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THE INVERSE PROBLEM OF DETERMINING THE PERMEABILITY REDUCTION IN FLOW OF WATER WITH PARTICLES IN POROUS MEDIA

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Abstract. *Most of the oil in the world is produced by injecting water in some wells and recovering oil in other wells. In offshore fields sea water containing organic and mineral inclusions is injected. This practice curtails the well's injectivity because the particles suspended in the fluid are trapped while passing through the porous rock.*

In this work, we study the deep filtration during the injection of water containing solid particles to predict the loss of injectivity in wells. The mathematical model for the filtration process is characterized by the filtration and the permeability reduction functions which describe properties of the porous medium where the flow occurs. We develop a recovery method for determining the permeability reduction function indirectly from the pressure drop along the core, assuming that the filtration function has been determined previously by a separate procedure. For this recovery of permeability, we derive an integral equation of Volterra type for the rock formation damage function $k(\sigma)$ *and we discuss conditions for well-posedness of the operator equation. Finally, we describe a numerical implementation to calculate* $k(\sigma)$ *within an appropriate subset of feasible solutions. The classical Tikhonov-Phillips regularization is used to reduce the ill-posed Volterra equation of first kind to a well-posed problem.*

Keywords: *Formation Damage, Deep Bed Filtration, Inverse Problem, Tikhonov Regularization*

1. INTRODUCTION

The physical model used here for the flow of water with suspended particles suffering retention in porous media was developed in Bedrikovetsky et al. (2001) based on Herzig et al. (1970). During the flow, gradually the suspended particles are retained, reducing the permeability of the medium. For constant flow rate, this phenomenon, called *deep bed filtration with formation damage*, is modelled by the system of non-dimensional equations:

$$\frac{\partial}{\partial T}(c+\sigma) + \frac{\partial c}{\partial X} = 0, \tag{1}$$

$$\frac{\partial \sigma}{\partial T} = \lambda(\sigma)c, \tag{2}$$

$$1/k(\sigma) = -\frac{\partial p}{\partial X}. \tag{3}$$

The first equation represents mass conservation of solid particles. The second equation represents the rate at which suspended particles are retained. The concentrations of dispersed and deposited particles are $c(X,T)$ and $\sigma(X,T)$, which have values between 0 and 1. The dependence of the retention rate on σ is expressed by $\lambda(\sigma)$, which is called the *filtration function*. The physical domain is $0 \leq X \leq 1$ and $T > 1$.

We assume that permeability reduction is due to particle retention, and that it is a decreasing function of the retained concentration σ. Eq. (3) is a form of Darcy's law, where $k(\sigma)$ is the permeability reduction due to particle retention σ; when expressed as a function of σ, it is called the *formation damage function*. It is normalized so that $k(0) = 1$, i.e., it is one for clean porous rock.

Methods for determining the permeability reduction function $k(\sigma)$ from the pressure drop history $\Delta p(T)$ were presented in Pang and Sharma (1994), Wennberg (1997) and Bedrikovetsky et al. (2003) for constant coefficient and a one parameter family of solutions.

In this work, we present a more general method for determining $k(\sigma)$ indirectly from $\Delta p(T)$, using Tikhonov regularization and assuming that $\lambda(\sigma)$ has been previously determined by a separate method, such as those found in Alvarez et al. (2005) and Marchesin et al. (2004). We also present the regularization by parametrization method, which generalizes the one developed in Al-Abduwani et al. (2004) for functions of two parameters.

1.1 Boundary and measured data

As initial data at $T = 0$, we assume that the rock is clean and contains water with no particles, i.e., $\sigma(X, 0) = 0$, and $c(X, 0) = 0$. We assume that the solid particle concentration entering the porous medium is given, i.e., at $X = 0:\ c(0,T) = c_i(T) > 0,\ t > 0$. The pressure drop $\Delta p = p(1,T) - p(0,T)$ is measured in laboratory experiments, while the quantity $\sigma(0,T)$ needs to be determined by the model. Along the line $X = 0$, it follows from Eq. (2) that

$$\frac{d\sigma(0,T)}{dT} = \lambda(\sigma(0,T))c_i(T). \tag{4}$$

Given $\sigma(0, 0) = 0$, integrating Eq. (4) provides $\sigma(0,T)$, which is positive increasing.

2. Filtration function

In Alvarez (2005), two methods were developed to obtain the filtration function from given injected and effluent concentrations. Once we have recovered it, the deposition of particles σ can be determined by solving the system of equations (1)–(2). The well-posedness of this boundary/initial-value problem was established in Alvarez (2005), assuming that the filtration function $\lambda(\sigma)$ is piecewise C^1, with $\lambda(\sigma) > 0$ for $\sigma \in [0, 1]$, and numerical methods were developed to solve this system of equations.

3. The integral equation for the permeability reduction

For one-dimensional flow in a rock core with non-dimensional length 1, we integrate Eq. (3) and obtain the following relationship between deposited particle distribution and pressure drop history:

$$\int_0^1 f(\sigma(X,T))dX = g(T), \text{ for all } T \in [1, A], \tag{5}$$

in the physical domain $\mathcal{D} = [0, 1] \times [1, A]$, where A is the last time value considered, $f(\sigma) = 1/k(\sigma)$ is an unknown continuous function and $g(T) = -\Delta p$ is a given non-negative continuous function. A similar integral equation was obtained in Bedrikovetsky (1993). We begin by describing the method for finding the general solution of Eq. (5) in the continuous case, then we show how to solve the problem numerically for discrete data.

