irrational rotation number are *combinatorially equivalent*: for each $n \in \mathbb{N}$ the first n elements of an orbit are ordered in the same way for any homeomorphism with a given rotation number. This implies that circle homeomorphisms with irrational rotation number are semi-conjugate to the corresponding rigid rotation and, therefore, they admit a unique invariant Borel probability measure.

By Denjoy's theorem [7], any two C^2 circle diffeomorphisms with the same irrational rotation number are conjugate to each other by a C^0 homeomorphism (actually we just need C^1 maps such that the logarithm of the modulus of the derivative has bounded variation). This implies that C^2 diffeomorphisms with irrational rotation number are minimal, and therefore, the support of its unique invariant probability measure is the whole circle.

By a fundamental result of Herman [20], improved by Yoccoz [62], any two $C^{2+\varepsilon}$ circle diffeomorphisms whose common rotation number ρ satisfies the Diophantine condition:

$$\left|\rho - \frac{p}{q}\right| \geq \frac{C}{q^{2+\delta}}\,, \tag{0.1.1}$$

for some $\delta \in [0,1)$ and $C > 0$, and for every positive coprime integers p and q, are conjugate to each other by a circle diffeomorphism. More precisely, if $0 \leq \delta < \varepsilon \leq 1$ and $\varepsilon - \delta \neq 1$, any such diffeomorphism is conjugate to the corresponding rigid rotation by a $C^{1+\varepsilon-\delta}$ diffeomorphism [27]. This implies that its invariant probability measure is absolutely continuous with respect to Lebesgue, with Hölder continuous density with exponent $\varepsilon-\delta$. Moreover, any two C^∞ circle diffeomorphisms with the same Diophantine rotation number are C^∞-conjugate to each other, and real-analytic diffeomorphisms with the same Diophantine rotation number are conjugate to each other by a real-analytic diffeomorphism [45, Chapter I, Section 3].

These are examples of *rigidity* results: lower regularity of conjugacy implies higher regularity under certain conditions.

Since rigidity is totally understood in the setting of circle diffeomorphisms we continue in this thesis the study of rigidity problems for critical circle maps developed by de Faria, de Melo, Yampolsky, Khanin and Teplinsky among others.

By a *critical circle map* we mean an orientation preserving C^3 circle homeomorphism with exactly one non-flat critical point of odd type (for simplicity, and for being the generic case, we will assume in this thesis that the critical point is of cubic type). As usual, a critical point c is called *non-flat* if in a neighbourhood of c the map f can be written as $f(t) = \pm\big|\phi(t)\big|^d + f(c)$, where ϕ is a C^3 local diffeomorphism with $\phi(c) = 0$, and $d \in \mathbb{N}$ with $d \geq 2$. The *criticality* (or *type*, or *order*) of the critical point c is d.

The main reference for background in real one-dimensional dynamics is the book of de Melo and van Strien [45].

0.1.2 The Arnold family

Classical examples of critical circle maps are obtained from the two-parameter family $\widetilde{f}_{a,b} : \mathbb{C} \to \mathbb{C}$ of entire maps in the complex plane:

$$\widetilde{f}_{a,b}(z) = z + a - \left(\frac{b}{2\pi}\right)\sin(2\pi z) \quad \text{for } a \in [0,1) \text{ and } b \geq 0. \tag{0.1.2}$$

Since each $\widetilde{f}_{a,b}$ commutes with unitary horizontal translation, it is the lift of a holomorphic map of the punctured plane $f_{a,b} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ via the holomorphic universal cover $z \mapsto e^{2\pi i z}$. Since $\widetilde{f}_{a,b}$ preserves the real axis, $f_{a,b}$ preserves the unit circle $S^1 = \big\{z \in \mathbb{C} : |z| = 1\big\}$ and therefore induces a two-parameter family of real-analytic maps of the unit circle. This classical family was introduced by Arnold in [3], and is called the *Arnold family*.

For $b = 0$ the family $f_{a,b} : S^1 \to S^1$ is the family of rigid rotations $z \mapsto e^{2\pi i a} z$, and for $b \in (0,1)$ the family is still contained in the space of real-analytic circle diffeomorphisms.

For $b = 1$ each $\widetilde{f}_{a,b}$ still restricts to an increasing real-analytic homeomorphism of the real line, that projects to an orientation-preserving real-analytic circle homeomorphism, presenting one critical point of cubic type at 1, the projection of the integers. Denote by $\rho(a)$ the rotation number of the circle homeomorphism $f_{a,1}$. It is well-known that $a \mapsto \rho(a)$ is continuous, non-decreasing, maps $[0,1)$ onto itself and is such that the interval $\rho^{-1}(\theta) \subset [0,1)$ degenerates to a point whenever $\theta \in [0,1) \setminus \mathbb{Q}$ (see [20]). Moreover the set $\big\{a \in [0,1) : \rho(a) \in \mathbb{R} \setminus \mathbb{Q}\big\}$ has zero Lebesgue measure, see [56]. For $0 \leq p < q$ coprime integers we know that $\rho^{-1}\big(\{\frac{p}{q}\}\big)$ is always a non-degenerate closed interval. In the interior of this interval we find critical circle maps with two periodic orbits (of period q), one attracting and one repelling, which collapse to a single parabolic orbit in the boundary of the interval, see [9].

For $b > 1$ the maps $f_{a,b} : S^1 \to S^1$ are not invertible any more (they present two critical points of even degree). These examples show how critical circle maps arise as bifurcations from circle diffeomorphisms to endomorphisms, and in particular, from zero to positive topological entropy (compare with infinitely renormalizable unimodal maps [45, Chapter VI]). This is one of the main reasons why critical circle maps attracted the attention of physicists and mathematicians interested in understanding the *boundary of chaos* ([8], [15], [23], [30], [31], [34] [35], [46], [49], [50], [51], [54]).

0.1.3 Further examples

Another important class of critical circle maps is provided by the one-parameter family $f_\gamma : \mathbb{C} \to \mathbb{C}$ of Blaschke products in the complex plane:

$$f_\gamma(z) = e^{2\pi i \gamma} z^2 \left(\frac{z-3}{1-3z}\right) \quad \text{for } \gamma \in [0,1). \tag{0.1.3}$$

Every map in this family leaves invariant the unit circle (Blaschke products are the rational maps leaving invariant the unit circle), and its restriction to S^1 is a real-analytic homeomorphism with a unique critical point at 1, which is of cubic type (see Figure 1). Furthermore, for each irrational number θ in $[0,1)$ there exists a unique γ in $[0,1)$ such that the rotation number of $f_\gamma|_{S^1}$ is θ. With this family at hand, the developments on rigidity of critical circle maps were very useful in the study of local connectivity and Lebesgue measure of Julia sets associated to generic quadratic polynomials with Siegel disks ([47], [40], [58], [48]).

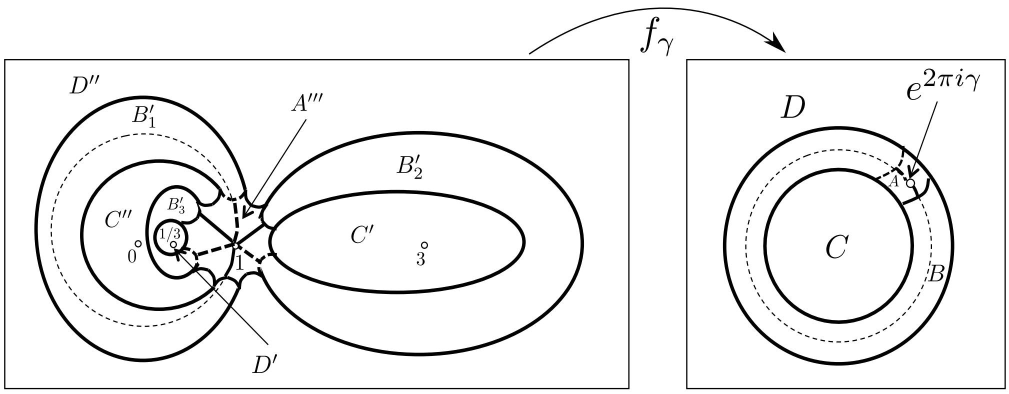

Figure 1: Topological behaviour of the Blaschke product f_γ (0.1.3) around the unit circle, for γ approximately equal to 1/8. At the left of Figure 1 we see the preimage under f_γ of the annulus around the unit circle drawn at the right (in both planes, the unit circle is dashed). The complement of the annulus $A \cup B$ in the complex plane has two connected components, C and D. The preimage of C is the union $C' \cup C''$, where the notation C' means that $f_\gamma : C' \to C$ has topological degree 1 (equivalently $f_\gamma : C'' \to C$ has topological degree 2). In the same way, the preimage of D is the union $D' \cup D''$, the preimage of B is $B'_1 \cup B'_2 \cup B'_3$ and the preimage of A is A'''.

0.2 Statement of the results

Since our goal is to study smoothness of conjugacies we will focus on critical circle maps without periodic orbits, that is, the ones with irrational rotation number. In [63] Yoccoz proved that the rotation number is the unique invariant of the topological classes. More precisely, any C^3 orientation preserving circle homeomorphism presenting only non-flat critical points (maybe more than one) and with irrational rotation number is topologically conjugate to the corresponding rigid rotation. For the sake of completeness we present Yoccoz's proof in Appendix A (Theorem A.4.2), where we also state and prove Denjoy's result (Theorem A.2.9) as an introduction to the techniques.

From the topological rigidity we get that any C^3 critical circle map with irrational rotation number is minimal and therefore the support of its unique invariant Borel probability measure is the whole circle. However let us point out that this invariant measure is always singular with respect to Lebesgue measure (see [25, Theorem 4, page 182] or [17, Proposition 1, page 219]). We remark also that the condition of non-flatness on the critical points cannot be removed: in [19] Hall was able to construct C^∞ homeomorphisms of the circle with no periodic points and no dense orbits.

Recall that an irrational number is of *bounded type* if it satisfies the Diophantine condition (0.1.1) for $\delta = 0$, that is, θ in $[0,1]$ is of bounded type if there exists $C > 0$ such that:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^2}\,,$$

for any integers p and $q \neq 0$. On one hand this is a respectable class: the set of numbers of bounded type is dense in $[0,1]$, with Hausdorff dimension equal to one. On the other hand, from the metrical viewpoint, this is a rather restricted class: while Diophantine numbers have full Lebesgue measure in $[0,1]$ (see Lemma A.1.8), the set of numbers of bounded type has zero Lebesgue measure (see Lemma A.1.3).

Since a critical circle map cannot be smoothly conjugate to a rigid rotation, in order to study smooth-rigidity problems we must restrict to the class of critical circle maps. Numerical observations ([15], [46], [54]) suggested in the early eighties that smooth critical circle maps with rotation number of bounded type are geometrically rigid. This was posed as a conjecture in several works by Lanford ([30], [31]), Rand ([49], [50] and [51], see also [46]) and Shenker ([54], see also [15]) among others:

Rigidity Conjecture. *Any two C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism, for some $\alpha > 0$.*

The conjecture has been proved by de Faria and de Melo for real-analytic critical circle maps [13] and nowadays (after the work of Yampolsky, Khanin and Teplinsky) it is understood without any assumption on the irrational rotation number: inside each topological class of real-analytic critical circle maps the degree of the critical point is the unique invariant of the C^1-conjugacy classes. In the following result we summarize many contributions of the authors quoted above:

Theorem A (de Faria-de Melo, Khmelev-Yampolsky, Khanin-Teplinsky)**.** *Let f and g be two real-analytic circle homeomorphisms with the same irrational rotation number and with a unique critical point of the same odd type. Let h be the conjugacy between f and g (given by Yoccoz's result) that maps the critical point of f to the critical point of g (note that this determines h). Then:*

1. *h is a C^1 diffeomorphism.*

2. *h is $C^{1+\alpha}$ at the critical point of f for a universal $\alpha > 0$.*

3. *For a full Lebesgue measure set of rotation numbers (that contains all bounded type numbers) h is globally $C^{1+\alpha}$.*

On one hand, the presence of the critical point gives us more rigidity than in the case of diffeomorphisms: smooth conjugacy is obtained for all irrational rotation numbers, with no Diophantine conditions. On the other hand, there exist examples ([4], [12]) showing that h may not be globally $C^{1+\alpha}$ in general, even for real-analytic dynamics.

Item (1) of Theorem A was proved by Khanin and Teplinsky in [26], building on earlier work of de Faria, de Melo and Yampolsky ([10], [11], [12], [13], [58], [59], [60], [61]). Item (2) was proved in [29] and Item (3) is obtained combining [12] with [61]. The proof of Theorem A relies on methods coming from complex analysis and complex dynamics ([39], [41]), and that is why rigidity is well understood for real-analytic critical circle maps, but nothing was known yet for smooth ones (even in the C^∞ setting). In this thesis we take the final step and solve positively the Rigidity Conjecture:

Theorem B (Main result)**.** *Any two C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism, for some universal $\alpha > 0$.*

The novelties of this thesis in order to transfer rigidity from real-analytic dynamics to (finitely) smooth ones are two: the first one is a *bidimensional* version of the *glueing procedure* (first introduced by Lanford [30], [31]) developed in Chapter 5, and the second one is the notion of *asymptotically holomorphic maps*, to be defined in Chapter 4 (Definition 4.1.2). Asymptotically holomorphic maps were already used in one-dimensional dynamics by Graczyk, Sands and Świątek in [16], but as far as we know never for critical circle maps.

Let us discuss the main ideas of the proof of Theorem B: a C^3 critical circle map f with irrational rotation number generates a sequence $\{\mathcal{R}^n(f) = (\eta_n, \xi_n)\}_{n\in\mathbb{N}}$ of commuting pairs of interval maps, each one being the renormalization of the previous one (see Definition 1.4.1). To prove Theorem B we need to prove the exponential convergence of the orbits generated by two critical circle maps with a given combinatorics of bounded type (see Theorem 0.3.1).

Our main task (see Theorem D in Chapter 2) is to show the existence of a sequence $\{f_n = (\widetilde{\eta}_n, \widetilde{\xi}_n)\}_{n\in\mathbb{N}}$ that belongs to a universal C^ω-compact set of real-analytic critical commuting pairs, such that $\mathcal{R}^n(f)$ is C^0-exponentially close to f_n at a universal rate, and both have the same rotation number. In Chapter 2, using the exponential contraction of the renormalization operator on the space of real-analytic critical commuting pairs (see Theorem 0.3.3), we conclude the exponential contraction of the renormalization operator in the space of C^3 critical commuting pairs with bounded combinatorics (see Theorem C in Section 0.3), and therefore the $C^{1+\alpha}$ rigidity as stated in Theorem B.

To realize the main task we extend the initial commuting pair to a pair of C^3 maps in an open complex neighbourhood of each original interval (the so-called *extended lift*, see Definition 4.1.4), that are asymptotically holomorphic (see Definition 4.1.2), each having a unique cubic critical point at the origin.

Using the real bounds (see Theorem 1.1.1), the Almost Schwarz inclusion (see Proposition 4.2.1) and the asymptotic holomorphic property we prove that for all $n \in \mathbb{N}$, greater or equal than some n_0, both η_n and ξ_n extend to a definite neighbourhood of their interval domains in the complex plane, giving rise to maps with a unique cubic critical point at the origin, and with exponentially small conformal distortion (see Theorem 4.0.4). Theorem 4.0.4 gives us also some geometric control that will imply the desired compactness (we wont study the dynamics of these extensions, just their geometric behaviour).

Using Ahlfors-Bers theorem (see Proposition 3.3.2) we construct for each $n \geq n_0$ a C^3 diffeomorphism Φ_n, exponentially close to the identity in definite domains around the dynamical intervals, that conjugates (η_n, ξ_n) to a C^3 critical commuting pair $(\widehat{\eta_n}, \widehat{\xi_n})$ exponentially close to (η_n, ξ_n), and such that $\widehat{\eta_n}^{-1} \circ \widehat{\xi_n}$ is an holomorphic diffeomorphism between complex neighbourhoods of the endpoints of the union of the dynamical intervals (see Section 5.1). Using this holomorphic diffeomorphism to glue the ends of a band around the union of the dynamical intervals we obtain a Riemann surface conformally equivalent to a rounds annulus A_{R_n} around the unit circle. This identification gives rise to a holomorphic local diffeomorphism P_n mapping the band onto the annulus and such that, via P_n, the pair $(\widehat{\eta_n}, \widehat{\xi_n})$ induces a C^3 map G_n from an annulus in A_{R_n} to A_{R_n}, having exponentially small conformal distortion, that restricts to a critical circle map on S^1 (see Proposition 5.1.7). The commuting condition of each pair $(\widehat{\eta_n}, \widehat{\xi_n})$ is equivalent to the continuity of the corresponding G_n, and that is why we project to the annulus A_{R_n}. The topological behaviour of each G_n on its annular domain is the same as the restriction of the Blaschke product f_γ (0.1.3) to the annulus $A''' \cup B_1'$, as depicted in Figure 1.

Using again Ahlfors-Bers theorem we construct a holomorphic map H_n, on a smaller but definite annulus around the unit circle, that is exponentially close to G_n and restricts to a real-analytic critical circle map with the same combinatorics as the restriction of G_n to S^1 (see Proposition 5.2.1 for much more properties).

Finally, using the projection P_n, we lift each H_n to a real-analytic critical commuting pair $f_n = (\widetilde{\eta}_n, \widetilde{\xi}_n)$ exponentially close to $(\widehat{\eta}_n, \widehat{\xi}_n)$, having the same combinatorics and with complex extensions C^0-exponentially close to the ones of $\mathcal{R}^n(f)$ produced in Theorem 4.0.4 (see Proposition 5.3.1). Compactness follows then from the geometric properties obtained in Theorem 4.0.4 (see Lemma 5.3.2).

0.2.1 Some geometric consequences of the main result

For a C^3 critical circle map f with irrational rotation number θ and critical point c in S^1, define the *n-th scaling ratio* as:

$$s_n(f) = \frac{d\big(f^{q_{n+1}}(c), c\big)}{d\big(f^{q_n}(c), c\big)}\,,$$

where $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of return times given by the continued fraction expansion of θ (see Chapter 1 and Appendix A) and d denote the standard distance in S^1.

The smoothness of the conjugacy leads to a geometric classification since, being essentially affine at small scales, the conjugacy preserves the small-scale properties of the dynamics. Some examples of these geometric properties are the following:

Corollary 0.2.1. *If f and g are C^3 critical circle maps with the same irrational rotation number of bounded type and the same odd degree at the critical point, we have* asymptotic geometric rigidity*:*

$$\lim_{n\to+\infty}\big(s_n(f) - s_n(g)\big) = 0$$

For real-analytic critical circle maps, Corollary 0.2.1 was first obtained by de Faria in his PhD thesis (see [10] and [11]).

Corollary 0.2.2. *Let μ be the unique invariant Borel probability of a C^3 critical circle map with rotation number of bounded type θ, and let $HD(\mu)$ denote the Hausdorff dimension of μ (the infimum of the dimensions of full measure sets). Then $HD(\mu)$ only depends on θ and the degree at the critical point.*

See [17] for some estimates.

0.3 A first reduction of the main result

As in the case of unimodal maps, the main tool in order to obtain smooth conjugacy between critical circle maps is the use of *renormalization* group methods [42]. As it was already clear in the early eighties ([15], [46]) it is convenient to construct a renormalization operator $\mathcal{R}$ (see Definition 1.4.1) acting not on the space of critical circle maps but on a suitable space of *critical commuting pairs* (see Definition 1.2.1).

Just as in the case of unimodal maps (see for instance [45, Chapter VI, Theorem 9.4]), the principle that *exponential convergence of the renormalization operator is equivalent to smooth conjugacy* also holds for critical circle maps. The following result is due to de Faria and de Melo [12, First Main Theorem, page 341]. For any $0 \leq r < \infty$ denote by d_r the C^r metric in the space of critical commuting pairs (see Definition 1.3.1):

Theorem 0.3.1 (de Faria-de Melo 1999)**.** *There exists a set $\mathbb{A}$ in $[0,1]$, having full Lebesgue measure and containing all irrational numbers of bounded type, for which the following holds: let f and g be two C^3 critical circle maps with the same irrational rotation number in the set $\mathbb{A}$ and with the same odd type at the critical point. If $d_0\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big)$ converge to zero exponentially fast when n goes to infinity, then f and g are $C^{1+\alpha}$ conjugate to each other for some $\alpha > 0$.*

Roughly speaking, the full Lebesgue measure set $\mathbb{A}$ is composed by irrational numbers in $[0,1]$ whose coefficients in the continued fraction expansion may be unbounded, but their growth is less than quadratic (see Chapter 6 or [12, Appendix C] for the precise definition). In sharp contrast with the case of diffeomorphisms, let us point out that $\mathbb{A}$ does not contain all Diophantine numbers, and contains some Liouville numbers (again see Chapter 6). The remaining cases were more recently solved by Khanin and Teplinsky [26, Theorem 2, page 198]:

Theorem 0.3.2 (Khanin-Teplinsky 2007)**.** *Let f and g be two C^3 critical circle maps with the same irrational rotation number and the same odd type at the critical point. If $d_2\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big)$ converge to zero exponentially fast when n goes to infinity, then f and g are C^1-conjugate to each other.*

To obtain the smooth conjugacy (Item (1) of Theorem A), Khanin and Teplinsky combined Theorem 0.3.2 with the following fundamental resut:

Theorem 0.3.3 (de Faria-de Melo 2000, Yampolsky 2003)**.** *There exists a universal constant λ in $(0,1)$ with the following property: given two real-analytic critical commuting pairs ζ_1 and ζ_2 with the same irrational rotation* *number and the same odd type at the critical point, there exists a constant* $C > 0$ *such that:*

$$d_r\big(\mathcal{R}^n(\zeta_1), \mathcal{R}^n(\zeta_2)\big) \leq C\lambda^n$$

for all $n \in \mathbb{N}$ *and for any* $0 \leq r < \infty$*. Moreover given a* C^ω*-compact set* $\mathcal{K}$ *of real-analytic critical commuting pairs, the constant* C *can be chosen the same for any* ζ_1 *and* ζ_2 *in* $\mathcal{K}$*.*

Theorem 0.3.3 was proved by de Faria and de Melo [13] for rotation numbers of bounded type, and extended by Yampolsky [61] to cover all irrational rotation numbers.

With Theorem 0.3.1 at hand, our main result (Theorem B) reduces to the following one:

Theorem C. *There exists* $\lambda \in (0, 1)$ *such that given* f *and* g *two* C^3 *critical circle maps with the same irrational rotation number of bounded type and the same criticality, there exists* $C > 0$ *such that for all* $n \in \mathbb{N}$*:*

$$d_0\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big) \leq C\lambda^n\,,$$

where d_0 *is the* C^0 *distance in the space of critical commuting pairs.*

This thesis is devoted to proving Theorem C. Of course it would be desirable to obtain Theorem C for C^3 critical circle maps with any irrational rotation number, but we have not been able to do this yet (see Chapter 6 for more comments).

Let us fix some notation that we will use along this thesis: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denotes respectively the set of natural, integer, rational, real and complex numbers. The real part of a complex number z will be denoted by $\Re(z)$, and its imaginary part by $\Im(z)$. $B(z, r)$ denotes the Euclidean open ball of radius $r > 0$ around a complex number z. $\mathbb{H}$ and $\widehat{\mathbb{C}}$ denotes respectively the upper-half plane and the Riemann sphere. $\mathbb{D} = B(0, 1)$ denotes the unit disk in the complex plane, and $S^1 = \partial\,\mathbb{D}$ denotes its boundary, that is, the unit circle. $\text{Diff}^3_+(S^1)$ denotes the group (under composition) of orientation-preserving C^3 diffeomorphisms of the unit circle. $\text{Leb}(A)$ denotes the Lebesgue measure of a Borel set A in the plane, and $\text{diam}(A)$ denotes its Euclidean diameter. Given a bounded interval I in the real line we denote its Euclidean length by $|I|$. Moreover, for any $\alpha > 0$, let:

$$N_\alpha(I) = \big\{z \in \mathbb{C} : d(z, I) < \alpha|I|\big\},$$

where d denotes the Euclidean distance in the complex plane.

The organization of this thesis is the following: in Section above 0.2 we stated some geometric consequences of the main result, while in Section 0.3

we reduced Theorem B to Theorem C, which states the exponential convergence of the renormalization orbits of C^3 critical circle maps with the same bounded combinatorics. In Chapter 1 below we introduce the renormalization operator in the space of critical commuting pairs, and review its basic properties. In Chapter 2 we reduce Theorem C to Theorem D, which states the existence of a C^ω-compact piece of real-analytic critical commuting pairs such that for a given C^3 critical circle map f, with any irrational rotation number, there exists a sequence $\{f_n\}$, contained in that compact piece, such that $\mathcal{R}^n(f)$ is C^0-exponentially close to f_n at a universal rate, and both have the same rotation number. In Chapter 3 we state a corollary of Ahlfors-Bers theorem (Proposition 3.3.2) that will be fundamental in Chapter 5 (its proof will be given in Appendix D). In Chapter 4 we construct the *extended lift* of a C^3 critical circle map (see Definition 4.1.4), and then we state and prove Theorem 4.0.4 as described above. In Chapter 5 we develop a bidimensional glueing procedure in order to prove Theorem D. Finally in Chapter 6 we review further questions and open problems in the area.

This thesis has four appendices: in Appendix A we briefly review one of the main tools in real one-dimensional dynamics, namely the distortion of the cross-ratio, and then we apply it in order to prove topological rigidity of critical circle maps (Yoccoz's theorem). In Appendix B we give the proof of Theorem 1.5.1, stated at the end of Chapter 1 and used in Chapter 2, in Appendix C we review some basic fact about Riemann surfaces used along the text, while in Appendix D we apply Ahlfors-Bers theorem in order to prove Proposition 3.3.2.

CHAPTER 1

Renormalization of critical commuting pairs

In this chapter we define the space of C^3 critical commuting pairs (Definition 1.2.1), and we endow it with the C^3 metric (Definition 1.3.1). This metric space, which is neither compact nor locally-compact, contains the phase space of the renormalization operator (Definition 1.4.1). Each C^3 critical circle map with irrational rotation number gives rise to an infinite renormalization orbit in this phase space, and the asymptotic behaviour of these orbits is the subject of this thesis.

We remark that, since there is no canonical differentiable structure (like a Banach manifold structure) in the space of C^3 critical commuting pairs endowed with the C^3 metric, we cannot apply the standard machinery from hyperbolic dynamics (see for instance [24, Chapters 6, 18 and 19]) in order to obtain exponential convergence as stated in Theorem C.

As we said in the introduction, a *critical circle map* is an orientation-preserving C^3 circle homeomorphism f, with exactly one critical point $c \in S^1$ of odd type. For simplicity, and for being the generic case, we will assume in this thesis that the critical point is of cubic type. Suppose that the rotation number $\rho(f) = \theta$ in $[0,1)$ is irrational, and let $\left[a_0, a_1, ..., a_n, a_{n+1}, ...\right]$ be its continued fraction expansion (see Definition A.1.1):

$$\theta = \lim_{n \to +\infty} \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_n}}}}}$$

We define recursively the *return times* of θ by:

$$q_0 = 1, \qquad q_1 = a_0 \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1} \quad \text{for} \quad n \geq 1.$$

Recall that the numbers q_n are also obtained as the denominators of the truncated expansion of order n of θ:

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, ..., a_{n-1}] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_{n-1}}}}}}$$

Let R_θ be the rigid rotation of angle $2\pi\theta$ in the unit circle. The arithmetical properties of the continued fraction expansion (see Section A.1.1) imply that the iterates $\{R_\theta^{q_n}(c)\}_{n\in\mathbb{N}}$ are the closest returns of the orbit of c under the rotation R_θ:

$$d\big(c, R_\theta^{q_n}(c)\big) < d\big(c, R_\theta^{j}(c)\big) \quad \text{for any} \quad j \in \{1, ..., q_n - 1\}$$

where d denote the standard distance in S^1. The sequence of return times $\{q_n\}$ increase at least exponentially fast as $n \to \infty$, and the sequence of return distances $\{d(c, R_\theta^{q_n}(c))\}$ decrease to zero at least exponentially fast as $n \to \infty$. Moreover the sequence $\{R_\theta^{q_n}(c)\}_{n\in\mathbb{N}}$ approach the point c alternating the order:

$$R_\theta^{q_1}(c) < R_\theta^{q_3}(c) < ... < R_\theta^{q_{2k+1}}(c) < ... < c < ... < R_\theta^{q_{2k}}(c) < ... < R_\theta^{q_2}(c) < R_\theta^{q_0}(c)$$

By Poincaré's result quoted at the beginning of the introduction, this information remains true at the combinatorial level for f: for any $n \in \mathbb{N}$ the interval $[c, f^{q_n}(c)]$ contains no other iterates $f^j(c)$ for $j \in \{1, ..., q_n - 1\}$, and if we denote by μ the unique invariant Borel probability of f we can say that $\mu\big([c, f^{q_n}(c)]\big) < \mu\big([c, f^j(c)]\big)$ for any $j \in \{1, ..., q_n - 1\}$. A priori we cannot say anything about the usual distance in S^1.

We say that $\rho(f)$ is of *bounded type* if there exists a constant $M \in \mathbb{N}$ such that $a_n < M$ for any $n \in \mathbb{N}$ (it is not difficult to see that this definition is equivalent with the one given in the introduction, see [28, Chapter II, Theorem 23]). As we said in the introduction, the set of numbers of bounded type has zero Lebesgue measure in $[0, 1]$ (see Lemma A.1.3).

1.1 Dynamical partitions

Denote by I_n the interval $[c, f^{q_n}(c)]$ and define $\mathcal{P}_n$ as:

$$\mathcal{P}_n = \left\{I_n, f(I_n), ..., f^{q_{n+1}-1}(I_n)\right\} \bigcup \left\{I_{n+1}, f(I_{n+1}), ..., f^{q_n-1}(I_{n+1})\right\}$$

A crucial combinatorial fact is that $\mathcal{P}_n$ is a partition (modulo boundary points) of the circle for every $n \in \mathbb{N}$. We call it the *n-th dynamical partition* of f associated with the point c. Note that the partition $\mathcal{P}_n$ is determined by the piece of orbit:

$$\{f^j(c) : 0 \leq j \leq q_n + q_{n+1} - 1\}$$

The transitions from $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$ can be described in the following easy way: the interval $I_n = [c, f^{q_n}(c)]$ is subdivided by the points $f^{jq_{n+1}+q_n}(c)$ with $1 \leq j \leq a_{n+1}$ into $a_{n+1} + 1$ subintervals. This sub-partition is spreaded by the iterates of f to all the $f^j(I_n) = f^j([c, f^{q_n}(c)])$ with $0 \leq j < q_{n+1}$. The other elements of the partition $\mathcal{P}_n$, which are the $f^j(I_{n+1})$ with $0 \leq j < q_n$, remain unchanged.

As we are working with critical circle maps, our partitions in this thesis are always determined by the critical orbit. A major result for critical circle maps is the following:

Theorem 1.1.1 (real bounds)**.** *There exists $K > 1$ such that given a C^3 critical circle map f with irrational rotation number there exists $n_0 = n_0(f)$ such that for all $n \geq n_0$ and for every pair I, J of adjacent atoms of $\mathcal{P}_n$ we have:*

$$K^{-1}|I| \leq |J| \leq K|I|.$$

Moreover, if Df denotes the first derivative of f, we have:

$$\frac{1}{K} \leq \frac{\left|Df^{q_n-1}(x)\right|}{\left|Df^{q_n-1}(y)\right|} \leq K \quad \textit{for all } x, y \in f(I_{n+1}) \textit{ and for all } n \geq n_0, \textit{ and:}$$

$$\frac{1}{K} \leq \frac{\left|Df^{q_{n+1}-1}(x)\right|}{\left|Df^{q_{n+1}-1}(y)\right|} \leq K \quad \textit{for all } x, y \in f(I_n) \textit{ and for all } n \geq n_0.$$

Theorem 1.1.1 was proved by Świątek and Herman (see [21], [56], [18] and [12]). The control on the distortion of the return maps follows from Koebe distortion principle (see [12, Section 3]). Note that for a rigid rotation we have $|I_n| = a_{n+1}|I_{n+1}| + |I_{n+2}|$. If a_{n+1} is big, then I_n is much larger than I_{n+1}. Thus, even for rigid rotations, real bounds do not hold in general.

1.2 Critical commuting pairs

The first return map of the union of adjacent intervals $I_n \cup I_{n+1}$ is given respectively by $f^{q_{n+1}}$ and f^{q_n}. This pair of interval maps:

$$\left(f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}}\right)$$

motivates the following definition:

Definition 1.2.1. A *critical commuting pair* $\zeta = (\eta, \xi)$ consists of two smooth orientation-preserving interval homeomorphisms $\eta : I_\eta \to \eta(I_\eta)$ and $\xi : I_\xi \to \xi(I_\xi)$ where:

1. $I_\eta = [0, \xi(0)]$ and $I_\xi = [\eta(0), 0]$;
2. There exists a neighbourhood of the origin where both η and ξ have homeomorphic extensions (with the same degree of smoothness) which commute, that is, $\eta \circ \xi = \xi \circ \eta$;
3. $\big(\eta \circ \xi\big)(0) = \big(\xi \circ \eta\big)(0) \neq 0$;
4. $\eta'(0) = \xi'(0) = 0$;
5. $\eta'(x) \neq 0$ for all $x \in I_\eta \setminus \{0\}$ and $\xi'(x) \neq 0$ for all $x \in I_\xi \setminus \{0\}$.

Any critical circle map f with irrational rotation number θ induces a sequence of critical commuting pair in a natural way: let $\widetilde{f}$ be the lift of f to the real line (for the canonical covering $t \mapsto e^{2\pi i t}$) satisfying $\widetilde{f}'(0) = 0$ and $0 < \widetilde{f}(0) < 1$. For each $n \geq 1$ let $\widetilde{I_n}$ be the closed interval in the real line, adjacent to the origin, that projects to I_n. Let $T : \mathbb{R} \to \mathbb{R}$ be the translation $x \mapsto x + 1$ and define $\eta : \widetilde{I_n} \to \mathbb{R}$ and $\xi : \widetilde{I_{n+1}} \to \mathbb{R}$ as:

$$\eta = T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}} \quad \text{and} \quad \xi = T^{-p_n} \circ \widetilde{f}^{q_n}$$

Then the pair $(\eta|_{\widetilde{I_n}}, \xi|_{\widetilde{I_{n+1}}})$ form a critical commuting pair that we denote by $(f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}})$ to simplify notation.

A converse of this construction was introduced by Lanford ([30], [31]) and it is known as *glueing procedure*: the map $\eta^{-1} \circ \xi$ is a diffeomorphism from a small neighbourhood of $\eta(0)$ onto a neighbourhood of $\xi(0)$. Identifying $\eta(0)$ and $\xi(0)$ in this way we obtain from the interval $\big[\eta(0), \xi(0)\big]$ a smooth compact boundaryless one-dimensional manifold M. The discontinuous piecewise smooth map:

$$f_\zeta(t) = \begin{cases} \xi(t) & \text{for } t \in \big[\eta(0), 0\big) \\ \eta(t) & \text{for } t \in \big[0, \xi(0)\big] \end{cases}$$

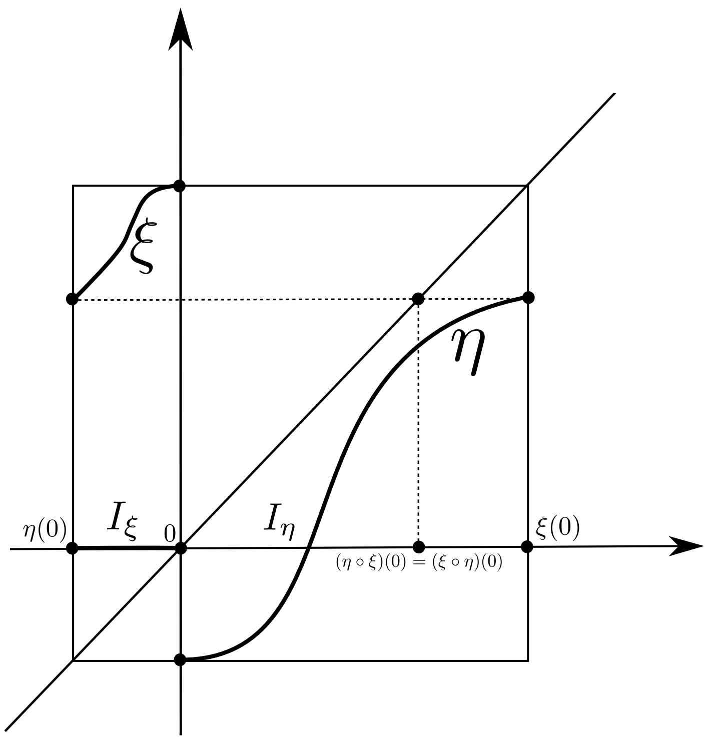

Figure 1.1: A commuting pair.

projects to a smooth homeomorphism on the quotient manifold M. By choosing a diffeomorphism $\psi : M \to S^1$ we obtain a critical circle map in S^1, just by conjugating with ψ. Although there is no canonical choice for the diffeomorphism ψ, any two different choices give rise to smoothly-conjugate critical circle maps in S^1. Therefore any critical commuting pair induces a whole *smooth conjugacy class* of critical circle maps. In Chapter 5 we propose a bidimensional extension of this procedure, in order to prove our main result (Theorem B).

1.3 The C^r metric

We endow the space of C^3 critical commuting pairs with the C^3 metric. Given two critical commuting pairs $\zeta_1 = (\eta_1, \xi_1)$ and $\zeta_2 = (\eta_2, \xi_2)$ let A_1 and A_2 be the Möbius transformations such that for $i = 1, 2$:

$$A_i\big(\eta_i(0)\big) = -1, \quad A_i(0) = 0 \quad \text{and} \quad A_i\big(\xi_i(0)\big) = 1\,.$$

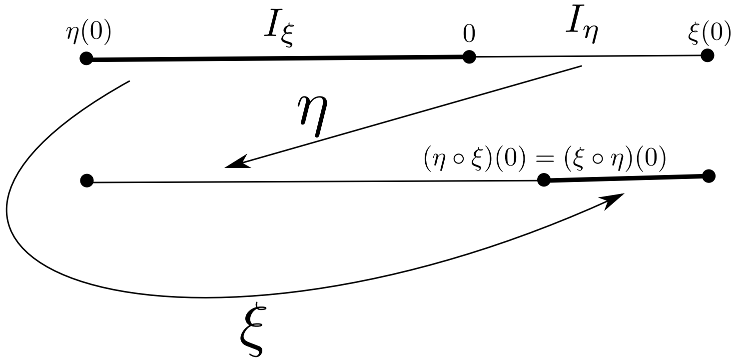

Figure 1.2: Scheme of a commuting pair.

Definition 1.3.1. For any $0 \le r < \infty$ define the C^r metric on the space of C^r critical commuting pairs in the following way:

$$d_r(\zeta_1, \zeta_2) = \max\left\{\left|\frac{\xi_1(0)}{\eta_1(0)} - \frac{\xi_2(0)}{\eta_2(0)}\right|, \left\|A_1 \circ \zeta_1 \circ A_1^{-1} - A_2 \circ \zeta_2 \circ A_2^{-1}\right\|_r\right\}$$

where $\|\cdot\|_r$ is the C^r-norm for maps in $[-1,1]$ with one discontinuity at the origin, and ζ_i is the piecewise map defined by η_i and ξ_i:

$$\zeta_i : I_{\xi_i} \cup I_{\eta_i} \to I_{\xi_i} \cup I_{\eta_i} \quad \text{such that} \quad \zeta_i|_{I_{\xi_i}} = \xi_i \quad \text{and} \quad \zeta_i|_{I_{\eta_i}} = \eta_i$$

Note that d_r is a pseudo-metric since it is invariant under conjugacy by homotheties: if α is a positive real number, $H_\alpha(t) = \alpha t$ and $\zeta_1 = H_\alpha \circ \zeta_2 \circ H_\alpha^{-1}$ then $d_r(\zeta_1, \zeta_2) = 0$. In order to have a metric we need to restrict to *normalized* critical commuting pairs: for a commuting pair $\zeta = (\eta, \xi)$ denote by $\widetilde{\zeta}$ the pair $(\widetilde{\eta}|_{\widetilde{I_\eta}}, \widetilde{\xi}|_{\widetilde{I_\xi}})$ where tilde means rescaling by the linear factor $\lambda = \frac{1}{|I_\xi|}$. Note that $|\widetilde{I_\xi}| = 1$ and $\widetilde{I_\eta}$ has length equal to the ratio between the lengths of I_η and I_ξ. Equivalently $\widetilde{\eta}(0) = -1$ and $\widetilde{\xi}(0) = \frac{|I_\eta|}{|I_\xi|} = \xi(0)/\big|\eta(0)\big|$.

When we are dealing with real-analytic critical commuting pairs, we consider the C^ω-topology defined in the usual way: we say that $\big(\eta_n, \xi_n\big) \to \big(\eta, \xi\big)$ if there exist two open sets $U_\eta \supset I_\eta$ and $U_\xi \supset I_\xi$ in the complex plane and $n_0 \in \mathbb{N}$ such that η and η_n for $n \ge n_0$ extend continuously to $\overline{U_\eta}$, are holomorphic in U_η and we have $\big\|\eta_n - \eta\big\|_{C^0(\overline{U_\eta})} \to 0$, and such that ξ and ξ_n for $n \ge n_0$ extend continuously to $\overline{U_\xi}$, are holomorphic in U_ξ and we have $\big\|\xi_n - \xi\big\|_{C^0(\overline{U_\xi})} \to 0$. We say that a set $\mathcal{C}$ of real-analytic critical commuting pairs is closed if every time we have $\{\zeta_n\} \subset \mathcal{C}$ and $\{\zeta_n\} \to \zeta$, we have $\zeta \in \mathcal{C}$. This defines a Hausdorff topology, stronger than the C^r-topology for any $0 \le r \le \infty$.

