Geometric properties of the Markov and Lagrange spectra

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IMPA

1 Introduction and statement of the results

Let α be an irrational number. According to Dirichlet's theorem, the inequality $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ has infinitely many rational solutions $\frac{p}{q}$. Hurwitz improved this result by proving that $|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$ also has infinitely many rational solutions $\frac{p}{q}$ for any irrational α, and that $\sqrt{5}$ is the largest constant that works for any irrational α. However, for particular values of α we can improve this constant.

More precisely, if we define $k(\alpha) := \sup\{k > 0 \mid |\alpha - \frac{p}{q}| < \frac{1}{kq^2}$ has infinitely many rational solutions $\frac{p}{q}\} = \limsup_{p,q\to+\infty} (q|q\alpha - p|)^{-1}$. We have $k(\alpha) \geq \sqrt{5}$, $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $k\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$. We can define the set $L = \{k(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q},\, k(\alpha) < +\infty\}$.

This set is called the Lagrange spectrum. Hurwitz theorem determines the first element of L, which is $\sqrt{5}$. This set L encodes many diophantine properties of real numbers. It is a classical subject the study of the geometric structure of L. Markov proved in 1879 ([Ma]) that

$$L \cap (-\infty, 3) = \{k_1 = \sqrt{5} < k_2 = 2\sqrt{2} < k_3 = \frac{\sqrt{221}}{5} < \dots\}$$

where k_n is a sequence (of irrational numbers whose squares are rational) converging to 3, which means that the "beginning" of the set L is discrete. This is not true for the whole set L. Indeed, M. Hall proved in 1947 ([H]) that L contains a whole half line (for instance $[6, +\infty)$), and G. Freiman determined in 1975 ([F]) the biggest half line that is contained in L, which is $[c, +\infty)$, with

$$c = \frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616\dots.$$

These last two results are based on the study of sums of regular Cantor sets, whose relationship with the Lagrange spectrum will be explained below.

If the continued fraction of α is $\alpha = [a_0, a_1, a_2, \dots]$ then we have the following formula:

$$k(\alpha) = \limsup_{n\to\infty}(\alpha_n + \beta_n),$$

where $\alpha_n = [a_n, a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = [0, a_{n-1}, a_{n-2}, \dots, a_1]$.

This follows from the equality

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{(\alpha_{n+1} + \beta_{n+1})q_n^2}\ , \quad \forall n \in \mathbb{N},$$

which can be easily proved by induction.

This formula for $k(\alpha)$ implies that we have the following alternative definition of the Lagrange spectrum L:

Let $\Sigma = \mathbb{N}^{\mathbb{Z}}$ be the set of all bi-infinite sequences of positive integers. If $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$, let $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \dots], \forall n \in \mathbb{Z}$. We define $f(\underline{\theta}) = \alpha_0 + \beta_0 = [a_0; a_1, a_2, \dots] + [0; a_{-1}, a_{-2}, \dots]$. We have $L = \{\limsup_{n\to\infty} f(\sigma^n\underline{\theta}), \underline{\theta} \in \Sigma\}$, where $\sigma\colon \Sigma \to \Sigma$ is the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

Let us define the Markov spectrum M by $M = \{\sup_{n\in\mathbb{Z}} f(\sigma^n\underline{\theta}), \underline{\theta} \in \Sigma\}$. It also has an arithmetical interpretation, namely

$$M = \{(\inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|)^{-1}, \quad f(x,y) = ax^2 + bxy + cy^2, \quad b^2 - 4ac = 1\}.$$

It is well-known (see [CF]) that M and L are closed sets of the real line and $L \subset M$.

We have the following results about the Markov and Lagrange spectra:

Theorem 1: *Given $t \in \mathbb{R}$ we have*

$$HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t)) =: d(t)$$

(here HD denotes Hausdorff dimension), and $d(t)$ is a continuous surjective function from $\mathbb{R}$ to $[0,1]$. Moreover:

i) $d(t) = \min\{1, 2D(t)\}$, where $D(t) := HD(k^{-1}(-\infty, t)) = HD(k^{-1}(-\infty, t])$ is a continuous function from $\mathbb{R}$ to $[0,1)$.

ii) $\max\{t \in \mathbb{R} \mid d(t) = 0\} = 3$

iii) $d(\sqrt{12}) = 1$.

(indeed ii) and iii) are consequences of i)).

This theorem solves affirmatively Problem 3 of [B]. It also gives some answers to Problem 5 of the same paper: the continuous function $d(t) = HD(L \cap (-\infty, t))$, which coincides (for $t > 0$) with $\sigma(1/t)$, in the notation of [B], is a Cantor stair function - it is constant in the connected components of the complement of $L \cap (-\infty, t_1]$, where $t_1 := \min\{t \in \mathbb{R} \mid d(t) = 1\}$; notice that $L \cap (-\infty, t_1]$ is a compact set with zero Lebesgue measure, and so with empty interior. On the other hand, by its definition, $d(t)$ cannot be Hölder continuous with any exponent, since, for $\varepsilon > 0$ small, it sends the set $L \cap (-\infty, 3+\varepsilon]$, whose Hausdorff dimension $d(3+\varepsilon)$ is a small positive number to the nontrivial interval $[0, d(3+\varepsilon)]$ - this implies that any Hölder exponent $\alpha > 0$ for the function $d(t)$ should satisfy $\alpha \leq d(3+\varepsilon), \forall \varepsilon > 0$, and thus $\alpha = 0$, a contradiction.

The proof of Theorem 1 is based on the idea of approximating parts of the spectra from inside and from outside by sums of regular Cantor sets. Theorem 1 uses techniques developed in a joint work with J.C. Yoccoz about sums of Cantor sets that implies that the sum of two non essentially affine regular Cantor sets have Hausdorff dimension equal to the minimum between one and the sum of their Hausdorff dimensions. This result will be discussed in the next section.

Theorem 2: $\lim_{t\to\infty} HD(k^{-1}(t)) = 1$.

This in particular solves affirmatively Problem 4 of [B].

We also prove a result on the topological structure of the Lagrange spectrum L:

Theorem 3: *L' is a perfect set, i.e., $L'' = L'$.*

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2 A dimension formula for arithmetic sums of regular Cantor sets

We say that $K \subset \mathbb{R}$ is a *regular Cantor set of class C^k*, $k \geq 1$, if:

i) there are disjoint compact intervals $I_1, I_2, \dots, I_r$ such that $K \subset I_1 \cup \dots \cup I_r$ and the boundary of each I_j is contained in K;

ii) there is a C^k expanding map ψ defined in a neighbourhood of $I_1 \cup I_2 \cup \cdots \cup I_r$ such that $\psi(I_j)$ is the convex hull of a finite union of some intervals I_s satisfying:

ii.1) for each j, $1 \le j \le r$ and n sufficiently big, $\psi^n(K \cap I_j) = K$;

ii.2) $K = \bigcap_{n \in \mathbb{N}} \psi^{-n}(I_1 \cup I_2 \cup \cdots \cup I_r)$.

We say that $\{I_1, I_2, \ldots, I_r\}$ is a *Markov partition* for K and that K is *defined* by ψ.

Let K, K' be regular Cantor sets of class C^2. Let ψ be the expansive function which defines K. It is a general fact that, given a periodic point p of period r of ψ, there is a C^2 diffeomorphism h of the support interval I of K such that $h^{-1} \circ \psi^r \circ h$ is affine in $h^{-1}(J)$, where J is the connected component of the domain of ψ^r which contains p. Defining $\tilde{\psi} := h^{-1} \circ \psi \circ h$, we say that K is *non essentially affine* if $(\tilde{\psi^r})''(x) \neq 0$ for some $x \in h^{-1}(K)$.

