LINEAR RESPONSE FORMULA FOR PIECEWISE EXPANDING UNIMODAL MAPS

VIVIANE BALADI AND DANIEL SMANIA

ABSTRACT. The average $\mathcal{R}(t) = \int \varphi \, d\mu_t$ of a smooth function φ with respect to the SRB measure μ_t of a smooth one-parameter family f_t of piecewise expanding interval maps is not always Lipschitz [4], [17]. We prove that if f_t is tangent to the topological class of f, and if $\partial_t f_t|_{t=0} = X \circ f$, then $\mathcal{R}(t)$ is differentiable at zero, and $\mathcal{R}'(0)$ coincides with the resummation proposed in [4] of the (a priori divergent) series $\sum_{n=0}^{\infty} \int X(y) \partial_y (\varphi \circ f^n)(y) \, d\mu_0(y)$ given by Ruelle's conjecture. In fact, we show that $t \mapsto \mu_t$ is differentiable within Radon measures. It is the first time that a linear response formula is obtained in a setting where structural stability does not hold. Violation of causality [25] reflects the fact that f_t may be transversal to the topological class of f.

1. INTRODUCTION

Let us call SRB measure for a dynamical system $f : \mathcal{M} \to \mathcal{M}$, on a manifold $\mathcal{M}$ endowed with Lebesgue measure, an f-invariant ergodic probability measure μ so that the set $\{x \in \mathcal{M} \mid \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi \, d\mu\}$ has positive Lebesgue measure, for continuous observables φ. (In fact this defines a *physical measure,* see e.g. [28].) If f_t is a smooth one-parameter family with $f_0 = f$, and each f_t admits a unique SRB measure μ_t, it is natural to ask how μ_t depends on t. More precisely, one studies, for fixed smooth enough φ, the function $\mathcal{R}(t) = \int \varphi \, d\mu_t$.

If f is a sufficiently smooth uniformly hyperbolic diffeomorphism restricted to a transitive attractor, Ruelle [20]–[23] proved that $\mathcal{R}(t)$ is differentiable at $t = 0$ and gave an explicit formula for $\mathcal{R}'(0)$, depending on f_t only through its linear part (the "infinitesimal deformation") $v = \partial_t f_t|_{t=0}$. For obvious reasons, this formula is called the *linear response formula.* We refer to the introductions of [8], [7], [4] for a discussion of more references regarding linear response for hyperbolic dynamical systems, including [7], [6], [11], and applications to statistical mechanics [10].

A much more difficult situation consists in studying nonuniformly hyperbolic interval maps f, e.g. smooth unimodal maps. For some of these maps, in particular those which satisfy the Collet-Eckmann condition, there exists a unique SRB measure μ. Two new difficulties are that structural stability does not hold, and that f_t will not always have an SRB measure even if f has one. In this setting, Ruelle ([24], [25]) has outlined a program, for infinitesimal deformations of the form $v = X \circ f$.

Date: June 5, 2007.

V.B. is partially supported by ANR-05-JCJC-0107-01. D.S. is partially supported by CNPq 470957/2006-9 and 310964/2006-7, FAPESP 2003/03107-9. Both authors thank the organisers of the workshop Topologia e Dinâmica at UFF, Niteroi, Brasil, February 2007, where this work was started. V.B. gratefully acknowledges the warm hospitality of Universidad de la República, Montevideo, Uruguay, where part of this work was done.

He proposed $\Psi(1)$, where

$$\Psi(z) = \sum_{n=0}^{\infty} \int z^n X(y) \partial_y (\varphi \circ f^n)(y) \, d\mu_0(y) \, , \tag{1}$$

is the "susceptibility function," [1] as a candidate for the derivative, in the sense of Whitney's extension, of $\mathcal{R}(t)$ at $t = 0$. (We refer e.g. to the introduction of [4] for more details.) Beware that the series (1) may diverge at $z = 1$ so that $\Psi(1)$ needs to be suitably interpreted.

In this paper, just like in [4], we consider a simpler situation which exhibits however a similar bifurcation structure (in particular structural stability does not hold and infinitely many symbols may be required to code the dynamics): piecewise expanding interval maps. For such maps, it has been known for some time that μ_t exists for all t, and, under mild assumptions, that $\mathcal{R}(t)$ has modulus of continuity $O(t \ln |t|)$ (see (7) below and the references given there). We view the setting of piecewise expanding interval maps as a laboratory in which to test our ideas about smooth deformations. The arguments are free from technicalities, but exhibit most of the features that will appear in the Collet-Eckmann case.

Let us recall now recent results in this piecewise expanding setting. Assuming that $\partial_t f_t|_{t=0} = X \circ f$, a function $(f, X) \mapsto \mathcal{J}(f, X)$ was introduced in [4] (see (41)). There exist ([4], [17]) examples of piecewise expanding unimodal interval maps f_t so that $\mathcal{R}(t)$ is not Lipschitz. For these counterexamples, it turns out that $\mathcal{J}(f, X) \neq 0$. The function $\Psi(z)$ is holomorphic [4] in the open unit disc. In addition, if $\mathcal{J}(f, X) = 0$ and f is Markov (i.e., the postcritical orbit is finite) then $\Psi(z)$ is holomorphic at $z = 1$ ([4]). If $\mathcal{J}(f, X) = 0$ but f is not Markov a resummation Ψ_1 was devised [4] for the possibly divergent series $\Psi(1)$ (see Proposition 4.3 below). In view of the above facts (see also [4, Remark 4.5]), a modification of Ruelle's conjecture, was proposed in [4, Conjecture A] for perturbations of piecewise expanding or Collet-Eckmann f, assuming in addition that each f_t *is topologically conjugated to* f.

The main result of this paper is the proof of Conjecture A from [4] in the piecewise expanding setting. In fact, we prove a slightly stronger result (Theorem 5.1): It is enough to assume that f_t is *tangent* to the topological class of f (see §2.1). Also, the observable φ need only be continuous, so that in fact we prove that $t \mapsto \mu_t$ is differentiable into Radon measures. The interpretation of $\Psi(1)$ in Theorem 5.1 is in the sense of Ψ_1 from [4], and we find a more compact expression for Ψ_1.

Our approach to prove Theorem 5.1 is a perturbative spectral analysis (via resolvents) of transfer operators, on suitable spaces, adapted from those in [4]. [2] (In spirit, this is somewhat similar to the work of Butterley-Liverani [7].) To perform this analysis, we use the Keller-Liverani [14] results together with tools which are classical in dynamics, but not in this framework: the smooth motions (Proposition 2.4) and the twisted cohomological equation for f and $X \circ f$. The novelty of this work resides in the combination of these two ingredients. A key new ingredient in the implementation of our ideas is the use of the isometty G_t in the proof of Theorem 5.1: this isometry is the device which allows us to use the same Banach

[1] Since $\Psi(e^{i\omega})$ is the Fourier transform of the "linear response" [21], it is natural to consider the variable ω, but we prefer to work with the variable $z = e^{i\omega}$.

[2] The spaces in [4] were inspired by what Ruelle told us about his ongoing work on the nonuniformly expanding case [26].

space for the transfer operators of all perturbations, by forcing the singularities (here, jumps) to lie on a prescribed set.

We next summarise informally the picture for piecewise expanding, piecewise smooth unimodal maps (see § 2.1 for assumptions). If the critical point is not periodic, noting $f^0 = \mathrm{id}$, we say that v is horizontal for f if $\sum_{j=0}^{\infty} \frac{v(f^j(c))}{(f^j)'(f(c))} = 0$ (see (9) for the periodic case). Then:

(i) $\mathcal{J}(f, X) = 0$ if and only if X is horizontal for f (Corollary 2.6).
(ii) $X \circ f$ is horizontal for f if and only if the candidate Ψ_1 from [4] for the derivative is well-defined (Proposition 4.3 from [4], Proposition 4.5).
(iii) If f_t is tangent to the topological class of f then $\partial_t f_t|_{t=0}$ is horizontal for f (Corollary 2.6).
(iv) If v is horizontal for f, then any f_t with $\partial_t f_t|_{t=0} = v$ is tangent to the topological class of f. (Theorem 2.8 below, to appear in [5].)
(v) If f_t is stably mixing [3] and tangent to the topological class of f with $\partial_t f_t|_{t=0} = X \circ f$, then $\mathcal{R}(t)$ is differentiable at $t = 0$, and the linear response formula $\mathcal{R}'(0) = \Psi_1$ holds (Theorem 5.1).
(vi) If $\partial_t f_t|_{t=0}$ is not horizontal there are examples where $\mathcal{R}(t)$ is non differentiable ([4], [17], see Remark 5.2).

In view of the results of the present paper, we expect that the following strengthening of Conjecture A [4] in the Collet-Eckmann case holds:

Conjecture A′. Let f be a mixing smooth Collet-Eckmann unimodal map with a nonflat critical point. Let f_t be a smooth perturbation, with $f_0 = f$ and $\partial_t f_t|_{t=0} = X \circ f$, which is *tangent to the topological class of* f (i.e., so that there exists $\tilde{f}_t$ such that $|\tilde{f}_t - f_t| = O(t^2)$ and each $\tilde{f}_t$ is topologically conjugated to f). Then $\mathcal{R}(t)$ is differentiable at 0 *in the sense of Whitney* for all smooth observables φ, and $\mathcal{R}'(0) = \Psi(1)$ (the infinite sum being suitably interpreted).

In particular, if f_t remains in the topological class of a Collet-Eckmann map f, Conjecture A' is just [4, Conjecture A], where differentiability of $\mathcal{R}(t)$ is foreseen in the usual sense. We expect (see Conjecture B in [4]) that paths f_t which are *not* tangent to conjugacy classes give rise to $\mathcal{R}(t)$ which are in general Hölder but not Lipschitz in the sense of Whitney. We believe that in the Markov Collet-Eckmann case a pole of $\Psi(z)$ in the unit disc, i.e., violation of causality [25], holds if and only if $X \circ f$ is not horizontal if and only if f_t is not tangent to the topological class of f. Note that topological classes are called hybrid classes in this context, and they form a well understood lamination for smooth maps with a quadratic critical point (see [15], [2] and references therein).

We hope that injecting in our argument tools analogous to those developed by Ruelle [26] in the nonrecurrent nonuniformly hyperbolic setting (in particular spaces of sums of smooth functions and sums of "spikes") should eventually give a proof of Conjecture A'.

This work is about the linear response. One can also wonder about formulas for the derivatives of higher order of $\mathcal{R}(t)$ (see [22]). Indeed, we expect that a suitable modification of the proof of Theorem 5.1 will give, if f_t is a C^{r_0, r_0+1} perturbation, tangent up to order $r_0 - 1$ to the topological class of a stably mixing piecewise

[3] Beware that if f is not stably mixing, then there exist f_t with $\partial_t f_t|_{t=0} = X \circ f$ horizontal and $\Psi(z)$ holomorphic at 0, but $\mathcal{R}(t)$ not Lipschitz.

expanding unimodal map f (i.e., we replace $|f_t - \tilde{f}_t| = 0(t^2)$ by $O(t^{r_0})$ for $r_0 \geq 3$ in § 2.1), that $\mathcal{R}(t)$ has a Taylor series of degree $r_0 - 1$ at 0, with explicit coefficients (in the spirit of [22]). The coefficients will be related to twisted cohomological equations for derivatives of higher order of h_t (see the proof of Proposition 2.4). In the Collet-Eckmann setting, if f_t is tangent to the hybrid class of f up to order $r_0 - 1$, then we expect that higher order derivatives and Taylor series of degree $r_0 - 1$ should be attainable, of course in the sense of Whitney. (If f_t lies in the hybrid class, we expect a Taylor series in the usual calculus sense.)

The paper is organised as follows: Section 2 contains definitions, and the essential result on the "smooth motions" $h_t(x)$ (Proposition 2.4). The infinitesimal conjugacy $\alpha = \partial_t h_t|_{t=0}$ is introduced there. In Section 3, we recall the decomposition of the invariant density from [4], we adapt results from [14] on the transfer operators to reduce from families tangent to the topological class to families within the topological class (Proposition 3.3), and we introduce appropriate spaces $\mathcal{B}_t$ for transfer operators (Subsection 3.3) of sums of a "smooth" function with a sum of jumps along the postscritical orbit. In Section 4, we recall information from [4] on the susceptibility function $\Psi(z)$ and the candidate Ψ_1 for the derivative of $\mathcal{R}(t)$. We prove Theorem 5.1 in Section 5, combining the main ingredients (Proposition 2.4, Proposition 3.3, and the spectral analysis on the function spaces $\mathcal{B}_t$ from Subsection 3.3). The proof uses strongly the perturbation theory from Keller and Liverani [14] (we need to extend their result slightly, see Appendix B). Finally, Section 6 contains (Theorem 6.2) a simpler formula for $\mathcal{R}'(0)$, which is true if and only if α is absolutely continuous (a rare event).

2. The setting, the twisted cohomological equation and the infinitesimal conjugacy α

2.1. **Piecewise expanding C^r unimodal maps and their perturbations.** If $K \subset \mathbb{R}$ is a compact interval and $\ell \geq 0$, we let $C^\ell(K)$ denote the set of functions on K which extend to C^ℓ functions in an open neighbourhood of K. In this work, we consider the following objects:

Definition. For an integer $r \geq 1$, a *piecewise expanding C^r unimodal map* is a continuous map $f : I \to I$, where $I = [a, b]$, so that f is strictly increasing on $I_+ = [a, c]$, strictly decreasing on $I_- = [c, b]$ ($a < c < b$), with $f(a) = f(b) = a$; and for $\sigma = \pm$, the map $f|_{I_\sigma}$ extends to a C^r map on a neighbourhood of I_σ, with [4] $\inf |f'|_{I_\sigma}| > 1$.
A piecewise expanding C^r unimodal map f is *good* if either c is not periodic under f or $\inf |(f^{n_1})'| > 2$, where $n_1 \geq 2$ is the minimal period of c; it is *mixing* if f is topologically mixing on $[f^2(c), f(c)]$.

Beware that a piecewise expanding C^r unimodal map f is only continuous, and never C^1 (it is piecewise C^r). We restrict to unimodal (as opposed to multimodal) to avoid unessential combinatorical difficulties.

Given a piecewise expanding C^r unimodal map f, we shall use the following notation: The point c will be called the *critical point* of f. We write $c_k = f^k(c)$ for $k \geq 0$. We say that c is *preperiodic* if it is not periodic but there exist $n_0 \geq 1$ and $n_1 \geq 1$ so that c_{n_0} is periodic of minimal period n_1 (we take n_0 minimal for

[4] A prime denotes derivation with respect to $x \in I$, a priori in the sense of distributions.

this property and our assumptions imply $n_0 \ge 2$). If c is periodic for f of minimal period $n_1 \ge 2$ we set (by convention) $n_0 = 1$. If c is preperiodic or periodic for f, we set

$$N_f := n_0 + n_1 - 1 \ge 2 \,. \tag{2}$$

(If c is periodic we have $N_f = n_1$.) If c is neither preperiodic nor periodic for f, we set $N_f = \infty$.

