Suboptimal Deterministic Contracts: Examples.

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Abstract

In this paper we show that even when the virtual surplus is strictly concave, an stochastic contract may be desirable. The results are established in the Principal-Agent model context with a continuum type set.

Keywords: stochastic contracts, deterministic contracts, optimality

JEL Codes: D42, D82.

1. Introduction

The literature in theory of contracts deals mainly with deterministic contracts. In this paper we present three examples where the deterministic contract is suboptimal, even though the virtual surplus in these examples is strictly concave, which would suggest that any randomization is undesirable.

In the Principal-Agent model, the principal's optimization problem consists in choosing a contract (q, t) that maximizes her expected utility subject to the agent's individual rationality (IR) and incentive-compatibility (IC) constraints. The standard method to solve it is to derive a new expression for the objective function eliminating the monetary transfer t and then define a relaxed version of the original problem, using this new expression and ignoring the (IC) constraints. Now observe that if q_r is implementable (i.e. if we can find a monetary transfer t_r such that (q_r, t_r) satisfies the (IC) constraints), then (q_r, t_r) is a solution of the original problem. Strausz [8], in a model with finite type set, proved that this contract (q_r, t_r) is optimal even if we allow for stochastic contracts. We extend this result for a continuum of types $\Theta = [\underline{\theta}, \overline{\theta}]$.

Therefore, if we want to find examples where the deterministic contract is suboptimal, we should look for situations where the relaxed solution is not implementable. In the first example, we are under single-crossing condition CS_+ but the relaxed solution is strictly decreasing. In the second example, the single-crossing condition is not valid. Finally, in the third example, the agent's

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type has a mass point.[2] In all examples, the virtual surplus is strictly concave in the q variable.

2. Model

We analyse the optimality of deterministic contracts in the unidimensional screening problem using the Principal-Agent setup. The principal and agent interaction is mediated by a contract, a pair (q, t) where $q \in Q \subset \mathbb{R}$ represents the agent's decision and $t \in \mathbb{R}$ is the monetary transfer. The agent's private information is represented by his type $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$, a random variable with probability distribution function $F(\theta)$. The principal and the agent have quasi-linear preferences given respectively by

$$U(q, t, \theta) = u(q, \theta) - t \text{ and } V(q, t, \theta) = v(q, \theta) + t.$$

We assume that $u(\cdot, \theta)$ is continuous, $v(q, \theta)$ is C^3 with $v_\theta > 0$[3] and that $Q \subset \mathbb{R}$ is compact. The function $F(\theta)$ is absolute continuous with density $f(\theta) > 0$.[4]

We consider stochastic and deterministic contracts. For stochastic ones, let $\mathcal{Q}$ denote the Borel $\sigma-$algebra on Q and ΔQ the set of probability measures on $\mathcal{Q}$. Then a stochastic contract consists in a pair of functions $S = (\mu, t) : \Theta \to (\Delta Q, \mathbb{R})$ associating for each θ type agent a probability measure $\mu(\theta)$ and a monetary transfer $t(\theta)$.[5]

On the other hand, a deterministic contract is simply a pair of functions $D = (q, t) : \Theta \to (Q, \mathbb{R})$. Notice that any deterministic contract $D = (q, t)$ is equivalent to an stochastic contract $S = (\mu_d, t)$ where $\mu_d(\theta)$, for all $A \in \mathcal{Q}$, is defined by

$$\mu_D(\theta)(A) = \begin{cases} 1, & \text{if } q(\theta) \in A, \\ 0, & \text{if } q(\theta) \notin A. \end{cases}$$

Using the *'Revelation Principle'*[6] we can state the principal's maximization problem. In the stochastic case, it consists in choosing the pair $(\mu, t) : \Theta \to$

[2]This example is inspired by Hellwig [3], where we can find a methodology for dealing with more general specifications for the type distribution function.

[3]The analysis here is also valid when $v_\theta < 0$. We only have to change the expression for the virtual surplus.

[4]In Example 3, we will consider a more general distribution function that is discontinuous.

[5]As observed by [9], under quasilinear preferences we may assume without loss of generality that the monetary transfer is deterministic.