In [Mo], we use the Scale recurrence Lemma of [MY] in order to prove the following

Theorem. *If K and K' are regular Cantor sets of class C^2 and K is non essentially affine, then $HD(K + K') = \min\{HD(K) + HD(K'), 1\}$.*

3 Regular Cantor sets defined by the Gauss map

The Gauss map is the map $g\colon (0,1] \to [0,1]$ given by $g(x) = \{\frac{1}{x}\} = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$, $\forall\, x \in (0,1]$. Regular Cantor sets defined by the Gauss map (or iterates of it) restricted to some finite union of intervals are closely related to continued fractions with bounded partial quotients. We will often consider such regular Cantor sets associated to *complete shifts*. A complete shift is associated to finite sets of finite sequences of positive integers. Given a finite set $B = \{\beta_1, \beta_2, \ldots, \beta_m\}$, $m \ge 2$, where $\beta_j \in (\mathbb{N}^*)^{r_j}$, $r_j \in \mathbb{N}^*$, $1 \le j \le m$ and β_i does not begin by β_j for $i \neq j$, the complete shift associated to B is the set $\Sigma(B) \subset (\mathbb{N}^*)^{\mathbb{N}}$ of the finite sequences obtained by concatenations of elements of B, that is, $\Sigma(B) = \{(\alpha_0, \alpha_1, \alpha_2, \ldots) \mid \alpha_j \in B,\ \forall\, j \in \mathbb{N}\}$. Here (and in the rest of the paper), we use the following notation for concatenations of finite sequences: if $\alpha_j = (a_j^{(1)}, a_j^{(2)}, \ldots, \alpha_j^{(m_j)})$ then $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ means the sequence $(a_0^{(1)}, a_0^{(2)}, \ldots, \alpha_0^{(m_0)}, a_1^{(1)}, a_1^{(2)}, \ldots, \alpha_1^{(m_1)}, a_2^{(1)}, a_2^{(2)}, \ldots, \alpha_2^{(m_2)}, \ldots)$. In some cases some of the α_j are finite sequences and some cases are single numbers, which are viewed as one-element sequences for this notation. Associated to $\Sigma(B)$ is the Cantor set $K(B) \subset [0,1]$ of the real numbers whose continued fractions are of the form $[0;\gamma_1,\gamma_2,\gamma_3,\dots]$, where $\gamma_j \in B$, $\forall\, j \geq 1$. It is a regular Cantor set. Indeed, if a_j and b_j are respectively the smallest and the largest elements of $K(B)$ whose continued fractions begin by $[0;\beta_j]$, for $1 \leq j \leq m$, and $I_j = [a_j, b_j]$ then $K(B)$ is the regular Cantor set defined for the map ψ with domain $\bigcup_{j=1}^{m} I_j$ given by $\psi|_{I_j} = g^{r_j}$, $1 \leq j \leq m$.

We have the following

Proposition. *The Cantor sets $K(B)$ defined by the Gauss map associated to complete shifts are non essentially affine.*

Proof: Let $B = \{\beta_1, \beta_2, \dots, \beta_m\}$, $\beta_j \in (\mathbb{N}^*)^{r_j}$, $1 \leq j \leq m$. For each $j \leq m$, let $x_j = [0;\beta_j,\beta_j,\beta_j,\dots] \in I_j$ be the fixed point of $\psi|_{I_j} = g^{r_j}$. Moreover, according to the classical theory of continued fractions, if $p_k^{(j)}/q_k^{(j)} := [0; b_1^{(j)}, b_2^{(j)}, \dots, b_k^{(j)}]$, for $1 \leq j \leq m$, $1 \leq k \leq r_j$, we have $I_j \subset \{[0;\beta_j,\alpha],\ \alpha \geq 1\}$ and $\psi|_{I_j}(x)$ is given by

$$\psi|_{I_j(x)} = \frac{q_n^{(j)} x - p_n^{(j)}}{-q_{n-1}^{(j)} x + p_{n-1}^{(j)}};$$

so x_j is the positive root of $q_{n-1}^{(j)} x^2 + (q_n^{(j)} - p_{n-1}^{(j)})x - p_n^{(j)} = 0$ (since x_j is the fixed point of $\psi|_{I_j}$).

For each $j \leq m$, since $\psi|_{I_j}$ is a Möbius function with a hyperbolic fixed point x_j, there is a Möbius function $\alpha_j(x) = \frac{a_j x + b_j}{a_j x + d_j}$ with $\alpha_j(x_j) = x_j$, $\alpha_j'(x_j) = 1$ such that $\alpha_j \circ (\psi|_{I_j}) \circ \alpha_j^{-1}$ is an affine map. If we show that the Möbius functions $\alpha_1 \circ (\psi|_{I_2}) \circ \alpha_1^{-1}$ is not affine then we are done, since the second derivative of a non-affine Möbius function never vanishes.

Suppose by contradiction that $\alpha_1 \circ (\psi|_{I_2}) \circ \alpha_1^{-1}$ is affine. Since $\alpha_1 \circ (\psi|_{I_1}) \circ \alpha_1^{-1}$ is also affine these two functions have a common fixed point at ∞, so $\alpha_1^{-1}(\infty) = -d_1/c_1$ is a common fixed point of $\psi|_{I_2}$ and $\psi|_{I_1}$, which implies that $\alpha_1^{-1}(\infty)$ is a common root of $q_n^{(1)} x^2 + (q_{n-1}^{(1)} - p_n^{(1)})x - p_{n-1}^{(1)}$ and $q_n^{(2)} x^2 + (q_{n-1}^{(2)} - p_n^{(2)})x - p_{n-1}^{(2)}$. Since these polynomials of $\mathbb{Z}[x]$ are monic and irreducible (indeed x_1 and x_2 are irrational), they must coincide, and so their remaining roots x_1 and x_2 must coincide, which is a contradiction. $\square$

Definition: If $\beta = (b_1, b_2, \dots, b_{n-1}, b_n)$ then $\beta^t := (b_n, b_{n-1}, \dots, b_2, b_1)$. Given a set of finite sequences B, we define $B^t := \{\beta^t, \beta \in B\}$.

Proposition. $HD(K(B)) = HD(K(B^t)),\ \forall\ B$.

Proof: This follows from $q_n(\beta) = q_n(\beta^t), \quad \forall \beta$ (see the appendix of [CF] on properties of continuants). $\square$

Corollary. $HD(K(B) + K(B^t)) = \min\{1, 2 \cdot HD(K(B))\}$, for every set B of finite sequences of positive integers.

4 Fractal dimensions of the spectra

We recall that the Lagrange spectrum is given by $L = \{\ell(\underline{\theta}),\ \underline{\theta} \in \Sigma\}$, where $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ and, for $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$, $\ell(\underline{\theta}) := \limsup_{n\to+\infty}(\alpha_n + \beta_n)$, where α_n and β_n are defined as the continued fractions $\alpha_n := [a_n, a_{n+1}, a_{n+2}, \dots]$ and $\beta_n := [0, a_{n-1}, a_{n-2}, \dots]$, while the Markov spectrum is given by $M = \{m(\underline{\theta}), \underline{\theta} \in \Sigma\}$, where $m(\underline{\theta}) = \sup\{\alpha_n + \beta_n,\ n \in \mathbb{Z}\}$.

Let $\Sigma = (\mathbb{N}^*)^{\mathbb{N}} = \Sigma^- \times \Sigma^+$, where $\Sigma^- = (\mathbb{N}^*)^{\mathbb{Z}_-}$ and $\Sigma^+ = (\mathbb{N}^*)^{\mathbb{N}}$, and $\sigma\colon \Sigma \to \Sigma$ the shift given by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$. We will work with a one-parameter family of subshifts of Σ given by $\Sigma_t = \{\underline{\theta} \in \Sigma \mid m(\underline{\theta}) \le t\}$; for $t \in \mathbb{R}$ (in fact we will take $t \ge 3$).

