A Franks' lemma that preserves invariant manifolds

Nikolaz Gourmelon

March 31, 2008

Résumé

Un célèbre lemme de John Franks dit que toute perturbation de la différentielle d'un difféomorphisme f le long d'une orbite périodique peut être réalisée par une C^1-perturbation g du difféomorphisme sur un voisinage arbitrairement petit de ladite orbite. Ce lemme cependant ne donne aucune information sur le comportement des variétés invariantes de l'orbite périodique après perturbation.

Dans cet article nous montrons que si la perturbation de la dérivée peut être jointe à la dérivée initiale par un chemin, alors la distance C^1 entre f et g peut être trouvée arbitrairement proche du diamètre du chemin. De plus, si des directions stables ou instables d'indices fixés existent le long du chemin, alors les variétés invariantes correspondantes peuvent être préservées en-dehors d'un voisinage arbitrairement petit de l'orbite.

Abstract

A well-known lemma by John Franks asserts that one can realise any perturbation of the derivative of a diffeomorphism f along a periodic orbit by a C^1-perturbation g of the whole diffeomorphism on an arbitrarily small neighbourhood of the periodic point. However, that lemma does not provide any information on the behaviour of the invariant manifolds of the periodic point for g.

In this paper we show that if the perturbated derivative can be joined from the initial derivative by a continuous path, then the C^1-distance between f and g can be found arbitrarily close to the diameter of the path. Moreover, if strong stable or unstable directions of some indices exist along that path, then the corresponding invariant manifolds can be preserved outside a small neighbourhood of the orbit.

1 Introduction

To study the dynamics of C^1-generic diffeomophisms on compact manifolds, that is, diffeomorphisms of a residual subset of the set $\mathrm{Diff}^1(M)$ of C^1 diffeomorphisms, one heavily relies on a few C^1-specific perturbation tools and ideas.

On the one hand, closing and connecting lemmas create periodic points and connecting homoclinically saddle. The C^1-Closing Lemma of Pugh [Pug67] states that a recurent orbit can be closed by an arbitrarily small C^1-perturbation. This was eventually generalized into the C^1-ergodic closing Lemma by Mañé [Mañ82]. Using similar ideas, the connecting lemma of Hayashi [Hay97] states that if the unstable manifold of a saddle point accumulates on a point of the stable manifold of another saddle, then a C^1 perturbation creates a transverse intersection between the two manifolds. That result is further generalized in [WX00], [Arn01] and finally in [BC04] and [Cro06], where remarkable generic consequences are obtained.

On the other hand we have tools to perturb the derivative along a periodic orbit, or to create local dynamical patterns by C^1-perturbations around periodic orbit. John Franks' introduced in [Fra71] a very simple lemma that allows to realise the perturbation of the derivative along a periodic orbit as a C^1-perturbation of the whole diffeomorphism on an arbitrarily small neighbourhood of that orbit. This is the very lemma that systematically allows to reduce C^1-perturbation problems along periodic orbits to linear algebra.

Another perturbation result around a periodic orbit and a consequence of Franks' lemma is for instance the first step of their proof of the Palis C^1-density conjecture in dimension 2 (the union of hyperbolic diffeomorphisms and diffeomorphisms admitting a homoclinic tangency is C^1-dense in the set of diffeomorphisms). Pujals and Sambarino [PS00] first proved that if the dominated splitting between the stable and unstable directions of a saddle point is not strong enough, then a C^1-perturbation of the derivative along the orbit induces a small angle between the two eigendirections. They apply the Franks' Lemma and finally, they do another perturbation to obtain a tangency between the two manifolds. Wen [Wen02] generalized somewhat that first step in dimension greater than 2 under similar non-domination hypothesis.

These perturbations results rely on the Franks' lemma which unfortunately fails to yield any information on the behaviour of the invariant manifolds of the periodic point. In particular, one does not control a priori what homoclinic class the periodic point will belong to, what strong connections it may have after perturbation, and it may not be possible to apply a connecting lemma in order to recreate a broken homoclinic relation. Therefore we naturally ask whether the Franks' perturbation lemma can be tamed into preserving more or less the invariant manifolds of the saddle point.

In [Gou06], a technique is found to preserve any fixed finite set in the invariant manifolds of a periodic point for particular types of perturbations along a periodic orbit. In particular it implies that one can create of homoclinic tangencies inside homoclinic classes on which there is no stable/unstable uniform dominated splitting. This technique however is very complex and difficult to adapt to other contexts.

In this paper, we find a very simple and general context in which we have good control of the invariant manifolds of a saddle point after a perturbation of its derivative. We first state the so-called Franks Lemma:

Lemma (Franks)**.** *Let f be a diffeomorphism. For all $\epsilon > 0$ there is $\delta > 0$ such that, for any periodic orbit X of f, for any δ-perturbation A of the derivative $df_{|X}$ along the orbit X, one finds a C^1 ϵ-perturbation g of f on an arbitrarily small neighbourhood of X such that $dg_{|X} = A$.*

In this paper we provide a perturbation theorem that extends the Franks' Lemma, controlling both the behaviour of the invariant manifolds of X, and the size ϵ of the C^1-perturbation we need to obtain the derivative A. Precisely, we prove that if the perturbation A of $df_{|X}$ is done along a path along which the strong stable/unstable directions of some indices allways exist, then the diffeomorphism g can be chosen in order to preserve the corresponding strong stable/unstable manifolds outside an arbitrarily small neighbourhood. Moreover, the size of the perturbation is given by the length of the path. That theorem is precisely stated in section 2.

We state in section 5 further foreseen generalisations of our perturbation Lemma. If we do not require that the flags of stable/unstable be entirely preserved outside a small neighbourhood of X, but only almost entirely preserved, then one allows the eigenvalues to cross each other along the perturbation path (it is not required any more that the strong stable/unstable directions of fixed indices exist all along the path). Moreover, these times at which several eigenvalues have same moduli enable as many freedoms of choice for the strong stable and unstable manifolds.

Finally in section 6 we claim that requiring the existence of "good" paths is not so constraining. Indeed, many of the cocycle perturbations techniques that we know of are adaptable to building such paths. A few cocycle perturbation statements are proposed as examples. We point out that our result allows another proof of [Gou06].

