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CONSTANT MEAN CURVATURE FOLIATIONS AND SCALAR CURVATURE RIGIDITY OF THREE-MANIFOLDS

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Abstract

This work contains essentially two rigidity results.

For compact surfaces Σ with boundary properly embedded in a Riemannian three-manifold (M, g) with mean convex boundary that are local minima for the free boundary problem for the area, we prove that a geometric invariant constructed from the infimum of the scalar curvature of M, the infimum of the mean curvature of ∂M, the area of Σ and the length of $\partial\Sigma$ is bounded from above by a constant which depends only on the topology of Σ, with equality (under additional hypotheses) if and only if Σ has constant Gaussian curvature, $\partial\Sigma$ has constant geodesic curvature and (M, g) locally splits as a product $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$ in a neighborhood of Σ. This generalizes previous results about locally area-minimizing surfaces without boundary.

For asymptotically hyperbolic three-manifolds (M, g) that have a minimal boundary, scalar curvature greater than or equal to -6 and that are sufficiently small perturbations of the Anti-de Sitter-Schwarzschild spaces of positive mass, we prove that the Hawking mass of ∂M (which is a function of the area of ∂M only) is bounded from above by the mass of (M, g) (which depends only on the geometry at infinity), with equality if and only if (M, g) is isometric to the Anti-de Sitter-Schwarzschild space of same mass. This proves the Penrose Conjecture for this class of asymptotically hyperbolic manifolds.

In the proofs of these results, the construction of foliations of (M, g) by constant mean curvature surfaces and the monotonicity of suitable functionals along these families play a fundamental role.

Keywords: Minimal surfaces, free boundary surfaces, area-minimizing surfaces, CMC foliations, asymptotically hyperbolic manifolds, Penrose Conjecture, scalar curvature, rigidity.

Resumo

Este trabalho contém essencialmente dois resultados de rigidez.

Considerando superfícies compactas Σ, com bordo, propriamente mergulhadas em uma variedade Riemanniana (M^3, g) com bordo convexo em média e que são pontos de mínimo local para o problema de minimização de área em M com bordo livre em ∂M, provamos que um invariante geométrico construído a partir do ínfimo da curvatura escalar de M, do ínfimo da curvatura média de ∂M, da área de Σ e do comprimento de $\partial\Sigma$ é limitado superiormente por uma constante que depende apenas da topologia de Σ, valendo a igualdade (com hipóteses adicionais) se, e somente se, Σ tem curvatura Gaussiana constante, $\partial\Sigma$ tem curvatura geodésica constante e (M, g) se decompõe localmente em produto $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$ em vizinhança de Σ. Isto generaliza resultados anteriores sobre superfícies sem bordo localmente minimizantes de área.

Considerando variedades assintoticamente hiperbólicas (M^3, g) que têm bordo mínimo, curvatura escalar maior que ou igual a -6 e que são perturbações suficientemente pequenas dos espaços Anti-de Sitter-Schwarzschild de massa positiva, provamos que a massa de Hawking de ∂M (que é função apenas de sua área) é limitada superiormente pela massa de (M, g) (que depende apenas de sua geometria no infinito), valendo a igualdade se, e somente se, (M, g) é isométrica ao espaço Anti-de Sitter-Schwarzschild de mesma massa. Isto demonstra a Conjectura de Penrose para esta classe de variedades assintoticamente hiperbólicas.

Nas demonstrações destes resultados, a construção de folheações de (M, g) por superfícies com curvatura média constante e a monotonicidade de funcionais apropriados ao longo dessas famílias tem um papel fundamental.

Palavras-chave: Superfícies mínimas, superfícies com bordo livre, superfícies minimizantes de área, folheaçoes CMC, variedades assintoticamente hiperbólicas, Conjectura de Penrose, curvatura escalar, rigidez.

Contents

Introduction

This thesis is based on our papers [1] and [2], in which we studied very different problems. As a general introduction to the present work, we would like to explain very briefly a common background in which our results can be understood.

The underlying question is to understand the relations between the scalar curvature of a Riemannian manifold (M, g) and the other geometric invariants of (M, g). In particular, one finds situations where it is impossible to *increase* the scalar curvature of a given Riemannian manifold (M_0, g_0) without *changing* some of its geometric invariants. In this case, one might talk about *rigidity* of the model space (M_0, g_0).

A prototypical example of this phenomenon is the following beautiful result about the Euclidean space $(\mathbb{R}^n, \delta)$.

Theorem. *Let g be a Riemannian metric on $\mathbb{R}^n$ that coincides with the Euclidean metric δ outside a compact set. If the scalar curvature of g is nonnegative, then $(\mathbb{R}^n, g)$ is isometric to $(\mathbb{R}^n, \delta)$.*

This result is a consequence of the celebrated Positive Mass Theorem (proven by R. Schoen and S.T. Yau [31] and E. Witten [35]). The Riemannian manifolds $(\mathbb{R}^n, g)$ above are asymptotically flat manifolds with zero mass, where, loosely speaking, the mass of an asymptotically flat manifold is a geometric invariant determined by the behavior of the metric near the infinity in comparison to the Euclidean metric. The Positive Mass Theorem says that all asymptotically flat manifolds with nonnegative scalar curvature have nonnegative mass, and the only one with zero mass is, up to isometries, the Euclidean space $(\mathbb{R}^n, \delta)$. The theorem follows.

There are many other examples of rigidity and also non-rigidity phenomena involving the scalar curvature, and many results on the subject can be found in recent literature. We refer the reader to the survey [7]. Some of them inspired our work, and our contribution in [1] and [2] was to prove some new scalar curvature rigidity results for three-manifolds.

Regarding the proofs of our results, we can also point out a common important step: the construction of smooth foliations by constant mean curvature surfaces. This is done with the aid of the inverse function theorem. A remarkable utilization of these families of surfaces in relation to scalar curvature rigidity problems can be found in the work of H. Bray [3], where a fundamental idea was that, if the scalar curvature is bounded from below, one can get monotonicity of certain functionals along these families and use this monotonicity to conclude rigidity. Our proofs were also guided by this beautiful idea.

This work is organized as follows. After fixing the terminology and notations in a preliminary chapter, we present with minor modifications and no substantial addition the contents of [1] and [2] in two separated chapters. Chapter 1 contains the rigidity results for area-minimizing free boundary surfaces in mean convex three-manifolds. Chapter 2 contains the rigidity results for small perturbations of the Anti-de Sitter-Schwarzschild spaces of positive mass. Both chapters contain an introductory material explaining the context and the statements of our main results – Theorems A, B and C in Chapter 1, and Theorem D in Chapter 2. Finally, some calculations can be found in the Appendix.

Basic material

The aim of this chapter is to fix the terminology and the notations by recalling some basic facts about the geometry of surfaces in a Riemannian three-manifold.

Throughout this work, (M^3, g) will denote a Riemannian three-manifold manifold with boundary ∂M, which we allow to be empty. In case ∂M is not empty, we denote by X its outward pointing unit normal vector field.

Let ∇ denote the Levi-Civita connection of (M, g). We follow the most usual conventions regarding the definition of the Riemann tensor of (M, g). Ric will denote the Ricci tensor and R the scalar curvature of g.

Let Σ^2 be a compact connected surface. Σ is said to be properly immersed (respect., embedded) in M when there is a smooth immersion (respect., embedding) $i : \Sigma \to M$ such that $i(\Sigma) \cap \partial M = i(\partial\Sigma)$. When Σ is closed, i.e., when Σ has no boundary, this condition only means that Σ is immersed (or embedded) in $M \setminus \partial M$. All surfaces considered in this work will be properly immersed compact surfaces. Therefore we will sometimes simply say that Σ is a surface in M.

We always consider the surfaces Σ in (M, g) with the induced metric. Regarding the intrinsic geometry, let K denote the Gaussian curvature of (Σ, g) and let k denote the geodesic curvature of $\partial\Sigma$ in (Σ, g). Let ν be the co-normal of $\partial\Sigma$, i.e., the unit vector field along $\partial\Sigma$ that is tangent to Σ, normal to $\partial\Sigma$ and points outward Σ. If T is a unit vector field tangent to $\partial\Sigma$, we have $k = g(\nabla_T \nu, T)$.

A fundamental result about the geometry of compact surfaces (Σ, g) is, of course, the

Gauss-Bonnet Theorem.

$$\int_\Sigma K dA + \int_{\partial\Sigma} k dL = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

A surface Σ in (M, g) is said to be two-sided when its normal bundle is trivial. This topological condition is always verified, for example, by simply connected surfaces. In any case, in this work we consider only two-sided surfaces in (M, g). A unit vector field normal to a surface Σ in (M, g) will be denoted by N, and we always choose it pointing toward the unbounded component of $M \setminus \Sigma$ (whenever this condition makes sense).

The extrinsic geometry of Σ in (M, g) is encoded in the second fundamental form B, which is the two-tensor on Σ defined by

$$B(Y, W) = g(\nabla_Y N, W)$$

for every pair of vectors Y, W tangent to Σ. If ∇^Σ is the Levi-Civita connection of (Σ, g), then $\nabla_Y W = \nabla^\Sigma_Y W - B(Y, W)N$ for every pair of vectors Y, W tangent to Σ.

The norm of the second fundamental form will be denoted by $|B|$, its trace by H and its traceless part by $\mathring{B}$. H is called the mean curvature of Σ. We say that the ambient manifold (M, g) has mean convex boundary when $H^{\partial M} \geq 0$.

The equations relating the geometry of (M, g) and $\Sigma \subset M$ used in this work are listed bellow.

Fundamental equations. *Let Σ be surface in (M^3, g). Then*

1) (Contracted Gauss equation)

$$2K = R - 2Ric(N, N) + H^2 - |B|^2. \tag{0.1}$$

2) (Contracted Codazzi equation)

$$\mathrm{div}_\Sigma B - dH = Ric(N, -). \tag{0.2}$$

3) Let $\{T, T^\perp, X\}$ be an orthonormal referential along $\partial\Sigma \subset \partial M$, where T is tangent to $\partial\Sigma$ and X is the outward normal to ∂M. Then

$$\begin{aligned} H^{\partial M} &= B^{\partial M}(T, T) + B^{\partial M}(T^\top, T^\top) \\ &= g(X, \nu)k + g(X, N)B^\Sigma(T, T) + B^{\partial M}(T^\perp, T^\perp). \end{aligned} \tag{0.3}$$

A surface Σ in M is called minimal when $H = 0$, totally umbilic when $\mathring{B} = 0$ and totally geodesic when $B = 0$. Σ is called a free boundary surface when $\partial\Sigma$ meets ∂M orthogonally, i.e., when $\nu = X$ along $\partial\Sigma$. In particular, on a free boundary surface Σ its normal N is tangent to ∂M along $\partial\Sigma$.

Given a surface Σ in (M, g), let $|\Sigma|$ denote its area and $|\partial\Sigma|$ denote the length of its boundary. The above definitions of minimal and free boundary surfaces are related to the variational theory of the area functional.

An admissible variation of Σ is a smooth map $f : \Sigma \times (-\epsilon, \epsilon) \to M$ such that $f_t : \Sigma \to M$ is a proper immersion in (M, g) for every $t \in (-\epsilon, \epsilon)$ and $f_0(\Sigma) = \Sigma$. Sometimes we will forget the function f and simply say that $\{\Sigma_t := f_t(\Sigma)\}$ is a variation of $\Sigma = \Sigma_0$. The variational vector field $V = \partial_t f$ will be usually decomposed in its normal and tangent part on Σ,

$$V = V^\top + \rho N.$$

The function ρ is called the lapse function or the normal velocity of the variation. When each Σ_t is properly embedded and ρ has a sign, the family of surfaces $\{\Sigma_t\}$ gives a foliation of a region of M.

Given these definitions, we have the following formula for the first derivative of the area of the surfaces along a variation.

First variation of area. *Given a variation $\{\Sigma_t\}$ of Σ,*

$$\frac{d}{dt}_{|t=0}|\Sigma_t| = \int_\Sigma H\rho dA + \int_{\partial\Sigma} g(\nu, V)dL.$$

Since the admissible variations we consider are such that the variational vector field V is tangent to ∂M along $\partial\Sigma$, a surface Σ in M is a critical point of the area under all admissible variations if and only if Σ is a free boundary minimal surface.

The operator

$$L_\Sigma = \Delta_\Sigma + Ric(N, N) + |B^\Sigma|^2$$

is called the Jacobi operator of Σ, where Δ_Σ denotes the Laplace-Beltrami operator of (Σ, g). The Jacobi operator is related to the first variation of the mean curvature of the surfaces of a variation.

First variation of the mean curvature. *Given a variation $\{\Sigma_t\}$ of Σ,*

$$(\partial_t H_t)_{|t=0} = dH(V^\top) - L_\Sigma(\rho). \tag{0.4}$$

Another useful formula describes the variation of the normal vectors to the surfaces Σ_t.

First variation of the unit normal field. *Given a variation $\{\Sigma_t\}$ of Σ,*

$$(\nabla_{\partial_t} N_t)_{|t=0} = \nabla_{V^\top} N - \nabla^\Sigma \rho, \tag{0.5}$$

where $\nabla^\Sigma \rho$ denotes the gradient on Σ of the lapse function ρ.

Given a critical point of the area functional, i.e., given a free boundary minimal surface Σ, one can use the above formulas to calculate the second derivative of area along variations for which $V^\top = 0$ along Σ.

Second variation of area. *Let Σ be a free boundary minimal surface. For every admissible variation $\{\Sigma_t\}$ of Σ with $V = \rho N$ on Σ,*

$$\frac{d^2}{d^2t}_{|t=0}|\Sigma_t| = -\int_\Sigma L_\Sigma(\rho)\rho dA + \int_{\partial\Sigma}\left(\frac{\partial\rho}{\partial\nu} - B^{\partial M}(N,N)\rho\right)\rho dL$$
$$= \int_\Sigma |\nabla^\Sigma\rho|^2 - (Ric(N,N) + |B^\Sigma|^2)\rho^2 dA - \int_{\partial\Sigma} B^{\partial M}(N,N)\rho^2 dL.$$

Given a free boundary minimal surface Σ in (M,g), we consider the quadratic form Q on $C^\infty(\Sigma)$ associated to the second variation of area,

$$Q(\phi,\psi) = -\int_\Sigma L_\Sigma(\phi)\psi dA + \int_{\partial\Sigma}\left(\frac{\partial\phi}{\partial\nu} - B^{\partial M}(N,N)\phi\right)\psi dL.$$

A free boundary minimal surface Σ in (M,g) is called free boundary stable when the second variation of area is nonnegative for all admissible variations. In view of the second variation of area formula, this is equivalent to the analytical definition that the quadratic form Q is nonnegative. In the case of closed surfaces, in particular, a minimal surface Σ is called stable if and only if its Jacobi operator L_Σ has only nonnegative eigenvalues.

A properly immersed surface Σ in (M,g) will be called locally area-minimizing when every nearby properly immersed surface has area greater than or equal to the area of Σ. In particular, locally area-minimizing surfaces are free boundary stable minimal surfaces.