Define $J := (-\infty, f(c)]$ and $\chi : \mathbb{R} \to \{0, 1, 1/2\}$ by

$$\chi(x) = 0 \text{ if } x \notin J\,, \quad \chi(x) = 1 \text{ if } x \in \operatorname{int} J\,, \quad \chi(f(c)) = \frac{1}{2}\,. \tag{3}$$

The two inverse branches of f, a priori defined on $[f(a), f(c)]$ and $[f(b), f(c)]$, may be extended to maps $\psi_+ : J \to \mathbb{R}_-$ and $\psi_- :\to \mathbb{R}_+$ in $C^r(J)$, with $\sup|\psi'_\sigma| < 1$ for $\sigma = \pm$. We set

$$\lambda_0 = \lim_{n\to\infty} (\sup(|(f^{-n})'|))^{1/n}\,, \quad \Lambda_0 = \lim_{n\to\infty} (\sup|(f^n)'|)^{1/n}\,. \tag{4}$$

and choose

$$\lambda \in (\lambda_0, 1)\,, \quad \Lambda > \Lambda_0\,.$$

Definition. Let $r \ge r_0 \ge 2$ be integers. For a piecewise expanding C^r unimodal map f, a $C^{r_0,r}$ *perturbation* of f is a family of piecewise expanding C^r unimodal maps $f_t : I \to I$, $|t| < \epsilon$, with $f_0 = f$, and satisfying the following properties: There exists a neighbourhood $\mathcal{I}_\sigma$ of I_σ, $\sigma = \pm$, so that the C^r norm of the extension of $f_t|_{I_\sigma}$ to $\mathcal{I}_\sigma$ is uniformly bounded for small $|t|$, and so that

$$\|(f - f_t)|_{\mathcal{I}_\sigma}\|_{C^{r-1}} = O(t)\,. \tag{5}$$

The map $(x,t) \mapsto f_t(x)$, extends to a C^{r_0} function on a neighbourhood of $(I_+ \cup I_-) \times \{0\}$. The *infinitesimal deformation* of the perturbation f_t is defined by

$$v = \partial_t f_t|_{t=0}\,. \tag{6}$$

Our assumptions imply that the infinitesimal deformation satisfies $v(a) = v(b) = 0$ and, if $f(c) = b$, also $v(c) = 0$.

If f_t is a $C^{2,2}$ perturbation of a piecewise expanding C^2 unimodal map, then each f_t (for small enough t) admits an absolutely continuous invariant probability measure (see e.g. [4] for references), with a density ρ_t which is of bounded variation. In fact, there is only one absolutely continuous invariant probability measure. Each ρ_t is continuous on the complement of the at most countable set $\{f_t^k(c) \mid k \ge 1\}$, and it is supported in $[f_t^2(c), f_t(c)] \subset [a,b]$ (we extend it by zero on $\mathbb{R}$). If f is good and mixing, then f_t is mixing and the absolutely continuous invariant measure is mixing. (If f is mixing, but not good, f_t need not be mixing.) In other words, assuming that f is good and mixing implies that f is stably mixing (we do not claim the converse), in addition, denoting by $|\varphi|_{L^1(Leb)}$ the $L^1(\mathbb{R}, \text{Lebesgue})$ norm of φ, by [14, Prop. 7] (by uniform Lasota-Yorke estimates, see [14, Remarks 1, 5]), we have

$$|\rho_t - \rho_0|_{L^1(Leb)} = 0(t \ln|t|)\,. \tag{7}$$

If f is not good, the function $t \mapsto \rho_t$ need not be continuous. (This is germane to the fact that mixing is not necessarily preserved if f is not good. See [13] for an illuminating multimodal example.) See also Remark 3.4.

Remark 2.1. Note that Ruelle's conjecture offers a candidate for the derivative of

$$\mathcal{R}(t) = \int \varphi\, \rho_t\, dx \tag{8}$$

only if $\partial_t f_t|_{t=0} = X \circ f$. (See also Remark 4.1.)

Definition. For integers $r \geq r_0 \geq 2$, and a piecewise expanding C^r unimodal map f, a $C^{r_0,r}$ *perturbation of* f *tangent to the topological class of* f is a $C^{r_0,r}$ perturbation f_t of f so that there exist a $C^{2,2}$ perturbation $\tilde f_t$ of f with

$$\sup_x |\tilde f_t(x) - f_t(x)| = O(t^2)$$

and homeomorphisms h_t with $h_t(c) = c$ and $\tilde f_t = h_t \circ f \circ h_t^{-1}$.

Clearly, if f_t is a $C^{2,2}$ perturbation of f tangent to the topological class of f, then $v = \partial_t f_t|_{t=0} = \partial_t \tilde f_t|_{t=0}$. We shall see (Corollary 2.6) that the infinitesimal deformations v of tangent perturbations are *horizontal* for f:

Definition. A continuous $v : I \to \mathbb{R}$ is *horizontal* [5] for a piecewise expanding C^1 unimodal map f if, setting $M_f = n_1$ if c is periodic of minimal period $n_1 \geq 2$, and $M_f = +\infty$ otherwise,

$$\sum_{j=0}^{M_f - 1} \frac{v(c_j)}{(f^j)'(c_1)} = 0\,. \tag{9}$$

See also Subsection 2.3 for a discussion of perturbations f_t tangent to the topological class of f.

When considering $C^{2,2}$ perturbations f_t, we have in particular $\sup_x |f_t'(x) - f'(x)| = o(1)$ (considering the extensions to neighbourhoods of I_σ) and we shall implicitly restrict to ϵ small enough so that

$$\begin{aligned} &\sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup(|(f_t^{-n})'|))^{1/n} < \lambda\,, \quad \sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup(|(\tilde f_t^{-n})'|))^{1/n} < \lambda\,, \\ &\sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup|(f_t^{n})'|)^{1/n} < \Lambda\,, \quad \sup_{|t|<\epsilon} \lim_{n\to\infty} (\sup|(\tilde f_t^{n})'|)^{1/n} < \Lambda\,. \end{aligned} \tag{10}$$

2.2. **The twisted cohomological equation, the smooth motions $h_t(x)$, and the infinitesimal conjugacy α.** In this section, we discuss the following *twisted cohomological equation* (TCE, see e.g. [27]) for piecewise expanding unimodal f and bounded v:

$$v(x) = \alpha(f(x)) - f'(x)\alpha(x)\,, \quad \forall x \in I\,, x \neq c\,. \tag{11}$$

Let us start with an easy lemma:

Lemma 2.2. *Assume that f is a piecewise expanding C^1 unimodal map and that v is a bounded function on I. Then for every $\omega \in \mathbb{R}$ the unique bounded solution $\alpha_{(\omega)}$ to (11) which satisfies $\alpha_{(\omega)}(c) = \omega$ is given by:*

$$\alpha_{(\omega)}(x) = \begin{cases} -\sum_{j=0}^{\infty} \frac{v(f^j(x))}{(f^{j+1})'(x)}\,, & \text{if } f^j(x) \neq c\,, \forall j \geq 0\,, \\ \frac{\omega}{(f^\ell)'(x)} - \sum_{j=0}^{\ell-1} \frac{v(f^j(x))}{(f^{j+1})'(x)} & \text{if } \exists \ell \geq 1 \text{ s.t. } f^\ell(x) = c\,. \end{cases} \tag{12}$$

[5] See [15], [2] and references therein for a motivation of this terminology.

Remark 2.3. If (11) admits a continuous solution α, it is easy to see by taking limits as $x \to c$ from the left and from the right that $\alpha(c) = 0$ and $v(c) = \alpha(c_1)$. (In particular, there is at most one continuous solution to (11).) We shall not use this.

Proof. For x so that $f^\ell(x) \neq c$ for all $\ell \geq 0$ (12) defines a bounded solution uniquely on this set: Indeed any bounded solution satisfies $\beta = -v/f' - \ldots - v \circ f^{k-1}/(f^k)' + \beta \circ f^{k+1}/(f^k)'$; if $\beta(x) \neq \alpha_{(\omega)}(x)$, then we take k so that $K/(f^k)' < (\beta(x) - \alpha_{(\omega)}(x))/3$ with $K = \max(\sup|\beta|, \sup|\alpha_{(\omega)}|)$, and we get a contradiction. If $\beta(c) = \omega$, then for each x so that $f^\ell(x) = c$ we must have $\beta(x) = \alpha_{(\omega)}(x)$ as defined in (12). $\square$

When v is the infinitesimal deformation of a perturbation f_t tangent to the topological class of f we shall relate solutions to (11) to the conjugacies h_t. The key ingredient for this is the following information about the smoothness of $t \mapsto h_t$:

Proposition 2.4. *Let $r_0 \geq 2$ be an integer. Assume that $\tilde f_t$ is a C^{r_0,r_0} perturbation of a piecewise expanding C^{r_0} unimodal map f, so that for each small t there exists a homeomorphism h_t with $h_t(c) = c$ and $\tilde f_t = h_t \circ f \circ h_t^{-1}$. Then for small enough ϵ, the maps $t \mapsto h_t(x)$ are $C^{r_0-1+Lip}$ on $[-\epsilon, \epsilon]$, uniformly in $x \in I$. (I.e. $\sup_x \|h_\cdot(x)\|_{C^{r_0-1+Lip}([-\epsilon,\epsilon])} < \infty$.)*

Remark 2.5. Although the $h_t(x)$ cannot be called "holomorphic motions" (see e.g. [2]) they certainly be called "smooth motions"! Beware that the maps $t \mapsto h_t^{-1}(x)$ are in general not C^{1+Lip}, although it is easy to see that the map $t \mapsto h_t^{-1}(x)$ is differentiable at $t = 0$ with derivative $-\alpha(x)$ for all $x \in I$. Also, the maps $x \mapsto h_t(x)$, $x \mapsto h_t^{-1}(x)$ are in general not absolutely continuous (see Section 6).

It will then be easy to show:

Corollary 2.6. *Under the assumptions of Proposition 2.4 the bounded function $\alpha : I \to \mathbb{R}$ defined by $\alpha(x) = \partial_t h_t(x)|_{t=0}$ satisfies the TCE (11) for $v = \partial_t f_t|_{t=0}$. In addition, $\alpha(c) = 0$ and $v(c) - \alpha(c_1) = 0$, so that v is horizontal for f.*

Definition. Under the assumptions of Proposition 2.4, the function $\alpha = \partial_t h_t|_{t=0}$ is the *infinitesimal conjugacy* associated to the infinitesimal deformation v of f_t.

Remark 2.7. It follows from Corollary 2.6 that if f_t is a perturbation of f and $v = \partial_t f_t|_{t=0}$ is not horizontal for f, then there exist arbitrarily small t so that f and f_t are *not* topologically conjugated, in particular f is not structurally stable. See [1] for an analogous statement about rational maps.

Proof of Proposition 2.4. To simplify notation, we assume that $c = 0$ in this proof. Let $\mathcal{P}_t$ be the set of points which are either periodic or eventually periodic for $\tilde f_t$, and whose forward orbit under $\tilde f_t$ does not contain the turning point c. It is easy to see that $\mathcal{P}_t$ is dense in I. Let $\theta = \sup_{x,t} |\tilde f_t'(x)|^{-1}$. We first prove that $(t, x) \to h_t(x)$ is continuous. Fix (x_0, t_0) and let $\kappa > 0$. Pick $n \in \mathbb{N}$ and $\delta > 0$ such that $\theta^n + \frac{\delta}{1-\theta} < \kappa$. Choose $\eta_0 < \epsilon/2$ small enough such that if $|t - t_0| < \eta_0$ then

$$\sup_x |\tilde f_t(x) - \tilde f_{t_0}(x)| < \delta\,,$$

and let η_1 be such that $|x - x_0| < \eta_1$ implies $f^k(x) \cdot f^k(x_0) \geq 0$, for every $k \geq n$. So $\tilde f_t^k(h_t(x)) \cdot \tilde f_t^k(h_t(x_0)) \geq 0$, for every $k \geq n$ and t. Of course $\tilde f_t^k(h_t(x_0)) \cdot$

$\tilde f^k_{t_0}(h_{t_0}(x_0)) \ge 0$. By Lemma A.1, for every $(t,x) \in \{|t-t_0|<\eta_0\}\times\{|x-x_0|<\eta_1\}$ we have

$$|h(t,x) - h(t_0,x_0)| \le \kappa\,.$$

In the remainder of this proof, $\partial^i_t h_t$ denotes $\partial^i_s h_s|_{s=t}$. The implicit function theorem tells us that if $p \in \mathcal{P}_0$ then $t \to h_t(p)$ is a C^{r_0} function. Differentiating the equation $h_t \circ f(p) = \tilde f_t \circ h_t(p)$ with respect to t we obtain

$$\partial_t h_t \circ f(p) = \partial_t \tilde f_t \circ h_t(p) + \tilde f'_t(h_t(p))\partial h_t(p)\,. \tag{13}$$

In other words

$$\partial_t h_t \circ f(p) - \tilde f'_t(h_t(p))\partial h_t(p) = \partial_t \tilde f_t \circ h_t(p) = F_1(p)\,.$$

Next, differentiating (13) r_0 times, we can easily prove that for each $i \le r_0$

$$\partial^i_t h_t \circ f(p) - \tilde f'_t(h_t(p))\partial^i_t h_t(p) = F_i(p)\,, \tag{14}$$

where the function F_i is a polynomial combination of compositions of (all) partial derivatives of $\tilde f_t(x)$ up to order i, including mixed ones, with the function h_t, and partial derivatives $\partial^j_t h_t$, for $j = 1, \ldots, i-1$.

For every $q \in \mathcal{P}_t$, we have $q = h_t(p)$, with $p \in \mathcal{P}_0$. Define

$$\alpha^i_t(q) := \partial^i_t h_t(h^{-1}_t(q)).$$

Define $Q_i(q) = F_i(h^{-1}_t(q))$. From (14) we obtain the twisted cohomological equation

$$Q_i(q) = \alpha^i_t(\tilde f_t(q)) - \tilde f'_t(q)\cdot \alpha^i_t(q)\,. \tag{15}$$

Let call this equation TCE_i.

Note that F_1 is bounded on $\mathcal{P}_0$. We claim that

$$|F_i|_\infty < \infty$$

for every $i \le r_0$. Indeed, suppose by induction that F_ℓ and $\partial^{\ell-1}_t h_t$ are bounded functions on $\mathcal{P}_0$, for every $\ell \le i < r_0$. Then Q_i is bounded on $\mathcal{P}_t$, and the unique solution for TCE_i on $\mathcal{P}_t$ is given by the expression

$$\alpha^i_t(q) = -\sum_{j=0}^{\infty} \frac{Q_i(\tilde f^j_t(q))}{(\tilde f^{j+1}_t)'(q)}\,.$$

The uniqueness of the solution follows from the fact that every point in $\mathcal{P}_t$ is eventually periodic.

In particular

$$\sup_{q\in\mathcal{P}_t} |\alpha^i_t(q)| \le \frac{|Q_i|_\infty}{1-\sup_x|\tilde f'_t(x)|^{-1}}\,. \tag{16}$$

It follows that $\partial^i_t h_t$ is bounded on $\mathcal{P}_0$, and hence F_i is bounded in the same domain. This concludes the inductive argument.

Then from (16) we have an upper bound for $|\partial^i_t h_t|$, for $i \le r_0$, which is uniform on $t \in [-\epsilon,\epsilon]$ (up to taking a smaller ϵ). So the family of functions $t \to h_t(p)$, with $p \in \mathcal{P}_0$ and $t \in [-\epsilon,\epsilon]$, is a bounded subset of $C^{r_0}([-\epsilon,\epsilon])$.