[6]The *'Revelation Principle'* has been enunciated in Gibbard [2].

$\Delta Q \times \mathbb{R}$ that solves

$$(P_s)\begin{cases} \max_{S=(\mu,t)} V(S) = \int_\Theta \{E_{\mu(\theta)}[u(q,\theta)] - t(\theta)\} dF(\theta), & (1) \\ \textit{subject to the individual-rationality constraints} \\ \quad E_{\mu(\theta)}[v(q,\theta)] + t(\theta) \geq 0, \quad \forall \theta \in \Theta, & (2) \\ \textit{and the incentive compatibility constraints} \\ \quad E_{\mu(\theta)}[v(q,\theta)] + t(\theta) \geq E_{\mu(\theta')}[v(q,\theta)] + t(\theta'), \quad \forall \theta, \theta' \in \Theta. & (3) \end{cases}$$

On the other hand, in the deterministic case, the principal has to choose a pair of functions $(q,t) : \Theta \to \mathbb{R} \times \mathbb{R}$ that solves

$$(P_d)\begin{cases} \max_{D=(q,t)} V(D) = \int_\Theta [u(q(\theta),\theta) - t(\theta)] dF(\theta), & (4) \\ \textit{subject to the individual-rationality constraints} \\ \quad v(q(\theta),\theta) + t(\theta) \geq 0, \quad \forall \theta \in \Theta, & (5) \\ \textit{and the incentive compatibility constraints} \\ \quad v(q(\theta),\theta) + t(\theta) \geq v(q(\theta'),\theta) + t(\theta'), \quad \forall \theta, \theta' \in \Theta. & (6) \end{cases}$$

Observe that if S^* and D^* are maximizers of (P_s) and (P_d) respectively, then the valuation satisfies $V(D^*) \leq V(S^*)$. Indeed, as we saw before, the set of deterministic contracts is contained in the set of stochastic contracts.

2.1. Implementability

We say that $q : \Theta \to Q$ is implementable when there exists a monetary transfer $t(\theta)$ such that the pair (q,t) satisfies (6). In the deterministic case, the single-crossing condition gives a characterization of implementability.

Definition 1 (Single-crossing). *The function $v(q,\theta)$ satisfies the single-crossing condition when we have either*

$$\forall\ (q,\theta)\ \textit{in}\ Q \times \Theta : v_{q\theta} > 0, \qquad (CS_+)$$

or

$$\forall\ (q,\theta)\ \textit{in}\ Q \times \Theta : v_{q\theta} < 0. \qquad (CS_-)$$

Under the single-crossing condition, implementability is equivalent to the monotonicity of the decision function, with $q(\cdot)$ increasing if CS_+ or decreasing if CS_-.[7]

In the stochastic case the definition of implementability is similar. Thus a probability measure $\mu(\theta)$ is implementable if one can find a monetary transfer

[7] See [7] or [9], chapter 7.

$t(\theta)$ such that the constraints (3) are satisfied. If we restrict the set of utility functions, we also have a simple characterization of the implementable stochastic decisions. Indeed, let us consider the class of separable utility functions with the form

$$v(q, \theta) = \ell(\theta)h(q),$$

where $\ell'(\theta) > 0$. With this functions, the agent's expected utility is

$$E_{\mu(\theta)}[v(q, \theta)] = \ell(\theta)E_{\mu(\theta)}[h(q)].$$

Then, defining

$$\tilde{v}(x, \theta) = \ell(\theta)x \text{ and } x(\theta) = E_{\mu(\theta)}[h(q)],$$

we can see that $\tilde{v}$ satisfies the single-crossing condition (CS_+). Therefore $x(\theta)$ is implementable if and only if it is nondecreasing. Finally, the essential observation is that $x(\theta)$ implementability is equivalent to $\mu(\theta)$ implementability. This criterion will be used in our examples and is summarized in the next Proposition.