1.4 The renormalization operator

Let $\zeta = (\eta, \xi)$ be a C^3 critical commuting pair according to Definition 1.2.1, and recall that $\big(\eta \circ \xi\big)(0) = \big(\xi \circ \eta\big)(0) \neq 0$. Let us suppose that $\big(\xi \circ \eta\big)(0) \in I_\eta$ (just as in both Figure 1.1 and Figure 1.2 above) and define the *height* $\chi(\zeta)$ of the commuting pair $\zeta = (\eta, \xi)$ as r if:

$$\eta^{r+1}(\xi(0)) \leq 0 \leq \eta^r(\xi(0))$$

and $\chi(\zeta) = \infty$ if no such r exists (note that in this case the map $\eta|_{I_\eta}$ has a fixed point, so when we are dealing with commuting pairs induced by critical circle maps with irrational rotation number we have finite height). Note also that the height of the pair $(f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}})$ induced by a critical circle maps f is exactly a_{n+1}, where $\rho(f) = [a_0, a_1, a_2, ..., a_n, a_{n+1}, ...]$ (because the combinatorics of f are the same as for the rigid rotation $R_{\rho(f)}$).

For a pair $\zeta = (\eta, \xi)$ with $\big(\xi \circ \eta\big)(0) \in I_\eta$ and $\chi(\zeta) = r < \infty$ we see that the pair:

$$\big(\eta|_{[0, \eta^r(\xi(0))]}, \eta^r \circ \xi|_{I_\xi}\big)$$

is again a commuting pair, and if $\zeta = (\eta, \xi)$ is induced by a critical circle map:

$$\zeta = (\eta, \xi) = \big(f^{q_{n+1}}|_{I_n}, f^{q_n}|_{I_{n+1}}\big)$$

we have that:

$$\big(\eta|_{[0, \eta^r(\xi(0))]}, \eta^r \circ \xi|_{I_\xi}\big) = \big(f^{q_{n+1}}|_{I_{n+2}}, f^{q_{n+2}}|_{I_{n+1}}\big)$$

This motivates the following definition (the definition in the case $\big(\xi \circ \eta\big)(0) \in I_\xi$ is analogue):

Definition 1.4.1. Let $\zeta = (\eta, \xi)$ be a critical commuting pair with $\big(\xi \circ \eta\big)(0) \in I_\eta$. We say that ζ is *renormalizable* if $\chi(\zeta) = r < \infty$. In this case we define the *renormalization* of ζ as the critical commuting pair:

$$\mathcal{R}(\zeta) = \Big(\widetilde{\eta}|_{[0, \widetilde{\eta^r(\xi(0))}]}, \widetilde{\eta^r \circ \xi}|_{\widetilde{I_\xi}}\Big)$$

A critical commuting pair is a special case of a *generalized interval exchange map* of two intervals, and the renormalization operator defined above is just the restriction of the *Zorich accelerated version* of the *Rauzy-Veech renormalization* for interval exchange maps (see for instance [64]). However we will keep in this thesis the classical terminology for critical commuting pairs.

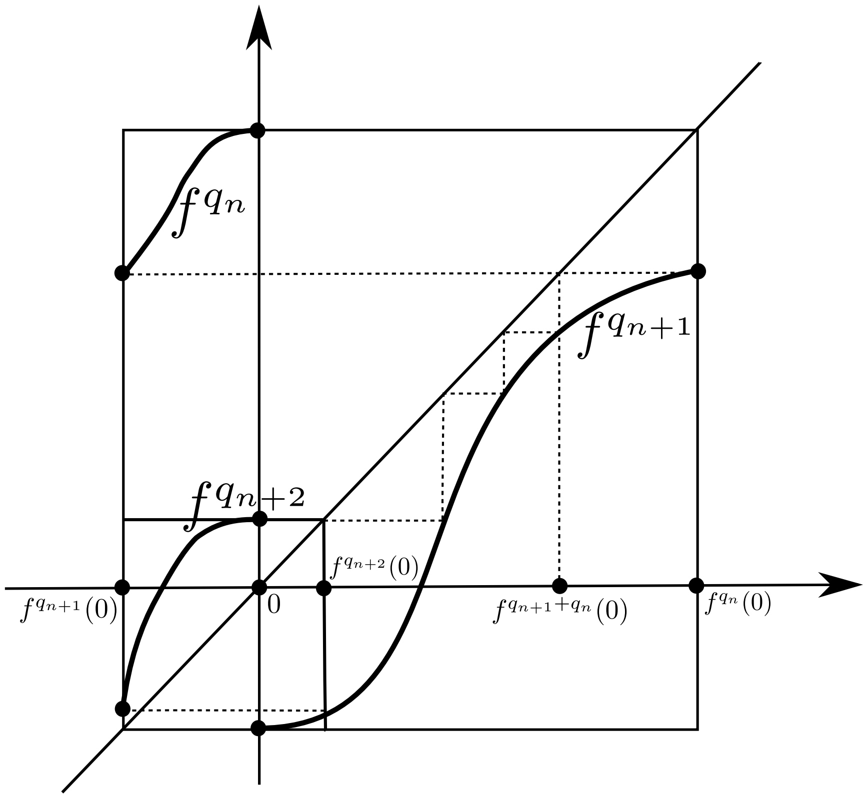

Figure 1.3: Two consecutive renormalizations of f, without rescaling (recall that f^{q_n} means $T^{-p_n} \circ \widetilde{f^{q_n}}$). In this example $a_{n+1} = 4$.

Definition 1.4.2. Let ζ be a critical commuting pair. If $\chi(\mathcal{R}^j(\zeta)) < \infty$ for $j \in \{0, 1, ..., n-1\}$ we say that ζ is *n-times renormalizable*, and if $\chi(\mathcal{R}^j(\zeta)) < \infty$ for all $j \in \mathbb{N}$ we say that ζ is *infinitely renormalizable*. In this case the irrational number θ whose continued fraction expansion is equal to:

$$\big[\chi(\zeta), \chi\big(\mathcal{R}(\zeta)\big), ..., \chi\big(\mathcal{R}^n(\zeta)\big), \chi\big(\mathcal{R}^{n+1}(\zeta)\big), ...\big]$$

is called the *rotation number* of the critical commuting pair ζ, and denoted by $\rho(\zeta) = \theta$.

The rotation number of a critical commuting pair can also be defined with the help of the glueing procedure described above, just as the rotation number of any representative of the conjugacy class obtained after glueing and uniformizing.

An immediate but very important remark is that when ζ is induced by a critical circle map with irrational rotation number, the pair ζ is automatically infinitely renormalizable (and both notions of rotation number coincide): any C^3 critical circle map f with irrational rotation number gives rise to a well defined orbit $\{\mathcal{R}^n(f)\}_{n\geq 1}$ of infinitely renormalizable C^3 critical commuting pairs defined by:

$$\mathcal{R}^n(f) = \left(\widetilde{f^{q_n}}|_{\widetilde{I_{n-1}}}, \widetilde{f^{q_{n-1}}}|_{\widetilde{I_n}}\right) \quad \text{for all} \quad n \geq 1.$$

For any positive number θ denote by $\lfloor\theta\rfloor$ the *integer part* of θ, that is, $\lfloor\theta\rfloor \in \mathbb{N}$ and $\lfloor\theta\rfloor \leq \theta < \lfloor\theta\rfloor + 1$. Recall that the *Gauss map* $G : [0,1] \to [0,1]$ is defined by (see Section A.1.1):

$$G(\theta) = \frac{1}{\theta} - \left\lfloor \frac{1}{\theta} \right\rfloor \quad \text{for} \quad \theta \neq 0 \quad \text{and} \quad G(0) = 0\,,$$

and note that ρ *semi-conjugates* the renormalization operator with the Gauss map:

$$\rho\big(\mathcal{R}^n(\zeta)\big) = G^n\big(\rho(f)\big)$$

for any ζ at least n-times renormalizable. In particular the renormalization operator acts as a *left shift* on the continued fraction expansion of the rotation number: if $\rho(\zeta) = [a_0, a_1, ...]$ then $\rho\big(\mathcal{R}^n(\zeta)\big) = [a_n, a_{n+1}, ...]$.

1.5 Lipschitz continuity along the orbits

For $K > 1$ and $r \in \{0, 1, ..., \infty, \omega\}$ denote by $\mathcal{P}^r(K)$ the space of C^r critical commuting pairs $\zeta = (\eta, \xi)$ such that $\eta(0) = -1$ (they are normalized) and $\xi(0) \in [K^{-1}, K]$. Recall also that T denotes the translation $t \mapsto t + 1$ in the real line. Let $K_0 > 1$ be the universal constant given by the real bounds. In the next chapter we will use the following:

Lemma 1.5.1. *Given $M > 0$ and $K > K_0$ there exists $L > 1$ with the following property: let f be a C^3 critical circle map with irrational rotation number $\rho(f) = [a_0, a_1, ...]$ satisfying $a_n < M$ for all $n \in \mathbb{N}$. There exists $n_0 = n_0(f) \in \mathbb{N}$ such that for any $n \geq n_0$ and any renormalizable critical commuting pair $\zeta = (\eta, \xi)$ satisfying:*

1. $\zeta, \mathcal{R}(\zeta) \in \mathcal{P}^3(K)$,
2. $$\left\lfloor \frac{1}{\rho(\zeta)} \right\rfloor = a_n\,,$$

3. If $\big(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}}\big)(0) < 0 < \big(T^{-p_n} \circ \widetilde{f}^{q_n}\big)(0)$ *then:*

$$\left|\left|\frac{\big(T^{-p_n} \circ \widetilde{f}^{q_n}\big)(0)}{\big(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}}\big)(0)}\right| - \xi(0)\right| < \left(\frac{1}{K^2}\right)\left(\frac{K+1}{K-1}\right).$$

Otherwise, if $\big(T^{-p_n} \circ \widetilde{f}^{q_n}\big)(0) < 0 < \big(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}}\big)(0)$*, then:*

$$\left|\left|\frac{\big(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}}\big)(0)}{\big(T^{-p_n} \circ \widetilde{f}^{q_n}\big)(0)}\right| - \xi(0)\right| < \left(\frac{1}{K^2}\right)\left(\frac{K+1}{K-1}\right), \quad \textit{and}$$

4. $\big(\eta \circ \xi\big)(0)$ *and* $\big(T^{-p_{n+1}-p_n} \circ \widetilde{f}^{q_{n+1}+q_n}\big)(0)$ *have the same sign,*

then we have that:

$$d_0\left(\mathcal{R}^{n+1}(f), \mathcal{R}(\zeta)\right) \leq L \cdot d_0\left(\mathcal{R}^n(f), \zeta\right),$$

where d_0 *is the* C^0 *distance in the space of critical commuting pairs.*

We postpone the proof of Lemma 1.5.1 until Appendix B.

CHAPTER 2

Reduction of Theorem C

In this chapter we reduce Theorem C to the following:

Theorem D. *There exist a C^ω-compact set $\mathcal{K}$ of real-analytic critical commuting pairs and a constant $\lambda \in (0,1)$ with the following property: given a C^3 critical circle map f with any irrational rotation number there exist $C > 0$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ contained in $\mathcal{K}$ such that:*

$$d_0\big(\mathcal{R}^n(f), f_n\big) \leq C\lambda^n \quad \textit{for all } n \in \mathbb{N},$$

and such that the pair f_n has the same rotation number as the pair $\mathcal{R}^n(f)$ for all $n \in \mathbb{N}$.

Note that $\mathcal{K}$ is C^r-compact for any $0 \leq r \leq \infty$ (see Section 1.3). Note also that Theorem D is true for *any* combinatorics. The following argument was inspired by [43]:

Proof that Theorem D implies Theorem C. Let $\mathcal{K}$ be the C^ω-compact set of real-analytic critical commuting pairs given by Theorem D. By the real bounds there exists a uniform constant $n_0 \in \mathbb{N}$ such that $\mathcal{R}^n(\zeta) \in \mathcal{P}^\omega(K_0)$ for all $\zeta \in \mathcal{K}$ and all $n \geq n_0$. Therefore there exists $K > K_0$ such that $\mathcal{R}^n(\zeta) \in \mathcal{P}^\omega(K)$ for all $\zeta \in \mathcal{K}$ and all $n \geq 1$. Let $M > \max_{n\in\mathbb{N}}\{a_n\}$ where $\rho(f) = \rho(g) = [a_0, a_1, ...]$, and let $L > 1$ given by Lemma 1.5.1.

By Theorem D there exist constants $\lambda_1 \in (0,1)$, $C_1(f), C_1(g) > 0$ and two sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ contained in $\mathcal{K}$ such that for all $n \in \mathbb{N}$ we have $\rho(f_n) = \rho(g_n) = [a_n, a_{n+1}, ...]$ and:

$$d_0\big(\mathcal{R}^n(f), f_n\big) \leq C_1(f)\lambda_1^n \quad \text{and} \quad d_0\big(\mathcal{R}^n(g), g_n\big) \leq C_1(g)\lambda_1^n. \tag{2.0.1}$$

Let $n_0(f), n_0(g) \in \mathbb{N}$ given by Lemma 1.5.1, and consider $n_0 = \max\big\{n_0(f), n_0(g)\big\}$ and also $C_1 = \max\big\{C_1(f), C_1(g)\big\}$. Fix $\alpha \in (0,1)$ such that $\alpha > \frac{\log L}{\log L - \log \lambda_1}$, and for all $n > (1/\alpha)n_0$ let $m = \lfloor \alpha n \rfloor$. By the choice of $K > K_0$, and since $f_m, g_m \in \mathcal{K}$ for all $m \in \mathbb{N}$, we have that $\mathcal{R}^j(f_m) \in \mathcal{P}^3(K)$ for all $j \in \mathbb{N}$. By the real bounds:

$$\left|\frac{\big(T^{-p_{n+1}-p_n} \circ \widetilde{f}^{q_{n+1}+q_n}\big)(0)}{\big(T^{-p_{n+1}} \circ \widetilde{f}^{q_{n+1}}\big)(0)}\right| \in \left[\frac{1}{K}, K\right] \quad \text{for all } n \geq n_0\,,$$

and by (2.0.1) we have Item (3) and Item (4) of Lemma 1.5.1 for $\zeta = f_n$, by taking n_0 big enough. Applying Lemma 1.5.1 we obtain:

$$\begin{aligned} d_0\big(\mathcal{R}^n(f), \mathcal{R}^{n-m}(f_m)\big) &\leq L^{n-m} \cdot d_0\big(\mathcal{R}^m(f), f_m\big) \\ &\leq C_1 L^{n-m}\lambda_1^m \end{aligned}$$

and by the same reasons:

$$\begin{aligned} d_0\big(\mathcal{R}^n(g), \mathcal{R}^{n-m}(g_m)\big) &\leq L^{n-m} \cdot d_0\big(\mathcal{R}^m(g), g_m\big) \\ &\leq C_1 L^{n-m}\lambda_1^m\,. \end{aligned}$$

Let $\lambda_2 = L^{1-\alpha}\lambda_1^{\alpha}$, and note that $\lambda_2 \in (0,1)$ by the choice of α. Consider also $C_2 = 2C_1(1/\lambda_1)L > 0$. Since f_m and g_m are real-analytic and they have the same combinatorics, we know by Yampolsky's result (Theorem 0.3.3) that there exist constants $\lambda_3 \in (0,1)$ and $C_3 > 0$ (uniform in $\mathcal{K}$) such that:

$$d_0\big(\mathcal{R}^{n-m}(f_m), \mathcal{R}^{n-m}(g_m)\big) \leq C_3\lambda_3^{n-m} \quad \text{for all } n \in \mathbb{N}.$$

Finally consider $C = C_2 + C_3 > 0$ and $\lambda = \max\{\lambda_2, \lambda_3^{1-\alpha}\} \in (0,1)$. By the triangle inequality:

$$d_0\big(\mathcal{R}^n(f), \mathcal{R}^n(g)\big) \leq C\lambda^n \quad \text{for all } n \in \mathbb{N}.$$

$\square$

Even that Theorem D is true for any irrational rotation number, we have been able to prove that it implies Theorem C only for bounded type rotation number (see Chapter 6 for more comments).

CHAPTER 3

Approximation by holomorphic maps.

3.1 The Beltrami equation

Until now we were working on the real line, now we start to work on the complex plane. We assume that the reader is familiar with the notion of *quasiconformality* (the book of Ahlfors [1] and the one of Lehto and Virtanen [32] are classical references of the subject).

If we interpret the formulas $z = x + iy$ and $\overline{z} = x - iy$ as a change of variables in the complex plane, and we apply the chain rule (as if z and $\overline{z}$ were independent variables) we obtain the two basic differential operators of complex calculus:

$$\frac{\partial}{\partial z} = \left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

The second one, the $\frac{\partial}{\partial \overline{z}}$ derivative, is the most important for our purposes. By the Cauchy-Riemann equations it vanish precisely at the holomorphic maps, and in this case the $\frac{\partial}{\partial z}$ derivative is the usual one. The kernel of the $\frac{\partial}{\partial z}$ derivative is the set of *antiholomorphic* maps: $\frac{\partial F}{\partial z} \equiv 0$ if and only if $\overline{F}$ is holomorphic.

Instead of $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial \overline{z}}$ we will use the more compact notation ∂F and $\overline{\partial} F$ respectively. To be more explicit, let $\Omega \subset \mathbb{C}$ be a domain and let $F : \Omega \to F(\Omega)$ be a C^1 diffeomorphism. The isomorphism between $\mathbb{C}^2$ and the vector space of real linear transformations in the complex plane $\mathcal{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ that identify the pair (a, b) with the linear map $z \mapsto az + b\overline{z}$ can be used to define $\big(\partial F(w), \overline{\partial} F(w)\big)$ at any $w \in \Omega$ as the preimage of the derivative of F at w, that is, $\big(DF(w)\big)(z) = \partial F(w)z + \overline{\partial} F(w)\overline{z}$ for any $w \in \Omega$ and any $z \in \mathbb{C}$.

3.2 Quasiconformal homeomorphisms

We introduce first quasiconformal *diffeomorphisms* (Definition 3.2.2 below) and later quasiconformal *homeomorphisms* (Definition 3.2.3 below). Let $\Omega \subset \mathbb{C}$ be a domain and let $G : \Omega \to G(\Omega)$ and $F : G(\Omega) \to F\big(G(\Omega)\big)$ be two orientation-preserving C^1 diffeomorphisms (since both preserve orientation we have $\partial F \neq 0$ and $\partial G \neq 0$ everywhere). The complex derivatives satisfy the following chain rules:

$$\partial(F \circ G)(z) = \partial F\big(G(z)\big)\partial G(z) + \overline{\partial} F\big(G(z)\big)\partial\overline{G}(z). \tag{3.2.1}$$

$$\overline{\partial}(F \circ G)(z) = \partial F\big(G(z)\big)\overline{\partial} G(z) + \overline{\partial} F\big(G(z)\big)\overline{\partial G}(z). \tag{3.2.2}$$

If G is holomorphic, equations (3.2.1) and (3.2.2) become:

$$\partial(F \circ G)(z) = \partial F\big(G(z)\big)G'(z). \tag{3.2.3}$$

$$\overline{\partial}(F \circ G)(z) = \overline{\partial} F\big(G(z)\big)\overline{G'(z)}. \tag{3.2.4}$$

In particular:

$$\frac{\overline{\partial}(F \circ G)(z)}{\partial(F \circ G)(z)} = \left(\frac{\overline{\partial} F\big(G(z)\big)}{\partial F\big(G(z)\big)}\right)\left(\frac{\overline{G'(z)}}{G'(z)}\right) \quad \text{and} \quad \left|\frac{\overline{\partial}(F \circ G)(z)}{\partial(F \circ G)(z)}\right| = \left|\frac{\overline{\partial} F\big(G(z)\big)}{\partial F\big(G(z)\big)}\right|.$$

If F is holomorphic, equations (3.2.1) and (3.2.2) become:

$$\partial(F \circ G)(z) = F'\big(G(z)\big)\partial G(z). \tag{3.2.5}$$

$$\overline{\partial}(F \circ G)(z) = F'\big(G(z)\big)\overline{\partial} G(z). \tag{3.2.6}$$

In particular:

$$\frac{\overline{\partial}(F \circ G)(z)}{\partial(F \circ G)(z)} = \frac{\overline{\partial} G(z)}{\partial G(z)}\,.$$

This motivates the following definition:

Definition 3.2.1. Let $\Omega \subset \mathbb{C}$ be a domain and let $F : \Omega \to F(\Omega)$ be an orientation-preserving C^1 diffeomorphism. The *Beltrami coefficient* of F in Ω is the continuous function $\mu_F : \Omega \to \mathbb{C}$ defined by:

$$\mu_F(z) = \frac{\overline{\partial} F(z)}{\partial F(z)} \quad \text{for any } z \in \Omega.$$

Note that F is conformal in Ω if and only if $\mu_F \equiv 0$ in Ω. From:

$$\det\big(DF(z)\big) = |\partial F(z)|^2 - \big|\overline{\partial} F(z)\big|^2$$

and the fact that F preserves orientation we see at once that $\mu_F(\Omega) \subset \mathbb{D}$.

Definition 3.2.2. Let $\Omega \subset \mathbb{C}$ be a domain and let $F : \Omega \to F(\Omega)$ be an orientation-preserving C^1 diffeomorphism. If there exists $k \in [0,1)$ such that $\big|\mu_F(z)\big| \leq k < 1$ for every $z \in \Omega$ we say that F is K-*quasiconformal* in Ω, where $K \in [1, +\infty)$ is defined by $K = \frac{1+k}{1-k}$.

In particular F is conformal if and only if it is 1-quasiconformal (since we are still taking about diffeomorphisms, this is just the Cauchy-Riemann equations). The constant $K > 1$ measures how near a map is to being conformal: the closer K is to 1, the more *nearly conformal* the map is. The geometric meaning of this is the following: the differential of F at any point $z \in \Omega$ maps circles centred at the origin into similar ellipses. The ratio of the major to the minor axis (the *eccentricity* of the ellipse) is given by:

$$\frac{|\partial F(z)| + \big|\overline{\partial} F(z)\big|}{|\partial F(z)| - \big|\overline{\partial} F(z)\big|} = \frac{1 + \big|\mu_F(z)\big|}{1 - \big|\mu_F(z)\big|}.$$

Therefore a K-quasiconformal C^1 diffeomorphism is an orientation-preserving diffeomorphism whose derivative at any point maps circles centred at the origin into similar ellipses with eccentricity at most K.

Note that $K(k) = \frac{1+k}{1-k}$ is an orientation-preserving real-analytic diffeomorphism between $[0,1)$ and $[1, +\infty)$, with inverse given by $k(K) = \frac{K-1}{K+1}$.

Of course when restricted to a compactly contained open set in Ω, every C^1 diffeomorphism is K-quasiconformal for some $K \geq 1$.

As we saw above $\mu_{F \circ G} = \mu_G$ if F is holomorphic, and $\big|\mu_{F \circ G}\big| = \big|\mu_F \circ G\big|$ if G is holomorphic. In particular, if G is holomorphic and F is K-quasiconformal, both $F \circ G$ and $G \circ F$ are K-quasiconformal. More general, if F is K_1-quasiconformal and G is K_2-quasiconformal, the composition $F \circ G$ is $K_1 K_2$-quasiconformal. Note also that:

$$\big|\mu_{F^{-1}}\big| = \big|\mu_F \circ F^{-1}\big|.$$

In particular, F and F^{-1} are simultaneously K-quasiconformal diffeomorphisms.

Now we define quasiconformal *homeomorphisms*. Recall that a continuous real function $h : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous* if it has derivative at Lebesgue almost every point, its derivative is integrable and $h(b) - h(a) =$

$\int_a^b h'(t)dt$. A continuous function $F : \Omega \subset \mathbb{C} \to \mathbb{C}$ is *absolutely continuous on lines* in Ω if its real and imaginary parts are absolutely continuous on Lebesgue almost every horizontal line, and Lebesgue almost every vertical line.

Definition 3.2.3. Let $\Omega \subset \mathbb{C}$ be a domain and let $K \geq 1$. An orientation-preserving homeomorphism $F : \Omega \to F(\Omega)$ is K-*quasiconformal* (from now on K-q.c.) if F is absolutely continuous on lines and:

$$\left|\overline{\partial} F(z)\right| \leq \left(\frac{K-1}{K+1}\right)\left|\partial F(z)\right| \quad \text{a.e. in } \Omega.$$

This is the *analytic* definition of quasiconformal homeomorphisms, see Definition C.2.5 in Appendix C for the *geometric* definition.

Most of the properties quoted above for quasiconformal diffeomorphisms are still true for quasiconformal homeomorphisms. For instance a K-q.c. homeomorphism is conformal if and only if $K = 1$, F and F^{-1} are simultaneously K-q.c., the composition of a K_1-q.c. and a K_2-q.c. homeomorphism is a K_1K_2-q.c. homeomorphism (again we refer the reader to [1] and [32]).

Given a K-q.c. homeomorphism $F : \Omega \to F(\Omega)$ we define its *Beltrami coefficient* as the measurable function $\mu_F : \Omega \to \mathbb{D}$ given by:

$$\mu_F(z) = \frac{\overline{\partial} F(z)}{\partial F(z)} \quad \text{for a.e. } z \in \Omega.$$

Note that μ_F belongs to $L^\infty(\Omega)$ and satisfy $\|\mu_F\|_\infty \leq (K-1)/(K+1) < 1$. Conversely any measurable function from Ω to $\mathbb{C}$ with L^∞ norm less than one is the Beltrami coefficient of a quasiconformal homeomorphism:

Theorem 3.2.4 (Morrey 1938)**.** *Given any measurable function $\mu : \Omega \to \mathbb{D}$ such that $|\mu(z)| \leq (K-1)/(K+1) < 1$ almost everywhere in Ω for some $K \geq 1$, there exists a K-q.c. homeomorphism $F : \Omega \to F(\Omega)$ which is a solution of the* Beltrami equation*:*

$$\partial F(z)\, \mu(z) = \overline{\partial} F(z) \text{ a.e..}$$

The solution is unique up to post-composition with conformal diffeomorphisms. In particular, if Ω is the entire Riemann sphere, there is a unique solution (called the normalized *solution) that fix 0, 1 and ∞.*

See [1, Chapter V, Section B] or [32, Chapter V] for the proof. Note that Theorem 3.2.4 not only states the existence of a solution of the Beltrami equation, but also the fact that any solution is an homeomorphism.

Proposition 3.2.5. *If $\mu_n \to 0$ in the unit ball of L^∞, the normalized quasiconformal homeomorphisms f^{μ_n} converge to the identity uniformly on compact sets of $\mathbb{C}$. In general if $\mu_n \to \mu$ almost everywhere in $\mathbb{C}$ and $\|\mu_n\|_\infty \leq k < 1$ for all $n \in \mathbb{N}$, then the normalized quasiconformal homeomorphisms f^{μ_n} converge to f^μ uniformly on compact sets of $\mathbb{C}$.*

See [1, Chapter V, Section C].

3.3 Ahlfors-Bers theorem

The Beltrami equation induces therefore a one-to-one correspondence between the space of quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ that fix 0, 1 and ∞, and the space of Borel measurable complex-valued functions μ on $\widehat{\mathbb{C}}$ for which $\|\mu\|_\infty < 1$. The following classical result expresses the analytic dependence of the solution of the Beltrami equation with respect to μ:

Theorem 3.3.1 (Ahlfors-Bers 1960)**.** *Let Λ be an open subset of some complex Banach space and consider a map $\Lambda \times \mathbb{C} \to \mathbb{D}$, denoted by $(\lambda, z) \mapsto \mu_\lambda(z)$, satisfying the following properties:*

1. *For every λ the function $\mathbb{C} \to \mathbb{D}$ given by $z \mapsto \mu_\lambda(z)$ is measurable, and $\|\mu_\lambda\|_\infty \leq k$ for some fixed $k < 1$.*
2. *For Lebesgue almost every $z \in \mathbb{C}$, the function $\Lambda \to \mathbb{D}$ given by $\lambda \mapsto \mu_\lambda(z)$ is holomorphic.*

For each λ let F_λ be the unique quasiconformal homeomorphism of the Riemann sphere that fix 0, 1 and ∞, and whose Beltrami coefficient is μ_λ (F_λ is given by Theorem 3.2.4). Then $\lambda \mapsto F_\lambda(z)$ is holomorphic for all $z \in \mathbb{C}$.

See [2] or [1, Chapter V, Section C] for the proof. In Chapter 5 we will make repeated use of the following corollary of Ahlfors-Bers theorem:

Proposition 3.3.2. *For any bounded domain U in the complex plane there exists a number $C(U) > 0$, with $C(U) \leq C(W)$ if $U \subseteq W$, such that the following holds: let $\big\{G_n : U \to G_n(U)\big\}_{n\in\mathbb{N}}$ be a sequence of quasiconformal homeomorphisms such that:*

- *The domains $G_n(U)$ are* uniformly bounded*: there exists $R > 0$ such that $G_n(U) \subset B(0, R)$ for all $n \in \mathbb{N}$.*

- *$\mu_n \to 0$ in the unit ball of L^∞, where μ_n is the Beltrami coefficient of G_n in U.*

Then given any domain V such that $\overline{V} \subset U$ there exist $n_0 \in \mathbb{N}$ and a sequence $\big\{H_n : V \to H_n(V)\big\}_{n \geq n_0}$ of biholomorphisms such that:

$$\|H_n - G_n\|_{C^0(V)} \leq C(U)\left(\frac{R}{d\big(\partial V, \partial U\big)}\right)\|\mu_n\|_\infty \quad \textit{for all } n \geq n_0,$$

where $d\big(\partial V, \partial U\big)$ denote the Euclidean distance between the boundaries of U and V (which are disjoint compact sets in the complex plane, since V is compactly contained in the bounded domain U).

We postpone the proof of Proposition 3.3.2 until Appendix D. In the next chapter we will also use the following extension of the classical Koebe's one-quarter theorem [6, Theorem 1.3]:

Proposition 3.3.3. *Given $\varepsilon > 0$ there exists $K > 1$ for which the following holds: let $f : \mathbb{D} \to f(\mathbb{D}) \subset \mathbb{C}$ be a K-quasiconformal homeomorphism such that $f(0) = 0$, $f\big((-1,1)\big) \subset \mathbb{R}$ and $f(\mathbb{D}) \subset B(0, 1/\varepsilon)$. Suppose that $f|_{(-1/2,1/2)}$ is differentiable and that $\big|f'(t)\big| > \varepsilon$ for all $t \in (-1/2, 1/2)$, where f' denotes the real one-dimensional derivative of the restriction of f to the interval $(-1/2, 1/2)$. Then:*

$$B(0, \varepsilon/16) \subset f(\mathbb{D}).$$

Proof. Suppose, by contradiction, that there exist $\varepsilon > 0$ and a sequence $\big\{f_n : \mathbb{D} \to f_n(\mathbb{D}) \subset \mathbb{C}\big\}_{n\in\mathbb{N}}$ of quasiconformal homeomorphisms with the following properties:

1. Each f_n is K_n-q.c. with $K_n \to 1$ as n goes to infinity.
2. $f_n(0) = 0$ and $f_n\big((-1,1)\big) \subset \mathbb{R}$ for all $n \in \mathbb{N}$.
3. $f_n(\mathbb{D}) \subset B(0, 1/\varepsilon)$ for all $n \in \mathbb{N}$.
4. $f_n|_{(-1/2,1/2)}$ is differentiable and $\big|f_n'(t)\big| > \varepsilon$ for all $t \in (-1/2, 1/2)$ and for all $n \in \mathbb{N}$.
5. $B(0, \varepsilon/16)$ is not contained in $f_n(\mathbb{D})$ for any $n \in \mathbb{N}$.

By compactness, since $K_n \to 1$ and $f_n(0) = 0$ for all $n \in \mathbb{N}$, we can assume by taking a subsequence that there exists $f : \mathbb{D} \to \mathbb{C}$ holomorphic such that $f_n \to f$ uniformly on compact sets of $\mathbb{D}$ as n goes to infinity (see for instance [32, Chapter II, Section 5]). Of course $f(0) = 0$ and $f\big((-1,1)\big) \subset \mathbb{R}$. We claim that $\big|Df(0)\big| > \varepsilon/2$, where Df denotes the complex derivative of the holomorphic map f. Indeed, note that Item (3) implies that:

$$\left(\frac{-\varepsilon}{m}, \frac{\varepsilon}{m}\right) \subset f_n\left(\left[\frac{-1}{m}, \frac{1}{m}\right]\right) \quad \text{for all } n, m \in \mathbb{N},$$

and then by the uniform convergence we have:

$$\left(\frac{-\varepsilon}{m}, \frac{\varepsilon}{m}\right) \subset f\left(\left[\frac{-1}{m}, \frac{1}{m}\right]\right) \quad \text{for all } m \in \mathbb{N}.$$

Since f is holomorphic this implies the claim. From the claim we see that f is univalent in $\mathbb{D}$, since the uniform limit of quasiconformal homeomorphisms is either constant or a quasiconformal homeomorphism (again see [32, Chapter II, Section 5]). Finally, by Koebe's one-quarter theorem we have $B(0, \varepsilon/8) \subset f(\mathbb{D})$, but this contradicts that $B(0, \varepsilon/16)$ is not contained in $f_n(\mathbb{D})$ for any $n \in \mathbb{N}$. $\square$

CHAPTER 4

Complex extensions of $\mathcal{R}^n(f)$

For every C^3 critical circle map, with any irrational rotation number, we will construct in this chapter a suitable extension to an annulus around the unit circle in the complex plane, with the property that, after a finite number of renormalizations, this extension have good geometric bounds and exponentially small Beltrami coefficient. In the next chapter we will perturb this extension in order to get a holomorphic map with the same combinatorics and also good bounds.

Recall that given a bounded interval I in the real line we denote its Euclidean length by $|I|$, and for any $\alpha > 0$ we denote by $N_\alpha(I)$ the $\mathbb{R}$-symmetric topological disk:

$$N_\alpha(I) = \big\{ z \in \mathbb{C} : d(z, I) < \alpha|I| \big\},$$

where d denotes the Euclidean distance in the complex plane. The goal of this chapter is the following:

Theorem 4.0.4. *There exist three universal constants $\lambda \in (0,1)$, $\alpha > 0$ and $\beta > 0$ with the following property: let f be a C^3 critical circle map with any irrational rotation number. For all $n \geq 1$ denote by (η_n, ξ_n) the components of the critical commuting pair $\mathcal{R}^n(f)$. Then there exist two constants $n_0 \in \mathbb{N}$ and $C > 0$ such that for each $n \geq n_0$ both ξ_n and η_n extend (after normalized) to $\mathbb{R}$-symmetric orientation-preserving C^3 maps defined in $N_\alpha\big([-1,0]\big)$ and $N_\alpha\big([0,\xi_n(0)]\big)$ respectively, where we have the following seven properties:*

1. *Both ξ_n and η_n have a unique critical point at the origin, which is of cubic type.*

2. *The extensions η_n and ξ_n commute in $B(0,\lambda)$, that is, both compositions $\eta_n \circ \xi_n$ and $\xi_n \circ \eta_n$ are well defined in $B(0,\lambda)$, and they coincide.*

3. $$N_\beta\big(\xi_n([-1,0])\big) \subset \xi_n\big(N_\alpha\big([-1,0]\big)\big).$$

4. $$N_\beta\big([-1,(\eta_n \circ \xi_n)(0)]\big) \subset \eta_n\big(N_\alpha\big([0,\xi_n(0)]\big)\big).$$

5. $$\eta_n\big(N_\alpha\big([0,\xi_n(0)]\big)\big) \cup \xi_n\big(N_\alpha\big([-1,0]\big)\big) \subset B(0,\lambda^{-1}).$$

6. $$\max_{z \in N_\alpha([-1,0]) \setminus \{0\}} \left\{ \frac{\big|\bar{\partial}\xi_n(z)\big|}{\big|\partial\xi_n(z)\big|} \right\} \leq C\lambda^n.$$

7. $$\max_{z \in N_\alpha([0,\xi_n(0)]) \setminus \{0\}} \left\{ \frac{\big|\bar{\partial}\eta_n(z)\big|}{\big|\partial\eta_n(z)\big|} \right\} \leq C\lambda^n.$$

In this chapter we prove Theorem 4.0.4 (see Section 4.3), and in Chapter 5 we prove Theorem D.

4.1 Extended lifts of critical circle maps

In this section we lift a critical circle map to the real line, and then we extend this lift in a suitable way to a neighbourhood of the real line in the complex plane (see Definition 4.1.4 below).

Let f and g be two C^3 critical circle maps with cubic critical points c_f and c_g, and critical values v_f and v_g respectively. Recall that $\mathrm{Diff}^3_+(S^1)$ denotes the group (under composition) of orientation-preserving C^3 diffeomorphisms of the unit circle, endowed with the C^3 topology. Let $\mathcal{A}$ and $\mathcal{B}$ in $\mathrm{Diff}^3_+(S^1)$ defined by:

$$\mathcal{A} = \left\{\psi \in \mathrm{Diff}^3_+(S^1) : \psi(c_f) = c_g\right\} \quad \text{and} \quad \mathcal{B} = \left\{\phi \in \mathrm{Diff}^3_+(S^1) : \phi(v_g) = v_f\right\}.$$

There is a canonical homeomorphism between $\mathcal{A}$ and $\mathcal{B}$:

$$\psi \mapsto R_{\theta_1} \circ \psi \circ R_{\theta_2},$$

where R_{θ_1} is the rigid rotation that takes c_g to v_f, and R_{θ_2} is the rigid rotation that takes v_g to c_f. We will be interested, however, in another identification between $\mathcal{A}$ and $\mathcal{B}$:

Lemma 4.1.1. *There exists a homeomorphism* $T : \mathcal{A} \to \mathcal{B}$ *such that for any* $\psi \in \mathcal{A}$ *we have:*

$$f = T(\psi) \circ g \circ \psi\,.$$

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "\psi"'] & S^1 \\ S^1 \arrow[r, "g"] & S^1 \arrow[u, "T(\psi)"'] \end{tikzcd}$$

The lemma is true precisely because the maps f and g have the same order at their respective critical points:

Proof. Let ψ in $\mathrm{Diff}^3_+(S^1)$ such that $\psi(c_f) = c_g$, and consider the orientation-preserving circle homeomorphism:

$$T(\psi) = f \circ \psi^{-1} \circ g^{-1},$$

that maps the critical value of g to the critical value of f. To see that $T(\psi)$ is in $\mathrm{Diff}^3_+(S^1)$ note that when $z \neq v_g$ we have that $T(\psi)$ is smooth at z, with non-vanishing derivative equal to:

$$\big(DT(\psi)\big)(z) = D\psi^{-1}\big(g^{-1}(z)\big)\left(\frac{Df\big(\big(\psi^{-1} \circ g^{-1}\big)(z)\big)}{Dg\big(g^{-1}(z)\big)}\right).$$

In the limit we have:

$$\lim_{z \to v_g}\left[D\psi^{-1}\big(g^{-1}(z)\big)\left(\frac{Df\big(\big(\psi^{-1} \circ g^{-1}\big)(z)\big)}{Dg\big(g^{-1}(z)\big)}\right)\right] = D\psi^{-1}(c_g)\left(\frac{\big(D^3 f\big)\big(c_f\big)}{\big(D^3 g\big)\big(c_g\big)}\right),$$

a well-defined number in $(0, +\infty)$. This proves that $T(\psi)$ is in $\mathcal{B}$ for every $\psi \in \mathcal{A}$. Moreover T is invertible with inverse $T^{-1} : \mathcal{B} \to \mathcal{A}$ given by $T^{-1}(\phi) = g^{-1} \circ \phi^{-1} \circ f$. □

Let $A : S^1 \to S^1$ be the map corresponding to the parameters $a = 0$ and $b = 1$ in the Arnold family (0.1.2), defined in the introduction of this thesis, and recall that the lift of A to the real line, by the covering $\pi : \mathbb{R} \to S^1 : \pi(t) = \exp(2\pi i t)$, fixing the origin is given by:

$$\widetilde{A}(t) = t - \left(\frac{1}{2\pi}\right)\sin(2\pi t).$$

The critical point of A in the unit circle is at 1, and it is of cubic type (the critical point is also a fixed point for A). Now let f be a C^3 critical circle map with a unique cubic critical point at 1, and let $\widetilde{f}$ be the unique lift of f to the real line under the covering π satisfying $\widetilde{f}'(0) = 0$ and $0 < \widetilde{f}(0) < 1$. By Lemma 4.1.1 we can consider two C^3 orientation preserving circle diffeomorphisms h_1 and h_2, with $h_1(1) = 1$ and $h_2(1) = f(1)$, such that the composition $h_2 \circ A \circ h_1$ agrees with the map f, that is, the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "h_1"'] & S^1 \\ S^1 \arrow[r, "A"] & S^1 \arrow[u, "h_2"'] \end{tikzcd}$$

For each $i \in \{1, 2\}$ let $\widetilde{h}_i$ be the lift of h_i to the real line under the covering π determined by $\widetilde{h}_i(0) \in [0, 1)$. In Proposition 4.1.3 below we will extend both $\widetilde{h_1}$ and $\widetilde{h_2}$ to complex neighbourhoods of the real line in a suitable way. For that purposes we recall the definition of asymptotically holomorphic maps:

Definition 4.1.2. Let I be a compact interval in the real line, let U be a neighbourhood of I in $\mathbb{R}^2$ and let $H : U \to \mathbb{C}$ be a C^1 map (not necessarily a diffeomorphism). We say that H is *asymptotically holomorphic* of order $r \geq 1$ in I if for every $x \in I$:

$$\overline{\partial} H(x, 0) = 0 \quad \text{and} \quad \frac{\overline{\partial} H(x, y)}{\big(d((x, y), I)\big)^{r-1}} \to 0$$

uniformly as $(x, y) \in U \setminus I$ converge to I. We say that H is $\mathbb{R}$-*asymptotically holomorphic* of order r if it is asymptotically holomorphic of order r in compact sets of $\mathbb{R}$.