We claim that $t \mapsto h_t(x)$ is $C^{r_0-1+Lip}$ for every $x \in I$. Indeed, let $p_n \in \mathcal{P}_0$ be a sequence which converges to x. Of course the sequence of functions $t \mapsto h_t(p_n)$ converges to the function $t \mapsto h_t(x)$. Since every sequence in a bounded subset of $C^{r_0}([-\epsilon,\epsilon])$ has a subsequence which converges to a function in $C^{r_0-1+Lip}$, we conclude that $t \mapsto h_t(x)$ is $C^{r_0-1+Lip}$. □

Proof of Corollary 2.6. By differentiating $\tilde f_t \circ h_t = h_t \circ f$ with respect to t at $t = 0$, we see that $\alpha(x)$ satisfies (11) at all $x \neq c$. Since $h_t(c) = c$ for all c we have $\alpha(c) = 0$. To prove $v(c) = \alpha(c_1)$, we use $\tilde f_t \circ h_t(c) = h_t \circ f(c)$: The derivative with respect to t of the right-hand-side at $t = 0$ is just $\alpha(c_1)$. This implies that the left-hand-side is differentiable at $t = 0$, and, using $h_t(c) = c$, the derivative is

$$\lim_{t\to 0} \frac{\tilde f_t(h_t(c)) - \tilde f_t(c)}{t} + \lim_{t\to 0} \frac{\tilde f_t(c) - f(c)}{t} = 0 + v(c)\,.$$

□

2.3. **Perturbations f_t tangent to the topological class of f.** For $r \geq 2$ and a fixed piecewise expanding C^r unimodal map f, we may pick $h_t(x)$ with $h_t(c) = c$, so that $(x,t) \mapsto h_t(x)$ is C^r, and define $\tilde f_t := h_t \circ f \circ h_t^{-1}$. Then $\tilde f_t$ is a $C^{r,r}$ perturbation of f in its topological class. If we assume in addition that $h_t(c+x) = \mathcal{S} h_t(c-x)$, where the ($C^r$) symmetry $\mathcal{S}$ is such that $f(c+x) = f(\mathcal{S}(c-x))$, we can ensure that the infinitesimal deformation is of the form $v = X \circ f$. Since $x \mapsto h_t(x)$ is a diffeomorphism in this construction, it gives a conjugacy between the invariant densities $\tilde\rho_t$ of $\tilde f_t$ and ρ_0 of f. Thus differentiability of $\widetilde{\mathcal{R}}(t) = \int \varphi \tilde\rho_t \, dx$ can be obtained by relatively easy perturbation theory arguments on the transfer operator. Theorem 5.1 applies to *all* smooth perturbations f_t which are tangent to $\tilde f_t$, and we may choose f_t in such a way as to ensure that f_t and f are not topologically conjugated (by modifying the kneading invariant), or are not smoothly conjugated (by acting on the multipliers [16]).

In view of a more general and systematic description of perturbations tangent to the topological class, recall that Corollary 2.6 implies that if a $C^{2,2}$ perturbation f_t of a C^2 map f is tangent to the topological class of f, then its infinitesimal deformation v is horizontal. In the smooth nonuniformly hyperbolic case (see [15], [2] and references therein) a converse to this statement holds. The proof of the converse in our setting will appear elsewhere:

Theorem 2.8. *(See [5]) For $r_0 \geq 2$, let f be a piecewise expanding C^{r_0} unimodal map and let $v \in C^{r_0}(I)$ be horizontal for f and satisfy $v(a) = 0$, $v(b) = 0$, and, if $f(c) = b$, also $v(c) = 0$. Then there exists a family of piecewise expanding C^{r_0} unimodal maps $\tilde f_t : I \to I$, $|t| < \epsilon$, with $\tilde f_0 = f$, so that the map $(x,t) \mapsto \tilde f_t(x)$, extends to a $C^{r_0 - 1 + Lip}$ function on a neighbourhood of $(I_+ \cup I_-) \times \{0\}$, and, in addition, $\partial_t \tilde f_t|_{t=0} = v$, and for each t there is a homeomorphism h_t with $h_t(c) = c$ and $\tilde f_t = h_t \circ f \circ h_t^{-1}$. The conjugacies h_t are in general not absolutely continuous.*

In particular, the above implies that any $C^{2,r}$ perturbation f_t of a piecewise expanding C^r unimodal map f ($r \geq 2$) so that $v = \partial_t f_t|_{t=0}$ is horizontal and $v \in C^2(I)$ is tangent to the topological class of f.

Note that there exist (many) $C^{2,r}$ perturbations f_t of mixing piecewise expanding C^r unimodal maps, and such that $v = \partial_t f_t|_{t=0}$ is C^r and horizontal (also if we require $v = X \circ f$). Indeed, the functional $L_f : v \mapsto v(c) - \alpha_{(0)}(c_1)$ is bounded and linear from $\{v \in C^r(I)\}$ to $\mathbb{R}$. So it has a codimension-one kernel.

3. Transfer operators and their spectra

3.1. **Definitions and previous results.** Recall that a point x is called regular for a function ϕ if $2\phi(x) = \lim_{y\uparrow x} \phi(y) + \lim_{y\downarrow x} \phi(y)$. If ϕ_1 and ϕ_2 are functions of bounded variation on $\mathbb{R}$ having at most regular discontinuities, the Leibniz formula says that $(\phi_1\phi_2)' = \phi_1'\phi_2 + \phi_1\phi_2'$, where both sides are a priori finite measures. (Viewing a function ϕ in BV as a measure means considering $\phi\, dx$.)

For a piecewise expanding C^2 unimodal map f, recalling (3), we introduce two linear operators:

$$\mathcal{L}_0\varphi(x) := \chi(x)\varphi(\psi_+(x)) - \chi(x)\varphi(\psi_-(x))\,, \tag{17}$$

and

$$\mathcal{L}_1\varphi(x) := \chi(x)\psi_+'(x)\varphi(\psi_+(x)) + \chi(x)|\psi_-'(x)|\varphi(\psi_-(x))\,. \tag{18}$$

Note that $\mathcal{L}_1$ is the usual (Perron-Frobenius) transfer operator for f, in particular, $\mathcal{L}_1\rho_0 = \rho_0$ and $\mathcal{L}_1^*(\text{Lebesgue}_{\mathbb{R}}) = \text{Lebesgue}_{\mathbb{R}}$. The operators $\mathcal{L}_0$ and $\mathcal{L}_1$ both act boundedly on the Banach space

$$BV = BV^{(0)} := \{\varphi : \mathbb{R} \to \mathbb{C} \mid \text{var}(\varphi) < \infty\,, \text{supp}(\varphi) \subset [a,b]\}/\sim\,,$$

endowed with the norm $\|\varphi\|_{BV} = \inf_{\phi\sim\varphi} \text{var}(\phi)$, where var denotes total variation and $\varphi_1 \sim \varphi_2$ if the bounded functions φ_1, φ_2 differ on an at most countable set. To get finer information on $\mathcal{L}_0$, we consider the smaller Banach space (see e.g. [19])

$$BV^{(1)} = \{\varphi : \mathbb{R} \to \mathbb{C} \mid \text{supp}(\varphi) \subset (-\infty, b]\,, \varphi' \in BV\}\,,$$

for the norm $\|\varphi\|_{BV^{(1)}} = \|\varphi'\|_{BV}$. If $\mathcal{L}$ is a bounded linear operator on a Banach space $\mathcal{B}$, we denote the spectrum of $\mathcal{L}$ by $\text{sp}(\mathcal{L})$, and we define $R_{ess}(\mathcal{L})$, the essential spectral radius of $\mathcal{L}$, to be

$$R_{ess}(\mathcal{L}) = \inf\{R \geq 0 \mid \text{sp}(\mathcal{L}) \cap \{|z| > R\} \text{ consists of isolated eigenvalues of finite multiplicity}\}\,.$$

Recalling the definition (4) of λ_0, we have the following key lemma (see [4], the claims on $\mathcal{L}_1$ on BV are classical):

Lemma 3.1. *Assume that f is a mixing piecewise expanding C^2 unimodal map. The essential spectral radius of $\mathcal{L}_1$ on BV is $\leq \lambda_0$. In addition, 1 is a maximal eigenvalue of $\mathcal{L}_1$, which is simple, for the eigenvector ρ_0, and there are no other eigenvalues of $\mathcal{L}_1$ of modulus 1 on BV. The spectral radius of $\mathcal{L}_0$ on BV is equal to 1. For any $\varphi \in BV^{(1)}$*

$$(\mathcal{L}_0\varphi)' = \mathcal{L}_1(\varphi')\,. \tag{19}$$

Finally, the spectrum of $\mathcal{L}_0$ on $BV^{(1)}$ and that of $\mathcal{L}_1$ on BV coincide.

For further use, associate to a mixing piecewise expanding C^2 unimodal map f

$$\tau_0 = \max\Big(\lambda_0\,,\ \sup\{|z| \mid z \in \text{sp}\,(\mathcal{L}_1|_{BV})\,,\ z \neq 1\}\Big)\,, \tag{20}$$

(note that $\tau_0 < 1$), and choose

$$\tau \in (\tau_0, 1)\,.$$

Set $H_u(x) = -1$ if $x < u$, $H_u(x) = 0$ if $x > u$ and $H_u(u) = -1/2$. If f is a piecewise expanding C^2 unimodal map, the invariant density of f is of bounded variation and thus decomposes uniquely [18] as $\rho_0 = \rho_{sal} + \rho_{reg}$ with ρ_{reg} continuous and ρ_{sal} the *saltus term* (recalling N_f from § 2.1):

$$\rho_{sal} = \sum_{n=1}^{N_f} s_n H_{c_n}\,, \tag{21}$$

with $s_n = \lim_{y\downarrow c_n} \rho_0(y) - \lim_{x\uparrow c_n} \rho_0(x)$. By [4, Prop. 3.3] we have[6]:

Proposition 3.2. *Let f be a mixing piecewise expanding C^3 unimodal map. Then ρ_{reg} from the decomposition (21) of the invariant density is an element of $BV^{(1)}$.*

3.2. **Comparing the invariant densities of two tangent perturbations.** Our main result is about perturbations f_t which are tangent to the topological class of f_0. In this subsection, we prove Proposition 3.3 (using classical Banach spaces, and tools from Keller-Liverani [14]) which will allow us to reduce from this assumption to the hypothesis that f_t lies in the topological class of f_0.

We need more notation. Let f_t be a $C^{2,r}$ perturbation of a piecewise expanding C^r unimodal map ($r \ge 2$) Define $J_t := (-\infty, f_t(c)]$ and $\chi_t : \mathbb{R} \to \{0, 1, 1/2\}$ by

$$\chi_t(x) = 0 \text{ if } x \notin J_t\,, \quad \chi_t(x) = 1 \text{ if } x \in \operatorname{int} J_t\,, \quad \chi_t(f_t(c)) = \frac{1}{2}\,.$$

The two inverse branches of f_t, a priori defined on $[f_t(a), f_t(c)]$ and $[f_t(b), f_t(c)]$, may be extended to maps $\psi_{t,+} : J_t \to (-\infty, c]$ and $\psi_{t,-} : J_t \to [c, \infty)$ in $C^r(J_t)$, with $\sup |\psi'_{t,\sigma}| < 1$ for $\sigma = \pm$. Put

$$\mathcal{L}_{1,t}\varphi(x) := \chi_t(x)\psi'_{t,+}(x)\varphi(\psi_{t,+}(x)) + \chi_t(x)|\psi'_{t,-}(x)|\varphi(\psi_{t,-}(x))\,. \tag{22}$$

Recall our choices $\lambda < 1$ from (4) and $\tau < 1$ from (20). Lemma 3.1 applies to $\mathcal{L}_{1,t}$. By [14] we may assume that t is small enough so that

$$\max\bigl(\lambda, \sup_t \sup\{|z| \mid z \in \operatorname{sp}(\mathcal{L}_{1,t}|_{BV})\,,\ z \ne 1\}\bigr) < \tau\,.$$

We may now state the new result of this subsection:

Proposition 3.3. *Let f be a good mixing piecewise expanding C^2 unimodal map. Then for any $C \ge 1$ and every pair (f_t, g_t) of $C^{2,2}$ perturbations of f, and so that*

$$\sup_x |f_t(x) - g_t(x)| \le Ct^2\,,\ \forall |t| \le \epsilon\,, \tag{23}$$

there exist $C_1 \ge 1$, $\epsilon_0 > 0$ and $\xi > 1$ so that, letting ρ_t and $\tilde\rho_t$ denote the respective invariant densities of f_t and g_t, we have

$$\|\rho_t - \tilde\rho_t\|_{L^1(Leb)} \le C_1|t|^\xi\,, \quad \forall |t| \le \epsilon_0\,.$$

Remark 3.4. The assumption that f is good is crucial in the above proposition since otherwise we do not have uniform Lasota-Yorke bounds (26) in general.

Proof. Recall $\lambda < 1$ from (4) (we require that (10) hold for g_t too). Denote by $\mathcal{L}_{1,t}$ the transfer operator of f_t, by $\widetilde{\mathcal{L}}_{1,t}$ the transfer operator of g_t, acting on BV. Each $\mathcal{L}_{1,t}$ and each $\widetilde{\mathcal{L}}_{1,t}$ has a simple maximal eigenvalue at $z = 1$ and essential spectral radius $\le \lambda$ for small enough t. Our assumptions ensure that

$$\|f_t(x)\|_{C^{1+Lip}(V)} \le C\,, \quad \|g_t(x)\|_{C^{1+Lip}(V)} \le C\,, \tag{24}$$

on a neighbourhood V of $(I_+ \cup I_-) \times \{0\}$. Also, there exist $\widetilde{C}$ and $\epsilon_1 > 0$ depending only on f and C so that (our assumptions imply that g_t and f_t satisfy (5))

$$\begin{aligned} &\sup_j \|\mathcal{L}^j_{1,t}\|_{L^1(Leb)} < \widetilde{C}\,, \quad \sup_j \|\tilde{\mathcal{L}}^j_{1,t}\|_{L^1(Leb)} < \widetilde{C}\,,\ \forall |t| \le \epsilon_1\,,\\ &\|\mathcal{L}_{1,t}(\varphi) - \mathcal{L}_{1,0}(\varphi)\|_{L^1(Leb)} \le \widetilde{C}|t|\|\varphi\|_{BV}\,,\ \forall \varphi \in BV\,,\ \forall |t| \le \epsilon_1\,,\\ &\|\tilde{\mathcal{L}}_{1,t}(\varphi) - \mathcal{L}_{1,0}(\varphi)\|_{L^1(Leb)} \le \widetilde{C}|t|\|\varphi\|_{BV}\,,\ \forall \varphi \in BV\,,\ \forall |t| \le \epsilon_1\,. \end{aligned} \tag{25}$$

[6]The proof there does not require that c is not periodic.

also, since f is good [14, Remark 5],

$$\max(\|\mathcal{L}^j_{1,t}\varphi\|_{BV}, \|\tilde{\mathcal{L}}^j_{1,t}\|_{BV}) \le \widetilde{C}\lambda^j\|\varphi\|_{BV} + \widetilde{C}\|\varphi\|_{L^1}\,,\ \forall\varphi\in BV\,,\ \forall|t|\le\epsilon_1\,, \tag{26}$$

finally, (24) and (23) imply $\|(f_t-g_t)|_{I_\sigma}\|_{C^1} = O(t^2)$, with a constant depending only on f and C, and thus

$$\|\mathcal{L}_{1,t}(\varphi)-\tilde{\mathcal{L}}_{1,t}(\varphi)\|_{L^1(Leb)} \le \widetilde{C}t^2\|\varphi\|_{BV}\,,\ \forall\varphi\in BV\,,\ \forall|t|\le\epsilon_1\,. \tag{27}$$

It follows from (25–26) for $\widetilde{\mathcal{L}}_{1,t}$, $\mathcal{L}_1$, and [14, Theorem 1] that for each small enough $\delta>0$ there are $\epsilon_2>0$ and $\widehat{C}\ge 1$, depending only on f and C so that

$$\|(z-\widetilde{\mathcal{L}}_{1,t})^{-1}\|_{BV}\le\widehat{C}\,,\ \forall|t|\le\epsilon_2\,,\forall z \text{ with } |z|\ge\tau+\delta \text{ and } |z-1|\ge\delta\,. \tag{28}$$

We claim that the above estimate together with (27) implies $\|\rho_t-\tilde\rho_t\|_{L^1(Leb)} = O(|t|^{2\eta})$ for any $\eta<1$. Taking η so that $2\eta>1$, the claim ends the proof.