In order to find a solution for Eq. (5), we analyze the inverse problem associated to the integral operator

$$K_\sigma \ : \ D(K_\sigma) \subset L^2[0, M] \ \to Y \subset L^2[1, A], \text{ where } M = \max_{\mathcal{D}} \sigma(X,T) \text{ and}$$
$$(K_\sigma f)(T) = \textstyle\int_0^1 f(\sigma(X,T))dX, \text{ for } 1 \le T \le A.$$

We want to obtain a procedure to approximate the inverse operator K_σ^{-1}. For this purpose, the issues of existence, uniqueness and stability of Eq. (5) must be studied. We know from the physics that its solution f must be a positive, non-decreasing continuous function: Assumption 1 arises naturally from the physical model (Alvarez (2005)):

Assumption 1 *The inequalities* $-\epsilon_1 < \frac{\partial \sigma(X,T)}{\partial X} < -\epsilon_2 < 0$ *hold uniformly on the characteristic lines* $X - T =$ *constant associated to* (1)–(2) *in the domain* $\mathcal{D}$.

This information is very useful for finding a class of functions where Eq. (5) is well-posed.

The recovery strategy requires additional information about the solution, such as smoothness and its values at the boundaries, i.e. $f(0)$ and $f(M)$. Since $k(0) = 1$, we have $f(0) = 1$. On the other hand, the value $f(M)$ must either be obtained from laboratory measurements or be evaluated numerically.

From the fact that the function $\sigma(X,T)$ is continuously differentiable in $[0, 1]$, for each T fixed, Assumption 1 and the implicit function theorem, it follows that it is possible to obtain the inverse function $X = \sigma^{-1}(y, T) = s(y, T)$ of $y = \sigma(X,T)$ restricted to $\mathcal{D}$. Now, we reformulate the problem in Eq. (5) as a Fredholm integral equation of the first kind. We change variables $y = \sigma(X,T)$ in Eq. (5), and rewrite it compactly as a linear operator equation

$$(K_\sigma f)(T) = \int_0^M K_\sigma(y,T) f(y)dy = g(T), \text{with } T \in [1,A] \text{ and}$$

$$K_\sigma(y,T) = \begin{cases} 0 & \text{if} \quad \sigma(0,T) < y \leq M; \\ \left[-\frac{\partial\sigma}{\partial X}(s(y,T),T)\right]^{-1} & \text{if} \quad \sigma(1,T) < y \leq \sigma(0,T); \\ 0 & \text{if} \quad 0 \leq y \leq \sigma(1,T). \end{cases} \tag{6}$$

Notice that the function K_σ is bounded due to Assumption 1, so the kernel $K_\sigma(y,T)$ belongs to $L^2([0,M] \times [1,A])$. Thus, K_σ is a Hilbert-Schmidt operator, compact from $L^2[0,M]$ to $L^2[1,A]$. The adjoint operator K_σ^* of K_σ is given by $(K_\sigma^* g)(y) = \int_1^A K_\sigma(y,T)g(T)\,dT$, where K_σ^* is defined from $L^2[1,A]$ to $L^2[0,M]$.

4. Conditions for existence, uniqueness and stability

In this section we transform the ill-posed problem given by Eq. (6) into a well-posed problem by means of regularization methods.

4.1 Ill-posedness of the inverse problem

It is known that the inverse of a compact operator with an infinite dimensional domain is not continuous. Thus, the inverse problem given by Eq. (6) is not well-posed.

To find the solution, or at least an approximation, we need to find an appropriate subset contained in a class of feasible solutions that includes additional information such as smoothness and positivity. We choose our set of feasible solution as

$$\mathcal{F} = \{f \in L^2[0,M]; f \text{ is non-decreasing and } 0 \leq f(x) \leq B \;\text{ a.e. in }\; [0,M]\}, \tag{7}$$

where B is a constant independent of x and f; here a.e. means "almost everywhere".

Remark 2 *Frequently, restrictions of the form $\phi_1(x) \leq f(x) \leq \phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ are given functions, are imposed to define the set of feasible solutions $\mathcal{F}$ that we are seeking. For example, in our case the feasible solutions must be the set of non-negative functions uniformly bounded by some constant B, so $\phi_1(x) = 0$ and $\phi_2(x) = B$. According to Helly's choice theorem (see Hildebrandt (1963), page 45), this subset $\mathcal{F}$ of feasible solutions is compact (see Goncarskii and Jagola (1969)).*

To obtain an approximate solution of Eq.(6) in $\mathcal{F}$ we should keep in mind that the solution may not exist if the function g behaves too roughly. Since the kernel K_σ and the set of feasible solutions $\mathcal{F}$ are non-negative functions, g is necessarily a non-negative function.

To obtain an approximate solution to the inverse problem associated to Eq. (6), we prove that it is well-posed in Tikhonov's sense. Let $N(K_\sigma) \stackrel{\text{def}}{=} \{h \in L^2[0,M]$ such that $K_\sigma h = 0\}$. The existence and uniqueness are guaranteed by the following

Theorem 3 *Let us take $\mathcal{D}(K_\sigma) = N(K_\sigma)^\perp$ and $Y = K_\sigma(X)$. Then the integral equation* (6) *has a solution and it is unique.*

The proof of Theorem 3 can be found in Kirsch (1996). Stability is a consequence of the lemma of Tikhonov (Tikhonov and Arsenin (1977)).

Theorem 4 *The inverse problem of Eq.* (6) *is well-posed in $\mathcal{F}$ in Tikhonov's sense.*

Proof: The existence and uniqueness are guaranteed by Theorem 3. On the other hand, the subset of feasible solutions $\mathcal{F}$ in Eq. (7) is contained in the domain X because if $f \in \mathcal{F}$ and $K_\sigma f = 0$ then $f \equiv 0$. Since $\mathcal{F}$ is a compact subset, the lemma of Tikhonov guarantees that the mapping $K_\sigma : \mathcal{F} \to Y$ is continuous and has continuous inverse. □

4.2 Regularization method

Following the Tikhonov regularization method, we transform the integral equation of the first kind (6) into a well-posed Volterra-type integral equation.