The sum or product of $\mathbb{R}$-asymptotically holomorphic maps is also $\mathbb{R}$-asymptotically holomorphic. The inverse of an asymptotically holomorphic diffeomorphism of order r is asymptotically holomorphic map of order r. Composition of asymptotically holomorphic maps is asymptotically holomorphic.

In the following proposition we suppose $r \geq 1$ even though we will apply it for $r \geq 3$. In the proof we follow the exposition of Graczyk, Sands and Świątek in [16, Lemma 2.1, page 623].

Proposition 4.1.3. *For $i = 1, 2$ there exists $H_i : \mathbb{C} \to \mathbb{C}$ of class C^r such that:*

1. *H_i is an extension of $\widetilde{h}_i$: $H_i|_{\mathbb{R}} = \widetilde{h}_i$;*

2. H_i commutes with unitary horizontal translation: $H_i \circ T = T \circ H_i$;

3. H_i is asymptotically holomorphic in $\mathbb{R}$ of order r;

4. H_i is $\mathbb{R}$-symmetric: $H_i(\bar{z}) = \overline{H_i(z)}$.

Moreover there exist $R > 0$ and four domains B_R, U_R, V_R and W_R in $\mathbb{C}$, symmetric about the real line, and such that:

- $B_R = \big\{z \in \mathbb{C} : -R < \Im(z) < R\big\}$;
- *H_1 is an orientation preserving diffeomorphism between B_R and U_R;*
- $\widetilde{A}(U_R) = V_R$;
- *H_2 is an orientation preserving diffeomorphism between V_R and W_R.*
- *Both $\inf_{z \in B_R} \big|\partial H_1(z)\big|$ and $\inf_{z \in V_R} \big|\partial H_2(z)\big|$ are positive numbers.*

Proof. For $z = x + iy \in \mathbb{C}$, with $y \neq 0$, let $P_{x,y}$ be the degree r polynomial map that coincide with $\widetilde{h_i}$ in the $r + 1$ real numbers:

$$\left\{x + \left(\frac{j}{r}\right) y\right\}_{j \in \{0,1,\dots,r\}}$$

Recall that $P_{x,y}$ can be given by the following linear combination (the so-called *Lagrange's form of the interpolation polynomial*):

$$\begin{aligned} P_{x,y}(z) &= \sum_{j=0}^{j=r} \widetilde{h}_i\big(x + (j/r)y\big) \prod_{\substack{l=0 \\ l \neq j}}^{l=r} \frac{z - \big(x + (l/r)y\big)}{\big(x + (j/r)y\big) - \big(x + (l/r)y\big)} \\ &= \sum_{j=0}^{j=r} \widetilde{h}_i\big(x + (j/r)y\big) \prod_{\substack{l=0 \\ l \neq j}}^{l=r} \frac{z - x - (l/r)y}{\big((j-l)/r\big)y} \end{aligned}$$

We define $H_i(x + iy) = P_{x,y}(x + iy)$, that is:

$$H_i(x + iy) = P_{x,y}(x + iy) = \sum_{j=0}^{j=r} \widetilde{h}_i\big(x + (j/r)y\big) \prod_{\substack{l=0 \\ l \neq j}}^{l=r} \frac{ir - l}{j - l}$$

After computation we obtain:

$$H_i(x + iy) = P_{x,y}(x + iy) = \frac{1}{N} \sum_{j=0}^{j=r} \left(\frac{(-1)^j \dbinom{r}{j}}{1 + i(j/r)} \right) \widetilde{h}_i\big(x + (j/r)y\big)$$

where:

$$N=\sum_{j=0}^{j=r}\left(\frac{(-1)^j\binom{r}{j}}{1+i(j/r)}\right)\neq 0$$

Note that H_i is as smooth as $\widetilde{h}_i$, and $H_i(x)=\widetilde{h}_i(x)$ for any real number x (item (1)). Since $\widetilde{h}_i$ is a lift we have for any $j\in\{0,1,...,r\}$ that $\widetilde{h}_i\big(x+1+(j/r)y\big)=\widetilde{h}_i\big(x+(j/r)y\big)+1$, but then $P_{x+1,y}\big(x+1+(j/r)y\big)=P_{x,y}\big(x+(j/r)y\big)+1$ for any $j\in\{0,1,...,r\}$ and this implies $P_{x+1,y}\circ T=T\circ P_{x,y}$ in the whole complex plane. This proves item (2).

To prove that H_i is asymptotically holomorphic of order r in $\mathbb{R}$ note that:

$$\overline{\partial}H_i(x+iy)=\frac{1}{2N}\sum_{j=0}^{j=r}(-1)^j\binom{r}{j}\widetilde{h}_i'\big(x+(j/r)y\big)$$

and for any $k\in\{0,...,r\}$:

$$\frac{\partial^k}{\partial y^k}\overline{\partial}H_i(x+iy)=\left(\frac{1}{2N}\right)\left(\frac{1}{r^k}\right)\sum_{j=0}^{j=r}(-1)^j j^k\binom{r}{j}\widetilde{h}_i^{(k+1)}\big(x+(j/r)y\big)$$

Now we claim that for any $k\in\{0,...,r-1\}$ we have $\sum_{j=0}^{j=r}(-1)^j j^k\binom{r}{j}=0$. Indeed, for any $j\in\{0,...,r\}$ we have $\frac{\partial^j}{\partial t^j}(1-t)^r=(-1)^j\left(\frac{r!}{(r-j)!}\right)(1-t)^{r-j}$, and this gives us the equality $(1-t)^r=\sum_{j=0}^{j=r}(-1)^j\binom{r}{j}t^j$ for $r\geq 1$. Putting $t=1$ we obtain the claim for $k=0$. Since $t\frac{\partial}{\partial t}(1-t)^r=\sum_{j=0}^{j=r}(-1)^j j t^j\binom{r}{j}$, we obtain the claim for $k=1$ if we put $t=1$. Putting $t=1$ in $t\frac{\partial}{\partial t}\big[t\frac{\partial}{\partial t}(1-t)^r\big]$ we obtain the claim for $k=2$, and so forth until $k=r-1$.

With the claim we obtain for any $x\in\mathbb{R}$ that:

$$\overline{\partial}H_i(x)=\left(\frac{1}{2N}\right)\widetilde{h}_i'(x)\sum_{j=0}^{j=r}(-1)^j\binom{r}{j}=0$$

and for any $k\in\{0,...,r-1\}$:

$$\frac{\partial^k}{\partial y^k}\overline{\partial}H_i(x)=\left(\frac{1}{2N}\right)\left(\frac{\widetilde{h}_i^{(k+1)}(x)}{r^k}\right)\sum_{j=0}^{j=r}(-1)^j j^k\binom{r}{j}=0$$

By Taylor theorem:

$$\lim_{y\to 0}\frac{\overline{\partial}H_i(x+iy)}{y^{r-1}}=0$$

uniformly on compact sets of the real line, and from this follows that H_i is asymptotically holomorphic of order r in $\mathbb{R}$ (item (3)). To obtain the symmetry as in item (4) we can take $z\mapsto\frac{H_i(z)+\overline{H_i(\bar{z})}}{2}$, since this preserves all the other properties.

Finally note that the Jacobian of H_i at a point x in $\mathbb{R}$ is equal to $|\widetilde{h}_i'(x)|^2\neq 0$. This gives us a complex neighbourhood of the real line where H_i is an orientation preserving diffeomorphism, and the positive constant R. Since we also have $\left|\partial H_i\right|=\left|\widetilde{h}_i'\right|$ at the real line, and each $\widetilde{h}_i$ is the lift of a circle diffeomorphism, we obtain the last item of Proposition 4.1.3. $\square$

These are the extensions that we will consider:

Definition 4.1.4. The map $F:B_R\to W_R$ defined by $F=H_2\circ\widetilde{A}\circ H_1$ is called the *extended lift* of the critical circle map f.

$$\begin{tikzcd} B_R \arrow[r, "F"] \arrow[d, "H_1"'] & W_R \\ U_R \arrow[r, "\widetilde{A}"] & V_R \arrow[u, "H_2"'] \end{tikzcd}$$

We have the following properties:

- F is C^r in the horizontal band B_R;
- $T\circ F=F\circ T$ in B_R;
- F is $\mathbb{R}$-symmetric (in particular F preserves the real line), and F restricted to the real line is $\widetilde{f}$;
- F is asymptotically holomorphic in $\mathbb{R}$ of order r;
- The critical points of F in B_R are the integers (the same as $\widetilde{A}$), and they are of cubic type.

We remark that the extended lift of a real-analytic critical circle map will be C^∞ in the corresponding horizontal strip, but not necessarily holomorphic.

The pre-image of the real axis under F consists of $\mathbb{R}$ itself together with two families of C^r curves $\{\gamma_1(k)\}_{k\in\mathbb{Z}}$ and $\{\gamma_2(k)\}_{k\in\mathbb{Z}}$ arising as solutions of $\Im(F(x+iy))=0$. Note that $\gamma_1(k)$ and $\gamma_2(k)$ meet at the critical point $c_k=k$.

Let $\gamma_i^+(k) = \gamma_i(k) \cap \mathbb{H}$ and $\gamma_i^-(k) = \gamma_i(k) \cap \mathbb{H}^-$ for $i = 1, 2$. We also denote $\gamma_i^+(0)$ just by γ_i^+.

Lemma 4.1.5. *We can choose R small enough to have that γ_1^+ is contained in $T = \big\{ \arg(z) \in \big(\frac{\pi}{6}, \frac{\pi}{2}\big)\big\} \cap B_R$ (that is, the open triangle with vertices 0, iR and $(\sqrt{3}+i)R$), γ_2^+ is contained in $-\overline{T}$, γ_1^- is contained in $-T$ and γ_2^- is contained in $\overline{T}$.*

Proof. The derivative of H_1 at real points is conformal, so the angle between γ_1 and γ_2 with the real line at zero is $\frac{\pi}{3}$. $\square$

4.2 Poincaré disks

Besides the notion of asymptotically holomorphic maps, the main tool in order to prove Theorem 4.0.4 is the notion of Poincaré disk, introduced into the subject by Sullivan in his seminal article [55].

Given an open interval $I = (a, b) \subset \mathbb{R}$ let $\mathbb{C}_I = \big(\mathbb{C} \setminus \mathbb{R}\big) \cup I = \mathbb{C} \setminus \big(\mathbb{R} \setminus I\big)$. For $\theta \in (0, \pi)$ let D be the open disk in the plane intersecting the real line along I and for which the angle from $\mathbb{R}$ to ∂D at the point b (measured anticlockwise) is θ. Let $D^+ = D \cap \{z : \Im(z) > 0\}$ and let D^- be the image of D^+ under complex conjugation.

Define the *Poincaré disk* of angle θ based on I as $D_\theta(a, b) = D^+ \cup I \cup D^-$, that is, $D_\theta(a, b)$ is the set of points in the complex plane that *view* I under an angle $\geq \theta$ (see Figure 4.1). Note that for $\theta = \frac{\pi}{2}$ the Poincaré disk $D_\theta(a, b)$ is the Euclidean disk whose diameter is the interval (a, b).

We denote by $\operatorname{diam}\big(D_\theta(a, b)\big)$ the Euclidean diameter of $D_\theta(a, b)$. For $\theta \in \big[\frac{\pi}{2}, \pi\big)$ the diameter of $D_\theta(a, b)$ is always $|b - a|$. When $\theta \in \big(0, \frac{\pi}{2}\big)$ we have that:

$$\frac{\operatorname{diam}\big(D_\theta(a, b)\big)}{|b - a|}$$

is an orientation-reversing diffeomorphism between $\big(0, \frac{\pi}{2}\big)$ and $\big(1, +\infty\big)$, which is real-analytic. Indeed, when $\theta \in \big(0, \frac{\pi}{2}\big)$ the center of D^+ is $\big(\frac{a+b}{2}\big) + i\big(\frac{b-a}{2\tan\theta}\big)$, and its radius is $\frac{b-a}{2\sin\theta}$, thus we obtain:

$$\begin{aligned}
\operatorname{diam}\big(D_\theta(a, b)\big) &= 2\left(\frac{b-a}{2\tan\theta}\right) + 2\left(\frac{b-a}{2\sin\theta}\right) \\
&= \left(\frac{1}{\tan\theta} + \frac{1}{\sin\theta}\right)(b - a) \\
&= \left(\frac{1+\cos\theta}{\sin\theta}\right)(b - a).
\end{aligned}$$

Therefore we have:

$$\frac{\operatorname{diam}\big(D_\theta(a,b)\big)}{|b-a|} = \frac{1+\cos\theta}{\sin\theta} \quad \text{for any} \quad \theta \in \left(0, \frac{\pi}{2}\right).$$

In particular when θ goes to zero the ratio $\operatorname{diam}\big(D_\theta(a,b)\big)/|b-a|$ goes to infinity like $2/\theta$.

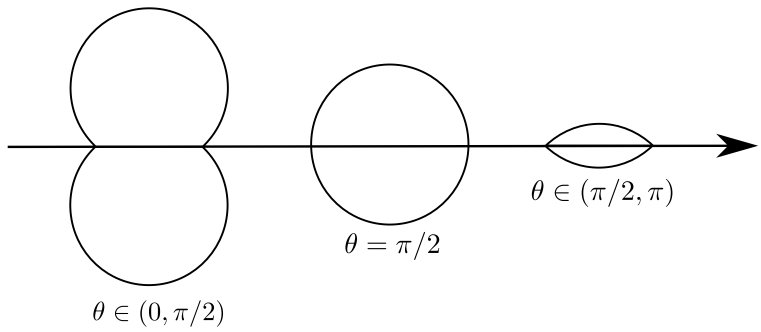

Figure 4.1: Poincaré disks.

Poincaré disks have a geometrical meaning: $\mathbb{C}_I$ is an open, connected and simply connected set which is not the whole plane. By the Riemann mapping theorem we can endow $\mathbb{C}_I$ with a complete and conformal Riemannian metric of constant curvature equal to -1, just by pulling back the Poincaré metric of $\mathbb{D}$ by any conformal uniformization. Note that I is always a hyperbolic geodesic by symmetry.

For a given $\theta \in (0,\pi)$ consider $\varepsilon(\theta) = \log\tan\left(\frac{\pi}{2} - \frac{\theta}{4}\right)$, which is an orientation-reversing real-analytic diffeomorphism between $(0,\pi)$ and $(0,+\infty)$. An elementary computation (see Lemma C.1.1) shows that the set of points in $\mathbb{C}_I$ whose hyperbolic distance to I is less than ε is precisely $D_\theta(a,b)$.

In particular we can state Schwarz lemma in the following way: let I and J be two intervals in the real line and let $\phi : \mathbb{C}_I \to \mathbb{C}_J$ be a holomorphic map such that $\phi(I) \subset J$. Then for any $\theta \in (0,\pi)$ we have that $\phi\big(D_\theta(I)\big) \subset D_\theta(J)$.

With this at hand (and a very clever inductive argument, see also [33]), Yampolsky was able to obtain *complex bounds* for critical circle maps in the *Epstein* class [58, Theorem 1.1]. The reason why we chose asymptotically holomorphic maps to extend our (finitely smooth) one-dimensional dynamics (see Proposition 4.1.3 and Definition 4.1.4 above) is the following asymptotic Schwarz lemma, obtained by Graczyk, Sands and Świątek in [16, Proposition 2, page 629] for asymptotically holomorphic maps:

Proposition 4.2.1 (Almost Schwarz inclusion)**.** *Let $h : I \to \mathbb{R}$ be a C^3 diffeomorphism from a compact interval I with non-empty interior into the real line. Let H be any C^3 extension of h to a complex neighbourhood of I, which is asymptotically holomorphic of order 3 on I. Then there exist $M > 0$ and $\delta > 0$ such that if $a, c \in I$ are different, $\theta \in (0, \pi)$ and $\operatorname{diam}\big(D_\theta(a, c)\big) < \delta$ then:*

$$H\big(D_\theta(a, c)\big) \subseteq D_{\tilde{\theta}}\big(h(a), h(c)\big)$$

where $\tilde{\theta} = \theta - M|c - a| \operatorname{diam}\big(D_\theta(a, c)\big)$. Moreover, $\tilde{\theta} > 0$.

Let us point out that a predecessor of this almost Schwarz inclusion, for real-analytic maps, already appeared in the work of de Faria and de Melo [13, Lemma 3.3, page 350].

4.3 Proof of Theorem 4.0.4

With Proposition 4.2.1 at hand, we are ready to start the proof of Theorem 4.0.4. We will work with $\widetilde{f}^{q_{n+1}}|_{I_n}$, the proof for $\widetilde{f}^{q_n}|_{I_{n+1}}$ being the same.

Proposition 4.3.1. *Let f be a C^3 critical circle map with irrational rotation number, and let F be its extended lift (according to Definition 4.1.4). There exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ there exist two numbers $K_n \geq 1$ and $\theta_n > 0$ satisfying $K_n \to 1$ and $\theta_n \to 0$ as $n \to +\infty$, and:*

$$\lim_{n \to +\infty} \left| \frac{\operatorname{diam}\big(D_{\theta_n/K_n}\big(\widetilde{f}(I_n)\big)\big)}{\big|\widetilde{f}(I_n)\big|} - \frac{\operatorname{diam}\big(D_{\theta_n}\big(\widetilde{f}^{q_{n+1}}(I_n)\big)\big)}{\big|\widetilde{f}^{q_{n+1}}(I_n)\big|} \right| = 0$$

with the following property: let $\theta \geq \theta_n$, $1 \leq j \leq q_{n+1}$ and let J be an open interval such that:

$$I_n \subseteq \overline{J} \subseteq \big(\widetilde{f}^{q_{n-1}-q_{n+1}}(0), \widetilde{f}^{q_n-q_{n+1}}(0)\big).$$

Then the inverse branch F^{-j+1} mapping $\widetilde{f}^j(J)$ back to $\widetilde{f}(J)$ is well defined over $D_\theta\big(\widetilde{f}^j(J)\big)$, and maps this neighbourhood diffeomorphically onto an open set contained in $D_{\theta/K_n}\big(\widetilde{f}(J)\big)$.

To simplify notation we will prove Proposition 4.3.1 for the case $J = I_n$ and $j = q_{n+1}$.

Proof. For each $n \in \mathbb{N}$ and $j \in \{1, ..., q_{n+1} - 1\}$ we know by combinatorics that $\widetilde{f}$ is a C^3 diffeomorphism from $\widetilde{f}^j(I_n)$ to $\widetilde{f}^{j+1}(I_n)$. Let $M_{j,n} > 0$ and $\delta_{j,n} > 0$ given by Proposition 4.2.1 applied to the corresponding inverse branch of the extended lift F. Moreover, let $M_n = \max_{j \in \{1,\dots,q_{n+1}-1\}}\{M_{j,n}\}$ and $\delta_n = \min_{j \in \{1,\dots,q_{n+1}-1\}}\{\delta_{j,n}\}$. For each $n \in \mathbb{N}$ let A_n and B_n be the affine maps given by:

$$A_n(t) = \big(1/\big|\widetilde{f}^{q_{n+1}}(I_n)\big|\big)\big(t - \widetilde{f}^{q_{n+1}}(0)\big) \text{ and } B_n(t) = \big(1/\big|\widetilde{f}(I_n)\big|\big)\big(t - \widetilde{f}(0)\big).$$

By the real bounds, the C^3 diffeomorphism $T_n : [0,1] \to [0,1]$ given by:

$$T_n = B_n \circ \widetilde{f}^{-q_{n+1}+1} \circ A_n^{-1}$$

has universally bounded distortion, and therefore:

$$\inf_{\substack{t \in [0,1] \\ n \geq n_0}} \big\{\big|T_n'(t)\big|\big\} > 0.$$

In particular $M = \sup_{n \geq n_0}\{M_n\}$ is finite, and $\delta = \inf_{n \geq n_0}\{\delta_n\}$ is positive. Let $d_n = \max_{1 \leq j \leq q_{n+1}} \big|\widetilde{f}^j(I_n)\big|$, and recall that by the real bounds the sequence $\{d_n\}_{n \geq 1}$ goes to zero exponentially fast when n goes to infinity. In particular we can choose a sequence $\big\{\alpha_n\big\}_{n \geq 1} \subset \big(0, \frac{\pi}{2}\big)$ also convergent to zero but such that:

$$\lim_{n \to +\infty} \left(\frac{d_n}{(\alpha_n)^3}\right) = 0\,.$$

Let $\psi : (0, \pi) \to [1, +\infty)$ defined by:

$$\psi(\theta) = \max\left\{1, \frac{1+\cos\theta}{\sin\theta}\right\} = \begin{cases} \frac{1+\cos\theta}{\sin\theta} & \text{for } \theta \in \big(0, \frac{\pi}{2}\big) \\ 1 & \text{for } \theta \in \big[\frac{\pi}{2}, \pi\big) \end{cases}$$

Note that ψ is an orientation-reversing real-analytic diffeomorphism between $\big(0, \frac{\pi}{2}\big)$ and $\big(1, +\infty\big)$. As we said before, for any $\theta \in (0, \pi)$ and any real numbers $a < b$, we have that $\operatorname{diam}\big(D_\theta(a,b)\big) = \psi(\theta)|b-a|$. Now define:

$$\theta_n = \alpha_n + \psi(\alpha_n)(\delta M) \sum_{j=0}^{q_{n+1}-1} \big|\widetilde{f}^{j+1}(I_n)\big|^2 > \alpha_n > 0$$

and:

$$K_n = \frac{\theta_n}{\alpha_n} = 1 + \left(\frac{\psi(\alpha_n)}{\alpha_n}\right)(\delta M) \sum_{j=0}^{q_{n+1}-1} \big|\widetilde{f}^{j+1}(I_n)\big|^2 > 1\,.$$

By the choice of α_n we have:

$$\lim_{n \to +\infty} \left(\frac{\psi(\alpha_n)}{\alpha_n}\right) d_n = 0\,,$$

and since:

$$\sum_{j=0}^{q_{n+1}-1} \left|\widetilde{f}^{j+1}(I_n)\right|^2 \leq d_n$$

we have that $\theta_n \to 0$ and $K_n \to 1$ when n goes to infinity. We also have:

$$\begin{aligned}
\left|\psi\big(\theta_n/K_n\big) - \psi\big(\theta_n\big)\right| &\leq \left(\max_{\theta\in\left[\theta_n/K_n,\theta_n\right]} \left|\psi'(\theta)\right|\right)\left|\theta_n - \theta_n/K_n\right| \\
&= \left|\psi'(\theta_n/K_n)\right|\left|\theta_n - \theta_n/K_n\right| \\
&= \left(\frac{\psi(\theta_n/K_n)}{\sin(\theta_n/K_n)}\right)\left|\theta_n - \theta_n/K_n\right| \\
&= \left(\frac{\psi(\alpha_n)}{\sin(\alpha_n)}\right)\left|\theta_n - \alpha_n\right| \\
&= (\delta M)\left(\frac{\big(\psi(\alpha_n)\big)^2}{\sin(\alpha_n)}\right)\sum_{j=0}^{q_{n+1}-1}\left|\widetilde{f}^{j+1}(I_n)\right|^2 \\
&\leq (\delta M)\left(\frac{\big(\psi(\alpha_n)\big)^2}{\sin(\alpha_n)}\right) d_n\,,
\end{aligned}$$

and this goes to zero by the choice of α_n. In particular:

$$\lim_{n\to+\infty}\left|\frac{\operatorname{diam}\left(D_{\theta_n/K_n}\big(\widetilde{f}(I_n)\big)\right)}{\left|\widetilde{f}(I_n)\right|} - \frac{\operatorname{diam}\left(D_{\theta_n}\big(\widetilde{f}^{q_{n+1}}(I_n)\big)\right)}{\left|\widetilde{f}^{q_{n+1}}(I_n)\right|}\right| = 0$$

as stated. We choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\psi(\alpha_n)d_n < \delta$.

Define inductively $\{\theta_j\}_{j=1}^{j=q_{n+1}}$ by $\theta_{q_{n+1}} = \theta_n$ and for $1 \leq j \leq q_{n+1}-1$ by:

$$\theta_j = \theta_{j+1} - M\left|\widetilde{f}^{j+1}(I_n)\right|\operatorname{diam}\left(D_{\theta_{j+1}}\big(\widetilde{f}^{j+1}(I_n)\big)\right) = \theta_{j+1} - M\psi(\theta_{j+1})\left|\widetilde{f}^{j+1}(I_n)\right|^2$$

We want to show that $\theta_j > \alpha_n = \frac{\theta_n}{K_n}$ for all $1 \leq j \leq q_{n+1}$. For this we claim that for any $1 \leq j \leq q_{n+1}$ we have that:

$$\theta_j \geq \alpha_n + \psi(\alpha_n)(\delta M)\sum_{k=0}^{j-1}\left|\widetilde{f}^{k+1}(I_n)\right|^2 > \alpha_n\,.$$

The claim follows by (reverse) induction in j (the case $j = q_{n+1}$ holds by definition). If the claim is true for $j+1$ we have $\psi(\theta_{j+1}) < \psi(\alpha_m)$, this implies $\theta_j > \theta_{j+1} - \psi(\alpha_n)(\delta M)\left|\widetilde{f}^{j+1}(I_n)\right|^2$ and with this the claim is true for j. It follows that:

$$\operatorname{diam}\left(D_{\theta_j}\big(\widetilde{f}^j(I_n)\big)\right) = \psi(\theta_j)\left|\widetilde{f}^j(I_n)\right| < \psi(\alpha_n)d_n < \delta \leq \delta_j \quad \text{for all} \quad 1 \leq j \leq q_{n+1}\,.$$

By Proposition 3.3.3 the inverse branch F^{-1} mapping $\widetilde{f}^{j+1}(I_n)$ back to $\widetilde{f}^{j}(I_n)$ is a well-defined diffeomorphism from the Poincaré disk $D_{\theta_{j+1}}\big(\widetilde{f}^{j+1}(I_n)\big)$ onto its image, and by Proposition 4.2.1 we know that:

$$F^{-1}\big(D_{\theta_{j+1}}\big(\widetilde{f}^{j+1}(I_n)\big)\big) \subseteq \big(D_{\theta_j}\big(\widetilde{f}^{j}(I_n)\big)\big).$$

The claim also gives us:

$$\theta_1 \geq \alpha_n + \psi(\alpha_n)(\delta M)\big|\widetilde{f}(I_n)\big|^2 > \alpha_n = \frac{\theta_n}{K_n}\,,$$

and this finish the proof. □

Corollary 4.3.2. *There exist constants $\alpha > 0$, $C_1, C_2 > 0$ and $\lambda \in (0,1)$ with the following property: let f be a C^3 critical circle map with irrational rotation number, and let F be its extended lift. There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists an $\mathbb{R}$-symmetric topological disk Y_n with:*

$$N_\alpha\big(\widetilde{f}(I_n)\big) \subset Y_n\,,$$

such that the composition $F^{q_{n+1}-1} : Y_n \to F^{q_{n+1}-1}(Y_n)$ is a well defined C^3 diffeomorphism and we have:

1. $$C_1 < \frac{\operatorname{diam}\big(F^{q_{n+1}-1}(Y_n)\big)}{\big|\widetilde{f}^{q_{n+1}}(I_n)\big|} < C_2\,, \quad \textit{and}$$

2. $$\sup_{z \in Y_n}\left\{\frac{\big|\overline{\partial} F^{q_{n+1}-1}(z)\big|}{\big|\partial F^{q_{n+1}-1}(z)\big|}\right\} \leq C_2 \lambda^n\,.$$

Proof. For each $n \in \mathbb{N}$ let:

- I_n be the closed interval whose endpoints are 0 and $\big(T^{-p_n} \circ \widetilde{f}^{q_n}\big)(0)$,
- J_n be the open interval containing the origin that projects to:

  $$\big(f^{q_{n+1}}(1), f^{q_n - q_{n+3}}(1)\big)$$

  under the covering $\pi(t) = e^{2\pi i t}$, and
- K_n be the open interval containing the origin that projects to

  $$\big(f^{q_{n-1}-q_{n+1}}(1), f^{q_n - q_{n+1}}(1)\big)$$

  under the covering π.

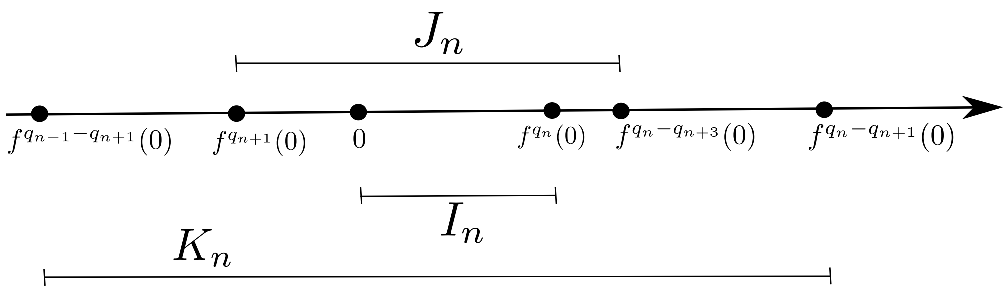

Figure 4.2: Relative positions of the relevant points in the proof of Corollary 4.3.2.

Note that $I_n \cup I_{n+1} \subset J_n \subset \overline{J_n} \subset K_n$ (see Figure 4.2). By combinatorics, the map $\widetilde{f} : \widetilde{f}^j(K_n) \to \widetilde{f}^{j+1}(K_n)$ is a diffeomorphism for all $j \in \{1, ..., q_{n+1} - 1\}$, and therefore all restrictions $\widetilde{f} : \widetilde{f}^j(J_n) \to \widetilde{f}^{j+1}(J_n)$ are diffeomorphisms for any $j \in \{1, ..., q_{n+1} - 1\}$ (just as in the proof of Proposition 4.3.1).

Recall that the extended lift $F : B_R \to W_R$ is given by the composition $F = H_2 \circ \widetilde{A} \circ H_1$ (see Definition 4.1.4). Let $n_0 \in \mathbb{N}$ given by Proposition 4.3.1, and for each $n \geq n_0$ let $K_n \geq 1$ and $\theta_n > 0$ also given by Proposition 4.3.1. Fix $\theta \in (0, \pi)$ such that $\theta > \theta_n$ for all $n \geq n_0$ and such that:

$$\left|\mu_{H_i}(z)\right| < \left(\frac{1}{2}\right)\left(d\big(z, \widetilde{f}^j(J_n)\big)\right)^2$$

for any $z \in D_{\theta/K_n}\big(\widetilde{f}^j(J_n)\big)$, any $j \in \{1, ..., q_{n+1} - 1\}$ and any $i \in \{1, 2\}$ (as before μ_{H_i} denotes the Beltrami coefficient of the quasiconformal homeomorphism H_i, and d denotes the Euclidean distance in the complex plane). The existence of such θ is guaranteed by Proposition 4.3.1, the fact that both H_i are asymptotically holomorphic in $\mathbb{R}$ of order 3, and the last item in Proposition 4.1.3.

Let $Y_n \subset F^{-q_{n+1}+1}\Big(D_\theta\big(\widetilde{f}^{q_{n+1}}(J_n)\big)\Big)$ be the preimage of $D_\theta\big(\widetilde{f}^{q_{n+1}}(J_n)\big)$ under $F^{q_{n+1}-1}$ given by Proposition 4.3.1, and note that:

- Y_n is an $\mathbb{R}$-symmetric topological disk,
- $\overline{\widetilde{f}(I_n)} \subset Y_n$,
- $\widetilde{f}(I_{n+1}) \subset Y_n$.
- By Proposition 4.3.1, $F^j(Y_n) \subset D_{\theta/K_n}\big(\widetilde{f}^{j+1}(J_n)\big)$ for all $j \in \{0, 1, ..., q_{n+1} - 1\}$.

Moreover:

$$\operatorname{diam}\big(F^{q_{n+1}-1}(Y_n)\big)=\operatorname{diam}\big(D_{\theta}\big(\widetilde{f}^{q_{n+1}}(J_n)\big)\big)=\psi(\theta)\big|\widetilde{f}^{q_{n+1}}(J_n)\big|\,,$$

and by the real bounds $\big|\widetilde{f}^{q_{n+1}}(J_n)\big|$ and $\big|\widetilde{f}^{q_{n+1}}(I_n)\big|$ are comparable (with universal constants independent of $n \geq n_0$). Again the map ψ is the same as in the proof of Proposition 4.3.1. This gives us Item (1), and now we prove Item (2). For each $n \geq n_0$ let $k_n \in [0,1)$ be the conformal distortion of $F^{q_{n+1}-1}$ at Y_n, that is:

$$k_n=\sup_{z\in Y_n}\left\{\frac{\big|\overline{\partial}F^{q_{n+1}-1}(z)\big|}{\big|\partial F^{q_{n+1}-1}(z)\big|}\right\}.$$

Moreover, for each $j \in \{1,...,q_{n+1}-1\}$ let $K_{n,j}$, $K_{n,j}(1)$ and $K_{n,j}(2)$ in $[1,+\infty)$ be the quasiconformality of F at $F^{j-1}(Y_n)$, of H_1 also at $F^{j-1}(Y_n)$, and of H_2 at $(\widetilde{A}\circ H_1)\big(F^{j-1}(Y_n)\big)$ respectively. Since $\widetilde{A}$ is conformal we have that:

$$\begin{aligned}
k_n &\leq \log\left(\prod_{j=1}^{q_{n+1}-1}K_{n,j}\right)=\sum_{j=1}^{q_{n+1}-1}\log K_{n,j}\\
&=\sum_{j=1}^{q_{n+1}-1}\big(\log K_{n,j}(1)+\log K_{n,j}(2)\big)\\
&\leq \sum_{j=1}^{q_{n+1}-1}M_0\,\big(\operatorname{diam}\big(F^{j-1}(Y_n)\big)\big)^2 \qquad \text{(for some } M_0>1)\\
&\leq \sum_{j=1}^{q_{n+1}-1}M_0\,\big(\operatorname{diam}\big(D_{\theta/K_n}(\widetilde{f}^j(J_n))\big)\big)^2\\
&=\sum_{j=1}^{q_{n+1}-1}M_0\,\big(\psi(\theta/K_n)\big)^2\big|\widetilde{f}^j(J_n)\big|^2<M_1\left(\sum_{j=1}^{q_{n+1}-1}\big|\widetilde{f}^j(J_n)\big|^2\right).
\end{aligned}$$

The last inequality follows from the fact that $K_n \to 1$ when n goes to ∞. By combinatorics the projection of the family $\big\{\widetilde{f}^j(J_n)\big\}_{j=1}^{q_{n+1}-1}$ to the unit circle has finite multiplicity of intersection (independent of $n \geq n_0$), and therefore:

$$\sum_{j=1}^{q_{n+1}-1}\left|\widetilde{f}^j(J_n)\right|^2<M_2\left(\max_{j\in\{1,\dots,q_{n+1}-1\}}\left|\widetilde{f}^j(J_n)\right|\right), \qquad (4.3.1)$$

where the constant $M_2>0$ only depends on the multiplicity of intersection of the projection of the family $\big\{\widetilde{f}^j(J_n)\big\}_{j=1}^{q_{n+1}-1}$ to the unit circle. By the real bounds, the right hand of (4.3.1) goes to zero exponentially fast at a universal rate (independent of f), and therefore we obtain constants $\lambda \in (0,1)$ and $C > 0$ such that:

$$k_n = \sup_{z \in Y_n} \left\{ \frac{\left|\overline{\partial} F^{q_{n+1}-1}(z)\right|}{\left|\partial F^{q_{n+1}-1}(z)\right|} \right\} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

To finish the proof of Corollary 4.3.2 we need to obtain definite domains around $\widetilde{f}(I_n)$ contained in Y_n. As in the proof of Proposition 4.3.1, for each $n \geq n_0$ let A_n and B_n be the affine maps given by:

$$A_n(z) = \big(1/\big|\widetilde{f}^{q_{n+1}}(I_n)\big|\big)\big(z - \widetilde{f}^{q_{n+1}}(0)\big) \text{ and } B_n(z) = \big(1/\big|\widetilde{f}(I_n)\big|\big)\big(z - \widetilde{f}(0)\big),$$

and also let $Z_n = A_n\big(D_\theta\big(\widetilde{f}^{q_{n+1}}(J_n)\big)\big)$. By the real bounds there exists a universal constant $\alpha_0 > 0$ such that:

$$N_{\alpha_0}\big([0,1]\big) \subset Z_n \quad \text{for all } n \geq n_0.$$

The $\mathbb{R}$-symmetric orientation preserving C^3 diffeomorphism $T_n : Z_n \to T_n(Z_n)$ given by:

$$T_n = B_n \circ F^{-q_{n+1}+1} \circ A_n^{-1}$$

induces a diffeomorphism in $[0,1]$ which, again by the real bounds, has universally bounded distortion. In particular there exists $\varepsilon > 0$ such that $\big|T_n'(t)\big| > \varepsilon$ for all $t \in [0,1]$ and for all $n \geq n_0$. By Proposition 3.3.3 there exists $\alpha > 0$ (only depending on α_0 and ε) such that (by taking n_0 big enough):

$$N_\alpha\big([0,1]\big) \subset T_n(Z_n) \quad \text{for all } n \geq n_0,$$

and therefore:

$$N_\alpha\big(\widetilde{f}(I_n)\big) \subset Y_n \quad \text{for all } n \geq n_0.$$

$\square$

Proposition 4.3.3. *There exist constants $\alpha > 0$, $C_1, C_2 > 0$ and $\lambda \in (0,1)$ with the following property: let f be a C^3 critical circle map with irrational rotation number, and let F be its extended lift. There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists an $\mathbb{R}$-symmetric topological disk X_n with:*

$$N_\alpha(I_n) \subset X_n\,, \quad \textit{where } I_n = \big[0, (T^{-p_n} \circ \widetilde{f}^{q_n})(0)\big],$$

such that the composition $F^{q_{n+1}}$ is well defined in X_n, it has a unique critical point at the origin, and we have:

1.
$$C_1 < \frac{\operatorname{diam}\big(F^{q_{n+1}}(X_n)\big)}{\big|\widetilde{f}^{q_{n+1}}(I_n)\big|} < C_2\,, \quad \textit{and}$$

2.
$$\sup_{z \in X_n \setminus \{0\}} \left\{ \frac{\big|\overline{\partial} F^{q_{n+1}}(z)\big|}{\big|\partial F^{q_{n+1}}(z)\big|} \right\} \leq C_2 \lambda^n\,.$$

Proof. From the construction of the extended lift F in Section 4.1 (see also Lemma 4.1.5) there exists a complex neighbourhood Ω of the origin such that the restriction $F : \Omega \to F(\Omega)$ is of the form $Q \circ \psi$, where $Q(z) = z^3 + \widetilde{f}(0)$, and $\psi : \Omega \to Q^{-1}\big(F(\Omega)\big)$ is an $\mathbb{R}$-symmetric orientation preserving C^3 diffeomorphism fixing the origin. In particular there exist $\varepsilon > 0$ and $\delta > 0$ such that if $t \in (-\delta, \delta)$ then $\big|(\psi^{-1})'(t)\big| > \varepsilon$, where $(\psi^{-1})'$ denotes the one-dimensional derivative of the restriction of ψ^{-1} to $Q^{-1}\big(F(\Omega)\big) \cap \mathbb{R}$. Let $K > 1$ given by Proposition 3.3.3 applied to $\varepsilon > 0$. Since ψ is asymptotically holomorphic of order 3 in Ω, we can choose Ω small enough in order to have that ψ is K-quasiconformal. By taking $n_0 \in \mathbb{N}$ big enough we can assume that $\big|\psi(I_n)\big| < \delta$ and $Y_n \subset F(\Omega)$ for all $n \geq n_0$, where the topological disk Y_n is the one given by Corollary 4.3.2. By Corollary 4.3.2 and elementary properties of the cube root map (see for instance [58, Lemma 2.2]) there exists a universal constant $\alpha_0 > 0$ such that for all $n \geq n_0$ we have that:

$$N_{\alpha_0}\big(\psi(I_n)\big) \subset Q^{-1}(Y_n)\,. \tag{4.3.2}$$

Define $X_n \subset \Omega$ as the preimage of Y_n under F, that is, $X_n = F^{-1}(Y_n) = \psi^{-1}\big(Q^{-1}(Y_n)\big)$. Item (1) follows directly from Item (1) in Corollary 4.3.2 since $F^{q_{n+1}}(X_n) = F^{q_{n+1}-1}(Y_n)$. By (4.3.2) and Proposition 3.3.3 there exists a universal constant $\alpha > 0$ such that:

$$N_\alpha(I_n) \subset X_n \subset \Omega \quad \text{for all } n \geq n_0.$$

To obtain Item (2) recall that by Item (2) in Corollary 4.3.2 we have:

$$\sup_{z \in Y_n} \left\{ \frac{\big|\overline{\partial} F^{q_{n+1}-1}(z)\big|}{\big|\partial F^{q_{n+1}-1}(z)\big|} \right\} \leq C \lambda^n\,.$$

Since Q is a polynomial, it is conformal at its regular points, and since $\big\|\mu_\psi\big\|_\infty \leq \frac{K-1}{K+1} < 1$ in Ω we have:

$$\sup_{z \in X_n \setminus \{0\}} \left\{ \frac{\big|\overline{\partial} F^{q_{n+1}}(z)\big|}{\big|\partial F^{q_{n+1}}(z)\big|} \right\} \leq C \lambda^n\,.$$

□

Theorem 4.0.4 follows directly from Proposition 4.3.3 and its analogue statement for $\widetilde{f}^{q_n}|_{I_{n+1}}$.

