MODULUS OF ANALYTIC CLASSIFICATION FOR UNFOLDINGS OF RESONANT DIFFEOMORPHISMS

JAVIER RIBÓN

1. Introduction

The dynamics of germs of complex analytic germs of diffeomorphism in the neighborhood of $0 \in \mathbb{C}$ is quite well-known. Either they are formally linealizable or some iterate belongs to $\text{Diff}_1(\mathbb{C}, 0) \setminus \{Id\}$ where $\text{Diff}_1(\mathbb{C}, 0)$ is the group of germs with identity linear part. In the former case the classical question is whether the mappings are analytically linearizable. The difficulty arises for diffeomorphisms whose linear part is a non-periodic rotation. The linearizability is guaranteed whenever the linear part satisfies some diophantine condition [30]. The optimal diophantine property has been introduced by Bruno [2] [3]; he proves that it is sufficient, the proof of the optimality corresponds to Yoccoz [32]. Moreover there is a dichotomy: either the dynamics is conjugated to a linear map or it is chaotic [24]. The other case is the resonant one and in one variable it can be reduced to the study of tangent to the identity diffeomorphism (i.e. elements of $\text{Diff}_1(\mathbb{C}, 0)$). The formal, topological [13] [4] and analytical invariants [8] [31] [15] are completely described.

Denote by $\text{Diff}(\mathbb{C}^n, 0)$ the group of germs of complex analytic germs of diffeomorphism in the neighborhood of $0 \in \mathbb{C}^n$ and let $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ be its formal completion. In this paper we are interested in 1-dimensional unfoldings of elements of $\text{Diff}(\mathbb{C}, 0)$, i.e. the elements of the group

$$\text{Diff}_p(\mathbb{C}^2, 0) = \{\varphi(x, y) \in \text{Diff}(\mathbb{C}^2, 0) \ : \ x \circ \varphi = x\}.$$

More precisely we study the set

$$\text{Diff}_{pr}(\mathbb{C}^2, 0) = \{\varphi(x, y) \in \text{Diff}_p(\mathbb{C}^2, 0) \ : \ j^1\varphi_{|x=0} \text{ is periodic but } \varphi_{|x=0} \text{ is not}\}.$$

In other words we deal with all the unfoldings of non-linearizable resonant diffeomorphisms. We provide for them a complete system of analytic invariants.

As a consequence of the Jordan-Chevalley decomposition in linear algebraic groups the analytic classification of elements of $\text{Diff}_{pr}(\mathbb{C}^2, 0)$ can be obtained by resolving the analogous problem in

$$\text{Diff}_{p1}(\mathbb{C}^2, 0) = \{\varphi \in \text{Diff}_p(\mathbb{C}^2, 0) \ : \ \varphi_{|x=0} \in \text{Diff}_1(\mathbb{C}, 0) \setminus \{Id\}\}.$$

A complete system of formal invariants for $\phi \in \text{Diff}_1(\mathbb{C}, 0)$ is composed by the ideal $I(\phi(y) - y)$ and the residue $Res\phi \in \mathbb{C}$. We can generalize the definition of residue for $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$, we obtain a function $Res\varphi : Fix\varphi \to \mathbb{C}$ which is a

Date: 15th June 2006.
Universidad de Valladolid, Departamento de Algebra, Geometria y Topologia, Paseo de la Magdalena s/n, Valladolid, Spain, 47011.
e-mail address: jfribon@impa.br.
MSC-class: 37F75, 32H02, 32S65, 32A05.

formal invariant. Now the relation $\sigma \circ \varphi_1 = \varphi_2 \circ \sigma$ for $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ and $\sigma \in \text{Diff}(\mathbb{C}^2, 0)$ implies

- $x \circ \sigma$ depends only on x, i.e. $\sigma^* dx \wedge dx = 0$ (σ preserves $dx = 0$).
- $I(y \circ \varphi_2 - y) \circ \sigma = I(y \circ \varphi_1 - y)$ (σ conjugates the fixed points sets).
- $(Res\varphi_2) \circ \sigma \equiv Res\varphi_1$ (σ conjugates the residue functions).

We denote $I(y \circ \varphi_2 - y) \circ \sigma = I(y \circ \varphi_1 - y)$ by $\sigma(Fix\varphi_1) = Fix\varphi_2$, in particular $Fix\varphi_1 = Fix\varphi_2$ means $(y \circ \varphi_1 - y) = (y \circ \varphi_2 - y)$. A complete system of formal invariants for $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ is given by the 3-uple $(dx = 0, Fix\varphi, Res\varphi)$ [21]. We define the group $SF(\varphi)$

$$SF(\varphi) = \{\varphi \in \text{Diff}(\mathbb{C}^2, 0) : \sigma^*(dx = 0, Fix\varphi, Res\varphi) = (dx = 0, Fix\varphi, Res\varphi)\}$$

of symmetries of the formal invariants of φ. We are interested on the action that these symmetries induce in the fixed points set $Fix\varphi$. We define an equivalence relation $\sim_1$ in $SF(\varphi)$ given by

$$\sigma_1 \sim_1 \sigma_2 \text{ if } \left\{ \begin{array}{c} x \circ \sigma_1 \equiv x \circ \sigma_2 \\ y \circ \sigma_1 - y \circ \sigma_2 \in \sqrt{(y \circ \varphi - y)} \end{array} \right.$$

The group $SF(\varphi)/\sim_1$ is reduced to $< Id >$ for generic φ. Moreover $SF(\varphi)/\sim_1$ is always a finite group except when $SF(\varphi)/\sim_1$ is isomorphic to $\text{Diff}(\mathbb{C}, 0)$. This pathology only happens if φ is formally conjugated to $(x, y+y^{\nu+1}+\lambda y^{2\nu+1})$ for some $(\nu, \lambda) \in \mathbb{N} \times \mathbb{C}$. Then up to a "essentially" unique preparation mapping conjugating analytic sets and holomorphic functions we can restrict ourselves to the level sets $((y \circ \varphi - y), Res\varphi) = (I, Res)$ and conjugating mappings $\sigma = (x, g(x, y))$ such that $g(x, y) - y \in \sqrt{I}$. Such mappings are called *special* (with respect to the set of zeros $V(I)$ of I). We denote $\varphi_1 \overset{sp}{\sim} \varphi_2$ if φ_1 and φ_2 are conjugated by a special element of $\text{Diff}(\mathbb{C}^2, 0)$. We say that $\kappa \in \vartheta(B(0, r))$ is r-*moderated* for some $r \in \mathbb{R}^+$ if κ is univalent in $B(0, r)$. Moreover we say that κ is x_0-*special* if $\kappa_{|V(I) \cap [x=x_0]} \equiv Id$. We say only special if the value of x_0 is implicit. We can introduce now the main theorem in this paper.

Theorem 1.1. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix\varphi_1 = Fix\varphi_2$. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if there exists $r \in \mathbb{R}^+$ such that $(\varphi_1)_{|x=x_0}$ is conjugated to $(\varphi_2)_{|x=x_0}$ by a special r-moderated mapping for all x_0 in a pointed neighborhood of 0.*

We prove this theorem by providing a complete system of analytic invariants for the elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ and then for those of $\text{Diff}_{pr}(\mathbb{C}^2, 0)$.

Let us focus for a moment in the elements $\phi \in \text{Diff}_1(\mathbb{C}, 0)$. A pointed neighborhood of the origin is divided in $\nu(\phi(y)-y)-1$ basins of attraction (attracting petals) and $\nu(\phi(y) - 1) - 1$ basins of repulsion (repulsing petals). Moreover $\nu(\phi(y) - y)$ determines the class of ϕ modulo topological conjugation [4].

A complete system of analytic invariants for the elements of $\text{Diff}_1(\mathbb{C}, 0)$ has been provided independently by Ecalle [7] and Voronin [31]. The space of orbits $orb_V(\phi)$ of $\phi \in \text{Diff}_1(\mathbb{C}, 0)$ by restriction to a petal V is biholomorphic to $\mathbb{C}^*$ by a mapping ρ_V. We can consider $orb_V(\phi) \sim \mathbb{P}^1(\mathbb{C})$ by adding the fixed point at 0 and ∞. The mapping ρ_V can be lifted to V and since

$$\frac{1}{2\pi i} \log \rho_V \circ \varphi = \frac{1}{2\pi i} \log \rho_V + 1$$

then the function $1/(2\pi i) \log \rho_V$ is a so-called *Fatou coordinate* of ϕ in V. Denote $\nu = \nu(\phi(y)-y)-1$; there are 2ν intersections of petals (as many as petals) producing changes of charts between copies of $\mathbb{P}^1(\mathbb{C})$ corresponding to different petals. The space $orb(\phi)$ determines the class of analytic conjugacy of ϕ. Such a class can be expressed in terms of the 2ν changes of charts. In this way a complete system of analytic invariants is obtained. This is the interpretation of Martinet-Ramis [17] of the Ecalle-Voronin invariants.

Consider two diffeomorphisms $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C},0)$ which are formally conjugated by some $\hat{\sigma} \in \mathrm{Diff}(\mathbb{C},0)$. A conjugation ζ from $orb(\phi)$ to $orb(\eta)$ is determined by $\zeta(orb_V(\phi))$ and $\zeta_{|orb_V(\phi)}$ for any attracting petal V of ϕ_1. The possible images of $orb_V(\phi)$ are ν (as many as attracting petals of η) and $\zeta_{|orb_V(\phi)} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is of the form λz for some $\lambda \in \mathbb{C}^*$. We have to check out for each element of a group isomorphic to $< e^{2\pi i/\nu} > \times \mathbb{C}^*$ whether or not it is compatible with the changes of charts. The elements of $\widehat{\mathrm{Diff}}(\mathbb{C},0)$ conjugating ϕ_1 and ϕ_2 are of the form $\hat{\rho} \circ \hat{\sigma}$ where $\hat{\rho}$ belongs to the formal centralizer $\hat{Z}(\eta)$ of η. The group $\hat{Z}(\eta)$ is isomorphic to $< e^{2\pi i/\nu} > \times \mathbb{C}$ and then we obtain $\hat{Z}(\eta)/ < \eta > \sim < e^{2\pi i/\nu} > \times \mathbb{C}^*$. Moreover there is a canonical bijection Γ from $\hat{Z}(\eta)/ < \eta >$ onto the space of candidates to conjugations of the spaces of orbits. The convergence of the elements of a class $\hat{\rho} \in \hat{Z}(\eta)/ < \eta >$ is equivalent to the compatibility with the changes of charts of $\Gamma(\hat{\rho})$. As a consequence we can determine the power series developpements of the analytic mappings conjugating ϕ and η in terms of their changes of charts. We carry the same program in $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$, whenever we have $\varphi_1 \overset{sp}{\sim} \varphi_2$ we identify the special analytic conjugations.

The study of deformations of elements $\phi \in \mathrm{Diff}_1(\mathbb{C},0)$ (parabolic implosion) is interesting to describe the evolution of the Julia sets when we deform a given rational mapping [12] [29] [23] [6]. Lavaurs, Shishikura and Oudkerk develop independently extensions of the Fatou coordinates of $\phi \in \mathrm{Diff}_1(\mathbb{C},0)$. Given an unfolding $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ these extensions are defined in some sectors in the parameter space. This point of view has been generalized recently by Mardesic-Roussarie-Rousseau [16] to obtain a complete system of analytic invariants for generic unfoldings of generic codimension 1 elements $\phi \in \mathrm{Diff}_1(\mathbb{C},0)$. In this paper we remove all the genericity conditions and the codimension 1 hypothesis. They study elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ of the form

$$\varphi(x,y) = (x, y - x + c_1(x)y^2 + o(y^2))$$

with $c_1(0) \neq 0$. The fixed point $(0,0)$ splits in two fixed points for the values of the parameter x close to 0. They apply a refinement of Shishikura's construction [29] to get extensions of the Fatou coordinates supported in Lavaurs sectors V_δ^L describing an angle as close to 4π as desired in the x-variable. Indeed the extensions are multivaluated around $x = 0$. Then they define analytic invariants a la Martinet-Ramis. More precisely they define a classifying space $\mathcal{M}$ and a mapping $m_\varphi : V_\delta^L \to \mathcal{M}$. Then φ and ζ are analytically conjugated if and only if $m_\varphi \equiv m_\zeta$. We skip here the details of the definition of m_φ but we stress that $m_\varphi(x_0)$ depends only on $\varphi_{|x=x_0}$. We generalize the definition of m_φ for all $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.

We say that $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ (resp. $\phi \in \mathrm{Diff}_1(\mathbb{C},0)$) is *analytically trivial* if φ (resp. ϕ) is the exponential of a germ of nilpotent vector field. We can classify the elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ depending on their rigidity properties with respect to the analytic conjugation.

- Flexible. φ is analytically trivial. In this case $Fix\varphi = Fix\zeta$, $Res\varphi \equiv Res\zeta$ and $m_\varphi \equiv m_\zeta$ imply $\varphi \overset{sp}{\sim} \zeta$. Moreover every special formal transformation conjugating φ and ζ is analytic. The analytic centralizer $Z(\varphi)$ contains a 1-parameter group.

- Rigid. φ is not analytically trivial but we still have that $Fix\varphi = Fix\zeta$, $Res\varphi \equiv Res\zeta$ and $m_\varphi \equiv m_\zeta$ imply $\varphi \overset{sp}{\sim} \zeta$.

The analytic special centralizer of $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ is a discrete group in the non-flexible setting. Moreover a mapping σ conjugating φ and ζ is determined by its restriction $\sigma_{|x=x_0}$ to a generic line $x = x_0$.

A sufficient condition for φ to be rigid is that $\varphi_{|x=0}$ is non-analytically trivial. In such a case the analytic centralizer $Z(\varphi_{|x=0})$ is discrete and such a property "extends" to the nearby values of the parameter. Therefore provided a good choice of κ conjugating $\varphi_{|x=0}$ and $\zeta_{|x=0}$ it can be extended in a unique way to obtain a special $\sigma \in \text{Diff}\,(\mathbb{C}^2, 0)$ such that $\sigma_{|x=0} \equiv \kappa$ and $\sigma \circ \varphi = \zeta \circ \sigma$.

The main theorem in [16] implies that in their context every unfolding is either flexible or rigid. This is not true, there is a small mistake in their proof. There exists a third possibility:

- Semi-rigid. φ is neither flexible nor rigid. A necessary condition to be semi-rigid is that φ is not analytically trivial but $\varphi_{|x=0}$ is.

We provide non-analytically trivial mappings $\varphi, \zeta \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ such that

1. $Fix\varphi = Fix\zeta$, $Res\varphi \equiv Res\zeta$ and $\varphi_{|x=0} \equiv \zeta_{|x=0}$.
2. φ and ζ are conjugated by an injective special analytic mapping σ defined in $|y| < C_0/\sqrt[\nu]{|\ln x|}$ such that $\sigma(e^{2\pi i}x, y) = \zeta \circ \sigma(x, y)$.

There exists always a mapping conjugating $\varphi_{|x=x_0}$ and $\zeta_{|x=x_0}$, namely $\sigma_{|x=x_0}$ if $x_0 \neq 0$ and Id for $x_0 = 0$. Since $m_\varphi(x_0)$ depends only on $\varphi_{|x=x_0}$ we obtain $m_\varphi \equiv m_\zeta$. Moreover the non-flexibility of φ implies $\varphi \overset{sp}{\not\sim} \zeta$ since otherwise we would have $\sigma \in \text{Diff}\,(\mathbb{C}^2, 0)$. Roughly speaking we have that $Z(\varphi_{|x=x_0})$ is discrete for x_0 generic but $Z(\varphi_{|x=0})$ contains a 1-parameter group. Thus we do not obtain that $\lim_{x_0 \to 0} \sigma_{|x=x_0}$ exists like in the rigid case since $\sigma_{|x=x_0}$ is no longer forced to adhere a discrete set when $x_0 \to 0$. The construction is rather flexible and we can suppose that $I(y \circ \varphi - y) = I(f)$ for every function $f \in \mathbb{C}\{x, y\}$ such that $f(0) = (\partial f/\partial y)(0) = 0$. In particular if we choose $f = y^2 - x$, we obtain a counter-example to the main theorem in [16]. Nevertheless their theorem remains true in the generic non-semi-rigid case. The example shows that we can not remove the moderated hypothesis in theorem 1.1; this condition can be expressed in terms of the changes of charts. Our complete system of analytic invariants for the elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ is the generalization of the Mardesic-Roussarie-Rousseau's system with a little twist to include the moderated hypothesis.

Next we explain briefly how to define Fatou coordinates for $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. In [16] it is a key tool to find "transversals" to the dynamics of φ. A priori this does not make much sense. Nevertheless they lift $\varphi_{|x=x_0}$ to a subset $C(x_0)$ of the universal covering of $\mathbb{P}^1(\mathbb{C})$ minus two points. The lifting's dynamics is flow-like, it is very similar to $\exp(\partial/\partial z) = z + 1$. We can choose straight lines transversal to $\mathbb{R}$. A transversal T and its image $\varphi(T)$ enclose a strip $S(T)$. The space of orbits of $\varphi_{|S(T)}$ is biholomorphic to $\mathbb{C}^*$. We can identify 0 and ∞ with the fixed points in the ends of T. The rigidity of the complex structure of $\mathbb{C}^*$ (or $\mathbb{P}^1(\mathbb{C})$) provides a Fatou coordinate in the neighborhood of $S(T)$ in $x = x_0$. It can be extended by iteration to the points of $C(x_0)$ whose orbits intersect $S(T)$. The construction depends holomorphically on x_0.

If the unfolded mapping $\phi \in \text{Diff}_1(\mathbb{C},0)$ is of codimension bigger than 1 the method can not be replicated. In general we have $\sharp(Fix\varphi \cap [x=x_0]) \geq 3$, it is much more difficult to find expressions for covering mappings of $\mathbb{P}^1(\mathbb{C}) \setminus (Fix\varphi \cap [x=x_0])$ such that the lifting of $\varphi_{|x=x_0}$ is flow-like.

We use a different point of view. Denote by $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ the set of germs of vector field of the form $f(x,y)\partial/\partial y$ whith $f(0)=(\partial f/\partial y)(0)=0$. Given $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2,0)$ there always exists $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ such that $y \circ \varphi - y \circ \exp(X) \in (y \circ \varphi - y)^2$. We say that $\exp(X)$ is a *convergent normal form* of φ since they are formally conjugated ($Fix\varphi = SingX$ and $Res\varphi \equiv Res(\exp(X))$) and the infinitesimal generator X of $\exp(X)$ is convergent. Fixed $x_0 \in B(0,\delta)$ the equation $X(\psi)=1$ defines a function in the universal covering of $B(0,\epsilon) \setminus (Fix\varphi \cap [x=x_0])$, it is unique up to a constant. We say that ψ is an *integral of the time form* of X. Clearly ψ is locally injective. Since we have

$$X(\psi)=1 \Rightarrow \psi \circ \exp(tX) = \psi + t \quad \forall t \in \mathbb{C}$$

then the dynamics of $\exp(X)$ in the ψ-coordinate is given by $z \to z+1$. It can be easily checked out that $\psi \circ \varphi \sim \psi + 1$. The natural candidates to transversals are the curves $\gamma : (-\infty,\infty) \to [x=x_0]$ given by $\gamma(t) = \exp(t\mu X)(x_0,y_0)$ where $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$. If we do not choose y_0 and μ carefully there is no guarantee that γ is defined for all $t \in (-\infty,\infty)$ or that the α and ω limits are fixed points. The equivalent tasks in [16] are trivial since there exists a precise expression of the covering transformation. Nevertheless this problem can be solved via the description of the dynamics of the real part of a vector field in $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ which can be implicitly found in [22]. This point of view can be used even if we do not work with unfoldings and just with discrete deformations of $\phi \in \text{Diff}_1(\mathbb{C},0) \setminus \{Id\}$ since there exists a universal theory of unfoldings of germs of vector field in one variable [11].