A closed surface Σ with constant mean curvature is said to be weakly stable when

$$-\int_\Sigma L_\Sigma(\phi)\phi d\Sigma \geq 0 \quad \text{for all} \quad \phi \in C^\infty(\Sigma) \quad \text{such that} \quad \int_\Sigma \phi d\Sigma = 0.$$

This analytical condition has also a geometric interpretation. For example, if Σ is a constant mean curvature closed surface enclosing a bounded domain in (M,g), Σ is weakly stable if and only if the second variation of its area is nonnegative for all variations preserving the amount of volume enclosed by Σ. A good example of such surfaces is given by an equator S^2 of the round three-sphere (S^3,g_0) of constant sectional curvature 1. It is a totally geodesic surface, its Jacobi operator is $\Delta_0 + 2$ and the eigenvalues of the Laplace-Beltrami operator Δ_0 of (S^2,g_0) are precisely $0,2,6,\ldots$

The calculations that lead to the above variation formulas can be found in the Appendix.

CHAPTER 1

Rigidity of area-minimizing free boundary surfaces in mean convex three-manifolds

1.1 Introduction

Let M be a Riemannian manifold with boundary ∂M. Free boundary minimal submanifolds arise as critical points of the area functional when one restricts to variations that preserve ∂M (but not necessarily leave it fixed). Many beautiful known results about closed minimal surfaces could guide the formulation of analogous interesting questions about free boundary minimal surfaces. Inspired by the rigidity theorems for area-minimizing closed surfaces proved in [5], [8], [25] and [29], we investigate rigidity of area-minimizing free boundary surfaces in Riemannian three-manifolds.

R. Schoen and S.T. Yau, in their celebrated joint work, discovered interesting relations between the scalar curvature of a three-dimensional manifold and the topology of stable minimal surfaces inside it, which emerge when one uses the second variation formula for the area, the Gauss equation and the Gauss-Bonnet theorem. An example is given by the following

Theorem 1 (R. Schoen and S.T. Yau). *Let M be an oriented Riemannian three-manifold with positive scalar curvature. Then M contains no immersed orientable closed stable minimal surface of positive genus.*

They used this result to prove that any Riemannian metric with non-negative scalar curvature on the three-torus must be flat. More generally, they proved the following theorem (see [32]).

Theorem 2 (R. Schoen and S.T. Yau). *Let M be a closed oriented three-manifold. If the fundamental group of M contains a subgroup isomorphic to* *the fundamental group of the two-torus, then any Riemannian metric on M with nonnegative scalar curvature must be flat.*

The hypothesis on the fundamental group implies that there exists a continuous map f from the two-torus to M that induces an injective homomorphism f_* on the fundamental groups. Then the idea is to apply a minimization procedure among maps that induce the same homomorphism f_* in order to obtain an immersed stable minimal two-torus in (M, g) for any Riemannian metric g. Since any non-flat Riemannian metric with nonnegative scalar curvature on a closed three-manifold can be deformed to a metric with positive scalar curvature (see [18]), the theorem follows.

In [12], D. Fischer-Colbrie and R. Schoen observed that an immersed, two-sided, stable minimal two-torus in a Riemannian three-manifold with nonnegative scalar curvature must be flat and totally geodesic, and conjectured that Theorem 2 would hold true if one merely assume the existence of an area-minimizing two-torus. This conjecture was established by M. Cai and G. Galloway [8]. More precisely, they proved that if M is a closed Riemannian three-manifold which contains a two-sided embedded two-torus that minimizes the area in its isotopy class, then M is flat. The fundamental step was the following local result.

Theorem 3 (M. Cai and G. Galloway)**.** *If a Riemannian three-manifold with nonnegative scalar curvature contains an embedded, two-sided, locally area-minimizing two-torus Σ, then the metric is flat in some neighborhood of Σ.*

In recent years, some similar results were proven for closed surfaces other than tori under different scalar curvature hypotheses. In particular, we mention the theorems of H. Bray, S. Brendle and A. Neves [5] and I. Nunes [29].

Theorem 4 (H. Bray, S. Brendle and A. Neves)**.** *Let (M, g) be a three-manifold with scalar curvature greater than or equal to 2. If Σ is an embedded two-sphere that is locally area-minimizing, then Σ has area less than or equal to 4π. Moreover, if equality holds, then Σ with the induced metric g_Σ has constant Gaussian curvature equal to 1 and there is a neighborhood of Σ in M that is isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$.*

Theorem 5 (I. Nunes)**.** *Let (M, g) be a three-manifold with scalar curvature greater than or equal to -2. If Σ is an embedded, two-sided, locally area-minimizing closed surface with genus γ greater than 1, then Σ has area greater than or equal to $4\pi(\gamma - 1)$. Moreover, if equality holds, then Σ with the induced metric g_Σ has constant Gaussian curvature equal to -1 and there is a neighborhood of Σ in M that is isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$.*

These local splitting theorems also imply interesting global theorems (see [5] and [29]).

Let us give a sketch of the proof of Theorems 4 and 5. In order to prove the inequalities for the area of the respective Σ in the statements above, one can follow R. Schoen and S.T. Yau, using the stability of Σ, the Gauss equation and the Gauss-Bonnet theorem. These inequalities also appeared in the work of Y. Shen and S. Zhu [33]. When the area of Σ achieves the equality stated in the respective theorems, there are more restrictions on the intrinsic and extrinsic geometries of Σ (recall D. Fischer-Colbrie and R. Schoen remark) which allow to construct a foliation of M around Σ by constant mean curvature surfaces. After this point, they prove that the leaves of the foliation have area not greater than that of Σ. This is achieved by very different means in [5] and [29]. Since Σ is area-minimizing, it follows that each leaf is area-minimizing and its area satisfies the equality stated in the respective theorems, an information that can be used to conclude the local splitting of (M, g) around Σ.

We remark that the use of foliations by constant mean curvature surfaces in relation to scalar curvature problems had already appeared in the work of G. Huisken and S.T. Yau [17] and H. Bray [3].

A very interesting unified approach to Theorems 3, 4 and 5 was provided by M. Micallef and V. Moraru [25], also based on foliations by constant mean curvature surfaces. Based on their method, we prove an analogous local rigidity theorem for free boundary surfaces.

When the scalar curvature of (M, g) and the mean curvature of ∂M are bounded from below, one can consider the following functional in the space of properly immersed compact surfaces Σ in (M, g):

$$I(\Sigma) = \frac{1}{2} \inf_{x \in M} R^M |\Sigma| + \inf_{x \in \partial M} H^{\partial M} |\partial \Sigma|.$$

The next proposition gives an upper bound to $I(\Sigma)$ when one assumes that Σ is a free boundary stable minimal surface:

Proposition 6. *Let (M, g) be a Riemannian three-manifold with boundary. Assume R^M and $H^{\partial M}$ are bounded from below. If Σ is a properly immersed, two-sided, free boundary stable minimal surface, then*

$$I(\Sigma) \leq 2\pi\chi(\Sigma), \tag{1.1}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Moreover, the equality holds if, and only if, Σ satisfies the following properties:

a) Σ is totally geodesic in M and $\partial\Sigma$ consists of geodesics of ∂M;

b) The scalar curvature R^M is constant along Σ and equal to $\inf R^M$, and the mean curvature $H^{\partial M}$ is constant along $\partial\Sigma$ and equal to $\inf H^{\partial M}$;

c) The normal vector field of Σ is in the nullity of Ric along Σ and in the nullity of $B^{\partial M}$ along $\partial\Sigma$.

In particular, a), b) and c) imply that Σ has constant Gaussian curvature $\inf R^M/2$ and $\partial\Sigma$ has constant geodesic curvature $\inf H^{\partial M}$ in Σ.

Inequality (1.1) relates the scalar curvature of M, the mean curvature of ∂M and the topology of the free boundary stable Σ, as in R. Schoen and S.T. Yau's Theorem 1. This connection has also been studied by J. Chen, A. Fraser and C. Pang [10].

For further reference, we will call *infinitesimally rigid* any properly embedded, two-sided, free boundary surface Σ in M that satisfies properties a), b) and c).

Example. It is interesting to have in mind the following model situation. In the Riemannian three-manifolds $(\mathbb{R}\times\Sigma^2, dt^2+g_0)$, where (Σ, g_0) is a compact Riemannian surface with constant Gaussian curvature whose boundary has constant geodesic curvature, all the slices $\{t\}\times\Sigma$ satisfy the hypotheses of Proposition 6 and are infinitesimally rigid. They also have two additional properties: they are in fact area-minimizing and each connected component of their boundary has the shortest possible length in its homotopy class inside the boundary of $\mathbb{R}\times\Sigma$. These properties are immediate consequences of the Maximum Principle.

Given an infinitesimally rigid surface Σ_0 we construct a foliation $\{\Sigma_t\}_{t\in I}$ around Σ_0 by free boundary constant mean curvature surfaces and then analyze the behavior of the area of the surfaces Σ_t following the unified approach of [25]. We prove that $|\Sigma_0| \geq |\Sigma_t|$ for every $t \in I$ (maybe for some smaller interval I) in two cases, depending on $\inf H^{\partial M}$ being zero or positive. As a consequence, we obtain a local rigidity theorem for area-minimizing free boundary surfaces in Riemannian three-manifolds with mean convex boundary.

Theorem A. *Let (M, g) be a Riemannian three-manifold with mean convex boundary. Assume that R^M is bounded from below.*

Let Σ be a properly embedded, two-sided, locally area-minimizing free boundary surface such that $I(\Sigma) = 2\pi\chi(\Sigma)$. Assume that one of the following hypotheses holds:

i) each component of $\partial\Sigma$ is locally length-minimizing in ∂M; or

ii) $\inf H^{\partial M} = 0$.

Then there exists a neighborhood of Σ in (M, g) that is isometric to $((-\epsilon,\epsilon)\times\Sigma, dt^2+g_\Sigma)$, where (Σ, g_Σ) has constant Gaussian curvature $\frac{1}{2}\inf R^M$ *and $\partial\Sigma$ has constant geodesic curvature* $\inf H^{\partial M}$ *in Σ.*

Remark. We emphasize that the meaning of hypothesis i) is that each component of $\partial\Sigma$ has the smallest length compared to every nearby closed curve in ∂M. Since we already know from Proposition 6 that the components of $\partial\Sigma$ are geodesics of ∂M, this hypothesis makes sense, but is stronger. We do not know if this hypothesis can be removed from Theorem A without affecting the result.

Theorem A can be used to prove some global rigidity results.

Let $\mathcal{F}_M$ be the set of all immersed disks in M whose boundaries are curves in ∂M that are homotopically non-trivial in ∂M. If $\mathcal{F}_M$ is non-empty, we define

$$\mathcal{A}(M,g) = \inf_{\Sigma\in\mathcal{F}_M} |\Sigma| \quad \text{and} \quad \mathcal{L}(M,g) = \inf_{\Sigma\in\mathcal{F}_M} |\partial\Sigma|.$$

Our first global rigidity theorem involves a combination of these geometric invariants.

Theorem B. *Let (M,g) be a compact Riemannian three-manifold with mean convex boundary. Assume that $\mathcal{F}_M$ is non-empty. Then*

$$\frac{1}{2}\inf R^M \mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g) \leq 2\pi. \tag{1.2}$$

Moreover, if equality holds, then the universal covering of (M,g) is isometric to $(\mathbb{R}\times\Sigma_0, dt^2+g_0)$, where (Σ_0,g_0) is a disk with constant Gaussian curvature $\inf R^M/2$ and $\partial\Sigma_0$ has constant geodesic curvature $\inf H^{\partial M}$ in (Σ_0,g_0).

The case $\inf R^M = 0$ and $\inf H^{\partial M} > 0$, which includes in particular mean convex domains of the Euclidean space, was treated by M. Li (see his preprint [20]). His approach is similar to the one in [5].

Our proof relies on the fact that $\mathcal{A}(M,g)$ can be realized as the area of a properly embedded free boundary minimal disk Σ_0, by a classical result of W. Meeks and S.T. Yau [23]. Since $H^{\partial M}\geq 0$, we can compare the invariant and $I(\Sigma_0)$, and hence inequality (1.2) follows from Proposition 6. When equality holds, Σ_0 must be infinitesimally rigid and the additional hypotheses of the local splitting Theorem A are verified. Using a continuation argument we conclude the global splitting of the universal covering.

When $\inf R^M$ is negative, we also prove a rigidity theorem for solutions of the Plateau problem, which is an immediate consequence of Theorem B.

As before, assume that (M,g) is a compact Riemannian three-manifold with mean convex boundary. Another classical result of W. Meeks and S.T. Yau [24] says that the Plateau problem has a properly embedded solution in M for any given closed embedded curve in ∂M that bounds a disk in M.

In particular, by considering solutions to the Plateau problem for homotopically non-trivial curves in ∂M that bound disks in M and have the shortest possible length among such curves, we prove the following

Theorem C. *Let (M,g) be a compact Riemannian three-manifold such that $\inf R^M = -2$ and $\inf H^{\partial M} > 0$. Assume that $\mathcal{F}_M$ is non-empty.*

If $\hat{\Sigma}$ is a solution to the Plateau problem for a homotopically non-trivial embedded curve in ∂M that bounds a disk in M and has length $\mathcal{L}(M,g)$, then

$$|\hat{\Sigma}| \geq \inf H^{\partial M}\mathcal{L}(M,g) - 2\pi. \tag{1.3}$$

Moreover, if equality holds in (1.3) for some $\hat{\Sigma}$, then the universal covering of (M,g) is isometric to $(\mathbb{R}\times\Sigma_0, dt^2+g_0)$, where (Σ_0,g_0) is a disk with constant Gaussian curvature -1 and $\partial\Sigma_0$ has constant geodesic curvature $\inf H^{\partial M}$ in Σ_0.

1.2 Infinitesimal rigidity

In begin by deducing some topological and geometrical consequences of the free boundary stability assumption, recall Proposition 6 in the Introduction. Inequality (1.1) follows from the second variation formula of area for free boundary minimal surfaces, the Gauss equation and the Gauss-Bonnet theorem.

Proof of Proposition 6. Let Σ be a properly immersed, two-sided, free boundary stable minimal surface. The free boundary stability hypothesis means that, for every $\phi \in C^\infty(\Sigma)$,

$$Q(\phi,\phi) = \int_\Sigma |\nabla^\Sigma \phi|^2 - (Ric(N,N) + |B^\Sigma|^2)\phi^2 dA - \int_{\partial\Sigma} B^{\partial M}(N,N)\phi^2 dL \geq 0.$$

By evaluating Q on the constant function 1, we get the inequalities

$$\begin{aligned} 0 &\geq \int_\Sigma (Ric(N,N) + |B^\Sigma|^2)dA + \int_{\partial\Sigma} B^{\partial M}(N,N)dL \\ &= \frac{1}{2}\int_\Sigma (R^M + |B^\Sigma|^2)dA - \int_\Sigma K dA - \int_{\partial\Sigma} k dL + \int_{\partial\Sigma} H^{\partial M} dL \\ &\geq \frac{1}{2}\inf R^M |\Sigma| + \inf H^{\partial M}|\partial\Sigma| - 2\pi\chi(\Sigma). \end{aligned}$$

where we used the Gauss equation (0.1) and equation (0.3) for the free boundary minimal surface Σ and the Gauss-Bonnet theorem. This proves inequality (1.1).