To obtain the claim, we revisit the proof of [14, Theorem 1]. Following Keller–Liverani, we put $\mathcal{Q}_t=(z-\mathcal{L}_{1,t})$ and $\widetilde{\mathcal{Q}}_t=(z-\widetilde{\mathcal{L}}_{1,t})$. In the sense of formal power series in z, we have for all $|t|\le\epsilon$

$$\mathcal{Q}_t^{-1}-\widetilde{\mathcal{Q}}_t^{-1}=\mathcal{Q}_t^{-1}(\mathcal{L}_{1,t}-\widetilde{\mathcal{L}}_{1,t})\widetilde{\mathcal{Q}}_t^{-1}\,. \tag{29}$$

By (28) and (27), the second part of the proof of [14, Theorem 1] gives that for any $\eta<1$ and $\gamma>0$, there are constants $\epsilon_0>0$, $\widetilde{A}\ge 1$, $\widetilde{B}\ge 1$, depending only on η, $\widetilde{C}$ and γ, so that for any z satisfying $|z|\ge\tau+\gamma$ and $|z-1|\ge\gamma$, all $\varphi\in BV$, and all $|t|\le\epsilon_0$,

$$\begin{aligned}\|\mathcal{Q}_t^{-1}(\varphi)\|_{L^1(Leb)} &\le 2(t^2)^\eta\big(\widetilde{A}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}+\widetilde{B}\big)\|\varphi\|_{BV}\\ &\quad + 2(t^2)^{\eta-1}\left(\widetilde{C}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}+\frac{\widetilde{C}}{1-\tau}\right)\|\varphi\|_{L^1(Leb)}\,.\end{aligned} \tag{30}$$

Applying the above estimate to $(\mathcal{L}_{1,t}-\widetilde{\mathcal{L}}_{1,t})\widetilde{\mathcal{Q}}_t^{-1}(\varphi)$ and using (29), we get

$$\begin{aligned}&\|(\mathcal{Q}_t^{-1}-\widetilde{\mathcal{Q}}_t^{-1})(\varphi)\|_{L^1}\\ &\quad\le 2|t|^{2\eta}(\|\mathcal{L}_{1,t}\|_{BV}+\|\widetilde{\mathcal{L}}_{1,t}\|_{BV})\big(\widetilde{A}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}+\widetilde{B}\big)\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}\|\varphi\|_{BV}\\ &\qquad + 2C|t|^{2\eta}\left(\widetilde{C}\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}+\frac{\widetilde{C}}{1-\tau}\right)\|\widetilde{\mathcal{Q}}_t^{-1}\|_{BV}\|\varphi\|_{BV}\,,\end{aligned} \tag{31}$$

for any $\varphi\in BV$. Writing the difference between the spectral projectors for the eigenvalue 1 of $\mathcal{L}_{1,t}$ and $\widetilde{\mathcal{L}}_{1,t}$ as a contour integral of the difference of the resolvents, this shows the claim. $\square$

3.3. **Spaces of sums of smooth functions and postscritical jumps.** In this subsection we shall introduce Banach spaces $\mathcal{B}_t\subset BV$ and $\mathcal{B}_t^{Lip}\subset BV$ of functions with controlled jumps along the postscritical orbit, on which the transfer operators $\mathcal{L}_{1,t}$ have essential spectral radius $\le\lambda$, in view of the proof of our main theorem in Section 5.

Let f be a mixing piecewise expanding C^3 unimodal map. Recall that $N_f = n_0+n_1-1$ if c is preperiodic, $N_f=n_1$ if c is periodic, and $N_f=\infty$ otherwise. Let $\widetilde{BV}$ be the Banach space of continuous functions of bounded variation supported in $[a, b]$, for the BV norm. Fix $\eta > 0$ small. Consider the Banach space $(\widehat{\mathcal{B}}, \|\cdot\|)$ of pairs $\phi = (\phi_{reg}, \phi_{sal})$ with $\phi_{reg} \in \widetilde{BV}$, and $\phi_{sal} = (u_k)_{k=1,\ldots,N_f}$, normed by

$$\|\phi\| = \|\phi_{reg}\|_{BV} + |\phi_{sal}|_\eta \text{ with } |\phi_{sal}|_\eta = \sup_{1\le k\le N_f} (1+\eta)^k |u_k| \,, \tag{32}$$

and so that, in addition,

$$\phi_{reg}(x) = \sum_{k=1}^{N_f} u_k \,,\ \forall x < a \,. \tag{33}$$

We define $\Gamma = \Gamma_0 : \widehat{\mathcal{B}} \to BV$ by

$$\Gamma(\phi_{reg}, (u_k)_{k\ge1}) = \phi_{reg} + \sum_{k=1}^{N_f} u_k H_{c_k} \,. \tag{34}$$

(In particular, $\mathrm{supp}(\Gamma(\phi)) \subset [a, b]$.) The map Γ is injective, and we define $\mathcal{B}_0 \subset BV$ to be the isometric image of $\widehat{\mathcal{B}}$ under Γ.

It is easy to see that $\rho_0 \in \mathcal{B}_0$. For $\phi = (\phi_{reg}, (u_k)_{k\ge1}) \in \widehat{\mathcal{B}}$, we may decompose $\tilde\varphi = \mathcal{L}_1(\Gamma(\phi)) \in BV$ into $\tilde\varphi = \tilde\varphi_{reg} + \tilde\varphi_{sal}$. Then, we have

$$\tilde\varphi_{sal} = \sum_{k\ge1} w_k H_{c_k} \,,$$

with (writing $f'(c_-) = \lim_{y\uparrow c} f'(y)$ and $f'(c_+) = \lim_{y\downarrow c} f'(y)$)

$$\begin{cases} w_k = \frac{u_{k-1}}{f'(c_{k-1})} \,, & k \ge 2 \,, \\ w_1 = -(\frac{1}{|f'(c_-)|} + \frac{1}{|f'(c_+)|})\big(\phi_{reg}(c) + \sum_{k\ge1, c_k>c} u_k\big) \,, \end{cases} \tag{35}$$

if the postscritical orbit is infinite (i.e., $N_f = \infty$), while

$$\begin{cases} w_k = \frac{u_{k-1}}{f'(c_{k-1})} \,, & 2 \le k \le N_f \,, k \ne n_0 \\ w_{n_0} = \frac{u_{n_0-1}}{f'(c_{n_0-1})} + \frac{u_{n_0+n_1-1}}{f'(c_{n_0+n_1-1})} \,, & \text{if } n_0 \ne 1 \,, \\ w_1 = -(\frac{1}{|f'(c_-)|} + \frac{1}{|f'(c_+)|})\big(\phi_{reg}(c) + \sum_{k\ge1, c_k>c} u_k\big) \,, \end{cases} \tag{36}$$

if $N_f < \infty$. Also, we find

$$\begin{aligned} \tilde\varphi_{reg} = {} & \mathcal{L}_1(\phi_{reg}) \\ & + H_{c_1}\left(\frac{1}{|f'(c_-)|} + \frac{1}{|f'(c_+)|}\right) \cdot \left(\phi_{reg}(c) + \sum_{1\le k\le N_f, c_k>c} u_k\right) \\ & + \sum_{k=2}^{N_f} u_{k-1}\left(\mathcal{L}_1(H_{c_{k-1}}) - \frac{H_{c_k}}{f'(c_{k-1})}\right) . \end{aligned} \tag{37}$$

It is thus not difficult to check that $\tilde\varphi \in \mathcal{B}_0$. We next prove that in fact $\mathcal{L}_1$ is bounded on $\mathcal{B}_0$ with essential spectral radius $\le \lambda$.

We shall use that if $\mathcal{L}$ is a bounded operator on a Banach space $\mathcal{B}$, and $\mathcal{K}$ is a compact operator on $\mathcal{B}$, then the essential spectral radii of $\mathcal{L}$ and $\mathcal{L} - \mathcal{K}$ coincide (see e.g. [9] or [12, Theorem IV.5.35]). This fact is behind most techniques to estimate the essential spectral radius: Lasota-Yorke or Doeblin-Fortet bounds, Hennion's theorem, the Nussbaum formula, see e.g. [3]. In view of this, recall that the BV-closed unit ball is compact for the $L^1(Leb)$ norm. (See e.g. [3, §3.2, Prop. 3.3] for a proof of this Arzelà-Ascoli type result). In view of obtaining compact perturbations if $N_f = \infty$, note that for any $\delta > 0$ there is $k_\delta = O(\ln(\delta^{-1}))$ so that for any $\phi = (\phi_{reg}, (u_k)_{k\ge 1}) \in \widehat{\mathcal{B}}$,

$$\sum_{k\ge k_\delta} |u_k| \le \delta \sup_{k\ge 1}\big((1+\eta)^k |u_k|\big) . \tag{38}$$

For $\varphi \in BV$, we write $\Pi_{reg}(\varphi) = \varphi_{reg} \in C^0$ and $\Pi_{sal}(\varphi) = \varphi_{sal}$. If $N_f \ne \infty$, the operator $\mathcal{K}_0(\varphi) = \Pi_{sal}(\mathcal{L}_1(\varphi))$ is finite rank on $\mathcal{B}_0$, and thus compact. If $N_f = \infty$, the operator

$$\mathcal{K}_0(\varphi) = -H_{c_1}(\varphi_{reg}(c) + \sum_{k\ge 1, c_k > c} u_k)(|f'(c_-)|^{-1} + |f'(c_+)|^{-1})$$

is rank one, and thus compact, while the operator $\Pi_{sal} \circ (\mathcal{L}_1 - \mathcal{K}_0)$ has norm bounded by $(1+\eta)\sup|f'|^{-1}$ by definition.

We next consider $\Pi_{reg} \circ \mathcal{L}_1$. If $N_f < \infty$, the second and third lines of (37) are finite rank contributions, which will be denoted by $\mathcal{K}_1(\phi)$. If $N_f = \infty$, since

$$\sup_{k\ge 2} \left\| \mathcal{L}_1(H_{c_{k-1}}) - \frac{H_{c_k}}{f'(c_{k-1})} \right\|_{BV} < \infty ,$$

then (38) implies that the second and third line of (37) give a compact contribution, also denoted by $\mathcal{K}_1(\phi)$.

Then, consider the Radon measure $(\Pi_{reg} \circ \mathcal{L}_1(\varphi) - \mathcal{K}_1(\varphi))'$. By the Leibniz formula we have, as Radon measures,

$$\begin{aligned}(\Pi_{reg} \circ \mathcal{L}_1(\varphi) - \mathcal{K}_1(\varphi))'(y) = \chi_J \bigg(&\frac{f''(\psi_+(y))}{(f'(\psi_+(y)))^2}\varphi(\psi_+(y)) - \frac{f''(\psi_-(y))}{(f'(\psi_-(y)))^2}\varphi(\psi_-(y)) \\ &+ \frac{\varphi'(\psi_+(y))}{(f'(\psi_+(y)))^2} - \frac{\varphi'(\psi_-(y))}{(f'(\psi_-(y)))^2} \bigg) .\end{aligned} \tag{39}$$

By the compact inclusion property mentioned above, the contribution φ_1 in the first line is compact, let us call $(\mathcal{K}_2(\varphi))' = \varphi_1$ the corresponding operator. Now, the operator $\varphi' \mapsto \mathcal{M}(\varphi') = (\Pi_{reg} \circ \mathcal{L}_1(\varphi) - \mathcal{K}_1(\varphi) - \mathcal{K}_2(\varphi))'$ is bounded on measures, with norm at most $\sup(|f'|^{-1})\|\mathcal{L}_1\|_\infty$ where $\|\mathcal{L}_1\|_\infty$ is the operator norm of $\mathcal{L}_1$ acting on bounded functions. Applying the above argument to $\mathcal{L}_1^j$, and using $\sup_j \|\mathcal{L}_1^j\|_\infty < \infty$, we obtain for each $j \ge 1$ a decomposition $\mathcal{L}_1^j = \mathcal{K}^{(j)} + \mathcal{M}^{(j)}$ where $\mathcal{K}^{(j)}$ is compact on $\mathcal{B}_0$, and $\|\mathcal{M}^{(j)}\|_{\mathcal{B}_0} \le C_0(1+\eta)^j \sup(|(f^j)'|^{-1})$. Therefore, the essential spectral radius of $\mathcal{L}_1$ on $\mathcal{B}_0$ is $\le \lambda$.

Consider now the Banach space $(\widehat{\mathcal{B}}^{Lip}, \|\cdot\|)$ of pairs $\phi = (\phi_{reg}, \phi_{sal})$ with $\phi_{reg} \in Lip((-\infty, b])$, and $\phi_{sal} = (u_k)_{k=1,\dots,N_f}$, normed by $\|\phi\| = \|\phi_{reg}\|_{Lip} + |\phi_{sal}|_\eta$ and so that $\phi_{reg}(x) = \sum_{k=1}^{N_f} u_k$ for all $x < a$ (in particular, ϕ_{reg} is constant on $(-\infty, a)$). Using Γ as above, we define a Banach space $\mathcal{B}_0^{Lip} \subset \mathcal{B}_0 \subset BV$. Since $\|\phi\|_{Lip} = \|\phi'\|_{L^\infty}$ and since the $Lip([a,b])$- closed unit ball is compact in the $L^\infty([a,b])$ topology, the same argument as above shows that $\mathcal{L}_1$ is bounded on $\mathcal{B}_0^{Lip}$, with essential spectral radius $\le \lambda$. Since $BV^{(1)} \subset Lip$, we have that $\rho_0 \in \mathcal{B}_0^{Lip}$.

If f_t is a $C^{2,3}$ perturbation of f we may define $\mathcal{B}_t$ and $\mathcal{B}_t^{Lip}$ for each t by taking the isometric image in BV of $\widehat{\mathcal{B}}$, respectively $\widehat{\mathcal{B}}^{Lip}$ under Γ_t defined by

$$\Gamma_t\Big(\phi_{reg}, (u_k)_{k\ge 1})\Big) = \phi_{reg} + \sum_{k=1}^{\infty} u_k H_{c_{k,t}} .$$

The argument above shows that $\mathcal{L}_{1,t}$ has essential spectral radius bounded by λ on $\mathcal{B}_t$ and $\mathcal{B}_t^{Lip}$. Since each $\mathcal{B}_t$ and each $\mathcal{B}_t^{Lip}$ is a subset of BV and since $\rho_t \in \mathcal{B}_t^{Lip} \subset \mathcal{B}_t$, we have proved that outside of the disc of radius τ the spectrum of $\mathcal{L}_{1,t}$ on $\mathcal{B}_t$ or on $\mathcal{B}_t^{Lip}$ consists in a simple eigenvalue at 1, with corresponding spectral projector $\varphi \mapsto \rho_t \int \varphi\, dx$.