We wish to find $f \in \mathcal{F}$ that minimizes the deviation $||K_\sigma f - g||$ in the $L^2[0, M]$ norm. We recall that the subset $\mathcal{F}$ is infinite dimensional and the operator K_σ is compact, so this minimization problem is ill-posed. Tikhonov regularization consists of penalizing the deviation in the sense of optimization theory, that is, one obtains a solution by minimizing Tikhonov's functional

$$\overline{\Phi_\alpha}(f) = ||K_\sigma f - g||^2 + \alpha_1||f||^2 + \alpha_2||\frac{d}{dx}f||^2, \tag{8}$$

with $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 \geq 0, \alpha_2 \geq 0$. The terms $||f||^2$ and $||\frac{d}{dx}f||^2$ are used as a penalization for the squares error $||K_\sigma f - g||^2$, to avoid large oscillations.

In previous sections we introduced integral equations of the first kind, which are defined by the linear compact operator K_σ: $X \to Y$, with $X \subset L^2[0, M]$ and $Y \subset L^2[1, A]$. These subspaces are chosen to select an adequate metric for the error on the RHS of Eq. (6) and to obtain an approximate solution in the subset of feasible solutions.

To find the solution in the Sobolev space $H^1[0, M]$, Eq. (6) is interpreted as an integral operator with smooth kernel mapping $H^1[0, M]$ on $L^2[1, A]$. All regularization theory, including convergence and regularity, remains applicable in this setting. Following Kress (1989) and Tikhonov and Arsenin (1977), the unique solution f_α that minimizes the functional in Eq. (8) satisfies the following integro-differential equation with boundary condition $f_\alpha'(0) = f_\alpha'(M) = 0$:

$$\alpha(f_\alpha - f_\alpha'') + K_\sigma^* K_\sigma f_\alpha = K_\sigma^* g, \tag{9}$$

Finally, the problem of finding a regularized solution f_α of Eq. (8) reduces to finding a solution of Eq. (9) satisfying the conditions $f_\alpha(0) = f_1$ and $f_\alpha(M) = f_2$, where f_1 and f_2 are known values. We assume that $f_\alpha(0) = 1$. In practice the values of $f_\alpha(M)$ can be estimated by the method of regularization by parametrization.

4.3 Choice of regularization parameter value

To solve the optimization problem in the regularization method, it is necessary to estimate the regularization parameter α in Eq. (8). There are many strategies that guarantee optimal convergence of the regularized solution starting from data polluted by errors (Kirsch (1996), Kress (1989), Plato (1990), Morozov (1984) and Tikhonov and Arsenin (1977)).

In this work, we combine the discrepancy principle and the L-curve methods for choosing the regularization parameter value. First, we obtain an approximate initial solution, fixing a prescribed analytical expression for the solution and estimating the deficiency of the possible solution. Here we choose a regularization parameter using the L-curve method. Second, we calculate a solution by Tikhonov regularization using the discrepancy principle for estimating the regularization parameter with previously specified tolerance.

The L-curve method The L-curve method is broadly discussed in Hansen (2001), and useful tools in the Matlab package are available, see Hansen (1994). Here we mention some fundamental ideas in the L-curve method. The L-curve is parametrized by the regularization parameter, i.e.,

$$\{\log(||Ax_\alpha - b||), \log(||L(x_\alpha - x_0)||); \quad \alpha \in \mathbb{R}^+\},$$

where the matrices A and L represent a discretized version of Tikhonov's functional and x_0 is a given reference value. In many application, such a curve takes a concave form, and the optimal value of the regularization parameter corresponds to the point of maximum curvature.

The discrepancy principle To make optimal a posteriori choices for the regularization parameters α, we approximate the solution of Eq. (6) for a given g^δ with a known error level from the unknown exact function g, i.e., satisfying $||g^\delta - g|| \leq \delta$, and use a perturbed right hand side to construct a reasonable approximation $f_{\alpha(\delta)}$ to the exact solution.

We compute the value $\alpha(\delta) > 0$ such that the corresponding Tikhonov solution $f_{\alpha(\delta)}$ of Eq.(6) (i.e., the minimum of the Tikhonov functional in Eq. (8)) satisfies the equality $||K_\sigma f_{\alpha(\delta)} - g^\delta|| = \delta$. The discrepancy principle guarantees that α is not too small, and that the error between the regularization solution and the known value g^δ is equal to δ (Morozov (1984)).

5. Collocation method for the integral equation

In this section we describe a numerical algorithm for obtaining the approximate solution of Eq. (9) in the space $H^1[0, M]$. We calculate a discrete approximation of f in Eq. (6) by means of a quadrature formula, for an appropriate non-uniform partition of interval $[0, M]$. Thus, the continuous linear problem is reduced to a linear system of equations.

5.1 Numerical algorithm

Here the problem of obtaining an approximate solution of Eq. (6) is reduced to solving a linear system of equations, obtained by the collocation method. We discretize the operator K_σ and its adjoint K_σ^*, then we obtain a discretization of $K_\sigma^* K_\sigma$ and use it for solving Eq. (9). Similar methods and examples can be found in Tikhonov and Arsenin (1977), de Hoog (1980), Varah (1983) Trummer (1984), Richter (1978), Anderssen et al. (1980) and Neubauer (1988).