When the equality holds in (1.1), every inequality above is in fact an equality. One immediately sees that Σ must be totally geodesic, *b*) holds and $Q(1,1) = 0$. By elementary considerations about bilinear forms, $Q(1,1) = 0$ and $Q(\phi,\phi) \geq 0$ for every $\phi \in C^\infty(\Sigma)$ imply $Q(1,\phi) = 0$ for every $\phi \in C^\infty(\Sigma)$. Choosing appropriately the arbitrary test function ϕ, we conclude that $Ric(N,N) = 0$ and $B^{\partial M}(N,N) = 0$.

Since Σ is totally geodesic, the Codazzi equation (0.2) then implies that in fact $Ric(N,Y)$ is zero for all vectors Y tangent to M along Σ. Moreover, $\nabla_T T$ and $\nabla_T X = \nabla_T \nu$ must be tangent to Σ. Hence, the geodesic curvature of $\partial\Sigma$ in ∂M given by $g(N, \nabla_T T)$ vanishes, and since $\nabla_T X$ is also orthogonal to X we conclude that $\nabla_T X$ is proportional to T, which means that T and therefore N are eigenvectors of ∇X along $\partial\Sigma$. The second part of *a*) and *c*) follow.

The final statement is then a consequence of the Gauss equation (0.1) and equation (0.3). The converse is immediate from the Gauss-Bonnet theorem. □

1.3 CMC foliation near an infinitesimally rigid surface

Given a properly embedded infinitesimally rigid surface Σ in M, there are smooth vector fields Z on M such that $Z(p) = N(p)\ \forall p \in \Sigma$ and $Z(p) \in T_p\partial M\ \forall p \in \partial M$. We fix $\phi = \phi(x,t)$ the flow of one of these vector fields and α a real number between zero and one.

The next proposition gives a family of free boundary constant mean curvature surfaces around any infinitesimally rigid surface.

Proposition 7. *Let (M,g) be a Riemannian three-manifold with boundary. Assume R^M and $H^{\partial M}$ are bounded from below. Let Σ be a properly embedded, two-sided, free boundary surface in (M,g).*

If Σ is infinitesimally rigid, then there exists $\epsilon > 0$ and a function $w : \Sigma \times (-\epsilon,\epsilon) \to \mathbb{R}$ such that, for every $t \in (-\epsilon,\epsilon)$, the set

$$\Sigma_t := \{\phi(x, w(x,t));\ x \in \Sigma\}$$

is a free boundary surface with constant mean curvature $H(t)$. Moreover, for every $x \in \Sigma$ and every $t \in (-\epsilon,\epsilon)$,

$$w(x,0) = 0, \quad \int_\Sigma (w(x,t) - t)\, dA = 0 \quad \textit{and} \quad \frac{\partial}{\partial t} w(x,t)\Big|_{t=0} = 1.$$

In particular, for some smaller ϵ, $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ gives a foliation of a neighborhood of $\Sigma_0 = \Sigma$ in M.

Proof. Given a function u in the Hölder space $C^{2,\alpha}(\Sigma)$, we define $\Sigma_u = \{\phi(x,u(x)); x \in \Sigma\}$, which is a properly embedded surface if the norm of u is small enough. We use the subscript u to denote the quantities associated to Σ_u. For example, H_u will denote the mean curvature of Σ_u, N_u will denote the unit normal vector field of Σ_u and X_u will denote the restriction of X to $\partial\Sigma_u$. In particular, $\Sigma_0 = \Sigma$, $H_0 = 0$ (since Σ is totally geodesic) and $g(N_0, X_0) = 0$ (since Σ_0 is free boundary).

Consider the Banach spaces

$$E = \{u \in C^{2,\alpha}(\Sigma); \int_\Sigma u dA = 0\} \quad \text{and} \quad F = \{u \in C^{0,\alpha}(\Sigma); \int_\Sigma u dA = 0\}.$$

Given small $\delta > 0$ and $\epsilon > 0$, we can define the map $\Phi : (-\epsilon,\epsilon) \times (B(0,\delta) \subset E) \to F \times C^{1,\alpha}(\partial\Sigma)$ given by

$$\Phi(t,u) = (\ H_{t+u} - \frac{1}{|\Sigma|}\int_\Sigma H_{t+u} dA,\ g(N_{t+u}, X_{t+u})\ ).$$

We claim that $D\Phi_{(0,0)}$ is an isomorphism when restricted to $0 \times E$.

In fact, for each $v \in E$, the map $f : (x,s) \in \Sigma \times (-\epsilon,\epsilon) \mapsto \phi(x, sv(x)) \in$

M gives a variation with variational vector field $\frac{\partial f}{\partial s} = vZ = vN$ on Σ. Since Σ is infinitesimally rigid, from the variation formulas (0.4) and (0.5) we get

$$D\Phi_{(0,0)}(0,v) = \frac{d}{ds}\Big|_{s=0}\Phi(0,sv) = (-\Delta_\Sigma v + \frac{1}{|\Sigma|}\int_{\partial\Sigma}\frac{\partial v}{\partial\nu}dL,\ -\frac{\partial v}{\partial\nu}).$$

The claim follows from classical results for Neumann type boundary conditions for the Laplace operator (see for example [19], page 137). A Freedholm alternative holds true and the kernel of the restriction of $D\Phi_{(0,0)}$ to $0\times E$ is trivial, since it consists of zero mean value functions satisfying the homogeneous Neumann problem on Σ.

Now we apply the implicit function theorem: for some smaller ϵ, there exists a function $t \in (-\epsilon,\epsilon) \mapsto u(t) \in B(0,\delta) \subset E$ such that $u(0) = 0$ and $\Phi(t,u(t)) = \Phi(0,0) = (0,0)$ for every t. In other words, the surfaces

$$\Sigma_{t+u(t)} = \{\phi(x, t+u(t)(x));\, x \in \Sigma\}$$

are free boundary constant mean curvature surfaces.

Let $w : (x,t) \in \Sigma\times(-\epsilon,\epsilon) \mapsto t + u(t)(x) \in \mathbb{R}$. By definition, $w(x,0) = u(0)(x) = 0$ for every $x \in \Sigma$ and $w(-,t) - t = u(t)$ belongs to $B(0,\delta) \subset E$ for every $t \in (-\epsilon,\epsilon)$. Observe that the map $G : (x,s) \in \Sigma\times(-\epsilon,\epsilon) \mapsto \phi(x,w(x,s)) \in M$ gives a variation with variational vector field on Σ given by $\left(\frac{\partial w}{\partial t}|_{t=0}\right)N$. Since for every t we have

$$0 = \Phi(t,u(t)) = (\ H_{w(-,t)} - \frac{1}{|\Sigma|}\int_\Sigma H_{w(-,t)}dA\ ,\ g(N_{w(-,t)}, X_{w(-,t)})\ ),$$

by taking the derivative at $t = 0$ we conclude that $\frac{\partial w}{\partial t}|_{t=0}$ satisfies the homogeneous Neumann problem. Therefore it must be constant on Σ. Since $\int_\Sigma (w(x,t) - t)\, dA = \int_\Sigma u(t)(x) dA = 0$ for every t, by differentiating at $t = 0$ we conclude that $\int_\Sigma \left(\frac{\partial w}{\partial t}|_{t=0}\right) dA = |\Sigma|$. Hence, $\frac{\partial w}{\partial t}|_{t=0} = 1$, as claimed.

Since $G_0(x) = \phi(x,0) = x$, $\partial_t G(x,0) = \frac{\partial w}{\partial t}|_{t=0}N(x) = N(x)$ for every $x \in \Sigma$ and Σ is properly embedded, by taking a smaller ϵ, if necessary, we can assume that G parametrizes a foliation of M around Σ. This finishes the proof of the proposition. □

1.4 Local rigidity

We consider infinitesimally rigid surfaces in a Riemannian three-manifold with mean convex boundary and scalar curvature bounded from below. First we analyze the behavior of the area of the surfaces in the family constructed in the previous section. This analysis is based on [25].

Proposition 8. *Let (M,g) be a Riemannian three-manifold with mean convex boundary and scalar curvature bounded from below. Let Σ_0 be a properly* *embedded, two-sided, free boundary, infinitesimally rigid surface.*

Assume that one of the following hypotheses holds:

i) each component of $\partial\Sigma_0$ is locally length-minimizing in ∂M; or

ii) $\inf H^{\partial M} = 0$.

Let $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ be as in Proposition 7. Then $|\Sigma_0| \geq |\Sigma_t|$ for every $t \in (-\epsilon,\epsilon)$ (maybe for a smaller ϵ).

Proof. Following the notation of Proposition 7, let $G : \Sigma_0 \times (-\epsilon,\epsilon) \to M$ given by $G_t(x) = \phi(x, w(x,t))$ parametrize the foliation $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ around the infinitesimally rigid Σ_0. After this point, we will use the subscript t to denote the quantities associated to $\Sigma_t = G_t(\Sigma_0)$.

For each $t \in (-\epsilon,\epsilon)$, the lapse function of the variation $\{\Sigma_t\}$ on Σ_t given by $\rho_t = g(\partial_t G, N_t)$ satisfies the equations

$$-H'(t) = \Delta_{\Sigma_t}\rho_t + (Ric(N_t,N_t) + |B_t|^2)\rho_t, \tag{1.4}$$

$$\frac{\partial\rho_t}{\partial\nu_t} = B^{\partial M}(N_t,N_t)\rho_t, \tag{1.5}$$

that follows from the general formulas (0.4) and (0.5) since we have a variation by free boundary constant mean curvature surfaces.

Furthermore, since $\partial_t G(x,0) = N_0(x)$ for every $x \in \Sigma_0$, we have $\rho_0 = 1$. Hence, we can assume $\rho_t > 0$ for all $t \in (-\epsilon,\epsilon)$. From equation (1.4) we have

$$H'(t)\frac{1}{\rho_t} = -(\Delta_{\Sigma_t}\rho_t)\frac{1}{\rho_t} - (Ric(N_t,N_t) + |B_t|^2).$$

Using the Gauss equation (0.1), we rewrite

$$H'(t)\frac{1}{\rho_t} = -(\Delta_{\Sigma_t}\rho_t)\frac{1}{\rho_t} + K_t - \frac{1}{2}(R^M_t + H(t)^2 + |B_t|^2).$$

Recalling that $H(t)$ is constant on Σ_t, we integrate by parts and use equation (1.5) in order to get

$$\begin{aligned} H'(t)\int_{\Sigma_t}\frac{1}{\rho_t}dA_t = &-\int_{\Sigma_t}\frac{|\nabla^{\Sigma_t}\rho_t|^2}{\rho_t^2}dA_t - \int_{\partial\Sigma_t}B^{\partial M}(N_t,N_t)dL_t \\ &+ \int_{\Sigma_t}K_t dA_t - \frac{1}{2}\int_{\Sigma_t}(R^M_t + H(t)^2 + |B_t|^2)dA_t. \end{aligned}$$

Since each Σ_t is free boundary, equation (0.3) and the Gauss-Bonnet theorem imply

$$\begin{aligned} H'(t)\int_{\Sigma_t}\frac{1}{\rho_t}dA_t = &-\int_{\Sigma_t}\frac{|\nabla^{\Sigma_t}\rho_t|^2}{\rho_t^2}dA_t - \frac{1}{2}\int_{\Sigma_t}(R^M_t + H(t)^2 + |B_t|^2)dA_t \\ &- \int_{\partial\Sigma_t}H^{\partial M}_t dL_t + 2\pi\chi(\Sigma_0). \end{aligned}$$

Finally, since Σ_0 is infinitesimally rigid, the Gauss-Bonnet theorem implies that $I(\Sigma_0) = 2\pi\chi(\Sigma_0)$. Hence, we have the following inequality:

$$H'(t)\int_{\Sigma_t}\frac{1}{\rho_t}dA_t \leq I(\Sigma_0) - I(\Sigma_t)$$
$$= \frac{1}{2}\inf R^M(|\Sigma_0| - |\Sigma_t|) + \inf H^{\partial M}(|\partial\Sigma_0| - |\partial\Sigma_t|).$$

By hypothesis, $\inf H^{\partial M} \geq 0$. If each boundary component is locally length-minimizing, the second term in the right hand side is less than or equal to zero, and in case $\inf H^{\partial M} = 0$, it is obviously zero. Therefore

$$H'(t)\int_{\Sigma_t}\frac{1}{\rho_t}dA_t \leq \frac{1}{2}\inf R^M(|\Sigma_0| - |\Sigma_t|) = -\frac{1}{2}\inf R^M\int_0^t\frac{d}{ds}|\Sigma_s|ds.$$

Since each Σ_t is free boundary, the first variation formula of area gives

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t}\rho_t H(t)dA_t = H(t)\int_{\Sigma_t}\rho_t dA_t. \tag{1.6}$$

Therefore

$$H'(t)\int_{\Sigma_t}\frac{1}{\rho_t}dA_t \leq -\frac{1}{2}\inf R^M\int_0^t H(s)\left(\int_{\Sigma_s}\rho_s dA_s\right)ds. \tag{1.7}$$

Claim: there exists $\epsilon > 0$ such that $H(t) \leq 0$ for every $t \in [0,\epsilon)$.

We consider three cases:

a) $\inf R^M = 0$.
Then it follows immediately from (1.7) that $H'(t) \leq 0$ for every $t \in [0,\epsilon)$. Since $H(0) = 0$, the claim follows.

b) $\inf R^M > 0$.
Let $\varphi(t) = \int_{\Sigma_t}\frac{1}{\rho_t}dA_t$ and $\xi(t) = \int_{\Sigma_t}\rho_t dA_t$. Inequality (1.7) can be rewritten as

$$H'(t) \leq -\frac{1}{2}\inf R^M\frac{1}{\varphi(t)}\int_0^t H(s)\xi(s)ds. \tag{1.8}$$

By continuity, we can assume that there exists a constant $C > 0$ such that $\frac{1}{\varphi(t)}\int_0^t\xi(s)ds \leq 2C$ for every $t \in [0,\epsilon]$.

Choose $\epsilon > 0$ such that $C\inf R^M\epsilon < 1$. Then $H(t) \leq 0$ for every $t \in [0,\epsilon)$. In fact, suppose that there exists $t_+ \in (0,\epsilon)$ such that $H(t_+) > 0$. By continuity, there exists $t_- \in [0,t_+]$ such that $H(t) \geq H(t_-)$ for every $t \in [0,t_+]$. Notice that $H(t_-) \leq H(0) = 0$. By the mean value theorem, there exists $t_1 \in (t_-, t_+)$ such that $H(t_+) - H(t_-) = H'(t_1)(t_+ - t_-)$. Hence, since $\inf R^M > 0$, inequality (1.8) gives

$$\begin{aligned}\frac{H(t_+) - H(t_-)}{t_+ - t_-} = H'(t_1) &\leq \frac{1}{2} \inf R^M \frac{1}{\varphi(t_1)} \int_0^{t_1} (-H(s))\xi(s)ds \\ &\leq \frac{1}{2} \inf R^M (-H(t_-)) \left(\frac{1}{\varphi(t_1)} \int_0^{t_1} \xi(s)ds \right) \\ &\leq \inf R^M (-H(t_-))C.\end{aligned}$$

It follows that $H(t_+) \leq H(t_-)(1 - C \inf R^M \epsilon)$, which is a contradiction since $H(t_+) > 0$ and $H(t_-) \leq 0$.

c) $\inf R^M < 0$.