Remerciements : *Je remercie vivement Flavio Abdenur, Christian Bonatti, Sylvain Crovisier et Lorenzo Díaz pour de nombreuses discussions, suggestions et encouragements ainsi que Marcelo Viana et l'IMPA - Instituto nacional de Matemática Pura e Aplicada (Rio de Janeiro) pour la confiance qui m'a été accordée et pour le soutien financier et matériel dans le cadre de mon Post-Doctorat.*

2 Definitions and statement of results.

In the following f is a C^1-diffeomorphism of a Riemannian manifold M of dimension d, and X is a periodic orbit for f. Let Σ be the vector space of cocycles σ on $TM_{|X}$ that project on $f_{|X}$, that is, such that the following diagram commutes:

$$\begin{array}{ccc} TM_{|X} & \xrightarrow{\sigma} & TM_{|X} \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{f} & X \end{array} .$$

We endow that vector space with the norm $\|\sigma\|_\Sigma = \sup\limits_{\substack{v \in TM \\ \|v\|=1}} \|\sigma(v)\|$. The *eigenvalues* of a cocycle are the eigenvalues of the first return map. When the eigenvalues $\lambda_1 \ldots, \lambda_d$ of a cocycle σ, counted with multiplicity and ordered by increasing moduli, are so that $|\lambda_i| < |\lambda_{i+1}|$ and $|\lambda_i| < 1$, the *i-strong stable direction of dimension i of σ* is the invariant bundle corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_i$. If the cocycle $\sigma = df_{|X}$ has a strong stable direction of dimension i, then the *i-strong stable manifold* of X for f is the unique f-invariant, i-dimensionnal manifold that is tangent to that direction. The *strong unstable manifolds* are naturally defined symmetrically.

Given two finite sets I, J of positive integers, we denote by $\Sigma_{I,J}$ the set of cocyles that are bijective and have a strong stable (resp. unstable) direction of dimension i for all $i \in I$ (resp. $i \in J$). We endow $\Sigma_{I,J}$ with the following distance: for all $\sigma, \tau \in \Sigma_{I,J}$, $\mathrm{dist}(\sigma, \tau) = \max\left(\|\sigma - \tau\|, \|\sigma^{-1} - \tau^{-1}\|\right)$.

Let f be in $\Sigma_{I,J}$. Let $\mathcal{U}$ be a neighbourhood of X.

The *local i-strong stable manifold of X inside $\mathcal{U}$* for f is the set of points of the i-strong stable manifold whose positive iterates remain in $\mathcal{U}$. We denote it by $W^{s,i}_{loc.\mathcal{U}}(f, X)$. The *local i-strong stable manifold* of f *outside* $\mathcal{U}$ is the set of points y of the i-strong stable manifold of f outside $\mathcal{U}$ whose positive orbit does not leave $\mathcal{U}$ once it entered it. We denote it by $W^{s,i}_{loc.\backslash\mathcal{U}}(f, X)$. The *local strong unstable manifolds* are naturally defined symmetrically.

Let g be a perturbation of f such that the cocycles $df_{|X}, dg_{|X}$ are in $\Sigma_{I,J}$. We say that *g preserves locally the i-strong stable manifold* of f *outside* $\mathcal{U}$ if and only if $W^{s,i}_{loc.\backslash\mathcal{U}}(f, X) = W^{s,i}_{loc.\backslash\mathcal{U}}(g, X)$. We say that it *preserves locally the i-strong unstable manifold* of f *outside* $\mathcal{U}$ if and only if $W^{u,i}_{loc.\backslash\mathcal{U}}(f, X) = W^{u,i}_{loc.\backslash\mathcal{U}}(g, X)$. We write that *$g$ preserves locally the (I, J)-strong* *stable/unstable manifolds* of f outside $\mathcal{U}$, if and only if for all $i \in I$ (resp. $j \in J$), it preserves the i-strong stable (resp. the j-strong unstable) manifolds of f outside $\mathcal{U}$.

We can now state the main theorem:

Theorem 2.1. *Assume that $df_{|X}$ is in $\Sigma_{I,J}$, and let $\gamma\colon [0,1] \to \Sigma_{I,J}$ be a path starting at $df_{|X}$. Let $r(\gamma)$ be the radius $\sup_{t\in[0,1]} \operatorname{dist}(df_{|X}, \gamma(t))$ of the path γ. Then there is a perturbation g of f on an arbitrarily small neighbourhood $\mathcal{U}$ of X such that*

- *$dg_{|X} = \gamma(1)$,*
- *the C^1-distance between g and f is arbitrarily close to the radius $r(\gamma)$,*
- *the (I, J)-strong stable/unstable manifolds are preserved locally outside $\mathcal{U}$.*

3 Perturbation propositions

In this section X is still the orbit of a periodic point of the diffeomorphism f. Given two diffeomophisms g, h of M, we write that $h = g$ *locally at* X if they are equal on a neighbourhood of X. Let us state two fundamental perturbation propositions:

Proposition 3.1 ($\mathcal{P}_{I,J}$)**.** *Let $g_k \in \operatorname{Diff}(M)$ be a sequence that converges to $g \in \operatorname{Diff}(M)$ for the C^1-topology, with $dg_{|X}$ and $dg_{k|X}$ in $\Sigma_{I,J}$. For all neighbourhood $\mathcal{U}$ of X, there exists a sequence of diffeomorphisms h_k that converges to g such that*

- *$h_k = g_k$ locally at X,*
- *$h_k = g$ outside $\mathcal{U}$,*
- *for k great enough, the (I, J)-invariant manifolds of g and h_k coincide locally outside $\mathcal{U}$.*

Proposition 3.2 ($\mathcal{P}'_{I,J}$)**.** *Let $g_k \in \operatorname{Diff}(M)$ be a sequence that converges to $g \in \operatorname{Diff}(M)$ for the C^1-topology, with $dg_{|X}$ and $dg_{k|X}$ in $\Sigma_{I,J}$. For all neighbourhood $\mathcal{U}$ of X, there exists a sequence of diffeomorphisms h_k that converges to g such that*

- *$h_k = g$ locally at X,*
- *$h_k = g_k$ outside $\mathcal{U}$,*
- *for k great enough, the (I, J)-invariant manifolds of g_k and h_k coincide locally outside $\mathcal{U}$.*

We are going to prove Propositions $\mathcal{P}_{I,J}$ and $\mathcal{P}'_{I,J}$ (Propositions 3.1 and 3.2) by induction for all pairs I, J of finite sets of strictly positive integers.