Discretization of the operator K_σ We partition the interval $[1, A]$ into $y_1, y_2, \ldots, y_{2r+1}$. Taking the kernel K_σ in Eq. (6) and using an appropriate quadrature formula we obtain

$$\int_0^M K_\sigma(s, y_i) f_\alpha(s) ds \approx \sum_{j=1}^{m_s} \omega_j K_\sigma(s_j, y_i) f_\alpha(s_j), \tag{10}$$

where ω_j, $j = 1, \ldots, m_s$ are weights to be determined, (s_j), is a partition of $[0, M]$, and (y_i), $i = 1, \ldots, 2r+1$ is a partition of $[1, A]$. In Alvarez (2005) a way to calculate the weight vector $W = (\omega_j)$ and the values $S = (s_j)$ is presented.

Remark 5 *Notice that $K_\sigma(s, T)$ is discontinuous on the curves $s = \sigma(0, T)$ and $s = \sigma(1, T)$. A non-uniform partition of $[0, M]$ is required to guarantee that the quadrature of Eq.* (6) *takes into account all possible discontinuities. If a uniform partition were used, its accuracy would be reduced to first order at the discontinuities.*

Discretizing the adjoint operator K_σ^* **with the operator** K_σ The product of the operators $K_\sigma^* K_\sigma$ is approximated as the product of two matrices by using an appropriate integration formula: the second term on the left hand side of Eq. (9) can be approximated as follows

$$\int_1^A K_\sigma(t_i, y) \left[\int_0^M K_\sigma(s, y) f_\alpha(s) ds \right] dy \approx \sum_{k=1}^{2r+1} A_k K_\sigma(t_i, y_k) \sum_{j=1}^{m_s} \omega_j K_\sigma(s_j, y_k) f_\alpha(s_j), \tag{11}$$

where (t_i), $i = 1, \ldots, m_s$ is a partition of $[0, M]$ and A_k are Simpson's integration weights. Then, setting $b_{ik} = A_k K_\sigma(t_i, y_k)$ and $c_{kj} = \omega_j K_\sigma(s_j, y_k)$, i.e., $B = (b_{ik})$ an m_s by $2r+1$ matrix and $C = (c_{kj})$ an $2r+1$ by m_s matrix, Eq. (11) can be rewritten as

$$\int_1^A K_\sigma(t_i, y) \left[\int_0^M K_\sigma(s, y) f_\alpha(s) ds \right] dy \approx \sum_{j=1}^{m_s} \left(\sum_{k=1}^{2r+1} b_{ik} c_{kj} \right) f_\alpha(s_j) = BC f_\alpha. \tag{12}$$

Notice that matrices C and B are discrete versions of the operators K_σ and K_σ^*: since the inverse problem in Eq. (6) is ill-posed, these matrices are ill-conditioned, and Tikhonov regularization for matrices is required to solve a discretization of Eq. (9) (Hansen (1994)).

Solution of the integro-differential equation Using the same notation as in Eq. (10), we set the values $s_0 = 0$ and $s_{m_s+1} = s_{m_s} + d_{m_s+1}$, where $d_{m_s+1} = s_{m_s} - s_{m_s-1}$, and obtain $d_k = s_{k+1} - s_k$, where $d_k \neq d_{k+1}$, for $k = 0, \ldots, m_s$. Thus, Eq. (9) is approximated by a system of linear equations for the unknowns f_i of the form:

$$-\tfrac{\alpha}{d_i d_{i-1}} f_{i+1} - \tfrac{\alpha}{d_{i-1}^2} f_{i-1} + \left(\tfrac{\alpha}{d_i d_{i-1}} + \tfrac{\alpha}{d_{i-1}^2} + \alpha \right) f_i + \sum_{j=1}^{m_s} \left(\sum_{k=1}^{2r+1} b_{ik} c_{kj} \right) f_j = y_i, \tag{13}$$

where $y_i = \sum_{k=1}^{2r+1} b_{ik} h_k$, $i = 1, \ldots, m_s$ (see Tikhonov and Arsenin (1977), pag 78), and $f_0 = f(0)$ and $f_{m_s+1} = f(M)$ are given. Denoting $\bar{f} = (f(s_1), \ldots, f(s_{m_s}))$, we rewrite the system (13) as

$$(\alpha V + BC)\bar{f} = \bar{y},$$

with B and C from Eq. (12), and the non-zero elements of the matrix $V = (V_{ij})$ given by the formulae:

$$\begin{aligned}
V_{ii} &= \tfrac{1}{d_i d_{i-1}} + \tfrac{1}{d_{i-1}^2} + 1, && \text{for} \quad i = 1, \ldots, m_s, \\
V_{i,i-1} &= -\tfrac{1}{d_{i-1}^2}, \quad V_{i,i+1} = -\tfrac{1}{d_i d_{i-1}}, && \text{for} \quad i = 2, \ldots, m_s - 1, \\
V_{1,2} &= -\tfrac{1}{d_2 d_1}, \quad V_{m_s,m_s-1} = -\tfrac{1}{d_{m_s}}.
\end{aligned}$$

The values of the vector $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_{m_s})^T$ are given by:

$$\begin{aligned}
&\bar{y}_i = y_i, \text{for } i = 2, \ldots, m_s - 1, \\
&\bar{y}_1 = y_1 + \tfrac{\alpha}{d_0^2} f_0, \quad \bar{y}_{m_s} = y_{m_s} + \tfrac{\alpha}{d_{m_s} d_{m_s-1}} f_{m_s+1}.
\end{aligned}$$

6. Regularization by parametrization

In the previous section we solved Eq. (5) under the assumption that $f(\sigma)$ belongs to a compact subset. No assumption was made about the analytical structure of the solution. This led to an ill-posed problem solved by the Tikhonov regularization method.

In this section we prescribe a parametric expression for the function $f(\sigma)$, i.e., we project the function $f(\sigma)$ on a finite dimensional subspace V_n. Even in this formulation, the problem remains ill-posed, but the procedure reduces to estimating a finite number of parameter values. This method is useful because we do not need a priori information on the deficiency of the solution δ to obtain the regularization parameter.