Given a finite sequence $\alpha = (a_1, a_2, \dots, a_n) \in (\mathbb{N}^*)^n$, we define its *size* by $s(\alpha) := |I(\alpha)|$, where $I(\alpha)$ is the interval $\{x \in [0,1] \mid x = [0; a_1, a_2, \dots, a_n, \alpha_{n+1}],\ \alpha_{n+1} \ge 1\}$. If we take $p_0 = 0$, $q_0 = 1$, $p_1 = 1$, $q_1 = a_1$ and, for $k \ge 0$, $p_{k+2} = a_{k+2}p_{k+1} + p_k$ and $q_{k+2} = a_{k+2}q_{k+1} + q_k$, we have that $I(\alpha)$ is the interval with extremities $[0; a_1, a_2, \dots, a_n] = p_n/q_n$ and $[0; a_1, a_2, \dots, a_{n-1}, a_n + 1] = \frac{p_n+p_{n-1}}{q_n+q_{n-1}}$, and so

$$s(\alpha) = \left|\frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right| = \frac{1}{q_n(q_n + q_{n-1})}$$

(since $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$). We define $r(\alpha) = \lfloor \log s(\alpha)^{-1} \rfloor$. We also define, for $r \in \mathbb{N}, P_r = \{\alpha = (a_1, a_2, \dots, a_n) \mid r(\alpha) \ge r \text{ and } r((a_1, a_2, \dots, a_{n-1})) < r\}$.

If $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$ then $m(\underline{\theta}) \ge \sup\{a_n, n \in \mathbb{Z}\}$. Thus, if $\sup\{a_n, n \in \mathbb{Z}\} \ge 5$, then $m(\underline{\theta})$ belongs to Hall's ray. So we will assume that $a_n \le 4, \forall\ n \in \mathbb{N}$, and $m(\underline{\theta}) < 5$. Given $t \in [3,5]$ and $r \in \mathbb{N}$, let $C(t,r)$ be the set $\{\alpha = (a_1, a_2, \dots, a_n) \in P_r \mid K_t \cap I(\alpha) \ne \emptyset\}$. Here $K_t := \{[0;\gamma], \gamma \in \pi_+(\Sigma_t)\}$, where $\pi_+\colon \Sigma \to \Sigma^+$ is the projection associated to the decomposition $\Sigma = \Sigma^- \times \Sigma^+$. We define $N(t,r) := |C(t,r)|$.

It is not difficult to show that for any finite sequences α, β and any $k \in \{1, 2, 3, 4\}$ we have $r(\alpha\beta k) \geq r(\alpha) + r(\beta)$, so if $C(t, r) = \{\alpha_1, \alpha_2, \ldots, \alpha_u\}$ and $C(t, s) = \{\beta_1, \beta_2, \ldots, \beta_v\}$, we may cover K_t by the $4uv = 4N(t, r)N(t, s)$ intervals $I(\alpha_i\beta_j k)$, $1 \leq i \leq u$, $1 \leq j \leq v$, $1 \leq k \leq 4$ which satisfy $r(\alpha_i\beta_j k) \geq r + s$, $\forall\, i, j, k$. Replacing, if necessary, some of these intervals by larger intervals $I(\gamma)$ with $r(\gamma) = r + s$, we conclude that $N(t, r + s) \leq 4N(t, r)N(t, s)$, so

$$\log(4N(t, r + s)) \leq \log(4N(t, r)) + \log(4N(t, s)), \quad \forall\, r, s.$$

This implies that

$$\lim_{m\to\infty} \frac{1}{m} \log(4N(t, m)) = \inf_{n\in\mathbb{N}^*} \frac{1}{m} \log(4N(t, m)) = \lim_{m\to\infty} \frac{1}{m} \log(N(t, m))$$

exists. We will call this limit $D(t)$ (which coincides with the box dimension of K_t, as follows easily from its definition).

Lemma 1. *Given $t \in [3, 5]$ and $\eta \in (0, 1)$ there is $\delta > 0$ and a Cantor set $K(B)$ defined by the Gauss map associated to a complete shift $\Sigma(B) \subset \{1, 2, 3, 4\}^{\mathbb{N}}$ such that $\Sigma(B) \subset \Sigma_{t-\delta}$ and $HD(K(B)) > (1 - \eta)D(t)$.*

Proof: Let $\tau = \eta/40$. Choose $r_0 \in \mathbb{N}$ large such that, for $r \geq r_0$, $|\frac{\log N(t,r)}{r} - D(t)| < \frac{\tau}{2}D(t)$. Let $B_0 := C(t, r_0)$ and $N_0 := N(t, r_0) = |B_0|$. Let $k = 8N_0^2\lceil 2/\tau\rceil$. Take $\tilde{B} = \{\beta = \beta_1\beta_2 \ldots \beta_k \mid \beta_j \in B_0, 1 \leq j \leq k \text{ and } K_t \cap I(\beta) \neq \emptyset\}$.

Given $\beta = \beta_1\beta_2 \ldots \beta_k \in \tilde{B}$ with $\beta_i \in B_0$, $1 \leq i \leq k$, we say that j, $1 \leq j \leq k$, is a *right-good* position of β if there are elements $\beta^{(s)} = \beta_1\beta_2 \ldots \beta_{j-1}\beta_j^{(s)}\beta_{j+1}^{(s)} \ldots \beta_k^{(s)}$, $s = 1, 2$ of $\tilde{B}$ such that we have the following inequality of continued fractions: $[0; \beta_j^{(1)}] < [0; \beta_j] < [0; \beta_j^{(2)}]$. We say that j is a *left-good* position if there are elements $\beta^{(s)} = \beta_1\beta_2 \ldots \beta_{j-1}\beta_j^{(s)}\beta_{j+1}^{(s)} \ldots \beta_k^{(s)}$, $s = 3, 4$ of $\tilde{B}$ such that $[0; (\beta_j^{(3)})^t] < [0; \beta_j^t] < [0; (\beta_j^{(4)})^t]$. Finally, we say that j is a *good* position if it is both right-good and left-good.

We will show that most positions of most words of $\tilde{B}$ are good. Let us first estimate $|\tilde{B}|$. It is not difficult to show that, for $\beta \in \tilde{B}$, $|I(\beta)| < (2e^{-r_0})^k < e^{-k(r_0-1)}$. Moreover, since $N(t, k(r_0 - 1)) \geq \frac{1}{4}e^{k(r_0-1)D(t)}$, $\{I(\beta); \beta \in \tilde{B}\}$ form a covering of K_t by intervals of size smaller than $e^{-k(r_0-1)}$ and the function $h : \tilde{B} \to C(t, k(r_0 - 1))$ defined by $h(\beta) = h((b_1b_2 \ldots b_k)) = (b_1b_2 \ldots b_j)$, where $j = min\{i\ ; i \leq k \text{ and } r((b_1b_2 \ldots b_i)) \geq k(r_0 - 1)\ \}$ is onto, we have:

$$
\begin{aligned}
|\tilde{B}| \geq \frac{1}{4} e^{k(r_0-1)D(t)} &> 2\, e^{k(r_0-2)D(t)} \\
&\geq 2\, e^{(1-\tau/2)r_0 k D(t)}, \quad \text{since } r_0 \text{ is large} \\
&> 2\, e^{(1-\tau)(1+\tau/2)r_0 k D(t)} \\
&> 2\, N_0^{(1-\tau)k}, \quad \text{since } N(t, r_0) < e^{(1+\frac{\tau}{2})D(t) r_0}.
\end{aligned}
$$

Now, let us estimate the number of words $\beta \in \tilde{B}$ such that at least $k/20$ positions of β are not right good: we have at most 2^k choices for the set of the $m \geq k/20$ positions which are not right-good. Once we choose this set of positions, if j is such a position and $\beta_1, \beta_2, \ldots, \beta_{j-1} \in B_0$ are already chosen, there are at most two (the largest and the smallest) choices for $\beta_j \in B_0$ such that for some $\beta = \beta_1\beta_2 \ldots \beta_{j-1}\beta_j\beta_{j+1} \ldots \beta_k \in \tilde{B}$ the position j is not right good. If j is any other position we have of course at most $N_0 = |B_0|$ possible choices for β_j, so we have at most $2^m \cdot N_0^{k-m} \leq 2^{k/20} N_0^{19k/20}$ words in $\tilde{B}$ with this chosen set of m positions which are not right-good. Therefore, the number of words $\beta \in \tilde{B}$ for which the number of positions which are not right-good is at least $k/20$ is bounded by $2^k \cdot 2^{k/20} \cdot N_0^{19k/20} = 2^{21k/20} \cdot N_0^{19k/20}$. Analogously, the number of words $\beta \in \tilde{B}$ for which there are at least $k/20$ positions which are not left-good is also bounded by $2^{21k/20} \cdot N_0^{19k/20}$.