CHAPTER 5

Proof of Theorem D

As its tittle indicates, this chapter is entirely devoted to the proof of Theorem D, and recall that Theorem D implies our main theorem (Theorem B in the introduction) as we saw in Chapter 2.

First let us fix some notation and terminology (see Appendix C for complete proofs and much more information). By a *topological disk* we mean an open, connected and simply connected set properly contained in the complex plane. Let $\pi : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be the holomorphic covering $z \mapsto \exp(2\pi i z)$, and let $T : \mathbb{C} \to \mathbb{C}$ be the unitary horizontal translation $z \mapsto z + 1$ (which is a generator of the group of automorphisms of the covering). For any $R > 1$ consider the *band*:

$$B_R = \big\{ z \in \mathbb{C} : -\log R < 2\pi\Im(z) < \log R \big\},$$

which is the universal cover of the *round annulus*:

$$A_R = \left\{ z \in \mathbb{C} : \frac{1}{R} < |z| < R \right\}$$

via the holomorphic covering π. Since B_R is T-invariant, the translation generates the group of automorphisms of the covering. The restriction $\pi : \mathbb{R} \to S^1 = \partial\mathbb{D}$ is also a covering map, the automorphism T preserves the real line, and again generates the group of automorphisms of the covering.

The map $U_R : \mathbb{D} \to B_R$ defined by:

$$U_R(z) = \left(\frac{\log R}{\pi^2} \right) \log \left(\frac{1+z}{1-z} \right)$$

is a biholomorphism sending $(-1,1)$ onto the real line, and preserving the sign of the imaginary part. Here log denotes the principal branch of the logarithm (for $z = re^{i\theta}$ with $0 \leq \theta < 2\pi$, we have $\log z = \log r + i\theta$).

In particular we can endow the band B_R with the Riemannian metric obtained by pushing the Poincaré metric from the unit disk, via the biholomorphism U_R. It is easy to check[1] that this metric is invariant under the translation T, and therefore we can project it to the annulus A_R with the holomorphic covering π. This hyperbolic metric in the annulus A_R induces a complete distance (by computing the infimum among the hyperbolic lengths of all piecewise smooth curves joining two points), that we denote by d_{A_R}.

More generally, an *annulus* is an open and connected set A in the complex plane whose fundamental group is isomorphic to $\mathbb{Z}$. By the Uniformization Theorem such an annulus must be conformally equivalent either to the punctured disk $\mathbb{D} \setminus \{0\}$, to the punctured plane $\mathbb{C} \setminus \{0\}$, or to some round annulus $A_R = \big\{z \in \mathbb{C} : 1/R < |z| < R\big\}$. In the last case the value of $R > 1$ is unique, and there exists a holomorphic covering from $\mathbb{D}$ to A whose group of deck transformations is infinite cyclic, and such that any generator is a Möbius transformation having exactly two fixed points at the boundary of the unit disk (for instance, $U_R^{-1} \circ T \circ U_R$ fixes the points -1 and 1).

Since the deck transformations are Möbius transformations, they are isometries of the Poincaré metric on $\mathbb{D}$ and therefore there exists a unique Riemannian metric on A such that the covering map provided by the Uniformization Theorem is a local isometry. This metric is complete, and in particular, any two points can be joined by a minimizing geodesic. There exists a unique simple closed geodesic in A, whose hyperbolic length is equal to $\pi^2/\log R$. The length of this closed geodesic is therefore a conformal invariant (all these statements are reviewed in full detail in Appendix C of this thesis).

We denote by Θ the antiholomorphic involution $z \mapsto 1/\bar{z}$ in the punctured plane $\mathbb{C} \setminus \{0\}$, and we say that a map is S^1-*symmetric* if it commutes with Θ. An annulus is S^1-*symmetric* if it is invariant under Θ (for instance, the round annulus A_R described above is S^1-symmetric). In this case, the unit circle is the *core curve* (the unique simple closed geodesic) for the hyperbolic metric in A. In this chapter we will deal only with S^1-symmetric annulus. In particular any time that some annulus A_0 is contained in some other annulus A_1, we have that A_0 separates the boundary components of A_1 (more technically, the inclusion is *essential* in the sense that the fundamental group $\pi_1(A_0)$ injects into $\pi_1(A_1)$).

[1]For instance by noting that $\big(U_R^{-1} \circ T \circ U_R\big)(z) = \frac{z-\alpha}{1-\alpha z}$ for all $z \in \mathbb{D}$, where $\alpha \in (-1,0)$ is equal to $(e^{-\pi^2/\log R} - 1)/(e^{-\pi^2/\log R} + 1)$.

Besides Theorem 4.0.4 (stated and proved in Chapter 4), the main tool in order to prove Theorem D is Proposition 3.3.2 (stated in Chapter 3, and proved in Appendix D as a corollary of Ahlfors-Bers Theorem). The proof of Theorem D will be divided in three sections. Along the proof, C will denote a positive constant (independent of $n \in \mathbb{N}$) and n_0 will denote a positive (big enough) natural number. At first, let $n_0 \in \mathbb{N}$ given by Theorem 4.0.4. Moreover let us use the following notation: $W_1 = N_\alpha\big([-1,0]\big)$, $W_2 = W_2(n) = N_\alpha\big([0,\xi_n(0)]\big)$, $W_0 = B(0,\lambda)$ and $\mathcal{V} = B(0,\lambda^{-1})$, where $\alpha > 0$ and $\lambda \in (0,1)$ are the universal constants given by Theorem 4.0.4. Recall that $\eta_n(0) = -1$ for all $n \geq 1$ after normalization.

5.1 A first perturbation and a bidimensional glueing procedure

From Theorem 4.0.4 we have:

Lemma 5.1.1. *There exists an $\mathbb{R}$-symmetric topological disk U with:*

$$-1 \in U \subset W_1 \setminus W_0,$$

such that for all $n \geq n_0$ the composition:

$$\eta_n^{-1} \circ \xi_n : U \to \big(\eta_n^{-1} \circ \xi_n\big)(U)$$

is an $\mathbb{R}$-symmetric orientation-preserving C^3 diffeomorphism.

For each $n \geq n_0$ denote by A_n the diffeomorphism $\eta_n^{-1} \circ \xi_n$. Note that $\|\mu_{A_n}\|_\infty \leq C\lambda^n$ in U for all $n \geq n_0$, and that the domains $\big\{A_n(U)\big\}_{n \geq n_0}$ are uniformly bounded since they are contained in $\cup_j W_2^j$. Fix $\varepsilon > 0$ and $\delta > 0$ such that the rectangle:

$$V = \big(-1-\varepsilon, -1+\varepsilon\big) \times \big(-i\delta, i\delta\big)$$

is compactly contained in U, and apply Proposition 3.3.2 to the sequence of $\mathbb{R}$-symmetric orientation-preserving C^3 diffeomorphisms:

$$\{A_n : U \to A_n(U)\}_{n \geq n_0}$$

to obtain a sequence of $\mathbb{R}$-symmetric biholomorphisms:

$$\big\{B_n : V \to B_n(V)\big\}_{n \geq n_0}$$

such that:

$$\big\|A_n - B_n\big\|_{C^0(V)} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

From the commuting condition we obtain:

Lemma 5.1.2. *For each $n \geq n_0$ there exist three $\mathbb{R}$-symmetric topological disks $V_i(n)$ for $i \in \{1,2,3\}$ with the following five properties:*

- $0 \in V_1(n) \subset W_0$;
- $\big(\eta_n \circ \xi_n\big)(0) = \big(\xi_n \circ \eta_n\big)(0) = \xi_n(-1) \in V_2(n) \subset W_2$;
- $\xi_n(0) \in V_3(n) \subset W_2$;
- *When restricted to $V_1(n)$, both η_n and ξ_n are orientation-preserving three-fold C^3 branched coverings onto V and $V_3(n)$ respectively, with a unique critical point at the origin;*
- *Both restrictions $\xi_n|_V$ and $\eta_n|_{V_3(n)}$ are orientation-preserving C^3 diffeomorphisms onto $V_2(n)$.*

In particular the composition $\eta_n^{-1} \circ \xi_n$ is an orientation-preserving C^3 diffeomorphism from V onto $V_3(n)$ for all $n \geq n_0$.

For each $n \geq n_0$ let $U_1(n)$, $U_2(n)$ and $U_3(n)$ be three $\mathbb{R}$-symmetric topological disks such that:

- $\overline{U_1(n)}$, $\overline{U_2(n)}$ and $\overline{U_3(n)}$ are pairwise disjoint;
- $V \bigcap U_j(n) = \emptyset$ and $V_i(n) \bigcap U_j(n) = \emptyset$ for $i, j \in \{1,2,3\}$;
- $\overline{U_1(n)} \subset W_1$ and $\overline{U_2(n)} \bigcup \overline{U_3(n)} \subset W_2$;

and such that:

$$\mathcal{U}_n = \text{interior}\left[V \bigcup \left(\bigcup_{i=1}^{i=3} V_i(n)\right) \bigcup \left(\bigcup_{j=1}^{j=3} \overline{U_j(n)}\right)\right]$$

is an $\mathbb{R}$-symmetric topological disk (see Figure 5.1). Note that:

$$\overline{I_{\xi_n} \cup I_{\eta_n}} \subset \mathcal{U}_n \subset W_1 \cup W_2 \quad \text{for all } n \geq n_0,$$

and that $\mathcal{U}_n \setminus \big(\overline{V \cup V_1(n) \cup V_2(n) \cup V_3(n)}\big)$ has three connected components, which are precisely $U_1(n)$, $U_2(n)$ and $U_3(n)$. By Theorem 4.0.4 we can choose $U_1(n)$, $U_2(n)$ and $U_3(n)$ in order to also have:

$$\overline{N_\delta\big([-1,0]\big) \cup N_\delta\big([0, \xi_n(0)]\big)} \subset \mathcal{U}_n \quad \text{for all } n \geq n_0,$$

for some universal constant $\delta > 0$, independent of $n \geq n_0$. Note also that each $\mathcal{U}_n$ is uniformly bounded since it is contained in $N_\alpha\big([-1, K]\big)$, where $\alpha > 0$ is given by Theorem 4.0.4, and $K > 1$ is the universal constant given by the real bounds.

For each $n \geq n_0$ let $\mathcal{T}_n$ be an $\mathbb{R}$-symmetric topological disk such that:

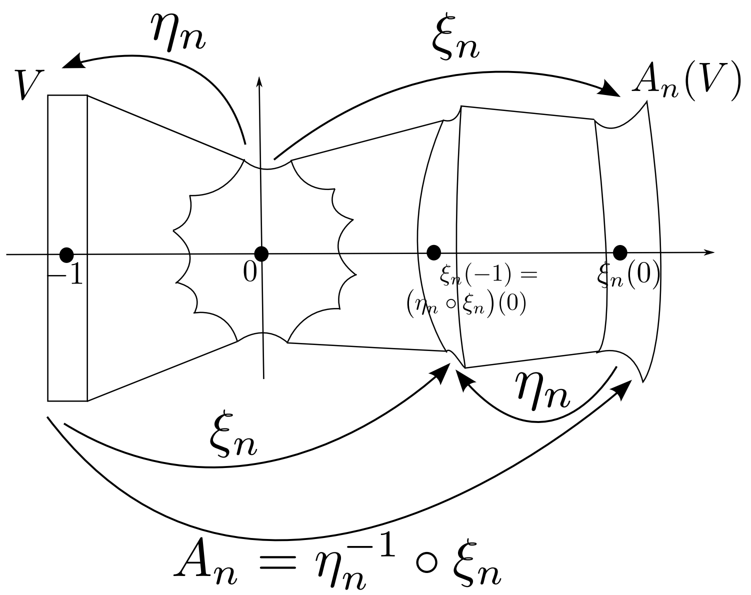

Figure 5.1: The domain $\mathcal{U}_n$.

- V, $V_1(n)$, $V_2(n)$ and $B_n(V)$ are contained in $\mathcal{T}_n$,
- $\mathcal{T}_n \setminus \big(V \cup B_n(V)\big)$ is connected and simply connected,
- The Hausdorff distance between $\overline{\mathcal{T}_n}$ and $\overline{\mathcal{U}_n}$ is less or equal than:

$$\big\|A_n - B_n\big\|_{C^0(V)} \leq C\lambda^n,$$

Lemma 5.1.3. *For each $n \geq n_0$ there exists an orientation-preserving $\mathbb{R}$-symmetric C^3 diffeomorphism $\Phi_n : \mathcal{U}_n \to \mathcal{T}_n$ such that:*

- *$\Phi_n \equiv Id$ in the interior of $V \cup \overline{U_1(n)} \cup V_1(n)$, in particular $\Phi_n(0) = 0$.*
- *$B_n = \Phi_n \circ \big(\eta_n^{-1} \circ \xi_n\big) \circ \Phi_n^{-1}$ in V, that is, $\Phi_n \circ A_n = B_n \circ \Phi_n$ in V.*
- *$\big\|\Phi_n - Id\big\|_{C^0(\mathcal{U}_n)} \leq C\lambda^n$.*
- *$\|\mu_{\Phi_n}\|_\infty \leq C\lambda^n$ in $\mathcal{U}_n$.*

Proof of Lemma 5.1.3. For each $n \geq n_0$ we have $\|A_n - B_n\|_{C^0(V)} \leq C\lambda^n$ and therefore:

$$\big\|Id - \big(B_n \circ A_n^{-1}\big)\big\|_{C^0\left(V_3(n)\right)} \leq C\lambda^n.$$

If we define $\Phi_n|_{V_3(n)} = B_n \circ A_n^{-1}$ we also have $\|\mu_{\Phi_n}\|_\infty = \|\mu_{A_n^{-1}}\|_\infty$ in $V_3(n)$, which is equal to $\|\mu_{A_n}\|_\infty$ in V. In particular $\|\mu_{\Phi_n}\|_\infty \leq C\lambda^n$ in $V_3(n)$, and then we define Φ_n in the whole $\mathcal{U}_n$ by interpolating $B_n \circ A_n^{-1}$ in $V_3(n)$ with the identity in the interior of $V \cup \overline{U_1(n)} \cup V_1(n)$. $\square$

Consider the seven topological disks:

$$X_1(n) = \text{interior}\big(V \cup \overline{U_1(n)} \cup V_1(n)\big) \subset W_1 \cap \mathcal{U}_n\,,$$

$$X_2(n) = \text{interior}\big(V_1(n) \cup \overline{U_2(n)} \cup V_2(n) \cup \overline{U_3(n)} \cup V_3(n)\big) \subset W_2 \cap \mathcal{U}_n\,,$$

$$\widehat{X}_1(n) = \{z \in X_1(n) : \xi_n(z) \in \mathcal{U}_n\}\,, \quad \widehat{X}_2(n) = \{z \in X_2(n) : \eta_n(z) \in \mathcal{U}_n\}\,,$$

$$\widehat{\mathcal{T}}_n = \Phi_n\big(\widehat{X}_1(n)\big) \cup \Phi_n\big(\widehat{X}_2(n)\big) \subset \mathcal{T}_n\,,$$

$$Y_1(n) = X_1(n) \cap \Phi_n\big(\widehat{X}_1(n)\big) \quad \text{and} \quad Y_2(n) = X_2(n) \cap \Phi_n\big(\widehat{X}_2(n)\big).$$

Note that V, $V_1(n)$ and $B_n(V)$ are contained in $\widehat{\mathcal{T}}_n$ for all $n \geq n_0$. Moreover, we have the following two corollaries of Theorem 4.0.4:

Lemma 5.1.4. *There exists $\delta > 0$ such that for all $n \geq n_0$ we have:*

$$N_\delta\big([-1,0]\big) \subset Y_1(n) \quad \textit{and} \quad N_\delta\big([0,\xi_n(0)]\big) \subset Y_2(n)\,.$$

Lemma 5.1.5. *Both:*

$$\sup_{n \geq n_0}\left\{\sup_{z \in Y_1(n)}\left\{\det\big(D\xi_n(z)\big)\right\}\right\} \quad \textit{and} \quad \sup_{n \geq n_0}\left\{\sup_{z \in Y_2(n)}\left\{\det\big(D\eta_n(z)\big)\right\}\right\}$$

are finite, where $\det(\cdot)$ *denotes the determinant of a square matrix.*

Let:

$$\widehat{\xi}_n : \Phi_n\big(\widehat{X}_1(n)\big) \to \big(\Phi_n \circ \xi_n\big)\big(\widehat{X}_1(n)\big) \text{ defined by } \widehat{\xi}_n = \Phi_n \circ \xi_n \circ \Phi_n^{-1},$$

and:

$$\widehat{\eta}_n : \Phi_n\big(\widehat{X}_2(n)\big) \to \big(\Phi_n \circ \eta_n\big)\big(\widehat{X}_2(n)\big) \text{ defined by } \widehat{\eta}_n = \Phi_n \circ \eta_n \circ \Phi_n^{-1}.$$

Since each Φ_n is an $\mathbb{R}$-symmetric C^3 diffeomorphism, the pair $\big(\widehat{\eta}_n, \widehat{\xi}_n\big)$ restrict to a critical commuting pair with the same rotation number as (η_n, ξ_n), and the same criticality (that we are assuming to be cubic, in order to simplify). Note also that $\widehat{\eta}_n(0) = -1$ for all $n \geq n_0$. Moreover, from Lemma 5.1.5 and $\big\|\Phi_n - Id\big\|_{C^0(\mathcal{U}_n)} \leq C\lambda^n$ we have:

$$\Big\|\xi_n - \widehat{\xi}_n\Big\|_{C^0\left(Y_1(n)\right)} \leq C\lambda^n \quad \text{and} \quad \|\eta_n - \widehat{\eta}_n\|_{C^0\left(Y_2(n)\right)} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

Therefore is enough to shadow the sequence $\big(\widehat{\eta}_n, \widehat{\xi}_n\big)$ in the domains $Y_1(n)$ and $Y_2(n)$, instead of (η_n, ξ_n) (the shadowing sequence will be constructed in Section 5.3 below). The main advantage of working with the sequence $\big(\widehat{\eta}_n, \widehat{\xi}_n\big)$ is precisely the fact that $\widehat{\eta}_n^{-1} \circ \widehat{\xi}_n$ is univalent in V for all $n \geq n_0$ (since it coincides with B_n). In particular we can choose each topological disk $\mathcal{U}_n$ and $\mathcal{T}_n$ defined above with the additional property that, identifying V with $B_n(V)$ via the biholomorphism B_n, we obtain from $\mathcal{T}_n$ an abstract annular Riemann surface $\mathcal{S}_n$ (with the complex structure induced by the quotient).

Let us denote by $p_n : \mathcal{T}_n \to \mathcal{S}_n$ the canonical projection (note that p_n is not a covering map, just a surjective local diffeomorphism). The projection of the real line, $p_n(\mathbb{R} \cap \mathcal{T}_n)$, is real-analytic diffeomorphic to the unit circle S^1. We call it the *equator* of $\mathcal{S}_n$.

Since complex conjugation leaves $\mathcal{T}_n$ invariant and commutes with B_n, it induces an antiholomorphic involution $F_n : \mathcal{S}_n \to \mathcal{S}_n$ acting as the identity on the equator $p_n(\mathbb{R} \cap \mathcal{T}_n)$. Note that F_n has a continuous extension to $\partial\mathcal{S}_n$ that switches the boundary components.

Since $\mathcal{S}_n$ is obviously not biholomorphic to $\mathbb{D} \setminus \{0\}$ neither to $\mathbb{C} \setminus \{0\}$ we have $\operatorname{mod}(\mathcal{S}_n) < \infty$ for all $n \geq n_0$, where $\operatorname{mod}(\cdot)$ denotes the conformal modulus of an annular Riemann surface (see Definition C.2.4). For each $n \geq n_0$ define a constant R_n in $(1, +\infty)$ by:

$$R_n = \exp\left(\frac{\operatorname{mod}\big(\mathcal{S}_n\big)}{2}\right),$$

that is, $\mathcal{S}_n$ is conformally equivalent to $A_{R_n} = \big\{z \in \mathbb{C} : R_n^{-1} < |z| < R_n\big\}$ (see Theorem C.2.3). Any biholomorphism between $\mathcal{S}_n$ and A_{R_n} must send the equator $p_n\big(\mathbb{R} \cap \mathcal{T}_n\big)$ onto the unit circle S^1 (because the equator is invariant under the antiholomorphic involution F_n, and the unit circle is invariant under the antiholomorphic involution $z \mapsto 1/\bar{z}$ in A_{R_n}, see Lemma C.2.1). Let $\Psi_n : \mathcal{S}_n \to A_{R_n}$ be the conformal uniformization determined by $\Psi_n\big(p_n(0)\big) = 1$ (again see Lemma C.2.1), and let $P_n : \mathcal{T}_n \to A_{R_n}$ be the holomorphic surjective local diffeomorphism:

$$P_n = \Psi_n \circ p_n.$$

See Figure 5.2. Note that $P_n(0) = 1$ and $P_n(\mathcal{T}_n \cap \mathbb{R}) = S^1$ for all $n \geq n_0$. Moreover $P_n(z)\overline{P_n(\overline{z})} = 1$ for all $z \in \mathcal{T}_n$ and all $n \geq n_0$. From now on we forget about the abstract cylinder $\mathcal{S}_n$.

Lemma 5.1.6. *There exist two constants $\delta > 0$ and $C > 1$ such that for all $n \geq n_0$ and for all $z \in N_\delta\big([-1, \widetilde{\xi}_n(0)]\big)$ we have $z \in \widehat{\mathcal{T}_n} \subset \mathcal{T}_n$ and:*

$$\frac{1}{C} < \big|P_n'(z)\big| < C\,.$$

Proof of Lemma 5.1.6. By the real bounds there exists a universal constant $C_0 > 1$ such that for each $n \geq n_0$ there exists $w_n \in \big[-1, \widetilde{\xi}_n(0)\big]$ such that:

$$\frac{1}{C_0} < \big|P_n'(w_n)\big| < C_0\,.$$

To prove Lemma 5.1.6 we need to construct a definite complex domain around $\big[-1, \widetilde{\xi}_n(0)\big]$ where P_n has universally bounded distortion. Again by the real bounds there exist $\delta > 0$ and $l \in \mathbb{N}$ with the following properties: for each $n \geq n_0$ there exists $z_1, z_2, ..., z_{k_n} \in \big[-1, \widetilde{\xi}_n(0)\big]$ with $k_n < l$ for all $n \geq n_0$ such that:

- $\big[-1, \widetilde{\xi}_n(0)\big] \subset \cup_{i=1}^{k_n} B(z_i, \delta)$.
- $B(z_i, 2\delta) \subset \widehat{\mathcal{T}_n} \subset \mathcal{T}_n$ for all $i \in \{1, ..., k_n\}$.
- $P_n|_{B(z_i,2\delta)}$ is univalent for all $i \in \{1, ..., k_n\}$.

By convexity we have for all $n \geq n_0$ and for all $i \in \{1, ..., k_n\}$ that:

$$\sup_{v,w \in B(z_i,\delta)} \left\{\frac{\big|P_n'(v)\big|}{\big|P_n'(w)\big|}\right\} \leq \exp\left(\sup_{w \in B(z_i,\delta)} \left\{\frac{\big|P_n''(w)\big|}{\big|P_n'(w)\big|}\right\}\right),$$

and by Koebe distortion theorem (see for instance [6, Section I.1, Theorem 1.6]) we have:

$$\sup_{w \in B(z_i,\delta)} \left\{\frac{\big|P_n''(w)\big|}{\big|P_n'(w)\big|}\right\} \leq \frac{2}{\delta} \quad \text{for all } n \geq n_0 \text{ and for all } i \in \{1, ..., k_n\}.$$

□

Now we project each commuting pair $(\widetilde{\eta}_n, \widetilde{\xi}_n)$ from $\widehat{\mathcal{T}_n}$ to the round annulus A_{R_n}.

Proposition 5.1.7 (Glueing procedure)**.** *The pair:*

$$\widehat{\xi}_n : \Phi_n\big(\widehat{X}_1(n)\big) \to \mathcal{T}_n \quad \textit{and} \quad \widehat{\eta}_n : \Phi_n\big(\widehat{X}_2(n)\big) \to \mathcal{T}_n$$

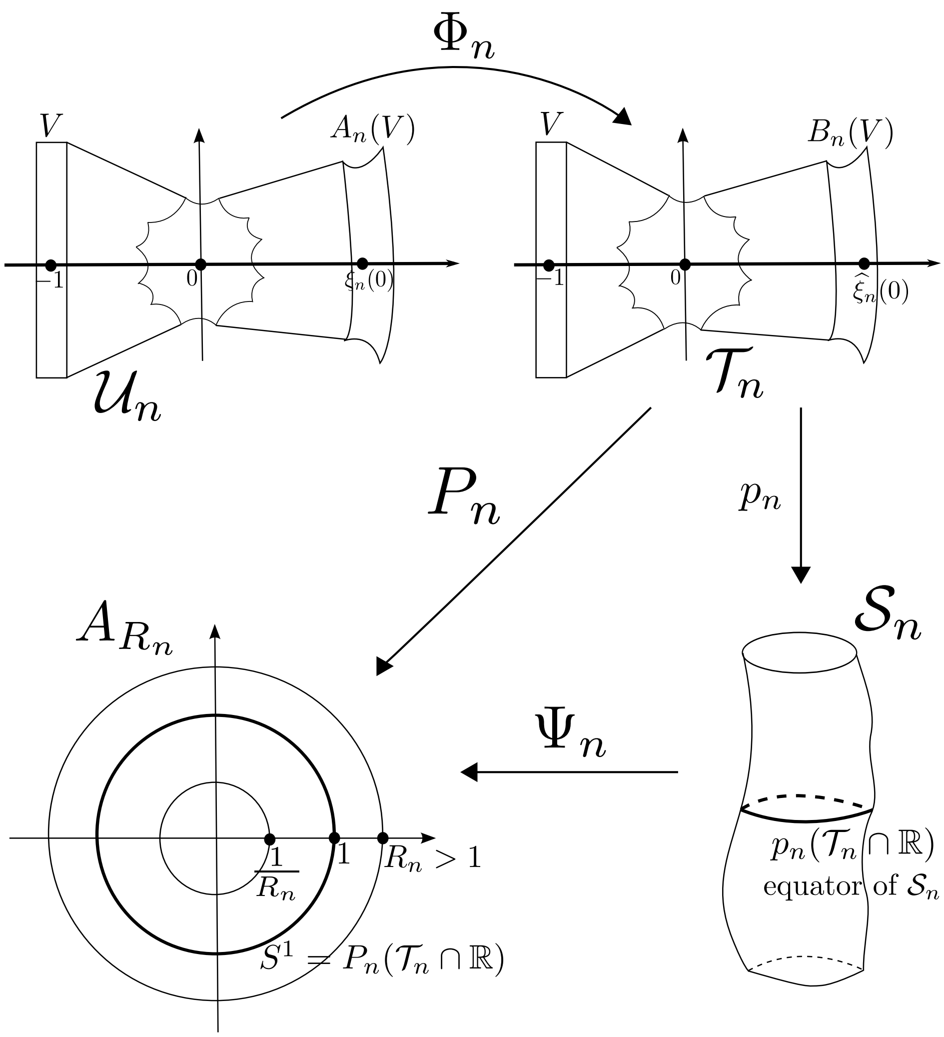

Figure 5.2: Bidimensional Glueing procedure.

projects under P_n to a well-defined orientation-preserving C^3 map:

$$G_n : P_n(\widehat{\mathcal{T}}_n) \subset A_{R_n} \to A_{R_n}.$$

For each $n \geq n_0$, $P_n(\widehat{\mathcal{T}}_n)$ is a Θ-invariant annulus with positive and finite modulus. Each G_n is S^1-symmetric, in particular G_n preserves the unit circle.

When restricted to the unit circle, G_n produce a C^3 critical circle map $g_n : S^1 \to S^1$ with cubic critical point at $P_n(0) = 1$, and with rotation *number* $\rho(g_n) = \rho\big(\mathcal{R}^n(f)\big) \in \mathbb{R} \setminus \mathbb{Q}$.

$$\begin{tikzcd} \widehat{\mathcal{T}}_n \subset \mathcal{T}_n \arrow[r, "(\widehat{\eta}_n, \widehat{\xi}_n)"] \arrow[d, "P_n"'] & \mathcal{T}_n \arrow[d, "P_n"] \\ P_n\big(\widehat{\mathcal{T}}_n\big) \subset A_{R_n} \arrow[r, "G_n"] & A_{R_n} \end{tikzcd}$$

Moreover the unique critical point of G_n *in* $P_n\big(\widehat{\mathcal{T}}_n\big)$ *is the one in the unit circle (at the point* 1*) and:*

$$\left|\overline{\partial} G_n(z)\right| \leq C\lambda^n \left|\partial G_n(z)\right| \quad \textit{for all } z \in P_n\big(\widehat{\mathcal{T}}_n\big) \setminus \{1\}, \textit{ that is:}$$

$$\|\mu_{G_n}\|_{\infty} \leq C\lambda^n \textit{ in } P_n\big(\widehat{\mathcal{T}}_n\big).$$

Proof of Proposition 5.1.7. This follows from:

- The construction of $\mathcal{U}_n$ and $\mathcal{T}_n$.
- The property $B_n = \Phi_n \circ \big(\eta_n^{-1} \circ \xi_n\big) \circ \Phi_n^{-1}$ in V.
- The commuting condition in $V_1(n)$.
- The symmetry $P_n(z)\overline{P_n(\overline{z})} = 1$ for all $z \in \mathcal{T}_n$ and all $n \geq n_0$.
- The fact that $P_n : \mathcal{T}_n \to A_{R_n}$ is holomorphic, $P_n(0) = 1$ and $P_n(\mathcal{T}_n \cap \mathbb{R}) = S^1$ for all $n \geq n_0$.

□

Note that each g_n belongs to the smooth conjugacy class obtained with the glueing procedure (described in Section 1.2) applied to the C^3 critical commuting pair $\big(\widehat{\eta}_n, \widehat{\xi}_n\big)$. As we said in the introduction, the topological behaviour of each G_n on its annular domain is the same as the restriction of the Blaschke product f_{γ} (0.1.3) to the annulus $A''' \cup B_1'$, as depicted in Figure 1. In the next section we will construct a sequence of real-analytic critical circle maps, with the desired combinatorics, that extend to holomorphic maps exponentially close to G_n in a definite annulus around the unit circle (see Proposition 5.2.1 below).

5.2 Main perturbation

The goal of this section is to construct the following sequence of perturbations:

Proposition 5.2.1 (Main perturbation)**.** *There exist a constant $r > 1$ and a sequence of holomorphic maps defined in the annulus A_r:*

$$\{H_n : A_r \to \mathbb{C}\}_{n \geq n_0}$$

such that for all $n \geq n_0$ the following holds:

- $A_r \subset P_n(\widehat{\mathcal{T}}_n) \subset P_n(\mathcal{T}_n) = A_{R_n}$.
- $\big\|H_n - G_n\big\|_{C^0(A_r)} \leq C\lambda^n$.
- $H_n(A_r) \subset \big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}}_n\big) \subset P_n(\mathcal{T}_n) = A_{R_n}$.
- *H_n preserves the unit circle and, when restricted to the unit circle, H_n produces a real-analytic critical circle map $h_n : S^1 \to S^1$ such that:*
  - *The unique critical point of h_n is at $P_n(0) = 1$, and is of cubic type.*
  - *The critical value of h_n coincide with the one of g_n, that is, $h_n(1) = g_n(1) \in P_n(V \cap \mathbb{R})$.*
  - $\rho(h_n) = \rho(g_n) = \rho\big(\mathcal{R}^n(f)\big) \in \mathbb{R} \setminus \mathbb{Q}$.
- *The unique critical point of H_n in A_r is the one in the unit circle.*

The remainder of this section is devoted to proving Proposition 5.2.1. We wont perturb the maps G_n directly (basically because they are non invertible). Instead, we will decompose them (see Lemma 5.2.2 below), and then we will perturb on their *coefficients* (see the definition after the statement of Lemma 5.2.2). Those perturbations will be done, again, with the help of Proposition 3.3.2 of Chapter 3.

Let $A : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ be the map corresponding to the parameters $a = 0$ and $b = 1$ in the Arnold family (0.1.2), defined in the introduction. The lift of A to the complex plane by the holomorphic covering $z \mapsto \exp(2\pi i z)$ is the entire map $\widetilde{A} : \mathbb{C} \to \mathbb{C}$ given by:

$$\widetilde{A}(z) = z - \left(\frac{1}{2\pi}\right)\sin(2\pi z).$$

Then A preserves the unit circle, and its restriction $A : S^1 \to S^1$ is a real-analytic critical circle map. The critical point of A in the unit circle is at 1, and is of cubic type (the critical point is also a fixed point for A). The following is a bidimensional version of Lemma 4.1.1 in Chapter 4:

Lemma 5.2.2. *For each $n \geq n_0$ there exist:*

- $S_n > 1$,
- *an S^1-symmetric orientation-preserving C^3 diffeomorphism $\psi_n : P_n\big(\widehat{\mathcal{T}_n}\big) \to A_{S_n}$ and*
- *an S^1-symmetric biholomorphism $\phi_n : A(A_{S_n}) \to \big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}_n}\big)$ such that:*

$$G_n = \phi_n \circ A \circ \psi_n \quad \text{in } P_n\big(\widehat{\mathcal{T}_n}\big).$$

The diffeomorphisms ψ_n and ϕ_n are called the *coefficients* of G_n in $P_n\big(\widehat{\mathcal{T}_n}\big)$.

$$\begin{tikzcd} P_n\big(\widehat{\mathcal{T}_n}\big) \arrow[r, "G_n"] \arrow[d, "\psi_n"'] & \big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}_n}\big) \\ A_{S_n} \arrow[r, "A"] & A(A_{S_n}) \arrow[u, "\phi_n"'] \end{tikzcd}$$

Proof of Lemma 5.2.2. For each $n \geq n_0$ let $S_n > 1$ such that $A(A_{S_n})$ is a Θ-invariant annulus with:

$$\operatorname{mod}\big(A(A_{S_n})\big) = \operatorname{mod}\Big(\big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}_n}\big)\Big).$$

In particular there exists a biholomorphism $\phi_n : A(A_{S_n}) \to \big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}_n}\big)$ that commutes with Θ. Each ϕ_n preserves the unit circle and we can choose it such that $\phi_n(1) = G_n(1)$, that is, ϕ_n takes the critical value of A into the critical value of G_n.

Since both G_n and A are three-fold branched coverings around their critical points and local diffeomorphisms away from them, the equation $G_n = \phi_n \circ A \circ \psi_n$ induces an orientation-preserving C^3 diffeomorphism $\psi_n : P_n\big(\widehat{\mathcal{T}_n}\big) \to A_{S_n}$, that commutes with Θ and such that $\psi_n(1) = 1$, that is, ψ_n takes the critical point of G_n into the one of A. The fact that ψ_n is smooth at 1 with non-vanishing derivative follows from the fact that the critical points of G_n and A have the same degree (see Lemma 4.1.1 in Chapter 4). $\square$

Note that, at the beginning of the proof of Lemma 5.2.2, we have used the fact that the image under the Arnold map A of a small round annulus around the unit circle is also an annulus. This is true, even that A has a critical point in the unit circle (placed at 1, and being also a fixed point of A). Even more is true: the conformal modulus of the annulus $A(A_s)$ depends continuously on $s > 1$ (and we also used this fact in the proof). The topological behaviour of the restriction of A to each round annulus A_{S_n} is the same as the restriction of the Blaschke product f_γ (0.1.3) to the annulus $A''' \cup B_1'$, as depicted in Figure 1 in the introduction of this thesis.

As we said, the idea in order to prove Proposition 5.2.1 is to perturb each diffeomorphism ψ_n with Proposition 3.3.2. In order to control the C^0 size of those perturbations we will need some geometric control, that we state in four lemmas, before entering into the proof of Proposition 5.2.1. From Lemma 5.1.6 we have:

Lemma 5.2.3.

$$1 < \inf_{n \geq n_0} \{R_n\} \quad \textit{and} \quad \sup_{n \geq n_0} \{R_n\} < +\infty.$$

Lemma 5.2.4. *For all $n \geq n_0$ both $P_n(\widehat{\mathcal{T}_n})$ and $\big(G_n \circ P_n\big)(\widehat{\mathcal{T}_n})$ are Θ-invariant annulus with finite modulus. Moreover there exists a universal constant $K > 1$ such that:*

$$\frac{1}{K} < \operatorname{mod}\big(P_n(\widehat{\mathcal{T}_n})\big) < K \quad \textit{for all } n \geq n_0.$$

Proof of Lemma 5.2.4. By Lemma 5.2.3 we know that $R = \sup_{n \geq n_0}\{R_n\}$ is finite, and since for all $n \geq n_0$ both $P_n(\widehat{\mathcal{T}_n})$ and $\big(G_n \circ P_n\big)(\widehat{\mathcal{T}_n})$ are contained in the corresponding A_{R_n}, we obtain at once that both $P_n(\widehat{\mathcal{T}_n})$ and $\big(G_n \circ P_n\big)(\widehat{\mathcal{T}_n})$ have finite modulus, and also that $\sup_{n \geq n_0}\big\{\operatorname{mod}\big(P_n(\widehat{\mathcal{T}_n})\big)\big\}$ is finite. Just as in Lemma 5.2.3, the fact that $\inf_{n \geq n_0}\big\{\operatorname{mod}\big(P_n(\widehat{\mathcal{T}_n})\big)\big\}$ is positive follows from Lemma 5.1.4 and Lemma 5.1.6. □

Lemma 5.2.5. *There exists a constant $r_0 > 1$ such that $\overline{A_{r_0}} \subset P_n\big(\widehat{\mathcal{T}_n}\big)$ for all $n \geq n_0$.*

Proof of Lemma 5.2.5. By the invariance with respect to the antiholomorphic involution $z \mapsto 1/\bar{z}$, the unit circle is the core curve (the unique closed geodesic for the hyperbolic metric) of each annulus $P_n\big(\widehat{\mathcal{T}_n}\big)$. Since:

$$\inf_{n \geq n_0}\big\{\operatorname{mod}\big(P_n(\widehat{\mathcal{T}_n})\big)\big\} > 0\,,$$

the statement is well-known, see for instance [38, Chapter 2, Theorem 2.5]. □

Lemma 5.2.6. *We have:*

$$s = \inf_{n \geq n_0} \{S_n\} > 1 \quad \text{and} \quad S = \sup_{n \geq n_0} \{S_n\} < +\infty.$$

Proof of Lemma 5.2.6. Since $\mu_{\psi_n} = \mu_{G_n}$ in $P_n\big(\widehat{\mathcal{T}_n}\big)$, we have $\|\mu_{\psi_n}\|_\infty \leq C\lambda^n$ in $P_n\big(\widehat{\mathcal{T}_n}\big)$ for all $n \geq n_0$. By the geometric definition of quasiconformal homeomorphisms (see Definition C.2.5 in Appendix C, or [32, Chapter I, Section 7]) we have:

$$\left(\frac{1 - C\lambda^n}{1 + C\lambda^n}\right) \operatorname{mod}\big(P_n(\widehat{\mathcal{T}_n})\big) \leq 2\log(S_n) \leq \left(\frac{1 + C\lambda^n}{1 - C\lambda^n}\right) \operatorname{mod}\big(P_n(\widehat{\mathcal{T}_n})\big)$$

for all $n \geq n_0$, and we are done by Lemma 5.2.4. □

With this geometric control at hand, we are ready to prove Proposition 5.2.1:

Proof of Proposition 5.2.1. Let $r_0 > 1$ given by Lemma 5.2.5 (recall that $\overline{A_{r_0}} \subset P_n\big(\widehat{\mathcal{T}_n}\big)$ for all $n \geq n_0$), and fix $r \in \big(1, (1 + r_0)/2\big)$. How small $r - 1$ must be will be determined in the course of the argument (see Lemma 5.2.7 below). For any $r \in \big(1, (1+r_0)/2\big)$ consider $\underline{r} = r_0 - (r-1) \in \big((1+r_0)/2, r_0\big)$.

The sequence of S^1-symmetric C^3 diffeomorphisms

$$\big\{\psi_n : A_{r_0} \to \psi_n(A_{r_0})\big\}_{n \geq n_0}$$

satisfy the hypothesis of Proposition 3.3.2 since:

- $\mu_{\psi_n} = \mu_{G_n}$ in $P_n\big(\widehat{\mathcal{T}_n}\big)$ and therefore $\|\mu_{\psi_n}\|_\infty \leq C\lambda^n$ for all $n \geq n_0$, and
- $\psi_n(A_{r_0}) \subset A_{S_n} \subset A_S$ for all $n \geq n_0$ (see Lemma 5.2.6 above).