We assume that the solution of Eq. (5) is given by the polynomial expression $f(\sigma) = 1 + \beta_1 \sigma + \ldots + \beta_n \sigma^n$, where $\beta_1, \ldots, \beta_n$ are parameters to be determined. Then Eq. (5) can be rewritten as $1 + \beta_1 S_1(T) + \ldots + \beta_n S_n(T) = g(T)$ where $S_k(T) = \int_0^1 (\sigma(X,T))^k dX$ are the moments of σ, for all $T \in [1, A]$. Since the values of $S_1(T), \ldots, S_n(T)$ and $g(T)$ for T equal to $T_1, \ldots, T_m \in [1, A]$ are known, solving Eq. (5) reduces to estimating the parameters $\beta_1, \beta_2, ..., \beta_n$ by finding the "best" solution of the linear system of equations $C\bar{\beta} = \bar{g}$ where $C = (c_{ij})$ is a $m \times n$ matrix given by $c_{ij} = S_j(T_i)$, $\bar{g} = (\bar{g}_i$ is a m vector given by $\bar{g}_i = g(T_i) - 1$ and $\bar{\beta} = (\beta_j)$ is the n vector of unknowns, which can be found using the Tikhonov regularization method (Hansen (2001)), i.e.,

$$\bar{\beta}_\alpha = argmin\{||C\bar{\beta} - \bar{g}||^2 + \alpha||L(\bar{\beta} - \bar{\beta}_0)||^2\}, \tag{14}$$

where the matrix L represents a discretization of the derivative operator, $\alpha > 0$ is the regularization parameter and $\bar{\beta}_0$ is a prescribed reference value. In this case, the L-curve method is used to estimate a value of the regularization parameter.

7. Applications of the algorithm to synthetic data

In this section we test the regularization methods with synthetic data. We discuss the method of regularization by parametrization and the collocation method applied to Eq. (5).

7.1 Synthetic data

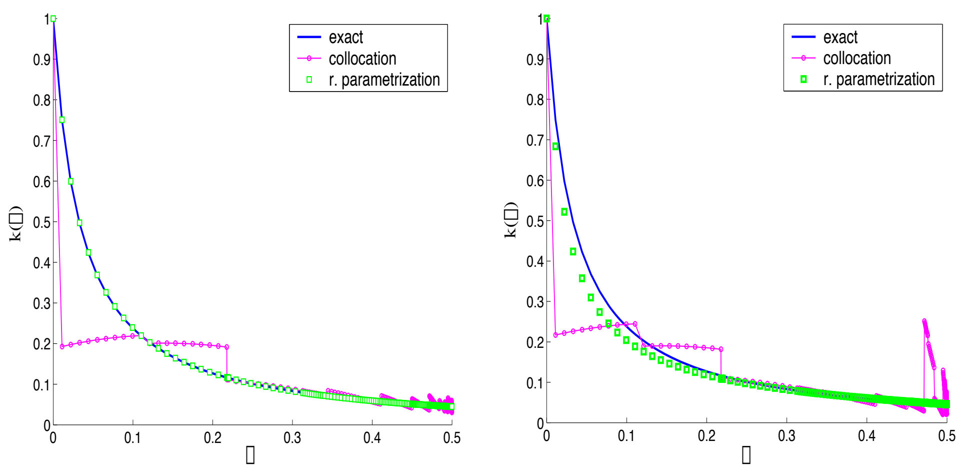
To create the synthetic data, we use the analytical solution obtained in Alvarez (2005). First, we fix the number of parameters and their values: in the following experiments, we use $k(\sigma) = (1 + 30\sigma + 20\sigma^2 + 10\sigma^3)^{-1}$. Then we calculate the right hand side of Eq. (5), and add random perturbations to simulate experimental error (Sun et al. (2001)). Thus, we obtain:

$$g^{\delta}(T_j) = \int_0^1 (k(\sigma(X, T_j)))^{-1} dX \pm \delta\nu, \text{for } j = 1, \ldots, m \text{ and } T_j \in [1, A].$$

where ν represents the standard Gaussian random variable with zero mean and unit standard deviation. In this study, we use the relative error τ to define the standard deviation, i.e., $\delta = \tau g(T_j)$. We present two numerical experiments to recover $k(\sigma)$, taking τ as 0.01 and 0.05.

7.2 Numerical results and discussion

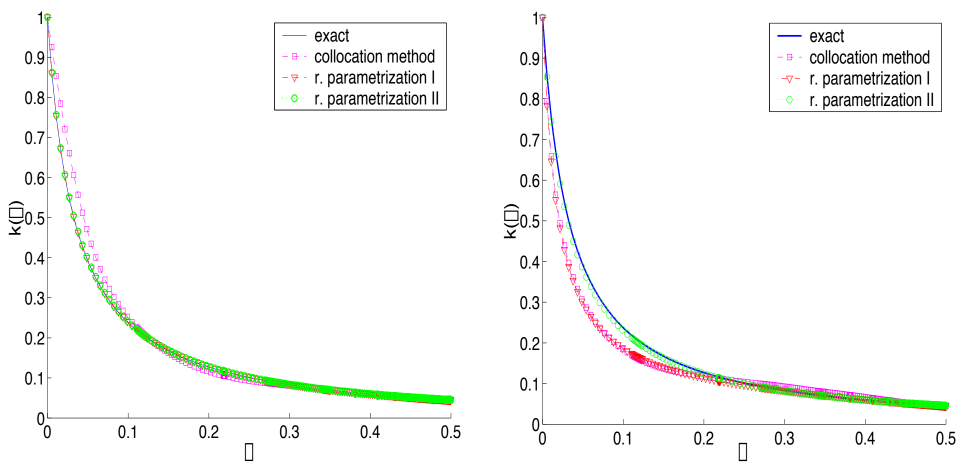
The regularized solution obtained with the regularization parameter α, using either the discrepancy principle or L-curve methods, is relatively stable with respect to data perturbations. Both methods are reasonable computational procedures, but ther sensitivity to perturbations are quite different. We start by examining the numerical solution



a) $\tau = 0.01$.　　b) $\tau = 0.05$

Figure 1: Solid line: exact solution. Solid line with circles: regularized solution by collocation method ($\alpha_1 \neq 0, \alpha_2 = 0$). Squares: solution by parametrization. ($\alpha_1 = \alpha_2 = 0$).