4. The susceptibility function and the candidate Ψ_1 for the derivative

The susceptibility function [25] associated to a piecewise expanding C^2 unimodal map f, a test function $\varphi \in C^1([a,b])$, and a deformation $v = X \circ f$ for $X \in C^1([a,b])$ is the formal power series

$$\Psi(z) = \sum_{n=0}^{\infty} \int z^n X(y)\rho_0(y)(\varphi \circ f^n)'(y)\, dy = \sum_{n=0}^{\infty} \int z^n \mathcal{L}_0^n (X\rho_0)(x)\varphi'(x)\, dx\,. \tag{40}$$

In this section, we recall in Proposition 4.3 the resummation Ψ_1 proposed in [4] for the a priori divergent series $\Psi(1)$ when $X \circ f$ is horizontal. In addition, we give in Lemma 4.4 an expression for Ψ_1 in terms of the infinitesimal conjugacy α from Section 2, and we show that Ψ_1 is not well-defined if $X \circ f$ is not horizontal (Proposition 4.5).

Remark 4.1. If the infinitesimal deformation v is not of the form $X \circ f$, the heuristic argument of Ruelle [21] suggests to define the susceptibility function as:

$$\Psi(z) = \sum_{n=0}^{\infty} \int z^n \mathcal{L}_1(v\rho_0)(y)(\varphi \circ f^n)'(y)\, dy = \sum_{n=0}^{\infty} \int z^n \mathcal{L}_0^n \big(\mathcal{L}_1(v\rho_0)\big)(x)\varphi'(x)\, dx\,.$$

The analysis of the above expressions produces additional difficulties, and will not be pursued here.

Since $X\rho_0 \in BV$, Lemma 3.1 implies that the power series $\Psi(z)$ extends to a holomorphic function in the open unit disc, and in this disc we have

$$\Psi(z) = \int (\mathrm{id} - z\mathcal{L}_0)^{-1}(X\rho_0)(x)\, \varphi'(x)\, dx\,.$$

Recalling the jumps s_n in the saltus term ρ_{sal} for ρ (see (21)), the weighted total jump of f defined in [4] is:

$$\mathcal{J}(f,X) = \sum_{n=1}^{N_f} s_n X(c_n)\,. \tag{41}$$

In [4], we resummed the possibly divergent series $\Psi(1)$ under the condition $\mathcal{J}(f,X) = 0$ (see Proposition 4.3 below). We have the following simple but enlightening lemma:

Lemma 4.2. *Assume that f is a piecewise expanding C^2 unimodal map f, and that $X : I \to \mathbb{R}$ is bounded. Define $\alpha_{(0)}(c_1)$ by (12) for $v = X \circ f$. Then*

$$\mathcal{J}(f,X) = s_1(X(c_1) - \alpha_{(0)}(c_1))\,.$$

Since $s_1 < 0$, the lemma implies $\mathcal{J}(f,X) = 0$ if and only if $\alpha_{(0)}(c_1) = X(c_1)$, i.e., if and only if $X \circ f$ is horizontal for f.

Proof. If c is neither periodic nor preperiodic, then $s_k = f'(c_k)s_{k+1}$ for $k \ge 1$, and thus

$$\mathcal{J}(f,X) = s_1(X(c_1) - \alpha_{(0)}(c_1)) = s_1 \sum_{j\ge 0} \frac{X(f^j(c_1))}{(f^j)'(c_1)} \tag{42}$$

(see [4, Rem. 4.5]). The case of periodic c is similar using $s_k = f'(c_k)s_{k+1}$ for $1 \le k \le n_1 - 1$ and $M_f = n_1$.

If c is preperiodic, using $s_k = f'(c_k)s_{k+1}$ for $1 \le k \le n_0 + n_1 - 2$, $k \ne n_0 - 1$, and

$$s_{n_0} = \frac{s_{n_0-1}}{f'(c_{n_0-1})} + \frac{s_{n_0+n_1-1}}{f'(c_{n_0+n_1-1})} = \frac{s_{n_0-1}}{f'(c_{n_0-1})} + \frac{s_{n_0}}{(f^{n_1})'(c_{n_0})}\,,$$

which implies $(1-(f^{n_1})'(c_{n_0}))s_{n_0} = s_{n_0-1}/(f'(c_{n_0-1}))$ and thus

$$s_{n_0+j} = \frac{s_1}{(f^{n_0+j-1})'(c_1)} \frac{1}{1-1/(f^{n_1})'(c_{n_0})}\,, \quad 0 \le j \le n_1 - 1\,,$$

we get

$$\begin{aligned}\mathcal{J}(f,X) &= s_1\Big(\sum_{n=0}^{n_0-2} \frac{X(f^n c_1)}{(f^n)'(c_1)} + \sum_{j=0}^{n_1-1} \frac{X(f^{n_0+j-1}(c_1))}{(f^{n_0+j-1})'(c_1)} \frac{1}{1-1/(f^{n_1})'(c_{n_0})}\Big)\\ &= s_1(X(c_1) - \alpha_{(0)}(c_1))\,.\end{aligned}$$

□

We next recall the candidate Ψ_1 for the derivative of $t \mapsto \mathcal{R}(t)$ from Ruelle's conjecture as interpreted in [4]. Note that if $X \in C^2(f(I))$ satisfies $X(a) = 0$ then the function $\widetilde{X}$ defined by $\widetilde{X}(x) := X(x)$ for $x \ge a$ and $\widetilde{X}(x) := 0$ for $x \le a$ is such that $\widetilde{X}'$ is of bounded variation, and $\widetilde{X}'\tilde\rho$ is supported in $[a,b]$ for any $\tilde\rho$ supported in $(-\infty, b]$. Recall M_f from (9). Then, by Proposition 3.2 and the properties of s_k from the proof of Lemma 4.2, putting together [4, Lemma 4.1, Proposition 4.4, Theorem 5.2] gives [7]:

Proposition 4.3. *Let f be a mixing piecewise expanding C^3 unimodal map. Let $X \in C^2(f(I))$ satisfy $X(a) = 0$ and $\mathcal{J}(f,X) = 0$. For $\varphi \in C^1([a,b])$ and $|z| < 1$:*

$$\begin{aligned}\Psi(z) = -\sum_{j=1}^{\infty} \varphi(c_j) &\sum_{k=1}^{\min(j,M_f)} z^{j-k} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)}\\ &- \int (\mathrm{id} - z\mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\varphi\, dx\,.\end{aligned} \tag{43}$$

The second term in (43) extends to a holomorphic function in the open disc of radius λ_0^{-1}. If c is periodic or preperiodic then the first term of (43) is a rational function which is holomorphic at $z = 1$.

In addition, the following is a well-defined complex number

$$\Psi_1 = -\sum_{j=1}^{M_f} \varphi(c_j) \sum_{k=1}^{j} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} - \int (\mathrm{id} - \mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\varphi\, dx\,. \tag{44}$$

[7] Theorem 5.2 in [4] also holds if c is periodic, with a similar proof.

Note that $\Psi_1 = \Psi_1(\varphi)$ is well-defined even if φ is only continuous. If c is preperiodic or periodic, Ψ_1 is just the value at 1 of the holomorphic extension of $\Psi(z)$, and we have $\Psi_1 = \lim_{z\to 1}\Psi(z)$. If c is neither periodic nor preperiodic we do not know if the resummation Ψ_1 for the possibly divergent series $\Psi(1)$ is Abelian, i.e., if $\Psi_1 = \lim_{z\in(0,1), z\to 1}\Psi(z)$.

We have the following simpler expression for the first term of Ψ_1:

Lemma 4.4. *Let f be a mixing piecewise expanding C^3 unimodal map. Let $X \in C^2(f(I))$ satisfy $X(a) = 0$ and $\mathcal{J}(f, X) = 0$, and let $\varphi \in C^1([a, b])$. Then, setting $\alpha = \alpha_{(0)}$ from (12) for f and $v = X \circ f$,*

$$\Psi_1 = -\int \alpha\varphi\, \rho'_{sal} - \int (\mathrm{id} - \mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\varphi\, dx\,. \tag{45}$$

Proof. By Lemma 4.2 $X(c_1) = \alpha(c_1)$. Thus, by (49) the first term of Ψ_1 from (44) may be rewritten as a Stieltjes integral

$$\begin{aligned} -s_1 \sum_{j=1}^{M_f} \varphi(c_j)\Big(X(c_1) - \alpha(c_1) + \frac{\alpha(c_j)}{(f^{j-1})'(c_1)}\Big) &= -s_1 \sum_{j=1}^{M_f} \varphi(c_j)\frac{\alpha(c_j)}{(f^{j-1})'(c_1)} \\ &= -\int \alpha\varphi\, \rho'_{sal}\,. \end{aligned} \tag{46}$$

$\square$

In fact, Ψ_1 is well-defined only if $\mathcal{J}(f, X) = 0$:

Proposition 4.5. *Let f be a mixing piecewise expanding C^3 unimodal map f, let $X \in C^2(f(I))$ satisfy $X(a) = 0$. For every $\varphi \in C^0([a, b])$ the following series converges*

$$-\sum_{j=1}^{\infty} \int \mathcal{L}_1^j((X\rho_{reg})')(x)\varphi(x)\, dx\,.$$

If $\mathcal{J}(f, X) \neq 0$ then Ψ_1 is not well-defined, in the following sense: There exists $\varphi \in C^\infty([a, b])$ so that, on the one hand, both series below diverge

$$-\sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{\min(j,M_f)} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} - \sum_{j=1}^{\infty} \int \mathcal{L}_1^j(X'\rho_{sal})(x)\varphi(x)\, dx\,, \tag{47}$$

and on the other hand, the following series diverges

$$-\sum_{j=1}^{\infty}\Big(\varphi(c_j) \sum_{k=1}^{\min(j,M_f)} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} + \int \mathcal{L}_1^j(X'\rho_{sal})(x)\varphi(x)\, dx\Big)\,. \tag{48}$$

Proof. Since $\int ((X\rho_{reg})')(x)\, dx = 0$, the proof of [4, Proposition 4.4], implies

$$\Big|-\int \mathcal{L}_1^j((X\rho_{reg})')(x)\varphi(x)\, dx\Big| \le C\tau^j\,,$$

which gives the first claim.

To fix ideas assume that $\mathcal{J}(f, X) > 0$. Recalling Lemma 4.2, note that if c is not periodic, then for each j

$$\sum_{k=1}^{j} s_k X(c_k) = X(c_1) - \alpha_{(0)}(c_1) + \frac{\alpha_{(0)}(c_j)}{(f^{j-1})'(c_1)} = \mathcal{J}(X, f) + \frac{\alpha_{(0)}(c_j)}{(f^{j-1})'(c_1)}\,. \tag{49}$$

By the proof of [4, Proposition 4.4],

$$|-\int \mathcal{L}_1^j(X'\rho_{sal})(x)\varphi(x)\,dx - \mathcal{J}(f,X)\int \varphi\rho_0\,dx| \le C\tau^j\,,$$

thus if $\int \varphi\rho_0\,dx > 0$ then the second term in (47) diverges to $+\infty$. If c is not periodic and, in addition, $\inf_j \varphi(c_j) > \int \varphi\rho_0\,dx > 0$ then the first term diverges to $-\infty$ (use (49)). Finally, for the same φ, if c is not periodic then (48) is $\mathcal{J}(f,X)\sum_j(-\varphi(c_j)+\int \varphi\rho_0\,dx)$, which clearly diverges to $-\infty$. The case of periodic c is similar. □

5. Proof of the main theorem

If f_t is a $C^{2,2}$ perturbation of a mixing piecewise expanding C^2 unimodal map f tangent to its topological class, then Corollary 2.6 gives that the infinitesimal deformation v is horizontal. If $v = X\circ f$, Lemma 4.2 thus implies that $\mathcal{J}(f,X) = 0$. Therefore, if $X \in C^2(f(I))$, a candidate Ψ_1 for the derivative is defined by Proposition 4.3 and Lemma 4.4. Our main result can now be stated:

Theorem 5.1. *Let f_t be a $C^{2,3}$ perturbation of a mixing piecewise expanding C^3 unimodal map f with infinitesimal deformation $v = X\circ f$ such that $X \in C^2(f(I))$. If f_0 is good and f_t is tangent to its topological class, or if $f_t = \tilde f_t$ lies in the topological class of f_0, then $t \mapsto \rho_t\,dx$ from $(-\epsilon,\epsilon)$ to Radon measures is differentiable at 0, and*

$$\partial_t(\rho_t\,dx)|_{t=0} = -\alpha\rho'_{sal} - (\mathrm{id}-\mathcal{L}_1)^{-1}(X'\rho_{sal} + (X\rho_{reg})')\,.$$

In particular, for any $\hat\varphi \in C^0([a,b])$, the map $\mathcal{R}(t) = \int \hat\varphi\,\rho_t\,dx$ is differentiable at $t=0$, and $\mathcal{R}'(0) = \Psi_1(\hat\varphi)$.

Remark 5.2. There exist perturbations f_t of good mixing piecewise expanding C^∞ unimodal maps f, with $v = X\circ f$ and $\mathcal{J}(f,X)\neq 0$ so that $\mathcal{R}(t)$ is not Lipschitz for some $\varphi \in C^\infty[a,b]$ ([4, §6] and [17]). Examples are given for c preperiodic (see also [4, Remark 6.3]). A more general theory of the lack of smoothness of $\mathcal{R}(t)$ for perturbations f_t so that the infinitesimal deformation is not horizontal is desirable.

Proof. Since $\tilde f_t = f_t$ if f is not good, we may assume without loss of generality by Proposition 3.3 that $\tilde f_t = f_t = h_t\circ f\circ h_t^{-1}$ for all t. Also, since each ρ_t is a probability measure, we may restrict to continuous functions $\hat\varphi$ so that $\int \hat\varphi\,d\rho_0 = 0$. The proof will then be divided in three steps.

Step 1: Perturbation theory via resolvents.