Proposition 1 (Implementability criterion). *Let the agent's utility function be $v(q, \theta) = \ell(\theta)h(q)$, with $\ell'(\cdot) > 0$,*[8] *and consider a probability measure $\mu(\cdot)$ in $\mathcal{Q}$. Then $\mu(\theta)$ is implementable if and only if $E_{\mu(\theta)}[h(q)]$ is nondecreasing.*

2.2. Relaxed Problem

Now we investigate when the deterministic contract is an optimal choice for the principal. Our analysis relies on Mirrlees [6] approach to the problem. The trick is to use the agent's indirect utility function to eliminate the monetary transfer function $t(\theta)$ from the principal's problem. After that, we introduce a relaxed version of problems (P_s) and (P_d), disregarding the incentive-compatibility constraints (3) and (6). Finally we have to check ex-post if the solution of these relaxed problems satisfies the incentive compatibility constraints.

Let us begin with an important definition. Suppose that the stochastic contract (μ, t) is incentive compatible. Then the agent's indirect utility is defined by

$$\mathcal{V}(\theta) = \max_{\theta' \in \Theta} E_{\mu(\theta')}[v(q, \theta)] + t(\theta') = E_{\mu(\theta)}[v(q, \theta)] + t(\theta), \tag{7}$$

and using the envelope theorem from [5], we can get its derivative

$$\mathcal{V}'(\theta) = E_{\mu(\theta)}[v_\theta(q, \theta)]. \tag{8}$$

[8] We have an analogous result when $\ell'(\cdot) < 0$. In this case implementability is equivalent to a nonincreasing $E_{\mu(\theta)}[h(q)]$.

We plug $\mathcal{V}(\cdot)$ into the objective function in (1) and, after that, an integration by parts gives us

$$\int_\Theta E_{\mu(\theta)}[u(q,\theta) + v(q,\theta) - \frac{1-F(\theta)}{f(\theta)} v_\theta(q,\theta)] f(\theta) d\theta. \tag{9}$$

The virtual surplus function is defined by

$$g(q,\theta) = (u(q,\theta) + v(q,\theta) - \frac{1-F(\theta)}{f(\theta)} v_\theta(q,\theta)). \tag{10}$$

Then, the relaxed version of the principal's stochastic problem is simply

$$\max_{\mu(\cdot)} \int_\Theta E_{\mu(\theta)}[g(q,\theta)] f(\theta) d\theta. \tag{P_s^r}$$

Repeating the same procedure, we can define relaxed version of the principal's deterministic problem as

$$\max_{q(\cdot)} \int_\Theta g(q,\theta) f(\theta) d\theta. \tag{P_d^r}$$

The solution of (P_d^r) is called the relaxed solution, denoted by $q_r(\theta)$. We can associate it to the probability measure $\mu_r(\theta)$ defined for all $A \in \mathcal{Q}$ by

$$\mu_r(\theta)(A) = \begin{cases} 1, \text{if } q_r(\theta) \in A, \\ 0, \text{if } q_r(\theta) \notin A, \end{cases} \tag{11}$$

Observe that $\mu_r(\theta)$ is a solution for problem (P_s^r). Moreover, when $q_r(\theta)$ is implementable, it solves problem (P_d) and (P_s). This is the result of the next proposition.[9]

Proposition 2 (Strausz-Jullien). *Let $q_r(\theta)$ be the relaxed solution. Then*

(i) μ_r solves problem (P_s^r).
(ii) When $q_r(\theta)$ is implementable, then $q_r(\theta)$ solves problem (P_d) and μ_r solves problem (P_s).

Proof.

(i) First, observe that $q_r(\theta)$ satisfies

$$q_r(\theta) \in \arg\max_{q \in Q} g(q,\theta).$$

Therefore,

$$\forall \mu(\theta) \in \Delta Q, \quad E_{\mu(\theta)}[g(q,\theta)] \leq g(q_r(\theta),\theta) = E_{\mu_r(\theta)}[g(q,\theta)].$$

[9]Strausz [8] established this result for a finite type set Θ and we are establishing its natural extension for a continuous type set. We mention that Jullien [4] derived a similar result in a more general context.