Choose $\epsilon > 0$ such that $-C \inf R^M \epsilon < 1$, where $C > 0$ is the same constant that appears in case b). Then $H(t) \leq 0$ for every $t \in [0, \epsilon)$. In fact, suppose that there exists $t_0 \in (0, \epsilon)$ such that $H(t_0) > 0$. Let

$$P = \{t \in [0, t_0];\ H(t) \geq H(t_0)\}.$$

Let $t^* \in [0, \epsilon]$ be the infimum of P. Observe that, by the definition of t^*, $H(t) \leq H(t_0) = H(t^*)$ for every $t \in [0, t^*]$.

If $t^* > 0$, then the mean value theorem implies that there exists $t_1 \in (0, t^*)$ such that $H(t^*) = H'(t_1)t^*$, since $H(0) = 0$. Hence, since $\inf R^M < 0$, inequality (1.8) gives

$$\begin{aligned}\frac{H(t^*)}{t^*} = H'(t_1) &\leq -\frac{1}{2} \inf R^M \frac{1}{\varphi(t_1)} \int_0^{t_1} H(s)\xi(s)ds \\ &\leq -\frac{1}{2} \inf R^M H(t^*) \left(\frac{1}{\varphi(t_1)} \int_0^{t_1} \xi(s)ds \right) \\ &\leq -\inf R^M H(t^*)C.\end{aligned}$$

It follows that $H(t^*)(1 + C \inf R^M H(t^*)\epsilon) \leq 0$. This is a contradiction since $H(t^*) = H(t_0) > 0$.

Hence $t^* = 0$, which is again a contradiction since in this case $H(0) \geq H(t_0) > 0$.

This proves the claim. By equation (1.6), we conclude that $|\Sigma_0| \geq |\Sigma_t|$ for every $t \in [0, \epsilon)$. The proof that $|\Sigma_0| \geq |\Sigma_t|$ for every $t \in (-\epsilon, 0]$ is analogous. □

We are now ready to prove the main result (recall Theorem A in the Introduction).

Proof of Theorem A. Since Σ is locally area-minimizing and $I(\Sigma) = 2\pi\chi(\Sigma)$, Σ is infinitesimally rigid. From Propositions 7 and 8 we obtain a foliation

$\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ around $\Sigma_0 = \Sigma$ such that $|\Sigma_0| \geq |\Sigma_t|$ for every $t \in (-\epsilon,\epsilon)$. Therefore $|\Sigma_t| = |\Sigma_0|$ and each Σ_t is also locally area-minimizing (at least if we take a possibly smaller ϵ).

It is immediate to see that when $\inf H^{\partial M} = 0$ or when the connected components of $\partial\Sigma_0$ are locally length-minimizing in ∂M,

$$2\pi = I(\Sigma_0) \leq I(\Sigma_t) \leq 2\pi,$$

which implies that each Σ_t is infinitesimally rigid. From equations (1.4) and (1.5) in Proposition 8, one sees that for each t the lapse function ρ_t satisfies the homogeneous Neumann problem. Hence ρ_t is a constant function on Σ_t.

Since we have a foliation, the normal fields of Σ_t define a vector field on M near Σ_0. Because all surfaces Σ_t are totally geodesic and ρ_t is constant, using the variation formula (0.5) one concludes that this field is parallel. In particular, its flow is a flow by isometries and therefore provides the local splitting: a neighborhood of Σ_0 is in fact isometric to the product $((-\epsilon,\epsilon)\times\Sigma_0, dt^2 + g_{\Sigma_0})$. Since Σ_0 is infinitesimally rigid, (Σ_0, g_{Σ_0}) has constant Gaussian curvature $\inf R^M/2$ and $\partial\Sigma_0$ has constant geodesic curvature $\inf H^M$ in Σ_0.

□

1.5 Global rigidity

We begin this section with the precise statement of the result of W. Meeks and S.T. Yau about the existence of area-minimizing free boundary disks that we want to use in order to prove global results. Given a three-manifold M, recall that we have denoted by $\mathcal{F}_M$ the set of immersed disks in M whose boundaries are curves in ∂M that are homotopically non-trivial in ∂M.

Theorem 9 (W. Meeks and S.T. Yau, [23])**.** *Let (M,g) be a compact Riemannian three-manifold with mean convex boundary. If $\mathcal{F}_M$ is non-empty, then:*

1) There exists an immersed minimal disk Σ_0 in M such that $\partial\Sigma_0$ represents a homotopically non-trivial curve in ∂M and $|\Sigma_0| = \inf_{\Sigma\in\mathcal{F}_M}|\Sigma|$.

2) Any such least area immersed disk is in fact a properly embedded free boundary disk.

We are now ready to prove our global results (recall Theorems B and C in the Introduction).

Proof of Theorem B. Since $\mathcal{F}_M$ is non-empty, Theorem 9 says that there exists a properly embedded free boundary minimal disk $\Sigma_0 \in \mathcal{F}_M$ such that $|\Sigma_0| = \mathcal{A}(M,g)$. Since Σ_0 is two-sided and free boundary stable, the inequality follows from Proposition 6:

$$\frac{1}{2}\inf R^M \mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g) \leq I(\Sigma_0) \leq 2\pi.$$

Assume the equality holds. In case $\inf H^{\partial M}$ is not zero, $\partial\Sigma_0$ must have length $\mathcal{L}(M,g)$, therefore it is length-minimizing. In any case, we can apply Theorem A to get a local splitting of (M,g) around Σ_0.

Let exp denote the exponential map of (M,g). Let S be the set all $t>0$ such that the map $\Psi : [-t,t]\times\Sigma_0 \to M$ given by $\Psi(s,x)=\exp_x(sN_0(x))$ is well-defined, $\Psi([-t,t]\times\partial\Sigma_0)$ is contained in ∂M and $\Psi : ((-t,t)\times\Sigma_0, ds^2+g_{\Sigma_0}) \to (M,g)$ is a local isometry.

We claim that $S=[0,+\infty)$. In order to prove this, first observe that S is non-empty, since we have the local splitting around Σ_0. From the definition of S, it is also clear that S must contain all the interval $[-t,t]$ if t belongs to S. Moreover, if $\{t_n\}$ is an increasing sequence in S converging to $t\in[0,+\infty)$, then $t\in S$. Therefore the claim will follow if we prove that $\sup S=+\infty$.

If $t^* = \sup S < +\infty$, since S contains t^* we can consider the immersed disks $\Sigma_{\pm t^*} = \{\Psi_{\pm t^*}(x);\, x\in\Sigma_0\}$ in M. Notice that $\partial\Sigma_{\pm t^*}$ are homotopically non-trivial curves in ∂M and that $|\Sigma_{\pm t^*}| = |\Sigma_0| = \mathcal{A}(M,g)$. It follows from Theorem 9 that $\Sigma_{\pm t^*}$ are properly embedded free boundary minimal disks. Comparing as before $I(\Sigma_0)$ and $I(\Sigma_{\pm t^*})$ we verify that we can apply the local rigidity Theorem A and then conclude that S contains $t' > t^*$, a contradiction.

Therefore we have a well-defined local isometry,

$$\Psi : (t,x) \in (\mathbb{R}\times\Sigma_0, dt^2+g_{\Sigma_0}) \mapsto \exp_x(tN_0(x)) \in (M,g),$$

such that $\Psi(\mathbb{R}\times\partial\Sigma_0)$ is contained in ∂M. Such Ψ is a covering map. This finishes the proof of Theorem B. $\square$

In order to prove Theorem C (see the Introduction), consider any $\hat{\Sigma}$ as in its statement. $\hat{\Sigma}$ has area at least $\mathcal{A}(M,g)$ and $\partial\hat{\Sigma}$ has length $\mathcal{L}(M,g)$. When $\inf R^M$ is negative,

$$I(\hat{\Sigma}) = \frac{1}{2}\inf R^M|\hat{\Sigma}| + \inf H^{\partial M}|\partial\hat{\Sigma}| \leq \frac{1}{2}\inf R^M\mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g).$$

and therefore Theorem C is an immediate corollary of Theorem B.

CHAPTER 2

On perturbations of the Anti-de Sitter-Schwarzschild spaces of positive mass

2.1 Introduction

Let (M^3, g) be a complete Riemannian three-manifold, possibly with boundary, with exactly one end. Assume that the complement of a compact subset of M is diffeomorphic to $\mathbb{R}^3$ minus a ball. Roughly speaking, if the metric g decays sufficiently fast to the hyperbolic metric when expressed in the spherical coordinates induced by this chart, (M, g) is called *asymptotically hyperbolic*. If an asymptotically hyperbolic manifold (M, g) has scalar curvature $R \geq -6$ and its boundary ∂M is empty or has mean curvature $H \leq 2$, it is well-defined a geometric invariant called the *total mass* of (M, g), a nonnegative number m that is zero if and only if (M, g) is isometric to the hyperbolic space. This is the content of the Positive Mass Theorem in the asymptotically hyperbolic setting, proved with spinorial methods by X. Wang [34] when ∂M is empty and in general by P. Chruściel and M. Herzlich [11] under weaker asymptotic conditions.

If an asymptotically hyperbolic manifold (M, g) has scalar curvature $R \geq -6$ and its boundary ∂M is a *connected minimal surface* that is *outermost*, i.e., if there are no other closed minimal surfaces in M, then it is conjectured that the total mass m of (M, g) is related to the area of ∂M by the following inequality:

$$\left(\frac{|\partial M|}{16\pi}\right)^{\frac{1}{2}} + 4\left(\frac{|\partial M|}{16\pi}\right)^{\frac{3}{2}} \leq m. \tag{2.1}$$

This statement is known as the *Penrose Conjecture* in the asymptotically hyperbolic setting. We refer the reader to the surveys [6] and [21], where he will find a comprehensive discussion on these types of inequalities in various settings. In the original *asymptotically flat* setting, the corresponding Penrose Conjecture has been proved with different techniques by G. Huisken and T. Ilmanen [16] and by H. Bray [4].

The Penrose Conjecture contains also a rigidity statement. There are important models, known as the *Anti-de Sitter-Schwarzschild spaces of positive mass*, that satisfy equality in (2.1). They are obtained as spherically symmetric metrics g_m on $M = [0, +\infty) \times S^2$ with constant scalar curvature -6, where the parameter m is a positive real number that coincides with the total mass of (M, g_m) above discussed (see Section 2.2 for more details). The Penrose Conjecture also asserts that the Anti-de Sitter-Schwarzschild spaces of positive mass are the unique asymptotically hyperbolic manifolds, with scalar curvature $R \geq -6$ and an outermost minimal boundary, that satisfy the equality in (2.1).

A special feature of the Anti-de Sitter-Schwarzschild spaces of positive mass is that they are foliated by constant mean curvature spheres which are *weakly stable*. This kind of foliation of an asymptotically hyperbolic manifold is interesting, among other reasons, because of a monotonicity result observed by H. Bray in [3]: if $R \geq -6$, the so-called Hawking mass functional, defined for closed surfaces Σ in (M, g) by

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma (H^2 - 4) d\Sigma\right),$$

is monotone non-decreasing along these foliations in the direction of increasing area of the leaves.

If ∂M is a minimal surface in (M, g), its Hawking mass is exactly the left-hand side of (2.1). This suggests an approach to prove the Penrose inequality (2.1), at least for the class of asymptotically hyperbolic manifolds (M, g), with $R \geq -6$ and an outermost minimal boundary, admitting this kind of foliation starting at ∂M and sweeping out all M: inequality (2.1) would follow if the Hawking mass of the leaves near the infinity converges to the total mass of (M, g).

In [27] and [28], A. Neves and G. Tian proved existence and uniqueness results for foliations by weakly stable CMC spheres of the complement of a compact set of certain asymptotically hyperbolic manifolds. See also the works of R. Rigger [30] and R. Mazzeo and F. Pacard [22] for other results of this nature in asymptotically hyperbolic settings. For the original asymptotically flat setting, see the pioneering work of G. Huisken and S.T. Yau [17].

We will consider metrics on $M = [0, +\infty) \times S^2$ that are small *global* perturbations of the Anti-de Sitter-Schwarzschild metric g_m in a suitable sense and show that the foliation constructed in [28] outside a compact set can be extended up to ∂M. Following the argument outlined above, we then show the Penrose inequality is true for these asymptotically hyperbolic manifolds.

In order to state our results more precisely, let us introduce some terminology. Given $m > 0$, the perturbations of g_m we consider belong to the space $\mathcal{M}(M, m)$ of Riemannian metrics g on $M := \{p = (s, x) \in [0, +\infty) \times S^2\}$ such that ∂M is a minimal surface in (M, g) and

$$d(g, g_m) := \sup_{p \in M} \left(\sum_{i=0}^{3} \exp(-4s) \|(\nabla^m)^i (g - g_m)\|_{g_m}(p) \right) < +\infty.$$

See Section 2.2 for the details. All these metrics are asymptotically hyperbolic with well-defined total mass m in the sense of [34] and [11].

Our main results can be summarized as follows (see Theorem 7 and Theorem 8):

Theorem D. *Let $m > 0$. Then there exists $\epsilon > 0$ such that for every metric $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ the following statements hold:*

i) There exists a foliation $\{\Sigma_t\}_{t \in [0, +\infty)}$ of (M, g) by weakly stable CMC spheres such that $\Sigma_0 = \partial M$ is an outermost minimal surface.

ii) (The Penrose inequality). If the scalar curvature of g is greater than or equal to -6, then

$$\left(\frac{|\partial M|}{16\pi} \right)^{\frac{1}{2}} + 4 \left(\frac{|\partial M|}{16\pi} \right)^{\frac{3}{2}} \leq \lim_{t \to +\infty} m_H(\Sigma_t) = m,$$

with equality if and only if (M, g) is isometric to the Anti-de Sitter-Schwarzschild space of mass m.

We remark that in the asymptotically hyperbolic setting there is also another form of the Penrose Conjecture where the boundary corresponds to some $H = 2$ surface (see [6] and [21]). In the end of this chapter we briefly discuss it and explain the immediate modifications of the previous results that establish it for small perturbations (see Theorem 9 and Theorem 10).

The approach to the Penrose Conjecture involving the monotonicity of the Hawking mass for a family of surfaces that interpolates the outermost boundary and the infinity was originally suggested in the asymptotically flat setting by R. Geroch [13], who proposed the inverse mean curvature flow to produce such family. This program was sucessfully implemented by G. Huisken and T. Ilmanen [16]. In the asymptotically hyperbolic setting, however, there are serious difficulties in using this approach, see the work of A. Neves [26].