The approach in [22] is topological. We want to identify what formal conjugations are analytic and to study the dependance of the domain of definition of a conjugation with respect to the parameter. A more analytical approach is required. We use some of the techniques in [22] like the dynamical splitting and also others like the study of polynomials vector fields related to deformations introduced by Douady-Estrada-Sentenac in [6]. The polynomial vector fields that we consider are different. Ours are related to the infinitesimal properties of the unfolding. They appear after blow-up transformations.

In order to describe the dynamics of $Re(\mu X)$ for $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ we are interested on undertanding the nature of the set $B_X \subset \mathbb{S}^1 \times \mathbb{S}^1$ given by $(\lambda_0,\mu_0) \not\in B_X$ if $Re(\mu X)_{|[0,\delta)\lambda}$ is stable in the neighborhood of (λ_0,μ_0). We do not give now the precise definition of stability since it involves infinitesimal properties of $Re(\mu X)$. Anyway, in particular we have that the points of $SingX$ are either attracting, repulsing or parabolic for $Re(\mu X)_{|[0,\delta)\lambda}$ and $(\mu,\lambda) \sim (\mu_0,\lambda_0)$. Then their basins of attraction and repulsion are open sets. We prove that the set $B_{X,\lambda}$ given by $\{\lambda\} \times B_{X,\lambda} = B_X \cap (\{\lambda\} \times \mathbb{S}^1)$ is finite and it depends continuously on λ.

We say that (K_1,μ_1), …, (K_l,μ_l) is an EV-covering of $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ if

- K_j is a compact connected subset of $\mathbb{S}^1$ for all $j \in \{1,\dots,l\}$.
- $\mu_j \in \mathbb{S}^1 \setminus \cup_{\lambda \in K_j} B_{X,\lambda}$ for all $j \in \{1,\dots,l\}$.
- $\mathbb{S}^1 = \cup_{j=1}^{l} \dot{K}_j$ (we denote the interior of K_j by $\dot{K}_j$).

An EV-covering always exists. An EV-covering for φ is an EV-covering for some $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ such that $\exp(X)$ is a convergent normal form of φ. The definition does not depend on the choice of X but only on $Fix\varphi$ and $Res\varphi$.

Let $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2,0)$ with convergent normal form $\exp(X)$. Consider an element (K,μ) of an EV-covering of $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$. We denote by $\alpha^{\mu X}$ and $\omega^{\mu X}$ the α and ω limits respectively with respect to the vector field $Re(\mu X)$ in a domain $B(0,\delta) \times B(0,\epsilon)$. We define the set $Reg(\epsilon,\mu X,K)$ of connected components of $(\alpha^{\mu X},\omega^{\mu X})^{-1}(SingX \times SingX) \setminus SingX$. The elements of $Reg(\epsilon,\mu X,K)$ are called regions of $Re(\mu X)$. They are open sets in $[0,\delta)K \times B(0,\epsilon)$ by stability. We choose transversals $T = \exp(\mathbb{R}\mu X)(x_0,y_0)$ for $(x_0,y_0) \in H \in Reg(\epsilon,\mu X,K)$.

Let $H \in Reg(\epsilon,\mu X,K)$. The set $H \cap [x=0]$ is a union of $c(H)$ connected components of $(\alpha^{\mu X},\omega^{\mu X})^{-1}(\{(0,0)\} \times \{(0,0)\}) \setminus \{(0,0)\}$ whereas $H \cap [x=x_0]$ is always connected for $x_0 \in (0,\delta)K$. We denote $H \in Reg_{c(H)}(\epsilon,\mu X,K)$.

Supposed $(\alpha^{\mu X})_{|H} \equiv (\omega^{\mu X})_{|H}$ then H is a topological product and $c(H)=1$. Otherwise $\alpha^{\mu X}(H(x))$ and $\omega^{\mu X}(H(x))$ are different points tending to a single one $(0,0)$ when $x \to 0$ in $(0,\delta)K$. This collapse splits $H \cap [x=0]$ in two connected components.

Denote $\nu = \nu(y \circ \varphi(0,y) - y) - 1$. Consider one of the 2ν connected components J of $(\alpha^{\mu X},\omega^{\mu X})^{-1}(\{(0,0)\} \times \{(0,0)\}) \setminus \{(0,0)\}$. Then J is contained in a unique region $H \in Reg(\epsilon,\mu X,K)$. Moreover the space of orbits of $\varphi_{|J \cup (H \setminus [x=0])}$ is homeomorphic to $[0,\delta)K \times \mathbb{P}^1(\mathbb{C})$ by a mapping holomorphic in the interior of H. The Ecalle-Voronin invariants of $\varphi_{|x=0}$ can be extended in a natural way to obtain 2ν changes of charts between different copies of $[0,\delta)K \times \mathbb{P}^1(\mathbb{C})$. Again they depend holomorphicaly on $x \in (0,\delta)\dot{K}$ and extend continuously to $x=0$. The variety obtained by taking 2ν copies of $[0,\delta)K \times \mathbb{P}^1(\mathbb{C})$ and doing the 2ν identifications corresponding to the changes of charts is called the μ-orbit space of φ at K.

The μ-space of orbits is not the space of orbits of φ. Given two different connected components J_1, J_2 of $H \cap [x=0]$ for some $H \in Reg_2(\epsilon,\mu X,K)$ we have that the spaces of orbits of $\varphi_{|J_1 \cup (H \setminus [x=0])}$ and $\varphi_{|J_2 \cup (H \setminus [x=0])}$ are identified outside of $x=0$. In this way we obtain $\sharp Reg_2(\epsilon,\mu X,K)$ identifications not contained in the μ-orbit space. Anyway, the structure of the orbit space of φ can be deduced from the structure of the μ-orbit space since the extra identifications depend only on X. Nevertheless the existence of return mappings makes non-evident that the μ-orbit space is an analytic invariant. We prove that this is the case. Basically the 2ν changes of charts that we obtain for all the elements of the EV-covering are the base for a complete system of invariants. Our construction, even if different, it is analogous to the Mardesic-Roussarie-Rousseau's one in many aspects.

We introduce next some of the analytic aspects of the construction. Given a region $H \in Reg(\epsilon,\mu X,K)$ there exists a unique vector field X_H^{φ} (the Lavaurs vector field) such that it is continuous in $\overline{H}$, holomorphic in $\dot{H}$ and fulfills $\varphi = \exp(X_H^{\varphi})$. Our construction implies that $(X_H^{\varphi}(y) - X(y))/(y \circ \varphi - y)$ is a continuous function in $\overline{H}$ vanishing at $\overline{H} \cap Fix\varphi$. The infinitesimal generator $\hat{X}$ of φ (i.e. the only formal nilpotent derivation such that $\varphi = \exp(\hat{X})$) satisfies $\hat{X}(y) - X(y) \in (y \circ \varphi - y)^2$. Then we deduce that $\hat{X}$ is the asymptotic development of X_H^{φ} in the neighborhood of $\overline{H} \cap Fix\varphi$ in H until the first non-zero term. This fact is a consequence of our improvement of the constructions in [29] and [16]. We introduce convergent normal forms not just to obtain a model for φ in the regions. This would be guaranteed by choosing $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ such that $Fix\varphi = SingX$ and $Res\varphi \equiv Res(\exp(X))$. We required $y \circ \varphi - y \circ \exp(X) \in (y \circ \varphi - y)^2$ with the hope of controlling the behavior of φ in the neighborhood of the fixed points. Moreover the introduction of normal forms has still another advantage; a Fatou coordinate of φ is defined up to an additive constant. Thus a normalizing condition is required in the construction. The classical choices are not invariant by iteration. We give an invariant by iteration condition by prescribing the behavior of the Fatou coordinate in the neighborhood of the fixed points. We define Fatou coordinates in strips but the definitions paste together. This property is interesting since otherwise to estimate the asymptotic behavior of a Fatou coordinate in a region it is necessary to do it in a strip and then propagating the estimates by iteration.

Since an element η in the formal special centralizer of φ such that $j^1\eta = Id$ is determined by the first non-zero term $((y \circ \eta - y)/(y \circ \varphi - y))_{|Fix\varphi}$ (or more formally by the class of $(y \circ \eta - y)/(y \circ \varphi - y)$ modulo $\sqrt{(y \circ \varphi - y)}$) then the extra term is going to be a key ingredient to identify what special formal transformations conjugating two elements $\varphi, \zeta \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ converge.

Given $\zeta \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix\varphi = Fix\zeta$ and $Res\varphi \equiv Res\zeta$ the property $m_\varphi(x_0) = m_\zeta(x_0)$ for $x_0 \in [0, \delta)K$ and a member of an EV-covering (K, μ) implies that they are conjugated by a transformation whose expression in $H \cap [x = x_0]$ for a region $H \in Reg(\epsilon, \mu X, K)$ is of the form

$$\exp(c(x_0) X_H^\zeta)_{|x=x_0} \circ \sigma_H(\varphi, \zeta)_{|x=x_0}$$

for some $c(x_0) \in \mathbb{C}$. These definitions paste together, we obtain a conjugation defined in the neighborhood of $Fix\varphi \cap [x = x_0]$. The mapping $\sigma_H(\varphi, \zeta)$ is continuous in $\overline{H}$ and holomorphic in $\dot{H}$. Its behavior is moderated-like. Thus the size of the domain of definition depends basically on $\exp(c(x_0) X_H^\zeta)$. Our asymptotic study proves that the latter map behaves like $\exp(c(x_0)X)$ where $\exp(X)$ is a convergent normal form of ζ. The moderated hypothesis in theorem 1.1 is equivalent to the boundness of $c(x)$ in a pointed neighborhood of 0. Roughly speaking once we fix moderated choices $\sigma_H(\varphi, \zeta)$ of mappings conjugating φ and ζ the choice of an element of the centralizer of ζ providing an analytic conjugation is bounded. Then we can conclude the proof of theorem 1.1 with an argument of Riemann's kind.

Finally let us remark that the study of germs of diffeomorphism is a useful tool to classify singular foliations. For instance consider codimension 1 complex analytic foliations defined in a 2-dimensional manifold. Up to a birrational transformation we can suppose that all the singularities are reduced. Denote by $\Omega_{red}(\mathbb{C}^2, 0)$ the set of germs of reduced codimension 1 complex analytic singularity in the neighborhood of $0 \in (\mathbb{C}^2, 0)$. Let $\omega \in \Omega_{red}(\mathbb{C}^2, 0)$; if the quotient of the eigenvalues $q(\omega)$ is in the domain of Poincaré (i.e. $q(\omega) \notin \mathbb{R}^- \cup \{0\}$) then ω is conjugated to its linear part. Anyway, the analytic class of $\omega \in \Omega_{red}(\mathbb{C}^2, 0)$ is determined by the analytic class of the holonomy of ω along a "strong" integral curve [20]. Such a holonomy is formally linearizable if $q(\omega) \in \mathbb{R}^- \setminus \mathbb{Q}^-$ and resonant whenever $q(\omega) \in \mathbb{Q}^- \cup \{0\}$. Traditionally a singularity $\omega \in \Omega_{red}(\mathbb{C}^2, 0)$ such that $q(\omega) \in \mathbb{Q}^-$ is called resonant whereas it is called a saddle-node if $q(\omega) = 0$. The modulus of analytic classification for both resonant and saddle-node singularities have been described by Martinet-Ramis [18] [17]. Then it is natural to study unfoldings of resonant diffeomorphisms in order to study unfoldings of resonant singularities and saddle-nodes. This point of view has been developped by Martinet, Ramis [25], Glutsyuk [9] and Mardesic-Roussarie-Rousseau [16]. Moreover Rousseau classifies generic unfoldings of codimension 1 saddle-nodes [28]. This program can not be carried in higher codimension without a complete system of analytic invariants for unfoldings of elements of $\text{Diff}_1(\mathbb{C}, 0)$ of codimension greater than 1. We remove such an obstacle in this paper.

We comment the structure of the paper. In section 3 we introduce the concepts of infinitesimal generator and convergent normal forms for germs of unipotent diffeomorphism. We prove that every element of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ has a convergent normal form. Section 4 is basically a quick survey about the topological, formal and analytic classifications of tangent to the identity germs of diffeomorphism in one variable. We study the formal properties of elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ in section 5. We describe the formal invariants and the structure of the formal special centralizer of an element of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$. We also reduce the problem of classifying unfoldings of resonant diffeomorphisms to the tangent to the identity case via the semisimple-unipotent decomposition. Section 6 deals with the special case of unfoldings in which the fixed points set is parameterized by x. We can use then a parameterized version of the Ecalle-Voronin theory. We introduce the main results of the paper in this simpler case and we prove slightly sharper versions. In section 7 we give a concept of stability for the real flows of elements of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and then we describe their topological behavior in the stable zones. In section 8 we give a quantitative mesure of how much $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ is similar to a convergent normal form. The estimates are a key ingredient in our refinement of the Shishikura-Mardesic-Roussarie-Rousseau's construction. In this way we obtain Fatou coordinates with controlled asymptotic behavior in the neighborhood of the fixed points. Finally in section 9 we define the analytic invariants, we describe its nature and compare with the ones in [16]. In section 10 we prove the main theorem, moreover we provide a complete system of analytic invariants in both the general and the particular non-semi-rigid cases. We prove the optimality of our results in section 11.

2. Notations and definitions

Let $\text{Diff}\,(\mathbb{C}^n, 0)$ be the group of complex analytic germs of diffeomorphism at $0 \in \mathbb{C}^n$. Consider coordinates $(x_1, \ldots, x_{n-1}, y) \in \mathbb{C}^n$. We say that $\varphi \in \text{Diff}\,(\mathbb{C}^n, 0)$ is a parameterized diffeomorphism if $x_j \circ \varphi = x_j$ for all $1 \leq j < n$. Denote by $\text{Diff}_u(\mathbb{C}^n, 0)$ the subgroup of $\text{Diff}\,(\mathbb{C}^n, 0)$ composed by unipotent diffeomorphisms, i.e. $\varphi \in \text{Diff}_u(\mathbb{C}^n, 0)$ if $j^1\varphi$ is unipotent. We define

$$\text{Diff}_{up}(\mathbb{C}^n, 0) = \text{Diff}_u(\mathbb{C}^n, 0) \cap \text{Diff}_p(\mathbb{C}^n, 0)$$

the group of germs of unipotent parameterized diffeomorphisms (or up-diffeomorphisms for shortness). The formal completions of the previous groups will be denoted with a hat, for instance $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is the formal completion of $\text{Diff}\,(\mathbb{C}^n, 0)$.

Let $\text{Diff}_1(\mathbb{C}, 0)$ be the subgroup of $\text{Diff}\,(\mathbb{C}, 0)$ of germs of tangent to the identity diffeomorphisms, i.e. $\varphi \in \text{Diff}\,(\mathbb{C}, 0)$ belongs to $\text{Diff}_1(\mathbb{C}, 0)$ if $(\partial\varphi/\partial y)(0) = 1$. We define the set

$$\text{Diff}_{p1}(\mathbb{C}^2, 0) = \{\varphi \in \text{Diff}_p(\mathbb{C}^2, 0) : \varphi_{|x=0} \in \text{Diff}_1(\mathbb{C}, 0) \setminus Id\}.$$

Then $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ is the set of one dimensional unfoldings of one dimensional tangent to the identity germs of diffeomorphism (excluding the identity).

We denote $\varphi_1 \sim \varphi_2$ if $\varphi_1, \varphi_2 \in \text{Diff}\,(\mathbb{C}^n, 0)$ are analytically conjugated.

We define a formal vector field $\hat{X}$ as a derivation of the maximal ideal of the ring $\mathbb{C}[[x_1, \ldots, x_{n-1}, y]]$. We also express $\hat{X}$ in the more conventional form

$$\hat{X} = \sum_{j=1}^{n-1} \hat{X}(x_j) \frac{\partial}{\partial x_j} + \hat{X}(y) \frac{\partial}{\partial y}.$$

We consider the set $\hat{\mathcal{X}}_N(\mathbb{C}^n,0)$ of nilpotent formal vector fields, i.e. the formal vector fields $\hat{X}$ such that $j^1\hat{X}$ is nilpotent. We denote by $\mathcal{X}(\mathbb{C}^n,0)$ the set of germs of analytic vector field at $0 \in \mathbb{C}^n$.

We denote the rings $\mathbb{C}\{x_1,\dots,x_{n-1},y\}$ and $\mathbb{C}[[x_1,\dots,x_{n-1},y]]$ by ϑ_n and $\hat{\vartheta}_n$ respectively. We denote $f \sim g$ if $f = O(g)$ and $g = O(f)$.

Let $\varphi \in \mathrm{Diff}\,(\mathbb{C}^n,0)$. Denote by $Fix\varphi$ the fixed points set of φ. Denote by $Z(\varphi)$ and $\hat{Z}(\varphi)$ the centralizer and the formal centralizer of φ, i.e. the centralizers of φ in $\mathrm{Diff}\,(\mathbb{C}^n,0)$ and $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ respectively.

3. The infinitesimal generator

In this section we associate a formal vector field to every element of $\mathrm{Diff}_{up}(\mathbb{C}^n,0)$. The properties of this object can be used to provide a complete system of formal invariants for the up-diffeomorphisms [21]. Here, we introduce the properties that we will use later on.

Let $X \in \mathcal{X}(\mathbb{C}^n,0)$; suppose that X is singular at 0. We denote by $\exp(tX)$ the flow of the vector field X, it is the unique solution of the differential equation

$$\frac{\partial}{\partial t}\exp(tX) = X(\exp(tX))$$

with initial condition $\exp(0X) = Id$. We define the exponential $\exp(X)$ of X as $\exp(1X)$. We can define the exponential operator for $\hat{X} \in \hat{\mathcal{X}}_N(\mathbb{C}^n,0)$. Moreover the definition coincides with the previous one if $\hat{X}$ is convergent. We define

$$\begin{array}{llll}\exp(\hat{X}): & \hat{\vartheta}_n & \rightarrow & \hat{\vartheta}_n \\ & g & \rightarrow & \sum_{j=0}^{\infty}\frac{\hat{X}^{\circ(j)}}{j!}(g).\end{array}$$

The nilpotent character of $\hat{X}$ implies that the power series $\exp(\hat{X})(g)$ converges in the Krull topology for all $g \in \hat{\vartheta}_n$. Moreover, since $\hat{X}$ is a derivation then $\exp(\hat{X})$ acts like a diffeomorphism, i.e.

$$\exp(\hat{X})(g_1g_2) = \exp(\hat{X})(g_1)\exp(\hat{X})(g_2)$$

for all $g_1, g_2 \in \hat{\vartheta}_n$. Then we can use the more conventional notation

$$\exp(\hat{X}) = \left(\sum_{j=0}^{\infty}\frac{\hat{X}^{\circ(j)}}{j!}(x_1),\dots,\sum_{j=0}^{\infty}\frac{\hat{X}^{\circ(j)}}{j!}(x_{n-1}),\sum_{j=0}^{\infty}\frac{\hat{X}^{\circ(j)}}{j!}(y)\right).$$

Moreover $j^1\exp(\hat{X}) = \exp(j^1\hat{X})$, thus $j^1\exp(\hat{X})$ is a unipotent linear isomorphism. The following proposition is classical.

Proposition 3.1. *The mapping* $\exp : \hat{\mathcal{X}}_N(\mathbb{C}^n,0) \to \widehat{\mathrm{Diff}}_u(\mathbb{C}^n,0)$ *is a bijection.*

Consider the inverse mapping $\log : \widehat{\mathrm{Diff}}_u(\mathbb{C}^n,0) \to \hat{\mathcal{H}}_N(\mathbb{C}^n,0)$. We can interpret $\varphi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n,0)$ as a linear operator $\varphi : \hat{\mathfrak{m}} \to \hat{\mathfrak{m}}$ where $\hat{\mathfrak{m}}$ is the maximal ideal of $\hat{\vartheta}_n$. Denote by Θ the operator $\varphi - Id$, we have

$$(\log\varphi)(g) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{\Theta^{\circ(j)}}{j}(g)$$

for all $g \in \hat{\vartheta}_n$. The power series in the right hand side converges in the Krull topology since φ is unipotent. Moreover $j^1(\log\varphi) = \log(j^1\varphi)$ is nilpotent and $\log\varphi$ satisfies the Leibnitz rule. We say that $\log\varphi$ is the *infinitesimal generator* of φ. The exponential mapping has a geometrical nature; next proposition claims that $\log\varphi$ preserves the orbits of $\partial/\partial y$ for $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^n,0)$ and also that $Sing(\log\varphi) = Fix\varphi$.