(ii) There are two affirmatives in this item. For the first affirmative, observe that problem (P_d^r) is less constrained than problem (P_d). So if the relaxed solution is implementable then it is also a solution of (P_r). For the second, we have

$$V(S) \leq \max_{\mu(\cdot)} \int_{\Theta} E_{\mu(\theta)}[g(q,\theta)]d\theta \leq \int_{\Theta} g(q_r(\theta),\theta)d\theta,$$

where the inequality at the left hand side is because problem (P_s^r) is less constrained than problem (P_s). The other inequality comes from item (i). Finally, as $\mu_r(\theta)$ is implementable (because $q_r(\theta)$ is implementable) the result follows.

■

The consequence of Proposition 2 is straightforward. When the relaxed solution is implementable, then the deterministic contract is optimal and there is no gain for the principal in considering stochastic contracts.

3. Examples

We are going to present three examples where the deterministic contract is suboptimal. As we know from Proposition 2, the relaxed solution $q_r(\cdot)$ in these examples cannot be implementable. Our strategy, in the first two examples, is to present a sequence of implementable random variables $\{\tilde{q}_k(\theta)\}_{k>1}$ satisfying

(i) $\lim_{k\to\infty} E[\tilde{q}_k(\theta)] = q_r(\theta)$,

(ii) $\lim_{k\to\infty} \int_0^1 E[g(\tilde{q}_k(\theta),\theta)]f(\theta)d\theta = \int_0^1 g(q_r(\theta),\theta)f(\theta)d\theta.$

In the third example, we consider a discontinuous distribution function, so the objective function is slightly different and we have to replace item *(ii)* by

(ii') $\lim_{k\to\infty} E[V(\tilde{q}_k)] = V(q_r)$,

where V is the new objective function in the relaxed problem, or simply the valuation.[10]

Example 1 (Strict concavity). *The agent's and principal's utility are respectively*

$$v(q,\theta) = (\theta-1)q^3 \text{ and } u(q,\theta) = -\frac{1}{2}q\left((\theta-1)(2+4q^2)+q\right).$$

[10] In the previous examples, the valuation is just $V(\tilde{q}(\cdot)) = \int_0^1 E[g(\tilde{q}(\theta),\theta)]f(\theta)d\theta$. In Example 1, as we are considering a discontinuous distribution function, the objective function will be different.

The agent's type θ is uniformly distributed in $[0,1]$. With these specifications, the virtual surplus

$$g(q,\theta) = (1-\theta)q - \frac{q^2}{2}$$

is strictly concave and the relaxed solution is

$$q_r(\theta) = 1 - \theta.$$

This function is not implementable because it is decreasing and only increasing functions are implementable when we are under single-crossing CS_+. We fix this problem using the ironing procedure. As $q_r(\cdot)$ is decreasing for all $\theta \in \Theta$, the solution will consist in a bunching represented by $q_i(\theta) = \hat{q}$. The optimal bunching is characterized by

$$\int_0^1 \frac{d}{dq} g(\hat{q},\theta) f(\theta) d\theta = 0. \tag{12}$$

Solving equation (12) we get $q_i(\theta) = \frac{1}{2}$. The payoff associated to $q_i(\cdot)$ is

$$\int_0^1 g(q_i(\theta),\theta) f(\theta) d\theta = \frac{1}{8}.$$

We can see in Fig. 1 the relaxed solution $q_r(\theta)$ and the solution of (P_r), denoted by $q_i(\theta)$.

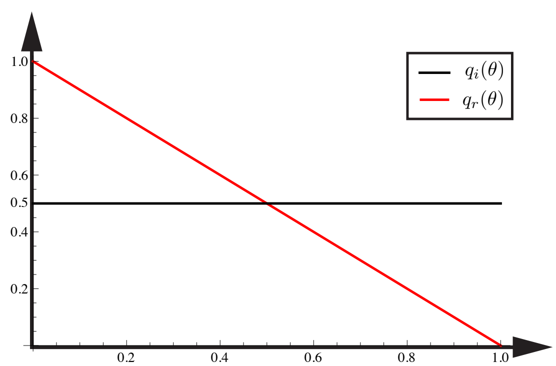

Figure 1: The solutions $q_r(\theta)$ and $q_i(\theta)$.