In the asymptotically hyperbolic setting, the Penrose Conjecture in its full generality is still an open problem. Using Bray's monotonicity as in [3], J. Corvino, A. Gerek, M. Greenberg and B. Krummel [9] proved it for asymptotically hyperbolic manifolds that are isometric to an Anti-de Sitter-Schwarzschild space of positive mass outside a compact set under restrictive hypotheses on the behavior of its isoperimetric surfaces. F. Girão and L. L. de Lima proved both cases and the higher dimensional analogues of them for another class of asymptotically hyperbolic manifolds, those that are certain graphical hypersurfaces of the hyperbolic space (see [14] and [15]).

2.2 The Anti-de Sitter-Schwarzschild spaces and its perturbations

Let m be a real number. Let $\rho_m : (r_0, +\infty) \to \mathbb{R}$ be the function given by $\rho_m(r) = \sqrt{1 + r^2 - 2m/r}$, where $r_0 = r_0(m)$ is the unique positive zero of ρ_m (if $m > 0$) or 0 (if $m \leq 0$). Let (S^2, g_0) be the round sphere of constant Gaussian curvature 1 and let dS^2 denote its volume element.

We call *Anti-de Sitter-Schwarzschild space of mass m* the metric completion of the Riemannian manifold $((r_0, +\infty) \times S^2, g_m)$, where using the natural r-coordinate the metric g_m is written as

$$g_m = \frac{dr^2}{1 + r^2 - \frac{2m}{r}} + r^2 g_0.$$

Although the expression of g_m in this coordinate system becomes singular at r_0, it can be proved that when $m > 0$ the metric g_m extends to a smooth Riemannian metric on $M = [r_0, \infty) \times S^2$. On the other hand a smooth extension is not possible if $m < 0$. Notice also that if we let the parameter m to be zero we recover the hyperbolic space (this can be easily seen by performing the coordinate change $r = \sinh s$).

We will call *coordinate spheres* the surfaces $S_r = \{r\} \times S^2 \subset M$. The following proposition describes the geometry of (M, g_m) and of its coordinate spheres.

Proposition 1. *(Geometry of the Anti-de Sitter-Schwarzschild space of mass m)*

i) The Ricci curvature of g_m is given by

$$Ric_m = (-2 - \frac{2m}{r^3})\frac{1}{\rho_m^2(r)}dr^2 + (-2 + \frac{m}{r^3})r^2 g_0.$$

ii) The scalar curvature of g_m is constant and equal to -6.

iii) The coordinate spheres S_r are totally umbilic surfaces with constant mean curvature given by

$$H_m(r) = \frac{2}{r}\sqrt{1 + r^2 - \frac{2m}{r}}.$$

iv) The Hawking mass of all coordinate spheres is m.

v) The Jacobi operator of the coordinate sphere S_r is given by

$$L_r = \frac{1}{r^2}\left(\Delta_0 + \left(2 - \frac{6m}{r}\right)\right),$$

where Δ_0 is the Laplacian operator of the round sphere (S^2, g_0).

Proof. A calculation in coordinates (see also the Appendix). □

Remark. Notice that when $m > 0$ the Jacobi operator of S_r is invertible except at $r = 3m$. In any case, it is invertible when restricted to the space of zero mean value functions. However, a degeneration occurs when r goes to infinity: up to normalization, it becomes $\Delta_0 + 2$, which is no more invertible in this restricted space.

From now on we assume $m > 0$. Let s be the function that gives the distance of a point on (M, g_m) to ∂M. Using $s \in [0, +\infty)$ as coordinate, one can write

$$g_m = \frac{dr^2}{1 + r^2 - \frac{2m}{r}} + r^2 g_0 = ds^2 + \sinh^2(s) v_m(s) g_0, \tag{2.2}$$

where v_m is a positive function defined on $[0, +\infty)$ that has the following expansion as s goes to infinity:

$$v_m(s) = 1 + \frac{2m}{3\sinh^3 s} + O(\exp(-5s)).$$

Although we have explicit formulas as in Proposition 1 only when we use the r-coordinate, it will be more convenient to use the s-coordinate. We will then consider g_m to be defined on $M = [0, +\infty) \times S^2$ by formula (2.2) above, and as a small abuse of notation we use s both for the first coordinate of a point $p \in M$ and for the function $r \in (r_0, +\infty) \mapsto s(r) \in (0, +\infty)$ that gives the coordinate change described above. It worths noticing that, as a function of s, the r coordinate expands as $r = \sinh(s)(1 + O(\exp(-3s)))$ as s goes to infinity. In particular, for example, the mean curvature of the coordinate spheres $S_s := S_{r(s)}$ behaves as

$$H_m(s) = 2\frac{\cosh s}{\sinh s} - \frac{2m}{\sinh^3 s} + O(\exp(-5s)) \quad \text{as } s \text{ goes to infinity.}$$

Now we define the class of metrics on $M = [0, +\infty) \times S^2$ we are going to work with. Fix some $\alpha \in (0, 1)$.

Definition 1. Given $m > 0$, let $\mathcal{M}(M, m)$ be the set of Riemannian metrics g on $M = [0, +\infty) \times S^2$ such that

a) ∂M is a minimal surface in (M, g); and

b) There exists a constant $C > 0$ such that for every $p = (s, x) \in M$,

$$\left(\|g - g_m\| + \|\nabla^m g\| + \|(\nabla^m)^2 g\| + \|(\nabla^m)^3 g\|\right)(p) \leq C\exp(-4s).$$

Here the norm is the $C^{0,\alpha}$-norm calculated with respect to the metric g_m and ∇^m denotes the Levi-Civita connection of g_m.

The space $\mathcal{M}(M, m)$ has a distance function

$$d(g_1, g_2) := \sup_{p\in M}\left(\sum_{i=0}^{3}\exp(-4s)\|(\nabla^m)^i(g_1 - g_2)\|(p)\right).$$

We remark that each metric in $\mathcal{M}(M, m)$ is asymptotically hyperbolic with total mass m, according to the definitions of [11] and even [34]. Observe also that we do not assume a priori that ∂M is outermost.

Given $g \in \mathcal{M}(M, m)$ one can calculate the expansions of its Ricci tensor, its scalar curvature and the mean curvature of the coordinate spheres in (M, g) as follows: one adds terms of order $O(\exp(-4s))$ to the expansion of the corresponding quantities of (M, g_m) in s-coordinate.

We finish this section by discussing surfaces in (M, g), $g \in \mathcal{M}(M, m)$. All surfaces we consider are closed surfaces $\Sigma \subset M = [0, +\infty) \times S^2$ such that $M \setminus \Sigma$ has two connected components, one of them containing ∂M. We define the inner radius and the outer radius of such Σ to be

$$\underline{s} = \inf\{s(x); x \in \Sigma\} \quad \text{and} \quad \overline{s} = \sup\{s(x); x \in \Sigma\},$$

respectively. We will frequently consider surfaces in (M, g) that are graphical over coordinate spheres S_s. Given some function f on S^2, we write

$$S_s(f) := \{(s + f(x), x) \in M; x \in S^2\}.$$

2.3 CMC foliations of compact regions

For metrics $g \in \mathcal{M}(M, m)$ that are close enough to g_m, we use the implicit function theorem to construct a family of weakly stable CMC spheres on compact regions of (M, g) containing ∂M.

Theorem 2. *Let $m > 0$. Given $S > 0$, there exists $\epsilon > 0$ and $\eta > 0$ with the following properties:*

For every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ and for every $s \in [0, S]$ there exists a unique function $u(s, g) \in C^{2,\alpha}(S^2)$ with $\|u(s, g)\|_{C^{2,\alpha}} < \eta$ and $\int_{S^2} u(s, g)dS^2 = 0$ such that the surface

$$\Sigma_s(g) := S_s(u(s, g)) = \{(s + u(s, g)(x), x) \in M; x \in S^2\}$$

has constant mean curvature with respect to the metric g.

Moreover, for every $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$, $\Sigma_0(g) = \partial M$ and the family $\{\Sigma_s(g)\}_{s\in[0,S]}$ gives a foliation of a compact region of (M,g) by weakly stable CMC spheres, with positive mean curvature if $s \in (0,S]$.

Finally, when $S > s(3m)$, given any constant $\kappa > 0$ and any compact interval $I \subset (s(3m),S]$, it is possible to choose the ϵ above in such way that for every $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ the mean curvature $H_g(s)$ of $\Sigma_s(g)$ in (M,g) is monotone decreasing on I and satisfies $|H_g(s) - H_m(s)| < \kappa$ for every $s \in I$.

Proof. For a minor technical reason, we fix some small $a > 0$ and consider $M \subset \tilde{M} = (-a,+\infty) \times S^2$. For each metric $g \in \mathcal{M}(M,m)$ we use the Taylor expansion of g in normal exponential coordinates based on ∂M to define an extension of g to $\tilde{M}$. This space of metrics inherits the distance of $\mathcal{M}(M,m)$. For simplicity we keep using the same notation $\mathcal{M}(M,m)$ for this space of metrics.

Consider the Banach spaces

$$E = \{u \in C^{2,\alpha}(S^2); \int_{S^2} u dS^2 = 0\} \text{ and } F = \{u \in C^{0,\alpha}(S^2); \int_{S^2} u dS^2 = 0\}.$$

Given $s \in [0,S]$ and $u \in E$ sufficiently small, we consider the surfaces

$$S_s(u) = \{(s + u(x), x) \in \tilde{M}; x \in S^2\}.$$

Denote by $H(s,u,g)$ the mean curvature of $S_s(u)$ with respect to a metric $g \in \mathcal{M}(M,m)$. Given a sufficiently small $\eta > 0$, we consider the map $\Phi : [0,S] \times \mathcal{M}(M,m) \times (B(0,\eta) \subset E) \to F$ given by

$$\Phi(s,g,u) = H(s,u,g) - \frac{1}{4\pi}\int_{S^2} H(s,u,g) dS^2.$$

By definition, $\Phi(s,g,u) = 0$ if and only if $S_s(u)$ is a CMC surface in $(\tilde{M},g)$. In particular, $\Phi(s,g_m,0) = 0$ for all $s \in [0,S]$.

We claim that, for every $s \in [0,S]$, $D\Phi_{(s,g_m,0)}$ is an isomorphism when restricted to E. In fact, for every $v \in E$, the family $t \mapsto S_s(tv)$ is a normal variation of the coordinate sphere S_s in (M,g_m) with speed v. Therefore, if L_s is the Jacobi operator of S_s with respect to g_m, we have

$$D\Phi_{(s,g_m,0)}(0,0,v) = \frac{d}{dt}_{|t=0}\Phi(s,g_m,tv) = -L_s(v) + \frac{1}{4\pi}\int_{S^2} L_s(v) dS^2 = -L_s(v).$$

The last equality follows because $L_s(v) = (1/r^2)(\Delta_0 + (2 - 6m/r))(v)$ (see Proposition 1, considering the coordinate change $r = r(s)$) and v has zero mean value. Since $m > 0$, $\Delta_0 + (2 - 6m/r)$ is an invertible operator from E to F for all $s \in [0,S]$ and the claim follows.

Therefore we can apply the implicit function theorem: there exists a small ball B around (s, g_m) in $[0,S]\times\mathcal{M}(M,m)$, some possibly smaller $\eta > 0$ and a function $(\tilde{s}, g) \in B \mapsto u(\tilde{s}, g) \in B(0,\eta) \subset E$ such that $u(s, g_m) = 0$ and $u(\tilde{s}, g)$ is uniquely defined in $B(0,\eta)$ by the equation $\Phi(\tilde{s}, g, u(\tilde{s}, g)) = 0$ for all $(\tilde{s}, g) \in B$.

By compactness, we can choose sufficiently small $\eta > 0$ and $\epsilon > 0$ such that u is uniquely defined on $[0,S] \times \{g \in \mathcal{M}(M,m); d(g,g_m) < \epsilon\}$ and takes values on $B(0,\eta) \subset E$. Therefore for each metric $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ we have constructed a family

$$\{\Sigma_s(g)\}_{s\in[0,S]} := \{S_s(u(s,g))\}_{s\in[0,S]}$$

of CMC spheres in $(\tilde{M}, g)$, where $u(s,g) \in E$ has norm $\|u(s,g)\|_{C^{2,\alpha}} < \eta$. Notice that $\{\Sigma_s(g_m)\}$ is precisely the foliation of M by coordinate spheres.

Since $\partial M = S_0(0)$ is minimal for all metrics $g \in \mathcal{M}(M,m)$, the uniqueness of the function u above constructed implies that $u(0,g) = 0$, i.e., $\Sigma_0(g) = \partial M$ for every $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$.

In order to prove that $\{\Sigma_s(g)\}$ is a foliation of some region of M, we have to analyze the sign of its lapse function, that is, its normal speed. Since for g_m the constructed family $\{\Sigma_s(g_m)\}$ is a foliation, its lapse function is positive on $[0,S]$, hence the lapse function of $\{\Sigma_s(g)\}_{s\in[0,S]}$ with respect to all $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ is also positive, at least when we choose a possibly smaller ϵ. Since each of these families starts at $\partial M = \{0\}\times S^2 \subset \tilde{M}$, the families $\{\Sigma_s(g)\}_{s\in[0,S]}$ foliate a compact region of M.

To see that $\Sigma_g(s)$ has positive mean curvature in (M,g) for all $s \in (0,S]$, observe that this is true for $\{\Sigma_s(g_m)\}$ in (M,g_m), and also that, by Proposition 1, $H_m'(0) = -(1/r_0^2)(2 - 6m/r_0) > 0$. Hence, by continuity we can arrange ϵ in such way that for every $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ the surface $\Sigma_s(g)$ has positive mean curvature in (M,g) for all $s \in (0,S]$.

Now we argue that the leaves are weakly stable. In fact, since $m > 0$, for every $s \in [0,S]$ the Jacobi operator L_s of $\Sigma_s(g_m)$ in (M,g_m) satisfies

$$-\int_{\Sigma_s(g_m)} L_s(\phi)\phi d\Sigma_s(g_m) = \int_{S^2} |\nabla_0\phi|^2 - \left(2 - \frac{6m}{r(s)}\right)\phi^2 dS^2 \geq \frac{6m}{r(S)}\int_{S^2}\phi^2 dS^2$$

for every $\phi \in C^\infty(\Sigma_s(g_m))$ with $\int \phi d\Sigma_s(g_m) = 0$. Arguing by contradiction we then conclude that for a possibly smaller ϵ there exists a constant $c > 0$ such that for every $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ and every $s \in [0,S]$ the Jacobi operator $L_{(s,g)}$ of the surface $\Sigma_s(g)$ in (M,g) is such that

$$-\int_{\Sigma_s(g)} L_{(s,g)}(\phi)\phi d\Sigma_s(g) \geq c\int_{\Sigma_s(g)}\phi^2 d\Sigma_s(g) \geq 0$$

for every $\phi \in C^\infty(\Sigma_s(g))$ with $\int \phi d\Sigma_s(g) = 0$, i.e., $\Sigma_s(g)$ is weakly stable.