Proposition 3.2. *Let $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^n,0)$. Then $\log\varphi$ is of the form $\hat{u}(y\circ\varphi - y)\partial/\partial y$ for some formal unit $\hat{u} \in \hat{\vartheta}_n$.*

Proof. Let $\Theta = \varphi - Id$. We have that $\log\varphi$ is of the form $\hat{f}\partial/\partial y$ since $\Theta(x_j) = 0$ and then $\Theta^{\circ(k)}(x_j) = 0$ for all $j \in \{1,\dots,n-1\}$ and all $k \in \mathbb{N}$. We have $\Theta(y) = y\circ\varphi - y$, moreover since

$$g\circ\varphi = g + \frac{\partial g}{\partial y}(y\circ\varphi - y) + \sum_{j=2}^{\infty} \frac{\partial^j g}{\partial y^j}\frac{(y\circ\varphi - y)^j}{j!} \tag{1}$$

we obtain that $\Theta^{\circ(2)}(y) \in (y\circ\varphi - y)\hat{\mathfrak{m}}$ where $\hat{\mathfrak{m}}$ is the maximal ideal of $\hat{\vartheta}_n$. Again by using the Taylor series expansion we can prove that $\Theta^{\circ(j)}(y) \in (y\circ\varphi - y)\hat{\mathfrak{m}}$ for all $j \geq 2$. Thus $\log\varphi = (\log\varphi)(y)\partial/\partial y$ is of the form $\hat{u}(y\circ\varphi - y)\partial/\partial y$ for some $\hat{u} \in \hat{\vartheta}_n$ such that $\hat{u}(0) = 1$. □

Let $\varphi = \exp(\hat{u}(y\circ\varphi - y)\partial/\partial y) \in \mathrm{Diff}_{up}(\mathbb{C}^n,0)$. We say that $\alpha \in \mathrm{Diff}_{up}(\mathbb{C}^n,0)$ is a *convergent normal form* of φ if $\log\alpha = u(y\circ\varphi - y)\partial/\partial y$ for some $u \in \vartheta_n$ and $y\circ\varphi - y\circ\alpha \in (y\circ\varphi - y)^2$. The last condition is equivalent to $\hat{u} - u \in (y\circ\varphi - y)$. If φ is a convergent normal form of itself, i.e. if $\log\varphi \in \mathcal{X}(\mathbb{C}^n,0)$ then we say that φ is *analytically trivial*.

Proposition 3.3. *Let $\varphi = \exp(\hat{u}(y\circ\varphi - y)\partial/\partial y) \in \mathrm{Diff}_{up}(\mathbb{C}^n,0)$. Then φ has a convergent normal form.*

Proof. Let $\Theta = \varphi - Id$. We have $(\log\varphi)(y) = \sum_{j=1}^{l}(-1)^{j+1}\Theta^{\circ(j)}(y)/j$. Consider the decomposition $f_1^{l_1}\dots f_p^{l_p}g_1\dots g_q$ of $y\circ\varphi - y \in \vartheta_n$ in irreducible factors where $l_j \geq 2$ for all $j \in \{1,\dots,p\}$. We define

$$u_2 = \frac{\ln(1+z)}{z}\circ\frac{\partial(y\circ\varphi - y)}{\partial y}.$$

Denote $f = y\circ\varphi - y$; by equation 1 we obtain that

$$(\log\varphi)(y)/(y\circ\varphi - y) - \left(1 - \frac{\partial f/\partial y}{2} + \frac{(\partial f/\partial y)^2}{3} + \dots\right) \in (f_1\dots f_p g_1\dots g_q).$$

We deduce that $\hat{u} - u_2$ belongs to $(g_1\dots g_p)$.

We claim that $\Theta^{\circ(k)}(y) \in (f_1^{l_1+k-1}\dots f_p^{l_p+k-1})$ for all $k \in \mathbb{N}$. The result is true for $k = 1$ by equation 1. Since

$$f_j\circ\varphi - f_j \in (f_j^2) \ \text{ and } \ h\circ\varphi - h \in (y\circ\varphi - y)$$

for all $h \in \hat{\vartheta}_n$ we deduce that

$$\Theta^{\circ(k)}(g) \in (f_j^{l_j+k-1}) \implies \Theta^{\circ(k+1)}(g) \in (f_j^{l_j+k}).$$

Denote $l = \max(l_1,\dots,l_p)$ and $u_1 = (\sum_{j=1}^{l}(-1)^{j+1}\Theta^{\circ(j)}(y)/j)/f$. We have that $\hat{u} - u_1 \in (f_1^{l_1}\dots f_p^{l_p})$. The function $u_1 - u_2$ belongs to the formal ideal $(f_1^{l_1}\dots f_p^{l_p}, g_1\dots g_q)$; by faithful flatness there exist $A, B \in \vartheta_n$ such that

$$u_1 - u_2 = Af_1^{l_1}\dots f_p^{l_p} + Bg_1\dots g_q.$$

We define $u = u_1 - A f_1^{l_1} \dots f_p^{l_p} = u_2 + B g_1 \dots g_q$. By construction it is clear that $\hat{u} - u$ belongs to $(f_1^{l_1} \dots f_p^{l_p}) \cap (g_1 \dots g_q)$ and then to $(y \circ \varphi - y)$. □

Let X be a holomorphic vector field defined in a connected domain $U \subset \mathbb{C}$ such that $X \neq 0$. Consider $P \in SingX$. There exists a unique meromorphic differential form ω in U such that $\omega(X) = 0$. We denote by $Res(X, P)$ the residue of ω at the point P. Given $Y = f(\underline{x}, y)\partial/\partial y$ and a point $P = (\underline{x}^0, y^0) \in SingX$ such that $SingX$ does not contain $\underline{x} = \underline{x}^0$ we define $Res(X, P) = Res(f(\underline{x}^0, y)\partial/\partial y, y^0)$.

Let $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^n, 0)$. Consider a convergent normal form α of φ. By definition $Res(\varphi, P) = Res(\log \alpha, P)$ for $P \in Fix\varphi$. The definition does not depend on the choice of α since given another convergent normal form β of φ we have that $dy/(\log \alpha)(y) - dy/(\log \beta)(y) \in \vartheta_n dy$. We denote the function $P \to Res(\varphi, P)$ defined in $Fix\varphi$ by $Res(\varphi)$.

Remark 3.1. *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Consider $X = u(x,y)(y \circ \varphi - y)\partial/\partial y$ for some unit $u \in \mathbb{C}\{x, y\}$. Suppose that $[y \circ \varphi = y] \cap [\partial(y \circ \varphi)/\partial y = 1] = \{(0,0)\}$; that is the generic situation. Then $Res(\varphi) \equiv Res(X)$ implies that $\exp(X)$ is a convergent normal form of φ. A 1-form with poles of order at most 1 and no residues has no poles at all.*

4. One variable theory

We introduce here for the sake of completeness some classical results concerning tangent to the identity complex analytic germs of diffeomorphism in one variable.

4.1. **Formal theory.** Let $\varphi \in \mathrm{Diff}_1(\mathbb{C}, 0) = \mathrm{Diff}_u(\mathbb{C}, 0)$. We define $\nu(\varphi)$ the order of φ as $\nu(\varphi) = \nu(\varphi(y) - y) - 1$.

Proposition 4.1. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{Id\}$. Then φ_1 is formally conjugated to φ_2 if and only if $\nu(\varphi_1) = \nu(\varphi_2)$ and $Res(\varphi_1) = Res(\varphi_2)$. In such a case if $\log \varphi_1$ and $\log \varphi_2$ are convergent then φ_1 and φ_2 are analytically conjugated.*

Supposed that φ_1, φ_2 are formally conjugated by $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$. Then every other formal conjugation can be expressed in the form $\hat{\tau} \circ \hat{\sigma}$ where $\hat{\tau}$ belongs to the formal centralizer $\hat{Z}(\varphi_2)$ of φ_2. As a consequence it is interesting to describe the structure of $\hat{Z}(\varphi)$ for classification purposes.

Proposition 4.2. *Let $\varphi \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{Id\}$. Then there exists $\hat{\tau}_0 \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ satisfying $(\partial \hat{\tau}_0/\partial y)(0) = e^{2i\pi/\nu(\varphi)}$ and $\hat{\tau}_0^{\circ(\nu(\varphi))} = Id$ such that*

$$\hat{Z}(\varphi) = \{\hat{\tau}_0^{\circ(r)} \circ \exp(t \log \varphi) \text{ for } r \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z}) \text{ and } t \in \mathbb{C}\}.$$

Moreover $\hat{Z}(\varphi)$ is a commutative group.

We denote $\hat{\tau}_0$ by $\hat{\tau}_0(\varphi)$. We say that $\hat{\tau}_0(\varphi)$ is the *generating symmetry* of φ. Let $\kappa_r = e^{2ir\pi/\nu(\varphi)}$. We denote the element $\hat{\tau}_0(\varphi)^{\circ(r)} \circ \exp(t \log \varphi)$ of $\hat{Z}(\varphi)$ by $Z_\varphi^{\kappa_r, t}$. The mapping $Z_\varphi^{\kappa, t} \mapsto (\kappa, t)$ is a bijection from $\hat{Z}(\varphi)$ to $< e^{2i\pi/\nu(\varphi)} > \times \mathbb{C}$.

4.2. **Topological behavior.** Let $\exp(X)$ be a convergent normal form of φ in $\mathrm{Diff}_1(\mathbb{C}, 0)$. The vector field X is of the form $X = (r_0 e^{i\theta_0} y^{\nu+1} + \sum_{j=\nu+2}^{\infty} a_j y^j)\partial/\partial y$ where $\nu = \nu(\varphi)$ and $r_0 \neq 0$. Consider the blow-up $\pi : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1 \to \mathbb{R}^2$ given by $\pi(r, e^{i\theta}) = re^{i\theta}$. We denote by $\tilde{X}$ the strict transform of $Re(X)$, we have $\tilde{X} = (\pi^* Re(X))/r^\nu$. We obtain that

$$\tilde{X} = r\left(r_0 Re(e^{i(\nu\theta+\theta_0)}) + O(r)\right)\frac{\partial}{\partial r} + \left(r_0 Re(-ie^{i(\nu\theta+\theta_0)}) + O(r)\right)\frac{\partial}{\partial\theta}.$$

We define $D_1(X) = \{\lambda \in \mathbb{S}^1 : \lambda^\nu e^{i\theta_0} = -1\}$ and $D_{-1}(X) = \{\lambda \in \mathbb{S}^1 : \lambda^\nu e^{i\theta_0} = 1\}$. We have that $\sharp D_1(X) = \sharp D_{-1}(X) = \nu$ and $Sing(\tilde{X}_{|r=0}) = D_1(X) \cup D_{-1}(X)$. Moreover, since

$$\tilde{X}_{|r=0} = (-r_0\nu s(\theta-\theta_1) + O((\theta-\theta_1)^2))\frac{\partial}{\partial\theta}$$

in the neighborhood of $e^{i\theta_1} \in D_s(X)$ then the points in $D_1(X)$ are attractif points for $\tilde{X}_{|r=0}$ whereas the points of $D_{-1}(X)$ are repulsif.

We define $\eta = -1/(r_0 e^{i\theta_0}\nu y^\nu)$, it satisfies $\tilde{X}(\eta) = 1+O(r)$. Consider $\lambda_1 \in D_1(X)$ and the set $S(r_1, \lambda_1) = [0 \leq r < r_1] \cap [\lambda \in \lambda_1 e^{(-i\pi/(4\nu), i\pi/(4\nu))}]$. We obtain $\eta(r,\lambda) \in e^{(-i\pi/4, i\pi/4)}/(\nu r_0 r^\nu)$ for all $(r,\lambda) \in S(r_1,\lambda_1)$. Since $\tilde{X}(\eta) = 1 + O(r)$ then the points in $S(r_1,\lambda_1)$ are attracted to $(0,\lambda_1)$ by the positive flow of $\tilde{X}$ for $r_1 > 0$ small enough. Analogously we can prove that $(0,\lambda_1)$ is a repulsif point for $\tilde{X}$ if $\lambda_1 \in D_{-1}(X)$.

The dynamics of φ is a small deformation of the dynamics of $\exp(X)$. We denote $D_s(\varphi) = D_s(X)$ for $s \in \{-1,1\}$ and $D(\varphi) = D_{-1}(\varphi) \cup D_1(\varphi)$. The definition of $D_s(\varphi)$ does not depend on the choice of the convergent normal form $\exp(X)$. Suppose that φ and $\varphi^{\circ(-1)}$ are holomorphic in an small enough open set $U \ni 0$. It is easy to prove that

$$V_\varphi^\lambda = \{P \in U \setminus \{0\} : \varphi^{\circ(sn)}(P) \in U \ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} \varphi^{\circ(sn)}(P) = (0,\lambda)\}$$

is an open set for all $\lambda \in D_s(\varphi)$. A domain V_φ^λ for $\lambda \in D_1(\varphi)$ is called an *attracting petal*. A domain V_φ^λ for $\lambda \in D_{-1}(\varphi)$ is called a *repulsing petal*.

We say that $V(\lambda,\theta)$ is a *sector* of direction $\lambda \in \mathbb{S}^1$ and angle $\theta \in \mathbb{R}^+$ if there exists $\mu \in \mathbb{R}^+$ such that $V(\lambda,\theta) = \lambda e^{i[-\theta/2,\theta/2]}(0,\mu]$. We say that $W(\lambda,\theta)$ is a *sectorial domain* of direction $\lambda \in \mathbb{S}^1$ and angle $\theta \in \mathbb{R}^+$ if it contains a sector of direction λ and angle θ' for all $\theta' \in (0,\theta)$.

The next proposition is a consequence of the previous discussion and the fact that φ is a small deformation of $\exp(X)$.

Proposition 4.3. *Let $\varphi \in \mathrm{Diff}_1(\mathbb{C},0)$. Fix a domain a domain of definition $0 \in U$. We have*

- *V_φ^λ is a sectorial domain of direction λ and angle $2\pi/\nu(\varphi)$ for all $\lambda \in D(\varphi)$.*
- *$\{0\} \cup \cup_{\lambda \in D(\varphi)} V_\varphi^\lambda$ is a neighborhood of 0.*
- *$V_\varphi^{\lambda_0} \cap V_\varphi^{\lambda_1} = \emptyset$ if $\lambda_1 \notin \{e^{-i\pi/\nu(\varphi)}\lambda_0, \lambda_0, e^{i\pi/\nu(\varphi)}\lambda_0\}$.*
- *$V_\varphi^{\lambda_0} \cap V_\varphi^{\lambda_1}$ is a sectorial domain of direction $\lambda_0 e^{i\pi/(2\nu(\varphi))}$ and angle $\pi/\nu(\varphi)$ for $\lambda_1 = e^{i\pi/\nu(\varphi)}\lambda_0$.*

4.3. **Analytic properties.** Next, we describe the analytic invariants of elements of $\mathrm{Diff}_1(\mathbb{C},0)$. Let $\varphi \in \mathrm{Diff}_1(\mathbb{C},0)$. Choose a normal form $\alpha \in \mathrm{Diff}_1(\mathbb{C},0)$ of φ. Consider the equation $(\log\alpha)(\psi_\alpha) = 1$. A holomorphic solution ψ_α is called a *Fatou coordinate* of α. Alternatively we also say that ψ_α is an *integral of the time form* (or dual form) of α. The function ψ_α is unique up to an additive constant. Indeed

ψ_α is of the form

$$\psi_\alpha = \frac{-1}{\nu(\varphi)a_{\nu(\varphi)+1}} \frac{1}{y^{\nu(\varphi)}} \left(1 + \sum_{j=1}^{\infty} b_j y^j \right) + Res(\varphi) \log y$$

where $\varphi = y + a_{\nu(\varphi)+1} y^{\nu(\varphi)+1} + O(y^{\nu(\varphi)+2})$. Let $\lambda \in D(\varphi)$; we say that $\eta \in \vartheta(V_\varphi^\lambda)$ is a Fatou coordinate of φ in V_φ^λ if $\eta \circ \varphi = \eta + 1$ and $\eta - \psi_\alpha$ is bounded. Clearly the definition does not depend on the choice of α.

Proposition 4.4. *Let $\varphi \in \mathrm{Diff}_1(\mathbb{C}, 0)$. Consider a convergent normal form α of φ and a direction $\lambda \in D(\varphi)$. Then there exists a unique Fatou coordinate ψ_φ^λ of φ in V_φ^λ such that $\lim_{y\to 0} \psi_\varphi^\lambda - \psi_\alpha = 0$ in every sector of direction λ and angle lesser than $2\pi/\nu(\varphi)$ contained in V_φ^λ. Moreover ψ_φ^λ is injective.*

We can provide a formula for ψ_φ^λ. We define $\Delta = \psi_\alpha \circ \varphi - (\psi_\alpha + 1)$. By Taylor's formula we obtain that

$$\Delta \sim \frac{\partial \psi_\alpha}{\partial y}(\varphi(y) - \alpha(y)) = O(y^{\nu(\varphi)+1}) \Rightarrow \Delta \in \mathbb{C}\{y\} \cap (y^{\nu(\varphi)+1}).$$

Since $(\psi_\varphi^\lambda - \psi_\alpha) - (\psi_\varphi^\lambda - \psi_\alpha) \circ \varphi = \Delta$ we can obtain $\psi_\varphi^\lambda - \psi_\alpha$ as a telescopic sum. More precisely let $\psi_\alpha^\lambda \in \vartheta(V_\varphi^\lambda)$ be a Fatou coordinate of α. We have

$$\psi_\varphi^\lambda = \psi_\alpha^\lambda + \sum_{j=0}^{\infty} \Delta \circ \varphi^{\circ(j)} \text{ and } \psi_\varphi^\lambda = \psi_\alpha^\lambda - \sum_{j=1}^{\infty} \Delta \circ \varphi^{\circ(-j)}$$

for $\lambda \in D_1(\varphi)$ and $\lambda \in D_{-1}(\varphi)$ respectively.

Let $\varphi \in \mathrm{Diff}_1(\mathbb{C}, 0)$ with convergent normal form α. Denote $\nu = \nu(\varphi)$. Consider that $\psi_\alpha^\lambda \in \vartheta(V_\varphi^\lambda)$ is chosen for all $\lambda \in D(\varphi)$. We define

$$\xi_\varphi^\lambda(z) = \psi_\varphi^{\lambda e^{i\pi/\nu}} \circ \left(\psi_\varphi^\lambda\right)^{\circ(-1)}(z)$$

for $\lambda \in D(\varphi)$. The dynamics of φ in every V_φ^λ is $z \mapsto z+1$ in the coordinate ψ_φ^λ. Then ξ_φ^λ is the change of chart which allow to glue two $z \mapsto z+1$ models corresponding to consecutive petals. In particular we have $\xi_\varphi^\lambda \circ (z+1) \equiv \xi_\varphi^\lambda(z) + 1$ for all $\lambda \in D(\varphi)$. Fix $\lambda_0 \in D(\varphi)$ and $\psi_\alpha^{\lambda_0}$. Denote $\lambda_j = \lambda_0 e^{i\pi j/\nu}$. There are several possible definitions for $\psi_\alpha^{\lambda_j}$. We consider *homogeneous coordinates*, supposed $\psi_\alpha^{\lambda_j}$ is defined we extend it to $V_\varphi^{\lambda_j} \cup V_\varphi^{\lambda_{j+1}}$ by analytic continuation. Then we define $\psi_\alpha^{\lambda_{j+1}} = \psi_\alpha^{\lambda_j} - \pi i Res(\varphi)/\nu$. Let us remark that $\psi_\varphi^{\lambda_0} = \psi_\varphi^{\lambda_{2\nu}}$. The definition of ξ_φ^λ depends on the choice of $\psi_\alpha^{\lambda_0}$. If we replace $\psi_\alpha^{\lambda_0}$ with $\psi_\alpha^{\lambda_0} + K$ for some $K \in \mathbb{C}$ then ξ_φ^λ becomes $(z+K) \circ \xi_\varphi^\lambda \circ (z-K)$ for all $\lambda \in D(\varphi)$. Denote $\zeta_\varphi = -\pi i Res(\varphi)/\nu(\varphi)$.