This implies that for at least $|\tilde{B}| - 2 \cdot 2^{21k/20} \cdot N_0^{19k/20} > 2N_0^{(1-\tau)k} - 2^{1+21k/20} \cdot N_0^{19k/20} > N_0^{(1-\tau)k}$ words of $\tilde{B}$, the number of good positions is at least $9k/10$. Let us call such a word of $\tilde{B}$ an *excellent* word.

If $\beta = \beta_1\beta_2 \ldots \beta_k \in \tilde{B}$ (with $\beta_j \in B_0$, $1 \leq j \leq k$) is an excellent word, we may find $\lceil 2k/5 \rceil$ positions $i_1, i_2, \ldots, i_{\lceil 2k/5 \rceil} \leq k$ with $i_{s+1} \geq i_s + 2$, $\forall\, s < \lceil 2k/5 \rceil$ such that the positions i_1, i_1+1, $i_2, i_2+1, \ldots, i_{\lceil 2k/5 \rceil}$, $i_{\lceil 2k/5 \rceil}+1$ are good. Since $k = 8N_0^2\lceil 2/\tau \rceil$, we may take, for $1 \leq s \leq 3N_0^2$, $j_s := i_{s\lceil 2/\tau \rceil}$ (notice that $3N_0^2\lceil 2/\tau \rceil < \frac{16}{5}N_0^2\lceil 2/\tau \rceil = 2k/5$), so we have $j_{s+1} - j_s \geq 2\lceil 2/\tau \rceil$, $\forall\, s < 3N_0^2$ and the positions j_s, j_{s+1} are good for $1 \leq s \leq 3N_0^2$.

Now, the number of possible choices of $(j_1, j_2, \ldots, j_{3N_0^2})$ is bounded by $\binom{k}{3N_0^2} < 2^k$ and, given $(j_1, j_2, \ldots, j_{3N^2})$ the number of choices of $(\beta_{j_0}, \beta_{j_0+1}, \ldots, \beta_{j_{3N_0^2}}, \beta_{j_{3N_0^2+1}})$ is bounded by $N_0^{6N_0^2}$. So, we may choose $\hat{\jmath}_1, \hat{\jmath}_2, \ldots, \hat{\jmath}_{3N^2}$ with $\hat{\jmath}_{s+1} - \hat{\jmath}_s \geq 2\lceil 2/\tau \rceil$, $\forall\, s < 3N_0^2$ and words $\hat{\beta}_{\hat{\jmath}_1}, \hat{\beta}_{\hat{\jmath}_1+1}, \hat{\beta}_{\hat{\jmath}_2}, \hat{\beta}_{\hat{\jmath}_2+1}, \ldots, \hat{\beta}_{\hat{\jmath}_{3N_0^2}}, \hat{\beta}_{\hat{\jmath}_{3N_0^2}+1} \in B_0$ such that the set $X := \{\beta = \beta_1\beta_2 \ldots \beta_k \in \tilde{B}$

excellent$\,|\,\hat{\jmath}_s,\hat{\jmath}_{s+1}$ are good positions and $\beta_{\hat{\jmath}_s}=\hat{\beta}_{\hat{\jmath}_s},\beta_{\hat{\jmath}_s+1}=\hat{\beta}_{\hat{\jmath}_s+1},\forall\, s\le 3N_0^2\}$ has at least $N_0^{(1-\tau)k}/2^k\cdot N_0^{6N_0^2}>N_0^{(1-2\tau)k}$ elements.

Since $N_0=|B_0|$, there are N_0^2 possible choices for the pairs $(\hat{\beta}_{\hat{\jmath}_s},\hat{\beta}_{\hat{\jmath}_s+1})$. We will consider, for $1\le s<t\le 3N_0^2$, the projections $\pi_{s,t}\colon X\to B_0^{\hat{\jmath}_t-\hat{\jmath}_s}$ given by $\pi_{s,t}(\beta_1\beta_2\dots\beta_k)=(\beta_{\hat{\jmath}_s+1},\beta_{\hat{\jmath}_s+2},\dots,\beta_{\hat{\jmath}_t})$.

For each pair (s,t) with $0\le s<t\le 3N_0^2$ such that $|\pi_{s,t}(X)|<N_0^{(1-10\tau)(\hat{\jmath}_t-\hat{\jmath}_s)}$, we will exclude from $\{1,2,\dots,3N_0^2\}$ the indices $s,s+1,\dots,t-1$. Let us estimate the total number of indices excluded: the set of excluded indices is the union of the intervals $[s,t)$ (intersected with $\mathbb{Z}$) over the pairs (s,t) as above. Now we use the elementary fact that, given a finite family of intervals, there is a subfamily of disjoint intervals whose sum of lenghts is at least half of the measure of the union of the intervals of the original family. We apply this fact to the above intervals $[s,t)$. Suppose that the total number of indices excluded is at least $2N_0^2$. Then, by the above fact, we may find a disjoint collection of intervals $[s,t)$ as above whose sum of lenghts is at least N_0^2. Let us call $\mathcal{P}$ the set of these pairs (s,t). Since $\hat{\jmath}_t-\hat{\jmath}_s\ge 2(t-s)\lceil 2/\tau\rceil,\forall t>s$, the sum of $(\hat{\jmath}_t-\hat{\jmath}_s)$ for $(s,t)\in\mathcal{P}$ is at least $2N_0^2\lceil 2/\tau\rceil$. Since for each pair $(s,t)\in\mathcal{P}$ we have $|\pi_{s,t}(X)|<N_0^{(1-10\tau)(\hat{\jmath}_t-\hat{\jmath}_s)}$, we get

$$\begin{aligned} N_0^{(1-2\tau)k}<|X| &< N_0^{(1-10\tau)\sum\limits_{(s,t)\in\mathcal{P}}(\hat{\jmath}_t-\hat{\jmath}_s)} N_0^{\#\{i;i\notin[\hat{\jmath}_s,\hat{\jmath}_t),\forall(s,t)\in\mathcal{P}\}}\\ &< N_0^{(1-10\tau)\cdot 2N_0^2\lceil 2/\tau\rceil}\cdot N_0^{k-2N_0^2\lceil 2/\tau\rceil}, \end{aligned}$$

since we have at most N_0 choices for β_i for each index i which does not belong to the union of the intervals $[\hat{\jmath}_s,\hat{\jmath}_t)$ associated to these pairs (s,t). However, this is a contradiction, since this inequality is equivalent to $N_0^{20\tau N_0^2\lceil 2/\tau\rceil}<N_0^{2\tau k}$, which cannot hold, because $2\tau k=16\tau N_0^2\lceil 2/\tau\rceil<20\tau N_0^2\lceil 2/\tau\rceil$. So, we proved that the total number of excluded indices is smaller than $2N_0^2$.

Now, there are at least N_0^2+1 indices which are not excluded. We will have two non-excluded indices $s<t$ such that $\hat{\beta}_{\hat{\jmath}_s}=\hat{\beta}_{\hat{\jmath}_t}$ and $\hat{\beta}_{\hat{\jmath}_s+1}=\hat{\beta}_{\hat{\jmath}_t+1}$. We claim that, for $B:=\pi_{s,t}(X)$, the shift $\Sigma(B)$ satisfies the conclusions of the statement.