The last statement of the theorem also follows by continuity, since $H_m'(s) < 0$ on $(s(3m), +\infty)$. □

Having in mind the gluing argument (see Section 2.6), we finish this section showing the existence and uniqueness of small graphs over coordinate spheres S_s with prescribed mean curvature $H_m(s)$ in (M,g), $g \in \mathcal{M}(M,m)$, at least when $d(g,g_m)$ is small enough and s is large enough. More precisely, we have:

Theorem 3. *Let $m > 0$. Given a compact interval $I \subset (s(3m), +\infty)$, there exists $\epsilon > 0$ and $\eta > 0$ with the following properties:*

For every $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ and for every $s \in I$ there exists a unique function $h(s,g) \in C^{2,\alpha}(S^2)$ with $\|h(s,g)\|_{C^{2,\alpha}} < \eta$ such that the surface

$$S_s(h(s,g)) = \{(s + h(s,g)(x), x) \in M; x \in S^2\}$$

has constant mean curvature $H_m(s)$ in (M,g).

Moreover, the family $\{S_s(h(s,g))\}_{s\in I}$ gives a foliation of a compact region of (M,g) by weakly stable CMC spheres.

Proof. Following the notations of Theorem 2, given $\eta > 0$ sufficiently small we consider the map $\Phi : I \times \mathcal{M}(M,m) \times (B(0,\eta) \subset C^{2,\alpha}(S^2)) \to C^{0,\alpha}(S^2)$ given by $\Phi(s,g,u) = H(s,u,g)$. Notice that $\Phi(s,g_m,0) = H_m(s)$ for all $s \in I$.

Given $s \in I$ and $g \in \mathcal{M}(M,m)$, we want to solve the equation $\Phi(s,g,u) = H_m(s)$ for some small $u \in C^{2,\alpha}(S^2)$. The linearization of Φ at $(s,g_m,0)$ is such that, for every $v \in C^{2,\alpha}(S^2)$,

$$D\Phi_{(s,g_m,0)}(0,0,v) = -L_s(v) = -\frac{1}{r^2}(\Delta_0 + (2 - \frac{6m}{r})),$$

where we use the coordinate $r = r(s)$, see Proposition 1. Since $I \subset (s(3m), +\infty)$, the map $v \in C^{2,\alpha}(S^2) \mapsto L_s(v) \in C^{0,\alpha}(S^2)$ is an isomorphism for all $s \in I$. Hence, we can apply the implicit function theorem. The last statement follows by the same arguments of Theorem 2. □

2.4 CMC foliation near the infinity

The next theorem is the version of the existence and uniqueness theorem of A. Neves and G. Tian [28] adapted to the asymptotically hyperbolic manifolds (M,g) where g belongs to the space of metrics $\mathcal{M}(M,m)$ (we refer the reader to Theorem 2.2 and the proof of Theorem 8.2 in [28]).

Theorem 4 (A. Neves and G. Tian)**.** *Let $m > 0$. Given $\epsilon_0 > 0$, there exists $\delta > 0$, $C > 0$ and $\underline{s}_0 > s(3m)$ with the following properties:*

1) *Given $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon_0$, for all $l \in (2, 2+\delta)$, there exists a unique sphere $\Sigma_l = \Sigma_l(g) \subset M$ such that*

a) $M \setminus \Sigma_l$ has two connected components, one of them containing ∂M;

b) Σ_l is a weakly stable constant mean curvature sphere in (M, g) with mean curvature $H = l$; and

c) The inner radius $\underline{s}_l$ and the outer radius $\overline{s}_l$ of Σ_l satisfy

$$\underline{s}_l \geq \underline{s}_0 \quad \text{and} \quad \overline{s}_l - \underline{s}_l \leq 1.$$

2) The family $\{\Sigma_l\}_{l\in(2,2+\delta)}$ gives a smooth foliation of the complement of a compact set of M and $\lim_{l\to 2} \underline{s}_l = +\infty$.

3) Given $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon_0$ and $l \in (2, 2 + \delta)$, if for the above surface Σ_l in (M, g) we define $\hat{s}_l$ by the equality

$$|\Sigma_l| = 4\pi \sinh^2 \hat{s}_l,$$

then:

a) If we set $w_l(p) = s(p) - \hat{s}_l$ for $p \in \Sigma_l$, then

$$\sup_{\Sigma_l} |w_l| \leq C \exp(-\underline{s}_l) \quad \text{and} \quad \int_{\Sigma_l} |\partial_s^\top|^2 d\Sigma_l \leq C \exp(-2\underline{s}_l).$$

b)

$$\int_{\Sigma_l} |\mathring{B}_l|^2 d\Sigma_l \leq C \exp(-4\underline{s}_l).$$

c) There exists a function $f \in C^2(S^2)$ with $\|f\|_{C^2} \leq C$ such that

$$\Sigma_l = S_{\hat{s}_l}(f) = \{(\hat{s}_l + f(x), x) \in M; x \in S^2\}.$$

Remark. Theorem 4 is proven by the continuity method. Two points are important for the gluing argument. First, for metrics $g \in \mathcal{M}(M, m)$, the *a priori* estimates are *uniform* in $d(g, g_m)$. Second, *for a fixed compact interval $I \subset (\underline{s}_0, +\infty)$, there exists $\epsilon \in (0, \epsilon_0)$ and $\eta > 0$ with the following property: for every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ and for every $s \in I$, there exists a unique function $h \in C^{2,\alpha}(S^2)$ with $\|h\|_{C^{2,\alpha}} < \eta$ such that the surface Σ_l in (M, g) with $l = H_m(s)$ given by Theorem 4 can be written as $\Sigma_l = S_s(h) = \{(s + h(x), x) \in M; x \in S^2\}$.* In other words, for a given compact interval I contained in $(s(3m), +\infty)$, all the surfaces of the foliation $\{\Sigma_l\}$ of (M, g) with mean curvature $l = H_m(s)$ for $s \in I$ are obtained by using the implicit function theorem as in Theorem 3, at least for $g \in \mathcal{M}(M, m)$ sufficiently close to g_m.

2.5 Limit of the Hawking mass

Let g be a metric in $\mathcal{M}(M,m)$ with scalar curvature $R \geq -6$. Recall that the Hawking mass of a closed surface Σ in (M,g) is

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma}(H^2-4)d\Sigma\right).$$

We want to calculate the limit of the Hawking mass of the weakly stable CMC spheres Σ_l in (M,g) given by Theorem 4 as they approach the infinity. In order to do this, we need the following consequence of Gauss equation (0.1), see [27].

Lemma 5. *Given $g \in \mathcal{M}(M,m)$, let $\{\Sigma_t\}_{t>t_0}$ be a family of constant mean curvature spheres in (M,g) such that $\underline{s}_t \to +\infty$ as t goes to infinity. Then*

$$(H_t^2-4)|\Sigma_t| = 16\pi - \int_{\Sigma_t}\frac{8m-12m|\partial_s^\top|^2}{\sinh^3 s}d\Sigma_t + 2\int_{\Sigma_t}|\mathring{B}_t|d\Sigma_t + |\Sigma_t|O(\exp(-4\underline{s}_t)).$$

Proof. The Gauss equation (0.1) for Σ_t in (M,g) can be written as

$$2K_t = (R+6) - 2(Ric(N_t,N_t)+2) + \frac{H_t^2-4}{2} - |\mathring{B}_t|^2.$$

By Proposition 1, for metrics $g \in \mathcal{M}(M,m)$, if $\{\partial_s, e_1, e_2\}$ is a g_m-orthonormal referential, then

$$\begin{aligned} Ric(\partial_s,\partial_s) &= -2 - \frac{2m}{\sinh^3 s} + O(\exp(-4s)),\\ Ric(e_i,e_j) &= (-2 + \frac{m}{\sinh^3 s})\delta_{ij} + O(\exp(-4s)),\\ Ric(\partial_s,e_i) &= O(\exp(-4s)), \quad \text{and}\\ R+6 &= O(\exp(-4s)). \end{aligned}$$

Considering the g_m-orthogonal decomposition $N_t = a\partial_s + X$, it follows that

$$4K_t = (H_t^2-4) + \left(\frac{8m-12m|X|^2_{g_m}}{\sinh^3 s}\right) - 2|\mathring{B}_t|^2 + O(\exp(-4s)). \qquad (2.3)$$

Observe that if ν is the unit normal of a coordinate sphere in (M,g), then $\partial_s = \nu + W$, where $g(\nu,W) = O(\exp(-4s))$ and $|W|_g = O(\exp(-4s))$. Hence, the g-orthogonal decomposition $N_t = b\nu + Y$ is such that $|X|^2_{g_m} = |Y|^2_g + O(\exp(-4s))$. On the other hand, if $\nu = bN_t + \nu^\top$ is the g-orthonormal decomposition of ν corresponding to the tangent space of Σ_t and its g-normal N_t, we have $|Y|^2_g = |\nu^\top|^2_g$. Therefore one can change $|X|^2_{g_m}$ by $|\partial_s^\top|^2_g$ in (2.3).

Since Σ_t is a sphere of constant mean curvature, the lemma follows after integration of (2.3). □

Proposition 6. *Let $m > 0$. Given $g \in \mathcal{M}(M,m)$ a metric with scalar curvature $R \geq -6$, the family $\{\Sigma_l\}_{l\in(2,2+\delta)}$ in (M,g) given by Theorem 4 is such that*

$$\lim_{\underline{s}_l\to+\infty} m_H(\Sigma_l) = \lim_{\underline{s}_l\to+\infty} \sqrt{\frac{|\Sigma_l|}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\Sigma_l}(H_l^2-4)d\Sigma_l\right) = m.$$

Proof. Let $\hat{s}_l$ be defined by $|\Sigma_l| = 4\pi\sinh^2\hat{s}_l$. We use the informations given by Theorem 4, item 3), and calculate all expansions as $\underline{s}_l$ goes to infinity.

Lemma 5 and the estimates of Theorem 4, item 3) on the behavior of $|\partial_s^\top|$ and $|\mathring{B}_l|$ implies

$$m_H(\Sigma_l) = \frac{|\Sigma_l|^{1/2}}{8\pi^{3/2}}\left(\int_{\Sigma_l}\frac{m}{\sinh^3 s}d\Sigma_l + |\Sigma_t|O(\exp(-4\underline{s}_l)) + O(\exp(-4\underline{s}_l))\right). \tag{2.4}$$

In order to analyze (2.4), observe first that $|\underline{s}_l - \hat{s}_l| = |\min_{x\in S^2}\{f(x)\}| \leq C$ for all l. Hence

$$|\Sigma_l| = 4\pi\sinh^2\hat{s}_l = O(\exp(2\underline{s}_l)). \tag{2.5}$$

On the other hand, for every $p=(s,x)\in\Sigma_l$, since $|w_l(p)| \leq C\exp(-\underline{s}_l)$,

$$\frac{\sinh\hat{s}_l}{\sinh s} = \frac{\sinh(s-w_l(p))}{\sinh s} = \cosh w_l(p) - \frac{\cosh s}{\sinh s}\sinh w_l(p) = 1 + O(\exp(-\underline{s}_l)).$$

Therefore

$$\int_{\Sigma_l}\frac{|\Sigma_l|^{1/2}}{\sinh^3(s)}d\Sigma_l = \frac{(4\pi)^{3/2}}{|\Sigma_l|}\int_{\Sigma_l}\left(\frac{\sinh\hat{s}}{\sinh s}\right)^3 d\Sigma_l = 8\pi^{3/2}(1+O(\exp(-\underline{s}_l))). \tag{2.6}$$

Combining (2.4), (2.5) and (2.6) we have $\lim_{\underline{s}_l\to+\infty} m_H(\Sigma_l) = m$. □

2.6 Gluing argument and properties of the global foliation

We now argue that when a metric $g \in \mathcal{M}(M,m)$ is sufficiently close to g_m it is possible to glue together the foliations of (M,g) obtained in Theorems 2 and 4.

Theorem 7. *Let $m > 0$. There exists $\epsilon > 0$ with the following property:*

If $g \in \mathcal{M}(M,m)$ is such that $d(g,g_m) < \epsilon$, then there exists a foliation $\{\Sigma_t\}_{t\in[0,+\infty)}$ of M such that:

i) Each Σ_t is a weakly stable CMC sphere in (M,g), with positive mean curvature when $t > 0$; and *ii)* $\Sigma_0 = \partial M$ *is an outermost minimal surface in* (M, g).

Moreover, if g has scalar curvature greater than or equal to -6*, then*

iii) $\lim_{t\to+\infty} m_H(\Sigma_t) = \lim_{t\to+\infty} \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t}(H_t^2 - 4)d\Sigma_t\right) = m.$

Proof. Given an arbitrary $\epsilon_0 > 0$, let $\delta > 0$, $C > 0$ and $\underline{s}_0 > s(3m)$ be given by Theorem 4. Recall that the function $H_m(s)$ is monotone decreasing on the interval $(s(3m), +\infty)$ and converges to 2 as s goes to infinity. Let $s_0 > \underline{s}_0$ be such that $H_m(s_0) < 2 + \delta$.

Let $S > s_0 + 1$. Given this choice of S, we can choose $\epsilon \in (0, \epsilon_0)$ and $\eta < 1$ sufficiently small in such way that Theorem 2 holds for S and Theorem 3 holds for the interval $[S-1, S]$ for every metric $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$.

In particular, we can assume that for every metric $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ and for every $s \in [S-1, S]$ the surface $\Sigma_l = \Sigma_l(g)$ in (M, g) described in Theorem 4 with $l = H_m(s)$ is given by

$$\Sigma_l = S_s(h),$$

where h is the unique function in $C^{2,\alpha}(S^2)$ with norm $< \eta$ such that its graph over S_s has constant mean curvature $H_m(s)$, see the remark after Theorem 4.

Given $\kappa \in (0, \eta)$, let $[c, d]$ be the image of the interval $[S - 3\kappa/4, S - \kappa/4]$ under the map $H_m(s)$. We can moreover assume that $\epsilon \in (0, \epsilon_0)$ is sufficiently small in such way that for every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ the foliation $\{\Sigma_s^1(g)\}_{s\in[0,S]}$ constructed in Theorem 2 has in particular the following properties:

a) For every $s \in [S-1, S]$, there exists a function $u(s, g) \in C^{2,\alpha}(S^2)$ with $\|u(s, g)\|_{C^{2,\alpha}} < \eta/2$ such that

$$\Sigma_s^1(g) = S_s(u(s, g)).$$

b) The mean curvature $H_g(s)$ of $\Sigma_s^1(g)$ in (M, g) is a decreasing function on the interval $[S-1, S]$ with

$$|H_g(s) - H_m(s)| < (d - c)/4$$

for all $s \in [S-1, S]$. Hence, there exists an interval $(a, b) \subset [S - 3\kappa/4, S - \kappa/4]$ such that for every $s \in (a, b)$ there exists a unique $\tilde{s} \in (S - 3\kappa/4, S - \kappa/4)$ with $H_g(s) = H_m(\tilde{s})$. This defines $\tilde{s}$ as a function of s.

We now prove that the theorem is true for this choice of ϵ.