Apply Proposition 3.3.2 to the bounded domain $A_{\underline{r}}$, compactly contained in A_{r_0}, to obtain a sequence of S^1-symmetric biholomorphisms

$$\big\{\widehat{\psi}_n : A_{\underline{r}} \to \widehat{\psi}_n(A_{\underline{r}})\big\}_{n \geq n_0}$$

such that:

$$\big\|\widehat{\psi}_n - \psi_n\big\|_{C^0(A_{\underline{r}})} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

Fix n_0 big enough to have $\widehat{\psi}_n(A_{\underline{r}}) \subset A_{S_n}$, and note that we can suppose that each $\widehat{\psi}_n$ fixes the point 1 (just as ψ_n) by considering:

$$z \mapsto \left(\frac{1}{\widehat{\psi}_n(1)}\right) \widehat{\psi}_n(z)\,.$$

Since $\left|\widehat{\psi}_n(z)\right| \leq S$ for all $z \in A_{\underline{r}}$ and for all $n \geq n_0$ (where $S \in (1, +\infty)$ is given by Lemma 5.2.6) and since $\left|\widehat{\psi}_n(1) - 1\right| \leq C\lambda^n$ for all $n \geq n_0$, we know that this new map (that we will still denote by $\widehat{\psi}_n$ to simplify) satisfy all the properties that we want for $\widehat{\psi}_n$, and also fixes the point $z = 1$.

For each $n \geq n_0$ consider the holomorphic map $H_n : A_{\underline{r}} \to \mathbb{C}$ defined by $H_n = \phi_n \circ A \circ \widehat{\psi}_n$. We have:

- $H_n(A_{\underline{r}}) \subset \big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}}_n\big) \subset A_{R_n}$.
- H_n is S^1-symmetric and therefore it preserves the unit circle.
- When restricted to the unit circle, H_n produces a real-analytic critical circle map $h_n : S^1 \to S^1$.
- The unique critical point of H_n in $A_{\underline{r}}$ is the one in the unit circle, which is at $P_n(0) = 1$, and is of cubic type.
- The critical value of H_n coincide with the one of G_n, that is, $H_n(1) = G_n(1) \in P_n(V \cap \mathbb{R})$.

We divide in three lemmas the rest of the proof of Proposition 5.2.1. We need to prove first that, for a suitable $r > 1$, H_n is C^0 exponentially close to G_n in the annulus A_r (Lemma 5.2.7 below), and then that we can choose each H_n with the desired combinatorics for its restriction h_n to the unit circle (Lemma 5.2.8 below). This last perturbation will change the critical value of each H_n (it wont coincide any more with the one of G_n). We will finish the proof of Proposition 5.2.1 with Lemma 5.2.9, that allow us to keep the critical point of H_n at the point $P_n(0) = 1$, and to place the critical value of H_n at the point $g_n(1)$ for all $n \geq n_0$. This will be important in the following section, the last one of this chapter.

Lemma 5.2.7. *There exists $r \in \big(1, (1 + r_0)/2\big)$ such that in the annulus A_r we have:*

$$\big\|H_n - G_n\big\|_{C^0(A_r)} \leq C\lambda^n \quad \textit{for all } n \geq n_0.$$

Proof of Lemma 5.2.7. The proof is divided in three claims:

First claim: There exists $\beta > 1$ such that $\overline{A_\beta} \subset A(A_{S_n})$ for all $n \geq n_0$.

Indeed, by Lemma 5.2.6 the round annulus $A_{(1+s)/2}$ is compactly contained in A_{S_n} for all $n \geq n_0$, and therefore the annulus $A\big(A_{(1+s)/2}\big)$ is contained in $A(A_{S_n})$ for all $n \geq n_0$. Thus we just take $\beta > 1$ such that $\overline{A_\beta} \subset A\big(A_{(1+s)/2}\big)$ and the first claim is proved.

From now on we fix $\alpha \in (1, \beta)$.

Second claim: There exists $r \in \big(1,(1+r_0)/2\big)$ close enough to one in order to simultaneously have $(A \circ \widehat{\psi}_n)(A_r) \subset A_\alpha$ and $(A \circ \psi_n)(A_r) \subset A_\alpha$ for all $n \geq n_0$.

Indeed, since $\overline{A_r} \subset A_{\underline{r}}$, $\widehat{\psi}_n$ is holomorphic, and $\widehat{\psi}_n(A_{\underline{r}}) \subset A_{S_n} \subset A_S$ for all $n \geq n_0$ (where $S \in (1,+\infty)$ is given by Lemma 5.2.6), we have by Cauchy derivative estimate that $\sup_{n \geq n_0}\left\{\big|\widehat{\psi}'_n(z)\big| : z \in A_r\right\}$ is finite. Since each $\widehat{\psi}_n$ preserves the unit circle, and since $\big\|\widehat{\psi}_n - \psi_n\big\|_{C^0(A_{\underline{r}})} \leq C\lambda^n$ for all $n \geq n_0$, the second claim is proved.

Another way to prove the second claim is by noting that, since $\overline{A_\alpha} \subset A_\beta \subset \overline{A_\beta} \subset A(A_{S_n})$ for all $n \geq n_0$, the hyperbolic metric on any annulus $A(A_{S_n})$ and the Euclidean metric are comparable in A_α with universal parameters, that is, there exists a constant $K > 1$ such that:

$$\left(\frac{1}{K}\right)|z-w| \leq d_{A(A_{S_n})}(z,w) \leq K|z-w|$$

for all $z, w \in A_\alpha$ and for all $n \geq n_0$, where $d_{A(A_{S_n})}$ denote the hyperbolic distance in the annulus $A(A_{S_n})$ (this is well-known, see for instance [6, Section I.4, Theorem 4.3]). Since each $A \circ \widehat{\psi}_n : A_{\underline{r}} \to A(A_{S_n})$ is holomorphic and preserves the unit circle, we know by Schwarz lemma that for all $z \in A_{\underline{r}}$ and for all $n \geq n_0$ we have:

$$d_{A(A_{S_n})}\left((A \circ \widehat{\psi}_n)(z), S^1\right) \leq d_{A_{\underline{r}}}\left(z, S^1\right),$$

where $d_{A_{\underline{r}}}$ denote the hyperbolic distance in the annulus $A_{\underline{r}}$. Since all distances $d_{A(A_{S_n})}$ are comparable with the Euclidean distance in A_δ with universal parameters, we have for all $z \in A_{\underline{r}}$ and for all $n \geq n_0$ that:

$$d\left((A \circ \widehat{\psi}_n)(z), S^1\right) \leq K d_{A_{\underline{r}}}\left(z, S^1\right),$$

where d is just the Euclidean distance in the plane. Fix $r \in \big(1,(1+r_0)/2\big)$ close enough to one (see Lemma C.2.2 in Appendix C for precise computations) in order to have that $z \in A_r$ implies $d_{A_{\underline{r}}}\left(z,S^1\right) < \frac{\alpha-1}{K\alpha}$ (and therefore $(A \circ \widehat{\psi}_n)(z) \in A_\alpha$ for all $n \geq n_0$). Again since $\big\|\widehat{\psi}_n - \psi_n\big\|_{C^0(A_{\underline{r}})} \leq C\lambda^n$ for all $n \geq n_0$, the second claim is proved.

Third claim: There exists a positive number M such that $\big|\phi'_n(z)\big| < M$ for all $z \in A_\alpha$ and for all $n \geq n_0$.

Indeed, recall that $\phi_n\big(A(A_{S_n})\big) = \big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}}_n\big) \subset A_{R_n}$ for all $n \geq n_0$. By Lemma 5.2.3 there exists a (finite) number Δ such that $\phi_n\big(A(A_{S_n})\big) \subset B(0,\Delta)$ for all $n \geq n_0$. Since $\overline{A_\alpha} \subset A_\beta \subset \overline{A_\beta} \subset A(A_{S_n})$ for all $n \geq n_0$, the third claim follows from Cauchy derivative estimate.

With the three claims at hand, Lemma 5.2.7 follows. $\square$

To control the combinatorics after perturbation we use the *monotonicity* of the rotation number:

Lemma 5.2.8. *Let f be a C^3 critical circle map and let g be a real-analytic critical circle map that extends holomorphically to the annulus:*

$$A_R = \left\{ z \in \mathbb{C} : \frac{1}{R} < \big|z\big| < R \right\} \quad \textit{for some} \quad R > 1.$$

There exists a real-analytic critical circle map h, with $\rho(h) = \rho(f)$, also extending holomorphically to A_R, where we have:

$$\big\| h - g \big\|_{C^0(A_R)} \leq d_{C^0(S^1)}\big(f, g\big).$$

In particular:

$$d_{C^r(S^1)}\big(h, g\big) \leq d_{C^0(S^1)}\big(f, g\big) \quad \textit{for any} \quad 0 \leq r \leq \infty.$$

Proof of Lemma 5.2.8. Let F and G be the corresponding lifts of f and g to the real line satisfying:

$$\rho(f) = \lim_{n \to +\infty} \frac{F^n(0)}{n} \quad \text{and} \quad \rho(g) = \lim_{n \to +\infty} \frac{G^n(0)}{n}\,.$$

Consider the band $B_R = \{z \in \mathbb{C} : -\log R < 2\pi\Im(z) < \log R\}$, which is the universal cover of the annulus A_R via the holomorphic covering $z \mapsto e^{2\pi i z}$. Let $\delta = \|F - G\|_{C^0(\mathbb{R})}$, and for any t in $[-1, 1]$ let $G_t : B_R \to \mathbb{C}$ defined as $G_t = G + t\delta$. Each G_t preserves the real line, and its restriction is the lift of a real-analytic critical circle map. Moreover, each G_t commutes with unitary horizontal translation in B_R.

Note that $\|G_t - G\|_{C^0(B_R)} = |t|\delta \leq \|F - G\|_{C^0(\mathbb{R})}$ for any $t \in [-1, 1]$. Moreover for any $x \in \mathbb{R}$ the family $\big\{G_t(x)\big\}_{t \in [-1,1]}$ is monotone in t, and we have $G_{-1}(x) \leq F(x) \leq G_1(x)$. In particular there exists $t_0 \in [-1, 1]$ such that:

$$\lim_{n \to +\infty} \frac{G_{t_0}^n(0)}{n} = \rho(F)\,,$$

and we define h as the projection of G_{t_0} to the annulus A_R. □

After the perturbation given by Lemma 5.2.8 we still have the critical point of h_n placed at 1, but its critical value is no longer placed at $g_n(1)$ (however they are exponentially close). To finish the proof of Proposition 5.2.1 we need to fix this, without changing the combinatorics of h_n in S^1. Until now each H_n is S^1-symmetric, in the sense that it commutes with $z \mapsto 1/\bar{z}$ in the annulus A_r. We will loose this property in the following perturbation, which turns out to be the last one.

Lemma 5.2.9. *For each $n \geq n_0$ consider the (unique) Möbius transformation M_n which maps the unit disk $\mathbb{D}$ onto itself fixing the basepoint $z = 1$, and which maps $H_n(1)$ to $G_n(1)$. Then there exists $\rho \in (1, r)$ such that $\overline{A_\rho} \subset M_n(A_r)$ for all $n \geq n_0$. Moreover for each $n \geq n_0$ we have:*

$$\left\| M_n \circ H_n \circ M_n^{-1} - G_n \right\|_{C^0(A_\rho)} \leq C\lambda^n.$$

Note that, when restricted to the unit circle, each M_n gives rise to an orientation-preserving real-analytic diffeomorphism which is, as Lemma 5.2.9 indicates, C^∞-exponentially close to the identity.

Proof of Lemma 5.2.9. Consider the biholomorphism $\psi : \mathbb{H} \to \mathbb{D}$ given by $\psi(z) = \frac{z-i}{z+i}$, whose inverse $\psi^{-1} : \mathbb{D} \to \mathbb{H}$ is given by $\psi^{-1}(z) = i\left(\frac{1+z}{1-z}\right)$. Note that ψ maps the vertical geodesic $\{z \in \mathbb{H} : \Re(z) = 0\}$ onto the interval $(-1, 1)$ in $\mathbb{D}$. Since ψ and ψ^{-1} are Möbius transformations, both extend uniquely to corresponding biholomorphisms of the entire Riemann sphere. The extension of ψ is a real-analytic diffeomorphism between the compactification of the real line and the unit circle, which maps the point at infinity to the point $z = 1$. For each $n \geq n_0$ consider the real number t_n defined by:

$$t_n = \psi^{-1}\big(G_n(1)\big) - \psi^{-1}\big(H_n(1)\big) = 2i\left(\frac{G_n(1) - H_n(1)}{\big(1 - G_n(1)\big)\big(1 - H_n(1)\big)}\right).$$

Each t_n is finite since for all $n \geq n_0$ both $G_n(1)$ and $H_n(1)$ are not equal to one. Moreover we claim that:

$$\inf_{n \geq n_0}\left\{\left|G_n(1) - 1\right|\right\} > 0 \quad \text{and} \quad \inf_{n \geq n_0}\left\{\left|H_n(1) - 1\right|\right\} > 0\,.$$

Indeed, since we have $\left|H_n(1) - G_n(1)\right| \leq C\lambda^n$ for all $n \geq n_0$, is enough to prove that $\inf_{n \geq n_0}\left\{\left|G_n(1) - 1\right|\right\} > 0$, and this follows by Lemma 5.1.6 since $1 = P_n(0)$ and $G_n(1) = P_n(-1)$ for all $n \geq n_0$. In particular, again using $\left|H_n(1) - G_n(1)\right| \leq C\lambda^n$ for all $n \geq n_0$, we see that $|t_n| \leq C\lambda^n$ for all $n \geq n_0$. From the explicit formula:

$$M_n(z) = \frac{(2i - t_n)z + t_n}{(2i + t_n) - t_n z} = \left(\frac{z - \left(\frac{t_n}{t_n - 2i}\right)}{1 - \left(\frac{t_n}{t_n + 2i}\right) z}\right)\left(\frac{2i - t_n}{2i + t_n}\right) \quad \text{for all } n \geq n_0,$$

we see that the pole of each M_n is at the point $z_n = 1 + i(2/t_n)$, and since $|t_n| \leq C\lambda^n$ for all $n \geq n_0$, we can take n_0 big enough to have that $z_n \in$

$\mathbb{C} \setminus \overline{B(0,2R)}$, where $R = \sup_{n \geq n_0}\{R_n\} < +\infty$ is given by Lemma 5.2.3. A straightforward computation gives:

$$\big(M_n - Id\big)(z) = \frac{t_n(z-1)^2}{(2i+t_n) - t_n z} \quad \text{for all } n \geq n_0,$$

and therefore:

$$\big\|M_n - Id\big\|_{C^0(A_R)} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

In particular for any fixed $\rho \in (1, r)$ we can choose n_0 big enough in order to have $\overline{A_\rho} \subset M_n(A_r)$ for all $n \geq n_0$. Moreover given any $z \in A_\rho$ we have:

$$\begin{aligned}\big(M_n \circ H_n \circ M_n^{-1} - G_n\big)(z) &= \big(M_n - Id\big)\big((H_n \circ M_n^{-1})(z)\big) + \big(H_n - G_n\big)(z)\\ &\quad + \big(H_n\big(M_n^{-1}(z)\big) - H_n(z)\big).\end{aligned}$$

In particular:

$$\begin{aligned}\big\|M_n \circ H_n \circ M_n^{-1} - G_n\big\|_{C^0(A_\rho)} &\leq \|M_n - Id\|_{C^0\left(H_n(A_r)\right)} + \|H_n - G_n\|_{C^0(A_\rho)}\\ &\quad + \|H_n\|_{C^1(A_r)} \big\|M_n^{-1} - Id\big\|_{C^0(A_\rho)}.\end{aligned}$$

Since $H_n(A_r) \subset A_R$ and $A_\rho \subset A_r \subset A_R$, the three terms $\|M_n - Id\|_{C^0\left(H_n(A_r)\right)}$, $\|H_n - G_n\|_{C^0(A_\rho)}$ and $\|M_n^{-1} - Id\|_{C^0(A_\rho)}$ are less or equal than $C\lambda^n$ for all $n \geq n_0$.

Finally, since each H_n is holomorphic and we have $\overline{A_r} \subset A_{\underline{r}}$ and $H_n(A_{\underline{r}}) \subset \big(G_n \circ P_n\big)\big(\widehat{\mathcal{T}_n}\big) \subset A_{R_n} \subset A_R$ for all $n \geq n_0$, we obtain from Cauchy derivative estimate that:

$$\sup_{n \geq n_0}\left\{\big\|H_n\big\|_{C^1(A_r)}\right\}$$

is finite, and therefore:

$$\big\|M_n \circ H_n \circ M_n^{-1} - G_n\big\|_{C^0(A_\rho)} \leq C\lambda^n \quad \text{for all } n \geq n_0.$$

$\square$

With Lemma 5.2.9 at hand we are done since $\big(M_n \circ H_n \circ M_n^{-1}\big)(1) = G_n(1)$. We have finished the proof of Proposition 5.2.1. $\square$

5.3 The shadowing sequence

This is the final section of Chapter 5, which is devoted to proving Theorem D. Let us recall what we have done: in Section 5.1 we constructed a suitable sequence $\{G_n\}_{n\geq n_0}$ of S^1-symmetric C^3 extensions of C^3 critical circle maps g_n to some annulus $P_n\big(\widehat{\mathcal{T}_n}\big)$. When lifted with the corresponding projection P_n (also constructed in Section 5.1) each g_n gives rise to a C^3 critical commuting pair $\big(\widehat{\eta}_n, \widehat{\xi}_n\big)$ exponentially close to $\mathcal{R}^n(f)$ and having the same combinatorics at each step (moreover, with complex extensions C^0-exponentially close to the ones of $\mathcal{R}^n(f)$ produced in Theorem 4.0.4, see Proposition 5.1.7 above for more properties).

In Section 5.2 we perturbed each G_n in a definite annulus A_r, in order to obtain a sequence of real-analytic critical circle maps, each of them having the same combinatorics as the corresponding $\mathcal{R}^n(f)$, that extend to holomorphic maps H_n exponentially close to G_n in A_r (see Proposition 5.2.1 above for more properties). Both the critical point and the critical value of each H_n coincide with the ones of the corresponding G_n, more precisely, the critical point of each H_n is at $P_n(0) = 1 \in P_n\big(V_1(n)\big) \cap S^1$, and its critical value is at $H_n(1) = G_n(1) \in P_n(V) \cap S^1 = P_n\big(B_n(V)\big) \cap S^1$. Recall also that $H_n(A_r) \subset P_n(\mathcal{T}_n)$ for all $n \geq n_0$.

In this section we lift each $H_n : A_r \to A_{R_n}$ via the holomorphic projection $P_n : \mathcal{T}_n \to A_{R_n}$ in the canonical way: let $\alpha > 0$ such that for all $n \geq n_0$ we have that:

$$\overline{N_\alpha\big([-1,0]\big) \cup N_\alpha\big([0, \widehat{\xi}_n(0)]\big)} \subset \widehat{\mathcal{T}}_n\,,$$

and that $P_n\Big(N_\alpha\big([-1,0]\big) \cup N_\alpha\big([0, \widehat{\xi}_n(0)]\big)\Big)$ is an annulus contained in A_r and containing the unit circle (the existence of such α is guaranteed by Lemma 5.1.4 and Lemma 5.1.6). Let us use the more compact notation $Z_1(n) = N_\alpha\big([-1,0]\big)$ and $Z_2(n) = N_\alpha\big([0, \widehat{\xi}_n(0)]\big)$. For each $n \geq n_0$ let $\widetilde{\eta}_n : Z_2(n) \to \mathcal{T}_n$ be the $\mathbb{R}$-preserving holomorphic map defined by the two conditions:

$$H_n \circ P_n = P_n \circ \widetilde{\eta}_n \text{ in } Z_2(n), \text{ and } \widetilde{\eta}_n(0) = -1\,.$$

In the same way let $\widetilde{\xi}_n : Z_1(n) \to \mathcal{T}_n$ be the $\mathbb{R}$-preserving holomorphic map defined by the two conditions:

$$H_n \circ P_n = P_n \circ \widetilde{\xi}_n \text{ in } Z_1(n), \text{ and } \widetilde{\xi}_n(0) = \widehat{\xi}_n(0)\,.$$

$$
\begin{tikzcd}
Z_1(n) \cup Z_2(n) \subset \mathcal{T}_n \arrow[r, "(\widetilde{\eta}_n, \widetilde{\xi}_n)"] \arrow[d, "P_n"'] & \mathcal{T}_n \arrow[d, "P_n"] \\
A_r \subset A_{R_n} \arrow[r, "H_n"] & A_{R_n}
\end{tikzcd}
$$

In the next proposition we summarize the main properties of this lift, which are all straightforward:

Proposition 5.3.1 (The shadowing sequence)**.** *For each $n \geq n_0$ the pair $f_n = (\widetilde{\eta}_n, \widetilde{\xi}_n)$ restricts to a real-analytic critical commuting pair with domains $I\big(\widetilde{\xi}_n\big) = \big[\widetilde{\eta}_n(0), 0\big] = [-1, 0]$ and $I\big(\widetilde{\eta}_n\big) = \big[0, \widetilde{\xi}_n(0)\big] = \big[0, \widehat{\xi}_n(0)\big]$, and such that $\rho(f_n) = \rho\big(\widehat{\eta}_n, \widehat{\xi}_n\big) = \rho\big(\mathcal{R}^n(f)\big) \in \mathbb{R} \setminus \mathbb{Q}$. Moreover $\widetilde{\xi}_n$ and $\widetilde{\eta}_n$ extend to holomorphic maps in $Z_1(n)$ and $Z_2(n)$ respectively where we have:*

- *$\widetilde{\xi}_n$ has a unique critical point in $Z_1(n)$, which is at the origin and of cubic type.*
- *$\widetilde{\eta}_n$ has a unique critical point in $Z_2(n)$, which is at the origin and of cubic type.*
- $\Big\|\widetilde{\xi}_n - \widehat{\xi}_n\Big\|_{C^0\left(Z_1(n) \cap \Phi_n(\widehat{X}_1(n))\right)} \leq C\lambda^n.$
- $\|\widetilde{\eta}_n - \widehat{\eta}_n\|_{C^0\left(Z_2(n) \cap \Phi_n(\widehat{X}_2(n))\right)} \leq C\lambda^n.$

With Proposition 5.3.1 at hand, Theorem D follows directly from the following consequence of Montel's theorem:

Lemma 5.3.2. *Let α be a constant in $(0,1)$ and let $\mathcal{V}$ be an $\mathbb{R}$-symmetric bounded topological disk such that $[-1, \alpha^{-1}] \subset \mathcal{V}$. Let W_1 and W_2 be topological disks whose closure is contained in $\mathcal{V}$ and such that $[-1, 0] \subset W_1$ and $[0, \alpha^{-1}] \subset W_2$. Denote by $\mathcal{K}$ the set of all normalized real-analytic critical commuting pairs $\zeta = (\eta, \xi)$ satisfying the following three conditions:*

- *$\eta(0) = -1$ and $\xi(0) \in [\alpha, \alpha^{-1}]$,*
- *$\alpha\big|\eta\big([0, \xi(0)]\big)\big| \leq \big|\xi\big([-1, 0]\big)\big| \leq \alpha^{-1}\big|\eta\big([0, \xi(0)]\big)\big|$,*
- *Both ξ and η extend to holomorphic maps (with a unique cubic critical point at the origin) defined in W_1 and W_2 respectively, where we have:*
  1. *$N_\alpha\big(\xi\big([-1, 0]\big)\big) \subset \xi(W_1)$;*

2. $N_{\alpha}\big(\eta\big([0,\xi(0)]\big)\big) \subset \eta(W_2)$*;*

3. $\xi(W_1) \cup \eta(W_2) \subset \mathcal{V}$*.*

Then $\mathcal{K}$ *is* C^{ω}*-compact.*

CHAPTER 6

Concluding remarks

The set $\mathbb{A} \subset [0,1]$ of de Faria and de Melo (see Theorem 0.3.1) is the set of rotation numbers $\rho = [a_0, a_1, ...]$ satisfying the following three properties:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j < \infty$$

$$\lim_{n\to\infty} \frac{1}{n}\log a_n = 0$$

$$\frac{1}{n}\sum_{j=k+1}^{k+n} \log a_j \leq \omega_\rho\left(\frac{n}{k}\right)$$

for all $0 < n \leq k$, where $\omega_\rho(t)$ is a positive function (that depends on the rotation number) defined for $t > 0$ such that $t\omega_\rho(t) \to 0$ as $t \to 0$ (for instante we can take $\omega_\rho(t) = C_\rho(1 - \log t)$ where $C_\rho > 0$ depends on the number).

The set $\mathbb{A}$ obviously contains all rotation numbers of bounded type, and it has full Lebesgue measure in $[0,1]$ (see Corollary A.1.6 in Section A.1.1 and [12, Appendix C]).

It is natural to ask: is there a condition on the rotation number equivalent to the $C^{1+\alpha}$ rigidity? This is not clear even in the real-analytic setting. We remark that $C^{1+\alpha}$ rigidity fails for some Diophantine rotation numbers (for instance with $\rho = [2, 2^2, 2^{2^2}, ..., 2^{2^n}, ...]$, see [12]).

As we said at the begining, it would be desirable to obtain Theorem B for C^3 critical circle maps with any irrational rotation number, but we have not been able to do this yet. The main difficulty is to control the distance of the successive renormalizations of two critical commuting pairs with a common unbounded type rotation number (compare Lemma 1.5.1). That is why we were able to prove that Theorem D implies Theorem C only for bounded type rotation numbers.

If we can prove Theorem C for any irrational rotation number, then (by Theorem 0.3.1) we can extend Theorem B to the full Lebesgue measure set $\mathbb{A}$, and using Theorem 0.3.2 (with essentially the same arguments as in [5] to obtain exponential convergence in the C^2 metric) we would be able to obtain C^1-rigidity for all rotation numbers.

Another difficult problem is the following: what can be said, in terms of smooth rigidity, for maps with finitely many non-flat critical points? More precisely, let f and g be two orientation preserving C^3 circle homeomorphisms with the same irrational rotation number, and with $k \geq 1$ non-flat critical points of odd type. Denote by $S_f = \{c_1, ..., c_k\}$ the critical set of f, by $S_g = \{c'_1, ..., c'_k\}$ the critical set of g, and by μ_f and μ_g their corresponding unique invariant measures. Beside the quantity and type of the critical points, new smooth conjugacy invariants appear: the condition $\mu_f\big([c_i, c_{i+1}]\big) = \mu_g\big([c'_i, c'_{i+1}]\big)$ for all $i \in \{1, ..., k-1\}$ is necessary (and sufficient) in order to have a conjugacy that sends the critical points of f to the critical points of g (the only one that can be smooth). Are those the unique smooth conjugacy invariants?

APPENDIX A

Topological Rigidity of Critical Circle Maps

A.1 Introduction

In this appendix we present a proof due to Jean-Christophe Yoccoz [63] of the fact that any C^3 orientation preserving circle homeomorphism, with irrational rotation number and such that all its critical points are non-flat, is topologically conjugate to the corresponding rigid rotation. This is an extension of the classical Denjoy's theorem that we state and prove before as an introduction to the techniques.

Let f be an orientation preserving circle homeomorphism, and assume that f has irrational rotation number θ. Recall that this is equivalent with the assumption that f has no periodic orbits, and this implies that the non-wandering set of f is minimal, being a Cantor set or the whole circle. As Poincaré showed f is *semi-conjugate* to the rigid rotation of angle θ (denoted by R_θ): there exists a continuous surjective map $h : S^1 \to S^1$ such that $h \circ f = R_\theta \circ h$. We can see this by taking a non-wandering point x in S^1 and considering its orbit $\mathcal{O}_f(x) = \big\{f^n(x)\big\}_{n\in\mathbb{Z}}$. The map $h_x\big(f^n(x)\big) = \exp(2\pi i n\theta)$ sends the point x to the point 1, and conjugates f with R_θ along the orbit of x.

A crucial point here is that f and R_θ are *combinatorially equivalent*, in the sense that for each $n \in \mathbb{N}$ the first n elements of the orbit of x under f are ordered in the same way as the first n elements of the orbit of 1 under the rotation R_θ (otherwise f has a periodic orbit). The combinatorial equivalence between f and R_θ implies that the map h_x extends continuously to the closure of $\mathcal{O}_f(x)$. This extension is surjective because any orbit of R_θ is dense in S^1, so we can extend h_x as a constant function in any connected component of the complement of $\overline{\mathcal{O}_f(x)}$. This gives us a semi-conjugacy h_x between f and R_θ that sends the point x to the point 1 (given any other point $z \in S^1$ we have $h_z = R_\beta \circ h_x$ with $\exp(2\pi i\beta) = 1/h_x(z)$). Note that for every $y \in S^1$ the set $h_x^{-1}\big(\{y\}\big)$ is either a closed interval or a single point.

If we know that f is minimal in the whole circle it follows that $\overline{\mathcal{O}_f(x)} = S^1$, so h_x is an homeomorphism and then f is topologically conjugate to the rotation R_θ.

In any case f is *uniquely ergodic*: there exists a unique Borel probability measure μ in S^1 such that $\mu(A) = \mu\big(f^{-1}(A)\big)$ for every Borel set $A \subset S^1$. Recall that the property of unique ergodicity is equivalent to the property that for every continuous function $\psi : S^1 \to \mathbb{R}$ the sequence of functions:

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f^j$$

converge uniformly to a constant [36, Chapter I, Section 9], that must be $\int \psi \, d\mu$. The measure μ is obtained by the pull-back of the normalized circular Lebesgue measure with any semi-conjugacy: $\mu(A) = \text{Leb}\big(h_x(A)\big)$, where Leb denote the Lebesgue measure (the Haar measure if we consider S^1 as the multiplicative group of complex numbers of modulus 1), and x is any point in the circle. Conversely, given any $x \in S^1$ we can obtain the semi-conjugacy h_x from the measure μ defining:

$$h_x(y) = \exp\big(2\pi i \mu([x, y])\big) = \int_x^y d\mu \pmod 1 .$$

Since μ is f-invariant and f has no periodic orbits, μ has no points of positive measure and this implies that h_x is continuous and surjective (of course we have $h_x \circ f = R_\theta \circ h_x$).

If we also know that f is minimal it follows that any open interval has positive μ-measure (since the support of μ is an f-invariant compact set it must be the whole circle), so h_x is an homeomorphism and f is topologically conjugate to the rotation R_θ.

Summarizing, an orientation preserving circle homeomorphism f with irrational rotation number θ is always semi-conjugate to the rigid rotation R_θ by a continuous surjective map h. If h is not a conjugacy there exists a point $y \in S^1$ such that $J = h^{-1}\big(\{y\}\big)$ is a non-degenerate closed interval. We call J a *wandering interval* since $f^n(J) \cap f^m(J) = \emptyset$ if $n \neq m \in \mathbb{Z}$, and since J is not contained in the basin of a periodic attractor.

The purpose of this appendix is to show some obstructions to the existence of wandering intervals, in terms of smoothness of the dynamics and non-flatness of the critical points. The organization is the following: in Section A.1.1 we recall the basic properties of the continued fraction expansion of an irrational number. In particular we prove that the coefficients of the continued fraction expansion of Lebesgue almost every real number in $[0,1]$ are unbounded, but their growth is less than quadratic. In Section A.1.2 we introduce the cross-ratio and prove that around a wandering interval the distortion of cross-ratio is arbitrary small as we iterate the dynamics (Lemma A.2.1). In Section A.2 we present a proof of the celebrated Denjoy's theorem stating that C^2 diffeomorphisms have no wandering intervals. In general C^1 circle diffeomorphisms may have wandering intervals, as we will see at the end of Section A.2. In Section A.3 we briefly recall the Schwarzian derivative, and in Section A.4 we prove Yoccoz's theorem and obtain the main result of his article [63] and this appendix: any C^3 orientation preserving circle homeomorphism, with irrational rotation number and such that all its critical points are non-flat, is topologically conjugate to the corresponding rigid rotation.

A.1.1 Continued fractions and return times

We briefly review in this section some classical facts about approximations of irrational numbers by continued fractions, and how this applies to circle dynamics. For any positive number θ denote by $\lfloor\theta\rfloor$ the *integer part* of θ:

$$\lfloor\theta\rfloor \in \mathbb{N} \quad \text{and} \quad \lfloor\theta\rfloor \leq \theta < \lfloor\theta\rfloor + 1$$

Define the *Gauss map* $G : [0,1] \to [0,1]$ by:

$$G(\theta) = \frac{1}{\theta} - \left\lfloor \frac{1}{\theta} \right\rfloor \quad \text{for} \quad \theta \neq 0 \quad \text{and} \quad G(0) = 0$$

For $k \geq 1$ consider $I_k = \left(\frac{1}{k+1}, \frac{1}{k}\right)$. Then G is an expanding orientation-reversing real-analytic diffeomorphism between each I_k and $(0,1)$, and the union $\bigcup_{k\geq 1} I_k$ is a Markov partition for G. After a well-known folklore theorem in one-dimensional dynamics (see [36, Chapter III, Theorem 1.2] or [45, Chapter V, Theorem 2.2]) the map G has a unique[1] invariant ergodic Borel probability ν (called the *Gauss measure*) which is equivalent to Lebesgue

[1] The map G has infinitely many invariant ergodic Borel probabilities since it has infinitely many periodic orbits (dense in $[0,1]$). Uniqueness comes from the equivalence with respect to Lebesgue measure.

measure (they share the same null sets). A straightforward computation shows that for any Borel set $A \subset [0,1]$ we have:

$$\nu(A) = \left(\frac{1}{\log 2}\right) \int_A \left(\frac{1}{1+\theta}\right) d\,\mathrm{Leb}$$

and from this explicit formula one can prove ergodicity (and even the mixing property) by hand.

Note that both $\mathbb{Q} \cap [0,1]$ and $[0,1] \setminus \mathbb{Q}$ are G-invariant. Under the action of G, all rational numbers in $[0,1]$ eventually land on the fix point at the origin, while the irrationals remain forever in the union $\bigcup_{k \geq 1} I_k$.

Definition A.1.1. The *continued fraction expansion* of an irrational number in $[0,1]$ is the sequence given by its itinerary under G according to the partition $\bigcup_{k \geq 1} I_k$.

More precisely, to any irrational number θ in $[0,1]$ we associate the sequence $\{a_n\}_{n \in \mathbb{N}}$ defined by $G^n(\theta) \in I_{a_n}$ for all $n \in \mathbb{N}$, that is:

$$a_n = \left\lfloor \frac{1}{G^n(\theta)} \right\rfloor \quad \text{for all} \quad n \in \mathbb{N}$$

Since G is expanding on the Markov partition $\bigcup_{k \geq 1} I_k$, the map h from $[0,1] \setminus \mathbb{Q}$ to $\mathbb{N}^{\mathbb{N}}$ (endowed with the product topology) that associates any irrational number to its itinerary is a well-defined homeomorphism, and therefore the action of G on $[0,1] \setminus \mathbb{Q}$ is topologically conjugate to the left shift $\sigma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ that sends $\{a_n\}_{n \in \mathbb{N}}$ to $\{a_{n+1}\}_{n \in \mathbb{N}}$:

$$\begin{tikzcd}
{[0,1] \setminus \mathbb{Q}} \arrow[r, "G"] \arrow[d, "h"'] & {[0,1] \setminus \mathbb{Q}} \arrow[d, "h"] \\
\mathbb{N}^{\mathbb{N}} \arrow[r, "\sigma"] & \mathbb{N}^{\mathbb{N}}
\end{tikzcd}$$

We will use the classical notation $\theta = \big[a_0, a_1, ..., a_n, a_{n+1}, ...\big]$. The natural numbers a_n are called the *partial quotients* of θ.

Definition A.1.2. We say that θ is of *bounded type* if there exists a constant $M > 0$ such that $a_n < M$ for all $n \in \mathbb{N}$.

Since periodic orbits of σ are dense in $\mathbb{N}^{\mathbb{N}}$ (endowed with the product topology), irrational numbers with periodic continued fraction expansion are dense in $[0,1]$, and therefore bounded type numbers are dense in $[0,1]$ as we said in the introduction of this thesis. We also mentioned, however, the following:

Lemma A.1.3. *The set of numbers of bounded type has zero Lebesgue measure in* $[0,1]$.

For a direct proof of Lemma A.1.3, with no ergodic arguments, see [28, Chapter III, Theorem 29].

Proof. Consider the increasing sequence $\{K_m\}_{m\in\mathbb{N}}$ of Cantor sets in $[0,1]$ defined by:

$$K_m = \big\{\theta \in [0,1] \setminus \mathbb{Q} : \theta = [a_0, a_1, ...] \quad \text{with} \quad a_n < m \quad \text{for all} \quad n \in \mathbb{N}\big\}.$$

It is enough to prove that $\mathrm{Leb}\big(K_m\big) = 0$ for each $m \in \mathbb{N}$, and since Gauss measure is equivalent with Lebesgue, it is enough to prove that $\nu\big(K_m\big) = 0$ for each $m \in \mathbb{N}$. This is just the ergodicity of ν under G since each K_m is G-invariant and contained in $\big(\frac{1}{m}, 1\big)$. □

The Birkhoff Ergodic Theorem [36, Chapter II, Theorem 1.1] gives us a much more precise statement:

Theorem A.1.4. *For Lebesgue almost every θ in $[0,1]$ we have that every integer $k \geq 1$ must appear infinitely many times in the continued fraction expansion of $\theta = \big[a_0, a_1, ...\big]$. Moreover if we define:*

$$\tau_n(\theta, k) = \left(\frac{1}{n}\right) \#\big\{0 \leq j < n : a_j = k\big\},$$

we have that $\big\{\tau_n(\theta,k)\big\}_{n\in\mathbb{N}}$ converges to the positive value:

$$\left(\frac{1}{\log 2}\right) \log\left(\frac{(k+1)^2}{k(k+2)}\right),$$

that only depends on k.

Proof. By definition of the continued fraction expansion:

$$\tau_n(\theta, k) = \left(\frac{1}{n}\right) \#\big\{0 \leq j < n : G^j(\theta) \in I_k\big\}$$

and by the Birkhoff Ergodic Theorem:

$$\lim_{n\to+\infty} \tau_n(\theta,k) = \nu(I_k) = \left(\frac{1}{\log 2}\right) \log\left(\frac{(k+1)^2}{k(k+2)}\right)$$

for ν almost every θ in $[0,1]$. Again by equivalence this is true for Lebesgue almost every θ in $[0,1]$. □

Since the asymptotic frequency is strictly decreasing in k one should expect that typical numbers, even having unbounded partial quotients, have slow growth:

Lemma A.1.5. *Let $\{b_n\}_{n\in\mathbb{N}}$ be any increasing sequence of positive real numbers such that $\sum_{n\in\mathbb{N}} 1/b_n < \infty$. For Lebesgue almost every $\theta = \big[a_0, a_1, ...\big]$ in $[0,1]$ we have $a_n < b_n$ for all n large enough.*

Proof. For each $n \in \mathbb{N}$ let:

$$U_n = \big\{\theta : a_n > b_n\big\} \quad \text{and} \quad V_n = \big\{\theta : a_0 > b_n\big\}$$

We want to prove that:

$$\operatorname{Leb}\left(\bigcap_{k\in\mathbb{N}}\bigcup_{n\geq k} U_n\right) = 0$$

Since $G^{-n}(V_n) = U_n$ and $V_n \subset \left(0, \frac{1}{b_n}\right)$ we get:

$$\nu(U_n) \leq \nu\left(0, \frac{1}{b_n}\right) = \left(\frac{1}{\log 2}\right)\log\left(1 + \frac{1}{b_n}\right) \leq \left(\frac{1}{\log 2}\right)\left(\frac{1}{b_n}\right)$$

for all n large enough. In particular:

$$\sum_{n\in\mathbb{N}} \nu(U_n) < \infty$$

and therefore the claim follows by Borel-Cantelli Lemma and the equivalence between Gauss and Lebesgue measures. □

Corollary A.1.6. *For Lebesgue almost every $\theta = \big[a_0, a_1, ..., a_n, a_{n+1}, ...\big]$ in $[0,1]$ we have:*

$$\lim_{n\to\infty} \frac{1}{n}\log a_n = 0 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j < \infty.$$

We also have:

$$\frac{1}{n}\sum_{j=k+1}^{k+n} \log a_j \leq C\big(\theta\big)\left[1 - \log\left(\frac{n}{k}\right)\right] \quad \textit{for all} \quad 0 < n \leq k,$$

where $C\big(\theta\big) > 0$ depends on θ.

Note that these are the three conditions in the definition of the set $\mathbb{A} \subset [0,1]$ introduced by de Faria and de Melo (defined in Chapter 6). The first and second condition follow straightforward from Lemma A.1.5 by taking, say, $b_n = n^{1+\varepsilon}$ for any $\varepsilon > 0$. The fact that the third condition holds Lebesgue almost everywhere also follows from Lemma A.1.5, but with more involved arguments [12, Proposition C.2, page 390].

Definition A.1.7. An irrational number in $[0,1]$ is said to be *Diophantine* if there exist constants $C > 0$ and $\delta \geq 0$ such that:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^{2+\delta}}\,, \tag{A.1.1}$$

for any natural numbers p and $q \neq 0$. Irrational numbers which are not Diophantine are called *Liouville* numbers.