obtained by the collocation method minimizing the penalized functional without derivatives ($\alpha_2 = 0$, in Eq. (8)) and the solution using parametrization without regularization ($\alpha_1 = \alpha_2 = 0$). In the first experiment we take a smaller relative error $\tau = 0.01$: Figure 1a shows the results. Notice that when only the norm of the function is used as penalization, the regularized solution has large oscillations around the exact solution. This suggests using the derivative to smooth the regularized solution (i.e. setting $\alpha_2 \neq 0$) so it becomes more accurate. On the other hand, the method of regularization by parametrization apparently gives an accurate solution. However, since no regularization is being used ($\alpha_1 = \alpha_2 = 0$ in Eq. (8)), this solution is more sensitive to experimental error. To illustrate this, Figure 1b shows the solutions of the same problem with $\tau = 0.05$. Notice that the parametrized solution changes with respect to the exact solution when τ increases, because the norm of the difference between the parametrized solution and the "true" solution is proportional to τ (Kress (1989)). Now we examine the results of minimizing Eq. (8)



a) $\tau = 0.01$ b) $\tau = 0.05$

Figure 2: Solid line: exact solution. Squares: regularized solution from collocation method with discrepancy principle. Triangles: solution from regularization by parametrization method with L-curve. Circles: solution from regularization by parametrization with discrepancy principle.

with both $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, shown in Figure 2. For the regularization by parametrization method, we estimate $\alpha = \alpha_1 = \alpha_2$ by both the L-curve method and the discrepancy principle. In the solution obtained by the collocation method, we use α estimated by the discrepancy principle. Figure 2a shows that all three solutions are reasonable approximations of the exact solution. In Figure 2b, the results of recovering the same solution with relative error $\tau = 0.05$ are shown. As in the first experiment, the regularized solution changes significantly relative to the exact solution when the value of τ is increased. We can conclude that the approximate solution is sensitive to the relative error τ.

In the numerical experiments with the method of regularization by parametrization, the matrices $L_1 = \text{diag}(1, -1)$ and $L_2 = \text{diag}(1, -2, 1)$ are used in the regularization term (see Eq. (14)). These matrices represent discretized versions of the derivative operator of first and second order respectively, and we use them to stabilize the least squares solution. The choice of order of the derivative depends on the degree of the ill-posed problem (Hansen (2001)): to obtain singular values with higher decrease rate, higher order derivative operators are required. More accurate and stable solutions were obtained as expected using higher order operators.

8. Convergence results

The numerical examples in the previous section suggest that it is possible to stabilize the integral equation of the first kind using only first derivative penalizing functionals (see Section 5.). In this section, we show that this is an intrinsic property of the operator kernel. First we present preliminary concepts and lemmas, then we prove stability results for the general linear equation (6).

8.1 Preliminaries

We use Tikhonov regularization for solving Eq. (6), that is, we minimize the functional:

$$\Phi_\alpha(f) = ||Kf - g||^2 + \alpha ||Lf||^2, \tag{15}$$

where L is an unbounded, self-adjoint linear operator. In the numerical experiment in the previous section we took $Lf = \frac{df}{dx}$, and obtained good numerical results. We now prove that taking one derivative in the penalizing functional $||Lf||$ guarantees that the regularized solution f_α of Eq. (15) converges in L^2-sense to the "true" solution $K^\dagger g$, i.e., to least square solutions of smallest norm. We assume that we have a Fredholm integral equation of the first kind on $[0,1]$ with $q \in L^2([0,1]^2)$ and $f \in L^2[0,1]$:

$$\int_0^1 q(x,t) f(x)\, dx = g(t), \tag{16}$$

Remark 6 *Equation* (6) *can be reduced to an equation of the form* (16) *by a trivial change of variables.*

We regularize Eq. (16) by using the Hilbert scale (Krein and Petunin (1966)) $(X_s)_{s\in\Re}$, which is induced by the operator $L : D(L) \subset L^2[0,1] \longrightarrow L^2[0,1]$ defined by

$$Lf = \sum_{n=1}^{\infty} n(f, v_n) v_n, \quad \text{with} \quad D(L) = \{f \in H^1[0,1] : f(0) = f(1) = 0\}, \tag{17}$$

where $v_n(s) = \sqrt{2}\sin(n\pi s)$ and $(\cdot,\cdot)$ is the inner product in $L^2[0,1]$. Notice that Eq. (17) is the spectral representation of the operator L with $D(L)$ defined in (17). Moreover, the operator L defined in this way is injective.