Now consider a discrete random variable $\tilde{q}_k(\theta)$ taking values in $Q_k = \{q_r(\theta), k\}$ with probabilities $p_k(\theta)$ and $1 - p_k(\theta)$ respectively.[11] The implementability requires that $E[\tilde{q}_k(\theta)^3]$, as a function of θ, is nondecreasing. This is achieved, for

[11]The notation we use is $p_k(\theta) = \Pr(\tilde{q}_k(\theta) = q_r(\theta))$ and $1 - p_k(\theta) = \Pr(\tilde{q}_k(\theta) = k)$.

example, by imposing

$$E[\tilde{q}_k(\theta)^3] = 1. \tag{13}$$

After that, if we solve (13) for $p_k(\theta)$ we find

$$p_k(\theta) = \frac{k^3 - 1}{(\theta - 1)^3 + k^3},$$

and using this probability, we get

$$E[\tilde{q}_k(\theta)] = \frac{-(\theta - 1)k^2 + (\theta - 1)^2 k + 1}{(\theta - 1)^2 + k^2 - \theta k + k}. \tag{14}$$

Taking the limit in (14) we get item(i).

For item(ii), observe that

$$E[g(\tilde{q}_k(\theta), \theta)] = \frac{-\theta + (\theta - 1)^2 k^2 + \left(-2\theta^3 + 6\theta^2 - 6\theta + 1\right) k + 1}{2\left((\theta - 1)^2 + k^2 - \theta k + k\right)},$$

and in the limit, we have

$$\lim_{k \to \infty} E[g(\tilde{q}_k(\theta), \theta)] f(\theta) = \frac{1}{2}(\theta - 1)^2$$

Using the dominated convergence theorem,

$$\lim_{k \to \infty} \int_0^1 E[g(\tilde{q}_k(\theta), \theta)] f(\theta) d\theta = \frac{1}{6} = \int_0^1 g(q_r(\theta), \theta) f(\theta) d\theta,$$

and item(ii) is established.

Example 2 (Non single-crossing). *The agent's and principal's utility are respectively*[12]

$$v(q, \theta) = \frac{1}{3}(\theta - 1)\left(4q^3 - 3q + 1\right), \text{ and}$$
$$u(q, \theta) = \frac{1}{6}\left(-4\theta - 16(\theta - 1)q^3 - 3q^2 + 6(\theta - 1)q + 4\right).$$

We assume that the agent's type θ is uniformly distributed in $[0, 1]$. Notice that the utility function $v(q, \theta)$ does not satisfy the single-crossing condition. Indeed, as we can see in Fig. 2, the (θ, q)-plane can be divided in two regions, CS_+ where $v_{q\theta} > 0$ and CS_- where $v_{q\theta} < 0$. The boundary between these regions is represented by the curve $q_0(\theta) = 1/2$.

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Using equation (10), we get the virtual surplus

$$g(q,\theta) = -\frac{1}{2}q(2\theta + q - 2),$$

and the relaxed solution is simply

$$q_r(\theta) = 1 - \theta.$$

The relaxed solution cannot be implementable because it is decreasing in CS_+, where $v_{q\theta} > 0$[13].

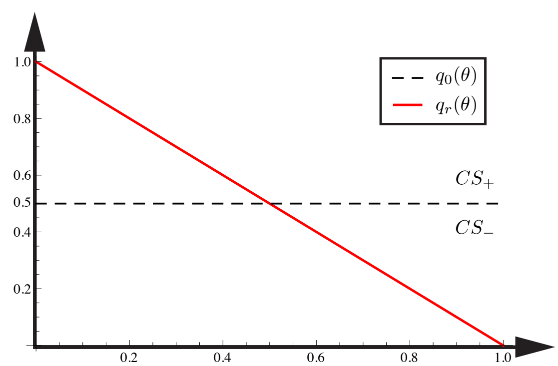

Figure 2: The relaxed solution $q_r(\theta)$ and the $q_0(\theta)$ curve.