Proof of $\mathcal{P}_{\emptyset,\emptyset}$ and $\mathcal{P}'_{\emptyset,\emptyset}$: These are slightly refined Franks' Lemmas. It is enough to take a unit partition $\mu + \nu = 1$ on M such that $\mu = 1$ outside a small neighbourhood of X and $\mu = 0$ in a smaller neighbourhood. Then follow the proof of Franks' Lemma. □

Given two finite sets I, J of strictly positive integers, if they exist, let i_0 and j_0 be respectively the least integer in I and J, and let $I^* = I \setminus \{i_0\}$ and $J^* = J \setminus \{j_0\}$.

Lemma 3.3. *For any subsets $I, J \in \mathbb{N} \setminus \{0\}$ such that $J \neq \emptyset$, Proposition $\mathcal{P}(I, J^*)$ implies Proposition $\mathcal{P}(I, J)$.*

Lemma 3.4. *For any subsets $I, J \in \mathbb{N} \setminus \{0\}$ such that $J \neq \emptyset$, Proposition $\mathcal{P}'(I, J^*)$ implies Proposition $\mathcal{P}'(I, J)$.*

By symmetry of statements, up to changing dynamics to inverse dynamics, we also have that $\mathcal{P}(I^*, J)$ implies $\mathcal{P}(I, J)$, and $\mathcal{P}'(I^*, J)$ implies $\mathcal{P}'(I, J)$. By induction, this implies Propositions 3.1 and 3.2 for all I, J.

Hence we are left to prove Lemmas 3.3 and 3.4. We first introduce a few notations and a regularity result on local invariant manifolds. Let $x \in X$. Given two neighbourhoods $\mathcal{V} \subset \mathcal{U}$ of the orbit X, the local i-strong unstable manifold of the point x inside $\mathcal{U}$ (resp. outside $\mathcal{V}$) is denoted by $W^{u,i}_{loc.\mathcal{U}}(g, x)$ (resp. $W^{u,i}_{loc.\setminus\mathcal{V}}(g, x)$). Their intersection is denoted by $W^{u,i}_{loc.\mathcal{U}\setminus\mathcal{V}}(g, x)$.

Remark 3.5. *While the strong unstable manifolds inside $\mathcal{V}$ are respectively included in the strong unstable manifolds inside $\mathcal{U}$, the strong unstable manifolds outside $\mathcal{V}$ in general do not respectively contain the strong unstable manifolds outside $\mathcal{U}$.*

Definition 3.6. Two neighbourhoods $\mathcal{V} \subset \mathcal{U}$ of the orbit X are said to be *regular for g* if

1. For any diffeomorphism h close enough to g, the local invariant manifolds of X for h outside $\mathcal{V}$ contain the local invariant manifolds of X for h outside $\mathcal{U}$, respectively.
2. The sets $W^{u,i}_{loc.\mathcal{U}\setminus\mathcal{V}}(g, x)$ are submanifolds (with boundary) of M that vary uniformly C^1-continuously by small perturbations of g.

Remark 3.7. *There exist arbitrarily small pairs of regular neighourhoods.*

3.1 Proof of Lemma 3.3

We assume $\mathcal{P}(I, J^*)$. Let $g_k \in \mathrm{Diff}(M)$ be a sequence that converges to $g \in \mathrm{Diff}(M)$ for the C^1-topology, with $dg_{|X}$ and $dg_{k|X}$ in $\Sigma_{I,J}$. In the following, j_0 is the least integer of J.

We can find an arbitrarily small regular pair of compact neighbourhoods $\mathcal{V}, \mathcal{U}$ of X, and a *"regular box"* $\mathcal{B}$ in $\mathcal{U} \setminus \mathcal{V}$ around a pair of consecutive fundamental domains of the j_0-strong unstable manifold of the point x. Precisely, there exist arbitrarily small regular neighbourhoods $\mathcal{U} \supset \mathcal{V} \ni X$ and a set $\mathcal{B} \subset \mathcal{U} \setminus \mathcal{V}$ identified to $\mathbb{S}^{j_0-1} \times [-1, 1[\times [-1, 1]^{d-j_0}$ through a C^1-diffeomorphism, such that

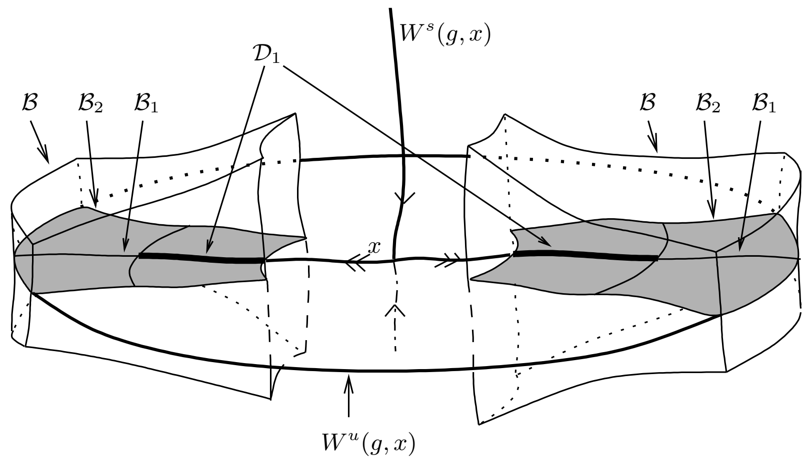
(i) the neighbourhoods $\mathcal{U}, \mathcal{V}$ satisfy the hypothesis of Lemma **??**,

(ii) for all $j \in J$, $\mathcal{B}_j = \mathbb{S}^{j_0-1} \times [-1, 1[\times [-1, 1]^{j-j_0} \times \{0\}^{d-j}$ is the intersection of $\mathcal{B}$ and the local unstable manifold $W^{u,j}_{loc.\mathcal{U}\setminus\mathcal{V}}(g, x)$ in $\mathcal{U}$ and outside $\mathcal{V}$,

(iii) the first half-box $\mathbb{S}^{j_0-1} \times [-1, 0[\times [-1, 1]^{d-j_0}$ is sent at the period on the second half-box by a translation: for all $(a, b, c) \in \mathbb{S}^{j_0-1} \times [-1, 0[\times [-1, 1]^{d-j_0}$, $g^p(a, b, c) = (a, b + 1, c)$, where p is the period of x. In particular $\mathcal{D}_{j_0} = \mathbb{S}^{j_0-1} \times [-1, 0[\times \{0\}^{d-j_0}$ is a fundamental domain of the j_0-strong unstable manifold,

(iv) the closure of $\mathcal{B}$ does not intersect the local stable manifold in $\mathcal{U}$,

Figure 1: Regular box for $J = \{1, 2\}$



When orientation on $W^{u,1}(g)$ is reversed, keep only one of the two connected components.