Proposition 4.5. *Let $\varphi \in \mathrm{Diff}_1(\mathbb{C}, 0)$ with convergent normal form α. Consider $\lambda \in D_s(\varphi)$. Then there exists $C \in \mathbb{R}^+$ such that*

- *ξ_φ^λ is defined in $sImgz < -C$ and $\xi_\varphi^\lambda \circ (z+1) \equiv (z+1) \circ \xi_\varphi^\lambda$.*
- *$\lim_{|Img(z)| \to \infty} \xi_\varphi^\lambda(z) - z = \zeta_\varphi$.*
- *$\xi_\varphi^\lambda = z + \zeta_\varphi + \sum_{j=1}^{\infty} a_{\lambda,j}^\varphi e^{-2\pi i s j z}$ for some $\sum_{j=1}^{\infty} a_{\lambda,j}^\varphi w^j \in \mathbb{C}\{w\}$.*

All the possible changes of charts can be realized.

Proposition 4.6. [31] [15] *Let $\alpha \in \mathrm{Diff}_1(\mathbb{C},0)$ such that $\log\alpha$ is convergent. Consider a series $\sum_{j=1}^{\infty} a_{\lambda,j} w^j \in \mathbb{C}\{w\}$ for all $\lambda \in D_\alpha$. Then there exists a mapping $\varphi \in \mathrm{Diff}_1(\mathbb{C},0)$ with convergent normal form α such that*

$$\xi_\varphi^\lambda = z + \zeta_\varphi + \sum_{j=1}^{\infty} a_{\lambda,j} e^{-2\pi i s j z}$$

in homogeneous coordinates for all $\lambda \in D_s(\alpha)$ and all $s \in \{-1,1\}$.

4.4. **Analytic classification.** Suppose that φ_1, φ_2 are formally conjugated. Consider convergent normal forms α_1 and α_2 of φ_1 and φ_2 respectively. We have that α_1 and α_2 are formally conjugated and then analytically conjugated by some $h \in \mathrm{Diff}(\mathbb{C},0)$ by proposition 4.1. Then up to replace φ_2 with $h^{\circ(-1)} \circ \varphi_2 \circ h$ we can suppose that φ_1 and φ_2 have common normal form $\alpha_1 = \alpha_2$. In particular we have that $\varphi_1(y) - \varphi_2(y) \in (y^{2(\nu(\varphi_1)+1)})$. Indeed φ_1 and φ_2 have common convergent normal form if and only if $\nu(\varphi_1) = \nu(\varphi_2)$ and $\varphi_1(y) - \varphi_2(y) \in (y^{2(\nu(\varphi_1)+1)})$.

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_1(\mathbb{C},0)$ with common convergent normal form α. There exists $\hat{\sigma}(\varphi_1,\varphi_2) \in \widehat{\mathrm{Diff}}(\mathbb{C},0)$ conjugating φ_1 and φ_2 such that $\hat{\sigma}(\varphi_1,\varphi_2)(y) - y \in (y^{\nu(\varphi)+2})$. Moreover $\hat{\sigma}(\varphi_1,\varphi_2)$ is unique. We say that $\hat{\sigma}(\varphi_1,\varphi_2)$ is the *privileged formal conjugation.* Choose $\lambda_0 \in D(\varphi_1) = D(\varphi_2)$ and $\psi_\alpha^{\lambda_0}$. The next couple of propositions are a consequence of Ecalle's theory. We always use homogeneous coordinates.

Proposition 4.7. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_1(\mathbb{C},0)$ with common convergent normal form α. Then for all $\lambda \in D(\varphi_1)$ there exists a unique holomorphic $\sigma_\lambda : V_{\varphi_1}^\lambda \to V_{\varphi_2}^\lambda$ conjugating φ_1 and φ_2 and such that $\hat{\sigma}(\varphi_1,\varphi_2)$ is a $\nu(\varphi_1)$-Gevrey asymptotic development of σ_λ in $V_{\varphi_1}^\lambda$. Moreover We have $\sigma_\lambda = (\psi_{\varphi_2}^\lambda)^{\circ(-1)} \circ \psi_{\varphi_1}^\lambda$.*

The expression $\sigma_\lambda : V_{\varphi_1}^\lambda \to V_{\varphi_2}^\lambda$ implies an abuse of notation. Rigorously $V_{\varphi_1}^\lambda$ and $V_{\varphi_2}^\lambda$ can be replaced by sectorial domains $W_{\varphi_1}^\lambda$ and $W_{\varphi_2}^\lambda$ of direction l and angle $2\pi/\nu(\varphi_1)$ and such that $\sigma_l : W_{\varphi_1}^l \to W_{\varphi_2}^l$ is a biholomorphism. For simplicity we keep this kind of notation throughout this section.

The elements of the centralizer $\hat{Z}(\varphi)$ of $\varphi \in \mathrm{Diff}_1(\mathbb{C},0)$ can be realized in the sectorial domains V_φ^λ for every $\lambda \in D_\varphi$.

Proposition 4.8. *Let $\varphi \in \mathrm{Diff}_1(\mathbb{C},0)$ with convergent normal form α. Consider an element $Z_\varphi^{\kappa,t}$ of $\hat{Z}(\varphi)$. Then for all $\lambda \in D_\varphi$ there exists a unique holomorphic $\tau_\lambda : V_\varphi^\lambda \to V_\varphi^{\lambda\kappa}$ such that $\varphi \circ \tau_\lambda = \tau_\lambda \circ \varphi$ and $Z_\varphi^{\kappa,t}$ is a $\nu(\varphi)$-Gevrey asymptotic development of τ_λ in V_φ^λ. Moreover we have $\tau_\lambda = (\psi_\varphi^{\lambda\kappa})^{\circ(-1)} \circ (\psi_\varphi^\lambda + t)$.*

We can combine propositions 4.7 and 4.8 to obtain:

Proposition 4.9. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_1(\mathbb{C},0)$ with common convergent normal form α. Consider $(\kappa,t) \in < e^{2i\pi/\nu(\varphi_1)} > \times\mathbb{C}$. Then for all $\lambda \in D_{\varphi_1}$ there exists a unique holomorphic $\sigma_\lambda^{\kappa,t} : V_{\varphi_1}^\lambda \to V_{\varphi_2}^{\lambda\kappa}$ conjugating φ_1 and φ_2 and such that $Z_{\varphi_2}^{\kappa,t} \circ \hat{\sigma}(\varphi_1,\varphi_2)$ is a $\nu(\varphi_1)$-Gevrey asymptotic development of $\sigma_\lambda^{\kappa,t}$ in $V_{\varphi_1}^\lambda$. Moreover $\sigma_\lambda^{\kappa,t} = (\psi_{\varphi_2}^{\lambda\kappa})^{\circ(-1)} \circ (\psi_{\varphi_1}^\lambda + t)$ in homogeneous coordinates.*

By uniqueness of the $\nu(\varphi_1)$-Gevrey sum in sectors of angle greater than $\pi/\nu(\varphi_1)$ we deduce that $Z_{\varphi_2}^{\kappa,t} \circ \hat{\sigma}(\varphi_1,\varphi_2)$ is analytic if and only if $\sigma_\lambda^{\kappa,t} = \sigma_{\lambda e^{i\pi/\nu(\varphi_1)}}^{\kappa,t}$ in $V_{\varphi_1}^\lambda \cap V_{\varphi_1}^{\lambda e^{i\pi/\nu(\varphi_1)}}$ for all $\lambda \in D_{\varphi_1}$. These conditions can be expressed in terms of the changes of charts.

Proposition 4.10. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_1(\mathbb{C},0)$ with common convergent normal form α. Then $\varphi_1 \sim \varphi_2$ if and only if there exists $(\kappa, t) \in < e^{2i\pi/\nu(\varphi_1)} > \times \mathbb{C}$ such that*

$$\xi_{\varphi_2}^{\lambda\kappa} \circ (z+t) \equiv (z+t) \circ \xi_{\varphi_1}^{\lambda} \quad \forall \lambda \in D(\varphi_1). \tag{2}$$

Indeed the equation 2 is equivalent to $Z_{\varphi_2}^{\kappa,t} \circ \hat{\sigma}(\varphi_1, \varphi_2) \in \mathrm{Diff}(\mathbb{C},0)$.

There is a quite common mistake in the study of tangent to the identity diffeomorphisms. We can find references in the litterature where it is claimed that given $\varphi_1, \varphi_2 \in \mathrm{Diff}_1(\mathbb{C},0)$ analytically conjugated and with common normal form then the conjugation can be choosen of the form $y + O(y^{\nu(\varphi_1)+2})$. In other words if φ_1 and φ_2 are analytically conjugated then $\hat{\sigma}(\varphi_1, \varphi_2) \in \mathrm{Diff}(\mathbb{C},0)$. This false statement is obtained by neglecting the role of the centralizer in the analytic conjugation. A reference can be found in [26].

Remark 4.1. *Let $\lambda \in D_s(\varphi_1)$. The condition $\xi_{\varphi_2}^{\lambda\kappa}(z+t) = (z+t) \circ \xi_{\varphi_1}^{\lambda}$ is equivalent to $a_{\lambda\kappa,j}^{\varphi_2} e^{-2\pi i s j t} = a_{\lambda,j}^{\varphi_1}$ for all $j \in \mathbb{N}$.*

Remark 4.2. *Let $\varphi \in \mathrm{Diff}_1(\mathbb{C},0)$ with convergent normal form α. Then $\log\varphi$ belongs to $\mathcal{X}(\mathbb{C},0)$ if and only if $\varphi \sim \alpha$ (prop. 4.1). Therefore $\log\varphi \in \mathcal{X}(\mathbb{C},0)$ if and only if $a_{\lambda,j}^{\varphi} = 0$ for all $\lambda \in D(\varphi)$ and all $j \in \mathbb{N}$.*

5. Formal conjugation

Part of this paper is devoted to explain the relations among formal conjugations, analytic conjugations and the centralizer when dealing with elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. In this section we study the formal properties of the diffeomorphisms.

5.1. **Formal invariants.** Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Suppose that there exists $\sigma \in \mathrm{Diff}(\mathbb{C}^2,0)$ such that $\sigma \circ \varphi_1 = \varphi_2 \circ \sigma$. We want to express σ as a composition $\sigma_1 \circ \sigma_2$ such that the action of σ on the formal invariants of φ_1 is the same action induced by σ_2. Moreover identifying a possible σ_2 is much simpler than finding σ.

The property $\sigma \circ \varphi_1 = \varphi_2 \circ \sigma$ implies that σ conjugates convergent normal forms of φ_1 and φ_2. We obtain:

Proposition 5.1. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Suppose that φ_1 and φ_2 are analytically conjugated by $\sigma \in \mathrm{Diff}(\mathbb{C}^2,0)$. Then*
- *$[(y \circ \varphi_2 - y) \circ \sigma]/(y \circ \varphi_1 - y)$ is a unit.*
- *$Res(\varphi_1, P) = Res(\varphi_2, \sigma(P))$ for all $P \in Fix\varphi_1$.*

Remark 5.1. *The residue functions are formal invariants* [21] *but for us it is enough to know that they are analytic invariants.*

We denote $\tau(Fix\varphi_1) = Fix\varphi_2$ if $[(y \circ \varphi_2 - y) \circ \tau]/(y \circ \varphi_1 - y)$ is a unit for some $\tau \in \widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$. In particular $Fix\varphi_1 = Fix\varphi_2$ means that $Id(Fix\varphi_1) = Fix\varphi_2$.

Consider a diffeomorphism $\tau \in \mathrm{Diff}(\mathbb{C}^2,0)$ satisfying the two conditions in proposition 5.1. By replacing φ_2 with $\tau^{\circ(-1)} \circ \varphi_2 \circ \tau$ we can suppose from now on that $Fix\varphi_1 = Fix\varphi_2$ and $Res(\varphi_1) \equiv Res(\varphi_2)$.

Consider a formal conjugation $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ between φ_1 and φ_2. We say that $\hat{\sigma}$ is *special* (with respect to $Fix\varphi_1$) if $x \circ \hat{\sigma} \equiv x$ and $y \circ \hat{\sigma} - y \in I(Fix\varphi_1)$. We say that $\hat{\sigma}$ is *good* (with respect to $Fix\varphi_1$) if $\hat{\sigma}$ is special and $y \circ \hat{\sigma} - y \in I(\gamma)^2$ for all irreducible component γ of $Fix\varphi_1$ such that $y \circ \varphi_1 - y \in I(\gamma)^2$. We denote $\varphi_1 \overset{sp}{\sim} \varphi_2$ if $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ are conjugated by a special element of $\mathrm{Diff}(\mathbb{C}^2,0)$.

We denote

$$\mathcal{X}_{p1}(\mathbb{C}^2,0) = \{X \in \mathcal{X}(\mathbb{C}^2,0) : \exp(X) \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)\}.$$

In particular the set of convergent normal forms of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is equal to $\exp(\mathcal{X}_{p1}(\mathbb{C}^2,0))$.

Proposition 5.2. *Let $\alpha_1, \alpha_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ such that $\log\alpha_j \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ for $j \in \{1,2\}$. Suppose that $Fix\alpha_1 = Fix\alpha_2$ and $Res(\alpha_1) \equiv Res(\alpha_2)$. Then $\alpha_1 \overset{sp}{\sim} \alpha_2$.*

Lemma 5.1. *Let $f \in \mathbb{C}\{x,y\}$ such that $f(0,y) \not\equiv 0$. Consider $A \in \mathbb{C}\{x,y\}$ such that $(A(x_0,y)/f(x_0,y))dy$ has vanishing residues for all x_0 in a neighborhood of 0. Then there exists a germ of meromorphic function β such that $\partial\beta/\partial y = A/f$ and $\beta f \in \sqrt{f} \subset \mathbb{C}\{x,y\}$.*

Proof. Let $P = (0,y_0) \neq (0,0)$ be a point close to the origin. Since $f(P) \neq 0$ there exists a unique holomorphic solution β defined in the neighborhood of P such that $\partial\beta/\partial y = A/f$ and $\beta(x,y_0) \equiv 0$. The residues vanish, then we extend β by analytic continuation to obtain $\beta \in \vartheta(U \setminus (f=0))$ for some neighborhood U of $(0,0)$.

Consider $Q \in (U \setminus \{(0,0)\}) \cap (f=0)$. Up to a change of coordinates $(x, y+h(x))$ we can suppose that $f = v(x,y)y^r$ in the neighborhood of Q where $y(Q) = 0 \neq v(Q)$ and $r \in \mathbb{N}$. The form $(A/f)dy$ is of the form $(\sum_{-1\neq j\geq -r} c_j(x)y^j)dy$. Then β is of the form $\sum_{0\neq j\geq -(r-1)} c_{j-1}(x)y^j/j + \beta_Q(x)$ for some β_Q holomorphic in a neighborhood of Q. As a consequence βf is holomorphic and vanishes at $f=0$ in a neighborhood of Q. Hence βf belongs to $\vartheta(U \setminus \{(0,0)\})$ and then to $\vartheta(U)$ since we can remove codimension 2 singularities. Clearly we have $\beta f \in I(f=0) = \sqrt{f}$. □

Proof of proposition 5.2. There exists $f \in \mathbb{C}\{x,y\}$ such that $\log\alpha_j = u_j f\partial/\partial y$ for some unit $u_j \in \mathbb{C}\{x,y\}$ and all $j \in \{1,2\}$. Let us use the path method (see [27] and [19]). We define

$$X_{1+z} = u_{1+z}f\frac{\partial}{\partial y} = \frac{u_1u_2f}{zu_1+(1-z)u_2}\frac{\partial}{\partial y}.$$

We have that $X_{1+z} \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ for all $z \in \mathbb{C}\setminus\{c\}$ where $c = u_2(0)/(u_2(0)-u_1(0))$. Moreover $SingX_{1+z}$ and $Res(X_{1+z})$ do not depend on z. It is enough to prove that $\log\alpha_1$ is analytically conjugated by a special diffeomorphism to $\log\alpha_2$ for $c \notin [0,1]$. If $c \in [0,1]$ we define

$$Y^1_{1+z} = \frac{u_1u_{1+i}f}{zu_1+(1-z)u_{1+i}}\frac{\partial}{\partial y} \quad \text{and} \quad Y^2_{1+z} = \frac{u_{1+i}u_2f}{zu_{1+i}+(1-z)u_2}\frac{\partial}{\partial y}.$$

Since $u_{1+i}(0)/(u_{1+i}(0)-u_1(0))$ and $u_1(0)/(u_1(0)-u_{1+i}(0))$ do not belong to $[0,1]$ then we obtain a special diffeomorphism conjugating α_1 and α_2 as a composition of special diffeomorphisms.

Suppose $c \notin [0,1]$. We look for $W \in \mathcal{X}(\mathbb{C}^3,0)$ of the form $h(x,y,z)f\partial/\partial y + \partial/\partial z$ such that $[W, X_{1+z}] = 0$. We ask hf to be holomorphic in a connected domain $V \times V' \subset \mathbb{C}^2 \times \mathbb{C}$ containing $\{(0,0)\} \times [0,1]$. We also require hf to vanish at $(f=0)\times V'$. Supposed that such a W exists then $\exp(W)_{|z=0}$ is a special mapping conjugating $\log\alpha_1$ and $\log\alpha_2$. The equation $[W, X_{1+z}] = 0$ is equivalent to

$$u_{1+z}f\frac{\partial(hf)}{\partial y} - hf\frac{\partial(u_{1+z}f)}{\partial y} = \frac{\partial(u_{1+z}f)}{\partial z}.$$

By simplifying we obtain

$$u_{1+z} f \frac{\partial h}{\partial y} - hf \frac{\partial u_{1+z}}{\partial y} = \frac{\partial u_{1+z}}{\partial z} \Rightarrow \frac{\partial (h/u_{1+z})}{\partial y} = \frac{1}{u_1 f} - \frac{1}{u_2 f}.$$

Let β be a solution of $\partial\beta/\partial y = 1/(u_1 f) - 1/(u_2 f)$ such that $\beta f \in \sqrt{f}$. Since $(1/(u_1 f) - 1/(u_2 f))dy$ has vanishing residues by hypothesis then such a solution exists by lemma 5.1. We are done by defining $h = u_{1+z}\beta$. $\square$

Suppose that $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ satisfy $Fix\varphi_1 = Fix\varphi_2$ and $Res(\varphi_1) \equiv Res(\varphi_2)$. Consider convergent normal forms α_1 and α_2 of φ_1 and φ_2 respectively. Then α_1 and α_2 are analytically conjugated by some special $\tau \in \mathrm{Diff}(\mathbb{C}^2, 0)$ (prop. 5.2). By replacing φ_2 with $\tau^{\circ(-1)} \circ \varphi_2 \circ \tau$ we can restrict ourselves to study elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with common normal form and analytic special conjugations.