In fact, fix some $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$. By item *a)* and *b)* above, given $s \in (a,b)$, if we define the function $\tilde{h} = s - \tilde{s}(s) + u(s,g) \in C^{2,\alpha}(S^2)$, then

$$\|\tilde{h}\|_{C^{2,\alpha}} \le |s - \tilde{s}(s)| + \|u\|_{C^{2,\alpha}} < \kappa/2 + \eta/2 < \eta$$

and the graph

$$S_{\tilde{s}(s)}(\tilde{h}) = S_s(u(s,g)) = \Sigma^1_s(g)$$

has constant mean curvature $H_g(s) = H_m(\tilde{s}(s))$. These are the conditions that uniquely characterize the function that gives Σ_l in (M,g) with $l = H_m(\tilde{s}(s))$ as a graph over $S_{\tilde{s}(s)}$. Therefore $\Sigma^1_s(g) = \Sigma_l(g)$ where $l = H_m(\tilde{s}(s))$ for all $s \in (a,b)$.

This proves that the foliations $\{\Sigma^1_s(g)\}$ and $\{\Sigma_l(g)\}$ given by Theorems 2 and 4, respectively, glue together. The obtained foliation, $\{\Sigma_t\}_{t\in[0,+\infty)}$, is a foliation of (M,g) by weakly stable CMC spheres, starting at the minimal $\Sigma_0 = \partial M$, such that each Σ_t has positive mean curvature for $t > 0$, and such that $\lim_{t\to+\infty} m_H(\Sigma_t) = m$ when g has scalar curvature $R \ge -6$ (see Theorem 2, Theorem 4 and Proposition 6). It remains only to prove that ∂M is an outermost minimal surface in (M,g). This is a consequence of the Maximum Principle, since we showed that $M \setminus \partial M$ is foliated by surfaces with positive mean curvature. □

2.7 The Penrose inequality

Using the foliation by weakly stable CMC spheres constructed above on (M,g), $g \in \mathcal{M}(M,m)$ sufficiently close to g_m, and the remark of H. Bray that the Hawking mass is monotone non-decreasing in such families (see [3]), we prove the Penrose inequality for this class of asymptotically hyperbolic manifolds.

Theorem 8. *Given $m > 0$, let $\epsilon > 0$ be given by Theorem 7. If $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ has scalar curvature $R \ge -6$, then*

$$\left(\frac{|\partial M|}{16\pi}\right)^{\frac{1}{2}} + 4\left(\frac{|\partial M|}{16\pi}\right)^{\frac{3}{2}} \le m. \tag{2.7}$$

Moreover, equality holds if and only if (M,g) is isometric to (M,g_m).

Proof. Let $\{\Sigma_t\}_{t\ge 0}$ be the foliation of (M,g) by weakly stable CMC spheres constructed in Theorem 7. We assume g has scalar curvature $R \ge -6$, so that $\{\Sigma_t\}$ has all the properties *i)*, *ii)* and *iii)* described there.

We claim that the Hawking mass of Σ_t,

$$m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t}(H_t^2 - 4)d\Sigma_t\right),$$

is monotone non-decreasing in t. In fact, we can be more precise:

Claim: $m_H^{'}(\Sigma_t) \geq 0$. Moreover, $m_H^{'}(\Sigma_t)$ is zero at $t > 0$ if and only if Σ_t satisfies the following properties:

a) R is constant and equal to -6 along Σ_t;

b) Σ_t is totally umbilic; and

c) Σ_t has constant Gaussian curvature.

The proof of the claim goes as follows. Choose a parametrization of this foliation by some function $G : [0, +\infty) \times S^2 \rightarrow M$ such that for each $t \in [0, +\infty)$, $G_t : S^2 \rightarrow M$ is a parametrization of Σ_t and $\partial_t G$ does not vanish. Let ρ_t be the lapse function, i.e., $\rho_t = g(N_t, \partial_t G)$ where N_t is the normal pointing toward infinity. Since we have a foliation, $\rho_t > 0$ on Σ_t for all t. We also define the mean value $\overline{\rho}_t = \int \rho_t d\Sigma_t / |\Sigma_t|$.

By the first variation formula of the Hawking mass (see the Appendix), we have

$$(16\pi)^{\frac{3}{2}} m_H^{'}(\Sigma_t) = 2|\Sigma_t|^{\frac{1}{2}} \int_{\Sigma_t} (\Delta_t H_t + Q_t H_t)\rho_t d\Sigma_t,$$

where

$$Q_t = \frac{1}{2}(R+6) + \left(\frac{4\pi}{|\Sigma_t|} - K_t\right) + \frac{1}{2}\left(|B_t|^2 - \frac{1}{2|\Sigma_t|}\int_{\Sigma_t} H_t^2 d\Sigma_t\right).$$

Since H_t is constant for each t,

$$\begin{aligned}
(16\pi)^{\frac{3}{2}} m_H^{'}(\Sigma_t) &= 2|\Sigma_t|^{\frac{1}{2}} H_t \int_{\Sigma_t} Q_t \rho_t d\Sigma_t \\
&= 2|\Sigma_t|^{\frac{1}{2}} H_t \int_{\Sigma_t} (\Delta_t + Q_t)(\rho_t - \overline{\rho}_t) d\Sigma_t + 2|\Sigma_t|^{\frac{1}{2}} H_t \overline{\rho}_t \int_{\Sigma_t} Q_t d\Sigma_t \\
&= 2|\Sigma_t|^{\frac{1}{2}} H_t \int_{\Sigma_t} L_t(\rho_t - \overline{\rho}_t) d\Sigma_t + 2|\Sigma_t|^{\frac{1}{2}} H_t \overline{\rho}_t \int_{\Sigma_t} Q_t d\Sigma_t.
\end{aligned}$$

In the last line, we used the Gauss equation (0.1) to see that $\Delta_t + Q_t$ and the Jacobi operator of Σ_t differ by a constant.

Since $\partial M = \Sigma_0$ is minimal, the derivative of $m_H(\Sigma_t)$ is zero at $t = 0$. When $t > 0$, Σ_t has positive mean curvature. Observe also that $\int_{\Sigma_t} Q_t d\Sigma_t \geq 0$, by Gauss-Bonnet Theorem and since $R \geq -6$. Therefore

$$(16\pi)^{\frac{3}{2}} m_H^{'}(\Sigma_t) \geq 2|\Sigma_t|^{\frac{1}{2}} H_t \int_{\Sigma_t} L_t(\rho_t - \overline{\rho}_t) d\Sigma_t.$$

Now we use the weak stability of Σ_t. Since H_t is constant for each t, $L_t(\rho_t) = -\partial_t H_t$ is also constant on Σ_t. Hence the stability inequality gives

$$0 \leq -\int_{\Sigma_t} L_t(\rho_t - \overline{\rho}_t)(\rho_t - \overline{\rho}_t) d\Sigma_t = \int_{\Sigma_t} L_t(\overline{\rho}_t)(\rho_t - \overline{\rho}_t) d\Sigma_t = \overline{\rho_t} \int_{\Sigma_t} L_t(\rho_t - \overline{\rho}_t) d\Sigma_t.$$

This implies that $m_H^{'}(\Sigma_t)$ is nonnegative.
If $m_H^{'}(\Sigma_t) = 0$ at $t > 0$, then

$$\int_{\Sigma_t} Q_t d\Sigma_t = 0 \quad \text{and} \quad \int_{\Sigma_t} L_t(\rho_t - \overline{\rho}_t)(\rho_t - \overline{\rho}_t) d\Sigma_t = 0.$$

The first equality implies that Σ_t satisfies a) and b). By the weak stability, the second equality implies that $L_t(\rho_t - \overline{\rho}_t)$ is constant, for it must be orthogonal to every function on Σ_t with zero mean value. Since $L_t(\rho_t)$ is constant, this implies that $L_t(\overline{\rho}_t)$ is also constant. Then *c*) follows from *a*), *b*) and the Gauss equation (0.1).

Once we proved the claim, inequality (2.7) follows immediately:

$$\left(\frac{|\partial M|}{16\pi}\right)^{\frac{1}{2}} + 4\left(\frac{|\partial M|}{16\pi}\right)^{\frac{3}{2}} = m_H(\Sigma_0) \leq \lim_{t\to+\infty} m_H(\Sigma_t) = m.$$

Now we analyze the equality. In this case $m_H(\Sigma_t)$ must be constant and equal to m. By the second part of the claim, we conclude that each Σ_t satisfies *a*), *b*) and *c*) for all $t > 0$. Hence, possibly after a change of the parametrization $G : [0, +\infty) \times S^2 \to M$, G^*g is a metric on $M = [0, +\infty) \times S^2$ that can be written in the form $ds^2 + \xi^2(s)g_0$, has constant scalar curvature $R = -6$, and is such that all slices $\{s\} \times S^2$ have Hawking mass m. These conditions uniquely characterize the metric g_m (see the Appendix). This finishes the proof. □

Another Penrose inequality. In Proposition 1, we saw that the mean curvature of the coordinates spheres in the Anti-de Sitter-Schwarzschild spaces of mass $m > 0$ is given by the function $H_m(r) = (2/r)\sqrt{1 + r^2 - 2m/r}$. Observe that $H_m(2m) = 2$ and that $H_m(r) > 2$ for all $r > 2m$. In particular, the Maximum Principle implies that there are no other closed surfaces with constant mean curvature 2 in $([2m, +\infty) \times S^2, g_m)$.

Let (M, g) be an asymptotically hyperbolic three-manifold with connected boundary ∂M. Assume that (M, g) has scalar curvature $R \geq -6$ and that ∂M is an outermost $H = 2$ surface, meaning that are no closed surfaces in M with constant mean curvature $H = 2$ other than ∂M. In this setting, the Penrose Conjecture is that the area of ∂M and the total mass m of (M, g) are related by the inequality

$$\left(\frac{|\partial M|}{16\pi}\right)^{\frac{1}{2}} \leq m,$$

and that equality holds if and only if (M, g) is isometric to the piece of the Anti-de Sitter-Schwarzschild space of mass m outside the domain bounded by the coordinate sphere of mean curvature 2.

Given $m > 0$, we analogously define the space $\mathcal{M}(M_2, m)$ of metrics g on $M_2 := [s(2m), +\infty) \times S^2$ such that ∂M_2 has constant mean curvature 2 in (M_2, g) and $d(g, g_m) < +\infty$. The analogous versions of Theorem 7 and Theorem 8 follows immediately by the same arguments.

Theorem 9. *Let $m > 0$. There exists $\epsilon > 0$ with the following property:*

If $g \in \mathcal{M}(M_2, m)$ is such that $d(g, g_m) < \epsilon$, then there exists a foliation $\{\Sigma_t\}_{t\in[0,+\infty)}$ of M_2 such that:

i) Each Σ_t is a weakly stable CMC sphere in (M_2, g), with mean curvature $H_t > 2$ when $t > 0$; and

ii) $\Sigma_0 = \partial M_2$ is an outermost $H = 2$ surface in (M_2, g).

Moreover, if g has scalar curvature $R \geq -6$, then

iii) $\lim_{t\to+\infty} m_H(\Sigma_t) = m$.

Theorem 10. *Given $m > 0$, let $\epsilon > 0$ be given by Theorem 9. If $g \in \mathcal{M}(M_2, m)$ with $d(g, g_m) < \epsilon$ has scalar curvature $R \geq -6$, then*

$$\left(\frac{|\partial M_2|}{16\pi}\right)^{\frac{1}{2}} \leq m.$$

Moreover, equality holds if and only if (M_2, g) is isometric to (M_2, g_m).

Appendix

For completeness, we present in this Appendix the relevant calculations. In particular, we prove the variation formulas contained in the basic material chapter, we deduce the first variation formula for the Hawking mass and we describe all warped product metrics on $I \times S^2$ of the form $dt^2 + \xi^2(t)g_0$, where (S^2, g_0) is the standard round sphere of constant curvature 1, that have constant scalar curvature -6.

Variation formulas: We work in the general setting. Let (M^{n+1}, g) be a Riemannian manifold with boundary ∂M and let X denote the unit normal vector field of ∂M that points outside M. Let Σ^n be a manifold with boundary $\partial\Sigma$ and assume Σ is immersed in M in such way that $\partial\Sigma$ is contained in ∂M. The outward pointing unit co-normal of $\partial\Sigma$ in Σ is denoted by ν. Let N be a local unit vector field normal to Σ, let B denote its second fundamental form defined by $B(Y,W) = g(\nabla_Y N, W)$ for every pair of vectors Y, W tangent to Σ and let $H = trB$ denote its mean curvature.

We consider variations of Σ given by smooth maps $f : \Sigma \times (-\epsilon, \epsilon) \to M$ with $\Sigma = f_0(\Sigma)$ such that $f_t : x \in \Sigma \mapsto f(x,t) \in M$ is an immersion of Σ in M and $f_t(\partial\Sigma)$ is contained in ∂M for every $t \in (-\epsilon, \epsilon)$.

The subscript t will be used to denote quantities associated to $\Sigma_t = f_t(\Sigma)$. For example, N_t will denote a local unit vector field normal to Σ_t and H_t will denote the mean curvature of Σ_t.

It will be useful for the computations to introduce local coordinates $x^1, \ldots, x^n$ in Σ. We will also use the simplified notation

$$\partial_t = \frac{\partial f}{\partial t} \quad \text{and} \quad \partial_i = \frac{\partial f}{\partial x_i},$$

where i runs from 1 to n. Regarding the indexes, we use the usual summation and notational conventions. For example, the mean curvature of Σ is given by $H = g^{ij}B_{ij}$, where $B_{ij} = B(\partial_i, \partial_j)$ for every $1 \leq i, j, \leq n$.

We decompose the variational vector field $V = \partial_t$ in its tangent and normal components along Σ_t,

$$\partial_t = \partial_t^\top + \rho_t N_t,$$

where ρ_t is the lapse function on Σ_t defined by $\rho_t = g(\partial_t, N_t)$.

Proposition 1.

$$\begin{aligned}\partial_t g_{ij} &= g(\nabla_{\partial_i}\partial_t, \partial_j) + g(\partial_i, \nabla_{\partial_j}\partial_t),\\ \partial_t g^{ij} &= -2g^{ik}g^{jl}g(\nabla_{\partial_k}\partial_t, \partial_l),\\ \partial_t \det[g_{ij}] &= (g^{ij}\partial_t g_{ij})\det[g_{ij}].\end{aligned}$$

Proof. The first equation is straightforward. The second follows from differentiating $g^{ik}g_{kl} = \delta^i_l$. The last equation is a consequence of the general formula for the derivative of the determinant at an invertible matrix U, $D(\det)(U)W = tr(U^{-1}W)\det(U)$ for all matrices W. □

Proposition 2.

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} H_t\rho_t dA_t + \int_{\partial\Sigma_t} g(\nu_t, \partial_t)dL_t.$$

Proof. In local coordinates $x^1, \ldots, x^n$, the area element of Σ is given by $dA_t = \sqrt{\det[g_{ij}]}dx^1...dx^n$. By Proposition 1,

$$\begin{aligned}\partial_t\sqrt{\det[g_{ij}]} &= \frac{1}{2}(g^{ij}\partial_t g_{ij})\sqrt{\det[g_{ij}]}\\ &= g^{ij}g(\nabla_{\partial_i}\partial_t, \partial_j)\sqrt{\det[g_{ij}]}\\ &= (g^{ij}g(\nabla_{\partial_i}\partial_t^\top, \partial_j) + g^{ij}g(\nabla_{\partial_i}N_t, \partial_j)\rho_t)\sqrt{\det[g_{ij}]}\\ &= (\mathrm{div}_{\Sigma_t}\partial_t^\top + H_t\rho_t)\sqrt{\det[g_{ij}]}.\end{aligned}$$

The first variation formula of area follows then by the divergence theorem. □

Proposition 3.

$$\begin{aligned}\nabla_{\partial_i}N_t &= g^{kl}B_{il}\partial_k,\\ \nabla_{\partial_t}N_t &= \nabla_{\partial_t^\top}N_t - \nabla^{\Sigma_t}\rho_t.\end{aligned}$$

where $\nabla^\Sigma\rho_t$ is the gradient of the function ρ_t on Σ_t.