See a representation in dimension 3 in figure 1. To obtain such a box $\mathcal{B}$, build successively the subsets $\mathcal{B}_j$ knowing that for any pair $j < j'$ in J, the local j-strong unstable manifold of g is a C^1-submanifold of the local j'-strong stable manifold. Finally we obtain $\mathcal{B}$ from the fact that the local unstable manifold of g is a C^1-submanifold of M.

Remark 3.8. *In the particular case $j_0 = 1$ and g preserves orientation on the 1-strong unstable manifold, the 1-fundamental domain $\mathcal{D}_1$ has two connected components (see figure 1), and so has $\mathcal{B}$.*

When $j_0 = 1$ and g reverses orientation on the 1-strong unstable manifold we actually only have one connected component. In the end of this section, we briefly show how to adapt the definition of regular box and the rest of our proof to that particular case.

By $\mathcal{P}(I, J^*)$, there is a sequence of diffeomorphisms h_k^* that converges to g such that

- $h_k^* = g_k$ locally at X,
- $h_k^* = g$ outside $\mathcal{V}$,
- for k great enough, the (I, J^*)-invariant manifolds of g and h_k^* coincide locally outside $\mathcal{V}$.

We will now push the local j_0-strong unstable manifold of h_k^* in $\mathcal{U} \setminus \mathcal{V}$ to coincide with that of g, by a small perturbation on $\mathcal{B}$. We will ensure that this perturbation can be done preserving the (I, J^*)-invariant manifolds of h_k^*. Let j_1 be the least integer in J^*. If J^* is empty, then let $j_1 = d$ and write $\mathcal{B} = \mathcal{B}_d$. Each of the following items hold, for k great enough:

(v) for all $j \in J^*$, $\mathcal{B}_j$ is in the j-strong local unstable manifold $W^{u,j}_{loc.\mathcal{U}\setminus\mathcal{V}}(h_k^*, x)$ in $\mathcal{U}$ and outside $\mathcal{V}$.

(vi) The intersection of the local j_0-strong unstable manifold $W^{u,j}_{loc.\mathcal{U}\setminus\mathcal{V}}(h_k^*, x)$ of h_k^* and $\mathcal{B}_{j_1} = \mathbb{S}^{j_0-1} \times [-1, 1[\times [-1, 1]^{j_1-j_0}$ is the graph of a C^1-function $\phi_k \colon \mathbb{S}^{j_0-1} \times [-1, 1[\to [-1, 1]^{j_1-j_0}$.

(*vii*) For all $(a,b)\in\mathbb{S}^{j_0-1}\times[-1,0[$, we have $\phi_k(a,b)=\phi_k(a+1,b)$.

(*viii*) The closure of $\mathcal{B}$ does not intersect the local stable manifold of x in $\mathcal{U}$ for h_k^*.

Proof : (*v*) comes from (*ii*) and from the (I,J^*)-invariant manifolds of g and h_k^* coinciding locally outside $\mathcal{V}$. (*vi*) is a consequence of (*v*), of the regularity of the neighbourhoods $\mathcal{V}\subset\mathcal{U}$ for g and of h_k^* tending C^1 to g. (*vii*) is a consequence of (*iii*) and of the equality $g=h_k^*$ outside $\mathcal{V}$, in particular on $\mathcal{B}$. (*viii*) is a consequence of (*iv*), of the compactness of $\mathcal{U}$ and of h_k^* tending C^1 to g. □

Since h_k^* tends to g, ϕ_k tends to 0 for the C^1-topology. Let $\rho+\sigma=1$ be a unit partition on $[0,1]$ such that $\rho=0$ on a neighbourhood of -1 and $\rho=1$ on a neighbourhood of 0. Define the map

$$\psi_k\colon\left\{\begin{array}{c}\mathbb{S}^{j_0-1}\times[-1,1[\to[-1,1]^{j_1-j_0}\\ \psi_k(a,b)=-\rho(a).\phi_k(a,b),\text{ for }a\in[-1,0[\\ \psi_k(a,b)=-\sigma(a+1).\phi_k(a,b),\text{ for }a\in[0,1[\end{array}\right.$$

It is well defined and C^1 on $\mathbb{S}^{j_0-1}\times[-1,1[$. Let $\theta\colon\mathcal{B}\to[0,1]$ be a C^1 map such that $\theta=1$ on $\mathbb{S}^{j_0-1}\times[-1,1[\times[-1/4,1/4]^{d-j_0}$ and $\theta=0$ outside $\mathbb{S}^{j_0-1}\times[-1,1[\times[-1/2,1/2]^{d-j_0}$. Then define

$$\Phi_k\colon\left\{\begin{array}{c}\mathcal{B}=\mathbb{S}^{j_0-1}\times[-1,1[\times[-1,1]^{j_1-j_0}\times[-1,1]^{d-j_1}\to\mathcal{B}\\ \Phi_k(a,b,c,d)=(a,b,c+\theta(a,b,c,d).\psi_k(a,b),d)\end{array}\right.$$

For k great enough Φ_k is well defined and is a diffeomorphism of $\mathcal{B}$ that extends the identity map on $M\setminus\mathcal{B}$. Finally we define $h_k=\Phi_k\circ h_k^*$. The sequence h_k tends to g for the C^1 topology and coincides with g outside $\mathcal{U}$.

By (*vi*) and (*vii*), for k great enough, the j_0-strong unstable manifold of X for h_k coincides locally with that of g on the strictly positive iterates of the first half box $\mathbb{S}^{j_0-1}\times[-1,0[\times[-1,1]^{d-j_0}$ until first return in $\mathcal{U}$ (since $g=h_k$ outside $\mathcal{V}\cup\mathcal{B}$). Therefore the local j_0-strong unstable manifold of X for h_k coincides with that of g locally outside $\mathcal{U}$.

Besides, by (*v*) and (*viii*), Φ_k leaves invariant the local (I,J^*)-invariant manifolds of g, for k great enough. Since the (I,J^*)-invariant manifolds of g and h_k^* coincide locally outside $\mathcal{V}$, they coincide outside $\mathcal{U}$ (by regularity of the pair $\mathcal{U},\mathcal{V}$ for g). Therefore the (I,J^*)-invariant manifolds of g and h_k also coincide outside $\mathcal{U}$. Hence the three following items are satisfied:

- $h_k=g_k$ locally at X,
- $h_k=g$ outside $\mathcal{U}$,
- for k great enough, the (I,J)-invariant manifolds of g and h_k coincide locally outside $\mathcal{U}$.