Proposition 5.3. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with common convergent normal form α. Let $f \in \mathbb{C}\{x, y\}$ such that $(y \circ \varphi_1 - y)/f$ is a unit and denote $\hat{u}_j = (log\varphi_j)(y)/f$ for $j \in \{1, 2\}$. Then φ_1 and φ_2 are formally conjugated by the good transformation*

$$\hat{\tau} = \exp\left(\hat{\beta}\frac{\hat{u}_1\hat{u}_2}{z\hat{u}_1 + (1-z)\hat{u}_2} f\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)_{|z=0}$$

where $\hat{\beta}$ can be any solution of $\partial\hat{\beta}/\partial y = 1/(\hat{u}_1 f) - 1/(\hat{u}_2 f)$ in $\mathbb{C}[[x, y]]$.

Proof. We have that $1/(\hat{u}_1 f) - 1/(\hat{u}_2 f) \in \mathbb{C}[[x, y]]$ since φ_1 and φ_2 have convergent common normal form. Let $\beta_k \in \mathbb{C}\{x, y\}$ such that $\hat{\beta} - \beta_k \in (x, y)^k$. We choose $u_{1,k} \in \mathbb{C}\{x, y\}$ such that $\hat{u}_1 - u_{1,k} \in (f)(x, y)^k$; this is possible by proposition 3.3. We define $u_{2,k} \in \mathbb{C}\{x, y\} \setminus (x, y)$ such that $\partial\beta_k/\partial y = 1/(u_{1,k} f) - 1/(u_{2,k} f)$. Now $\exp(u_{1,k} f\partial/\partial y)$ and $\exp(u_{2,k} f\partial/\partial y)$ are formally conjugated by

$$\tau_k \stackrel{def}{=} \exp\left(\beta_k \frac{u_{1,k}u_{2,k}}{zu_{1,k} + (1-z)u_{2,k}} f\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)_{|z=0}$$

by proposition 5.2. It is straightforward to check out that $u_{j,k} \to \hat{u}_j$ and $\tau_k \to \hat{\tau}$, the limits considered in the Krull topology. Thus $\hat{\tau}$ conjugates φ_1 and φ_2. $\square$

5.2. **Formal centralizer.** Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Next, we study the groups $\hat{Z}_{sp}(\varphi)$ of formal special auto-conjugations of φ and $\hat{Z}_{up}(\varphi) = \hat{Z}(\varphi) \cap \widehat{\mathrm{Diff}}_{up}(\mathbb{C}^2, 0)$. We say that $Fix\varphi$ is of *trivial type* if $I(Fix\varphi)$ is of the form (f) for some $f \in \mathbb{C}\{x, y\}$ such that $(\partial f/\partial y)(0, 0) \neq 0$. Let us remark that $Fix\varphi$ is of trivial type if and only if it has a unique smooth irreducible component transversal to $\partial/\partial y$.

Lemma 5.2. *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Then*

$$\hat{Z}_{up}(\varphi) = \{\exp(\hat{c}(x)\log\varphi) \text{ for some } \hat{c}(x) \in \mathbb{C}[[x]]\}.$$

In particular $\hat{Z}_{up}(\varphi)$ is commutative and all its elements are good. Moreover we have $\hat{Z}_{sp}(\varphi) = \hat{Z}_{up}(\varphi)$ if $Fix\varphi$ is not of trivial type.

Proof. We have that $\hat{\tau} \in \hat{Z}_{up}(\varphi)$ is equivalent to $[\log\varphi, \log\hat{\tau}] = 0$. Thus $\log\hat{\tau}$ is of the form $(\log\hat{\tau})(y)\partial/\partial y$ by the same arguments than in the proof of proposition 3.2. We obtain

$$[\log\varphi, \log\hat{\tau}] = 0 \Leftrightarrow \frac{\partial}{\partial y}\left(\frac{(\log\varphi)(y)}{(\log\hat{\tau})(y)}\right) = 0.$$

Since $(\log\varphi)(0, y) \not\equiv 0$ then $\log\hat{\tau} = \hat{c}(x)\log\varphi$ for some $\hat{c}(x) \in \mathbb{C}[[x]]$. We proved $\hat{Z}_{up}(\varphi) \subset \hat{Z}_{sp}(\varphi)$, we always have $\hat{Z}_{sp}(\varphi) \subset \hat{Z}_{up}(\varphi)$ in the non-trivial type case. $\square$

We define the order $\nu(\varphi)$ of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ as the order of $\varphi_{|x=0} \in \mathrm{Diff}_1(\mathbb{C}, 0)$. We define $\nu(X) = \nu(\exp(X)) = \nu(X(y)(0, y)) - 1$ for $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$.

Lemma 5.3. *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Suppose that $Fix\varphi$ is of trivial type. Then*

$$\hat{Z}_{sp}(\varphi) = \{\hat{\tau}_0^{\circ(r)} \circ \exp(\hat{c}(x) \log \varphi) \text{ for some } r \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z}) \text{ and } \hat{c}(x) \in \mathbb{C}[[x]]\}$$

where $\hat{\tau}_0 \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^2, 0)$ is periodic and $(\partial(y \circ \hat{\tau}_0)/\partial y)(0, 0) = e^{2\pi i/\nu(\varphi)}$. Moreover $\hat{Z}_{sp}(\varphi)$ is a commutative group.

Denote $\hat{\tau}_0(\varphi) = \hat{\tau}_0$. We say that $\hat{\tau}_0(\varphi)$ is the *generating symmetry* of φ. We denote $\exp(c(x) \log \varphi)$ by $Z_\varphi^{1,c}$ whereas we denote $\hat{\tau}_0(\varphi)^{\circ(r)} \circ \exp(c(x) \log \varphi)$ by $Z_\varphi^{\kappa,c}$ where $\kappa = e^{2\pi i r/\nu(\varphi)}$.

Proof. Let $\nu = \nu(\varphi_1)$. Up to a change of coordinates $(x, h(x, y))$ we can suppose that $y \circ \varphi - y = v(x, y)y^{\nu+1}$ where $v \in \mathbb{C}\{x, y\}$ is a unit and $\nu = \nu(\varphi)$. Consider a convergent normal form $\alpha = \exp(w(x, y)y^{\nu+1}\partial/\partial y)$ of φ. Let us remark that since

$$Res(\varphi, (x, 0)) = \frac{1}{2\pi i} \int_{y \in \partial B(0,\epsilon)} \frac{dy}{w(x, y)y^{\nu+1}}$$

for $\epsilon > 0$ small enough then $Res(\varphi)$ is a holomorphic function of $y = 0$. We define

$$X = \frac{y^{\nu+1}}{1 + y^\nu Res(\varphi, (x, 0))} \frac{\partial}{\partial y}.$$

By construction $Fix\varphi = Fix(\exp(X))$ and $Res(\varphi) \equiv Res(\exp(X))$. Up to a special change of coordinates we can suppose that $\exp(X)$ is a convergent normal form of φ (prop. 5.2). Let $\hat{\beta} \in \mathbb{C}[[x, y]]$ be the solution of $\partial\beta/\partial y = 1/X(y) - 1/(\log \varphi)(y)$ such that $\hat{\beta}(x, 0) \equiv 0$. We define

$$\hat{\xi} = \exp\left(\hat{\beta} \frac{u\hat{u}}{zu + (1 - z)\hat{u}} y^{\nu+1} \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)_{|z=0}$$

where $u = 1/(1 + Res(\varphi, (x, 0))y^\nu)$ and $\hat{u} = (\log \varphi)(y)/y^{\nu+1}$. Then $\hat{\xi}$ conjugates $\exp(X)$ and φ by proposition 5.3. We remark that $(x, e^{2\pi i/\nu}y)^*X = X$; hence $\hat{\tau}_0 = \hat{\xi} \circ (x, e^{2\pi i/\nu}y) \circ \hat{\xi}^{\circ(-1)} \in \hat{Z}_{sp}(\varphi)$ is periodic and $(\partial\hat{\tau}_0/\partial y)(0, 0) = e^{2\pi i/\nu}$. Given $\hat{\tau} \in \hat{Z}_{sp}(\varphi)$ there exists $r \in \mathbb{Z}$ such that $(\hat{\tau}_0^{\circ(-r)}\hat{\tau})_{|x=0}$ is tangent to the identity by proposition 4.2. As a consequence $\hat{\tau}_0^{\circ(-r)}\hat{\tau}$ is tangent to the identity. We obtain $\hat{\tau} = \hat{\tau}_0^{\circ(r)} \circ \exp(\hat{c}(x) \log \varphi)$ for some $\hat{c}(x) \in \mathbb{C}[[x]]$ by lemma 5.2. Moreover $\hat{Z}_{sp}(\varphi)$ is commutative since $(x, e^{2\pi i/\nu}y)^*X = X$ implies $\hat{\tau}_0^* \log \varphi = \log \varphi$. □

Next we stress that special and good conjugations are the same in the non-trivial type case.

Lemma 5.4. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with common convergent normal form α. Suppose that φ_1 and φ_2 are formally conjugated by a special $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ and that $Fix\varphi_1$ is not of trivial type. Then $\hat{\sigma}$ is good.*

Proof. We have that α and φ_j are conjugated by a good $\hat{\tau}_j \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ (prop. 5.3). Then it is enough to prove that $\hat{\tau}_2^{\circ(-1)} \circ \hat{\sigma} \circ \hat{\tau}_1 \in \hat{Z}_{sp}(\alpha)$ is good. This is a consequence of lemma 5.2. □

Let $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$. We denote by $Sing_V X$ the set of irreducible components of $SingX$ which are parameterized by x. Consider $\gamma \in Sing_V X$; we denote by $\nu_X(\gamma)$ the only element of $\mathbb{N} \cup \{0\}$ such that $X(y) \in I(\gamma)^{\nu_X(\gamma)+1} \setminus I(\gamma)^{\nu_X(\gamma)+2}$.

Proposition 5.4. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with common normal form* $\exp(X)$*. Consider $\gamma \in Sing_V X$. Then there exists a unique special $\hat{\sigma}(\varphi_1, \varphi_2, \gamma) \in \widehat{\text{Diff}}\,(\mathbb{C}^2, 0)$ conjugating φ_1 and φ_2 and such that $y \circ \hat{\sigma}(\varphi_1, \varphi_2, \gamma) - y \in I(\gamma)^{\nu_X(\gamma)+2}$.*

By definition the transformation $\hat{\sigma}(\varphi_1, \varphi_2, \gamma)$ is the *privileged formal conjugation* between φ_1 and φ_2 with respect to γ.

Proof. There exists a unique solution $\hat{\beta}$ of $\partial\hat{\beta}/\partial y = 1/(\log\varphi_1)(y) - 1/(\log\varphi_2)(y)$ such that $\hat{\beta}_{|\gamma} \equiv 0$. The formula in proposition 5.3 provides a special $\hat{\sigma}(\varphi_1, \varphi_2, \gamma) = \hat{\tau}$ conjugating φ_1 and φ_2 and such that $y \circ \hat{\sigma}(\varphi_1, \varphi_2, \gamma) - y \in I(\gamma)^{\nu_X(\gamma)+2}$.

Suppose $\hat{\sigma}(\varphi_1, \varphi_2, \gamma)$ is not unique. Thus we have $y \circ \hat{h} - y \in I(\gamma)^{\nu_X(\gamma)+2}$ for some $\hat{h} \in \hat{Z}_{up}(\varphi_1) \setminus \{Id\}$. By lemma 5.2 the transformation $\hat{h}$ is of the form $Z_{\varphi_1}^{1,c}$ for some $c \in \mathbb{C}[[x]]$. Since $(\log\hat{h})(y)$ belongs to $I(\gamma)^{\nu_X(\gamma)+2}$ then $c \equiv 0$ and $\hat{h} \equiv Id$. We obtain a contradiction. □

5.3. **Unfolding of diffeomorphisms** $y \to e^{2\pi i p/q} y + O(y^2)$**.** Consider the sets

$$\text{Diff}_{prs}(\mathbb{C}^2, 0) = \{\varphi \in \text{Diff}_p(\mathbb{C}^2, 0) : j^1\varphi_{|x=0} \text{ is periodic}\}$$

and

$$\text{Diff}_{pr}(\mathbb{C}^2, 0) = \{\varphi \in \text{Diff}_p(\mathbb{C}^2, 0) : j^1\varphi_{|x=0} \text{ is periodic but } \varphi_{|x=0} \text{ is not periodic}\}.$$

Given $\varphi \in \text{Diff}_{prs}(\mathbb{C}^2, 0)$ we denote by $q(\varphi)$ the smallest element of $\mathbb{N}$ such that $(\partial\varphi/\partial y)(0,0)^{q(\varphi)} = 1$. Clearly $\varphi \in \text{Diff}_{pr}(\mathbb{C}^2, 0)$ implies $\varphi^{\circ(q(\varphi))} \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. In this paper we classify analytically the elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$. We obtain for free a complete system of analytic invariants for the elements of $\text{Diff}_{pr}(\mathbb{C}^2, 0)$.

Proposition 5.5. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{prs}(\mathbb{C}^2, 0)$. Then $\varphi_1 \sim \varphi_2$ if and only if $(\partial\varphi_1/\partial y)(0,0) = (\partial\varphi_2/\partial y)(0,0)$ and $\varphi_1^{\circ(q(\varphi_1))} \sim \varphi_2^{\circ(q(\varphi_1))}$.*

Proof. The sufficient condition is obvious. Every mapping $\varphi \in \text{Diff}_u(\mathbb{C}^n, 0)$ admits a unique formal Jordan decomposition

$$\varphi = \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s$$

in semisimple $\varphi_s \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ and unipotent $\varphi_u \in \widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$ parts. Semisimple is equivalent to formally linearizable. The decomposition is compatible with the filtration in the space of jets, i.e. $j^k\varphi = j^k\zeta$ implies $j^k\varphi_s = j^k\zeta_s$ and $j^k\varphi_u = j^k\zeta_u$. Moreover we have $\varphi_s, \varphi_u \in \widehat{\text{Diff}}_p(\mathbb{C}^n, 0)$ for all $\varphi \in \text{Diff}_p(\mathbb{C}^n, 0)$.

Denote $q = q(\varphi_1)$ and $\upsilon = (\partial\varphi_1/\partial y)(0,0)$, we can suppose $\upsilon \neq 1$. Suppose $\varphi_1^{\circ(q)} \equiv Id$. This implies $\varphi_2^{\circ(q)} \equiv Id$. Denote by η_k the unipotent diffeomorphism

$$\frac{(x,y) + (x,\upsilon y)^{\circ(-1)} \circ \varphi_k + \ldots (x,\upsilon y)^{\circ(-(q-1))} \circ \varphi_k^{\circ(q-1)}}{q}.$$

By construction $\eta_k \circ \varphi_k = (x, \upsilon y) \circ \eta_k$ for $k \in \{1,2\}$. The diffeomorphism $\eta_2^{\circ(-1)} \circ \eta_1$ conjugates φ_1 and φ_2.

Suppose $\varphi_1^{\circ(q)} \not\equiv Id$. We have that $j^1\varphi_k$ is conjugated to $(x, \upsilon y)$ by a linear isomorphism and then semisimple for $k \in \{1,2\}$. Thus we obtain $j^1\varphi_{k,s} = j^1\varphi_k$, moreover since $\varphi_{k,s}$ is formally linearizable then $\varphi_{k,s}^{\circ(q)} \equiv Id$ for $k \in \{1,2\}$. We deduce that $\varphi_k^{\circ(q)} = \varphi_{k,u}^{\circ(q)}$ for $k \in \{1,2\}$. Hence $\log \varphi_{k,u}$ is of the form $\hat{f}_k \partial/y$ for some $\hat{f}_k \in \mathbb{C}[[x,y]] \setminus \{0\}$ and all $k \in \{1,2\}$.

Let σ be a diffeomorphism conjugating $\varphi_1^{\circ(q)}$ and $\varphi_2^{\circ(q)}$. The mapping σ conjugates $q \log \varphi_{1,u}$ and $q \log \varphi_{2,u}$ by uniqueness of the infinitesimal generator and then $\sigma \circ \varphi_{1,u} = \varphi_{2,u} \circ \sigma$. Denote $\eta = \sigma^{\circ(-1)} \circ \varphi_{2,s} \circ \sigma$. We claim that σ conjugates φ_1 and φ_2, it is enough to prove that $\varphi_{1,s} = \eta$. We have $x \circ \varphi_{1,s} = x \circ \eta = x$ and $(\partial \varphi_{1,s}/\partial y)(0,0) = (\partial \eta/\partial y)(0,0)$. As a consequence $\eta^{\circ(-1)} \circ \varphi_{1,s}$ is unipotent. Since both η and $\varphi_{1,s}$ commute with $\varphi_{1,u}$ then $(\eta^{\circ(-1)} \circ \varphi_{1,s}) \circ \varphi_{1,u} = \varphi_{1,u} \circ (\eta^{\circ(-1)} \circ \varphi_{1,s})$. We deduce that $[\log(\eta^{\circ(-1)} \circ \varphi_{1,s}), \log \varphi_{1,u}] = 0$. Since $\log(\eta^{\circ(-1)} \circ \varphi_{1,s})(x) = 0$ then we obtain $\log(\eta^{\circ(-1)} \circ \varphi_{1,s}) = (\hat{c}(x)/x^m) \log \varphi_{1,u}$ for some $\hat{c} \in \mathbb{C}[[x]]$ and $m \in \mathbb{Z}_{\geq 0}$. The equations $x \circ \eta = x$ and $\eta_* \log \varphi_{1,u} = \log \varphi_{1,u}$ imply that η commutes with $\eta^{\circ(-1)} \circ \varphi_{1,s}$. This leads us to $(\eta^{\circ(-1)} \circ \varphi_{1,s})^{\circ(q)} \equiv Id$. In particular $\hat{c}$ is identically 0, we obtain $\eta = \varphi_{1,s}$. □

Remark 5.2. *The techniques in this paper can be used to classify analytically the diffeomorphisms $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^2,0)$ such that $(y \circ \varphi - y)$ is not of the form (x^m) or $(x^m y)$ for $m \in \mathbb{N}$ up to a change of coordinates and then all resonant diffeomorphisms having an iterate in such a set. We work with elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ for the sake of simplicity. A complete system of analytic invariants for the case $(y \circ \varphi - y) = (x^m)$ has been provided by Voronin* [10].

6. Ecalle-Voronin invariants. Trivial type case

We present a complete system of analytic invariants for $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ in the trivial type case. We establish the link between the analytic classes of the one-variable diffeomorphisms $(\varphi)_{|x=x_0}$ for x_0 in a neighborhood of 0 and the analytic class of φ.

We suppose throughout this section that $I(Fix\varphi) = (y)$ for all $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ such that $Fix\varphi$ is of trivial type. This is possible up to change of coordinates of the form $(x, y + h(x))$.

Lemma 6.1. *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ such that $Fix\varphi$ is of trivial type. Then $(\log \varphi)(y)$ belongs to $\vartheta(B(0,\delta))[[y]]$ for some $\delta \in \mathbb{R}^+$.*

Proof. Suppose that φ is defined in $B(0,\delta) \times B(0,\epsilon)$. Let Θ be the operator $\varphi - Id$. By the proof of proposition 3.3 we have

$$(\log \varphi)(y) - \sum_{j=1}^{l} (-1)^{j+1} \frac{\Theta^{\circ(j)}(y)}{j} \in (y^{\nu(\varphi)+l+1}).$$

We are done since $\Theta^{\circ(j)}(y)$ is holomorphic in the neighborhood of $B(0,\delta) \times \{0\}$ for all $j \in \mathbb{N}$. □

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with common convergent normal form such that $Fix\varphi_1$ is of trivial type. We define $\hat{\sigma}(\varphi_1,\varphi_2) = \hat{\sigma}(\varphi_1,\varphi_2,Fix\varphi_1)$. We say that $\hat{\sigma}(\varphi_1,\varphi_2)$ is the *privileged formal conjugation* between φ_1 and φ_2. By construction we obtain that $y \circ \hat{\sigma}(\varphi_1,\varphi_2) - y \in (y^{\nu(\varphi_1)+2})$.