Recall the spaces $\mathcal{B}_t = \Gamma_t(\widehat{\mathcal{B}})$ from Subsection 3.3, for a fixed $\eta > 0$, and define linear isometries $G_t = \Gamma_0\circ\Gamma_t^{-1} : \mathcal{B}_t \to \mathcal{B}_0$. We decompose

$$\rho_t - \rho_0 = (G_t(\rho_t) - \rho_0) + (\rho_t - G_t(\rho_t))\,. \tag{50}$$

The second term may be analysed directly, noting that (as Radon measures)

$$\lim_{t\to 0}\frac{\rho_t - G_t(\rho_t)}{t} = \lim_{t\to 0}\frac{\rho_{sal,t} - \rho_{sal,t}\circ h_t}{t} = -\sum_{k=1}^{N_f}\alpha(c_k)s_k\delta_{c_k} = -\alpha\rho'_{sal}\,.$$

(We used that $c_{k,t} = h_t(c_k)$ implies $H_{c_k} = H_{c_{k,t}}\circ h_t$.) To study the first term in (50), set

$$\mathcal{P}_t = G_t\circ\mathcal{L}_{1,t}\circ G_t^{-1}\,,\quad \widehat{\mathcal{Q}}_t = \widehat{\mathcal{Q}}_t(z) = z - \mathcal{P}_t\,.$$

(Of course $\mathcal{P}_0 = \mathcal{L}_1$ and $\widehat{\mathcal{Q}}_0 = z - \mathcal{L}_1$.) The operator $\mathcal{P}_t$ on $\mathcal{B}_0$ is conjugated to $\mathcal{L}_{1,t}$ on $\mathcal{B}_t$ and therefore has the same spectrum. The fixed point of $\mathcal{P}_t$ is $G_t(\rho_t)$ and the fixed point of $\mathcal{P}_t^*$ is $\nu_t(\varphi) = \int G_t^{-1}(\varphi)\,dx$. We denote by $\widehat{\Pi}_t(\varphi) = G_t(\rho_t)\nu_t(\varphi)$ the corresponding spectral projector. Our strategy will be to use, as in Proposition 3.3,

$$\widehat{\mathcal{Q}}_t^{-1} - \widehat{\mathcal{Q}}_0^{-1} = \widehat{\mathcal{Q}}_t^{-1}(\mathcal{P}_t - \mathcal{P}_0)\widehat{\mathcal{Q}}_0^{-1}\,,$$

in order to write $G_t(\rho_t)\nu_t(\varphi_0) - \rho_0 \int \varphi_0\,dx$ as a difference of spectral projectors applied to $\varphi_0 \in \widetilde{\mathcal{B}}_0$, where

$$\widetilde{\mathcal{B}}_0 = \{\varphi \in \mathcal{B}_0 \mid \varphi'_{reg} \in \mathcal{B}_0^{Lip}\} \text{ with the norm } \|\varphi'_{reg}\|_{\mathcal{B}_0^{Lip}} + \|\varphi\|_{\mathcal{B}_0}\,.$$

In fact, we do not need to perform the spectral analysis of $\mathcal{L}_1$ on $\widetilde{\mathcal{B}}_0$, since we shall work exclusively with $\rho_0 \in \widetilde{\mathcal{B}}_0$ (the fact that $\rho'_{reg} \in \mathcal{B}_0^{Lip}$, i.e., that all discontinuities of ρ'_{reg} lie on the postscritical orbit, that the jump at c_k is $O(\lambda^k)$, and that $(\rho_{reg})'_{reg} \in Lip$ is an easy consequence of the proof of [4, Proposition 3.3], noting in particular the uniform bound for $\Delta'_n(x)$ there – see also (65) and (66)).

Since $\int \rho_0\,dx = 1$, noting that $\widehat{\mathcal{Q}}_0^{-1}(\rho_0) = \rho_0/(z-1)$, we find

$$\begin{aligned} G_t(\rho_t)\nu_t(\rho_0) - \rho_0 &= -\frac{1}{2i\pi}\oint \frac{\widehat{\mathcal{Q}}_t^{-1}(z)}{z-1}(\mathcal{P}_t - \mathcal{P}_0)(\rho_0)\,dz \\ &= (\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)(\mathcal{P}_t - \mathcal{P}_0)(\rho_0)\,, \end{aligned} \tag{51}$$

where the contour is a circle centered at 1, outside of the disc of radius τ.

We shall also use the following norms on $\mathcal{B}_0$, for $j \geq 0$

$$|\varphi|_{weak,j} = \frac{\|\varphi_{reg}\|_{L^1(Leb)}}{2} + \frac{\max\{|\varphi_{reg}(y)| \mid y \in \cup_{0\leq \ell\leq j} f^{-\ell}(c)\}}{2} + |\Gamma^{-1}(\varphi_{sal})|_\eta\,.$$

We have $|\varphi|_{weak,j} \leq \|\varphi\|_{\mathcal{B}_0}$ for all $j \geq 0$. It is not difficult to see by adapting the estimates in Subsection 3.3 that there exist $\epsilon > 0$ and $C \geq 1$ so that, for all $|t| \leq \epsilon$ all j, ℓ, all $\varphi \in \mathcal{B}_0$,

$$|\mathcal{P}_t^j(\varphi)|_{weak,\ell} \leq C|\varphi|_{weak,\ell+j}\,, \quad \|\mathcal{P}_t^j(\varphi)\| \leq C\lambda^j\|\varphi\| + C|\varphi|_{weak,j}\,. \tag{52}$$

(Uniformity in t of the constant C in the Lasota-Yorke estimate follows from the fact that each f_t is conjugated to f. The reason why $\sup_{\ell\leq j}|\varphi_{reg}(f^{-\ell}(c))|$ appears in the weak norm is to take into account the compact operators $\mathcal{K}_0(\mathcal{L}_1^j)$ from the decomposition in § 3.3.) We shall see in Step 3 that for any fixed $j \geq 0$ there is a modulus of continuity $\delta_j(t) \geq 0$ (i.e., $\limsup_{t\to 0}\delta_j(t) = 0$) so that for each $\varphi \in \mathcal{B}_0$

$$|\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)|_{weak,j} \leq \delta_j(t)\|\varphi\|_{\mathcal{B}_0}\,. \tag{53}$$

Therefore, the proof of [14, Theorem 1] (see Appendix B) gives $\epsilon_0 > 0$ so that

$$A_{\epsilon_0} := \sup_{|t|<\epsilon_0}\|(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)\|_{\mathcal{B}_0} < \infty\,. \tag{54}$$

Beware that it is not clear whether $|(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)(\varphi) - (\mathrm{id} - \mathcal{P}_0)^{-1}(\varphi)|_{weak,0}$ tends to zero uniformly in $\|\varphi\|_{\mathcal{B}_0} \leq 1$ as $t \to 0$. This is why we next consider $\mathcal{P}_t$ acting on $\mathcal{B}_0^{Lip}$: By § 3.3, the essential spectral radius is $\leq \lambda$, and the spectrum outside of the disc of radius τ consists in the eigenvalue 1, with projector $\widehat{\Pi}_t$. We introduce a weak norm on $\mathcal{B}_0^{Lip}$:

$$|\varphi|_{weak,\infty} = \|\varphi_{reg}\|_{L^\infty(Leb)} + |\Gamma^{-1}(\varphi_{sal})|_\eta\,.$$

Applying again the argument in § 3.3, we see that (52) holds for $\ell = \infty$. Clearly, $|\varphi|_{weak,j} \le |b-a||\varphi|_{weak,\infty}$. In Step 3, we shall find $\widetilde{C} \ge 1$ so that for each $\varphi \in \mathcal{B}_0^{Lip}$

$$|\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)|_{weak,\infty} \le \widetilde{C}|t|\|\varphi\|_{\mathcal{B}_0^{Lip}} . \tag{55}$$

Then, setting

$$\mathcal{N}_t = (\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t) - (\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0) ,$$

(52) and (55) imply by [14, Theorem 1, Corollary 1] that there are $\widehat{C} \ge 1$ and $\xi > 0$ so that for each $\varphi \in \mathcal{B}_0^{Lip}$

$$|\mathcal{N}_t(\varphi)|_{weak,\infty} \le \widehat{C}|t|^\xi \|\varphi\|_{\mathcal{B}_0^{Lip}} . \tag{56}$$

If we knew that there existed $\mathcal{D} \in \mathcal{B}_0^{Lip}$ so that [8]

$$\|\mathcal{P}_t(\rho_0) - \mathcal{P}_0(\rho_0) - t\mathcal{D}\|_{\mathcal{B}_0} = O(t^2) , \tag{57}$$

uniformly in small t (this will be shown in Step 2), then (51) and (56) would give

$$\partial_t(G_t(\rho_t)\nu_t(\rho_0))|_{t=0} = (\mathrm{id} - \mathcal{L}_1)^{-1}(\mathrm{id} - \widehat{\Pi}_0)(\mathcal{D}) , \tag{58}$$

in $L^\infty(Leb)$: Indeed, write $(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t) = \mathcal{N}_t + (\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0)$ and note that (54) implies

$$\begin{aligned} G_t(\rho_t)\nu_t(\rho_0) - \rho_0 &= (\mathcal{N}_t + (\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0))(t\mathcal{D} + O_{\mathcal{B}_0}(t^2)) \\ &= t\mathcal{N}_t(\mathcal{D}) + t(\mathrm{id} - \mathcal{P}_0)^{-1}(\mathrm{id} - \widehat{\Pi}_0)(\mathcal{D}) + A_{\epsilon_0} O(t^2) . \end{aligned}$$

Dividing by t and letting $t \to 0$, (56) gives the claim (58).

Note that $t \mapsto \nu_t(\rho_0)$ is differentiable at 0: As $\nu_t(\rho_0) = \int \rho_{sal} \circ h_t^{-1}\, dx + \int \rho_{reg}\, dx$, one easily sees that $\partial_t \nu_t(\rho_0)|_{t=0} = -\sum_{k=1}^{N_f} \alpha(c_k) s_k$. Then (58) implies

$$\partial_t(G_t\rho_t)|_{t=0} = \partial_t(G_t(\rho_t)\nu_t(\rho_0))|_{t=0} + \rho_0\, \partial_t(\nu_t(\rho_0))|_{t=0} .$$

Since our test functions satisfy $\int \hat\varphi\, d\rho_0 = 0$, we can ignore scalar multiples of ρ_0, and it only remains to show (53), (55), and (57) with

$$(\mathrm{id} - \widehat{\Pi}_0)(\mathcal{D}) = -X'\rho_0 - X\rho'_{reg} . \tag{59}$$

Step 2: Analysing the derivative of $t \mapsto \mathcal{P}_t(\rho_0)$.

In this step, we prove (57) and (59). By definition, for any $\varphi \in \mathcal{B}_0$

$$\mathcal{P}_t(\varphi) = (\mathcal{L}_{1,t}(\varphi_{sal} \circ h_t^{-1} + \varphi_{reg}))_{sal} \circ h_t + (\mathcal{L}_{1,t}(\varphi_{sal} \circ h_t^{-1} + \varphi_{reg}))_{reg} . \tag{60}$$

From now on, we assume that the postscritical orbit is infinite, to fix ideas. (The case of finite postcritical orbit is similar.) Recall (35). Noting that $c_k > c$ if and only if $c_{k,t} = f_t^k(c) > c$, and writing $\varphi_{sal} = \sum_k u_k H_{c_k}$, the contribution to $\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)$ from the first term in the right-hand-side of (60), i.e., $(\mathcal{P}_t(\varphi))_{sal} - \mathcal{P}_0(\varphi)_{sal}$, is just

$$\begin{aligned} &\sum_{k=2}^{N_f} u_{k-1}\left(\frac{1}{f_t'(c_{k-1,t})} - \frac{1}{f'(c_{k-1})}\right) H_{c_k} \\ &\qquad + (\varphi_{reg}(c) + \sum_{c_k > c} u_k)\left(\frac{1}{f_t'(c_-)} - \frac{1}{f'(c_-)} - \frac{1}{f_t'(c_+)} + \frac{1}{f'(c_+)}\right) H_{c_1} . \end{aligned} \tag{61}$$

[8] We emphasize that the norm in (57) is in $\mathcal{B}_0$, and a priori not in $\mathcal{B}_0^{Lip}$.

Next, we find by (37) that the derivative of the second term $((\mathcal{P}_t(\varphi))_{reg} - \mathcal{P}_0(\varphi)_{reg})$ of (60), which is an atomless measure, coincides with

$$
\begin{aligned}
(62)\qquad & (\mathcal{L}_{1,t}(\varphi_{reg}))'|_{(a,c_{1,t})} - (\mathcal{L}_1(\varphi_{reg}))'|_{(a,c_1)} \\
&\quad + \sum_{k=2, c_{k-1}>c}^{N_f} u_{k-1}\big((\mathcal{L}_{1,t}(H_{c_{k-1,t}}))'|_{(c_{k,t},c_{1,t})} - (\mathcal{L}_1(H_{c_{k-1}}))'|_{(c_k,c_1)}\big) \\
&\quad + \sum_{k=2}^{N_f} u_{k-1}\big((\mathcal{L}_{1,t}(H_{c_{k-1,t}}))'|_{(a,c_{k,t})} - (\mathcal{L}_1(H_{c_{k-1}}))'|_{(a,c_k)}\big)\,.
\end{aligned}
$$

Put $\varphi = \rho_0$, and consider first (61). Note that $c_{k,t} = h_t(c_k)$. Write

$$
\frac{1}{f_t'(h_t(w))} - \frac{1}{f'(w)} = \frac{f'(w) - f_t'(h_t(w))}{f_t'(h_t(w))f'(w)}\,,
$$

and decompose $f'(w) - f_t'(h_t(w) = f'(w) - f_t'(w) + f_t'(w) - f_t'(h_t(w))$, with $f'(w) - f_t'(w) = -tX'(f(w))f'(w) + O(t^2)$, and $f_t'(w) - f_t'(h_t(w)) = -tf_t''(w)\alpha(w) + O(t^2)$. Thus, we find, by using $(\mathcal{L}_1(\rho))_{sal} = \rho_{sal}$ and (11), that

$$
\begin{aligned}
\lim_{t\to 0}\frac{(\mathcal{P}_t(\rho_0))_{sal} - (\rho_0)_{sal}}{t} &= -\sum_{k=1}^{N_f} X'(c_k)s_k H_{c_k} - \sum_{k=2}^{N_f}\frac{\alpha(c_{k-1})s_{k-1}f''(c_{k-1})}{(f'(c_{k-1}))^2}H_{c_k} \\
&= -\sum_{k=1}^{N_f} X'(c_k)s_k H_{c_k} + \sum_{k=2}^{N_f}\frac{(X(c_k) - \alpha(c_k))s_{k-1}f''(c_{k-1})}{(f'(c_{k-1}))^3}H_{c_k} \\
(63)\qquad &= -(X'\rho)_{sal} + \sum_{k=1}^{N_f}(X(c_k) - \alpha(c_k))E_k\,,
\end{aligned}
$$

where we used $X(c_1) = \alpha(c_1)$ with (the choice of E_1 will become clear later on)

$$
\begin{aligned}
(64)\qquad E_k &= \frac{s_{k-1}f''(c_{k-1})}{(f'(c_{k-1}))^3}\,,\ k \ge 2\,, \\
E_1 &= \left(-\frac{\rho_{reg}(c)f''(c_-)}{(f'(c_-))^3} + \frac{\rho_{reg}(c)f''(c_+)}{(f'(c_+))^3}\right) \\
&\qquad\qquad + \sum_{k\ge 2, c_{k-1}>c} s_{k-1}\left(\frac{f''(c_-)}{(f'(c_-))^3} - \frac{f''(c_+)}{(f'(c_+))^3}\right).
\end{aligned}
$$

It will turn out essential to study $((\rho_{reg})')_{sal} = \sum_{k=1}^{N_k} s_k' H_{c_k}$. If $x \in [a, c_1)$ is not along the critical orbit we have

$$
(65)\quad (\rho_{reg})'(x) = (\rho_0)'(x) = (\mathcal{L}_1(\rho_0))'(x) = \sum_{f(y)=x}\frac{(\rho_{reg})'(y)}{|f'(y)|f'(y)} - \frac{\rho_0(y)f''(y)}{|f'(y)|(f'(y))^2}\,.
$$

(We used $(\rho_{reg})'(y) = (\rho_0)'(y)$ if y is not along the postscritical orbit.) Taking the difference between $(\rho_{reg})'(x)$ for $x \uparrow c_k$ and $x \downarrow c_k$, and recalling E_k from (64), we easily get from the previous identity that [9]

(66)

$$
s_k' = E_k' - E_k\,,\ \text{with } E_k' = \frac{s_{k-1}'}{(f'(c_{k-1})^2)}\,,\, k \ge 2\,,\ E_1' = -\frac{(\rho_{reg})'(c)}{(f'(c_-))^2} + \frac{(\rho_{reg})'(c)}{(f'(c_+))^2}\,.
$$

[9] If c is periodic then $(\rho_{reg})'(c)$ may be undefined, but $(\rho_{reg})'(c_\pm)$ are both defined.