Indeed, since s and t are not excluded, we have $|B|\ge N_0^{(1-10\tau)(\hat{\jmath}_t-\hat{\jmath}_s)}$. Moreover, for $\alpha\in B$, $|I(\alpha)|>(100e^{r_0})^{-(\hat{\jmath}_t-\hat{\jmath}_s)}>e^{-(\hat{\jmath}_t-\hat{\jmath}_s)(r_0+5)}$. So, the Hausdorff dimension of $K(B)$ is at least

$$\frac{(1-10\tau)\log N_0}{r_0+5}>\frac{(1-10\tau)r_0}{r_0+5}\cdot(1-\frac{\tau}{2})D(t)>(1-12\tau)D(t)>(1-\eta)D(t).$$

On the other hand, if $\tilde{k} := \hat{\jmath}_t - \hat{\jmath}_s$, $\gamma_1 := \hat{\beta}_{\hat{\jmath}_s+1}$ and $\gamma_2 := \hat{\beta}_{\hat{\jmath}_t} = \hat{\beta}_{\hat{\jmath}_s}$, all elements of B are of the form $\gamma_1\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2$, where $\gamma_1, \beta_2, \beta_3, \dots, \beta_{\tilde{k}-1}, \gamma_2 \in B_0$ and there are γ_1', γ_1'', $\gamma_2', \gamma_2'' \in B_0$ with $[0;\gamma_2'] < [0;\gamma_2] < [0;\gamma_2'']$; $[0;(\gamma_1')^t] < [0;\gamma_1^t] < [0;(\gamma_1'')^t]$ such that

$$I(\gamma_1'\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2\gamma_1) \cap K_t \neq \emptyset, \quad I(\gamma_1''\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2\gamma_1) \cap K_t \neq \emptyset,$$
$$I(\gamma_2\gamma_1\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2') \cap K_t \neq \emptyset, \quad I(\gamma_2\gamma_1\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2'') \cap K_t \neq \emptyset.$$

We will show that this implies the existence of $\delta > 0$ such that $\Sigma(B) \subset \Sigma_{t-\delta}$. Let $\gamma_1^t = (c_1, c_2, \dots, c_{m_1})$, with $c_j \in \{1,2,3,4\}$, $\forall\, j \le m_1$ and $\gamma_2 = (d_1, d_2, \dots, d_{m_2})$ with $d_j \in \{1,2,3,4\}$, $\forall\, j \le m_2$. Let $\gamma_1\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2 \in B$ where $\beta_2\beta_3\dots\beta_{\tilde{k}-1} = a_1a_2\dots a_{\tilde{m}}$ with $\tilde{a}_j \in \{1,2,3,4\}, \forall\, j \le \tilde{m}$. We want to estimate sums of continued fractions beginning by $[a_j; a_{j+1}, \dots, a_{\tilde{m}}, \gamma_2, \gamma_1, \dots] + [0; a_{j-1}, \dots, a_1, \gamma_1^t, \gamma_2^t, \dots]$. Let us assume, without loss of generality, that $q_{m_2+\tilde{m}-j}(a_{j+1}, ..., a_{\tilde{m}}, \gamma_2) \le q_{m_1+j-1}(a_{j-1}, ..., a_1, \gamma_1^t)$ (the other case, when the reverse inequality $q_{m_1+j-1}(a_{j-1}, ..., a_1, \gamma_1^t) \le q_{m_2+\tilde{m}-j}(a_{j+1}, ..., a_{\tilde{m}}, \gamma_2)$ holds, is symmetric). Assume also that $[a_j; a_{j+1}, ..., a_{\tilde{m}}, \gamma_2] < [a_j; a_{j+1}, ..., a_{\tilde{m}}, \gamma_2']$ (otherwise we change γ_2' by γ_2''). This allows us to exhibit $\delta > 0$ such that, for any $\underline{\theta}^{(i)} \in \{1,2,3,4\}^{\mathbb{N}}$, $1 \le i \le 4$,

$$\begin{aligned}&[a_j; a_{j+1}, \dots, a_{\tilde{m}}, \gamma_2, \underline{\theta}^{(1)}] + [0; a_{j-1}, \dots, a_1, \gamma_1^t, \gamma_2^t, \underline{\theta}^{(2)}] < \\ &\quad < [a_j; a_{j+1}, \dots, a_{\tilde{m}}, \gamma_2', \underline{\theta}^{(3)}] + [0; a_{j-1}, \dots, a_1, \gamma_1^t, \gamma_2^t, \underline{\theta}^{(4)}] - \delta.\end{aligned}$$

Indeed,

$$[a_j; a_{j+1}, \dots, a_{\tilde{m}}, \gamma_2', \underline{\theta}^{(3)}] - [a_j; a_{j+1}, \dots, a_{\tilde{m}}, \gamma_2, \underline{\theta}^{(1)}] > \frac{1}{12 q_{m_2+\tilde{m}-j}(a_{j+1}, ..., a_{\tilde{m}}, \gamma_2)^2} \quad \text{and}$$

$$\begin{aligned}&\left\|[0; a_{j-1}, \dots, a_1, \gamma_1^t, \gamma_2^t, \underline{\theta}^{(4)}] - [0; a_{j-1}, \dots, a_1, \gamma_1^t, \gamma_2^t, \underline{\theta}^{(2)}]\right\| < \\ &\quad \frac{1}{q_{m_1+m_2+j-1}(a_{j-1}, ..., a_1, \gamma_1^t, \gamma_2^t)^2} < \frac{1}{(F_{m_2+1} q_{m_1+j-1}(a_{j-1}, ..., a_1, \gamma_1^t))^2} \le \\ &\frac{1}{(F_{m_2+1} q_{m_2+\tilde{m}-j}(a_{j+1}, ..., a_{\tilde{m}}, \gamma_2))^2} \le \frac{1}{64 q_{m_2+\tilde{m}-j}(a_{j+1}, ..., a_{\tilde{m}}, \gamma_2)^2}\end{aligned}$$

(here we use $m_2 \ge 5$; (F_n) denotes Fibonacci's sequence, given by $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \ge 0$).

So, the inequality holds with

$$\delta := \frac{1}{5^{2(m_1+m_2+\tilde{m})}} < \frac{1}{20q_{m_2+\tilde{m}-j}(a_{j+1},...,a_{\tilde{m}},\gamma_2)^2}.$$

On the other hand, $I(\gamma_2\gamma_1, \beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2') \cap K_t \neq \emptyset$, so there are $\underline{\theta}^{(3)}$ and $\underline{\theta}^{(4)}$ such that $(\underline{\theta}^{(4)})^t\gamma_2\gamma_1\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2'\underline{\theta}^{(3)} \in \Sigma_t$, and thus $[a_j; a_{j+1}, \dots, a_{\tilde{m}}, \gamma_2', \underline{\theta}^{(3)}] + [0; a_{j-1}, \dots, a_1, \gamma_1^t, \gamma_2^t, \underline{\theta}^{(4)}] \leq t$, which implies that, for any $\underline{\theta}^{(i)} \in \{1,2,3,4\}^{\mathbb{N}}, i = 1, 2$, $[a_j; a_{j+1}, \dots, a_{\tilde{m}}, \gamma_2, \underline{\theta}^{(1)}] + [0; a_{j-1}, \dots, a_1, \gamma_1^t, \gamma_2^t, \underline{\theta}^{(2)}] < t - \delta$.