One can show (Neubauer (1988)) that for $s > 0$ there is a Hilbert scale defined as:

$$X_s = D(L^s) = \left\{ f \in H^s[0,1] : f^{2e}(0) = f^{2e}(1) = 0, e = 0, 1, \ldots, [s/2 - 1/4] \right\},$$

with inner product in X_s given by

$$(f,g)_k = (L^k f, L^k g) = \pi^{-2k} \left(\frac{d^k f}{dx^k}, \frac{d^k g}{dx^k} \right).$$

8.2 Auxiliary lemmas

Lemma 7 *There exist $b \geq 0$ and $d > 0$ such that*

$$||Lf|| \geq d ||f||_b. \tag{18}$$

Lemma 8 *Let $K : L^2[0,1] \to L^2[0,1]$ be a linear operator defined as*

$$Kf := \int_0^1 q(x,\cdot) f(x) dx.$$

Let $s_1(t)$ and $s_2(t)$ be monotone increasing continuous functions such that

$$0 \leq s_1(t) \leq s_2(t) < 1 \quad on \quad [0,1].$$

Let $S(x,t)$ be a positive continuous function positive on $s_1(t) \leq x \leq s_2(t)$ with $t \in [0,1]$. Assume that the kernel is given by $q(x,t) = \begin{cases} 0 & s_2(t) \leq x \leq 1 \\ S(x,t) & s_1(t) < x < s_2(t) \\ 0 & 0 \leq x \leq s_1(t). \end{cases}$

Then there exists a constant $v > 0$ such that $||Kf|| \geq v||f||_{-1}$, where

$$||f||_{-1}^2 = \sum_{j=1}^{\infty} |(f, e_j)|^2 / j^2.$$

The proofs of the previous Lemmas can be found in Alvarez (2005).

8.3 Convergence of the regularized solution

We show that Eq. (6) is a particular case of the family of operators treated in Lemma 8, and that its regularized solution converges to the least square solution $K^{\dagger}g$ penalized with a first derivative term. Choosing the regularization parameter α by the discrepancy principle, the following theorem is valid

Theorem 9 *Let $K : X \to Y$ be a compact linear operator, satisfying the assumptions in Lemma 8. Let $\hat{x}$ be the generalized solution belonging to the set $M_\rho = \{x \in D(L) : ||Lx|| < \rho\}$. Then the regularized solution $\tilde{x}_\alpha$ satisfies*

$$||\hat{x} - \tilde{x}_\alpha|| \leq 2\rho^{1/2}(\delta/v)^{1/2}.$$

Proof: It follows from the application of the fundamental theorem in Nair (1999) and Lemmas 7 and 8. $\square$

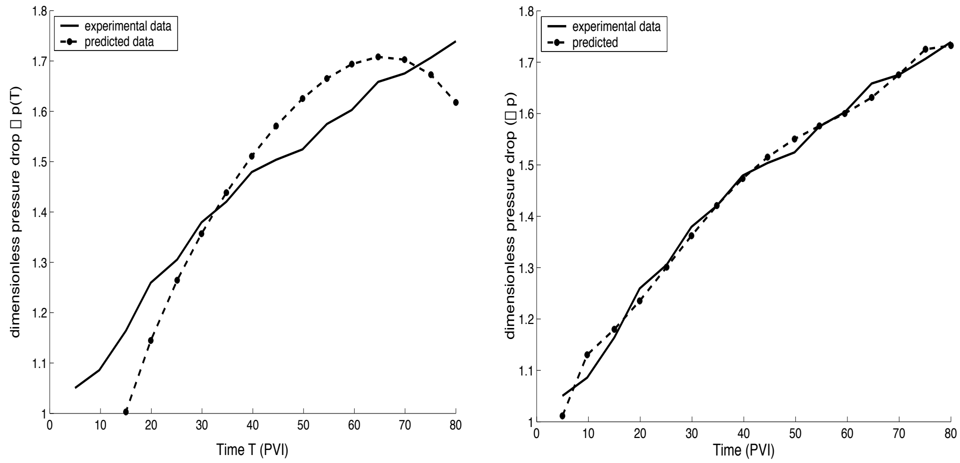
Remark 10 *If $\delta \to 0$ when $\alpha \to 0$ then $\tilde{x}_\alpha \to \hat{x}$.*

Now it is easy to see that the operator defined in Eq. (6) is a particular case of the class defined in Lemma 8: it is enough to use the change of variables $t = T/A$ and $x = y/M$ in the integral equation (6) and to choose $s_1(t) = \sigma(1,t)/M$, $s_2(t) = \sigma(0,t)/M$ and $S(x,t) = \left(\frac{\partial \sigma}{\partial x}(s(x,t),t)\right)^{-1}$. With these new variables, the operator K_σ in Eq. (6) satisfies the assumptions of Lemma 8, and the convergence result obtained here is valid.

9. Experimental data

Soma and Papadopoulos (1995) performed experiments injecting oil-in-water emulsions into quartz sand. They performed four similar experiments varying the ionic strength of the emulsion, and measured the effluent concentration and permeability reduction. We apply the recovery procedure developed in this work to one of their experiments.

The permeability reduction function is recovered from experimental pressure drop data using the regularization by parametrization method. We use the method described in Alvarez (2005) to recover the filtration function and to obtain the deposited particle



a) regularized solution with three parameters

b) regularized solution with seven parameters

Figure 3: Predicted and experimental pressure drop histories

Table 1: Values obtained for α and associated errors for various the number of parameters.

n	α	relative error
2	1.01×10^{-2}	0.240
3	2.80×10^{-3}	0.120
4	8.76×10^{-4}	0.071
5	2.77×10^{-4}	0.044
6	8.34×10^{-5}	0.026
7	2.28×10^{-5}	0.014

concentration $\sigma(X,T)$. Figure 3 shows the experimental and predicted pressure drop histories. We can see in Fig. 3a that predicted values for three parameters is not a good approximation of the experimental data. Better results are obtained when the number of parameters is increased (see Fig. 3b). As can be seen in Table 1 both the regularization parameter and the relative error decrease, as expected from the convergence results.

10. Conclusion

The method described here is a viable procedure estimating the parameters of the empirical formation damage function from pressure drop history. The model is concise and yields good match between forward and inverse problems for experimental data.