We now consider $\lim_{t\to 0} \frac{1}{t}((\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg})'$. We get two sorts of contributions to (62): For

(67)
$$x \in [\min(c_k, c_{k,t}), \max(c_k, c_{k,t})] \text{ or } x \in [\min(c_k, f_t(c_{k-1})), \max(c_k, f_t(c_{k-1}))]\,,$$

an atom may appear at c_k in the limit, we call such x singular points. For the other values of x, which we call the regular points, the limit will be a function. Recalling (64) and (66), we claim that the contribution of the singular points to $\lim_{t\to 0}((\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg})|_{(a,b)})'/t$ is

$$\sum_{k=1}^{N_f} (\alpha(c_k) E_k - X(c_k) E'_k)\delta_{c_k}\,. \tag{68}$$

Indeed, if $k \geq 2$ and $c_{k,t} < c_k$ and $c_{k-1} < c$, we must consider the Radon measure

$$\varphi \mapsto -\frac{s_{k-1}}{t} \int_{c_{k,t}}^{c_k} \frac{f''(\psi_-(x))}{(f'(\psi_-(x)))^3} \varphi(x)\, dx = \alpha(c_k) s_{k-1} \frac{f''(c_{k-1})}{(f'(c_{k-1}))^3} \varphi(c_k) + O(t)\,,$$

coming from $-(\mathcal{L}_1(H_{c_{k-1}}))'$ (we used $h_t(c_k) = c_{k,t}$). If $k \geq 2$, $c_{k,t} < c_k$, and $c_{k-1} > c$, we must consider the Radon measure

$$\varphi \mapsto -\frac{s_{k-1}}{t} \int_{c_{k,t}}^{c_k} \frac{f_t''(\psi_{t,+}(x))}{(f_t'(\psi_{t,+}(x)))^3} \varphi(x)\, dx = \alpha(c_k) s_{k-1} \frac{f''(c_{k-1})}{(f'(c_{k-1}))^3} \varphi(c_k) + O(t),$$

from $(\mathcal{L}_{1,t}(H_{c_{k-1,t}}))' - (\mathcal{L}_1(H_{c_{k-1}}))'$ (the corresponding term for the branches ψ_- and $\psi_{t,-}$ vanishes in the limit). For $k = 1$ and $c_{1,t} < c_1$ we must consider the three contributions given by, firstly,

$$\varphi \mapsto -\frac{1}{t} \int_{c_{1,t}}^{c_1} \frac{(\rho_{reg})'(\psi_-(x))}{(f'(\psi_-(x)))^2} \varphi(x)\, dx = \alpha(c_1) \frac{(\rho_{reg})'(c)}{(f'(c_-))^2} \varphi(c_1) + O(t)\,,$$

(recall also that $c_{1,t} = f(c_1)$ and $\alpha(c_1) = X(c_1)$), secondly,

$$\varphi \mapsto \frac{1}{t} \int_{c_{1,t}}^{c_1} \frac{\rho_{reg}(\psi_-(x)) f''(\psi_-(x))}{(f'(\psi_-(x)))^3} \varphi(x)\, dx = \alpha(c_1) \frac{-\rho_{reg}(c) f''(c_-)}{(f'(c_-))^3} \varphi(c_1) + O(t)\,,$$

and thirdly, by the sum over those $j \geq 2$ so that $c_{j-1} > c$ of

$$\varphi \mapsto -\frac{s_{j-1}}{t} \int_{c_{1,t}}^{c_1} \frac{f''(\psi_-(x))}{(f'(\psi_-(x)))^3} \varphi(x)\, dx = \alpha(c_1) s_{j-1} \frac{f''(c_-)}{(f'(c_-))^3} \varphi(c_1) + O(t)\,,$$

as well as the corresponding three contributions for ψ_+. The cases $c_{k,t} > c_k$ are similar. For $k \geq 2$, we must also deal with the jump terms from $(\mathcal{L}_{1,t}(\rho_{reg}))' - (\mathcal{L}_1(\rho_{reg}))'$ (one at $f_t(c_{k-1})$ the other at c_k), which give, using $f_t(c_{k-1}) - f(c_{k-1}) = tX(c_k) + O(t^2)$:

$$\varphi \mapsto \frac{1}{t} \int_{f_t(c_{k-1})}^{c_k} \frac{s'_{k-1}}{(f'(c_{k-1}))^2} \varphi(x)\, dx = -X(c_k) \frac{s'_{k-1}}{(f'(c_{k-1}))^2} \varphi(c_k) + O(t)\,.$$

We move to the regular points: For small t, let $k_t \geq 2$ be so that $\sum_{k \geq k_t} |s_{k-1}| \leq t^2$ (clearly, $k_t = O(\ln|t|)$), and take I_t to be the union of the $O(k_t)$ intervals of singular points associated to $k \leq k_t$ via (67) (in particular, the Lebesgue measure of I_t is an $O(t \ln|t|)$). We have by definition

$$\|(\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg} - (\mathcal{L}_{1,t}(\rho_0) - \mathcal{L}_1(\rho_0))_{reg}\|_{\mathcal{B}_0(I \setminus I_t)} = O(t^2)\,, \tag{69}$$

where $\|\phi_{reg}\|_{\mathcal{B}_0(I\setminus I_t)}$ is the norm of Radon measure $(\phi_{reg})'$ on the metric set $I \setminus I_t$. (Use that

$$\sum_{k\ge k_t} |s_{k-1}| \|\mathcal{L}_{1,t}(H_{c_{k-1,t}}) - \mathcal{L}_{1,t}(H_{c_{k-1}})\|_{\mathcal{B}_0} = O(t^2)\,,$$

and $\mathcal{L}_{1,t}(H_{c_{k-1,t}})(x) - \mathcal{L}_{1,t}(H_{c_{k-1}})(x) = 0$ for $k \le k_t$ and $x \notin I_t$.) The contribution (68) takes care of $\|(\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg}\|_{\mathcal{B}_0(I_t)}$ (note that $\sum_{k\ge k_t} |\alpha(c_k)E_k| + |X(c_k)E'_k| = O(t^2)$) so that we may concentrate on $(\mathcal{L}_{1,t}(\rho_0) - \mathcal{L}_1(\rho_0))_{reg}$ on $I \setminus I_t$.

Note that

$$f^{-1}(x) - f_t^{-1}(x) = t\frac{X(x)}{f'(f^{-1}(x))} + O(t^2)\,, \tag{70}$$

where we choose the same inverse branch for f_t and f. It follows that

$$\begin{aligned}\frac{\varphi(f_t^{-1}(x))}{|f_t'(f_t^{-1}(x))|} - \frac{\varphi(f^{-1}(x))}{|f'(f^{-1}(x))|} &= -tX'(x)\frac{\varphi(f^{-1}(x))}{|f'(f^{-1}(x))|}\\ &- tX(x)\Big(\frac{\varphi'(f^{-1}(x))}{f'(f^{-1}(x))|f'(f^{-1}(x))|} + \frac{\varphi(f^{-1}(x))f''(f^{-1}(x))}{(f'(f^{-1}(x)))^2|f'(f^{-1}(x))|}\Big) + O(t^2)\,,\end{aligned}$$

if φ is C^{1+Lip} at $f^{-1}(x)$, which gives, after summing over the two inverse branches,

$$-tX'(x)\mathcal{L}_1(\varphi)(x) - tX(x)(\mathcal{L}_1(\varphi))'(x) + O(t^2)\,. \tag{71}$$

Therefore, if $x \notin I_t$, and $x \ne c_k$ and $x \ne c_{k,t}$ for all $k \ge 1$, we have, decomposing $\rho_0 = \rho_{reg} + \sum_k s_k H_{c_k}$,

$$\begin{aligned}(\mathcal{L}_{1,t}(\rho_0) - \mathcal{L}_1(\rho_0))_{reg}(x) &= -t(X'\rho_0 - X(\rho_0)')_{reg}(x) + O(t^2)\\ &= -t(X'\rho_0)_{reg}(x) - t(X(\rho_{reg})')_{reg}(x) + O(t^2)\,.\end{aligned} \tag{72}$$

(The $O(t^2)$ term is in $\mathcal{B}_0$, not $\mathcal{B}_0^{Lip}$.) By continuity, (72) holds for all $x \notin I_t$.

The regular contribution to $\lim_{t\to 0}\big((\mathcal{P}_t(\rho_0))_{reg} - (\rho_0)_{reg}\big)/t$ is thus

$$-\big(X'\rho_0 - (X'\rho_0)_{sal}\big) - \big(X(\rho_{reg})' - (X(\rho_{reg})')_{sal}\big)\,. \tag{73}$$

All together, we find from (63–68–73) and (66) (differentiating in $\mathcal{B}_0$)

$$\partial_t(\mathcal{P}_t(\rho_0))|_{t=0} = -X'\rho_{sal} - X'\rho_{reg} - X(\rho_{reg})' \in \mathcal{B}_0^{Lip}\,.$$

This establishes (57) and (59) (note that $\int X'\rho_{sal} + (X\rho_{reg})'\,dx = 0$).

Step 3: Proving the weak norm bounds necessary for [14]**.**

It remains to prove the bounds (53) and (55) for $\mathcal{P}_t(\varphi) - \mathcal{P}_0(\varphi)$. We start with (53). For the term corresponding to (61), since φ is not necessarily a fixed point of $\mathcal{L}_1$, we get in addition to (63) a term

$$(|\varphi_{reg}(c)| + \sum_{c_k > c} |u_k|)O(t) = O(t)|\varphi|_{weak,0}\,.$$

Next, consider (62). For the $L^1(Leb)$ norm of $(\mathcal{P}_t - \mathcal{P})_{reg}$, the singular contributions produce an $O(t\ln|t|)$ term: Indeed, by (38), up to an error $O(t)$ we may restrict to a finite set of c_ks, where the cardinality of this finite set is an $O(\ln|t|)$; for this finite set, the total Lebesgue measure of the intervals of singular points is an $O(t\ln|t|)$. For the regular contributions, although $\mathcal{L}_1(\varphi)$ is not equal to φ in general, and φ_{reg}

is only continuous and of bounded variation, we get an $O(t)\|\varphi\|_{\mathcal{B}_0}$ contribution to the $L^1(Leb)$ norm of $(\mathcal{P}_t - \mathcal{P})_{reg}$: Indeed, the only delicate terms are of the form

$$\int h(y)(\varphi_{reg}(\psi_{+,t}(y)) - \varphi_{reg}(\psi_+(y)))\, dy\,,$$

with $|h| \le \|f\|_{C^{1+Lip}}$, and similarly with ψ_-. Now we exploit that if $\phi \in BV$ and Ψ_t is C^2 with $|\Psi_t(x) - x| \le C|t|$ and $|\Psi_t'(x) - 1| \le C|t|$ then (use [13, Lemma 11] as in [13, Lemma 13])

$$\int |\phi(y) - \phi(\Psi_t(y))|\, dy = O(t)\|\phi\|_{BV}\,.$$

We must still bound $|\mathcal{P}_t(\varphi)_{reg}(y) - \mathcal{P}_0(\varphi)_{reg}(y)|$ for $y \in \mathcal{S}_j = \cup_{0\le \ell \le j} f^{-\ell}(c)$. We make no distinction between regular and singular points here. The contribution corresponding to differences between derivatives of f of f_t gives $O(t)$. Next, φ_{reg} is continuous by definition of $\mathcal{B}_0$. Writing $\tilde\delta_j(\cdot)$ for its worse modulus of continuity on the finite set $\mathcal{S}_j$, we get since $|c_k - c_{k,t}| = O(t)$ that

$$\sup_{y\in\mathcal{S}_j} |\mathcal{P}_t(\varphi)_{reg}(y) - \mathcal{P}_0(\varphi)_{reg}(y)| = O(\tilde\delta_j(t) + |t|)\,.$$

Finally, (55) can be proved by using the Lipschitz assumption on φ_{reg}, to simplify the argument for (53): The uniform modulus of continuity $\delta(t) = O(t)$ of φ_{reg} allows us to deal with the L^∞ norm in $|\cdot|_{weak,\infty}$. □

6. The derivative in terms of the infinitesimal conjugacy α

Let f_t be a $C^{2,2}$ perturbation tangent to the topological class of a mixing piecewise expanding C^2 unimodal map. We do not know whether $x \mapsto h_t(x)$ is quasisymmetric, as in the smooth expanding case. Note however that in general it is *not* absolutely continuous (see [16] for the nonuniformly expanding case). For similar reasons, $\alpha = \partial_t h_t|_{t=0}$ is in general not absolutely continuous. In this section, we shall see that absolute continuity of α is equivalent to a remarkable formula for $\Psi_1 = \mathcal{R}'(0)$ which can be "guessed" from the following easy lemma:

Lemma 6.1. *Assume that f_t is a $C^{2,2}$ perturbation tangent to the topological class of a piecewise expanding C^2 unimodal map f, with infinitesimal perturbation $v = X\circ f$. Then recalling $\alpha = \partial_t h_t|_{t=0}$ from Corollary 2.6, we have*

$$(\mathrm{id} - \mathcal{L}_0)(\alpha\rho_0) = X\rho_0\,, \tag{74}$$

and $\sum_{k=0}^n \mathcal{L}_0^k(X\rho_0) = \alpha\rho_0 - \mathcal{L}_0^{n+1}(\alpha\rho_0)$.

The lemma gives that the partial sum of order n for the series $\Psi(z)$ at $z = 1$ is

$$\sum_{k=0}^n \int \mathcal{L}_0^k(X\rho_0)\varphi'\, dx = \int \varphi'\alpha\rho_0 - \int \varphi'\mathcal{L}_0^{n+1}(\alpha\rho_0)\, dx\,.$$

We do not claim that $\int \varphi'\mathcal{L}_0^{n+1}(\alpha\rho_0)\, dx$ converges as $n\to\infty$.

Proof. We know that $X(y) = \alpha(y) - f'(\psi(y))\alpha(\psi(y))$ where ψ is an arbitrary inverse branch of f. Multiply this by the positive number $\rho_0(\psi(y))/|f'(\psi(y))|$ and sum over inverse branches. Since ρ_0 is the invariant density, the sum of these positive numbers is $\rho_0(y)$, which gives the first claim. A telescopic sum gives the second claim. □

Theorem 6.2. *Assume that f_t is a $C^{2,3}$ perturbation tangent to the topological class of a mixing piecewise expanding C^3 unimodal map f with infinitesimal perturbation $v = X \circ f$ (in particular $\mathcal{J}(f, X) = 0$) so that $X \in C^2(f(I))$. If $\alpha = \partial_t h_t|_{t=0}$ is absolutely continuous then*

$$\Psi_1 = \int \varphi' \alpha \rho_0 \, dx \,, \quad \forall \varphi \in C^1([a, b]) \,. \tag{75}$$

Conversely, if (75) holds then $\alpha \in BV^{(1)}$ (in particular, α is absolutely continuous).

Theorem 6.2 will easily imply:

Corollary 6.3 (Derivative of the TCE)**.** *Under the assumptions of Theorem 6.2, if α is absolutely continuous, then*

$$(-\mathrm{id} + \mathcal{L}_1)(\alpha' \rho_0 + \alpha(\rho_{reg})') = X' \rho_0 + X(\rho_{reg})' \,. \tag{76}$$

Note that the proofs of Theorem 6.2 and Corollary 6.3 use the results from [4] (in particular Lemma 4.1, Prop. 4.4 there), Proposition 2.4, and the easy Lemma 6.1 but do not require any information from Sections 3, 4 or 5 of the present paper.