We also want to estimate sums of continued fractions beginning by $[d_j; d_{j+1}, ..., d_{m_2}, \gamma_1, ...] + [0; d_{j-1}, ..., d_1, a_{\tilde{m}}, ..., a_1, \gamma_1^t, ...]$ (and, symmetrically, sums of continued fractions beginning by $[0; c_{j+1}, ..., c_{m_1}, \gamma_2^t, ...] + [c_j; c_{j-1}, ..., c_1, a_1, ..., a_{\tilde{m}}, \gamma_2, ...]$). We have:

$$q_{m_2-j+m_1}(d_{j+1}, ..., d_{m_2}, \gamma_1) \leq q_{j-1+\tilde{m}+m_1}(d_{j-1}, ..., d_1, a_{\tilde{m}}, ..., a_1, \gamma_1^t).$$

Assume that $[d_j; d_{j+1}, ..., d_{m_2}, \gamma_1] < [d_j; d_{j+1}, ..., d_{m_2}, \gamma_1']$ (otherwise we change γ_1' by γ_1''). Since $I(\gamma_2\gamma_1\beta_2\beta_3\dots\beta_{\tilde{k}-1}\gamma_2\gamma_1') \cap K_t \neq \emptyset$, estimates analogous to the previous ones imply that, for any $\underline{\theta}^{(i)} \in \{1,2,3,4\}^{\mathbb{N}}$, $i = 1, 2$, we have $[d_j; d_{j+1}, ..., d_{m_2}, \gamma_1, \underline{\theta}^{(1)}] + [0; d_{j-1}, ..., d_1, a_{\tilde{m}}, ..., a_1, \gamma_1^t, \gamma_2^t, \underline{\theta}^{(2)}] < t - \delta$.
This implies that the complete shift $\Sigma(B)$ satisfies the conditions of the statement, which concludes the proof of the Lemma. $\square$

Lemma 2. *Given a complete shift $\Sigma(X) \subset \{1,2,3,4\}^{\mathbb{N}}$ (where X is a finite set of finite sequences whose terms belong to $\{1,2,3,4\}$) we have $HD(\ell(\Sigma(X))) = HD(m(\Sigma(X))) = min\{2 \cdot HD(K(X)), 1\}$.*

Proof: First of all we clearly have $\ell(\Sigma(X)) \subset m(\Sigma(X)) \subset \bigcup_{a=1}^{4}(a + K(X) + K(X))$, so $HD(\ell(\Sigma(X)) \leq HD(m(\Sigma(X)) \leq \min\{2 \cdot HD(K(X)), 1\}$.

Let $\varepsilon > 0$ be given. We will show that there are regular Cantor sets K, K' defined by iterates of the Gauss map with $HD(K), HD(K') > HD(K(X)) - \varepsilon$ such that $K + K' \subset \ell(\Sigma(X)) \subset m(\Sigma(X))$. Since, by the dimension formula stated in section 2, $HD(K+K') = \min\{HD(K) + HD(K'), 1\} > \min\{2 \cdot HD(K(X)), 1\} - 2\varepsilon$, and $\varepsilon > 0$ is arbitrary, the result will follow.

Given a positive integer n, let $X^n = \{(\gamma_1, \gamma_2, \dots, \gamma_n) | \gamma_j \in X, \forall j \leq n\}$. We have $\Sigma(X^n) = \Sigma(X)$ and $K(X^n) = K(X)$. Replacing X by X^n for some n large, we may assume without loss of generality that for any $A \subset X$ (resp. $A^t \subset X^t$) with $|A| \le 2$ (resp. $|A^t| \le 2$), we have $HD(K(X \setminus A)) > HD(K(X)) - \varepsilon$ (resp. $HD(K(X^t \setminus A^t)) > HD(K(X^t)) - \varepsilon = HD(K(X)) - \varepsilon$).

We will order X and X^t in the following way: given $\gamma, \tilde{\gamma} \in X$ (resp. $\gamma, \tilde{\gamma} \in X^t$), we say that $\gamma < \tilde{\gamma}$ if and only if $[0;\gamma] < [0;\tilde{\gamma}]$.

Suppose that the maximum of $m(\Sigma(X))$ is attained at $\underline{\tilde{\theta}} = (\dots, \tilde{\gamma}_{-1}, \tilde{\gamma}_0, \tilde{\gamma}_1, \dots), \tilde{\gamma}_i \in X, \forall i \in \mathbb{Z}$, in a position corresponding to the sequence $\tilde{\gamma}_0$. Let $X^* = X \setminus \{\min X, \max X\}$, $X^t = \{\gamma^t, \gamma \in X\}$, $(X^t)^* = X^t \setminus \{\min X^t, \max X^t\}$. For each positive integer m, let C^m be the set of sequences

$$(\dots \gamma_{-m-2}, \gamma_{-m-1}, \tilde{\gamma}_{-m}, \tilde{\gamma}_{-m+1}, \dots, \tilde{\gamma}_{-1}, \tilde{\gamma}_0, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{m-1}, \tilde{\gamma}_m, \gamma_{m+1}, \gamma_{m+2}, \dots)$$

where $\gamma_k \in X^*$ for $k \ge m+1$, $\quad \gamma_k^t \in (X^t)^*$ for $k \le -m-1$. Then, for m large enough, there is $\eta > 0$ such that for each $\underline{\theta} \in C^m$, $\sup(\alpha_n + \beta_n) = m(\underline{\theta})$ is attained only for values of n corresponding to the piece $\tau = \tilde{\gamma}_{-m}, \tilde{\gamma}_{-m-1}, \dots, \tilde{\gamma}_{-1}, \dots \tilde{\gamma}_0, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{m-1}, \tilde{\gamma}_m$ of $\underline{\theta}$, and, if n does not correspond to the piece τ, then $\alpha_n + \beta_n < m(\underline{\theta}) - \eta$. Indeed, if it is not the case, we may assume without loss of generality that there are a sequence m_k tending to $+\infty$ and, for each k, $\underline{\theta}^{(k)} \in C^{m_k}$ and n_k corresponding to a piece $\gamma_{r(k)}$, with $r(k) > m_k$ such that $\alpha_{n_k}(\underline{\theta}^{(k)}) + \beta_{n_k}(\underline{\theta}^{(k)}) > m(\underline{\theta}^{(k)}) - 1/k$. Since $\underline{\theta}^{(k)}$ converges to $\underline{\tilde{\theta}}$, which maximizes m in $\Sigma(X)$, $m(\underline{\theta}^{(k)})$ converges to $m(\underline{\tilde{\theta}})$, and, by compacity, if N_k denotes the size of the sequence $\tilde{\gamma}_0, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{m_k-1}, \tilde{\gamma}_{m_k}, \gamma_{m_k+1}, \gamma_{m_k+2}, \dots \gamma_{r(k)-1}$, $(\sigma^{N_k}(\underline{\theta}^{(k)}))$ has a subsequence which converges to some $\underline{\hat{\theta}} = (\dots, \hat{\gamma}_{-1}, \hat{\gamma}_0, \hat{\gamma}_1, \dots) \in \Sigma(X)$, with $\hat{\gamma}_i \in X^*, \forall i \ge 0$, such that $\sup(\alpha_n + \beta_n) = m(\underline{\hat{\theta}}) = m(\underline{\tilde{\theta}})$ is attained for some n corresponding to the piece $\hat{\gamma}_0$. This is a contradiction, since $m(\underline{\tilde{\theta}})$ is the maximum of $m(\Sigma(X))$, and, changing $\hat{\gamma}_1$ by $\min X$ or $\max X$, we increase strictly the value of $m(\underline{\hat{\theta}})$. Notice that the same argument shows that for any $\underline{\theta} \in C^m$ and $\underline{\theta}^* \in \Sigma(X^*)$, we have $m(\underline{\theta}^*) < m(\underline{\theta}) - \eta$ (for m large enough).