As we said in the introduction of this thesis, an irrational number is of bounded type if it satisfies condition (A.1.1) for $\delta = 0$, that is, θ in $[0,1]$ is of bounded type if there exists $C > 0$ such that:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^{2}}\,,$$

for any natural numbers p and $q \neq 0$ (for the equivalence between this definition and the one above see [28, Chapter II, Theorem 23]). As we saw in Lemma A.1.3 the set of numbers of bounded type has zero Lebesgue measure. However for any small $\delta > 0$ in condition (A.1.1) we capture Lebesgue almost every real number in $[0,1]$:

Lemma A.1.8. *Given any $\delta > 0$ the set:*

$$D_\delta = \left\{\theta \in [0,1] : \exists\, C > 0 \quad \textit{such that} \quad \left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^{2+\delta}} \quad \forall\, p, q \in \mathbb{N}\right\}$$

has full Lebesgue measure in $[0,1]$. In particular the set of Diophantine numbers in $[0,1]$ has full Lebesgue measure.

Proof. Fix some decreasing sequence $\{C_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $C_n \to 0$ when $n \to +\infty$, and consider:

$$U_n = \left\{\theta \in [0,1] : \exists\, p, q \in \mathbb{N} \quad \text{such that} \quad \left|\theta - \frac{p}{q}\right| < \frac{C_n}{q^{2+\delta}}\right\}.$$

Note that $\{U_n\}_{n\in\mathbb{N}}$ is a decreasing sequence and:

$$\bigcap_{n\in\mathbb{N}} U_n = [0,1] \setminus D_\delta.$$

Therefore is enough to prove that $\lim_{n\to+\infty} \text{Leb}(U_n) = 0$. For that fix $n \in \mathbb{N}$ and consider for any $q \in \mathbb{N} \setminus \{0\}$:

$$U_n(q) = \left\{ \theta \in [0,1] : \exists\, p \in \{0,1,...,q-1,q\} \quad \text{such that} \quad \left| \theta - \frac{p}{q} \right| < \frac{C_n}{q^{2+\delta}} \right\}.$$

Since:

$$U_n = \bigcup_{q \in \mathbb{N}\setminus\{0\}} U_n(q) \quad \text{and} \quad \text{Leb}\big(U_n(q)\big) = \frac{2C_n}{q^{1+\delta}},$$

we obtain:

$$\text{Leb}(U_n) \leq 2C_n \left(\sum_{q \in \mathbb{N}\setminus\{0\}} \frac{1}{q^{1+\delta}} \right)$$

and this goes to zero when n goes to infinity by the choice of $\{C_n\}_{n\in\mathbb{N}}$ and the fact that $\delta > 0$. □

Incidentally we have proved that $[0,1] \setminus D_\delta$ is a residual set, in the Baire sense, since each U_n is open and dense in $[0,1]$. This proves, with the obvious adaptations, that the set of Liouville numbers is a residual set in $[0,1]$, and in particular is uncountable and dense in $[0,1]$.

Now we recall without proofs the basic arithmetical properties of the continued fraction expansion (a classical reference is the monograph [28]) and its consequences in the dynamics of circle homeomorphisms with irrational rotation number.

Definition A.1.9. We define recursively the *return times* of an irrational number θ in $[0,1]$ by:

$$q_0 = 1, \quad q_1 = a_0 = \left\lfloor \frac{1}{\theta} \right\rfloor \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1} \text{ for } n \geq 1.$$

The numbers q_n are also obtained as the denominators of the truncated expansion of order n of θ:

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, ..., a_{n-1}] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, \cfrac{1}{a_{n-1}}}}}}$$

The sequence $\left\{\frac{p_n}{q_n}\right\}_{n\in\mathbb{N}}$ converge to θ exponentially fast (see Theorem 9 and 13 in [28, Chapter I]):

$$\frac{1}{q_n(q_n+q_{n+1})}<\left|\theta-\frac{p_n}{q_n}\right|\leq\frac{1}{q_nq_{n+1}}\quad\text{for all}\quad n\in\mathbb{N},$$

and that is why the rational numbers p_n/q_n are called the *convergents* of the irrational θ. Moreover each p_n/q_n is the best possible approximation to θ by fractions with denominator at most q_n [28, Chapter II, Theorem 15]:

$$\text{If } 0<q<q_n \text{ then } \big|\theta-p_n/q_n\big|<\big|\theta-p/q\big| \text{ for any } p\in\mathbb{N}.$$

Now fix any point $x\in S^1$. The arithmetical properties of the continued fraction expansion described above imply that the iterates $\{R_\theta^{q_n}(x)\}_{n\in\mathbb{N}}$ are the closest returns of the orbit of x under the rigid rotation R_θ:

$$d\big(x,R_\theta^{q_n}(x)\big)<d\big(x,R_\theta^{j}(x)\big)\quad\text{for any}\quad j\in\{1,...,q_n-1\}$$

where d denote the standard distance in S^1. The sequence of return times $\{q_n\}$ increase at least exponentially fast as $n\to\infty$ (since $q_{n+1}=a_nq_n+q_{n-1}\geq 2q_{n-1}$), and the sequence of return distances $\{d(x,R_\theta^{q_n}(x))\}$ decrease to zero at least exponentially fast as $n\to\infty$. Moreover the sequence $\{R_\theta^{q_n}(x)\}_{n\in\mathbb{N}}$ approach the point x alternating the order:

$$R_\theta^{q_1}(x)<R_\theta^{q_3}(x)<...<R_\theta^{q_{2k+1}}(x)<...<x<...<R_\theta^{q_{2k}}(x)<...<R_\theta^{q_2}(x)<R_\theta^{q_0}(x)$$

By Poincaré's result this information remains true at the combinatorial level for any circle homeomorphism f with rotation number θ: for any $x\in S^1$ the interval $[x,f^{q_n}(x)]$ contains no other iterates $f^j(x)$ for $j\in\{1,...,q_n-1\}$, and if we denote by μ the unique invariant Borel probability of f we can say that $\mu\big([x,f^{q_n}(x)]\big)<\mu\big([x,f^{j}(x)]\big)$ for any $j\in\{1,...,q_n-1\}$. A priori we cannot say anything about the usual distance in S^1.

A.1.2 Cross-ratio distortion

Let $a<b<c<d$ be four distinct points in the real line. Let S_1 be the Möbius transformation determined by $S_1(a)=0$, $S_1(c)=1$ and $S_1(d)=\infty$. Note that S_1 has real coefficients since it preserves the real line. Define $\mathrm{Cr}_1(a,b,c,d)\in(0,1)$ as $\mathrm{Cr}_1(a,b,c,d)=S_1(b)$, that is:

$$\mathrm{Cr}_1(a,b,c,d)=\left(\frac{d-c}{c-a}\right)\left(\frac{b-a}{d-b}\right).$$

If we denote by $T=(a,d)$ and by $M=(b,c)$ we have that:

$$\mathrm{Cr}_1(a,b,c,d)=\left(\frac{|L|}{|L|+|M|}\right)\left(\frac{|R|}{|R|+|M|}\right),$$

where L and R are the components of $T\setminus M$, and $|I|$ denote the length of an interval I.

The choice of the Möbius transformation S_1 is quite arbitrary. We can consider, for instance, the Möbius transformation S_2 determined by $S_2(a)=-1$, $S_2(b)=0$ and $S_2(d)=\infty$, and define $\mathrm{Cr}_2(a,b,c,d)\in(0,+\infty)$ as $\mathrm{Cr}_2(a,b,c,d)=S_2(c)$, that is:

$$\mathrm{Cr}_2(a,b,c,d)=\left(\frac{d-a}{b-a}\right)\left(\frac{c-b}{d-c}\right).$$

As before, if we denote by $T=(a,d)$ and by $M=(b,c)$ we have that:

$$\mathrm{Cr}_2(a,b,c,d)=\frac{|M||T|}{|L||R|}.$$

Several different definitions of cross-ratio can be found in the literature, depending on the purposes of the authors. The first definition given here is the one used in [56], while the second definition is the one chosen in [37] and [12]. Of course both definitions are related by a Möbius transformation. Indeed, consider the orientation reversing real-analytic diffeomorphism $S:(0,1)\to(0,+\infty)$ given by $S(x)=\frac{1-x}{x}$, whose inverse is given by $S^{-1}(x)=\frac{1}{1+x}$. Then we have $S\big(\mathrm{Cr}_1(a,b,c,d)\big)=\mathrm{Cr}_2(a,b,c,d)$ for all $a<b<c<d$ in $\mathbb{R}$. Note that both $\mathrm{Cr}_1(a,b,c,d)$ and $\mathrm{Cr}_2(a,b,c,d)$ are invariant under Möbius transformations, that is, if S is any Möbius transformation and $a<b<c<d$ are four distinct real numbers, we have $\mathrm{Cr}_i\big(S(a),S(b),S(c),S(d)\big)=\mathrm{Cr}_i(a,b,c,d)$ for $i=1,2$. In this appendix we will work with the second definition given above. More precisely:

Definition A.1.10. Given intervals $M\subsetneq T\subset S^1$ we define the *cross-ratio* of M in T as:

$$\mathrm{Cr}[M,T]=\frac{|M||T|}{|L||R|},$$

where L and R are the components of $T\setminus M$, and $|I|$ denote the length of an interval I. Suppose now that f is an homeomorphism in T, we define the *distortion of cross-ratio* of f in M and T as:

$$\mathrm{Cr}(f,M,T)=\frac{\mathrm{Cr}[f(M),f(T)]}{\mathrm{Cr}[M,T]}.$$

We have that f preserve cross-ratio if and only if any lift to the real line $\tilde{f}$ is a real Möbius transformation: there exist real numbers a,b,c,d such that $\tilde{f}(t)=(at+b)/(ct+d)$.

A.2 Denjoy's theory for smooth diffeomorphisms

In his classical article of 1932 [7], Denjoy proved the following well-known rigidity result: any C^2 circle diffeomorphism with irrational rotation number is topologically conjugate to the corresponding rigid rotation.

Actually the original result of Denjoy (see Theorem A.2.9 below) is for C^1 diffeomorphisms such that $\log Df$ has *bounded variation*: there exists a positive constant $V(f) \in \mathbb{R}$ such that given any ordered finite partition $\{x_0, x_1, ..., x_n\}$ of the circle we have that:

$$\sum_{i=0}^{n-1} \big| \log Df(x_{i+1}) - \log Df(x_i) \big| \leq V(f) = \mathrm{var}(\log Df)\,.$$

In this case we say that f is C^{1+bv}, or that $f \in \mathrm{Diff}_+^{1+bv}(S^1)$. If $f \in \mathrm{Diff}_+^2(S^1)$ we see at once that $f \in \mathrm{Diff}_+^{1+bv}(S^1)$ by taking:

$$V(f) \geq \frac{\max_{x \in S^1} \big|f''(x)\big|}{\min_{x \in S^1} \big|f'(x)\big|}\,, \text{ or even better } V(f) \geq \int_{S^1} \left|\frac{f''(x)}{f'(x)}\right| dx\,.$$

In this section we give a proof of Denjoy's result. The proof wont be the easiest one, but it has the flavour of Yoccoz's proof in Section A.4. The following result tells us that in order to prove that $f \in \mathrm{Diff}_+^{1+bv}(S^1)$ with irrational rotation number is conjugate to the corresponding rigid rotation, we need to obtain a lower bound for the distortion of the cross-ratio of high iterates.

Lemma A.2.1. *Let $f \in \mathrm{Diff}_+^{1+bv}(S^1)$ with irrational rotation number. Suppose that f is not conjugate to the corresponding rigid rotation, and let J be a maximal wandering interval. There exists a decreasing sequence $\{T_n\}_{n\in\mathbb{N}}$ of open intervals such that:*

- $\overline{J} = \bigcap_{n\in\mathbb{N}} T_n$,
- *The family $\big\{T_n, f(T_n), ..., f^{q_{n+1}-1}(T_n)\big\}$ has multiplicity of intersection 2 for all $n \in \mathbb{N}$, and:*
- $\lim_{n\to\infty} \mathrm{Cr}(f^{q_{n+1}}, J, T_n) = 0$.

The sequence $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of return times given by the rotation number of f (see Section A.1.1).

As usual, we say that a family of intervals has *multiplicity of intersection* $k \in \mathbb{N}$ if the maximum number of intervals from the family that has non-empty intersection is k. Before entering into the proof of Lemma A.2.1 let us point out three technical results:

Lemma A.2.2 (Denjoy-Koksma inequality). *Let $f \in \mathrm{Hom}_+(S^1)$ with $\rho(f) = \theta \in \mathbb{R} \setminus \mathbb{Q}$, and let μ be its unique invariant Borel probability measure. Let $\{q_n\}_{n\in\mathbb{N}}$ be the sequence of return times given by θ, the rotation number of f. For any $\psi : S^1 \to \mathbb{R}$ (non necessarily continuous) with finite total variation* $\mathrm{var}(\psi)$ *we have:*

$$\left|\sum_{j=0}^{q_n-1} \psi\big(f^j(x)\big) - q_n \int_{S^1} \psi \, d\mu\right| \leq \mathrm{var}(\psi) \quad \textit{for all } x \in S^1 \textit{ and all } n \in \mathbb{N}.$$

Proof of Lemma A.2.2. Fix $x \in S^1$ and $n \in \mathbb{N}$. By combinatorics there exist q_n pairwise disjoint open intervals $\{I_0, I_1, ..., I_{q_n-1}\}$ in the unit circle such that $R_\theta^j(1) = e^{2\pi i j\theta} \in \overline{I_j}$ and $\mathrm{Leb}(I_j) = 1/q_n$ for all $j \in \{0, 1, ..., q_n - 1\}$ (just take the intervals determined by the q_n-roots of unity, and label them in order to have $e^{2\pi i j\theta} \in \overline{I_j}$ for all $j \in \{0, 1, ..., q_n - 1\}$). Let $h = h_x$ be the semi-conjugacy between f and R_θ that maps the point x to the point 1 (see the introduction of this appendix), and for each $j \in \{0, 1, ..., q_n - 1\}$ let $J_j = h^{-1}(I_j)$. Note that $f^j(x) \in J_j$ and $\mu(J_j) = 1/q_n$ for all $j \in \{0, 1, ..., q_n - 1\}$. Moreover $\big\{\overline{J_j}\big\}_{j=0}^{q_n-1}$ is a partition of the unit circle (modulo boundary points, whose μ-measure is zero since μ is f-invariant and f has no periodic orbits). Therefore:

$$\begin{aligned}
\left|\sum_{j=0}^{q_n-1} \psi\big(f^j(x)\big) - q_n \int_{S^1} \psi \, d\mu\right| &= \left|\sum_{j=0}^{q_n-1} \left(\psi\big(f^j(x)\big) - q_n \int_{J_j} \psi \, d\mu\right)\right| \\
&\leq \sum_{j=0}^{q_n-1} \left|\psi\big(f^j(x)\big) - q_n \int_{J_j} \psi \, d\mu\right| \\
&= q_n \sum_{j=0}^{q_n-1} \left|\int_{J_j} \big(\psi\big(f^j(x)\big) - \psi\big) d\mu\right| \\
&\leq q_n \sum_{j=0}^{q_n-1} \int_{J_j} \big|\psi\big(f^j(x)\big) - \psi\big| d\mu \\
&\leq \sum_{j=0}^{q_n-1} \sup_{y \in J_j} \big|\psi\big(f^j(x)\big) - \psi(y)\big| \leq \mathrm{var}(\psi)\,.
\end{aligned}$$

□

Lemma A.2.3. *Let $f \in \mathrm{Diff}^1_+(S^1)$ with irrational rotation number, and let μ be its unique invariant Borel probability measure. Then:*

$$\int_{S^1} \log Df \, d\mu = 0 \,.$$

Proof of Lemma A.2.3. If $f \in \mathrm{Diff}^1_+(S^1)$ the function $\psi : S^1 \to \mathbb{R}$ defined by $\psi = \log Df$ is a continuous function and therefore, by the unique ergodicity of f, the sequence of functions:

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f^j$$

converge uniformly to a constant [36, Chapter I, Section 9], that must be $\int_{S^1} \log Df \, d\mu$. By the chain rule:

$$\sum_{j=0}^{n-1} \psi \circ f^j = \log(Df^n) \,. \tag{A.2.1}$$

Therefore the sequence of continuous functions $\log(Df^n)/n$ converge to the constant $\int_{S^1} \log Df \, d\mu$ uniformly in S^1. Since f^n is a diffeomorphism for all $n \in \mathbb{N}$, this constant must be zero. □

If $f \in \mathrm{Diff}^{1+bv}_+(S^1)$ we can put together Lemma A.2.2 and Lemma A.2.3 to obtain that the sequence of iterates $\{f^{q_n}\}_{n\in\mathbb{N}}$ is uniformly Lipschitz (and in particular equicontinuous) on the whole circle:

Corollary A.2.4. *If $f \in \mathrm{Diff}^{1+bv}_+(S^1)$ with irrational rotation number, then $e^{-V(f)} \leq \big(Df^{q_n}\big)(x) \leq e^{V(f)}$ for all $x \in S^1$ and all $n \in \mathbb{N}$, where $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of return times given by the rotation number of f.*

Proof of Corollary A.2.4. Apply Lemma A.2.2 with $\psi = \log Df$, and use (A.2.1) and Lemma A.2.3. □

With this at hand we are ready to prove Lemma A.2.1:

Proof of Lemma A.2.1. Let $J = (a, b)$ be a maximal wandering interval of f, and let $n \in \mathbb{N}$. The complement of the union of $\overline{J}$ and $f^{q_n}(\overline{J})$ are two open intervals. By combinatorics, the interval $f^{q_{n+1}+q_n}(J)$ is contained in one of them, and the interval $f^{q_{n+1}}(J)$ is contained in the other one. Since J is maximal as a wandering interval both a and b are recurrent for the future, and therefore the distance between $\overline{J}$ and $f^{q_n}(\overline{J})$ goes to zero as n goes to infinity. Let us suppose that for the fixed integer n we have that $f^{q_{n+1}+q_n}(J)$

is contained in the small component (this depends if n is even or odd, and the other case can be treated in the same way). Let $L_n = \big(f^{-q_n}(a), a\big)$, $R_n = \big(b, f^{q_n}(a)\big)$ and:

$$T_n = L_n \cup \overline{J} \cup R_n = \big(f^{-q_n}(a), f^{q_n}(a)\big).$$

By definition:

$$\mathrm{Cr}(f^{q_{n+1}}, J, T_n) = |L_n||R_n||f^{q_{n+1}}(T_n)| \left(\frac{|f^{q_{n+1}}(J)|}{|f^{q_{n+1}}(L_n)|}\right)\left(\frac{1}{|f^{q_{n+1}}(R_n)||J||T_n|}\right).$$

The key combinatorial point is that $J \subset f^{q_{n+1}}(R_n)$, and so we have $|f^{q_{n+1}}(R_n)| \geq |J|$. Since we also have $|T_n| \geq |J|$ and $|f^{q_{n+1}}(T_n)| \leq 1$ we obtain:

$$\mathrm{Cr}(f^{q_{n+1}}, J, T_n) \leq |R_n||f^{q_{n+1}}(J)| \left(\frac{|L_n|}{|f^{q_{n+1}}(L_n)|}\right)\left(\frac{1}{|J|^3}\right). \tag{A.2.2}$$

Estimate (A.2.2) holds for any homeomorphism f with irrational rotation number. Now we use the smoothness condition. By Corollary A.2.4 the sequence:

$$\frac{|L_n|}{|f^{q_{n+1}}(L_n)|}$$

is bounded from above. Since $\sum_{n\in\mathbb{Z}} |f^n(J)| \leq 1$ we have that $|f^{q_{n+1}}(J)| \to 0$ as n goes to infinity, and since J is maximal (as a wandering interval) we have that $|R_n| \to 0$ as n goes to infinity. This proves that:

$$\lim_{n\to\infty} \mathrm{Cr}(f^{q_{n+1}}, J, T_n) = 0\,.$$

Since T_n is of the form $(f^{-q_n}(a), f^{q_n}(a))$, we know by combinatorics that the family:

$$\big\{T_n, f(T_n), ..., f^{q_{n+1}-1}(T_n)\big\}$$

has multiplicity of intersection 2 for all $n \in \mathbb{N}$. □

A.2.1 The weak version

Theorem A.2.5. *Let f be a C^1 orientation preserving circle diffeomorphism such that $\log Df$ is a Lipschitz function. If f has irrational rotation number θ then f is topologically conjugate to the rigid rotation of angle θ.*

The proof of Theorem A.2.5 that we are giving below is based on Corollary A.2.7, which is contained in the work of Schwartz [53] of 1963. As Denjoy, Schwartz was interested in the dynamics of smooth flows on compact surfaces, where one-dimensional dynamics appear when considering first-return maps of local transverse sections.

Lemma A.2.6 (Bounded Distortion)**.** *Let f be a C^1 orientation preserving circle diffeomorphism such that $\log Df$ is a Lipschitz function of constant $C > 0$. Then:*

$$\exp\left(-C\sum_{i=0}^{n-1}|f^i(x)-f^i(y)|\right) \leq \frac{|Df^n(x)|}{|Df^n(y)|} \leq \exp\left(C\sum_{i=0}^{n-1}|f^i(x)-f^i(y)|\right)$$

for any $x, y \in S^1$ and for any $n \in \mathbb{N}$.

The proof of this lemma is an easy computation. What is important for us is the following consequence:

Corollary A.2.7. *Let f be a C^1 orientation preserving circle diffeomorphism such that $\log Df$ is a Lipschitz function. Let $J \subset S^1$ be an interval such that:*

$$\sum_{n\in\mathbb{N}}|f^n(J)| < \infty$$

Then there exists an open interval $T \supsetneq \overline{J}$ such that:

$$\sum_{n\in\mathbb{N}}|f^n(T)| < \infty$$

Proof. We will construct an open interval $T \supsetneq \overline{J}$ such that $|f^n(T)| \leq 2|f^n(J)|$ for all $n \in \mathbb{N}$. Fix some $\delta > 0$ and take $T \supsetneq \overline{J}$ such that $|T| \leq (1+\delta)|J|$. We claim that it is enough to take:

$$0 < \delta \leq \exp\left(-2C\sum_{n\in\mathbb{N}}|f^n(J)|\right) < 1$$

where $C > 0$ is the Lipschitz constant of $\log Df$. The proof goes by induction on n (the case $n = 0$ is done since $\delta < 1$): fix some $n \in \mathbb{N}$ and suppose that $|f^i(T)| \leq 2|f^i(J)|$ for every $i \in \{0, 1, ..., n-1\}$. By the Mean Value Theorem $|f^n(T \setminus J)| \leq |Df^n(x)||T \setminus J|$ for some $x \in T \setminus J$, and $|f^n(J)| = |Df^n(y)||J|$ for some $y \in J$. With this we have:

$$|f^n(T \setminus J)| \leq |Df^n(x)||T \setminus J| = \frac{|Df^n(x)|}{|Df^n(y)|}\frac{|f^n(J)|}{|J|}|T \setminus J|$$

By Lemma A.2.6 (Bounded Distortion Lemma):

$$\begin{aligned}
|f^n(T \setminus J)| &\leq \exp\left(C\sum_{i=0}^{n-1}|f^i(x)-f^i(y)|\right)\frac{|f^n(J)|}{|J|}|T\setminus J| \\
&= \exp\left(C\sum_{i=0}^{n-1}|f^i(x)-f^i(y)|\right)\frac{|T\setminus J|}{|J|}|f^n(J)| \\
&\leq \delta\exp\left(C\sum_{i=0}^{n-1}|f^i(T)|\right)|f^n(J)| \\
&\leq \delta\exp\left(2C\sum_{i=0}^{n-1}|f^i(J)|\right)|f^n(J)| \\
&\leq \delta\exp\left(2C\sum_{i\in\mathbb{N}}|f^i(J)|\right)|f^n(J)| \\
&\leq |f^n(J)|
\end{aligned}$$

and so we have $|f^n(T)| = |f^n(J)| + |f^n(T \setminus J)| \leq 2|f^n(J)|$. $\square$

From Lemma A.2.6 we also have the following easy corollary:

Corollary A.2.8 (Cross-ratio distortion principle)**.** *Let f be a C^1 orientation preserving circle diffeomorphism such that $\log Df$ is a Lipschitz function of constant $C > 0$. Let $J \subsetneq T \subset S^1$ intervals and let $n \in \mathbb{N}$. Then:*

$$\operatorname{Cr}(f^n, J, T) \geq \exp\left(-2C\sum_{i=0}^{n-1}|f^i(T)|\right)$$

Proof. By the Mean Value Theorem we have four points $x \in J, y \in T, z, w \in T \setminus J$ such that:

$$\operatorname{Cr}(f^n, J, T) = \frac{|Df^n(x)||Df^n(y)|}{|Df^n(z)||Df^n(w)|}$$

Now we apply the Bounded Distortion Lemma to obtain:

$$\begin{aligned}
\operatorname{Cr}(f^n, J, T) &\geq \exp\left(-C\sum_{i=0}^{n-1}|f^i(x)-f^i(z)| - C\sum_{i=0}^{n-1}|f^i(y)-f^i(w)|\right) \\
&\geq \exp\left(-2C\sum_{i=0}^{n-1}|f^i(T)|\right)
\end{aligned}$$

$\square$

We remark that these kind of estimates (and even better) hold under less regularity conditions in the dynamics [45, Chapter IV, Section 2]. We are ready to prove Theorem A.2.5:

Proof of Theorem A.2.5. Suppose that f is not conjugated to the rotation and let J be a maximal wandering interval of f. In particular we have $\sum_{n\in\mathbb{Z}}|f^n(J)| \leq 1 < \infty$. Let $\{T_n\}_{n\in\mathbb{N}}$ be any sequence such that: $\bigcap_{n\in\mathbb{N}} T_n = \overline{J}$, and let T be the interval given by Corollary A.2.7 (note that we can suppose that $T_n \subset T$ for all $n \in \mathbb{N}$). By the corollary above:

$$\begin{aligned}\operatorname{Cr}(f^{q_{n+1}}, J, T_n) &\geq \exp\left(-2C\sum_{i=0}^{q_{n+1}-1}|f^i(T_n)|\right)\\ &\geq \exp\left(-2C\sum_{i=0}^{q_{n+1}-1}|f^i(T)|\right)\\ &\geq \exp\left(-2C\sum_{i\in\mathbb{N}}|f^i(T)|\right)\end{aligned}$$

and this is a positive constant by Corollary A.2.7. This contradicts Lemma A.2.1. □

Actually we do not need cross-ratio arguments to prove Theorem A.2.5: let $J = (a, b)$ be a maximal wandering interval of f, and note that a and b are recurrent since the non-wandering set of f is minimal. Let T be the interval given by Corollary A.2.7 and let $n \in \mathbb{N}$ such that $f^n(a) \in T$. Since we can take n as big as we want eventually we will have $f^n(T) \subset T$ and this give us a contradiction since f has no periodic orbits.

A.2.2 The strong version

As we said before, the original result of Denjoy is for C^1 diffeomorphisms such that $\log Df$ has bounded variation. Note that if $\log Df$ is a Lipschitz function of constant $C > 0$, then $\log Df$ has bounded variation with constant $V = C$. A classical example of an homeomorphism of bounded variation which is not Lipschitz is given by $t \mapsto \sqrt{t}$ in $[0, 1]$, so the following is a stronger result than Theorem A.2.5:

Theorem A.2.9 (Denjoy, 1932). *Let f be a C^1 orientation preserving circle diffeomorphism such that $\log Df$ has bounded variation. If f has irrational rotation number θ then f is topologically conjugate to the rigid rotation of angle θ.*

The proof is obtained combining Lemma A.2.1 with the following estimate (compare with Lemma A.4.5 in Section A.4):

Lemma A.2.10. *Let f be a C^1 orientation preserving circle diffeomorphism with irrational rotation number and such that $\log Df$ has bounded variation with constant $V > 0$. There exists a positive constant $\delta = \delta(V) > 0$ such that given any interval J and any sequence $\{T_n\}_{n \in \mathbb{N}}$ of intervals containing J such that for any $n \in \mathbb{N}$ the first $q_{n+1} - 1$ iterates of T_n have multiplicity of intersection 2, we have that:*

$$\operatorname{Cr}(f^k, J, T_n) \geq \delta \quad \text{for any} \quad k \in \{0, \dots, q_{n+1}\}$$

Proof. Fix $n \in \mathbb{N}$ and $k \in \{0, \dots, q_{n+1}\}$, and note the following chain rule:

$$\operatorname{Cr}\big(f^k, J, T_n\big) = \prod_{i=0}^{k-1} \operatorname{Cr}\big(f, f^i(J), f^i(T_n)\big).$$

In particular:

$$\big|\log\big(\operatorname{Cr}(f^k, J, T_n)\big)\big| \leq \sum_{i=0}^{k-1} \big|\log\big(\operatorname{Cr}(f, f^i(J), f^i(T_n))\big)\big|.$$

By the Mean Value Theorem for any $i \in \{0, \dots, k-1\}$ there exist four points: $x_i \in f^i(J), y_{i,n} \in f^i(T_n), z_{i,n}, w_{i,n} \in f^i(T_n) \setminus f^i(J)$ such that:

$$\operatorname{Cr}\big(f, f^i(J), f^i(T_n)\big) = \frac{|Df(x_i)||Df(y_{i,n})|}{|Df(z_{i,n})||Df(w_{i,n})|}.$$

Therefore:

$$\begin{aligned}
\big|\log\big(\operatorname{Cr}(f^k, J, T_n)\big)\big| &\leq \sum_{i=0}^{k-1} \big|\log Df(x_i) + \log Df(y_{i,n}) - \log Df(z_{i,n}) - \log Df(w_{i,n})\big| \\
&\leq \sum_{i=0}^{k-1} \big|\log Df(x_i) - \log Df(z_{i,n})\big| + \big|\log Df(w_{i,n}) - \log Df(y_{i,n})\big|
\end{aligned}$$

Now consider the finite partition $\mathcal{P}_k = \big\{x_i, y_{i,n}, z_{i,n}, w_{i,n}\big\}_{i=0}^{k-1}$. Since the family $\big\{T_n, f(T_n), \dots, f^{q_{n+1}-1}(T_n)\big\}$ has multiplicity of intersection 2, the last term is less or equal than the double of the total variation of $\log Df$ in $\mathcal{P}_k$, and so we are done by taking $\delta = \exp(-2V)$. □

In [7] Denjoy also proved that some assumptions on the first derivative are needed: given any irrational number θ there exists a C^1 circle diffeomorphism with rotation number θ and wandering intervals [45, Chapter I, Section 2]. We remark that there are counterexamples even if the derivative is Hölder continuous [20]. See also [22] for weaker conditions on the first derivative.

A.2.3 The geometric classification

We finish Section A.2 with some remarks: let f be a circle homeomorphism topologically conjugate to R_θ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let μ be the unique Borel probability in S^1 invariant under f. One motivation to understand the measure μ is given by the Birkhoff Ergodic Theorem [36, Chapter II, Theorem 1.1]: given any point $x \in S^1$ and any interval $A \subset S^1$ we have that:

$$\lim_{n \to +\infty} \left[\left(\frac{1}{n} \right) \# \big\{ j : 0 \leq j < n \quad \text{and} \quad f^j(x) \in A \big\} \right] = \mu(A).$$

Note that if f is C^1 we have the following dichotomy: either μ is *absolutely continuous* with respect to Lebesgue, or μ is *singular* (otherwise we have a decomposition $\mu = \nu_1 + \nu_2$, where ν_1 is absolutely continuous with respect to Lebesgue, ν_2 is singular, and both are non-zero. Since f is C^1 it preserves sets of Lebesgue measure zero, and therefore both ν_1 and ν_2 are f-invariant, which contradicts the unique ergodicity of f).

As we saw in the introduction of this appendix $\mu(A) = \operatorname{Leb}\big(h(A)\big)$ for any Borel set $A \subset S^1$, where h is a circle homeomorphism that conjugates f with R_θ. In particular μ has no atoms (i.e. no points of positive measure) and gives positive measure to any open set. If h is absolutely continuous[2] there exists a Lebesgue integrable function h' such that $\mu(A) = \int_A h' d \operatorname{Leb}$, but in general this is not true: in 1961 Arnold gave examples ([3], see also [45, Chapter I, Section 5]) of real-analytic circle diffeomorphisms that are minimal, but where the invariant probability μ is not absolutely continuous with respect to Lebesgue measure. In these examples the rotation number is Liouville (see Definition A.1.7), and any conjugacy with the corresponding rigid rotation maps a set of zero Lebesgue measure in a set of positive Lebesgue measure.

In the same work Arnold showed (using KAM methods) that any real-analytic diffeomorphism with Diophantine rotation number, which is a small perturbation of a rigid rotation, is conjugate to the corresponding rigid rotation by a real-analytic diffeomorphism, and therefore the conjugacy can be taken holomorphic in a neighbourhood of the circle. He also conjectured that no restriction on being close to a rotation is needed. As we said in the introduction of this thesis, this was proved by Herman in 1979 [20] for a large class of Diophantine numbers, and extended by Yoccoz in 1984 [62] for all Diophantine numbers. Nowadays the picture is completely clear: any $C^{2+\varepsilon}$

[2] We say that $h : I \to \mathbb{R}$ is *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |h(\beta_i) - h(\alpha_i)| < \varepsilon$ for any $n \in \mathbb{N}$ and any disjoint collection of segments $(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)$ in I whose lengths satisfy $\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta$. In our context this is equivalent to the statement that h maps sets of Lebesgue measure zero to sets of Lebesgue measure zero.

circle diffeomorphism with rotation number θ satisfying:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^{2+\delta}}$$

for every positive coprime integers p and q, with $C > 0$ and $\delta \in [0, 1)$, is conjugated to the corresponding rigid rotation by a C^1 circle diffeomorphism. Even more, if $0 \leq \delta < \varepsilon \leq 1$ and $\varepsilon - \delta \neq 1$, the conjugacy is $C^{1+\varepsilon-\delta}$ (see also [27] and the references given there). This implies, in particular, that its unique invariant probability is Lebesgue absolutely continuous, and its density is Hölder continuous with exponent $\varepsilon - \delta$. Moreover C^∞ diffeomorphisms with the same Diophantine rotation number are C^∞-conjugate, and real-analytic diffeomorphisms with the same Diophantine rotation number are conjugate by a real-analytic diffeomorphism [45, Chapter I, Section 3].

Now for any $x \in S^1$ and $n \in \mathbb{N}$ consider the *n-th scaling ratio* of f and R_θ in x defined as:

$$s_n(f) = \frac{d\big(f^{q_{n+1}}(x), x\big)}{d\big(f^{q_n}(x), x\big)} \quad \text{and} \quad s_n(\theta) = \frac{d\big(R_\theta^{q_{n+1}}(x), x\big)}{d\big(R_\theta^{q_n}(x), x\big)}$$

where d denote the standard distance in S^1 (note that for all $n \geq 1$ we have $s_{n-1}(\theta) = \frac{1}{a_n + s_n(\theta)}$). Herman's result also implies the following *asymptotic geometric rigidity*:

$$\lim_{n \to +\infty} \big(s_n(f) - s_n(\theta)\big) = 0$$

provided that θ is Diophantine. See Corollary 0.2.1 for the analogue result in the context of critical circle maps.

A.3 Schwarzian derivative

In this section we briefly recall some basic definitions.

A.3.1 Non-flat critical points

Let I be a compact interval or the whole circle, and let $f : I \to I$ be a C^1 map. As usually we say that $c \in I$ is a *critical point* of f if $Df(c) = 0$. If for every $d \in \mathbb{N}$ we have that:

$$\lim_{t \to 0} \frac{Df(c+t)}{t^{d-1}} = 0 \tag{A.3.1}$$

we say that c is a *flat* critical point of f. This implies that f is C^∞ at the critical point and $(D^d f)(c) = 0$ for all $d \geq 1$, where $D^d f$ denote the derivative of f of order d (in particular flat critical points do not exist for non-constant real-analytic maps). A classical example of this phenomena is given by $f: \mathbb{R} \to \mathbb{R}$ defined as:

$$f(t) = \begin{cases} \exp(-1/t^2) & \text{for } t \in \mathbb{R} \setminus \{0\} \\ 0 & \text{for } t = 0 \end{cases}$$

Note that f is a C^∞ map with a flat critical point at the origin (the holomorphic extension of $\exp(-1/t^2)$ to the punctured plane $\mathbb{C} \setminus \{0\}$ presents an *essential* singularity: in any neighbourhood of the origin the function takes any complex value different from zero).

Otherwise we say that c is a *non-flat* critical point of order d, where d is the minimum integer such that:

$$\lim_{t \to 0} \frac{Df(c+t)}{t^{d-1}} \neq 0 \tag{A.3.2}$$

Note that d is also the minimum integer such that:

$$\int_{U \setminus \{c\}} \big(Df(t)\big)^{-\frac{1}{d}} dt < \infty$$

for some neighbourhood U of the critical point.

This implies that f is C^d at the critical point where we have $(D^n f)(c) = 0$ for all $n \in \{1, ..., d-1\}$ and $(D^d f)(c) \neq 0$. In this case the limit in A.3.2 is equal to $\frac{(D^d f)(c)}{(d-1)!}$ and we can say more:

Lemma A.3.1. *The critical point c is non-flat of order d if and only if for any $C > 1$ there exists an open neighbourhood U of c such that for any $t_0, t_1 \in U \setminus \{c\}$ we have:*

$$C^{-1} \left| \frac{t_0 - c}{t_1 - c} \right|^d \leq \left| \frac{f(t_0) - f(c)}{f(t_1) - f(c)} \right| \leq C \left| \frac{t_0 - c}{t_1 - c} \right|^d$$

Proof. For any given $C > 1$ consider:

$$\varepsilon(C) = \left(\frac{\big|(D^d f)(c)\big|}{d!} \right) \left(\frac{C-1}{C+1} \right)$$

Since $(D^d f)(c) \neq 0$ this is an orientation preserving real-analytic diffeomorphism between $(1, +\infty)$ and $\left(0, \frac{\left|(D^d f)(c)\right|}{d!}\right)$ with inverse given by:

$$C(\varepsilon) = \frac{\frac{\left|(D^d f)(c)\right|}{d!} + \varepsilon}{\frac{\left|(D^d f)(c)\right|}{d!} - \varepsilon} = \frac{\big|(D^d f)(c)\big| + d!\varepsilon}{\big|(D^d f)(c)\big| - d!\varepsilon}$$

By Taylor theorem:

$$\lim_{t\to c}\left(\frac{\big|f(t)-f(c)\big|}{|t-c|^d}\right)=\frac{\big|(D^d f)(c)\big|}{d!}\neq 0$$

Now for any given $C>1$ let $\varepsilon(C)$ in $\left(0,\frac{\left|(D^d f)(c)\right|}{d!}\right)$ as above, and for this ε let U be an open neighbourhood of c in I such that for any $t\in U\setminus\{c\}$ we have:

$$\frac{\big|(D^d f)(c)\big|}{d!}-\varepsilon\leq\frac{\big|f(t)-f(c)\big|}{|t-c|^d}\leq\frac{\big|(D^d f)(c)\big|}{d!}+\varepsilon$$

Equivalently:

$$\left(\frac{\big|(D^d f)(c)\big|}{d!}-\varepsilon\right)|t-c|^d\leq\big|f(t)-f(c)\big|\leq\left(\frac{\big|(D^d f)(c)\big|}{d!}+\varepsilon\right)|t-c|^d$$

Now for any t_0,t_1 in $U\setminus\{c\}$ we have the desired estimates for the given C. For the converse fix a neighbourhood U of c such that for any t in U:

$$f(t)-f(c)=\sum_{n=1}^{n=r}\left(\frac{\big(D^n f\big)(c)}{n!}\right)(t-c)^n+r(t)$$

with $\lim_{t\to c}\left(\frac{r(t)}{(t-c)^d}\right)=0$. By hypothesis the ratio:

$$\frac{f(t)-f(c)}{(t-c)^d}=\sum_{n=1}^{n=r}\left(\frac{\big(D^n f\big)(c)}{n!}\right)(t-c)^{n-d}+\left(\frac{r(t)}{(t-c)^d}\right)$$

is bounded in $U\setminus\{c\}$. The fact that is not going to infinity implies that $\big(D^n f\big)(c)=0$ for $n\in\{1,...,d-1\}$, and the fact that is not going to zero implies that $\big(D^d f\big)(c)\neq 0$. $\square$

If the map is smooth enough (C^r for $r\geq d$) we can go further (with essentially the same computations as in the proof of Lemma 4.1.1 in Chapter 4):

Lemma A.3.2. *The critical point c is non-flat of order d if and only if there exist C^r local diffeomorphisms ϕ and ψ with $\phi(c)=\psi\big(f(c)\big)=0$ such that $\psi\circ f\circ\phi^{-1}$ is the map $t\mapsto t^d$ around zero.*

A.3.2 The Schwarzian derivative

Now suppose that $r \geq 3$ and define the *Schwarzian derivative* in every non-critical point of f as:

$$Sf(x) = \frac{\big(D^3 f\big)(x)}{\big(Df\big)(x)} - \frac{3}{2}\left|\frac{\big(D^2 f\big)(x)}{\big(Df\big)(x)}\right|^2$$

Maybe, the main motivation of Schwarz for consider this operator [52, Chapter 1] was the following:

Lemma A.3.3. *The kernel of the Schwarzian derivative is the group of Möbius transformations, that is, $Sf \equiv 0$ if and only if there exist real numbers a, b, c, d such that $f(t) = (at+b)/(ct+d)$.*

Proof of Lemma A.3.3. The fact that the Schwarzian derivative vanish at Möbius transformations is a straightforward computation. Now, given a C^3 map f without critical points on some interval I, consider the C^2 map g defined by $g = (Df)^{-1/2}$. A straightforward computation gives the identity:

$$Sf = -2\left(\frac{D^2 g}{g}\right).$$

In particular $Sf \equiv 0$ if and only if $D^2 g \equiv 0$, and therefore there exist real numbers a and b such that $g(t) = at + b$, that is, $Df(t) = 1/(at+b)^2$. By integration we get:

$$f(t) = \left(\frac{-1}{a}\right)\left(\frac{1}{at+b}\right) + c\,,$$

for some real number c. □

Now we recall well-known properties of the Schwarzian derivative, for proofs see [45, Chapter II, Section 6] and [45, Chapter IV, Section 1]:

- For $k \geq 1$ we have the following chain rule:

  $$Sf^k(x) = \sum_{j=0}^{k-1} Sf\big(f^j(x)\big) \cdot |Df^j(x)|^2 = \sum_{j=0}^{k-1} Sf\big(f^j(x)\big) \cdot \prod_{i=0}^{j-1} |Df\big(f^i(x)\big)|^2$$

  In particular $Sf^k(x)$ depends only on the values of Sf and Df along the first k iterates of the point x under the map f.