Proof of Corollary 6.3. Putting together (75) and (46) we get

$$\begin{aligned} \Psi_1 + \int \alpha \varphi (\rho_{sal})' &= \int \alpha \varphi' \rho_0 \, dx + \int \alpha \varphi (\rho_{sal})' \\ &= \int (\mathrm{id} - \mathcal{L}_1)^{-1} (X' \rho_{sal} + (X \rho_{reg})') \varphi \, dx \,. \end{aligned}$$

And, since the boundary term in the integration by parts vanishes,

$$\begin{aligned} \int \alpha \varphi' \rho_0 \, dx + \int \alpha \varphi (\rho_{sal})' &= \int \alpha \varphi (-\rho_0' + (\rho_{sal})') - \int \alpha' \varphi \rho_0 \, dx \\ &= - \int \alpha \varphi (\rho_{reg})' \, dx - \int \alpha' \varphi \rho_0 \, dx \,. \end{aligned}$$

□

Proof of Theorem 6.2. We suppose that c is neither periodic nor preperiodic (the other cases are easier). The expressions (44) and (46) allow us to write Ψ_1 as

$$\Psi_1 = - \int \varphi \beta' \,, \tag{77}$$

where β' is a Stieltjes measure. In fact,

$$\beta' = \alpha (\rho_{sal})' + (\mathrm{id} - \mathcal{L}_1)^{-1} (X' \rho_{sal} + (X \rho_{reg})') \, dx \,.$$

The above implies that β' is the sum of an absolutely continuous measure with density of bounded variation, and a weighted sum of diracs along the postcritical orbit. Now by [4, Lemma 4.1], we know that $(\mathrm{id} - f_*)(\alpha \rho_{sal}') = X \rho_{sal}'$. Thus

$$(\mathrm{id} - f_*)(\beta') = X(\rho_{sal})' + X' \rho_{sal} + (X \rho_{reg})' = (X \rho_0)' \,. \tag{78}$$

Integrating (77) by parts, we get (there are no boundary terms, see e.g. [4, Proof of Prop. 4.4, Theorem 5.1]),

$$\Psi_1 = \int \varphi'(x) B(x) \, dx \,,$$

where B is a function of bounded variation, supported in $[a, b]$, satisfying $B' = \beta'$. In particular, B is the sum of an element B_1 of $BV^{(1)}$ with a function with prescribed jumps along the postcritical orbit. It is easy to check that this function is in fact just the saltus of $\alpha\rho_{sal}$ (or, equivalently, the saltus of $\alpha\rho_0$). By (78) (and the fact that both $B(x)$ and $\rho_0(x)$ vanish for $x \ge b$) we get that

$$(\mathrm{id} - \mathcal{L}_0)B = X\rho_0 \,. \tag{79}$$

Now, Lemma 6.1 implies that

$$(\mathrm{id} - \mathcal{L}_0)(\alpha\rho_0) = X\rho_0 \,. \tag{80}$$

Putting together (79–80) and $B = B_1 + (\alpha\rho_0)_{sal}$, we get that

$$(\mathrm{id} - \mathcal{L}_0)(B_1 - (\alpha\rho_0)_{reg}) = 0 \,. \tag{81}$$

After these preliminaries, we move on to the proof.

If α is absolutely continuous then $(\alpha\rho_0)_{reg}$ is absolutely continuous (because $\alpha \in BV$ and $((\alpha\rho_0)_{reg})' = \alpha'\rho_0 + \alpha(\rho_{reg})'$ is in $L^1(Leb)$). B_1 is absolutely continuous because it is in $BV^{(1)}$. The operator $\mathcal{L}_1$ acting on $L^1(Leb)$ has ρ_0 as unique fixed point, and thus $\mathcal{L}_0$ on the Banach space of absolutely continuous functions supported in $(-\infty, b]$ has $R_0(x) = -1 + \int_{-\infty}^x \rho_0(y)\, dy$ as unique fixed point. Thus (81) implies that $B_1 = (\alpha\rho_0)_{reg} + \kappa R_0$, so that $B = \alpha\rho_0 + \kappa R_0$. Since $B(x) = \alpha(x)\rho_0(x) = 0$ for $x \le a$ (use that $\int (X'\rho_{sal} + (X\rho_{reg})')dx = 0$ by $\mathcal{J}(f, X) = 0$), we have that $\kappa = 0$, proving (75).

We next prove the converse. If (75) holds then $B = \alpha\rho_0 = \alpha\rho_{sal} + \alpha\rho_{reg}$ is in BV by the preliminary remarks. Since ρ_0 is bounded from below on $[c_2, c_1]$, this implies that $\alpha|_{[c_2,c_1]}$ is in BV. The preliminaries also give $B - (\alpha\rho_0)_{sal} = (\alpha\rho_0)_{reg} \in BV^{(1)}$, i.e., $\alpha'\rho_0 + \alpha\rho_{reg} \in BV$, which implies that $\alpha'\rho_0 \in BV$ (since $\alpha \in BV$). Using again $\inf_{[c_2,c_1]} \rho_0 > 0$ we get that $\alpha' \in BV$, i.e., $\alpha \in BV^{(1)}$. □

Appendix A. An auxiliary lemma

Lemma A.1. *Let f and g be two piecewise expanding C^1 unimodal maps and assume that $c = 0$. If $\sup_x\{1/|f'(x)|,\ 1/|g'(x)|\} \le \theta$ and $\sup_x |f(x) - g(x)| \le \delta$, then for all points x_f and x_g such that*

$$f^k(x_f) \cdot g^k(x_g) \ge 0 \,, \forall k \le n \,, \tag{82}$$

we have $|x_f - x_g| < \theta^n + \frac{\delta}{1-\theta}$.

Proof. We can extend the inverse branches of f and g, denoted ψ^f_σ, ψ^g_σ, for $\sigma \in \{+, -\}$, to C^1 diffeomorphisms defined on $f(I) \cup g(I)$, so that they also have derivatives bounded from above by θ and

$$\max_{\sigma=+,-} \sup_{y \in f(I)\cup g(I)} |\psi^f_\sigma(y) - \psi^g_\sigma(y)| < \delta \,.$$

Condition (82) implies that there exists a sequence $\sigma_k \in \{+, -\}$, $k \le n$, such that

$$\psi^f_{\sigma_1} \circ \cdots \circ \psi^f_{\sigma_n}(f^n(x_f)) = x_f \text{ and } \psi^g_{\sigma_1} \circ \cdots \circ \psi^g_{\sigma_n}(f^n(x_g)) = x_g \,.$$

The lemma then follows from

$$\begin{aligned}
&|f^k(x_f) - g^k(x_g)| = |\psi^f_{\sigma_{k+1}}(f^{k+1}(x_f)) - \psi^g_{\sigma_{k+1}}(g^{k+1}(x_g))| \\
&\le |\psi^f_{\sigma_{k+1}}(f^{k+1}(x_f)) - \psi^f_{\sigma_{k+1}}(g^{k+1}(x_g))| + |\psi^f_{\sigma_{k+1}}(g^{k+1}(x_g)) - \psi^g_{\sigma_{k+1}}(g^{k+1}(x_g))| \\
&\le \theta|f^{k+1}(x_f) - g^{k+1}(x_g)| + \delta \,.
\end{aligned}$$

□

Appendix B. Keller-Liverani bounds for sequences of weak norms

We explain how (52) and (53) imply that for each $\gamma > 0$, there exist $\epsilon_0 > 0$ and $K \geq 1$ so that

$$\|(z - \mathcal{P}_t)^{-1}\|_{\mathcal{B}_0} \leq K\,, \quad \forall |t| < \epsilon_0\,,\ \text{if } |z| \geq \tau \text{ and } |z-1| \geq \gamma\,, \tag{83}$$

by adapting the proof of [14, Theorem 1] of Keller and Liverani. Since we have

$$(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)(\varphi) = -\frac{1}{2i\pi}\oint \frac{1}{z-1}(z-\mathcal{P}_t)^{-1}(\varphi)\, dz\,, \quad \forall \varphi \in \mathcal{B}_0\,,$$

(on any contour $|z-1| = \gamma$ with $\gamma \in (0, 1-\tau)$), the bound (83) implies that $\|(\mathrm{id} - \mathcal{P}_t)^{-1}(\mathrm{id} - \widehat{\Pi}_t)\|_{\mathcal{B}_0}$ is bounded uniformly in $|t| < \epsilon_0$, i.e., (54).

Fix $\lambda < \tau < 1$ as after (20). The first remark is that [14, Lemma 1] is replaced by the claim that there exist ϵ_1, n_1 and C_1, depending only on C from (52) and on τ, so that for any $|z| \geq \tau$, all $\varphi \in \mathcal{B}_0$, all $|t| \leq \epsilon_1$

$$\|\varphi\|_{\mathcal{B}_0} \leq C_1 \|\widehat{Q}_t(z)\varphi\|_{\mathcal{B}_0} + C_1 |\varphi|_{weak,n_1}\,. \tag{84}$$

Now, the beginning of the proof of [14, Theorem 1] gives that that (52) and (53) imply for all $m \geq 0$, $n \geq 0$ and all $|z| \geq \tau$, we have (see [14, (12)])

$$\begin{aligned} |\widehat{\mathcal{Q}}_t(z)^{-1}\varphi|_{weak,m} \leq \Bigg(& \|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0} C(2C + |z|) \left(\frac{\lambda}{\tau}\right)^n \\ & + \big(\|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0} C + \frac{C}{1-\tau}\big)(C\delta_{m+n}(t)) \left(\frac{1}{\tau}\right)^n \Bigg) \|\varphi\|_{\mathcal{B}_0} \\ & + \big(\|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0} C + \frac{C}{1-\tau}\big) \left(\frac{1}{\tau}\right)^n |\varphi|_{weak,m+n}\,. \end{aligned} \tag{85}$$

Fix $\gamma > 0$, write $H = \sup_{|z| \geq \tau, |z-1| > \gamma} \|\widehat{\mathcal{Q}}_0^{-1}(z)\|_{\mathcal{B}_0}$, and take

$$n_2 = \left[\frac{\ln(4C_1 HC(2C+2)}{\ln(\tau/\lambda)})\right]\,.$$

Then two applications of (84) as in the proof of [14, (15)] (taking $m = n_1$, $n = n_2$ in (85)) show that, taking,

$$\epsilon_0 = \sup\left\{ |t| \mid \delta_{n_1+n_2}(t)\left(HC + \frac{C}{1-\tau}\right)\left(\frac{1}{\tau}\right)^{n_2} \leq \frac{1}{4C_1}\right\}\,,$$

we have

$$\begin{aligned} \|\varphi\|_{\mathcal{B}_0} &\leq 2C_1 \|\widehat{\mathcal{Q}}_t(z)(\varphi)\|_{\mathcal{B}_0} + \frac{1}{2\delta_{n_1+n_2}(\epsilon_2)} |\widehat{\mathcal{Q}}_t(z)(\varphi)|_{weak,n_1+n_2} \\ &\leq K \|\widehat{\mathcal{Q}}_t(z)(\varphi)\|_{\mathcal{B}_0}\,, \end{aligned}$$

for all $|t| \leq \epsilon_0$, and any $|z| \in [\tau, 2]$ with $|z-1| > \gamma$, proving (83).

References

1. A. Avila, *Infinitesimal perturbations of rational maps,* Nonlinearity **15** (2002) 695–704.
2. A. Avila, M. Lyubich, and W. de Melo, *Regular or stochastic dynamics in real analytic families of unimodal maps,* Invent. Math. **154** (2003) 451–550.
3. V. Baladi, *Positive transfer operators and decay of correlations,* Advanced Series in Nonlinear Dynamics, Vol. 16, Singapore, World Scientific, 2000.
4. V. Baladi, *On the susceptibility function of piecewise expanding interval maps,* preprint (2007), to appear Comm. Math. Phys.

5. V. Baladi and D. Smania, *Smooth deformations of piecewise expanding unimodal maps,* (in preparation) 2007.
6. F. Bonetto, P. Falco, and G. Giuliani, *Analyticity of the SRB measure of a lattice of coupled Anosov diffeomorphisms of the torus,* Mathematical Physics Journal **45** (2004) 3282–3309.
7. O. Butterley and C. Liverani, *Smooth Anosov flows: correlation spectra and stability,* J. Modern Dynamics **1** (2007) 301–322.
8. D. Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems,* Invent. Math. **155** (2004) 389–449.
9. N. Dunford and J.T. Schwartz, *Linear Operators Part I. General theory,* with the assistance of W. G. Bade and R. G. Bartle, (reprint of the 1958 original), Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988.
10. G. Gallavotti, *Chaotic hypothesis: Onsager reciprocity and fluctuation-dissipation theorem,* J. Stat. Phys. **84** (1996) 899–926.
11. M. Jiang and R. de la Llave, *Linear response function for coupled hyperbolic attractors,* Comm. Math. Phys. **261** (2006) 379–404.
12. T. Kato, *Perturbation Theory for Linear Operators,* Second corrected printing of the second edition, Springer-Verlag, Berlin, 1984.
13. G. Keller, *Stochastic stability in some chaotic dynamical systems,* Monatshefte Math. **94** (1982) 313–333.
14. G. Keller and C. Liverani, *Stability of the spectrum for transfer operators,* Annali Scuola Normale Superiore di Pisa, **28** (1999) 141–152.
15. M. Lyubich, *Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture,* Ann. of Math. **149** (1999) 319–420.
16. M. Martens and W. de Melo, *The multipliers of periodic points in one-dimensional dynamics,* Nonlinearity **12** (1999) 217–227.
17. M. Mazzolena, *Dinamiche espansive unidimensionali: dipendenza della misura invariante da un parametro,* Master's Thesis, Roma 2, Tor Vergata (2007).
18. F. Riesz and B. Sz.-Nagy, *Functional analysis,* 1990 (reprint of the 1955 original), Dover Books on Advanced Mathematics, Dover, New York.
19. D. Ruelle, *Sharp zeta functions for smooth interval maps,* In: International Conference on Dynamical Systems (Montevideo 1995), Longman, 1996, 188–206.
20. D. Ruelle, *Differentiation of SRB states,* Comm. Math. Phys. **187** (1997) 227–241.
21. D. Ruelle, *General linear response formula in statistical mechanics, and the fluctuation-dissipation theorem far from equilibrium,* Phys. Lett. A **245** (1998) 220–224.
22. D. Ruelle, *Nonequilibrium statistical mechanics near equilibrium: computing higher-order terms,* Nonlinearity **11** (1998) 5–18.
23. D. Ruelle, *Differentiation of SRB states: Corrections and complements,* Comm. Math. Phys. **234** (2003) 185–190.
24. D. Ruelle, *Application of hyperbolic dynamics to physics: some problems and conjectures,* Bull. Amer. Math. Soc. **41** (2004) 275–278.
25. D. Ruelle, *Differentiating the a.c.i.m. of an interval map with respect to f,* Comm. Math. Phys. **258** (2005) 445–453.
26. D. Ruelle, Work in progress (personal communication, 2007).
27. D. Smania, *On the hyperbolicity of the period-doubling fixed point,* Trans. Amer. Math. Soc. **358** (2006) 1827–1846.
28. L.-S. Young, *What are SRB measures, and which dynamical systems have them?* J. Statist. Phys. **108** (2002) 733–754.

UMI 2924 CNRS-IMPA, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil; Permanent address: CNRS, UMR 7586, Institut de Mathématique de Jussieu, Paris

E-mail address: `baladi@math.jussieu.fr`

Departamento de Matemática, ICMC-USP, Caixa Postal 668, São Carlos-SP, CEP 13560-970 São Carlos-SP, Brazil

E-mail address: `smania@icmc.usp.br`