QED.

We now explain how to adapt the regular box for the particular case mentionned in Remark 3.8: we assume $j_0=1$ and g reverses orientation on the 1-strong unstable manifold. Then $\mathcal{B}$ is identified to $[-1,1[\times[-1,1]^{d-j_0}$ and we have to change (*ii*) and (*iii*) into

(*ii'*) for all $j\in J$, $\mathcal{B}_j=[-1,1[\times[-1,1]^{j-j_0}\times\{0\}^{d-j}$ is the intersection of $\mathcal{B}$ and the local unstable $W^{u,j}_{loc.\mathcal{U}\setminus\mathcal{V}}(g,x)$ in $\mathcal{U}$ and outside $\mathcal{V}$,

(iii') the first half-box $[-1,0[\times[-1,1]^{d-j_0}$ is sent at the double period on the second half-box by a translation: for all $(a,b,c) \in [-1,0[\times[-1,1]^{d-j_0}$, $g^{2p}(b,c)=(b+1,c)$, where p is the period of x. In particular $\mathcal{D}_{j_0} = [-1,0[\times\{0\}^{d-j_0}$ is a fundamental domain of the j_0-strong unstable manifold.

The rest of the proof is easily adapted. This concludes the proof of Lemma 3.3.

3.2 Proof of Lemma 3.4

The proof of Lemma 3.4, is very similar to that of Lemma 3.3. We only sketch it. We build again a regular box $\mathcal{B}$ around the j_0-strong stable manifold of x for g. Then instead of pushing local j_0-strong unstable manifold of h_k^* in that box to meet the strong unstable manifold of g, we push it to meet the j_0-strong unstable manifold of g_k.

Remember that g_k tends to g for the C^1-topology, therefore we find for each k a regular box $\mathcal{B}_k$ for g_k, so that the sequence of boxes $\mathcal{B}_k$ tends uniformly to $\mathcal{B}$ for the C^1-topology. Then, the same way as in the previous section, for k great enough we can perturb h_k^* on $\mathcal{B}_k$ into h_k, pushing its j_0-strong unstable manifold on that of g_k, and preserving its strong unstable manifolds of greater dimensions and its strong stable manifolds.

It is easily checked that, since $\mathcal{B}_k$ converges to $\mathcal{B}$ and h_k^* converges to g for the C^1-topology, the size of the perturbation tends to zero, as k-tends to ∞. Therefore the sequence h_k tends to g. This ends the proof of Lemma 3.4.

4 Proof of the main theorem

We fix a family of charts $\{\phi_x \colon B_x \to \mathbb{R}^d\}_{x\in X}$, where B_x is an open ball containing x and the closures $\overline{B_x}$ are pairwise disjoint. We denote by $\mathcal{D}$ the union of these balls. We endow $\mathcal{D}$ with the corresponding canonical linear structure and Euclidean metric. We endow $\mathrm{Diff}^1(M)$ with a Reimannian metric that extends that Euclidean metric.

We have this useful corollary of Proposition 3.1:

Lemma 4.1 (Linearisation)**.** *Let $g \in \mathrm{Diff}(M)$ such that $dg_{|X} \in \Sigma_{I,J}$. then there is an arbitrarily small perturbation h of g on an arbitrarily small neighbourhood $\mathcal{U}$ of X such that $dh = dg$ on X, such that h is locally linear and the (I,J)-invariant manifolds of g and h coincide locally outside $\mathcal{U}$.*

Proof : By unit partitions we find a sequence h_k of locally linear diffeomorphism that tends to g, and such that $dh_k = dg$ on X. Then we apply Proposition 3.1. □

For any cocycle $\sigma \in \Sigma_{I,J}$, we denote by $\widehat{\sigma}$ the linear diffeomorphism it induces from a neighbourhood $\mathcal{C}_\sigma \subset \mathcal{D}$ of X to its image $\mathcal{D}_\sigma \subset \mathcal{D}$. We say that the *$(I,J)$-quasidistance from σ to $\tau \in \Sigma_{I,J}$* is less than $\epsilon > 0$ if for any neighbourhood $\mathcal{U}$ of X there exists a diffeomorphism h from $\mathcal{C}_\sigma$ to $\mathcal{D}_\sigma$ that satisfies the following:

- dh is locally equal to $\widehat{\tau}$ at X,
- h is equal to $\widehat{\sigma}$ outside $\mathcal{U}$,
- The (I,J)-invariant manifolds of $\widehat{\sigma}$ and h coincide outside $\mathcal{U}$.

- the C^1-distance between h and $\widehat{\sigma}$ is less than ϵ.

We denote the infimum of these ϵ by $d_{I,J}(\sigma \to \tau)$. This is a quasidistance: it is positive, separate and satisfies the triangle inequality. Proposition 3.3 implies the following result:

Lemma 4.2. *For all $\sigma \in \Sigma_{I,J}$, for all $\epsilon > 0$, there is a neighbourhood $\Omega \subset \Sigma_{I,J}$ of σ such that the quasidistance $d_{I,J}(\sigma \to \tau)$ is less than ϵ, for any $\tau \in \Omega$.*

Let $\sigma \in \Sigma_{I,J}$ and let $\mathcal{U}$ be a neighbourhood of X in $\mathcal{C}_\sigma$ whose boundary does not intersect that of $\mathcal{C}_\sigma$. Then, if a diffeomorphism h from $\mathcal{C}_\sigma$ to $\mathcal{D}_\sigma$ is equal to $\widehat{\sigma}$ outside $\mathcal{U}$, one can locally conjugate it by a homothety as follows: for any $0 < \lambda < 1$ we denote by h_λ the diffeomorphism from $\mathcal{C}_\sigma$ to $\mathcal{D}_\sigma$ that is equal to $\lambda.Id \circ h \circ \lambda^{-1}.Id$ on $\lambda.\mathcal{U}$ and equal to $\widehat{\sigma}$ outside, where Id is the linear diffeomorphism induced on $\mathcal{C}_\sigma$ by the identical cocycle.