- From the chain rule we see that if a map has negative Schwarzian derivative, all its iterates also have negative Schwarzian derivative.

- Trivial but important, any quadratic polynomial has negative Schwarzian derivative.

- If c is a non-flat critical point of f, there exists an open neighbourhood U of c such that $Sf(x) < 0$ for all $x \in U \setminus \{c\}$.

- If f is monotone on an interval $J \subset I$, and $Sf < 0$ in J, then $|Df|$ does not have a positive local minimum on J. In particular, if f have no critical points in J and $x < y < z$ are in J then $|Df(y)| > \min\{|Df(x)|, |Df(z)|\}$.

- From Lemma A.3.3 we see that a C^3 diffeomorphism f preserve cross-ratio (see Section A.1.2) if and only if $Sf \equiv 0$. The relation between Schwarzian derivative and cross-ratio distortion is deeper: f increase cross-ratio if and only if $Sf < 0$, and decrease cross-ratio if and only if $Sf > 0$.

A.4 Yoccoz's proof

In terms of distortion estimates, the importance of the non-flatness condition on the critical points become clear if we restate Lemma A.3.1 for critical circle maps:

Corollary A.4.1 (Corollary of Lemma A.3.1)**.** *Let f be a C^r critical circle map, $r \geq 3$, with irrational rotation number, and let $c \in S^1$ be a non-flat critical point of f of order $d \leq r$. For any $C > 1$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$:*

$$C^{-1}\big(s_n(f)\big)^d \leq \frac{\big|f(I_{n+1})\big|}{\big|f(I_n)\big|} \leq C\big(s_n(f)\big)^d$$

where $s_n(f) = \frac{|I_{n+1}|}{|I_n|} = \frac{d\left(f^{q_{n+1}}(c),c\right)}{d\left(f^{q_n}(c),c\right)}$ is the n-th scaling ratio *of f (see Section A.2.3), and the sequence $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of return times given by the rotation number of f (see Section A.1.1).*

In 1984 Yoccoz [63] extended Denjoy's result to critical circle maps with only non-flat critical points:

Theorem A.4.2. *Let f be a C^1 orientation preserving circle homeomorphism with irrational rotation number θ and with $k \geq 1$ critical points $c_1, c_2, ..., c_k$. Suppose that:*

1. $\log Df$ *has bounded variation in any compact interval that contains no critical points.*

2. *Given any* $j \in \{1, 2, ..., k\}$ *there exist constants* $\varepsilon_j, A_j, B_j > 0$ *and* $s_j \in \mathbb{N}$ *such that:*

   - $(Df)^{-1/2}$ *is a convex*[3] *function in* $(c_j - \varepsilon_j, c_j)$ *and in* $(c_j, c_j + \varepsilon_j)$.
   - *Given* $|t| < \varepsilon_j$ *we have that* $A_j|t|^{s_j} \leq (Df)(c_j + t) \leq B_j|t|^{s_j}$.

Then f is topologically conjugate to the rigid rotation of angle θ.

Some remarks about the second condition:

- If f is C^3 then $g = (Df)^{-1/2}$ is C^2 away from the critical points of f, and $D^2g = (-1/2)gSf$, where Sf denote the Schwarzian derivative of f (see Lemma A.3.3). In particular g is a strictly convex function if and only if $Sf < 0$, and this always happens in a neighbourhood of a non-flat critical point, as we said in the previous section.

- As we also saw in the previous section, if f is C^r and c_j is a non-flat critical point of order d for some $j \in \{1, 2, ..., k\}$ and some $2 \leq d \leq r$, then the numbers A_j and B_j can be chosen defining a small open neighbourhood of $\frac{(D^d f)(c_j)}{(d-1)!}$, and the exponent s_j is equal to $d - 1$ (the last derivative vanishing at the critical point).

With this we conclude the following:

Corollary A.4.3. *Any C^3 orientation-preserving circle homeomorphism with irrational rotation number and only non-flat critical points is topologically conjugate to the corresponding rigid rotation.*

In particular any C^3 critical circle map (with irrational rotation number and only non-flat critical points) is minimal and so the support of its unique invariant Borel probability μ is the whole circle. As we said in the introduction of this thesis, however, in the case of exactly one critical point it has been proved that the measure μ is always *singular* with respect to Lebesgue measure: there exists a Borel set $A \subset S^1$ such that $\mu(A) = 1$ and $\text{Leb}(A) = 0$ (see [25, Theorem 4, page 182] or [18, Proposition 1, page 219]). We recall again that the condition of non-flatness on the critical points cannot be removed: in [19] Hall constructs C^∞ homeomorphisms of the circle with no periodic

[3] We say that $g : I \to \mathbb{R}$ is a *convex* function if for any $a < b \in I$ and any $t \in [a, b]$ we have the inequality: $g(t) - g(a) \leq (t - a)\left(\frac{g(b)-g(a)}{b-a}\right)$.

points and no dense orbits (those examples present two critical points which are flat).

Since flat critical points do not exist for non-constant real-analytic maps we have:

Corollary A.4.4. *Any real-analytic orientation preserving circle homeomorphism with irrational rotation number is topologically conjugate to the corresponding rigid rotation.*

A.4.1 Degenerated cross-ratio

Given an interval $J \subset S^1$ with boundary points a and b we define:

$$M(f, J) = \frac{|f(J)|}{|J|}\big(Df(a)Df(b)\big)^{-1/2}$$

with the convention that $M(f, J) = +\infty$ if a or b are critical points of f.

Let $\varepsilon > 0$ and let $J_\varepsilon \supsetneq J$ such that both components of $J_\varepsilon \setminus J$ (call them L_ε and R_ε) has length ε. Then:

$$\mathrm{Cr}(f, J, J_\varepsilon) = \frac{|f(J)||f(J_\varepsilon)|}{|J|\big(|J| + 2\varepsilon\big)Df(x_\varepsilon)Df(y_\varepsilon)}$$

where $x_\varepsilon \in L_\varepsilon$ and $y_\varepsilon \in R_\varepsilon$ are given by the Mean Value Theorem. Since f is C^1 we have that $\lim_{\varepsilon\to 0} \mathrm{Cr}(f, J, J_\varepsilon) = \big(M(f, J)\big)^2$, and that is why we call M the *degenerated cross-ratio*.

The main point in the proof of Theorem A.4.2 is the following estimate on the distortion of the degenerated cross-ratio of high iterates: close to the critical points $\log Df$ has unbounded variation, but still the map f does not contract the degenerated cross-ratio too much, provided that the critical points are non-flat.

Lemma A.4.5. *Let f as in Theorem A.4.2. There exists a positive constant $\delta > 0$ such that for any $n \geq 1$, any $p \in \{0, ..., q_{n+1}\}$, any $x \in S^1$ and any interval $J = (a, b)$ contained in the arc $\big(f^{-q_n}(x), f^{q_n}(x)\big)$ we have that: $M(f^p, J) \geq \delta$.*

Proof. Following Yoccoz we split the family $\big\{J, f(J), ..., f^{p-1}(J)\big\}$ in four disjoint families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ as follows:

- $\mathcal{F}_1$ contains the intervals $f^i(J)$ that are disjoint of the intervals:

  $$\left[c_j - \frac{\varepsilon_j}{2}, c_j + \frac{\varepsilon_j}{2}\right]$$

  for any $j \in \{1, ..., k\}$.

- $\mathcal{F}_2$ contains the intervals $f^i(J)$ that contain some interval of the form:
$$\left[c_j - \varepsilon_j, c_j - \frac{\varepsilon_j}{2}\right]$$
or some interval of the form:
$$\left[c_j + \frac{\varepsilon_j}{2}, c_j + \varepsilon_j\right]$$
Note that $f^i(J)$ may contain the critical point c_j.

- $\mathcal{F}_3$ contains the intervals $f^i(J)$ that are contained on an interval $(c_j - \varepsilon_j, c_j + \varepsilon_j)$, intersects $[c_j - \varepsilon_j/2, c_j + \varepsilon_j/2]$ but not contain the critical point c_j.

- $\mathcal{F}_4$ contains the intervals $f^i(J)$ that are contained in some interval $(c_j - \varepsilon_j, c_j + \varepsilon_j)$ and contain the critical point c_j.

Note that:
$$M(f^p, J) = \prod_{i=0}^{p-1} M\big(f, f^i(J)\big) = \prod_{l=1}^{4}\left(\prod_{f^i(J)\in\mathcal{F}_l} M\big(f, f^i(J)\big)\right)$$

Note also that $\#\mathcal{F}_2 \leq 4k$ and $\#\mathcal{F}_4 \leq 2k$ since the family $\big\{J, f(J), ..., f^{q_{n+1}-1}(J)\big\}$ has multiplicity of intersection 2.

- The intervals in the family $\mathcal{F}_1$ are treated in the same way as in Lemma A.2.10: by the Mean Value Theorem for each $f^i(J) \in \mathcal{F}_1$ there exists a point $x_i \in f^i(J)$ such that:
$$\log M\big(f, f^i(J)\big) = \log Df(x_i) - \frac{1}{2}\log Df(f^i(a)) - \frac{1}{2}\log Df(f^i(b))$$
Let $V > 0$ be the total variation of $\log Df$ in the compact set:
$$K = \bigcap_{j=1}^{j=k}\left(c_j - \frac{\varepsilon_j}{2}, c_j + \frac{\varepsilon_j}{2}\right)^c$$
Then:
$$\prod_{f^i(J)\in\mathcal{F}_1} M\big(f, f^i(J)\big) \geq \exp(-2V)$$

- Let $\varepsilon = \min\{\varepsilon_1, ..., \varepsilon_k\}$, $d = \min_{z \in K}\{Df(z)\} > 0$ with K as defined above, and let $D = \max_{z \in S^1}\{Df(z)\}$. For any interval $f^i(J)$ in $\mathcal{F}_2$ we have $|f^i(J)| \geq \frac{\varepsilon}{2}$ and so $|f(f^i(J))| \geq \frac{d\varepsilon}{2}$. In particular $M\big(f, f^i(J)\big) \geq \left(\frac{d}{D}\right)\left(\frac{\varepsilon}{2}\right)$. Then:

$$\prod_{f^i(J) \in \mathcal{F}_2} M\big(f, f^i(J)\big) \geq \left(\frac{d}{D}\right)^{4k}\left(\frac{\varepsilon}{2}\right)^{4k}$$

- Since $(Df)^{-1/2}$ is well defined and convex in any member of $\mathcal{F}_3$ we easily obtain that $M\big(f, f^i(J)\big) \geq 1$ for any $f^i(J) \in \mathcal{F}_3$: let g be the affine map that coincide with $(Df)^{-1/2}$ in the boundary points $f^i(a)$ and $f^i(b)$, and note that:

$$M\big(f, f^i(J)\big) = \left(\frac{g(f^i(a))g(f^i(b))}{|f^i(J)|}\right)\left(\int_{f^i(a)}^{f^i(b)} Df(t)dt\right)$$

Since $(Df)^{-1/2}$ is convex, $Df \geq \frac{1}{g^2}$ so:

$$M\big(f, f^i(J)\big) \geq \left(\frac{g(f^i(a))g(f^i(b))}{f^i(b) - f^i(a)}\right)\left(\int_{f^i(a)}^{f^i(b)} \frac{dt}{g^2(t)}\right)$$

and this is equal to 1 since g is affine. Then:

$$\prod_{f^i(J) \in \mathcal{F}_3} M\big(f, f^i(J)\big) \geq 1$$

In particular this prove the well-known fact that maps with negative Schwarzian derivative expand cross-ratio.

- Now we claim that if $f^i(J) \in \mathcal{F}_4$ we have that:

$$M\big(f, f^i(J)\big) \geq \frac{A_j}{2B_j(s_j + 1)}$$

where c_j is the critical point that belongs to $f^i(J)$. Indeed, let us suppose that $|f^i(a) - c_j| \leq |f^i(b) - c_j|$ and let $I = [c_j, f^i(b)]$. Note that:

$$\begin{aligned} Df(f^i(a))Df(f^i(b)) &\leq B_j^2|f^i(a) - c_j|^{s_j}|f^i(b) - c_j|^{s_j} \\ &\leq B_j^2|f^i(b) - c_j|^{2s_j} = \left(B_j|I|^{s_j}\right)^2 \end{aligned}$$

Also:

$$\begin{aligned} |f(I)| &= \int_{c_j}^{f^i(b)} Df(t)dt \geq A_j \int_{c_j}^{f^i(b)} |t-c_j|^{s_j} dt \\ &= A_j \left(\frac{|f^i(b)-c_j|^{s_j+1}}{s_j+1} \right) = \left(\frac{A_j}{s_j+1} \right) |I|^{s_j+1} \end{aligned}$$

Combining these two estimates we obtain:

$$\begin{aligned} M\big(f, f^i(J)\big) &= \frac{|f(f^i(J))|}{|f^i(J)|} \big(Df(f^i(a))Df(f^i(b))\big)^{-1/2} \\ &\geq \frac{|f(I)|}{2|I|} \big(Df(f^i(a))Df(f^i(b))\big)^{-1/2} \\ &\geq \frac{A_j}{2B_j(s_j+1)} \end{aligned}$$

In particular if:

$$\alpha = \min_{j \in \{1,\dots,k\}} \left\{ \frac{A_j}{2B_j(s_j+1)} \right\}$$

we have that:

$$\prod_{f^i(J) \in \mathcal{F}_4} M\big(f, f^i(J)\big) \geq \alpha^{2k}$$

We finish the proof taking $\delta = \exp(-2V) \left(\frac{d}{D}\right)^{4k} \left(\frac{\varepsilon}{2}\right)^{4k} \alpha^{2k}$. □

Note that the last estimates in the family $\mathcal{F}_4$ are false if we allow flat critical points as in Hall's examples. Since we are working with the degenerated cross-ratio instead of the usual one (defined in Section A.1.2), we cannot use Lemma A.2.1 and so we prove Theorem A.4.2 from Lemma A.4.5 directly:

Proof of Theorem A.4.2. Suppose that there exists a wandering interval $I \subset S^1$: $f^n(I) \cap f^m(I) = \emptyset$ for all $n \neq m \in \mathbb{Z}$.

Fix $n \geq 1$. By the Mean Value Theorem there exist $a \in f^{-q_n - q_{n+1}}(I)$ and $b \in f^{-q_{n+1}}(I)$ such that:

$$Df^{q_{n+1}}(a) = \frac{|f^{-q_n}(I)|}{|f^{-q_n - q_{n+1}}(I)|} \quad \text{and} \quad Df^{q_{n+1}}(b) = \frac{|I|}{|f^{-q_{n+1}}(I)|}$$

Let J be the compact interval with boundary points a and b that contains I, and let $x \in I$. By combinatorics we know that the intervals $f^{-q_n}(I)$,

$f^{-q_n-q_{n+1}}(I)$, I, $f^{-q_{n+1}}(I)$ and $f^{q_n}(I)$ are orderer in this way (or the opposite depending if n is even or odd). In any case $J \subset \big(f^{-q_n}(x), f^{q_n}(x)\big)$, and so Lemma A.4.5 give us $M(f^{q_{n+1}}, J) \geq \delta$, or:

$$\left(\frac{|f^{q_{n+1}}(J)|}{|J|}\right)^2 \frac{|f^{-q_n-q_{n+1}}(I)||f^{-q_{n+1}}(I)|}{|f^{-q_n}(I)||I|} \geq \delta^2$$

Equivalently:

$$\frac{|f^{-q_{n+1}}(I)|}{|f^{-q_n}(I)|} \geq \left(\frac{|J|}{|f^{q_{n+1}}(J)|}\right)^2 \left(\frac{|I|}{|f^{-q_n-q_{n+1}}(I)|}\right)\delta^2$$

Since I is a wandering interval there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that: $|f^{-q_n-q_{n+1}}(I)| \leq |I|^3\delta^2$. Using that $|f^{q_{n+1}}(J)| \leq 1$ for all $n \in \mathbb{N}$ we obtain:

$$\frac{|f^{-q_{n+1}}(I)|}{|f^{-q_n}(I)|} \geq \left(\frac{|J|}{|I|}\right)^2 > 1$$

since I is striclty contained in J. This contradicts the fact that the sequence $\{|f^{-q_n}(I)|\}_{n\in\mathbb{N}}$ goes to zero when n goes to infinity. □

Yoccoz's proof was the first time that estimates on cross-ratio distortion were successfully applied in one-dimensional dynamics. In 1992 Martens, de Melo and van Strien [37] went deeper with cross-ratio distortion techniques and extended the theory to non-invertible dynamics: they proved that any C^2 map of the circle or any compact interval with only non-flat critical points has no wandering interval. In other words: any open interval for which all positive iterates are mutually disjoint is contained in the basin of a periodic (maybe one-sided) attractor (see also [44] and [45, Chapter IV]).

APPENDIX B

Proof of Lemma 1.5.1

In this appendix we prove Lemma 1.5.1, stated at the end of Chapter 1 and used in Chapter 2. For that we need the following fact:

Lemma B.0.6. *Let $f_1, ..., f_n$ be C^1 maps with C^1 norm bounded by some constant $B > 0$, and let $g_1, ..., g_n$ be C^0 maps. Then:*

$$\left\| f_n \circ ... \circ f_1 - g_n \circ ... \circ g_1 \right\|_{C^0} \leq \left(\sum_{j=0}^{n-1} B^j \right) \max_{i \in \{1,...,n\}} \left\{ \left\| f_i - g_i \right\|_{C^0} \right\}$$

whereas the compositions makes sense.

Proof. The proof goes by induction on n (when $n = 1$ we have nothing to prove). Suppose that:

$$\left\| f_{n-1} \circ ... \circ f_1 - g_{n-1} \circ ... \circ g_1 \right\|_{C^0} \leq \left(\sum_{j=0}^{n-2} B^j \right) \max_{i \in \{1,...,n-1\}} \left\{ \left\| f_i - g_i \right\|_{C^0} \right\}.$$

Then for any t:

$$\begin{aligned}
\big|(f_n \circ ... \circ f_1 - g_n \circ ... \circ g_1)(t)\big| &\leq \big|f_n((f_{n-1} \circ ... \circ f_1)(t)) - f_n((g_{n-1} \circ ... \circ g_1)(t))\big| + \\
&+ \big|f_n((g_{n-1} \circ ... \circ g_1)(t)) - g_n((g_{n-1} \circ ... \circ g_1)(t))\big| \\
&\leq B\big|(f_{n-1} \circ ... \circ f_1 - g_{n-1} \circ ... \circ g_1)(t)\big| + \big\|f_n - g_n\big\|_{C^0} \\
&\leq B\left(\sum_{j=0}^{n-2} B^j\right) \max_{i \in \{1,...,n-1\}} \left\{\big\|f_i - g_i\big\|_{C^0}\right\} + \big\|f_n - g_n\big\|_{C^0} \\
&\leq \left(\sum_{j=0}^{n-1} B^j\right) \max_{i \in \{1,...,n\}} \left\{\big\|f_i - g_i\big\|_{C^0}\right\}.
\end{aligned}$$

□

For $K > 1$ and $r \in \{0, 1, ..., \infty, \omega\}$ recall from Chapter 1 that we denote by $\mathcal{P}^r(K)$ the space of C^r critical commuting pairs $\zeta = (\eta, \xi)$ such that $\eta(0) = -1$ (they are normalized) and $\xi(0) \in [K^{-1}, K]$.

Lemma B.0.7. *Given $M \in \mathbb{N}$, $B > 0$ and $K > 1$ there exists $L(M, B, K) > 1$ with the following property: let $\zeta_1 = (\eta_1, \xi_1)$ and $\zeta_2 = (\eta_2, \xi_2)$ be two renormalizable C^3 critical commuting pairs satisfying the following five conditions:*

1. *ζ_1, $\mathcal{R}(\zeta_1)$, ζ_2 and $\mathcal{R}(\zeta_2)$ belong to $\mathcal{P}^3(K)$.*
2. *The continued fraction expansion of both rotation numbers $\rho(\zeta_1)$ and $\rho(\zeta_2)$ have the same first term, say a_0, with $a_0 \leq M$. More precisely:*
$$\left\lfloor \frac{1}{\rho(\zeta_1)} \right\rfloor = \left\lfloor \frac{1}{\rho(\zeta_2)} \right\rfloor = a_0 \in \big\{1, ..., M\big\}.$$
3. *$\max\big\{\|\eta_1\|_{C^1}, \|\xi_1\|_{C^1}\big\} < B$.*
4. *$\big(\eta_1 \circ \xi_1\big)(0)$ and $\big(\eta_2 \circ \xi_2\big)(0)$ have the same sign.*
5. $$\big|\xi_1(0) - \xi_2(0)\big| < \left(\frac{1}{K^2}\right)\left(\frac{K+1}{K-1}\right).$$

Then we have:
$$d_0\big(\mathcal{R}(\zeta_1), \mathcal{R}(\zeta_2)\big) \leq L \cdot d_0(\zeta_1, \zeta_2),$$
where d_0 is the C^0 distance in the space of critical commuting pairs (see Section 1.3).

Proof. Suppose that both $\big(\eta_1 \circ \xi_1\big)(0)$ and $\big(\eta_2 \circ \xi_2\big)(0)$ are positive, and let $V \subset \mathbb{R}$ be the interval $\big[0, \max\big\{(\eta_1 \circ \xi_1)(0), (\eta_2 \circ \xi_2)(0)\big\}\big]$. For $\alpha > 0$ denote by T_α the (unique) Möbius transformation that fixes -1 and 0, and maps α to 1. Note that $p_\alpha = \alpha + \alpha\left(\frac{1+\alpha}{1-\alpha}\right)$ is the pole of T_α. If $\alpha > K/(K+2)$ then $p_\alpha \notin [1/K, K]$, and if $\alpha \in [1/K, K/(K+2)]$ then $p_\alpha - \alpha \geq \left(\frac{1}{K}\right)\left(\frac{K+1}{K-1}\right)$. By Item (5) in the hypothesis, and since ζ_1 and ζ_2 belong to $\mathcal{P}^3(K)$ by Item (1), there exists $L_0(K) > 1$ such that:

$$\big\|T_{\xi_1(0)}\big\|_{C^1(V)} \leq L_0\,,$$

$$\big\|T_{\xi_1(0)} - T_{\xi_2(0)}\big\|_{C^0(V)} \leq L_0\big|\xi_1(0) - \xi_2(0)\big| \leq L_0 \cdot d_0(\zeta_1, \zeta_2)\,,$$

$$\big|\eta_1^{a_0}\big(\xi_1(0)\big) - \eta_2^{a_0}\big(\xi_2(0)\big)\big| \leq L_0\big|\widetilde{\eta}_1^{a_0}(1) - \widetilde{\eta}_2^{a_0}(1)\big| \quad \text{and}$$

$$\big\|\widetilde{\eta}_1\big\|_{C^1([0,1])} \leq L_0\big\|\eta_1\big\|_{C^1([0,\xi_1(0)])} \leq L_0 B\,,$$

where $\widetilde{\eta}_i = T_{\xi_i(0)} \circ \eta_i \circ T_{\xi_i(0)}^{-1}$ for $i \in \{1, 2\}$. By Lemma B.0.6:

$$\big|\eta_1^{a_0}\big(\xi_1(0)\big) - \eta_2^{a_0}\big(\xi_2(0)\big)\big| \leq L_0^2\left(\sum_{j=0}^{a_0-1} B^j\right)\big\|\widetilde{\eta}_1 - \widetilde{\eta}_2\big\|_{C^0([0,1])}\,.$$

Defining $L_1(M, B, K) = L_0^2\left(\sum_{j=0}^{M-1} B^j\right)$ we obtain:

$$\big|\eta_1^{a_0}\big(\xi_1(0)\big) - \eta_2^{a_0}\big(\xi_2(0)\big)\big| \leq L_1 \cdot d_0(\zeta_1, \zeta_2)\,.$$

Therefore:

$$\begin{aligned}
\big|T_{\xi_1(0)}\big(\eta_1^{a_0}(\xi_1(0))\big) - T_{\xi_2(0)}\big(\eta_2^{a_0}(\xi_2(0))\big)\big| &\leq \big|T_{\xi_1(0)}\big(\eta_1^{a_0}(\xi_1(0))\big) - T_{\xi_1(0)}\big(\eta_2^{a_0}(\xi_2(0))\big)\big| \\
&\quad + \big|T_{\xi_1(0)}\big(\eta_2^{a_0}(\xi_2(0))\big) - T_{\xi_2(0)}\big(\eta_2^{a_0}(\xi_2(0))\big)\big| \\
&\leq L_0\big|\eta_1^{a_0}\big(\xi_1(0)\big) - \eta_2^{a_0}\big(\xi_2(0)\big)\big| + L_0 \cdot d_0(\zeta_1, \zeta_2) \\
&\leq (L_0 L_1 + L_0) \cdot d_0(\zeta_1, \zeta_2)\,.
\end{aligned}$$

Defining $L_2(M, B, K) = L_0 L_1 + L_0$ we obtain:

$$\big|T_{\xi_1(0)}\big(\eta_1^{a_0}(\xi_1(0))\big) - T_{\xi_2(0)}\big(\eta_2^{a_0}(\xi_2(0))\big)\big| \leq L_2 \cdot d_0(\zeta_1, \zeta_2)\,. \tag{B.0.1}$$

Moreover there exists $L_3(M, B, K) \geq L_2$ with the following four properties:

- From Item (1) in the hypothesis, both Möbius transformations:

$$T_{T_{\xi_i(0)}\left(\eta_i^{a_0}(\xi_i(0))\right)}$$

and also their inverses have C^1 norm bounded by L_3 in:

$$W=\left[0,\max\left\{T_{\xi_1(0)}\left(\eta_1^{a_0}(\xi_1(0))\right),T_{\xi_2(0)}\left(\eta_2^{a_0}(\xi_2(0))\right)\right\}\right].$$

- Both Möbius transformations:

$$T_{T_{\xi_i(0)}\left(\eta_i^{a_0}(\xi_i(0))\right)}$$

are at C^0-distance less or equal than $L_3\cdot d_0(\zeta_1,\zeta_2)$ in W (this follows from (B.0.1) and Item (1) in the hypothesis).

- The same with their inverses, that is, both Möbius transformations:

$$T^{-1}_{T_{\xi_i(0)}\left(\eta_i^{a_0}(\xi_i(0))\right)}$$

are at C^0-distance less or equal than $L_3\cdot d_0(\zeta_1,\zeta_2)$ in $[0,1]$ (again this follows from (B.0.1) and Item (1) in the hypothesis).

- The maps:

$$T_{\xi_1(0)}\circ\eta_1^{a_0}\circ\xi_1\circ T^{-1}_{\xi_1(0)}\quad\text{and}\quad T_{\xi_1(0)}\circ\eta_1\circ T^{-1}_{\xi_1(0)}$$

have C^1 norm bounded by L_3 in $[-1,0]$ and $[0,1]$ respectively (this follows from items (1), (2) and (3) in the hypothesis).

Note that for $i=\{1,2\}$ we have:

$$T_{\eta_i^{a_0}\left(\xi_i(0)\right)}\circ\eta_i^{a_0}\circ\xi_i\circ T^{-1}_{\eta_i^{a_0}\left(\xi_i(0)\right)}=$$

$$=T_{T_{\xi_i(0)}\left(\eta_i^{a_0}(\xi_i(0))\right)}\circ\left(T_{\xi_i(0)}\circ\eta_i^{a_0}\circ\xi_i\circ T^{-1}_{\xi_i(0)}\right)\circ T^{-1}_{T_{\xi_i(0)}\left(\eta_i^{a_0}(\xi_i(0))\right)}$$

in $[-1,0]$, and:

$$T_{\eta_i^{a_0}\left(\xi_i(0)\right)}\circ\eta_i\circ T^{-1}_{\eta_i^{a_0}\left(\xi_i(0)\right)}=$$

$$=T_{T_{\xi_i(0)}\left(\eta_i^{a_0}(\xi_i(0))\right)}\circ\left(T_{\xi_i(0)}\circ\eta_i\circ T^{-1}_{\xi_i(0)}\right)\circ T^{-1}_{T_{\xi_i(0)}\left(\eta_i^{a_0}(\xi_i(0))\right)}$$

in $[0,1]$. By Lemma B.0.6 and the four properties quoted above there exists $L_4(M,B,K) \geq L_3$ such that:

$$\left\|T_{\eta_1^{a_0}\left(\xi_1(0)\right)} \circ \eta_1^{a_0} \circ \xi_1 \circ T^{-1}_{\eta_1^{a_0}\left(\xi_1(0)\right)} - T_{\eta_2^{a_0}\left(\xi_2(0)\right)} \circ \eta_2^{a_0} \circ \xi_2 \circ T^{-1}_{\eta_2^{a_0}\left(\xi_2(0)\right)}\right\|_{C^0} \leq$$

$$\leq L_4 \max\left\{\left\|T_{T_{\xi_1(0)}\left(\eta_1^{a_0}(\xi_1(0))\right)} - T_{T_{\xi_2(0)}\left(\eta_2^{a_0}(\xi_2(0))\right)}\right\|_{C^0},\right.$$

$$\left. d_0(\zeta_1,\zeta_2), \left\|T^{-1}_{T_{\xi_1(0)}\left(\eta_1^{a_0}(\xi_1(0))\right)} - T^{-1}_{T_{\xi_2(0)}\left(\eta_2^{a_0}(\xi_2(0))\right)}\right\|_{C^0}\right\}$$

$$\leq L_3 L_4 \cdot d_0(\zeta_1,\zeta_2)\,.$$

in $[-1,0]$, and:

$$\left\|T_{\eta_1^{a_0}\left(\xi_1(0)\right)} \circ \eta_1 \circ T^{-1}_{\eta_1^{a_0}\left(\xi_1(0)\right)} - T_{\eta_2^{a_0}\left(\xi_2(0)\right)} \circ \eta_2 \circ T^{-1}_{\eta_2^{a_0}\left(\xi_2(0)\right)}\right\|_{C^0} \leq$$

$$\leq L_4 \max\left\{\left\|T_{T_{\xi_1(0)}\left(\eta_1^{a_0}(\xi_1(0))\right)} - T_{T_{\xi_2(0)}\left(\eta_2^{a_0}(\xi_2(0))\right)}\right\|_{C^0},\right.$$

$$\left. d_0(\zeta_1,\zeta_2), \left\|T^{-1}_{T_{\xi_1(0)}\left(\eta_1^{a_0}(\xi_1(0))\right)} - T^{-1}_{T_{\xi_2(0)}\left(\eta_2^{a_0}(\xi_2(0))\right)}\right\|_{C^0}\right\}$$

$$\leq L_3 L_4 \cdot d_0(\zeta_1,\zeta_2)\,.$$

in $[0,1]$. Therefore we are done by taking $L \geq L_3 L_4$. □

Proof of Lemma 1.5.1. Let f be a C^3 critical circle map with irrational rotation number $\rho(f) = [a_0, a_1, ..., a_n, a_{n+1}, ...]$, and recall that we are assuming that $a_n < M$ for all $n \in \mathbb{N}$. Let $n_0(f) \in \mathbb{N}$ given by the real bounds, and note that $\mathcal{R}^n(f) \in \mathcal{P}^3(K)$ for all $n \geq n_0$ since $K > K_0$ by hypothesis and therefore $\mathcal{P}^3(K) \supset \mathcal{P}^3(K_0)$. As a well-known corollary of the real bounds (see for instance [12, Theorem 3.1]) there exists a constant $B > 0$ such that the sequence $\{\mathcal{R}^n(f)\}_{n\in\mathbb{N}}$ is bounded in the C^1 metric by B, and we are done by taking $L > 1$ given by Lemma B.0.7. □

APPENDIX C

Annular Riemann Surfaces

In this appendix we briefly review some classical facts about Riemann surfaces (the book [14] is a major reference for the whole subject), we revisit the notion of Poincaré disk (that we have used in Chapter 4) and finally we study in some detail annular Riemann surfaces (in particular we give the precise definition of the modulus of an annular Riemann surface, that we have used extensively in Chapter 5).

C.1 Riemann surfaces

By a *Riemann surface* we mean a (connected) one-dimensional complex-analytic manifold.

Theorem (Uniformization Theorem)**.** *Any simply connected Riemann surface is conformally equivalent either to $\mathbb{D}$, to $\mathbb{C}$ or to $\widehat{\mathbb{C}}$.*

For a complete proof of this deep result, obtained independently by Klein, Poincaré and Koebe between 1882 and 1907, see [14, Chapter IV] (see also [52] for an historical account).

Given a Riemann surface S, denote by $\pi_1(S)$ its fundamental group, and by $\mathrm{Aut}(S)$ the group (under composition) of biholomorphisms of S. Since the field of meromorphic functions of $\widehat{\mathbb{C}}$ is isomorphic to the field of rational functions in the complex plane, we have that $\mathrm{Aut}(\widehat{\mathbb{C}})$ is isomorphic to the *normalized Möbius group*:

$$\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C} \quad \text{and} \quad ad-bc=1 \right\}.$$

Let $SL(2,\mathbb{C})$ be the group of 2×2 complex matrices of determinant equal to 1 (the complex *special linear* group), and let $PSL(2,\mathbb{C})$ the quotient of $SL(2,\mathbb{C})$ modulo the subgroup $\{\pm Id\}$. Both $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$ are 3-dimensional complex Lie groups, and $\mathrm{Aut}(\widehat{\mathbb{C}})$ is isomorphic to $PSL(2,\mathbb{C})$. The groups $\mathrm{Aut}(\mathbb{C})$ and $\mathrm{Aut}(\mathbb{D})$ are formed by the elements of $\mathrm{Aut}(\widehat{\mathbb{C}})$ that preserve $\mathbb{C}$ and $\mathbb{D}$ respectively (in particular both are Lie subgroups of $\mathrm{Aut}(\widehat{\mathbb{C}})$). It is easy to see that $\mathrm{Aut}(\mathbb{C})$ is the *affine group*:

$$\mathrm{Aut}(\mathbb{C}) = \big\{ z \mapsto az+b : a \in \mathbb{C}\setminus\{0\}, b \in \mathbb{C} \big\},$$

and therefore, $\mathrm{Aut}(\mathbb{C})$ is a 2-dimensional complex Lie group.

A straightforward application of Schwarz Lemma gives the identity:

$$\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta}\left(\frac{z-a}{1-\bar{a}z}\right) : a \in \mathbb{D}, \theta \in \mathbb{R} \right\},$$

that is, $\mathrm{Aut}(\mathbb{D})$ is diffeomorphic to the solid torus $\mathbb{D}\times S^1$, which is isomorphic to $PSL(2,\mathbb{R})$, a 3-dimensional real Lie group.

Using the general theory of covering spaces we obtain from the Uniformization Theorem a complete classification of Riemann surfaces (again see [14, Chapter IV] for a detailed proof):

Theorem (Uniformization of arbitrary Riemann surfaces)**.** *Any Riemann surface S is conformally equivalent to $\widetilde{S}/\Gamma$ where $\widetilde{S}$ is $\mathbb{D}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$, and $\Gamma \cong \pi_1(S)$ is a subgroup of $\mathrm{Aut}(\widetilde{S})$ acting discontinuously and without fixed points (the conformal structure of $\widetilde{S}/\Gamma$ is the one induced by $\widetilde{S}$ via the covering map).*

No other Riemann surface rather than $\widehat{\mathbb{C}}$ itself is covered by $\widehat{\mathbb{C}}$. The plane $\mathbb{C}$, the punctured plane $\mathbb{C}\setminus\{0\}$ (equivalently the cylinder $\mathbb{C}/\mathbb{Z}$) and any *torus* (that is, any surface diffeomorphic to $\mathbb{C}/\mathbb{Z}^2$) are covered by the plane. Any other Riemann surface not conformally equivalent with one of the above is covered by the unit disk $\mathbb{D}$. In particular any Riemann surface which is not homeomorphic to the sphere or torus (in the compact case), or homeomorphic to the plane or the punctured plane (in the noncompact case) is covered by the unit disk. This includes, for instance, compact Riemann surfaces with negative Euler characteristic (genus $g \geq 2$), and open sets in the Riemann sphere whose complement contains at least three points.

C.1.1 Riemannian metrics on surfaces

Let g_1 and g_2 be two Riemannian metrics on an orientable surface S. A smooth diffeomorphism $f:(S,g_1)\to(S,g_2)$ is said to be *conformal* if there exists a smooth function $\lambda:S\to(0,+\infty)$ such that given $z\in S$ and $v,w\in T_zS$ we have:

$$g_2\big(f(z)\big)\big(Df(z)v,Df(z)w\big)=\lambda(z)^2\cdot g_1(z)(v,w)$$

Equivalently, a conformal diffeomorphism is an angle-preserving diffeomorphism. In the complex plane the description of the group of conformal diffeomorphisms (for the Euclidean metric) is well-known: the orientation-preserving ones are the biholomorphisms, and the orientation-reversing ones are the antiholomorphic diffeomorphisms. An *isometry* between Riemannian metrics is a conformal diffeomorphism with $\lambda\equiv 1$. The group (under composition) of the isometries of a Riemannian metric g in S is denoted by $\mathrm{Iso}(S,g)$, or simply by $\mathrm{Iso}(S)$ if no confusion is possible. For instance, any element of $\mathrm{Iso}(\mathbb{C})$ is a translation composed with an *orthogonal* transformation, that is, a map of the form $z\mapsto A(z+b)$ for $A\in O(2)=\{A\in GL(2,\mathbb{R}):A^tA=AA^t=Id\}$, and b any complex number. Hence $\mathrm{Iso}(\mathbb{C})$ is a 3-dimensional real Lie group, with two (non-compact) connected components.

A Riemannian metric g on a Riemann surface S is said to be *conformal* if local charts are conformal with respect to the Euclidean metric in the plane, that is, given a local chart U in the complex plane there exists a smooth function $\lambda:U\to(0,+\infty)$ such that $g(z)(v,w)=\lambda(z)^2\,\langle v,w\rangle$ for any $z\in U$ and any $v,w\in T_zU$, where $\langle\cdot\rangle$ denote the Euclidean inner product in the complex plane. The smooth function λ is called the *coefficient of conformality* of the metric g. By Gauss *isothermal coordinates* any Riemannian metric on an oriented surface is locally conformally equivalent to the Euclidean metric in the plane [57, Section 2.5, Theorem 2.5.14]. One may think on Morrey's theorem (stated as Theorem 3.2.4 in Chapter 3, see also Ahlfors-Bers theorem, stated as Theorem 3.3.1) as the *measurable generalization* of Gauss coordinates.

In the sequel the word *curvature* means Gauss curvature. There is a special Riemannian metric in the unit disk [14, Section IV.8.4]:

Theorem (Poincaré metric)**.** *Up to multiplication by positive scalars, there exists a unique Riemannian metric in $\mathbb{D}$ whose group of isometries is the group of conformal diffeomorphisms of $\mathbb{D}$. Such a metric is complete (any two points in $\mathbb{D}$ are joined by a minimizing geodesic), conformal and has constant negative curvature. By normalization we can choose it with curvature -1. The geodesics of this metric are the circles which are orthogonal to the boundary of $\mathbb{D}$.*

Note that $\mathbb{C}$ and $\widehat{\mathbb{C}}$ also admit a Riemannian metric of constant curvature (zero and positive respectively). In the planar case this metric is of course the Euclidean metric, and since it is invariant under translations projects to any quotient. In particular any Riemann surface admits, by projecting from the holomorphic universal covering, a complete and conformal Riemannian metric with constant curvature. The coefficients of conformality in the simply connected models are:

- $\lambda(z) = \frac{2}{1+|z|^2}$ in $\widehat{\mathbb{C}}$;
- $\lambda(z) = 1$ in $\mathbb{C}$;
- $\lambda(z) = \frac{2}{1-|z|^2}$ in $\mathbb{D}$.

In the case of the sphere the definition is valid away from the point at infinity, where we extend the definition with the local chart $w = 1/z$. Still in this case, if we identify $\widehat{\mathbb{C}}$ with the unit sphere S^2 in $\mathbb{R}^3$, we obtain that $\mathrm{Iso}(\widehat{\mathbb{C}})$ is a 3-dimensional real compact Lie group, isomorphic to the *orthogonal* group $O(3)$.