Repeating what we did in Example 1, we will define a sequence of implementable random variables $\{\tilde{q}_k(\theta)\}_{k>1}$ taking values in $Q_k = \{q_r(\theta), k\}$ with probabilities $p_k(\theta)$ and $1 - p_k(\theta)$ respectively. If we take

$$p_k(\theta) = \frac{-4k^3 + 3k + 1}{-4\theta^3 + 12\theta^2 - 9\theta - 4k^3 + 3k + 1}.$$

then we get

$$E[\tilde{q}_k(\theta)^3] = \frac{2}{3}.$$

As $E[\tilde{q}_k(\theta)^3]$ is constant, using Proposition 1 we conclude that this sequence of random variables is implementable. The expected value is

$$E[\tilde{q}_k(\theta)] = \frac{-4(\theta-1)k^2 + 4(\theta-1)^2k + 1}{4\theta^2 - 8\theta + 4k^2 - 4(\theta-1)k + 1}.$$

[13] Following [1] one can see that the relaxed solution $q_r(\theta)$ violates a necessary implementability condition.

Taking the limit we get item *(i)*,

$$\lim_{k\to\infty} E[\tilde{q}_k(\theta)] = q_r(\theta)$$

The expected value of the virtual surplus is

$$E[g(\tilde{q}_k(\theta),\theta)] = \frac{-\theta + 4(\theta-1)^2k^2 + \left(-8\theta^3 + 24\theta^2 - 21\theta + 4\right)k + 1}{8\theta^2 - 16\theta + 8k^2 - 8(\theta-1)k + 2}$$

Again, in the limit we have

$$\lim_{k\to\infty} E[g(\tilde{q}_k(\theta),\theta)] = \frac{1}{2}(\theta-1)^2 = g(q_r(\theta),\theta).$$

Finally, using the dominated convergence theorem we get item (ii).

Example 3 (Distribution with atoms). *The agent's and principal's utility are respectively*

$$v(q,\theta) = (\theta-1)q^3 \text{ and } u(q,\theta) = \begin{cases} q^3 - \frac{q^2}{2} + \frac{q}{2}, & \text{if } \theta = 0, \\ \\ -\frac{1}{2}q\left(-2\theta + 4(\theta-1)q^2 + q\right), & \text{if } 0 < \theta \le 1. \end{cases}$$

The probability distribution function is

$$F(\theta) = \begin{cases} 0, & \text{if } \theta < 0, \\ \frac{\theta}{2} + \frac{1}{2}, & \text{if } 0 \le \theta < 1, \\ 1, & \text{if } \theta \ge 1, \end{cases}$$

with a mass point $\theta = 0$. This function can be decomposed as a sum $F(\theta) = F_{ac}(\theta) + F_J(\theta)$, where F_{ac} is absolute continuous with density f_{ac} and F_J is the jump function

$$F_J(\theta) = \begin{cases} 0, & \text{if } \theta < 0, \\ \frac{1}{2}, & \text{if } 0 \le \theta. \end{cases}$$

Now, if we repeat the same derivation we did in subsection 2.2, we get the following

$$V(q(\theta)) = \int_0^1 \hat{g}(q(\theta),\theta) f_{ac}(\theta) d\theta + \hat{g}_0(q) F_J(0),$$

where

$$\hat{g}(q,\theta) = [v(q,\theta) + u(q,\theta) - \frac{(1-F(\theta))}{f_{ac}(\theta)} v_{q\theta}(q,\theta)] = -\frac{1}{4}q(q-2\theta),$$

and

$$\hat{g}_0(q) = v(q,0) + u(q,0) = \frac{q}{2} - \frac{q^2}{2}.$$

To find the relaxed solution, we have to maximize the $\hat{g}(\cdot,\theta)$ and $\hat{g}_0(\cdot)$ in q, and we get

$$q_r(\theta) = \begin{cases} \frac{1}{2}, & \text{if } \theta = 0, \\ \theta, & \text{if } 0 < \theta \le 1. \end{cases}$$

As we are under single-crossing CS_+, the relaxed solution is not implementable. We need to use an ironing procedure to find the optimal solution for the deterministic problem. In this case, the resulting decision is characterized by

$$q_i(\theta) = \begin{cases} q_i, & \text{if } \theta \le q_i, \\ \theta, & \text{if } \theta \ge q_i, \end{cases}$$

and to find q_i we have to solve

$$\int_0^1 \frac{d}{dq}\hat{g}(q_i,\theta) f_{ac}(\theta) d\theta + \frac{d}{dq}\hat{g}_0(q_i) F_J(0) = 0. \tag{15}$$

and solving (15) we get the optimal choice for $q_i = \sqrt{6} - 2$. In Fig. 3 we depict $q_r(\theta)$ and $q_i(\theta)$[14].