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REFERENCES

Al-Abduwani, F. A. H., Farajzadeh, R., van den, B. W. M. G. T., & K., C. P., 2004. Utilizing static filtration experiments to test existing filtration theories for conformance. In *Second Produced Water Workshop. Aberdeen, UK.*

Alvarez, A., 2005. *Inverse problems for deep bed filtration in porous media*. PhD thesis, IMPA. Brasil.

Alvarez, A. C., Bedrikovetsky, P. G., Hime, G., Marchesin, D., & Rodriguez, J. R., 2005. A fast inverse solver for the filtration function for flow of water with particles in porous media. *in preparation*.

Anderssen, R. S., de Hoog F. R., & A., L. M., 1980. *The application and numerical solution of integral equations*. Sijthoff & Noordhoff.

Bedrikovetsky, P., 1993. *Mathematical Theory of Oil and Gas Recovery*. Kluwer.

Bedrikovetsky, P., Marchesin, D., Shecaira, F., Serra, A. L., & Resende, E., 2001. Characterization of deep bed filtration system from laboratory pressure drop measurements. *Journal of Petroleum Science and Engineering*, vol. 64, n. 3, pp. 167–177.

Bedrikovetsky, P., Tran, K., Van den Broek, W., Marchesin, D., Rezende, E., Siqueira, A., Serra, A., & Shecaira, F., 2003. Damage characterization of deep bed filtration from pressure measurements. *Journal SPE PF*, vol. , n. 3, pp. 119–128.

de Hoog, F., 1980. Review of fredholm equations of the first kind, application and numerical solution of integral equations. *(R.S.Anderssen, F.de Hoog and M.A.Lucas, eds.), Sijthoff & Noordhoff, Alphen aan den Rijn, Germantown*, pp. 119–134.

Goncarskii, A. V. & Jagola, A. G., 1969. Uniform approximation of monotonic solutions of noncorrect problems. *Dokl. Akad. Nauk SSSR*, vol. 184, n. 4, pp. 1–2.

Hansen, P. C., 1994. Regularization tools: A matlab package for analysis and solution of discrete ill-posed problems. *Numerical Algorithms*, vol. 6, pp. 236–240.

Hansen, P. C., 2001. The L-curve and its use in the numerical treatment of inverse problems. *Computational Inverse Problems in Electrocardiology, WIT Press, Southampton*, pp. 119–142.

Herzig, J. P., Leclerc, D., & Goff, P. L., 1970. Flow of suspensions through porous media-application to deep filtration. *Industrial and Engineering Chemistry*, vol. 65, n. 5, pp. 8–35.

Hildebrandt, T. H., 1963. *Introduction to the Theory of Integration*. Academic Press New York and London.

Kirsch, A., 1996. *An introduction to the mathematical theory of inverse problems*. Springer-Verlag, New York.

Krein, S. G. & Petunin, J. I., 1966. Scales of banach spaces. *Russian Math. Surveys*, vol. 21, pp. 85–160.

Kress, R., 1989. *Linear Equations*. Springer Verlag. Berlin Heidelberg.

Marchesin, D., Alvarez, A. C., Hime, G., Bedrikovetsky, P. G., & Al-Abduwani, F. A. H., 2004. The inverse problem of determining filtration function and permeability reduction in porous media. In *XXV CILAMCE. Iberian Latin American Congress on Computational Methods.Recife, Brasil*.

Morozov, V. A., 1984. *Methods of Solving Incorrectly Posed Problems*. Springer–Verlag, New York, Berlin Heidelberg.

Nair, M. T., 1999. On Morozov's method for Tikhonov regularization as an optimal order yielding algorithm. *Journal for Analysis and its Applications*, vol. 18, n. 1, pp. 37–46.

Neubauer, A., 1988. An a posteriori parameter choice for tikhonov regularization in Hilbert scales leading to optimal convergence rates. *SIAM J. Numer. Anal.*, vol. 25, pp. 1313–1326.

Pang, S. & Sharma, M., 1994. A model for predicting injectivity decline in water injection wells. In *69th Annual Technical Conference and Exhibition. New Orleans. Soc. of Petroleum Engineers. SPE 28489*.

Plato, R., 1990. Optimal algorithms for linear ill–posed problems yield regularization methods. *Numer. Funct. Anal. and Optim.*, vol. 11, pp. 111–118.

Richter, G. R., 1978. Numerical solution of integral equations of the first kind nonsmooth kernels. *SIAM J. Numer. Anal.*, vol. 15, n. 3, pp. 511–522.

Soma, J. & Papadopoulos, K., 1995. Flow of dilute, sub-micron emulsions in granular porous media: effects of ph and ionic strength. *J. Colloids and Surfaces. Series A: Physicochemical and Engineering Aspects*, vol. 101, pp. 51–61.

Sun, N., Sun, N. Z., Elimelech, M., & Ryan, N., 2001. Sensitivity analysis and parameter identifiability for colloid transport in geochemically hererogeneous porous media. *Water Resources Research*, vol. 37, n. 2, pp. 209–222.

Tikhonov, A. & Arsenin, V., 1977. *Solutions of Ill-Posed Problems*. Wiley, New York.

Trummer, M. R., 1984. A method for solving ill-posed linear operator equations. *SIAM J. Numer. Anal.*, vol. 21, pp. 729–737.

Varah, J. M., 1983. Pitfalls in the numerical solution of linear ill–posed problems. *SIAM J. Sci. Stat. Comput.*, vol. 4, pp. 164–176.

Wennberg, K., S. M., 1997. Determination of the filtration coefficient and the transition time for water injection wells. In *Soc. of Petroleum Engineers. SPE 38181*.