In the three simply connected models, the group of isometries of the Riemannian metric of constant curvature acts *transitively*: given any two points there exists an isometry taking one into the other. We will see in Lemma C.2.1 that this is no longer true for typical annular Riemann surfaces.

C.1.2 The upper half-plane

Another model for the Poincaré metric which is often very useful is the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. This model is conformally equivalent to the unit disk via the Möbius transformation $z \mapsto \frac{i-z}{i+z}$, with inverse given by $z \mapsto i\left(\frac{1-z}{1+z}\right)$ (see also the proof of Lemma 5.2.9 in Chapter 5). The group $\mathrm{Aut}(\mathbb{H})$ is the group of Möbius transformations of the Riemann sphere that preserve the upper half-plane. In particular they preserve the real line and so they have real coefficients:

$$\mathrm{Aut}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \quad \text{and} \quad ad-bc = 1 \right\} \cong PSL(2,\mathbb{R}).$$

Note that $\mathrm{Aut}(\mathbb{H})$ already acts transitively in $\mathbb{H}$. Moreover, given $z_1, z_2 \in \mathbb{H}$, $v_1 \in T_{z_1}\mathbb{H}$ and $v_2 \in T_{z_2}\mathbb{H}$, with $\|v_1\|_{z_1} = \|v_2\|_{z_2} = 1$, there exists $\psi \in \mathrm{Aut}(\mathbb{H})$ such that $\psi(z_1) = z_2$ and $D\psi(z_1)v_1 = v_2$. From this follows that $\mathrm{Aut}(\mathbb{H})$ is also diffeomorphic to $T^1\mathbb{H}$, the unit tangent bundle of $\mathbb{H}$ for the hyperbolic metric. This is very useful in some other contexts, for instance when proving that the geodesic flow for the hyperbolic metric on compact surfaces with negative Euler characteristic is an *Anosov* flow [24, Section 17.5].

A straightforward computation shows that in the upper half-plane the Poincaré metric's coefficient of conformality is given by $\lambda(z) = \frac{1}{\Im(z)}$. The geodesics in the upper half-plane are the circles and lines orthogonals to the real axis. The group of isometries $\mathrm{Iso}(\mathbb{H})$ is generated by $\mathrm{Aut}(\mathbb{H})$ together with the symmetry around the imaginary axis $z \mapsto -\bar{z}$, which is an antiholomorphic involution.

C.1.3 The $\mathbb{C}_I$ spaces

A third model for the Poincaré metric, crucial in this thesis, is the following: given an open interval $I = (a, b) \subset \mathbb{R}$ let $\mathbb{C}_I = \big(\mathbb{C} \setminus \mathbb{R}\big) \cup I = \mathbb{C} \setminus \big(\mathbb{R} \setminus I\big)$. Note that $\mathbb{C}_I$ is an open, connected and simply connected set which is not the whole plane. By the Riemann mapping theorem we can endow $\mathbb{C}_I$ with a complete and conformal Riemannian metric of constant curvature equal to -1, just by pulling back the Poincaré metric of $\mathbb{D}$ by any conformal uniformization. Note that I is always a hyperbolic geodesic by symmetry.

We revisit in this section the notion of Poincaré disk, introduced into the subject by Sullivan in his seminal article [55]: for $\theta \in (0, \pi)$ let D be the open disk in the plane intersecting the real line along I and for which the angle from $\mathbb{R}$ to ∂D at the point b (measured anticlockwise) is θ. Let $D^+ = D \cap \{z : \Im(z) > 0\}$ and let D^- be the image of D^+ under complex conjugation.

Define the *Poincaré disk* of angle θ based on I as $D_\theta(a, b) = D^+ \cup I \cup D^-$, that is, $D_\theta(a, b)$ is the set of points in the complex plane that *view* I under an angle $\geq \theta$ (see Figure C.1). Note that for $\theta = \frac{\pi}{2}$ the Poincaré disk $D_\theta(a, b)$ is the Euclidean disk whose diameter is the interval (a, b).

In Chapter 4 we mentioned the following:

Lemma C.1.1. *Let $I = (a, b)$ fixed. For a given $\theta \in (0, \pi)$ let $\varepsilon(\theta) = \log \tan\left(\frac{\pi}{2} - \frac{\theta}{4}\right)$. The set of points in $\mathbb{C}_I$ whose hyperbolic distance to I is less than ε is the Poincaré disk $D_\theta(a, b)$.*

Proof. We work first in the upper half-plane. Fix some $\varepsilon > 0$ and denote by U_ε the set of points in $\mathbb{H}$ whose hyperbolic distance to the imaginary axis is less than ε. We claim that U_ε is the cone:

$$\left\{ z \in \mathbb{H} : \frac{\Re(z)}{\Im(z)} < \tan \alpha \right\},$$

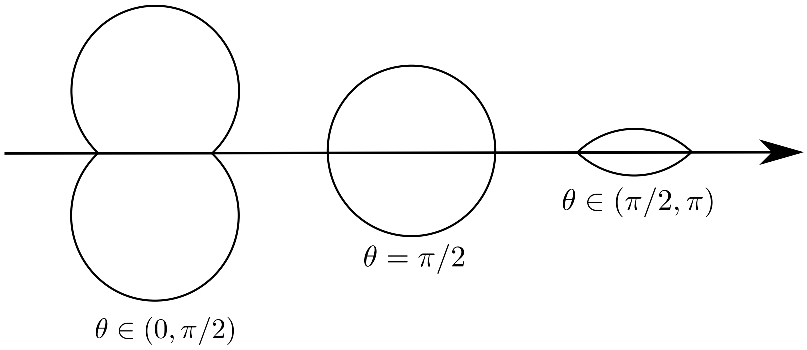

Figure C.1: Poincaré disk.

where the Euclidean angle α is related to ε by the formula:

$$\varepsilon = \left(\frac{1}{2}\right) \log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right).$$

Indeed, the geodesics orthogonal to the vertical axis are the (upper half part of the) circles centred at zero. Since the homotheties are isometries of the hyperbolic metric in $\mathbb{H}$, U_ε is a cone whose boundary is composed by two straight lines meeting the vertical geodesic $\{\Re(z) = 0\}$ at the origin and with the same Euclidean angle $\alpha \in \left(0, \frac{\pi}{2}\right)$. We will focus on the right sector, that is, the complex numbers with positive imaginary part and argument in $\left(\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right)$. All arcs of circles centred at the origin inside this part of the cone have the same length (again because the homotheties are isometries). We want to compute the angle α in terms of this length. Let $\gamma : \left(0, \frac{\pi}{2}\right) \to \mathbb{H}$ such that $\gamma(t) = e^{it}$. The distance between $\gamma(t)$ and i is:

$$\begin{aligned} d_{hyp}\big(\gamma(t), i\big) &= \int_t^{\frac{\pi}{2}} \|\gamma'(s)\|_{hyp}\, ds = \int_t^{\frac{\pi}{2}} \frac{\|\gamma'(s)\|_{euc}}{\Im\big(\gamma(s)\big)}\, ds \\ &= \int_t^{\frac{\pi}{2}} \frac{ds}{\sin s} = \int_{\cos t}^0 \frac{dx}{x^2-1} = \left(\frac{-1}{2}\right)\left(\int_{\cos t}^0 \frac{dx}{1+x} + \int_{\cos t}^0 \frac{dx}{1-x}\right) \\ &= \left(\frac{-1}{2}\right) \log\left(\frac{1-\cos t}{1+\cos t}\right) = \left(\frac{1}{2}\right) \log\left(\frac{1+\cos t}{1-\cos t}\right). \end{aligned}$$

Since $\alpha = \frac{\pi}{2} - t$, we have $\cos t = \sin\alpha$ and then:

$$\varepsilon = \left(\frac{1}{2}\right) \log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right),$$

as was claimed. Now we translate this information to the new model $\mathbb{C}_I$. Note that is enough to prove the lemma for the case $I = (-1, 1)$ (given $a < b$

in $\mathbb{R}$, the map $z \mapsto (1/2)\big((b-a)z+a+b\big)$ is an isometry between $\mathbb{C}_{(-1,1)}$ and $\mathbb{C}_{(a,b)}$ that also preserve Euclidean angles).

We need an explicit uniformization from the upper half-plane onto $\mathbb{C}_I$. In this case we cannot expect a Möbius transformation: any biholomorphism from $\mathbb{H}$ to $\mathbb{C}_I$ must have two (simple) critical points in $\partial\mathbb{H}$ whose images must be the points -1 and 1 (otherwise $\mathbb{C}_{(-1,1)}$ would contain a complex neighbourhood of -1 and 1). We want a uniformization sending the vertical geodesic $\{\Re(z) = 0\}$ onto $(-1,1)$, so 0 and ∞ will be its critical points. Consider the five rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ defined by:

- $P(z) = z^2$;
- $T_{-1}(z) = z - 1$;
- $I(z) = 1/z$;
- $T_{1/2}(z) = z + 1/2$;
- $M_2(z) = 2z$.

The first map is a quadratic polynomial, and the others four are Möbius transformations. Then $\Phi : \mathbb{H} \to \mathbb{C}_I$ defined by:

$$\Phi(z) = \big(M_2 \circ T_{1/2} \circ I \circ T_{-1} \circ P\big)(z) = \frac{z^2+1}{z^2-1}$$

is a biholomorphism that sends the imaginary axis (the part contained in $\mathbb{H}$) onto the interval $I = (-1,1)$. As we said Φ has two simple critical points at 0 and ∞ (the ones of P) whose images are, respectively, the points -1 and 1. Numbers with positive real part goes to numbers with negative imaginary part and viceversa. Points in $\mathbb{H}$ symmetric about the imaginary axis are mapped by Φ to conjugate points in $\mathbb{C}_I$.

Now we prove that $\Phi(U_\varepsilon)$ is a Poincaré disk, and compute its angle in terms of ε: since P sends lines passing through the origin to lines passing through the origin, and since we will apply the involution I to lines not containing the origin (the pole of I) we already know that the boundary of the cone in $\mathbb{H}$ is transformed by Φ onto the arc of a circle bounded by -1 and 1. By the symmetry of Φ we have that $\Phi(U_\varepsilon)$ is a Poincaré disk. We only need to compute its angle. Given $\alpha \in \big(0, \frac{\pi}{2}\big)$ define $c(\alpha)$ by:

$$c(\alpha) = \frac{1}{\tan(\pi - 2\alpha)} = \frac{-1}{\tan(2\alpha)}.$$

The map $\alpha \mapsto c(\alpha)$ is an orientation preserving real-analytic diffeomorphism between $\left(0, \frac{\pi}{2}\right)$ and the whole real line. Now we apply our composition Φ to the cone U_ε: since P double the angle between lines meeting at the origin (this is the only moment during the composition that angles are changed, since the other four maps involved are conformal diffeomorphisms), we need to apply the involution I to the straight line:

$$l = \left\{z \in \mathbb{C} : \Re(z) = c(\alpha)\lambda - 1, \Im(z) = \lambda \quad \text{for} \quad \lambda \in (0, +\infty)\right\}.$$

The angle at -1 between l and the interval $(-1, 0)$ is $\pi - 2\alpha$. Since I is conformal and -1 is a fix point, the angle at -1 between $I(l)$ and $\left(-\infty, -1\right)$ is also $\pi - 2\alpha$. This give us $\theta = \pi - 2\alpha$, now we want to relate θ directly with ε. The identities:

$$\frac{1+\sin\alpha}{1-\sin\alpha} = \left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2 = \tan^2\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = \tan^2\left(\frac{\pi}{2} - \frac{\theta}{4}\right),$$

give us:

$$\varepsilon = \left(\frac{1}{2}\right) \log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right) = \log\tan\left(\frac{\pi}{2} - \frac{\theta}{4}\right),$$

and this finishes the proof. $\square$

C.2 Annular Riemann surfaces

We say that a Riemann surface S is *annular* if its fundamental group $\pi_1(S)$ is isomorphic to $\mathbb{Z}$. For instance, any open and connected set in the Riemann sphere $\widehat{\mathbb{C}}$ whose complement has exactly two connected components is an annular Riemann surface. Besides $\mathbb{C} \setminus \{0\}$ and $\mathbb{D} \setminus \{0\}$, the canonical models are the following:

Lemma C.2.1. *Let $R > 1$ and let $A_R = \left\{z \in \mathbb{C} : R^{-1} < |z| < R\right\}$ endowed with the conformal structure induced by the complex plane.*

- *The group $\mathrm{Aut}(A_R)$ is generated by the rigid rotations and the conformal involution around the unit circle $z \mapsto 1/z$. In particular $\mathrm{Aut}(A_R)$ is isomorphic to $O(2)$, and so is a non-abelian 1-dimensional real compact Lie group with two connected components.*

- *The group of isometries of the hyperbolic metric in A_R is generated by $\mathrm{Aut}(A_R)$ together with the antiholomorphic involution $z \mapsto \bar{z}$. In particular the action of $\mathrm{Iso}(A_R)$ is non-transitive, since the orbit of a point $z_0 \in A_R$ under the action is the union of the two circles $\{z \in A_R : |z| = |z_0|\}$ and $\{z \in A_R : |z| = 1/|z_0|\}$.*

It follows that any biholomorphism or antiholomorphic diffeomorphism of A_R preserves the unit circle.

Proof. Let $\pi : \mathbb{H} \to A_R$ be the holomorphic universal covering map given by:

$$\pi(z) = \left(\frac{1}{R}\right) \exp\left[\left(\frac{2\log R}{i\pi}\right)\log z\right]$$

where $\log z$ denote the principal branch of the logarithm (for $z = re^{i\theta}$ with $0 \le \theta < 2\pi$, we have $\log z = \log r + i\theta$). Since the fundamental group of A_R is isomorphic to the integers, the group of automorphisms of the covering is generated by a single conformal diffeomorphism of the upper-half plane, which in this case is the homothety $T : \mathbb{H} \to \mathbb{H}$ such that $T(z) = \lambda z$, where $\lambda > 1$ is given by:s

$$\lambda = \exp\left(\frac{\pi^2}{\log R}\right).$$

Any biholomorphism (or antiholomorphic diffeomorphism) of A_R must lift to a biholomorphism (antiholomorphic diffeomorphism) of $\mathbb{H}$, so it must be a finite composition of the following four models:

- *Parabolic*: $z \mapsto z + b$ with $b \in \mathbb{R}$ (no fixed points and no invariant geodesics);
- *Hyperbolic*: $z \mapsto az$ with $a > 0$ (no fixed points, a unique invariant geodesic at the imaginary axis);
- *Elliptic*: $z \mapsto -1/z$ (with a single fixed point at i);
- *Orientation-reversing*: $z \mapsto -\bar{z}$ (symmetry around the imaginary axis).

Note that horizontal translations (parabolic model) do not project to the annulus A_R (they do not belong to the normalizer of the group generated by T in $\mathrm{Aut}(\mathbb{H})$). The homotheties (hyperbolic model) project to rigid rotations ($z \mapsto e^{i\theta}z$ with $\theta = (-2/\pi)\log R \log a$) and the elliptic model $z \mapsto -1/z$ projects to $z \mapsto 1/z$. Finally, the orientation-reversing model projects to $z \mapsto 1/\bar{z}$ in A_R (geometric inversion around the unit circle).

Summarizing, if $G : A_R \to A_R$ is a biholomorphism or an antiholomorphic diffeomorphism in the annulus A_R it must be a finite composition of the geometric involution around the unit circle $z \mapsto 1/\bar{z}$, rigid rotations and the conformal involution around the unit circle $z \mapsto \frac{1}{z}$. Finally, to see that $\mathrm{Aut}(A_R) \cong O(2)$, consider the isomorphism that send the involution $z \mapsto 1/z$ to the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and any rotation $z \mapsto e^{i\theta}z$ to the matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

□

The unit circle is the unique simple closed geodesic for the hyperbolic metric in A_R, and that is why any isometry must preserve it. Since $[i, \lambda i]$ is a fundamental domain for the covering of the unit circle by π, the hyperbolic length of S^1 in A_R is equal to the hyperbolic distance between i and λi in $\mathbb{H}$, which is precisely $\log\lambda = \frac{\pi^2}{\log R}$ (consider the parametrization by arc-length of the imaginary axis $\gamma : \mathbb{R} \to \mathbb{H}$ given by $\gamma(t) = ie^t$). In particular, $\log\lambda$ is a conformal invariant of A_R, and so it is R. In Chapter 5 we mentioned the following estimate:

Lemma C.2.2. *Given $1 < r < R$ consider $\alpha = \alpha(r,R) \in \left(0, \frac{\pi}{2}\right)$ and $\varepsilon = \varepsilon(r,R) > 0$ defined by:*

$$\alpha = \left(\frac{\pi}{2}\right)\left(\frac{\log r}{\log R}\right) \quad \textit{and} \quad \varepsilon = \left(\frac{1}{2}\right)\log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right).$$

Then $A_r = \left\{z \in A_R : d_{A_R}(z, S^1) < \varepsilon\right\}$, where d_{A_R} is the hyperbolic distance in the annulus A_R.

Proof. As in the proof of Lemma C.2.1 above consider the holomorphic covering $\pi : \mathbb{H} \to A_R$ given by:

$$\pi(z) = \left(\frac{1}{R}\right)\exp\left[\left(\frac{2\log R}{i\pi}\right)\log z\right].$$

The lift of the unit circle by π is the vertical geodesic of equation $\left\{\Re(z) = 0\right\}$, and the restriction of π to the cone $\{z \in \mathbb{H} : \Re(z) < \tan(\alpha)\,\Im(z)\}$ covers the annulus A_r. In the proof of Lemma C.1.1 we have shown that this cone is precisely the set of points in $\mathbb{H}$ at hyperbolic distance at most ε from the vertical geodesic $\left\{\Re(z) = 0\right\}$. □

From the Uniformization Theorem we are able to classify all annular Riemann surfaces:

Theorem C.2.3 (Uniformization of annular Riemann surfaces)**.** *Any annular Riemann surface is conformally equivalent either to* $\mathbb{C} \setminus \{0\}$*,* $\mathbb{D} \setminus \{0\}$ *or to an annulus* $A_R = \big\{z \in \mathbb{C} : R^{-1} < |z| < R\big\}$*. In the last case the value of* $R > 1$ *is unique.*

Proof. Let S be an annular Riemann surface, and suppose that S is not biholomorphic to the punctured plane $\mathbb{C} \setminus \{0\}$. By the Uniformization Theorem S is conformally equivalent to a quotient $\mathbb{H}/\Gamma$ where $\Gamma \cong \pi_1(S)$ is a subgroup of the group of Möbius transformations of $\mathbb{H}$ acting discontinuously and without fixed points. Since $\pi_1(S) \cong \mathbb{Z}$ the group Γ is generated by a single Möbius transformation $\psi : \mathbb{H} \to \mathbb{H}$, and since ψ has no fixed points in $\mathbb{H}$ and has at most two in $\overline{\mathbb{H}}$, we have two cases to consider:

1. If ψ has only one fix point in $\overline{\mathbb{H}}$ we can conjugate ψ with a suitable Möbius transformation $\widetilde{\phi} : \mathbb{H} \to \mathbb{H}$ taking this fix point to ∞ to obtain for all $z \in \mathbb{H}$ that $\widetilde{\phi}\big(\psi(z)\big) = \widetilde{\phi}(z) + 1$. Then $\widetilde{\phi}$ projects to a biholomorphism ϕ from S to the quotient of $\mathbb{H}$ by the group generated with the horizontal translation $z \mapsto z + 1$, and this quotient is conformally equivalent to $\mathbb{D} \setminus \{0\}$, via the holomorphic universal covering map $\pi : \mathbb{H} \to \mathbb{D} \setminus \{0\}$ defined by $z \mapsto \exp(2\pi i z)$.

2. If ψ has two fixed points in $\overline{\mathbb{H}}$ we can conjugate ψ with a suitable Möbius transformation $\widetilde{\phi} : \mathbb{H} \to \mathbb{H}$ taking these fixed points to 0 and ∞ to obtain for all $z \in \mathbb{H}$ that $\widetilde{\phi}\big(\psi(z)\big) = \lambda\widetilde{\phi}(z)$ for some $\lambda > 1$. Then $\widetilde{\phi}$ projects to a biholomorphism ϕ from S to the quotient of $\mathbb{H}$ by the group generated with the homothety $z \mapsto \lambda z$, and this quotient is conformally equivalent to A_R if (and only if) $R = \exp\left(\frac{\pi^2}{\log \lambda}\right)$, as we saw in Lemma C.2.1.

□

Any Riemann surface whose universal covering is $\mathbb{H}$, not conformally equivalent to $\mathbb{H}$ itself, to $\mathbb{D} \setminus \{0\}$ or to some annulus A_R, has a non-abelian fundamental group (if $\widetilde{S} = \mathbb{H}$ and $\pi_1(S)$ is abelian, it must be cyclic. See [14, Section IV.6.8]). Conversely any Riemann surface with non-abelian fundamental group is covered by $\mathbb{H}$ (see the discussion after the statement of the Uniformization Theorem for arbitrary Riemann surfaces).

Let S be an annular Riemann surface not conformally equivalent to $\mathbb{D}\setminus\{0\}$ neither to $\mathbb{C} \setminus \{0\}$, and let $R > 1$ such that $S \cong A_R$. While the hyperbolic metric in $\mathbb{D} \setminus \{0\}$ has no closed geodesics, there exists a unique simple closed geodesic for the hyperbolic metric in S whose length, equal to $\pi^2/\log R$, is a conformal invariant of S. This motivates the following definition:

Definition C.2.4. Let S be an annular Riemann surface not conformally equivalent to $\mathbb{D} \setminus \{0\}$ neither to $\mathbb{C} \setminus \{0\}$. Define the *conformal modulus* of S by $\operatorname{mod}(S) = 2 \log R$, where the constant $R > 1$ is given by Theorem C.2.3.

With this definition the length of the unique simple closed geodesic of S is equal to $2\pi^2 / \operatorname{mod}(S)$. In Chapter 3 we gave the *analytic* definition of quasiconformal homeomorphisms (see Definition 3.2.3). We give now the *geometric* definition:

Definition C.2.5. Let $K \geq 1$, and let U and V be two open and connected sets in the complex plane. An orientation-preserving homeomorphism $f : U \to V$ is K-*quasiconformal* if for any annulus A compactly contained in U we have:

$$\frac{1}{K} \operatorname{mod}(A) \leq \operatorname{mod}\big(f(A)\big) \leq K \operatorname{mod}(A).$$

We wont prove here that both definitions are equivalent, see [32, Chapter I, Section 7]. To compute the modulus of an annulus is a very hard problem, a classical approach is by using *extremal length* methods: for $R > 1$ let $A_R = \big\{z \in \mathbb{C} : R^{-1} < |z| < R\big\}$ be a round annulus symmetric around the unit circle. Let Γ_1 be the family of connected piecewise C^1 arcs contained in the annulus A_R which *join* the boundary circles of A_R, and let Γ_2 be the family of connected piecewise C^1 closed curves contained in the annulus A_R which *separate* the boundary circles (they have non-zero winding number about the origin).

For any given measurable function $\rho : \mathbb{C} \to [0, +\infty)$ denote by $A(\rho)$ the *area* of the conformal metric $\rho|dz|$, that is:

$$A(\rho) = \iint_{\mathbb{C}} \big(\rho(x+iy)\big)^2 dxdy\,.$$

We consider only those ρ such that $A(\rho) \in (0, +\infty)$. For any $\gamma : (a,b) \to \mathbb{C}$ denote by $L_\gamma(\rho)$ its *length* with respect to $\rho|dz|$, that is:

$$L_\gamma(\rho) = \int_\gamma \rho|dz| = \int_a^b \rho\big(\gamma(t)\big)\big|\gamma'(t)\big|dt$$

if $t \mapsto \rho\big(\gamma(t)\big)$ is measurable, and $L_\gamma(\rho) = +\infty$ otherwise. Finally, for $i \in \{1,2\}$ let:

$$L_i(\rho) = \inf_{\gamma \in \Gamma_i} \big\{L_\gamma(\rho)\big\}.$$

We claim that for any conformal metric $\rho|dz|$ we have:

$$\frac{L_1^2(\rho)}{A(\rho)} \leq \frac{\operatorname{mod}(A_R)}{2\pi} \leq \frac{A(\rho)}{L_2^2(\rho)}\,. \tag{C.2.1}$$

Indeed, fix $\theta \in [0, 2\pi)$ and let $\gamma \in \Gamma_1$ be the *ray* given by $\gamma(r) = re^{i\theta}$ for $r \in (1/R, R)$. Then we have:

$$L_1(\rho) \leq L_\gamma(\rho) = \int_\gamma \rho|dz| = \int_{\frac{1}{R}}^{R} \rho(re^{i\theta})dr$$

for all $\theta \in [0, 2\pi)$. By Cauchy-Schwarz inequality we get:

$$\begin{aligned}
\big(2\pi L_1(\rho)\big)^2 &\leq \left(\int_0^{2\pi}\int_{\frac{1}{R}}^{R} \rho(re^{i\theta})drd\theta\right)^2\\
&\leq \left(\int_0^{2\pi}\int_{\frac{1}{R}}^{R} \big(\rho(re^{i\theta})\big)^2 r\,drd\theta\right)\left(\int_0^{2\pi}\int_{\frac{1}{R}}^{R} \frac{dr}{r}\,d\theta\right)\\
&= \left(\iint_{A_R} \big(\rho(x+iy)\big)^2 dxdy\right) 2\pi \operatorname{mod}(A_R)\\
&\leq A(\rho) 2\pi \operatorname{mod}(A_R).
\end{aligned}$$

Therefore:

$$\frac{\operatorname{mod}(A_R)}{2\pi} \geq \frac{L_1^2(\rho)}{A(\rho)}\,,$$

as was claimed. On the other hand, for any circle γ centered at the origin with radius $r \in (1/R, R)$ we have:

$$L_2(\rho) \leq L_\gamma(\rho) = r\int_0^{2\pi} \rho(re^{i\theta})d\theta$$

since parametrizing with $\gamma(\theta) = re^{i\theta}$ gives $\big|\gamma'(\theta)\big| = r$. Then we have:

$$\frac{L_2(\rho)}{r} \leq \int_0^{2\pi} \rho(re^{i\theta})d\theta, \text{ and then } L_2(\rho)\int_{\frac{1}{R}}^{R}\frac{dr}{r} \leq \int_{\frac{1}{R}}^{R}\int_0^{2\pi} \rho(re^{i\theta})d\theta dr\,.$$

Again by Cauchy-Schwarz inequality we get that:

$$\begin{aligned}
L_2^2(\rho)\big(\operatorname{mod}(A_R)\big)^2 &\leq \left(\int_{\frac{1}{R}}^{R}\int_0^{2\pi} \rho(re^{i\theta})d\theta dr\right)^2\\
&\leq \left(\int_{\frac{1}{R}}^{R}\int_0^{2\pi} \big(\rho(re^{i\theta})\big)^2 r\,d\theta dr\right)\left(\int_{\frac{1}{R}}^{R}\int_0^{2\pi} \frac{1}{r}\,d\theta dr\right)\\
&= \left(\iint_{A_R} \big(\rho(x+iy)\big)^2 dxdy\right) 2\pi \operatorname{mod}(A_R)\\
&\leq A(\rho) 2\pi \operatorname{mod}(A_R).
\end{aligned}$$

Therefore:

$$\frac{\operatorname{mod}(A_R)}{2\pi} \leq \frac{A(\rho)}{L_2^2(\rho)}\,,$$

as was claimed. This proves inequalities (C.2.1).

A conformal metric $\rho^{ext}|dz|$ is said to be an *extremal metric* of the families Γ_1 and Γ_2 in the annulus A_R if both inequalities in (C.2.1) are equalities. The shortest curves in Γ_2 for such a metric must be the circles around the origin, which must have all the same length. Moreover ρ^{ext} must be an scalar multiple of $|z|^{-1}$ (to have equality when applying Cauchy-Schwarz). In particular ρ^{ext} must be constant when restricted to circles around the origin. If we normalize with the condition that those circles have length equal to one, we obtain the conformal metric in the complex plane $\rho^{ext}|dz|$ defined by $\rho^{ext}(z) = \frac{1}{2\pi|z|}$ for any $z \in A_R$, and $\rho^{ext} \equiv 0$ in $\mathbb{C} \setminus A_R$. Let us show that indeed both inequalities in (C.2.1) are equalities for $\rho^{ext}|dz|$. Note first that:

$$\begin{aligned}
A(\rho^{ext}) &= \iint_{A_R} \big(\rho^{ext}(x+iy)\big)^2 dxdy \\
&= \int_{\frac{1}{R}}^{R}\int_0^{2\pi} \big(\rho^{ext}(re^{i\theta})\big)^2 r\, d\theta dr \\
&= \int_{\frac{1}{R}}^{R}\int_0^{2\pi} \frac{r}{(2\pi r)^2}\, d\theta dr \\
&= \frac{\operatorname{mod}(A_R)}{2\pi}\,.
\end{aligned}$$

From the definition of ρ^{ext} we see at once that the shortest curves joining the boundary circles of A_R are the rays $\gamma(r) = re^{i\theta}$ for $r \in (1/R, R)$ and fixed $\theta \in [0, 2\pi)$. For any such ray γ we have:

$$L_\gamma(\rho^{ext}) = \int_\gamma \rho^{ext}|dz| = \int_{\frac{1}{R}}^{R} \rho^{ext}(re^{i\theta})dr = \frac{1}{2\pi}\int_{\frac{1}{R}}^{R}\frac{dr}{r} = \frac{\operatorname{mod}(A_R)}{2\pi}\,,$$

and therefore:

$$\frac{L_1^2(\rho^{ext})}{A(\rho^{ext})} = \frac{\operatorname{mod}(A_R)}{2\pi}\,,$$

as was claimed. On the other hand, for any $\gamma \in \Gamma_2$ we have:

$$L_\gamma(\rho^{ext}) = \int_\gamma \rho^{ext}|dz| = \frac{1}{2\pi}\int_\gamma \frac{|dz|}{|z|} \geq \frac{1}{2\pi}\left|\int_\gamma \frac{dz}{z}\right| = \left|\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}\right| \geq 1\,,$$

since curves in Γ_2 have non-zero winding number about the origin. But for any circle γ centered at the origin with radius $r \in (1/R, R)$ we have the equality $L_{\gamma}(\rho^{ext}) = 1$, and then $L_2(\rho^{ext}) = 1$, and then:

$$\frac{A(\rho^{ext})}{L_2^2(\rho^{ext})} = A(\rho^{ext}) = \frac{\mathrm{mod}(A_R)}{2\pi}\,.$$

Therefore the conformal metric $\rho^{ext}|dz|$ is the extremal metric (unique up to multiplication by a constant) for both families Γ_1 and Γ_2 in the annulus A_R as was claimed. For more about extremal length methods we refer the reader to the books [1, Chapter I, Section D] and [32, Chapter I, Section 6].

APPENDIX D

Proof of Proposition 3.3.2

In this appendix we give the proof of Proposition 3.3.2 of Chapter 3:

Proof of Proposition 3.3.2. Assume that each μ_n is defined in the whole complex plane, just by extending as zero in the complement of the domain U, that is:

$$\mu_n(z)\,\partial G_n(z) = \overline{\partial} G_n(z) \text{ for a.e. } z \in U, \text{ and } \mu_n(z) = 0 \text{ for all } z \in \mathbb{C} \setminus U.$$

Fix $n \in \mathbb{N}$. If $\mu_n \equiv 0$ we take $H_n = G_n|_V$, so assume that $\|\mu_n\|_\infty > 0$ and fix some small $\varepsilon \in \big(0, 1 - \|\mu_n\|_\infty\big)$. Denote by Λ the open disk $B\big(0, (1-\varepsilon)/\|\mu_n\|_\infty\big)$ centred at the origin and with radius $(1-\varepsilon)/\|\mu_n\|_\infty$ in the complex plane (note that $\overline{\mathbb{D}} \subset \Lambda$). Consider the one-parameter family of Beltrami coefficients $\big\{\mu_n(t)\big\}_{t \in \Lambda}$ defined by:

$$\mu_n(t) = t\mu_n.$$

Note that for all $t \in \Lambda$ we have $\big\|\mu_n(t)\big\|_\infty < 1-\varepsilon < 1$. Denote by $f^{\mu_n(t)}$ the solution of the Beltrami equation with coefficient $\mu_n(t)$, given by Theorem 3.2.4, normalized to fix 0, 1 and ∞. Note that $f^{\mu_n(0)}$ is the identity and that, by uniqueness, there exists a biholomorphism $H_n : f^{\mu_n(1)}(U) \to G_n(U)$ such that:

$$G_n = H_n \circ f^{\mu_n(1)} \text{ in } U.$$

By Ahlfors-Bers theorem (Theorem 3.3.1) we know that for any $z \in \mathbb{C}$ the curve $\big\{f^{\mu_n(t)}(z) : t \in [0,1]\big\}$ is smooth, that is, the derivative of $f^{\mu_n(t)}$ with respect to the parameter t exists at any $z \in \mathbb{C}$ and any $s \in [0,1]$. Following Ahlfors [1, Chapter V, Section C], we use the notation:

$$\dot{f}_n(z,s) = \lim_{t \to 0} \frac{f^{\mu_n(s+t)}(z) - f^{\mu_n(s)}(z)}{t}\,.$$

The limit exists for every $z \in \mathbb{C}$ and every $s \in [0,1]$ (actually for every $s \in \Lambda$), and the convergence is uniform on compact sets of $\mathbb{C}$. Then we have:

$$\left\| f^{\mu_n(1)} - Id \right\|_{C^0(U)} = \sup_{z \in U} \left\{ \left| f^{\mu_n(1)}(z) - z \right| \right\} \leq \sup_{z \in U} \left\{ \int_0^1 \left| \dot{f}_n(z,s) \right| ds \right\}.$$

Moreover, $\dot{f}_n$ has the following integral representation (see [1, Chapter V, Section C, Theorem 5] for the explicit computation):

$$\dot{f}_n(z,s) = -\left(\frac{1}{\pi}\right) \iint_{\mathbb{C}} \mu_n(w)\, S\big(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z)\big) \left(\partial f^{\mu_n(s)}(w)\right)^2 dxdy\,,$$

for every $z \in \mathbb{C}$ and every $s \in [0,1]$, where $w = x + iy$ and:

$$S(w,z) = \frac{1}{w-z} - \frac{z}{w-1} + \frac{z-1}{w} = \frac{z(z-1)}{w(w-1)(w-z)}\,.$$

Since each μ_n is supported in U we have:

$$\dot{f}_n(z,s) = -\left(\frac{1}{\pi}\right) \iint_{U} \mu_n(w)\, S\big(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z)\big) \left(\partial f^{\mu_n(s)}(w)\right)^2 dxdy\,.$$

From the formula:

$$\left|\partial f^{\mu_n(s)}(w)\right|^2 = \left(\frac{1}{1 - |s|^2 |\mu_n(w)|^2}\right) \det\left(Df^{\mu_n(s)}(w)\right)$$

we obtain:

$$\begin{aligned}
\left|\dot{f}_n(z,s)\right| &\leq \frac{1}{\pi} \iint_U \left(\frac{|\mu_n(w)|}{1 - |s|^2 |\mu_n(w)|^2}\right) \det\left(Df^{\mu_n(s)}(w)\right) \left| S\big(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z)\big)\right| dxdy \\
&\leq \frac{1}{\pi} \left(\frac{\|\mu_n\|_\infty}{1 - |s|^2 \|\mu_n\|_\infty^2}\right) \iint_U \det\left(Df^{\mu_n(s)}(w)\right) \left| S\big(f^{\mu_n(s)}(w), f^{\mu_n(s)}(z)\big)\right| dxdy \\
&= \frac{1}{\pi} \left(\frac{\|\mu_n\|_\infty}{1 - |s|^2 \|\mu_n\|_\infty^2}\right) \iint_{f^{\mu_n(s)}(U)} \left| S\big(w, f^{\mu_n(s)}(z)\big)\right| dxdy\,.
\end{aligned}$$

Therefore the length of the curve $\left\{ f^{\mu_n(t)}(z) : t \in [0,1] \right\}$ is less or equal than:

$$\frac{1}{\pi} \int_0^1 \left[\left(\frac{\|\mu_n\|_\infty}{1 - |s|^2 \|\mu_n\|_\infty^2}\right) \iint_{f^{\mu_n(s)}(U)} \left| S\big(w, f^{\mu_n(s)}(z)\big)\right| dxdy \right] ds \leq$$

$$\leq \left(\frac{1}{\pi}\right)\left(\frac{\|\mu_n\|_\infty}{1-\|\mu_n\|_\infty^2}\right)\int_0^1\left[\iint_{f^{\mu_n(s)}(U)}\big|S\big(w,f^{\mu_n(s)}(z)\big)\big|dxdy\right]ds\,.$$

If we define:

$$M_n(U)=\left(\frac{1}{\pi}\right)\sup_{z\in U}\left\{\int_0^1\left[\iint_{f^{\mu_n(s)}(U)}\big|S\big(w,f^{\mu_n(s)}(z)\big)\big|dxdy\right]ds\right\},$$

we get:

$$\big\|f^{\mu_n(1)}-Id\big\|_{C^0(U)}\leq\left(\frac{\|\mu_n\|_\infty}{1-\|\mu_n\|_\infty^2}\right)M_n(U).$$

We have two remarks:

First remark: since $\mu_n\to 0$ in the unit ball of L^∞, we know by Proposition 3.2.5 that for any $s\in[0,1]$ the normalized quasiconformal homeomorphisms $f^{\mu_n(s)}$ converge to the identity uniformly on compact sets of $\mathbb{C}$, in particular on $\overline{U}$. Therefore the sequence $M_n(U)$ converge to:

$$\left(\frac{1}{\pi}\right)\sup_{z\in U}\left\{\iint_U\big|S\big(w,z\big)\big|dxdy\right\}<\left(\frac{1}{\pi}\right)\sup_{z\in U}\left\{\iint_{\mathbb{C}}\big|S\big(w,z\big)\big|dxdy\right\}<\infty.$$

For fixed $z\in\mathbb{C}$ we have that $S(w,z)$ is in $L^1(\mathbb{C})$ since it has simple poles at 0, 1 and z, and is $O\big(|w|^{-3}\big)$ near ∞. The finiteness follows then from the compactness of $\overline{U}$.

Second remark: $x\mapsto x/(1-x^2)$ is an orientation-preserving real-analytic diffeomorphism between $(-1,1)$ and the real line, which is tangent to the identity at the origin. In fact $x/(1-x^2)=x+o(x^2)$ in $(-1,1)$.

With this two remarks we obtain $n_1\in\mathbb{N}$ such that for all $n\geq n_1$ we have:

$$\big\|f^{\mu_n(1)}-Id\big\|_{C^0(U)}\leq M(U)\|\mu_n\|_\infty,$$

where:

$$M(U)=\left(\frac{2}{\pi}\right)\sup_{z\in U}\left\{\iint_U\big|S\big(w,z\big)\big|dxdy\right\}.$$

Since V is compactly contained in the bounded domain U, the boundaries ∂V and ∂U are disjoint compact sets. Let $\delta>0$ be its Euclidean distance, that is, $\delta=d\big(\partial V,\partial U\big)=\min\big\{|z-w|\,:\,z\in\partial V,w\in\partial U\big\}$. Again by Proposition 3.2.5 we know, since $\mu_n\to 0$, that there exists $n_0\geq n_1$ in $\mathbb{N}$ such that for all $n\geq n_0$ we have $V\subset f^{\mu_n(1)}(U)$ and moreover:

$$f^{\mu_n(1)}(U)\supseteq B(z,\delta/2)\quad\text{for all } z\in V.$$

If we consider the restriction of H_n to the domain V we have:

$$\begin{aligned}\|H_n-G_n\|_{C^0(V)}&\leq\|H_n'\|_{C^0(V)}\,\big\|f^{\mu_n(1)}-Id\big\|_{C^0(U)}\\&\leq\|H_n'\|_{C^0(V)}\,M(U)\|\mu_n\|_\infty.\end{aligned}$$

By Cauchy's derivative estimate we know that for all $z \in V$:

$$\begin{aligned}
\big|H_n'(z)\big| &= \left|\frac{1}{2\pi i}\int_{\partial B(z,\delta/2)}\frac{H_n(w)}{(w-z)^2}\,dw\right|\\
&\leq \frac{2\|H_n\|_{C^0(f^{\mu_n(1)}(U))}}{\delta}\\
&= \frac{2\|G_n\|_{C^0(U)}}{\delta}\\
&\leq \frac{2R}{\delta}\quad \text{for all } n \geq n_0.
\end{aligned}$$

That is:

$$\big\|H_n'\big\|_{C^0(V)} \leq \frac{2R}{d\big(\partial V, \partial U\big)}\quad \text{for all } n \geq n_0,$$

and we obtain that for all $n \geq n_0$:

$$\frac{\big\|H_n - G_n\big\|_{C^0(V)}}{\|\mu_n\|_\infty} \leq \left(\frac{R}{d\big(\partial V, \partial U\big)}\right)\left(\frac{4}{\pi}\right)\sup_{z\in U}\left\{\iint_U \big|S\big(w,z\big)\big|dxdy\right\}.$$

Therefore is enough to consider:

$$C(U) = \left(\frac{4}{\pi}\right)\sup_{z\in U}\left\{\iint_U \big|S\big(w,z\big)\big|dxdy\right\}.$$

$\square$

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