Remark 4.3. *The C^1-distance between h_λ and $\widehat{\sigma}$ is less or equal to the C^1-distance between h and $\widehat{\sigma}$.*

The images of the (I, J)-invariant manifolds of h by the homothety $\lambda.Id$ are in the respective (I, J)-invariant manifolds of h_λ.

Remark 4.4. *Assume that there are fundamental domains of the invariant manifolds of $\widehat{\sigma}$ in $\mathcal{C}_\sigma$ outside $\cup_{0\leq\lambda\leq 1}\lambda.\mathcal{U}$. Then if the (I, J)-invariant manifolds coincide for $\widehat{\sigma}$ and h outside $\mathcal{U}$, the (I, J)-invariant manifolds also coincide for $\widehat{\sigma}$ and h_λ outside $\lambda.\mathcal{U}$.*

Proof of Lemma 4.2 : Let $\mathcal{U} \subset \mathcal{C}_\sigma$ be a neighbourhood of X that satisfies the assumptions of Remark 4.4. It can obviously be chosen arbitrarily small. Let σ_k be a sequence that tends to σ and let $\epsilon > 0$. We have to show that for all k great enough, the quasidistance $d_{I,J}(\sigma \to \tau)$ is less than ϵ.

Let g be a diffeomorphism that extends $\widehat{\sigma}$ on M. By a unit partition we build a sequence g_k of diffeomorphisms that tends to g, such that $dg_k = \sigma_k$ on X. We apply Proposition 3.3 to find a sequence h_k that tends to g such that

- $dh_k = \sigma_k$ on X,
- the restriction $h_{k|\mathcal{C}_\sigma}$ is equal to $\widehat{\sigma}$ outside $\mathcal{U}$,
- for k great enough, the (I, J)-invariant manifolds of $\widehat{\sigma}$ and $h_{k|\mathcal{C}_\sigma}$ coincide outside $\mathcal{U}$.

Let k_0 be such that, for all $k \geq k_0$, the three items above are satisfied and the C^1-distance between $\widehat{\sigma}$ and h_k is less than ϵ. Then, for any $k \geq k_0$, conjugating by any homothety of ratio $0 < \lambda < 1$ and by Remarks 4.3 and 4.4 we have

- $dh_{k,\lambda} = \sigma_k$ on X,
- $h_{k,\lambda|\mathcal{C}_\sigma} = \widehat{\sigma}$ outside $\lambda.\mathcal{U}$,
- the (I, J)-invariant manifolds of $\widehat{\sigma}$ and $h_{k|\mathcal{C}_\sigma}$ coincide outside $\lambda.\mathcal{U}$.
- the C^1-distance between $h_{k,\lambda}$ and $\widehat{\sigma}$ is less than ϵ.

Therefore, the distance the (I, J)-quasidistance from σ to σ_k is less than ϵ, for $k \geq k_0$. QED. □

Lemma 4.5. *For all $\tau \in \Sigma_{I,J}$, for all $\epsilon > 0$, there is a neighbourhood $\Omega \subset \Sigma_{I,J}$ of τ such that, for any $\sigma \in \Omega$, the quasidistance $d_{I,J}(\sigma \to \tau)$ is less than ϵ.*

Proof : The proof is the very similar to that of Lemma 4.2, applying Proposition 3.4 and Remarks 4.3 and 4.4. To be able to apply Remark 4.4 here, notice that for all σ there is a neighbourhood $\mathcal{U}$ of X that satisfies the assumptions of the remark, with respect to any σ' close enough to sigma. □

For all $\sigma, \tau \in \Sigma_{I,J}$, we define the distance $d_{I,J}(\sigma, \tau)$ to be the infimum of $d_{I,J}(\sigma \to \tau)$ and $d_{I,J}(\tau \to \sigma)$. As a direct consequence of Lemmas 4.2 and 4.5, we have

Lemma 4.6. *The metric $d_{I,J}$ is compatible with the topology on $\Sigma_{I,J}$ defined by the metric* dist *in section 2.*

Proof of Theorem 2.1 : Choose a path γ as in the assumptions of the theorem and fix $\epsilon > 0$ and a neighbourhood $\mathcal{U}$ of X. Let ρ be the radius $\sup_t\{\text{dist}(\gamma(t), df_{|X})\}$ of the path. By compactness of the path and Lemma 4.6, we find a sequence $\{\sigma_k\}_{k=0,\dots,n}$, such that $\sigma_0 = \gamma(0) = df_{|X}$ and $\sigma_n = \gamma(1)$, and such that the distance $d_{I,J}(\sigma_k, \sigma_{k+1})$ is strictly less than ϵ, for all $0 \leq k \leq n-1$.

By Lemma 4.1, we find an ϵ-perturbation f_0 of f on an arbitrarily small neighbourhood $\mathcal{U}_0 \subset \mathcal{U}$ of X such that $f_0 = \widehat{\sigma_0}$ on some neighbourhood $\mathcal{U}_1 \subset \mathcal{U}_0$ of X. Then, since $d_{I,J}(\sigma_k \to \sigma_{k+1}) \leq d_{I,J}(\sigma_k, \sigma_{k+1}) < \epsilon$, and applying each time Lemma 4.1, we can build by induction a sequence of open sets $\mathcal{U}_1 \supset ...\mathcal{U}_{n-1} \supset \mathcal{U}_n \supset K$ and a sequence of diffeomorphisms $f = f_0, f_1, ..., f_n$ such that, for all $1 \leq k \leq n$,

- $df_k = \widehat{\sigma_k}$ on $\mathcal{U}_{k+1}$,
- f_k is an ϵ-perturbation of f_{k-1} on $\mathcal{U}_k$ (thus an ϵ-perturbation of $\widehat{\sigma_{k-1}}$, by previous item),
- the (I, J)-invariant manifolds of the restrictions $f_{k-1|\mathcal{U}_0}$ and $f_{k|\mathcal{U}_0}$ coincide outside $\mathcal{U}_k$.

By a straightforward induction, for all k, f_k is a perturbation of f that preserves (I, J)-strong stable/unstable manifolds locally outside $\mathcal{U}_0$.