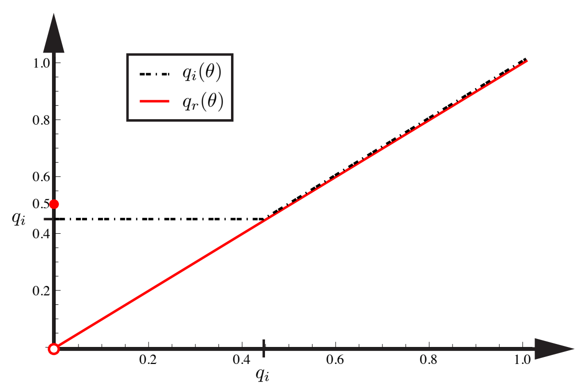

Figure 3: The relaxed solution $q_r(\theta)$ and the solution $q_i(\theta)$ resulting from the ironing procedure.

Now we will define a sequence of implementable random variables $\{\tilde{q}_k(\theta)\}_{k>1}$ taking values in $Q_k = \{q_r(\theta), k\}$ with probabilities $p_k(\theta)$ and $1 - p_k(\theta)$ respec-

[14]Observe that the shape of the q_i solution is just the same as in Hellwig [3] Fig.1.

tively. The probability we use is

$$p_k(\theta) = \begin{cases} 1, & \text{if } \theta = 0, \\ \frac{8k^3-1}{8(k^3-\theta^3)}, & \text{if } 0 < \theta \leq \frac{1}{2}, \\ 1, & \text{if } \theta > \frac{1}{2}. \end{cases}$$

With this specification, we have the implementability of our sequence, because

$$E[\tilde{q}_k(\theta)^3] = \begin{cases} \frac{1}{8}, & \text{if } \theta \leq \frac{1}{2}, \\ \theta^3, & \text{if } \theta > \frac{1}{2}, \end{cases}$$

is increasing. The expected value of $\tilde{q}_k(\theta)$ is

$$E[\tilde{q}_k(\theta)] = \begin{cases} \frac{1}{2}, & \text{if } \theta = 0, \\ \frac{8\theta k^2+8\theta^2 k+1}{8(\theta^2+k^2+\theta k)}, & \text{if } 0 < \theta \leq \frac{1}{2}, \\ \theta, & \text{if } \theta > \frac{1}{2}. \end{cases}$$

For item(i) we take the limit and we get

$$\lim_{k\to\infty} E[\tilde{q}_k(\theta)] = q_r(\theta).$$

For item(ii), we observe that

$$E[V(\tilde{q}_k(\theta)] = \int_0^{1/2} \frac{\theta + 8\theta^2 k^2 + \left(16\theta^3 - 1\right) k}{32\left(\theta^2 + k^2 + \theta k\right)} d\theta + \int_{1/2}^1 \frac{\theta^2}{4} d\theta + \frac{1}{16}.$$

The only term that depends on k satisfies

$$\lim_{k\to\infty} \frac{\theta + 8\theta^2 k^2 + \left(16\theta^3 - 1\right) k}{32\left(\theta^2 + k^2 + \theta k\right)} = \frac{\theta^2}{4}.$$

and using the dominated convergence theorem

$$\lim_{k\to\infty} E[V(\tilde{q}_k(\theta))] = \int_0^1 \frac{\theta^2}{4} d\theta + \frac{1}{16} = V(q_r(\theta)).$$

4. Conclusion

We presented three examples where the deterministic contract is suboptimal. To build them, we considered situations where the relaxed solution is not implementable. In all the examples, the virtual surplus is strict concave. However, this characteristic does not precludes the use of an stochastic contract by the principal. The reason is that randomization allows for a relaxation of the (IC) constraints.

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