Proof. Since $g(N_t, N_t) = 1$, $\nabla_{\partial_i} N_t$ and $\nabla_{\partial_t} N_t$ are tangent to Σ_t. The first equation is just the expression of $\nabla_{\partial_i} N_t$ in the basis $\{\partial_k\}$. On the other hand, since $g(N_t, \partial_i) = 0$, we have

$$\nabla_{\partial_t} N_t = g^{ik} g(\nabla_{\partial_t} N_t, \partial_k)\partial_i = -g^{ik} g(N_t, \nabla_{\partial_t}\partial_k)\partial_i = -g^{ik} g(N_t, \nabla_{\partial_k}\partial_t)\partial_i.$$

In local coordinates, the gradient of ρ_t in Σ_t is given by $\nabla^{\Sigma_t}\rho_t = (g^{ij}\partial_j\rho_t)\partial_i$. Therefore

$$\begin{aligned}\nabla_{\partial_t} N_t &= -g^{ik} g(N_t, \nabla_{\partial_k}\partial_t^\top)\partial_i - g^{ik} g(N_t, \nabla_{\partial_k}(\rho_t N_t))\partial_i \\ &= g^{ik} B_t(\partial_k, \partial_t^\top) - (g^{ik}\partial_k\rho_t)\partial_i \\ &= \nabla_{\partial_t^\top} N_t - \nabla^{\Sigma_t}\rho_t\end{aligned}$$

□

Proposition 4.

$$\partial_t H_t = dH_t(\partial_t^\top) - L_{\Sigma_t}\rho_t.$$

where $L_{\Sigma_t} = \Delta_{\Sigma_t} + Ric(N_t, N_t) + |B_t|^2$ *is the Jacobi operator of* Σ_t.

Proof. Since $H_t = g^{ij} g(\nabla_{\partial_i} N_t, \partial_j)$,

$$\begin{aligned}\partial_t H_t &= \partial_t g^{ij} g(\nabla_{\partial_i} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_t}\nabla_{\partial_i} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_t}\partial_j) \\ &= -2g^{ik} g^{jl} g(\nabla_{\partial_k}\partial_t, \partial_l) g(\nabla_{\partial_i} N_t, \partial_j) + g^{ij} g(R(\partial_t, \partial_i)N_t, \partial_j) \\ &\quad + g^{ij} g(\nabla_{\partial_i}\nabla_{\partial_t} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j}\partial_t) \\ &= -2g^{ik} g(\nabla_{\partial_k}\partial_t, \nabla_{\partial_i} N_t) - Ric(\partial_t, N_t) \\ &\quad + g^{ij} g(\nabla_{\partial_i}\nabla_{\partial_t} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j}\partial_t) \\ &= -g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j}\partial_t) - Ric(\partial_t, N_t) \\ &\quad + g^{ij} g(\nabla_{\partial_i}(\nabla_{\partial_t^\top} N_t), \partial_j) - g^{ij} g(\nabla_{\partial_i}(\nabla^{\Sigma_t}\rho_t), \partial_j).\end{aligned}$$

Now we use the contracted Codazzi equation (0.2):

$$\begin{aligned}Ric(\partial_t^T, N_t) &= g^{ij}(\nabla^{\Sigma_t}_{\partial_i} B)(\partial_t^\top, \partial_j) - dH(\partial_t^\top) \\ &= g^{ij}\partial_i g(\nabla_{\partial_t^\top} N_t, \partial_j) - g^{ij} g(\nabla_{(\nabla_{\partial_i}\partial_t^\top)^\top} N_t, \partial_j) \\ &\quad - g^{ij} g(\nabla_{\partial_t^\top} N_t, (\nabla_{\partial_i}\partial_j)^\top) - dH(\partial_t^\top) \\ &= g^{ij}(\partial_i g(\nabla_{\partial_t^\top} N_t, \partial_j) - g(\nabla_{\partial_t^\top} N_t, \nabla_{\partial_i}\partial_j)) \\ &\quad - g^{ij} g(\nabla_{\partial_j} N_t, (\nabla_{\partial_i}\partial_t^\top)^\top) - dH(\partial_t^\top) \\ &= g^{ij} g(\nabla_{\partial_i}(\nabla_{\partial_t^\top} N_t), \partial_j) - g^{ij} g(\nabla_{\partial_j} N_t, \nabla_{\partial_i}\partial_t^\top) - dH(\partial_t^\top).\end{aligned}$$

Canceling out the corresponding terms, we have

$$\begin{aligned}\partial_t H_t = &- g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} N_t)\rho_t - Ric(N_t, N_t)\rho_t \\ &+ dH(\partial_t^\top) - g^{ij} g(\nabla_{\partial_i}(\nabla^{\Sigma_t}\rho_t), \partial_j).\end{aligned}$$

The proposition follows. □

Proposition 5. *Let Σ be a free boundary minimal surface. For all variations of Σ with variational vector field $\partial_{t|t=0} = \rho N$ along Σ,*

$$\frac{d^2}{dt^2}_{|t=0}|\Sigma_t| = -\int_\Sigma L_\Sigma(\rho)\rho dA + \int_{\partial\Sigma}\left(\frac{\partial\rho}{\partial\nu} - B^{\partial M}(N,N)\rho\right)\rho dL.$$

Proof. By assumption, $H_t = 0$ and $g(\nu_t, \partial_t) = 0$ at $t = 0$. From Proposition 4, if we differentiate the first variation formula of area at $t = 0$ for a variation with $\partial_t^\top = 0$ along $\Sigma_0 = \Sigma$ we get

$$\frac{d^2}{dt^2}_{|t=0}|\Sigma_t| = -\int_\Sigma L_\Sigma(\rho)\rho dA + \int_{\partial\Sigma}(\partial_t g(\nu_t, \partial_t))_{|t=0}\, dL.$$

To analyze the boundary integral, observe that

$$g(\nu_t, N_t) = 0 \quad \Rightarrow \quad g(\nabla_{\partial_t}\nu_t, N_t) = -g(\nu_t, \nabla_{\partial_t}N_t).$$

Since $X = \nu$ and $\partial_{t|t=0} = vN$ is tangent to ∂M along $\partial\Sigma$, from Proposition 3 we get

$$\begin{aligned}(\partial_t g(\nu_t, \partial_t))_{|t=0} &= g((\nabla_{\partial_t}\nu_t)_{|t=0}, N)\rho + g(X, (\nabla_{\partial_t}\partial_t)_{|t=0}) \\ &= g(\nu, \nabla^\Sigma\rho)\rho - B^{\partial M}(N,N)\rho^2.\end{aligned}$$

□

The Hawking mass: Let (M^3, g) be a Riemannian manifold with $\inf R > -\infty$. Given a closed surface Σ in M, we define its Hawking mass to be

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma (H^2 + \frac{2}{3}\inf R)d\Sigma\right).$$

Remark. Since the scalar curvature of asymptotically hyperbolic manifolds converges to -6 at infinity, in general one has $\inf R \leq -6$. Therefore, when such manifolds have scalar curvature greater than or equal to -6, the above formula coincides with the one used in Chapter 2.

Proposition 6. *Given a variation $\{\Sigma_t\}$ of Σ with variational vector field V on Σ,*

$$(16\pi)^{3/2}\frac{d}{dt}_{|t=0} m_H(\Sigma_t) = 2|\Sigma|^{1/2}\int_\Sigma (\Delta_\Sigma H + Q_\Sigma H) g(V, N) d\Sigma, \tag{2.8}$$

where

$$Q_\Sigma = \frac{1}{2}(R - \inf R) + \left(\frac{4\pi}{|\Sigma|} - K\right) + \frac{1}{2}\left(|B|^2 - \frac{1}{2|\Sigma|}\int_\Sigma H^2 d\Sigma\right).$$

Proof. Since the functional is invariant under reparametrizations of Σ we only need to consider variations with $V = \rho N$ on Σ. Using Propositions 1, 2 and 4 and Gauss equation (0.1), we calculate:

$$
\begin{aligned}
(16\pi)^{3/2}\frac{d}{dt}_{|t=0} m_H(\Sigma_t) = \\
&= \frac{1}{2|\Sigma|^{1/2}}\int_\Sigma H\rho d\Sigma\left(16\pi - \int_\Sigma\left(H^2+\frac{2}{3}\inf R\right)d\Sigma\right)\\
&+ |\Sigma|^{1/2}\left(\int_\Sigma 2HL_\Sigma(\rho)d\Sigma - \int_\Sigma\left(H^2+\frac{2}{3}\inf R\right)H\rho d\Sigma\right)\\
&= |\Sigma|^{1/2}\left(\int_\Sigma 2H\Delta_\Sigma\rho d\Sigma + \int_\Sigma\left(R+|B|+H^2-2K\right)H\rho d\Sigma\right)\\
&\quad - |\Sigma|^{1/2}\int_\Sigma\left(H^2+\frac{2}{3}\inf R\right)H\rho d\Sigma\\
&\quad + |\Sigma|^{1/2}\int_\Sigma\left(\frac{8\pi}{|\Sigma|}-\frac{1}{2|\Sigma|}\int_\Sigma H^2 d\Sigma - \frac{1}{3}\inf R\right)H\rho d\Sigma.
\end{aligned}
$$

After this point, formula (2.8) follows immediately. □

Using Gauss-Bonnet theorem, we conclude that $\int_\Sigma Q_\Sigma d\Sigma \geq 0$. Moreover, equality holds if and only if Σ is a topological sphere, is totally umbilic and R is constant and equal to $\inf R$ along Σ.

Spherically symmetric metrics of constant scalar curvature -6: Let (S^2, g_0) be the round sphere of constant curvature 1. Given an open interval $I \subset (0,+\infty)$, we want to consider metrics $\overline{g}$ on $I \times S^2$ that can be written as

$$\overline{g} = \rho^{-2}(r)dr^2 + r^2 g_0 \tag{2.9}$$

for some positive function $\rho : r \in I \mapsto \rho(r) \in (0,+\infty)$. After a coordinate change $t = t(r)$ we can write this metric as a warped product $dt^2 + \xi^2(t)g_0$, and reciprocally.

We calculate the scalar curvature of these metrics. This can be done with the aid of the Gauss equation (0.1).

The Gaussian curvature of S_r is given by r^{-2}, since $\overline{g}_{S_r}$ is a rescaling of the round metric g_0 by the factor r^2. By the spherical symmetry, every coordinate sphere S_r is umbilical and has constant mean curvature H_r. Since $|S_r| = 4\pi r^2$ and $N_r = \rho(r)\partial_r$, the first variation formula of area gives

$$H_r = \frac{2}{r}\rho(r).$$

Using these informations in Gauss equation (0.1) we get

$$R = 2\frac{(1-\rho^2(r))}{r^2} + 2Ric(\partial_r,\partial_r)\rho^2(r). \tag{2.10}$$

To calculate $Ric(\partial_r, \partial_r)$, we introduce local coordinates x^1, x^2 on S^2 and use the coordinates $x^0 = r, x^1, x^2$ on $I \times S^2$. Let us also make the useful convention that capital letters runs over the set $\{0, 1, 2\}$ and small letters runs over the set $\{1, 2\}$. For example, if we set $g_{ij} := g_0(\partial_i, \partial_j)$ we can write

$$\overline{g}_{0A} = \rho^{-2}\delta_{0A} \quad \text{and} \quad \overline{g}_{ij} = r^2 g_{ij}.$$

Now we calculate the Christoffel symbols of the metric $\overline{g}$,

$$\overline{\Gamma}_{AB}^{C} = \frac{1}{2}\overline{g}^{CD}(\partial_A \overline{g}_{BD} + \partial_B \overline{g}_{AD} - \partial_D \overline{g}_{AB}).$$

Observe that Γ_{AB}^{C} vanishes when exactly two indexes equal 0 and

$$\overline{\Gamma}_{00}^{0} = -\rho^{-1}\rho', \quad \overline{\Gamma}_{i0}^{k} = r^{-1}\delta_i^k, \quad \overline{\Gamma}_{ij}^{0} = -r\rho^2 g_{ij} \quad \text{and} \quad \overline{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k},$$

where Γ_{ij}^{k} are the Christoffel symbols of g_0.

The Ricci tensor is given in coordinates by

$$\overline{R}_{CB} = \partial_A \overline{\Gamma}_{BC}^{A} - \partial_B \overline{\Gamma}_{AC}^{A} + \overline{\Gamma}_{BC}^{P}\overline{\Gamma}_{AP}^{A} - \overline{\Gamma}_{AC}^{P}\overline{\Gamma}_{BP}^{A}.$$

In particular,

$$\begin{aligned}\overline{R}_{00} &= -\partial_0 \overline{\Gamma}_{i0}^{i} + \overline{\Gamma}_{00}^{0}\overline{\Gamma}_{i0}^{i} - \overline{\Gamma}_{i0}^{p}\overline{\Gamma}_{p0}^{i} \\ &= 2r^{-2} + (-\rho^{-1}\rho')(2r^{-1}) - (r^{-1}\delta_i^p)(r^{-1}\delta_p^i) \\ &= -2r^{-1}\rho^{-1}\rho'.\end{aligned}$$

Going back to equation (2.10) we deduce a formula for the scalar curvature R of $\overline{g}$ in terms of ρ,

$$\frac{R}{2} = \frac{1-\rho^2}{r^2} - \frac{2}{r}\rho\rho'. \tag{2.11}$$

The problem we want to solve is to find all metrics $\overline{g}$ as in (2.9) with constant scalar curvature -6. From equation (2.11), this is equivalent to the requirement that the function ρ must satisfy the first order ODE

$$2r\rho\rho' = 1 - \rho^2 + 3r^2. \tag{2.12}$$

We will be able to completely solve this ODE. In fact, consider the function

$$m(r) = \frac{r}{2}(1 - \rho^2 + r^2).$$

Observe that this function would give precisely the Hawking mass of the coordinate sphere S_r with respect to $\overline{g}$ if we knew this metric had constant scalar curvature equal to -6. Differentiating it,

$$2m'(r) = (1 - \rho^2 + r^2) + r(-2\rho\rho' + 2r) = (1 - \rho^2 + 3r^2) - 2r\rho\rho'$$

Therefore ρ satisfy equation (2.12) if and only if $m(r) = m$ is a constant function, in which case we can write

$$\rho(r) = 1 + r^2 - \frac{2m}{r}$$

and then conclude that $(I \times S^2, \overline{g})$ must be a piece of the Anti-de Sitter-Schwarzschild space of mass m.

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