Choosing $\mathcal{U}_0$ small enough, we may assume that the C^1-distance between $\widehat{\sigma_k}$ and $\widehat{\sigma_0}$ is less than $\delta + \epsilon$, by restriction to $\mathcal{U}_1$. Assume that f_k is a $(\delta + 2\epsilon)$-perturbation of f_0. By restriction to $\mathcal{U}_{k+1}$, it is equal to $\widehat{\sigma_k}$ therefore is $\delta + \epsilon$-close to f_0. Then the diffeomorphism f_{k+1}, which is an ϵ-perturbation of f_k on $\mathcal{U}_{k+1}$, is also a $(\delta + 2\epsilon)$-perturbation of f_0. By induction, we get that f_n is a $(\delta + 2\epsilon)$-perturbation of f_0 on $\mathcal{U}_1$, therefore a $(\delta + 3\epsilon)$-perturbation of f on $\mathcal{U}_0 \subset \mathcal{U}$. This ends the proof of Theorem 2.1. □

5 Further Results

In this paper, we have assumed that the i-strong stable/unstable directions exist at any time t of the homotopy, and we obtain a perturbation lemma that preserves the corresponding invariant manifolds entirely, locally outside of an arbitrarily small neighbourhood.

We now announce two generalisations of this result and a few consequences: assuming only that the i-strong stable/unstable manifold exists at the beginning and the end of the homotopy, and that at any time t the stable/unstable manifold has dimension $\geq i$, we have a perturbation result that preserves the corresponding invariant manifolds *almost* entirely, locally outside of an arbitrarily small neighbourhood.

We need some definitions. Let f be a C^1-diffeomorphism and X be a periodic saddle point for f. Given a fundamental domain of the stable/unstable manifold of x identified diffeomorphically to $\mathbb{S}^{i_s-1} \times [0,1[$, an *annulus* $A(f,X)$ is a subset of the form $\mathbb{S}^{i_s-1} \times [0,\rho[$, where $0<\rho<1$. We say that a perturbation g of f that coincides with f on X *leaves invariant the* (I,J)*-invariant manifolds of* f *on the annulus* A if and only if the (I,J)-invariant manifolds coincide for f and g by restriction to A.

Theorem . *Assume that* $df_{|X}$ *is in* $\Sigma_{I,J}$*, and let* $\gamma\colon [0,1] \to \Sigma$ *be a path starting at* $df_{|X}$ *such that* $\gamma(1) \in \Sigma_{I,J}$ *and for all* $0<t<1$ *the stable/unstable direction of* $\gamma(t)$ *has dimension greater than the maximum element of* I/J*. Then, for any (arbitrarily great) annulus* $A(f,X)$*, there is a perturbation* g *of* f *on an arbitrarily small neighbourhood* $\mathcal{U}$ *of* X *such that*

- $dg_{|X} = \gamma(1)$*,*
- *the* C^1*-distance between* g *and* f *is arbitrarily close to the radius* $r(\gamma)$ *of* γ*,*
- *the* (I,J)*-strong stable/unstable manifolds is preserved on the annulus* A*.*

As a consequence, the (I,J)-strong stable/unstable manifolds can be preserved almost entirely outside an arbitrarily small neighourhood of X.

We can generalise further and see that if at some time t of the path γ, some eigenvalues have same modulus, then one can prescribe the flags of strong stable/unstable manifolds within a range of possible admissible flags. Let γ be a path in Σ. Let $1 \le i_0 < i_1 < \ldots < i_\alpha$ and $1 \le j_0 < j_1 < \ldots < j_\beta$ be two sequences such that

- at any time t, the stable and unstable directions of $\gamma(t)$ have dimensions greater or equal to i_α and j_β, respectively.
- for all $0 \le k \le \alpha-1$ there exists t such that, counting the eigenvalues of $\gamma(t)$ with multiplicity and ordering them by increasing moduli, the i_k-th and the i_{k+1}-th (stable) eigenvalues of $\gamma(t)$ have same modulus.
- for all $0 \le k \le \beta-1$ there exists t such that, counting the eigenvalues of $\gamma(t)$ with multiplicity and ordering them by decreasing moduli, the j_k-th and the j_{k+1}-th (unstable) eigenvalues of $\gamma(t)$ have same modulus.

Such pair of sequences is called an *admissible pair* for the path $\gamma(t)$. Let f be a diffeomorphism such that $df_{|X} = \gamma(0)$. A *stable/unstable invariant flag* for f is a pair of sequences of f-invariant manifolds $\mathcal{W}^{s,1} \subset \ldots \subset \mathcal{W}^{s,k} \subset W^s(f,X)$ and $\mathcal{W}^{u,1} \subset \ldots \subset \mathcal{W}^{u,l} \subset W^u(f,X)$ such that $\mathcal{W}^{s,i}$ (resp. $\mathcal{W}^{u,i}$) is an injectively immersed submanifold of dimension i in $\mathcal{W}^{s,i+1}$ (resp. $\mathcal{W}^{u,i+1}$), topologically equal to $\mathbb{R}^i \times X$, containing X, and C^1 away from X.

A stable/unstable flag $\mathcal{W}^{s,1} \subset \ldots \mathcal{W}^{s,i_\alpha}$ and $\mathcal{W}^{u,1} \subset \ldots \mathcal{W}^{u,j_\beta}$ for (f,γ) is an *admissible stable/unstable flag* if there is an admissible pair $\{i_k\}_{k=0\ldots\alpha}, \{j_l\}_{k=0\ldots\beta}$ such that

- if $i_{k-1} \le i \le i_k$ then $\mathcal{W}^{s,i}$ contains the strong stable manifolds of (f,X) of dimension $\le i_{k-1}$ and is contained in the strong stable manifolds of dimension $\ge i_k$.
- if $j_{k-1} \le j \le j_k$ then $\mathcal{W}^{u,j}$ contains the strong unstable manifolds of (f,X) of dimension $\le j_{k-1}$ and is contained in the strong unstable manifolds of dimension $\ge j_k$.

The next theorem states that we can almost entirely realise any admissible stable/unstable flag for (f,γ), by a perturbation of size arbitrarily close to the radius of the path γ with derivative at X equal to $\gamma(1)$.