Lemma 6.2. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with common convergent normal form such that $Fix\varphi_1$ is of trivial type. Then $y \circ \hat{\sigma}(\varphi_1,\varphi_2) \in \vartheta(B(0,\delta))[[y]]$ for some $\delta \in \mathbb{R}^+$.*

Proof. We have $(\log \varphi_1)(y), (\log \varphi_2)(y) \in \vartheta(B(0,\delta))[[y]]$ for some $\delta \in \mathbb{R}^+$ by lemma 6.1. Consider $\hat{\beta} \in \mathbb{C}[[x,y]]$ such that $\partial\hat{\beta}/\partial y = 1/(\log \varphi_1)(y) - 1/(\log \varphi_2)(y)$ and $\hat{\beta}(x,0) \equiv 0$. We deduce that $\hat{\beta}$ and then $y \circ \hat{\sigma}(\varphi_1, \varphi_2)$ belong to $\vartheta(B(0,\delta))[[y]]$ by proposition 5.3. □

Lemma 6.3. *Let $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ such that $Fix\varphi$ is of trivial type. Then $y \circ \hat{\tau}_0(\varphi)$ belongs to $\vartheta(B(0,\delta))[[y]]$ for some $\delta \in \mathbb{R}^+$.*

Proof. Consider the notations in the proof of lemma 5.3. We have that

$$\hat{\tau}_0(\varphi) = \hat{\sigma}(\exp(X), \varphi) \circ (x, e^{2\pi i/\nu(\varphi)} y) \circ \hat{\sigma}(\exp(X), \varphi)^{\circ(-1)}.$$

Now $y \circ \hat{\sigma}(\exp(X), \varphi)$ belongs to $\vartheta(B(0,\delta))[[y]]$ for some $\delta \in \mathbb{R}^+$ by the previous lemma. Therefore $y \circ \hat{\tau}_0(\varphi)$ belongs to $\vartheta(B(0,\delta))[[y]]$. □

Let $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. Suppose that $Fix\varphi$ is of trivial type. We define φ_w as the germ of $\varphi_{|x=w}$ at the neighborhood of $y = 0$. We define $D_s(\varphi)$ as the continuous sections of $\cup_{w \in B(0,\delta)} \{w\} \times D_s(\varphi_w)$. The directions in $D_s(\varphi_w)$ vary continuously with respect to w. Thus the mapping $\lambda \mapsto \lambda(0)$ establishes a bijection from $D_s(\varphi)$ onto $D_s(\varphi_0)$. We define $D(\varphi) = D_{-1}(\varphi) \cup D_1(\varphi)$.

For $\lambda \in D(\varphi)$ we define the petal $V_\varphi^\lambda = \cup_{w \in B(0,\delta)} V_{\varphi_w}^{\lambda(w)}$. The set V_φ^λ is open. We say that $\eta \in \vartheta(V_\varphi^\lambda)$ is a Fatou coordinate of φ in V_φ^λ if $\eta_{|x=w}$ is a Fatou coordinate of $\varphi_{|x=w}$ in $V_{\varphi_w}^{\lambda(w)}$ for all w in a neighborhood of 0.

Fix a convergent normal form α of φ. Fix a direction $\lambda_0 \in D(\varphi)$ and a Fatou coordinate $\psi_\alpha^{\lambda_0} \in \vartheta(V_\alpha^{\lambda_0} \cup V_\varphi^{\lambda_0})$ of α. Now consider homogeneous coordinates, i.e. we exhibit for every $\lambda \in D(\varphi)$ an integral of the time form $\psi_\alpha^\lambda \in \vartheta(V_\alpha^\lambda \cup V_\varphi^\lambda)$ of α such that the system $\{(\psi_\alpha^\lambda)_{|x=w}\}_{\lambda \in D(\varphi)}$ provides homogeneous coordinates for all w in a neighborhood of 0. There exists a unique integral of the time form $\psi_\varphi^\lambda \in \vartheta(V_\varphi^\lambda)$ of φ for all $\lambda \in D(\varphi)$ such that $\lim_{y \to 0} (\psi_\varphi^\lambda - \psi_\alpha)(w,y) = 0$ in every sector of direction $\lambda(w)$ and angle lesser than $2\pi/\nu(\varphi)$ contained in $V_\varphi^\lambda \cap (x = w)$ for all w in a neighborhood of 0. Moreover $(x, \psi_\varphi^\lambda(x,z))$ is injective in V_φ^λ. The proof can be obtained like in subsection 4.3. Let $\lambda \in D_s(\varphi)$, we can define the change of charts

$$\xi_\varphi^\lambda(x,z) = \psi_\varphi^{\lambda e^{i\pi/\nu(\varphi)}} \circ \left(x, \psi_\varphi^\lambda(x,z)\right)^{\circ(-1)}(x,z).$$

We obtain that ξ_φ^λ is of the form

$$\xi_\varphi^\lambda(x,z) = z - \pi i Res(\varphi, (x,0))/\nu(\varphi) + \sum_{j=1}^{\infty} a_{\lambda,j}^\varphi(x) e^{-2\pi i s j z}$$

where $a_{\lambda,j}^\varphi$ is an analytic function for all $j \in \mathbb{N}$. Moreover $\sum_{j=1}^\infty a_{\lambda,j}^\varphi(x) w^j$ is an analytic function in a neighborhood of $(x,w) = (0,0)$. A different choice of convergent normal form or homogeneous coordinates provides new Fatou coordinates $\psi_\varphi^\lambda(x,z) + t(x)$ for some $t \in \mathbb{C}\{x\}$ independent of $\lambda \in D(\varphi)$. Thus the changes of charts are unique up to conjugation with $z + t(x)$ for some $t \in \mathbb{C}\{x\}$.

Let $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with common convergent normal form α. We always suppose that their Fatou coordinates are calculated with respect to a common system of homogeneous coordinates. Since $\hat{\tau}_0(\varphi_2)$ and $\hat{\sigma}(\varphi_1, \varphi_2)$ depend analytically on x by lemmas 6.3 and 6.2 then there are parameterized versions of the results in subsections 4.2, 4.3 and 4.4. We obtain:

Proposition 6.1. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with common convergent normal form α. Suppose that $Fix\varphi_1$ is of trivial type. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if there exists $(\kappa, t) \in < e^{2i\pi/\nu(\alpha)} > \times \mathbb{C}\{x\}$ such that*

$$\xi_{\varphi_2}^{\lambda\kappa}(x, z + t(x)) = (z + t(x)) \circ \xi_{\varphi_1}^{\lambda}(x, z) \quad \forall \lambda \in D(\varphi_1). \tag{3}$$

Indeed the equation 3 is equivalent to $Z_{\varphi_2}^{\kappa,t} \circ \hat{\sigma}(\varphi_1, \varphi_2) \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$.

Remark 6.1. *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Analogously to remark 4.2 the previous proposition implies that*

$$\log \varphi \in \mathcal{X}(\mathbb{C}^2, 0) \Leftrightarrow \xi_{\varphi}^{\lambda}(x, z) \equiv z - \pi i Res(\varphi, (x, 0))/\nu(\varphi) \ \forall \lambda \in D(\varphi).$$

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix\varphi_1 = Fix\varphi_2$. We say that $m_{\varphi_1}(w) = m_{\varphi_2}(w)$ if $(\varphi_1)_w \sim (\varphi_2)_w$. Next we analyze whether m_φ determines the analytic class of φ. It turns out that the analytic triviality of φ_0 plays a preeminent role. Let $\alpha \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be a convergent normal form of φ and consider homogeneous coordinates for the changes of charts of φ. Consider the set

$$E_s(\varphi) = \{(\lambda, k) \in D_s(\varphi) \times \mathbb{N} \ s.t. \ a_{\lambda,k}^{\varphi} \not\equiv 0\}$$

for $s \in \{-1, 1\}$. We define $E(\varphi) = E_{-1}(\varphi) \cup E_1(\varphi)$. We define

$$gd(\varphi) = \gcd\{j \in \mathbb{N} : \exists \lambda \in D(\varphi) \text{ s.t. } (\lambda, j) \in E(\varphi)\}.$$

The definitions of $gd(\varphi)$ and $E_s(\varphi)$ for $s \in \{-1, 1\}$ do not depend on the choice of homogeneous coordinates.

Proposition 6.2. *Let $\varphi_1 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ such that $\log \varphi_1$ is divergent. Suppose that $Fix\varphi_1$ is of trivial type. Then there exists $\varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with common convergent normal form with φ_1 such that*

- $m_{\varphi_1} \equiv m_{\varphi_2}$
- *φ_1 is not analytically conjugated to φ_2 by a special diffeomorphism*

if and only if

- $\log(\varphi_1)_0$ *belongs to $\mathcal{X}(\mathbb{C}, 0)$*
- $\{s \in \{-1, 1\} : a_{\lambda,j}^{\varphi_1} \in (x^{j/gd(\varphi_1)+1})$ for all $(\lambda, j) \in D_s(\varphi_1) \times \mathbb{N}\} \neq \emptyset$.

In such a case there exists $(r, s, q) \in \mathbb{Z}/(\nu(\varphi_1)\mathbb{Z}) \times \mathbb{C}\{x\} \times (\mathbb{Q} \setminus \{0\})$ such that φ_1 and φ_2 are conjugated by a transformation of the form

$$\exp\left(\left(\frac{q}{2\pi i}\log x + s(x)\right)\log \varphi_2\right) \circ \hat{\tau}_0(\varphi_2)^{\circ(r)} \circ \hat{\sigma}(\varphi_1, \varphi_2)$$

which is analytic in a domain of the form $|y| < C_0/\sqrt[\nu(\varphi_1)]{|\ln x|}$ for some $C_0 \in \mathbb{R}^+$.

Proof of the sufficient condition. Choose a convergent normal form α of φ_1. Consider Fatou coordinates ψ_α^λ in V_α^λ for all $\lambda \in D(\varphi_1)$. Suppose that $m_{\varphi_1} \equiv m_{\varphi_2}$. Then there exists $(\kappa(x), d(x)) \in < e^{2i\pi/\nu(\alpha)} > \times \mathbb{C}$ such that

$$\xi_{\varphi_2}^{\lambda\kappa(x)}(x, z + d(x)) = (z + d(x)) \circ \xi_{\varphi_1}^{\lambda}(x, z)$$

in homogeneous coordinates for all $\lambda \in D(\varphi_1)$ and all x in a neighborhood of 0. A priori the functions $\kappa(x)$ and $d(x)$ are not even continuous. There exists $\kappa \in < e^{2i\pi/\nu(\alpha)} >$ such that $W = [\kappa(x) = \kappa]$ is uncountable in every neighborhood of $x = 0$. Let $(\lambda, j) \in E_s(\varphi_1)$; we have $a_{\lambda,j}^{\varphi_1}(x) = a_{\lambda\kappa,j}^{\varphi_2}(x)e^{-2\pi s j d(x) i}$ for all $x \in W$ and then $a_{\lambda\kappa,j}^{\varphi_2} \not\equiv 0$. We denote $sg : D(\varphi_1) \to \{-1, 1\}$ the function such that

$sg(D_s(\varphi_1)) = \{s\}$ for $s \in \{-1,1\}$. We denote $h_{\lambda,j} = a_{\lambda\kappa,j}^{\varphi_2}/a_{\lambda,j}^{\varphi_1}$. The previous formula implies that

$$h_{\lambda,j}^{sg(\lambda)k} = h_{\mu,k}^{sg(\mu)j} \tag{4}$$

for all $(\lambda,j),(\mu,k) \in E(\varphi_1)$ and for all $x \in W \setminus (a_{\lambda,j}^{\varphi_1} a_{\mu,k}^{\varphi_1} = 0)$. The equation 4 is satisfied for all x in a neighborhood of 0 since W is uncountable. Consider $\nu(\lambda,j) \in \mathbb{Z}$ the order of vanishing of $h_{\lambda,j}$, i.e. $h_{\lambda,j}/x^{\nu(\lambda,j)}$ is a unit. Thus we have

$$\nu(\lambda,j) sg(\lambda) k = \nu(\mu,k) sg(\mu) j \tag{5}$$

for all $(\lambda,j),(\mu,k) \in E(\varphi_1)$.

Consider a point $x_0 \in W$ such that $a_{\lambda,j}^{\varphi_1}(x_0) \neq 0$ for all $(\lambda,j) \in E(\varphi_1)$. Choose $(\lambda_0,j_0) \in E(\varphi_1)$; we define $q = sg(\lambda_0)\nu(\lambda_0,j_0)/j_0$ and

$$s(x) = \frac{1}{2\pi i sg(\lambda_0) j_0} \ln \frac{h_{\lambda_0,j_0}(x)}{x^{\nu(\lambda_0,j_0)}}.$$

Denote $t(x) = q/(2\pi i)\ln x + s(x)$. We can suppose $t(x_0) = d(x_0)$ by choosing properly the determination of the logarithm. The equation 4 implies that t does not depend on $(\lambda_0,j_0) \in E(\varphi_1)$. Thus we obtain $e^{2\pi i sg(\lambda) j t(x)} a_{\lambda,j}^{\varphi_1} = a_{\lambda\kappa,j}^{\varphi_2}$ for all $(\lambda,j) \in E(\varphi_1)$. By proposition 4.10 we deduce that φ_1 and φ_2 are conjugated by an analytic mapping σ defined in a neighborhood of $(y=0) \setminus \{(0,0)\}$ and whose expression in each petal $V_{\varphi_1}^{\lambda}$ is given by

$$\left[\left(\psi_{\varphi_2}^{\lambda\kappa}\right)^{\circ(-1)} \circ \left(\psi_{\varphi_2}^{\lambda\kappa} + \frac{q}{2\pi i}\ln x \right) \right] \circ \left[\left(\psi_{\varphi_2}^{\lambda\kappa}\right)^{\circ(-1)} \circ \left(\psi_{\varphi_1}^{\lambda} + s(x) \right) \right]. \tag{6}$$

The condition $\varphi_1 \stackrel{sp}{\not\sim} \varphi_2$ implies that $q \neq 0$, otherwise $Z_{\varphi_2}^{\kappa,s} \circ \hat{\sigma}(\varphi_1,\varphi_2)$ is a special analytic conjugation by proposition 6.1. Moreover since $\psi_{\varphi_j}^{\lambda} - \psi_{\alpha}^{\lambda}$ is bounded for $j \in \{1,2\}$ then σ is defined in a domain very similar to the domain of definition of $\exp(q/(2\pi i)\ln x \log \alpha)$. We have $\psi_{\alpha}^{\lambda} \sim 1/y^{\nu(\alpha)}$, thus σ is defined in a domain of the form $|y| < C_0/\sqrt[\nu(\alpha)]{|\ln x|}$ for some $C_0 \in \mathbb{R}^+$.

The property $q \neq 0$ implies $\nu(\lambda,j) \neq 0$ for all $(\lambda,j) \in E(\varphi_1)$. Suppose that $\log(\varphi_1)_0$ does not belong to $\mathcal{X}(\mathbb{C},0)$. By remark 4.2 this implies the existence of $a_{\lambda_0,j_0}^{\varphi_1}$ such that $a_{\lambda_0,j_0}^{\varphi_1}(0) \neq 0$. Since $\xi_{\varphi_2}^{\lambda\kappa(0)}(0, z + d(0)) = (z + d(0)) \circ \xi_{\varphi_1}^{\lambda}(0,z)$ then $a_{\lambda_0\kappa(0),j_0}^{\varphi_2}(0) \neq 0$. We deduce that $\nu(\lambda_0,j_0) > 0 > \nu(\lambda_0\kappa(0)\kappa^{-1}, j_0)$, this inequality contradicts equation 5 since $sg(\lambda_0) = sg(\lambda_0\kappa(0)\kappa^{-1})$. Analogously we obtain $\log(\varphi_2)_0 \in \mathcal{X}(\mathbb{C},0)$.

To prove the second property we can suppose that $E_s(\varphi_1) \neq \emptyset$ for all $s \in \{-1,1\}$; otherwise the result is trivial. The equation 5 implies that either $\nu(E_{-1}(\varphi_1)) \subset \mathbb{Z}_{<0}$ and $\nu(E_{+1}(\varphi_1)) \subset \mathbb{N}$ or $\nu(E_{-1}(\varphi_1)) \subset \mathbb{N}$ and $\nu(E_{+1}(\varphi_1)) \subset \mathbb{Z}_{<0}$. We suppose that we are in the former case without lack of generality. Since $a_{\lambda,j}^{\varphi_1} \equiv a_{\lambda\kappa,j}^{\varphi_2}/h_{\lambda,j}$ we deduce that $a_{\lambda,j}^{\varphi_1} \in (x^{-\nu(\lambda,j)+1})$ for all $(\lambda,j) \in E_{-1}(\varphi_1)$. Let $(\mu,k) \in E(\varphi_1)$, the equation 5 implies that $-\nu(\lambda,j) \in j\mathbb{N}/\gcd(j,k)$. Therefore $-\nu(\lambda,j) \in j\mathbb{N}/gd(\varphi)$. This implies $a_{\lambda,j}^{\varphi_1} \in (x^{j/gd(\varphi)+1})$ for all $(\lambda,j) \in D_{-1}(\varphi_1) \times \mathbb{N}$. □

Proof of the necessary condition. We keep the notations in the previous proof. Suppose without lack of generality that $a_{\lambda,j}^{\varphi_1} \in (x^{j/gd(\varphi_1)+1})$ for all $(\lambda,j) \in D_{-1}(\varphi_1) \times \mathbb{N}$. We define $a_{\lambda,j}^{\varphi_2} = a_{\lambda,j}^{\varphi_1} x^{sj/gd(\varphi_1)}$ for all $(\lambda,j) \in D_s(\varphi_1) \times \mathbb{N}$. Let $\lambda \in D(\varphi_1)$; it is straightforward to prove that

$$\sum_{j=1}^{\infty} a_{\lambda,j}^{\varphi_1}(x) x^{sg(\lambda)j/gd(\varphi_1)} z^j \in \mathbb{C}\{x,z\} \Leftrightarrow \sum_{j=1}^{\infty} a_{\lambda,j}^{\varphi_1}(x) z^j \in \mathbb{C}\{x,z\}.$$

Now we can use a parameterized version of proposition 4.6 to obtain $\varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with convergent normal form α and such that

$$\xi_{\varphi_2}^{\lambda}(x,z) \equiv z - \pi i Res(\varphi_1,(x,0))/\nu(\varphi_1) + \sum_{j=1}^{\infty} a_{\lambda,j}^{\varphi_2}(x) e^{-2\pi i s j z}$$

for all $\lambda \in D_s(\varphi_1)$ and all $s \in \{-1,1\}$. Let us remark that $\log(\varphi_2)_0 \in \mathcal{X}(\mathbb{C},0)$ since $a_{\lambda,j}^{\varphi_2}(0) = 0$ for all $(\lambda,j) \in D(\varphi_1) \times \mathbb{N}$. Our choice of φ_2 implies that

$$\xi_{\varphi_2}^{\lambda}\left(x, z + \frac{1}{2\pi i g d(\varphi_1)} \ln x\right) \equiv \left(z + \frac{1}{2\pi i g d(\varphi_1)} \ln x\right) \circ \xi_{\varphi_1}^{\lambda}(x,z) \ \ \forall \lambda \in D(\varphi_1).$$

Therefore we get $m_{\varphi_1}(x) = m_{\varphi_2}(x)$ for all $x \neq 0$. Moreover $m_{\varphi_1}(0) = m_{\varphi_2}(0)$ since both $(\varphi_1)_0$ and $(\varphi_2)_0$ are analytically trivial.