Now, fixing m with the above properties and $\gamma^{(0)} \in X$ such that $(\gamma^{(0)})^t \in (X^t)^*$, we may associate to each $x = [0; \gamma_1(x), \gamma_2(x), \gamma_3(x), \dots] \in K(X^*)$, an element $\underline{\theta}(x) \in C^m$ given by

$$\underline{\theta}(x) = (\dots, \gamma^{(0)}, \gamma^{(0)}, \tilde{\gamma}_{-m}, \tilde{\gamma}_{-m+1}, \dots \tilde{\gamma}_{-1}, \tilde{\gamma}_0, \tilde{\gamma}_1, \dots \tilde{\gamma}_{m-1}, \tilde{\gamma}_m, \gamma_1(x), \gamma_2(x), \dots) =$$

$$= (\dots, \gamma^{(0)}, \gamma^{(0)}, \tau, \gamma_1(x), \gamma_2(x), \dots).$$

For each position n corresponding to the piece τ of $\underline{\theta}(x)$, we write $g_n(x) = \alpha_n(\underline{\theta}(x)) + \beta_n(\underline{\theta}(x))$; in fact $\beta_n(\underline{\theta}(x))$ does not depend on x, so, for distinct values of n, the functions g_n are distinct rational maps of x. This implies that, except for finitely many values of x, the values of $g_n(x)$ for these values of n are all distinct. Let us fix such a value $x^\# = [0; \gamma_1^\#, \gamma_2^\#, \gamma_3^\#, \dots]$ of x (such that the values of $g_n(x^\#)$ for these values of n are all distinct). Since $\sup(\alpha_n + \beta_n) = m(\underline{\theta}(x^\#))$ is attained only for values of n corresponding to the piece τ of $\underline{\theta}(x^\#)$, take n_0 such that $\alpha_{n_0}(\underline{\theta}(x^\#)) + \beta_{n_0}(\underline{\theta}(x^\#))$ is maximum. For N large enough, taking $\tau^\# = ((\gamma^{(0)})^N, \tau, \gamma_1^\#, \gamma_2^\#, \dots, \gamma_N^\#)$, the following holds: if $\underline{\theta} = (\dots, \gamma_{-2}, \gamma_{-1}, \tau^\#, \gamma_1, \gamma_2, \dots)$, with $\gamma_k \in X^*, (\gamma_{-k})^t \in (X^t)^*, \forall k \geq 1$, writing $\sigma^{n_0}(\tau^\#) = (\overline{a}_{-N_1}, \dots, \overline{a}_{-2}, \overline{a}_{-1}, \overline{a}_0, \overline{a}_1, \overline{a}_2, \dots, \overline{a}_{N_2})$, we have $m(\underline{\theta}) = [\overline{a}_0; \overline{a}_1, \overline{a}_2, \dots, \overline{a}_{N_2}, \gamma_1, \gamma_2, \gamma_3, \dots] + [0; \overline{a}_{-1}, \overline{a}_{-2}, \dots, \overline{a}_{N_1}, {\gamma_{-1}}^t, {\gamma_{-2}}^t, {\gamma_{-3}}^t, \dots]$. It follows that, defining

$$K := \{[\overline{a}_0; \overline{a}_1, \overline{a}_2, \dots, \overline{a}_{N_2}, \gamma_1, \gamma_2, \gamma_3, \dots]; \gamma_j \in X^*, \forall j \geq 1\} \text{ and}$$

$$K' := [0; \overline{a}_{-1}, \overline{a}_{-2}, \dots, \overline{a}_{N_1}, {\gamma'_1}^t, {\gamma'_2}^t, {\gamma'_3}^t, \dots]; {\gamma'_j}^t \in (X^t)^*, \forall j \geq 1\},$$

we have $K + K' \subset \ell(\Sigma(X)) \subset m(\Sigma(X))$. In order to show this, given
$x = [\overline{a}_0; \overline{a}_1, \overline{a}_2, \dots, \overline{a}_{N_2}, \gamma_1, \gamma_2, \gamma_3, \dots] \in K$ and
$y = [0; \overline{a}_{-1}, \overline{a}_{-2}, \dots, \overline{a}_{N_1}, {\gamma'_1}^t, {\gamma'_2}^t, {\gamma'_3}^t, \dots] \in K'$, and defining, for each positive integer m,
$\tau^{(m)} = (\gamma'_m, \gamma'_{m-1}, \dots, \gamma'_1, \tau^\#, \gamma_1, \gamma_2, ..., \gamma_m)$, we have, for

$$\underline{\theta}^*(x,y) = (\dots, \gamma^{(0)}, \gamma^{(0)}, \tau^{(1)}, \tau^{(2)}, \tau^{(3)}, \dots), \text{ and } \hat{\underline{\theta}}(x,y) = (\dots, \gamma'_3, \gamma'_2, \gamma'_1, \tau^\#, \gamma_1, \gamma_2, \gamma_3, \dots),$$

$$\ell(\underline{\theta}^*(x,y)) = m(\hat{\underline{\theta}}(x,y)) = x + y.$$

Indeed, there is a sequence (s_k) with s_k corresponding to the piece $\tau^{(k)}$ of $\underline{\theta}^*(x,y)$ such that $\sigma^{s_k}(\underline{\theta}^*(x,y))$ converges to $\sigma^{n_0}(\hat{\underline{\theta}}(x,y))$, so $\alpha_{s_k}(\underline{\theta}^*(x,y)) + \beta_{s_k}(\underline{\theta}^*(x,y))$ converges to $\alpha_{n_0}(\hat{\underline{\theta}}(x,y)) + \beta_{n_0}(\hat{\underline{\theta}}(x,y)) = m(\hat{\underline{\theta}}(x,y)) = x + y$, and, in particular, $\ell(\underline{\theta}^*(x,y)) \geq m(\hat{\underline{\theta}}(x,y)) = x + y$. On the other hand, there are increasing sequences (m_k) and (r_k) such that the position m_k corresponds to the piece $\tau^{(r_k)}$ in $\underline{\theta}^*(x,y)$ and $\alpha_{m_k}(\underline{\theta}^*(x,y)) + \beta_{m_k}(\underline{\theta}^*(x,y))$ converges to $\ell(\underline{\theta}^*(x,y))$. Now, if $|m_k - s_{r_k}|$ has a bounded subsequence, then there is $b \in \mathbb{Z}$ such that $\sigma^{m_k}(\underline{\theta}^*(x,y))$ has a subsequence converging to $\sigma^b(\hat{\underline{\theta}}(x,y))$, so $\ell(\underline{\theta}^*(x,y)) = \lim(\alpha_{m_k}(\underline{\theta}^*(x,y)) + \beta_{m_k}(\underline{\theta}^*(x,y))) \leq m(\hat{\underline{\theta}}(x,y)) = x + y$. On the other hand, if $|m_k - s_{r_k}|$ is unbounded, there is $c \in \mathbb{Z}$ and a subsequence of $\sigma^{m_k}(\underline{\theta}^*(x,y))$ which converges to $\sigma^c(\underline{\theta}^*)$, where $\underline{\theta}^*$ is an element of $\Sigma(X^*)$, but in this case we would have $\ell(\underline{\theta}^*(x,y)) \le m(\underline{\theta}^*) < m(\hat{\underline{\theta}}(x,y)) - \eta$, which is a contradiction.

Finally, notice that K and K' are diffeomorphic respectively to $K(X^*)$ and $K((X^t)^*)$, so $HD(K) = HD(K(X^*)) > HD(K(X)) - \varepsilon$ and $HD(K') = HD(K((X^t)^*)) > HD(K(X^t)) - \varepsilon = HD(K(X)) - \varepsilon$. $\square$

Proof of Theorem 1: Applying Lemma 2 to the complete shift $\Sigma(B)$ obtained in Lemma 1, we get that, for any $\eta > 0$, there is $\delta > 0$ such that $\min\{2(1-\eta)D(t), 1\} \le HD(K(B)) \le HD(L \cap (-\infty, t-\delta)) \le HD(M \cap (-\infty, t-\delta)) \le HD(M \cap (-\infty, t)) \le \min\{2 \cdot HD(K_t), 1\} \le \min\{2 \cdot D(t), 1\}$, so $d(t) := HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t))$ satisfies $d(t) = \min\{2 \cdot D(t), 1\}$.