Theorem . *Let $\gamma\colon [0,1]\to\Sigma$ be a path starting at $df_{|X}$. Let $\{\mathcal{W}^{s,i}\}_{i=1,\dots,i_\alpha}$ and $\{\mathcal{W}^{u,j}\}_{j=1,\dots,j_\beta}$ be an admissible stable/unstable flag for (f,γ). Then for any annulus A for (f,X), there is a perturbation g of f on an arbitrarily small neighbourhood $\mathcal{U}$ of X such that*

- *$dg_{|X}=\gamma(1)$,*
- *the C^1-distance between g and f is arbitrarily close to the radius $r(\gamma)$,*
- *for all $1\le i\le i_\alpha$, the i-strong stable manifold of g coincides with the manifold $\mathcal{W}^{s,i}$ on the annulus A, if it exists.*
- *for all $1\le j\le j_\beta$, the j-strong stable manifold of g coincides with the manifold $\mathcal{W}^{u,j}$ on the annulus A, if it exists.*

We have for instance the following particular case:

Theorem . *Assume X is hyperbolic for f. Let γ be a path starting at $df_{|X}$ such that the index is constant along it. Assume that $\gamma(1)$ has pairwise distinct eigenvalues, and that $\gamma(1/2)$ has all stable eigenvalues with same modulus. Fix an stable/unstable invariant flag with respect to f.*

Then there is a perturbation g of size close to $r(\gamma)$, on an arbitrarily small neighbourhood of X such that $dg=\gamma_1$, and such that the strong stable/unstable manifolds of g coincide with manifolds of the fixed stable/unstable invariant flag on an arbitrarily great annulus

6 Hints for applications

In this section we announce that the perturbation techniques for linear cocycles as developed in [Mañ82], [BDP00], [Gan04], and [BGV04], successively, can be rewritten in order to take into account the need of a good path between the initial cocycle and the pertubation. The perturbations of cocycles obtained by the techniques of [BGV04] can indeed be done along paths whose size are small.

Let $K\subset M$ be a compact invariant set for a diffeomorphism f. We recall that an invariant splitting $TM_K=E^1\oplus\dots\oplus E^k$ for df is *N-dominated* if and only if for any pair u,v of unit vectors in consecutive bundles E^i,E^{i+1}, we have $\|df(u)\|\le 1/2\|df(v)\|$. A saddle orbit X for f is *N-dominated* if and only if its stable/unstable splitting $TM_X=E^s\oplus E^u$ is N-dominated.

Let us state a few of the foreseeable results:

Proposition . *Let $f\in \mathrm{Diff}^1(M)$, $\epsilon>0$. There exists $N>0$ and $P>0$ such that if a saddle point X of f is not N-dominated and has period greater than P, then the following holds:*

Let I,J be the biggest sets such that $df_{|X}\in\Sigma_{I,J}$. There is a path γ in $\Sigma_{I,J}$ that starts at $df_{|X}$ such that

- *the radius of γ is less than ϵ,*
- *the angle between the stable and unstable direction of $\gamma(1)$ is less than ϵ.*

Note that $\gamma(1)$ is hyperbolic with same index as $df_{|X}$ With theorem 2.1, this proposition allows to create small angles between the stable and unstable manifolds of a periodic saddle point of long period, while preserving its invariant manifolds outside an arbitrarily small neighbourhood of its orbit. In particular, this leads to another proof of the results of [Gou06].

Proposition . *Let $f \in \mathrm{Diff}^1(M)$, $\epsilon > 0$. There exists $N > 0$ such that if a saddle point X of f is not N-dominated, then the following holds:*

Let I, J be the biggest sets such that $df_{|X} \in \Sigma_{I,J}$. Let I^* and J^* be the sets I, J without their respective biggest element. Then there is a path γ that starts at $df_{|X}$ such that

- *the radius of γ is less than ϵ,*
- *either $\gamma(t)$ has constant index for $t \in [0, 1[$*
- *$\gamma(1)$ has an eigenvalue of modulus 1.*

This, with the first theorem announced, allows to create saddle nodes from periodic saddle points with weak domination and strong connections, under some assumptions of volume contraction or dilation along the center bundles.

A consequence of the second theorem that we announced in section 5 is for instance this:

Theorem . *Fix $\epsilon > 0$. There exists $N > 0$ such that if X is a saddle orbit for f with non-trivial homoclinic class, and the weakest and second weakest stable eigendirections (counted with multiplicity) of X are not mutually N-dominated, then there exists an ϵ-perturbation of f on an arbitrarily small neighbourhood of X for which X has a strong homoclinic connection.*

This translates as follows on homoclinic classes:

Corollary . *Let $Hom(X, f)$ be a non trivial homoclinic class, and let $TM_{Hom(X,f)} = E^s \oplus E^u$ be the stable/unstable oseledets splitting. Let $E^s = E^s_1 \oplus ... \oplus E^s_k$ be the finest dominated splitting on E^s. If the weakest stable bundle E^s_k has dimension greater than 2, then there exists an arbitrarily small perturbation that creates a strong connection in $Hom(X, f)$.*

References

[Arn01] Marie-Claude Arnaud. Création de connexions en topologie C^1. *Ergodic Theory Dynam. Systems*, 21(2):339–381, 2001.

[BC04] C. Bonatti and S. Crovisier. Récurrence et généricité. *Invent. Math.*, 158(1):33–104, 2004.

[BDP00] C. Bonatti, L.J. Diaz, and E.R. Pujals. A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math.*, 158(2):355–418, 2000.

[BGV04] C. Bonatti, N. Gourmelon, and T. Vivier. Perturbations of the derivative along periodic orbits. *Prépublication de l'Université de Bourgogne*, (376), 2004.

[Cro06] Sylvain Crovisier. Periodic orbits and chain-transitive sets of C^1-diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.*, (104):87–141, 2006.

[Fra71] J. Franks. Necessary conditions for stability of diffeomorphisms. *Transactions of the A.M.S*, 158(2):302–304, August 1971.

[Gan04] S. Gan. A necessary and sufficient condition for the existence of dominated splitting with a given index. *Trends in Math.*, 7:143–168, 2004.

[Gou06] N. Gourmelon. Creation de tangences homoclines. *En preparation.*, 2006.

[Hay97] Shuhei Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Ann. of Math. (2)*, 145(1):81–137, 1997.

[Mañ82] R. Mañé. An ergodic closing lemma. *Ann. of Math.*, 116:503–540, 1982.

[PS00] E.R. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Ann. of Math.*, 3:961–1023, 2000.

[Pug67] Charles C. Pugh. The closing lemma. *Amer. J. Math.*, 89:956–1009, 1967.

[Wen02] L. Wen. Homoclinic tangencies and dominated splittings. *Nonlinearity*, 15:1445–1469, 2002.

[WX00] Lan Wen and Zhihong Xia. C^1 connecting lemmas. *Trans. Amer. Math. Soc.*, 352(11):5213–5230, 2000.

Nikolaz Gourmelon
nicolas@impa.br
IMPA, Estrada Dona Castorina, 110
Rio de Janeiro, Brasil 22460-320