We define $\nu_k(\lambda,j)$ the order of vanishing of $a_{\lambda,j}^{\varphi_k}$ at 0, it is $-\infty$ if $a_{\lambda,j}^{\varphi_k} \equiv 0$. We have $\nu_2(\lambda,j) = \nu_1(\lambda,j) + sj/gd(\varphi_1)$ for all $(\lambda,j) \in D_s(\varphi_1) \times \mathbb{N}$. Suppose $\varphi_1 \overset{sp}{\sim} \varphi_2$; then there exists $(\kappa,d) \in < e^{2i\pi/\nu(\varphi_1)} > \times \mathbb{C}\{x\}$ such that $a_{\lambda,j}^{\varphi_1}(x) \equiv a_{\lambda\kappa,j}^{\varphi_2}(x) e^{-2\pi s j d(x) i}$ for all $(\lambda,j) \in D_s(\varphi_1) \times \mathbb{N}$ (prop. 6.1). Thus we obtain $\nu_2(\lambda\kappa,j) = \nu_1(\lambda,j)$ for all $(\lambda,j) \in D(\varphi_1) \times \mathbb{N}$. Choose $(\lambda_0,j_0) \in D(\varphi_1) \times \mathbb{N}$ such that $\nu_1(\lambda_0,j_0) \neq -\infty$. By remark 6.1 that is possible since $\log \varphi_1$ is divergent. We define

$$H = \{\lambda \in D_{sg(\lambda_0)}(\varphi_1) : \nu(\lambda,j_0) \neq -\infty\}.$$

Denote $c = sg(\lambda_0) j_0 / gd(\varphi_1)$. We obtain

$$\sum_{\lambda \in H} \nu_2(\lambda,j_0) = \sum_{\lambda \in H} (\nu_1(\lambda,j_0) + c) = c\sharp H + \sum_{\lambda \in \kappa H} \nu_2(\lambda,j_0) = c\sharp H + \sum_{\lambda \in H} \nu_2(\lambda,j_0).$$

This is impossible since $c \neq 0$ and $H \neq \emptyset$. Thus φ_1 and φ_2 are not conjugated by a special diffeomorphism. □

Corollary 6.1. *Let $\alpha \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ such that $\log \alpha \in \mathcal{X}(\mathbb{C}^2,0)$. Suppose that $Fix\alpha$ is of trivial type. Then there exist $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with convergent normal form α such that $m_{\varphi_1} \equiv m_{\varphi_2}$ but $\varphi_1 \overset{sp}{\not\sim} \varphi_2$.*

Corollary 6.2. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ such that $Fix\varphi_1 = Fix\varphi_2$. Suppose that $Fix\varphi_1$ is of trivial type and that $\log(\varphi_1)_0 \notin \mathcal{X}(\mathbb{C},0)$. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only $m_{\varphi_1} \equiv m_{\varphi_2}$.*

We say that η is a *r-moderated mapping* if η is a biholomorphism from $B(0,r)$ onto $\eta(B(0,r))$. If besides that $\eta(B(0,r))$ is contained in $B(0,R)$ then we say that η is rR-moderated.

Next proposition is intended to show that $\eta_w \circ (\varphi_1)_{|x=w} = (\varphi_2)_{|x=w} \circ \eta_w$ for all $w \neq 0$ and $\varphi_1 \overset{sp}{\not\sim} \varphi_2$ are not compatible if the domains of definition of the conjugations η_w have a regular behavior when $w \to 0$.

Proposition 6.3. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ such that $Fix\varphi_1 = Fix\varphi_2$. Suppose that $Fix\varphi_1$ is of trivial type. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if there exist $r \in \mathbb{R}^+$ and a r-moderated mapping η_x conjugating $(\varphi_1)_x$ and $(\varphi_2)_x$ for all x in a pointed neighborhood of 0.*

We do not ask the mappings η_x to have any kind of good dependance with respect to x.

Proof. By proposition 5.1 we have that $\nu(\varphi_1) = \nu(\varphi_2)$ and $Res(\varphi_1) \equiv Res(\varphi_2)$. Let α_j be a convergent normal form of φ_j for $j \in \{1,2\}$. Since $\nu(\alpha_j) = \nu(\varphi_j)$ and $Res(\alpha_j) \equiv Res(\varphi_j)$ for all $j \in \{1,2\}$ then there exists a special $\zeta \in \text{Diff}\,(\mathbb{C}^2, 0)$ conjugating α_1 and α_2 by proposition 5.2. Denote $\tilde{\varphi}_2 = \zeta^{\circ(-1)} \circ \varphi_2 \circ \zeta$. The mapping $\zeta_x^{\circ(-1)} \circ \eta_x$ conjugates $(\varphi_1)_x$ and $(\tilde{\varphi}_2)_x$, they share the convergent normal form $(\alpha_1)_x$. Then we obtain $\zeta_x^{\circ(-1)} \circ \eta_x = Z_{(\tilde{\varphi}_2)_x}^{\kappa,t} \circ \hat{\sigma}((\varphi_1)_x, (\tilde{\varphi}_2)_x)$ for some $(\kappa, t) \in < e^{2\pi i/\nu(\varphi_1)} > \times \mathbb{C}$. This implies $|(\partial(\zeta_x^{\circ(-1)} \circ \eta_x)/\partial y)(0)| = 1$ and then $|(\partial\zeta_x)/\partial y)(0)| = |(\partial\eta_x)/\partial y)(0)|$. Denote $b(x) = (\partial\eta_x/\partial y)(0)$. We obtain that $\eta_x(ry)/(rb(x))$ is a Schlicht function for all x in a pointed neighborhood of 0. By the Koebe's distortion theorem (see [5], page 65) we get

$$\sup_{y \in B(0,r_1)} |\eta_x(y)| \leq r|b(x)| \sup_{y \in B(0,r_1/r)} \left| \frac{\eta_x(ry)}{rb(x)} \right| \leq r \left| \frac{\partial(y \circ \zeta)}{\partial y}(x, 0) \right| \frac{r_1/r}{(1 - r_1/r)^2}$$

for all $r_1 < r$ and all x in a pointed neighborhood of 0. We deduce that $\zeta_x^{\circ(-1)} \circ \eta_x$ is rR-moderated for some $R \in \mathbb{R}^+$ by considering a smaller $r > 0$ if necessary. By replacing φ_2 with $\tilde{\varphi}_2$ and η_x with $\zeta_x^{\circ(-1)} \circ \eta_x$ we can suppose that φ_1 and φ_2 have common normal form.

Suppose that either $\log \varphi_1$ or $\log \varphi_2$ belongs to $\mathcal{X}(\mathbb{C}^2, 0)$. Since $(\varphi_1)_x$ is conjugated to $(\varphi_2)_x$ for all x in a pointed neighborhood of 0 then $a_{\lambda,j}^{\varphi_k} \equiv 0$ for all $(\lambda, j, k) \in D(\varphi_1) \times \mathbb{N} \times \{1, 2\}$ by remark 4.2. Thus $\log \varphi_k \in \mathcal{X}(\mathbb{C}^2, 0)$ for all $k \in \{1, 2\}$ by remark 6.1. The discussion in the previous paragraph implies that φ_1 and φ_2 are conjugated by a special diffeomorphism.

Now suppose that $\log \varphi_j \notin \mathcal{X}(\mathbb{C}^2, 0)$ for $j \in \{1, 2\}$. Since η_w conjugates $(\varphi_1)_w$ and $(\varphi_2)_w$ then η_w is of the form $Z_{(\varphi_2)_w}^{\kappa(w),d(w)} \circ \hat{\sigma}((\varphi_1)_w, (\varphi_2)_w)$ where $(\kappa(w), d(w))$ belongs to $< e^{2\pi i/\nu(\varphi_1)} > \times \mathbb{C}$ for all w in a pointed neighborhood of 0. We choose $\kappa \in < e^{2\pi i/\nu(\varphi_1)} >$ such that $(\kappa(x) = \kappa)$ is an uncountable set in every neighborhood of 0. Denote $W = [\kappa(x) = \kappa] \setminus \cup_{(\lambda,j) \in E(\varphi_2)} (a_{\lambda,j}^{\varphi_2} = 0)$. By the proof of proposition 6.2 we obtain that there exists $(q, s) \in \mathbb{Q} \times \mathbb{C}\{x\}$ such that

$$\exp\left(\frac{q}{2\pi i} \log x \log \varphi_2\right) \circ Z_{\varphi_2}^{\kappa,s} \circ \hat{\sigma}(\varphi_1, \varphi_2)$$

conjugates φ_1 and φ_2 in a neighborhood of $(y = 0) \setminus \{(0, 0)\}$. We deduce that $\exp((q/(2\pi i) \ln w + s(w) - d(w)) \log \varphi_2)_{|x=w}$ is in $Z((\varphi_2)_w)$ for all $w \in W \setminus \{0\}$. We obtain the equation

$$a_{\lambda,j}^{\varphi_2}(x) = a_{\lambda,j}^{\varphi_2}(x) e^{-2\pi i s j (q/(2\pi i) \ln x + s(x) - d(x))}$$

for all $(\lambda, j) \in E_s(\varphi_2)$ and $s \in \{-1, 1\}$ and $x \in W \setminus \{0\}$ (prop. 4.10). Since $E(\varphi_2) \neq \emptyset$ then $(q/2\pi i) \ln x + s(x) - d(x)$ belongs to $\mathbb{Q}$ for all $x \in W \setminus \{0\}$.

We want to estimate $d(x)$. We have

$$y \circ \eta_w - y \circ (Z_{\varphi_2}^{\kappa,0})_{|x=w} - \kappa d(w) \frac{(\log \varphi_2)(y)}{y^{\nu(\varphi_1)+1}}(w, 0) y^{\nu(\varphi_1)+1} \in (y^{\nu(\varphi_1)+2})$$

for all $w \in W \setminus \{0\}$. The series $[(\log \varphi_2)(y)/y^{\nu(\varphi_1)+1}](x, 0)$ is a unit of $\mathbb{C}\{x\}$ by lemma 6.1. We denote by $C(\eta_x)$ and $C(Z_{\varphi_2}^{\kappa,0})$ the coefficients of $y^{\nu(\varphi_1)+1}$ of η_x and

$Z_{\varphi_2}^{\kappa,0}$ respectively. We obtain that $C(\eta_x) \in \mathbb{C}$ for all x in a pointed neighborhood of 0 and $C(Z_{\varphi_2}^{\kappa,0}) \in \mathbb{C}\{x\}$ by lemma 6.3. We have that

$$C(\eta_x) = \frac{1}{2\pi i} \int_{|y|=r/2} \frac{y \circ \eta_x(y)}{y^{\nu(\varphi_1)+2}}$$

and then $|C(\eta_x)| \leq 2^{\nu(\varphi_1)+1} R / r^{\nu(\varphi_1)+1}$ for all x in a pointed neighborhood of 0. We deduce that there exists $K > 0$ such that $|d(x)| \leq K$ for all $x \in W \setminus \{0\}$ in a neighborhood of 0. Hence $Img((q/2\pi i)\ln x)$ is bounded for $x \in W \setminus \{0\}$ in a neighborhood of 0. This implies $q = 0$ since otherwise

$$\lim_{x \to 0} \left| Img\left(\frac{q}{2\pi i} lnx \right) \right| = \lim_{x \to 0} \frac{|q|}{2\pi} |\ln |x|| = \infty.$$

We obtain a special element $Z_{\varphi_2}^{\kappa,s} \circ \hat{\sigma}(\varphi_1, \varphi_2)$ of $\text{Diff}\,(\mathbb{C}^2, 0)$ conjugating φ_1 and φ_2 by proposition 6.1. □

Consider $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix\varphi_1 = Fix\varphi_2$ of trivial type. We denote $Inv(\varphi_1) \sim Inv(\varphi_2)$ if there exists $(\kappa(x), d(x)) \in < e^{2\pi i/\nu(\varphi_1)} > \times [|Img(z)| < I]$ such that

$$\xi_{\varphi_2}^{\lambda\kappa(x)}(x, z + d(x)) = (z + d(x)) \circ \xi_{\varphi_1}^{\lambda}(x, z) \ \ \forall \lambda \in D(\varphi_1)$$

in homogeneous coordinates for all $x \neq 0$ in a neighborhood of 0 and some $I \in \mathbb{R}^+$.

Proposition 6.4. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix\varphi_1 = Fix\varphi_2$ of trivial type. Then we have $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if $Inv(\varphi_1) \sim Inv(\varphi_2)$.*

The previous proposition provides a complete system of analytic invariants in the trivial type case. It is composed by the changes of charts modulo moderated changes of coordinates.

Proof. The condition $Inv(\varphi_1) \sim Inv(\varphi_2)$ implies in particular $Res(\varphi_1) \equiv Res(\varphi_2)$. Let α_j be a convergent normal form of φ_j for $j \in \{1, 2\}$. Thus α_1 and α_2 are conjugated by a good $\sigma \in \text{Diff}_p(\mathbb{C}^2, 0)$ by proposition 5.3. By replacing φ_2 with $\sigma^{\circ(-1)} \circ \varphi_2 \circ \sigma$ and $\xi_{\varphi_2}^{\lambda}(x, z)$ with $(z + t(x)) \circ \xi_{\varphi_2}^{\lambda} \circ (x, z - t(x))$ for all $\lambda \in D(\varphi_2)$ and some $t \in \mathbb{C}\{x\}$ we can suppose that φ_1 and φ_2 have common convergent normal form α. The proof of proposition 6.3 also works if we replace the moderated hypothesis with the boundness of $Img(d)$. □

Proposition 6.3 provides a geometrical interpretation of the system of invariants $Inv(\varphi)$. In this paper we define the analogue of Mardesic-Roussarie-Rousseau's [16] invariants of analytic classification for all $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ (theorem 10.2). We prove a rigidity theorem (analogous to corollary 6.2), a theorem making clear the relation among the analytic conjugation and the centralizer (analogous to proposition 6.1) and the moderated theorem 1.1 giving geometrical insight about the nature of the space of invariants.

7. Dynamics of the real flow of a normal form

From now on we deal with diffeomorphism $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ such that $Fix\varphi$ is not of trivial type. In particular the number $N(\varphi) = \sharp[(Fix\varphi) \cap (x = x_0)]$ where $x_0 \neq 0$ satisfies $N(\varphi) \geq 2$. Our goal is splitting a domain $|y| < \epsilon$ in several sets in which the dynamics are simpler to analyze. Afterwards we intend to analyze the sectors in the parameter space in which a vector field of the form

$Re(\lambda X)$ ($\lambda \in \mathbb{S}^1 \setminus \{-1,1\}$) has a stable behavior. The stability will provide well-behaved transversals to $Re(X)$. Such transversals are the base to construct Fatou coordinates of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ (with convergent normal form X) for all x in a neighborhood of 0.

Consider the function

$$\begin{array}{cccc} ag_X^\epsilon : & B(0,\delta) \times \partial B(0,\epsilon) & \to & \mathbb{S}^1 \\ & (x,y) & \mapsto & (X(y)/y)/|X(y)/y|. \end{array}$$

By lifting ag_X^ϵ to $\mathbb{R} = \widetilde{\mathbb{S}^1}$ we obtain a mapping $arg_X^\epsilon : B(0,\delta) \times \mathbb{R} \to \mathbb{R}$ such that $e^{2\pi i \theta} \circ arg_X^\epsilon(x,\theta) = ag_X^\epsilon(x, \epsilon e^{2\pi i \theta})$. It is easy to prove that $(\partial arg_X^\epsilon/\partial\theta)(0,\theta)$ tends uniformly to $\nu(X)$ when $\epsilon \to 0$. By continuity we obtain that $\partial arg_X^\epsilon/\partial\theta$ is very close to $\nu(X)$ for $0 < \epsilon << 1$ and $0 < \delta(\epsilon) << 1$.

Let $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ and fix $0 < \epsilon << 1$. We define the set $T_X^\epsilon(x_0)$ of tangent points between $Re(X)_{|x=x_0}$ and $\partial B(0,\epsilon)$ for $x_0 \in B(0,\delta(\epsilon))$. Denote the set $\cup_{x \in B(0,\delta)} \{x\} \times T_X^\epsilon(x)$ by T_X^ϵ. We say that a point $y_0 \in T_X^\epsilon(x_0)$ is *convex* if the germ of trajectory of $Re(X)_{|x=x_0}$ passing through y_0 is contained in $\overline{B}(0,\epsilon)$. Next lemma is a consequence of $\partial arg_X^\epsilon/\partial\theta \sim \nu(X)$ and $T_{\lambda X}^\epsilon(x_0) = ag_X^\epsilon(x_0,y)^{\circ(-1)}\{-i/\lambda, i/\lambda\}$.

Lemma 7.1. *Let $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$. There exist $\epsilon_0 > 0$ and $\delta_0 : (0,\epsilon_0) \to \mathbb{R}^+$ such that $T_{\lambda X}^\epsilon(x_0)$ is composed of $2\nu(X)$ convex points for all $\lambda \in \mathbb{S}^1$, $0 < \epsilon < \epsilon_0$ and $x_0 \in B(0,\delta_0(\epsilon))$. Moreover, each connected component of $\partial B(0,\epsilon) \setminus T_{\lambda X}^\epsilon(x_0)$ contains a unique point of $T_{\mu X}^\epsilon(x_0)$ for all $\mu \in \mathbb{S}^1 \setminus \{-\lambda,\lambda\}$.*

Remark 7.1. *Fix $\lambda \in \mathbb{S}^1$. We have $T_{\lambda X}^\epsilon(x) = \{T_{\lambda X}^{\epsilon,1}(x), \ldots, T_{\lambda X}^{\epsilon,2\nu(X)}(x)\}$ for all $x \in B(0,\delta_0(\epsilon))$ where $T_{\lambda X}^{\epsilon,j} : B(0,\delta_0(\epsilon)) \to T_X^\epsilon$ is continuous for all $1 \le j \le 2\nu(X)$.*

7.1. **Splitting the dynamics.** For simplicity of the notation we will consider the sets $\mathcal{X}_{tp1}(\mathbb{C}^2,0) \subset \mathcal{X}_{p1}(\mathbb{C}^2,0)$ and $\mathrm{Diff}_{tp1}(\mathbb{C}^2,0) \subset \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ whose elements satisfy that their singular or fixed points sets respectively are not of trivial type but they are a union of smooth curves transversal to $\partial/\partial y$. For all $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ there exists $k \in \mathbb{N}$ such that $(x^{1/k},y) \circ \varphi \circ (x^k,y)$ belongs to $\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. An element $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$ is of the form $u(x,y)\prod_{j=1}^N (y-\alpha_j(x))^{n_j} \partial/\partial y$ for some unit $u \in \mathbb{C}\{x,y\}$ and some $\alpha_j \in \mathbb{C}\{x\} \cap (x)$ for all $j \in \{1,\ldots,N(X)\}$. We have that $\nu(X) = n_1 + \ldots + n_N - 1 \ge 1$.

Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. We define $T_0 = (|y| \le \epsilon)$. Suppose that we have a sequence $\beta = \beta_0 \ldots \beta_k$ where $\beta \in \{0\} \times \mathbb{C}^k$ and $k \ge 0$ and a set $T_\beta = (|t| \le \eta)$ in coordinates (x,t) canonically associated to T_β. The coordinates (x,y) are canonically associated to T_0. Suppose also that

$$X = x^{d_\beta} v(x,t)(t-\gamma_1(x))^{s_1} \ldots (t-\gamma_p(x))^{s_p} \frac{\partial}{\partial t}$$

where $\gamma_1(0) = \ldots = \gamma_p(0) = 0$ and $(v=0) \cap T_\beta = \emptyset$. Denote $\nu(\beta) = s_1 + \ldots + s_p - 1$ and $N(\beta) = p$. Define $X_{\beta,E} = (X(t)/x^{d_\beta})\partial/\partial t$. Denote by $TE_{\mu X}^{\beta,\eta}(r,\lambda)$ the set of tangent points between $Re(\lambda^{d_\beta}\mu X_{\beta,E})_{|x=r\lambda}$ and $|t| = \eta$ for $(r,\lambda,\mu) \in \mathbb{R}_{\ge 0} \times \mathbb{S}^1 \times \mathbb{S}^1$. If $N(\beta) = 1$ then we define $E_\beta = T_\beta$, in other words we do not split T_β.