In order to show that $d(t) = \min\{2HD(k^{-1}(-\infty, t)), 1\}$, it is enough to show that $HD(k^{-1}(-\infty, t)) = D(t)$. We first notice that, in the notations of the end of the proof of Lemma 2 above, given $z = [0; \alpha_1, \alpha_2, \dots]; \alpha_j \in X^*, \forall j \ge 1\} \in K(X^*)$, we may define

$$\underline{\lambda}(z) = \underline{\lambda}_{x,y}(z) = (\alpha_{1!}, \tau^{(1)}, \alpha_{2!}, \tau^{(2)}, \alpha_3, \alpha_4, \alpha_5, \alpha_{3!}, \tau^{(3)}, \alpha_7, \dots, \alpha_{4!}, \tau^{(4)},$$
$$\alpha_{25}, \alpha_{26}, \dots, \alpha_{5!}, \tau^{(5)}, \ \dots \ , \alpha_{r!}, \tau^{(r)}, \alpha_{r!+1}, \dots),$$

and $h(z) = [0; \underline{\lambda}(z)]$. We have, as before, $k(h(z)) = \ell(\dots, \gamma^{(0)}, \gamma^{(0)}, \underline{\lambda}(z)) = x + y$. On the other hand, given any $\rho > 0$, we have $|z - z'| = O(|h(z) - h(z')|^{1-\rho})$ for $|z - z'|$ small, so $HD(k^{-1}(x+y)) \ge HD(K(X^*)) > HD(K(X)) - \varepsilon$. Taking $X = B$, as before, we get $HD(k^{-1}(-\infty, t)) \ge HD(k^{-1}(-\infty, t-\delta)) \ge HD(k^{-1}(x+y)) > HD(K(B)) - \varepsilon > (1-\eta)D(t) - \varepsilon$. So, since η and ε are arbitrary, $HD(k^{-1}(-\infty, t)) \ge D(t)$. Now, if $w \in k^{-1}(-\infty, t))$, we have $\limsup_{n\to\infty}(\alpha_n(w) + \beta_n(w)) = k(w) < t$, so there is $n_0 \in \mathbb{N}$ such that $n \ge n_0 \implies \alpha_n(w) + \beta_n(w) < t$. This implies that $k^{-1}(-\infty, t)) \subset \bigcup_{n \in \mathbb{N}}(g^{-n}(K_t))$, where g is the Gauss map, so $HD(k^{-1}(-\infty, t)) \ge D(t)$. On the other hand,

$$\begin{aligned} D(t) = HD(k^{-1}(-\infty, t)) &\le HD(k^{-1}(-\infty, t]) \le \lim_{s \to t+} HD(k^{-1}(-\infty, s)) \\ &= \lim_{s \to t+} D(s) = D(t). \end{aligned}$$

Then $HD(k^{-1}(-\infty, t]) = HD(k^{-1}(-\infty, t)) = D(t)$, and we conclude that $d(t) = \min\{2HD(k^{-1}(-\infty, t)), 1\} = \min\{2HD(k^{-1}(-\infty, t]), 1\}$

Now let us show that $d(t)$ is a continuous function: if we have $N(t, r) = |C(t, r)|$ as in the proof of Lemma 1, and for each $t > t_0$, r large, $\frac{\log N(t,r)}{r} > D(t_0) + \eta$ we would have

$D(t_0) \geq D(t_0) + \eta$, contradiction (indeed $C(t,r) \subset C(s,r)$ for $t \leq s$, and, by compacity, $C(t_0, r) = \bigcap_{t>t_0} C(t,r)$).

In order to conclude, we notice that, since, for each positive integer m, $\Sigma(\{21^{2m}2, 21^{2m+2}2\} \subset \Sigma_{3+2^{-m}}$, so $D(3+\varepsilon) > 0$ for every $\varepsilon > 0$ and, since $\Sigma_{\sqrt{12}} = \{1,2\}^{\mathbb{Z}}$, $HD(K_{\sqrt{12}}) = HD(K(\{1,2\})) = 0,5312... > 1/2$, so we have $d(\sqrt{12}) = 1$. $\square$

Proof of Theorem 2: Given $m \geq 2$, let $C_m = \{\alpha = [0; a_1, a_2, a_3, ...] \in [0,1]; a_k \leq m, \forall k \geq 1\}$. M. Hall proved in [H] that $C_4 + C_4 = \{\alpha+\beta; \alpha, \beta \in C_4\} = [\sqrt{2}-1, 4(\sqrt{2}-1)]$. On the other hand, we have $\lim_{m\to\infty} HD(C_m) = 1$. In fact, Jarník proved in [J] that $HD(C_m) > 1 - \frac{1}{m \cdot \log 2}, \forall m > 8$.

Let now $t \geq 6$ be given. Let $m = \lfloor t \rfloor - 2$. There are an integer $n \in \{m+1, m+2\}$ and $\alpha = [0; a_1, a_2, a_3, ...], \beta = [0; b_1, b_2, b_3, ...] \in C_4$ such that $t = n + \alpha + \beta$. For each $r \geq 1$, let $\tilde{\tau}^{(r)}$ be the sequence $(b_r, b_{r-1}, ..., b_2, b_1, n, a_1, a_2, ..., a_r)$. Consider now the map $\tilde{h} : C_m \to [0,1]$ given by

$$\begin{aligned}\tilde{h}(z) = \tilde{h}([0; c_1, c_2, c_3, ...]) = [0; c_{1!}, \tilde{\tau}^{(1)}, c_{2!}, \tilde{\tau}^{(2)}, c_3, c_4, c_5, c_{3!}, \tilde{\tau}^{(3)}, \\ c_7, c_8, \ldots, c_{4!}, \tilde{\tau}^{(4)}, c_{25}, \ldots, c_{5!}, \tau^{(5)}, \ \ldots \ , c_{r!}, \tilde{\tau}^{(r)}, c_{r!+1}, \ldots].\end{aligned}$$

It is easy to see that $k(\tilde{h}(z)) = t$ for every $z \in C_m$. On the other hand, given any $\rho > 0$, we have $|z - z'| = O(|\tilde{h}(z) - \tilde{h}(z')|^{1-\rho})$ for $|z-z'|$ small, so $HD(k^{-1}(t)) \geq HD(C_m)$, and, since $\lim_{m\to\infty} HD(C_m) = 1$, we are done. $\square$

Proof of Theorem 3: Let $x \in L'$. Consider a sequence x_n converging to x, $x_n \in L$, $x_n \neq x$. Chose $\underline{\theta}^{(n)} \in \Sigma$ such that $x_n = \ell(\underline{\theta}^{(n)})$. We have $\underline{\theta}^{(n)} = (b_j^{(n)})_{j\in\mathbb{Z}}$ (we will assume $b_j^{(n)} \leq 4$, $\forall\, j$, $\forall\, n$, since me may assume that the x_n are not in Hall's ray). We have $x_n = \limsup_{j\to\infty}(\alpha_j^{(n)} + \beta_j^{(n)})$. Given $\delta > 0$, $\exists\, n_0 \in \mathbb{N}$ large such that $n \geq n_0 \Rightarrow |\ell(\underline{\theta}^{(n)}) - x| < \delta$ and there are infinitely many $j \in \mathbb{N}$ such that $|\alpha_j^{(n)} + \beta_j^{(n)} - x| < \delta$. Let $N = \lceil \delta^{-1} \rceil$. Given such a pair (j, n) consider the finite sequence with $2N+1$ terms $(b_{j-N}^{(n)}, b_{j-N+1}^{(n)}, \ldots, b_j^{(n)}, \ldots, b_{j+N}^{(n)}) =: S(j,n)$. There is a sequence S such that for infinitely many values of n, S appears infinitely many times as $S(j,n)$, $j \in \mathbb{N}$, i.e., there are $j_1(n) < j_2(n) < \ldots$ with $S(j_i(n), n) = S$, $\forall\, i \geq 1$, for all n in some infinite set $L \subset \mathbb{N}$.

Consider the sequences $\beta(i,n)$ for $i \geq 1$, $n \in L$ given by

$$\beta(i,n) = (b_{j_i(n)+N+1}^{(n)}, b_{j_i(n)+N+2}^{(n)}, \ldots, b_{j_{i+1}(n)+N}^{(n)}).$$

There are (i_1, n_1) and (i_2, n_2) for which there is no sequence γ such that $\beta(i_1, n_1)$ and $\beta(i_2, n_2)$ are concatenations of copies of γ, otherwise x_n would be constant for $n \in L$. This implies that, taking $B = \{\beta(i_1, n_1)\beta(i_2, n_2), \beta(i_2, n_2)\beta(i_1, n_1)\}$, $K(B)$ is a regular Cantor set, so, as in Lemma 2, $\ell(K(B))$ contains a regular Cantor set $\hat{K}$ with $d(x, \hat{K}) \leq 2\delta$.

$\square$

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