Suppose $N(\beta) > 1$. Denote $S_\beta = \{(\partial\gamma_1/\partial x)(0), \ldots, (\partial\gamma_p/\partial x)(0)\}$. We define $t = xw$ and the sets $E_\beta = T_\beta \cap [|t| \ge |x|\rho]$ and $M_\beta = (|w| \le \rho)$ for some $\rho >> 0$. We denote $\dot{E}_\beta = [\rho|x| < |t| < \eta]$ if $N(\beta) > 1$, otherwise $\dot{E}_\beta = [|t| < \eta] \setminus Sing X_{\beta,E}$.

We have

$$X = x^{d_\beta + s_1 + \ldots + s_p - 1} v(x, xw) \left(w - \frac{\gamma_1(x)}{x} \right)^{s_1} \ldots \left(w - \frac{\gamma_p(x)}{x} \right)^{s_p} \frac{\partial}{\partial w}$$

in M_β, we define $m_\beta = d_\beta + \nu(\beta)$ and the polynomial vector field

$$X_\beta(\lambda) = \lambda^{m_\beta} v(0,0) \left(w - \frac{\partial \gamma_1}{\partial x}(0) \right)^{s_1} \ldots \left(w - \frac{\partial \gamma_p}{\partial x}(0) \right)^{s_p} \frac{\partial}{\partial w}$$

for $\lambda \in \mathbb{S}^1$. We define $I_\beta = (|w| \leq \rho) \setminus \cup_{\zeta \in S_\beta} (|w - \zeta| < r(\zeta))$ where $r(\zeta) > 0$ is small enough for all $\zeta \in S_\beta$. We define $X_{\beta,M} = (X(w)/x^{m_\beta})\partial/\partial w$; we denote by $TI_{\mu X}^{\beta,\rho}(r,\lambda)$ the set of tangent points between $Re(\lambda^{m_\beta} \mu X_{\beta,M})_{|x = r\lambda}$ and $|w| = \rho$. Finally we define $\dot{I}_\beta = (|w| < \rho) \setminus \cup_{\zeta \in S_\beta} (|w - \zeta| \leq r(\zeta))$.

Fix $\zeta \in S_\beta$. We define $d_{\beta\zeta} = m_\beta$. Consider the coordinate t' such that $w - \zeta = t'$. We denote $T_{\beta\zeta} = (|t'| \leq r(\zeta))$. We have

$$X = x^{d_{\beta\zeta}} h(x, t') \prod_{(\partial \gamma_j / \partial x)(0) = \zeta} \left(t' - \left(\frac{\gamma_j(x)}{x} - \zeta \right) \right)^{s_j} \frac{\partial}{\partial w}.$$

Every set M_β with $\beta \neq \emptyset$ is called a *magnifying glass set*. The sets E_β are called *exterior sets* whereas the sets I_β are called *intermediate sets*.

In the previous paragraph we introduced a method to divide $|y| \leq \epsilon$ in a union of exterior and intermediate sets.

Example: Consider $X = y(y - x^2)(y - x)\partial/\partial y$. We have

$$(|y| \leq \epsilon) = E_0 \cup I_0 \cup E_{01} \cup E_{00} \cup I_{00} \cup E_{000} \cup E_{001}.$$

We have $X_0(1) = w_1^2(w_1 - 1)\partial/\partial w_1$ and $X_{00}(1) = -w_2(w_2 - 1)\partial/\partial w_2$ where $y = xw_1$ and $y = x^2 w_2$. We also get $m_0 = 2$ and $m_{00} = 3$.

Remark 7.2. *Let $X_u = u(x,y) \prod_{j=1}^{N} (y - \alpha_j(x))^{n_j} \partial/\partial y \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ for k in $\{1, 2\}$. The polynomial vector field $(X_u)_\beta(\lambda)$ associated to a magnifying glass set M_β depend only on $u(0,0)$. The value $u(0,0)$ is a formal special invariant in the non-trivial type case. Thus the combinatorial data associated to $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ depends in particular on its class modulo special analytic conjugation.*

Lemma 7.2. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and an exterior set $E_\beta = [\eta \geq |t| \geq \rho|x|]$ associated to X with $0 < \eta << 1$. Then $TE_{\mu X}^{\beta,\eta}(r,\lambda)$ is composed of $2\nu(\beta)$ convex points for all $(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1$ and r close to 0. Each connected component of $\partial B(0,\eta) \setminus TE_{\mu X}^{\beta,\eta}(r,\lambda)$ contains a unique point of $TE_{\mu' X}^{\beta,\eta}(r,\lambda)$ $\forall \mu' \in \mathbb{S}^1 \setminus \{-\mu, \mu\}$.*

Lemma 7.3. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and an exterior set $E_\beta = [\eta \geq |t| \geq \rho|x|]$ associated to X with $N(\beta) > 1$ and $\rho >> 0$. Then $TI_{\mu X}^{\beta,\rho}(r,\lambda)$ is composed of $2\nu(\beta)$ convex points for all $(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1$ and r close to 0. Denote $t = xw$; each connected component of $[|w| = \rho] \setminus TI_{\mu X}^{\beta,\rho}(r,\lambda)$ contains a unique point of $TI_{\mu' X}^{\beta,\rho}(r,\lambda)$ for all $\mu' \in \mathbb{S}^1 \setminus \{-\mu, \mu\}$.*

Lemma 7.2 is the analogue of lemma 7.1 for exterior sets. Lemma 7.3 is deduced from the polynomial character of $X_\beta(1)$ since it implies that $\partial \arg_{X_\beta(1)}^{\rho}/\partial\theta \sim \nu(\beta)$ when $\rho \to \infty$.

Let $X \in \mathcal{X}(\mathbb{C}, 0)$. Consider a set $F \subset \mathbb{C}^n$ contained in the domain of definition of X. Denote by $\dot{F}$ the interior of F. We define $It(X, P, F)$ the maximal interval where $\exp(zX)(P)$ is well-defined and belongs to F for all $z \in It(X, P, F)$ whereas $\exp(zX)(P)$ belongs to $\dot{F}$ for all $z \neq 0$ in the interior of $It(X, P, F)$. We define

$$\partial It(X, P, F) = \{\inf(It(X, P, F)), \sup(It(X, P, F))\} \subset \mathbb{R} \cup \{-\infty, \infty\}.$$

We denote $\Gamma(X, P, F) = \exp(It(X, P, F)X)(P)$.

We will consider coordinates $(x, y) \in \mathbb{C} \times \mathbb{C}$ or $(r, \lambda, y) \in \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \times \mathbb{C}$ in $\mathbb{C}^2$. Given a set $F \subset \mathbb{C}^2$ we denote by $F(x_0)$ the set $F \cap [x = x_0]$ and by $F(r_0, \lambda_0)$ the set $F \cap [(r, \lambda) = (r_0, \lambda_0)]$. In the next subsections we analyze the dynamics in the exterior and intermediate sets.

7.2. **Parabolic exterior sets.** Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Suppose we have

$$X = x^{d_\beta} v(x, t)(t - \gamma_1(x))^{s_1} \ldots (t - \gamma_p(x))^{s_p} \partial/\partial t$$

in some exterior set $E_\beta = [\eta \geq |t| \geq |x|\rho]$ for some $\rho \geq 0$. We say that E_β is parabolic if $s_1 + \ldots + s_p \geq 2$. In particular E_0 is always parabolic since $N \geq 2$.

Lemma 7.4. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and a parabolic exterior set $E_\beta = [|t|] \leq \eta]$ associated to X with $0 < \eta << 1$. Consider $\mu \in \mathbb{S}^1$ and $t_0 \in TE_{\mu X}^{\beta,\eta}(r, \lambda)$. Then we have $It(\mu\lambda^{d_\beta} X_{\beta,E}, (r, \lambda, t_0), \overline{B}(0, \eta)) = \mathbb{R}$ and $\lim_{z \in \mathbb{R}, |z| \to \infty} \exp(z\mu\lambda^{d_\beta} X_{\beta,E})(r\lambda, t_0)$ is the point in $E_\beta(r, \lambda) \cap SingX_{\beta,E}$.*

Proof. Consider $\eta_0 > 0$ such that $TE_{\mu X}^{\beta,\eta}(r, \lambda)$ is composed of $2\nu(\beta)$ convex points for all $0 < \eta < \eta_0$, $(r, \lambda) \in [0, \delta(\eta)) \times \mathbb{S}^1$ and $\mu \in \mathbb{S}^1$.

Fix $0 < \eta < \eta_0$, $\mu \in \mathbb{S}^1$ and $(r, \lambda) \in [0, \delta(\eta)) \times \mathbb{S}^1$. Denote $Y = (\mu\lambda^{d_\beta} X_{\beta,E})_{x=r\lambda}$. We have that $SingY$ is a point $t = \gamma_0$. Let $\tilde{Y}$ be the strict transform of $Re(Y)$ by the blow-up $\pi : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1 \to \mathbb{C}$ of $t = \gamma_0$ given by $\pi(s, \gamma) = s\gamma + \gamma_0$. We consider the set $S \subset \pi^{-1}(B(0, \eta)) \setminus Sing\tilde{Y}$ of points (s, γ) such that $It(\tilde{Y}, (s, \gamma), B(0, \eta)) = \mathbb{R}$ and $\lim_{z \to \pm\infty} \exp(z\tilde{Y})(s, \gamma) \in Sing\tilde{Y}$. By the discussion in subsection 4.2 the set S has exactly $2\nu(\beta)$ connected components. More precisely every connected component of S contains exactly an arc $\{0\} \times e^{(i\theta, i(\theta + \pi/\nu(\beta)))}$ for $e^{i\theta} \in Sing\tilde{Y}$.

Consider a connected component C of S. We have $It(\tilde{Y}, t_0, B(0, \eta + c)) = \mathbb{R}$ for all $t_0 \in \partial C$. By Poincaré-Bendixon's theorem the α and ω limits of t_0 by $Re(Y)$ are either γ_0 or a cycle enclosing γ_0 since the points in $Sing\tilde{Y}$ are either attracting or repelling. The second possibility is excluded by Cartan's lemma. We deduce that there exists $t_C \in \partial B(0, \eta) \cap \partial C$. Clearly $t_C \in \overline{C}$ implies that $t_C \in TE_{\mu X}^{\beta,\eta}(r, \lambda)$ and that $\exp(zY)(t_C)$ belongs to $\overline{B}(0, \eta)$ for all $z \in \mathbb{R}$. Moreover we obtain $\lim_{|z| \to \infty} \exp(zY)(t_C) = (x_0, \gamma_0)$. The number of connected components of S coincides with $\sharp TE_{\mu X}^{\beta,\eta}(r, \lambda)$. We deduce that $\exp(zY)(t_C) \in B(0, \eta)$ for all $z \in \mathbb{R} \setminus \{0\}$ since $\overline{C}_1 \cap \overline{C}_2 = \emptyset$ for different connected components of S. $\square$

Proposition 7.1. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and let $E_\beta = [\eta \geq |t| \geq \rho|x|]$ be a parabolic exterior set associated to X. Consider $t_0 \in TE_{\mu X}^{\beta,\eta}(r, \lambda)$ and $\mu \in \mathbb{S}^1$. Then we have*

$$\lim_{z \in \mathbb{R}, z \to c} \exp(z\mu\lambda^{d_\beta} X_{\beta,E})(r, \lambda, t_0) \in (\partial E_\beta \cup SingX_{\beta,E}) \setminus [|t| = \eta]$$

for $c \in \partial It(\mu\lambda^{d_\beta} X_{\beta,E}, (r, \lambda, t_0), E_\beta)$.

Proof. If $N(\beta) = 1$ the result is true by lemma 7.4. Suppose $N(\beta) > 1$. Consider $\eta_0 > 0$ and $\rho_0 > 0$ such that $TE_{\mu X}^{\beta,\eta}(r, \lambda)$ and $TI_{\mu X}^{\beta,\rho}(r, \lambda)$ are both composed of $2\nu(\beta)$ convex points for all $0 < \eta < \eta_0$, $\rho > \rho_0$, $(r, \lambda) \in [0, \delta(\eta, \rho)) \times \mathbb{S}^1$ and $\mu \in \mathbb{S}^1$.

Fix $0 < \eta < \eta_0$ and $\rho > \rho_0$. We can suppose that $r\lambda \neq 0$ since otherwise the proof is analogous to the proof in lemma 7.4.

Fix $(r, \lambda) \in (0, \delta(\eta, \rho)) \times \mathbb{S}^1$ and $\mu \in \mathbb{S}^1$. Consider a point $t_1 \in TI_{\mu X}^{\beta,\rho}(r, \lambda)$. There exists exactly one connected component H_s of $[|w| = \rho] \setminus TI_{\mu X}^{\beta,\rho}(r, \lambda)$ such that $t_1 \in \overline{H_s}$ and $Re(s\mu X)$ points towards $|w| < \rho$ for $s \in \{-1, 1\}$. We define $S(t_1)$ as the set of points t in $\dot{E}_\beta(r\lambda)$ such that there exists $c_{-1}(t), c_1(t) \in \mathbb{R}^+$ satisfying that $\exp((-c_{-1}, c_1)\mu X)(r\lambda, t)$ is well-defined and contained in $\dot{E}_\beta$ whereas $\exp(sc_s\mu X)(r\lambda, t) \in H_s$ for $s \in \{-1, 1\}$. Clearly $S(t_1) \neq \emptyset$ since $t_1 \in \overline{S(t_1)}$.

Like in lemma 7.4 there exists a unique $t_0 \in \overline{S(t_1)} \cap TE_{\mu X}^{\beta,\eta}(r, \lambda)$. We deduce that $It = It(\mu X, (r\lambda, t_0), E_\beta)$ is compact. Moreover we have

$$\exp(h_I \mu \lambda^{d_\beta} X_{\beta,E})(r\lambda, t_0) \in H_{-1} \text{ and } \exp(h_S \mu \lambda^{d_\beta} X_{\beta,E})(r\lambda, t_0) \in H_1$$

where $It = [h_I/r^{d_\beta}, h_S/r^{d_\beta}]$. □

Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and $\mu \in \mathbb{S}^1$. We define $SC_{\mu X}^{\beta,\eta}(r, \lambda)$ the set of connected components of

$$[\eta > |t| > \rho r] \setminus \cup_{t \in TE_{\mu X}^{\beta,\eta}(r,\lambda)} \Gamma(\mu \lambda^{d_\beta} X_{\beta,E}, (r, \lambda, t), E_\beta).$$

The behavior of the trajectories passing through tangent points characterizes the dynamics of $Re(\mu X)$ in a parabolic exterior set. It is a topological product. The next results are a consequence of this fact.

Proposition 7.2. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and let $E_\beta = [\eta \geq |t| \geq \rho |x|]$ be a parabolic exterior set associated to X. Consider $t_0 \in \dot{E}_\beta(r, \lambda)$ and $\mu \in \mathbb{S}^1$. Then we have*

$$\lim_{z \in \mathbb{R}, z \to c} \exp(z\mu \lambda^{d_\beta} X_{\beta,E})(r, \lambda, t_0) \in \partial E_\beta \cup Sing X_{\beta,E}$$

for $c \in \partial It(\mu \lambda^{d_\beta} X_{\beta,E}, (r, \lambda, t_0), E_\beta)$.

Proof. Let $C \in SC_{\mu X}^{\beta,\eta}(r, \lambda)$. Consider the set L_C of points in C satisfying the result in the proposition. It is enough to prove that $C = L_C$ for all $C \in SC_{\mu X}^{\beta,\eta}(r, \lambda)$.

The points in C in the neighborhood of points in $TE_{\mu X}^{\beta,\eta}(r, \lambda)$ are contained in L_C by proposition 7.1 and continuity of the flow. We have that C is a simply connected open set such that $C \cap Sing X_{\beta,E} = \emptyset$. Moreover every trajectory of $Re(\mu \lambda^{d_\beta} X_{\beta,E})$ contained in E_β and intersecting the set $TE_{\mu X}^{\beta,\eta}(r, \lambda) \cup TI_{\mu X}^{\beta,\rho}(r, \lambda)$ is disjoint from C. Thus the set L_C is open and closed in C and then $L_C = C$. □

The next result can be proved like proposition 7.2, it is true in the neighborhood of the tangent points by lemma 7.4 and it defines an open and closed property in connected sets. We skip the proof.

Corollary 7.1. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and let $E_\beta = [\eta \geq |t| \geq \rho |x|]$ be a parabolic exterior set associated to X. Let $(\mu_0, r, \lambda, t_0) \in \mathbb{S}^1 \times [0, \delta) \times \mathbb{S}^1 \times \partial B(0, \eta)$ such that $Re(\mu_0 \lambda^{d_\beta} X_{\beta,E})(r\lambda, t_0)$ does not point towards $\mathbb{C} \setminus \overline{B}(0, \eta)$. Then there exists $c(\mu_0, r, \lambda, t_0) \in \mathbb{R}^+ \cup \{\infty\}$ such that $\exp((0, c)\mu_0 \lambda^{d_\beta} X_{\beta,E})(r\lambda, t_0) \in \dot{E}_\beta$ and $\lim_{z \to c} \exp(z\mu_0 \lambda^{d_\beta} X_{\beta,E})(r\lambda, t_0)$ belongs to $(\partial E_\beta \cup Sing X_{\beta,E}) \setminus [|t| = \eta]$.*

Let $X = x^{d_\beta} v(x, t) \prod_{j=1}^{N(\beta)} (t - \gamma_j(x))^{s_j} \partial/\partial t \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. We define

$$X_\beta^0 = v(0, t - \gamma_1(x))(t - \gamma_1(x))^{\nu(\beta)+1} \partial/\partial t.$$

Let $\psi_{\beta,E}^{0}$ be a holomorphic integral of the time form of X_{β}^{0} in the neighborhood of $E_{\beta} \setminus SingX$. We have $\psi_{\beta,E}^{0}(x, e^{2\pi i}y) - \psi_{\beta,E}^{0}(x,y) \equiv 2\pi i Res(X_{\beta}^{0}, (0,0))$, in general $\psi_{\beta,E}^{0}$ is multivaluated. Consider a holomorphic integral $\psi_{\beta,E}$ of the time form of $X_{\beta,E}$ in the neighborhood of $E_{\beta} \setminus SingX$ such that $\psi_{\beta,E}(0,y) \equiv \psi_{\beta,E}^{0}(0,y)$. Clearly $\psi_{X,\beta}^{0} = \psi_{\beta,E}^{0}/x^{d_{\beta}}$ and $\psi_{X,\beta} = \psi_{\beta,E}/x^{d_{\beta}}$ are integrals of the time forms of $x^{d_{\beta}}X_{\beta}^{0}$ and X respectively. We want to provide accurate estimates for $\psi_{X,\beta}$.

Lemma 7.5. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and let $E_{\beta} = [\eta \geq |t| \geq \rho|x|]$ be a parabolic exterior set associated to X. Consider $\zeta > 0$ and $\theta > 0$. Then $|\psi_{X,\beta}/\psi_{X,\beta}^{0} - 1| \leq \zeta$ in $E_{\beta} \cap [t - \gamma_1(x) \in \mathbb{R}^{+}e^{i[-\theta,\theta]}] \cap [x \in B(0, \delta(\zeta,\theta))]$ for $N(\beta) = 1$. The same inequality is true for $N(\beta) \geq 2$ if $\rho > 0$ is big enough.*

Proof. Consider the change of coordinates $(x,z) = (x, t - \gamma_1(x))$. The function $\psi_{\beta,E}^{0}$ is of the form

$$\psi_{\beta,E}^{0} = \frac{-1}{\nu(\beta)v(0,0)} \frac{1}{z^{\nu(\beta)}} + Res(X_{\beta}^{0}, (0,0)) \ln z + h(z) + b(x)$$

where h is a $O(1/z^{\nu(\beta)-1})$ meromorphic function and $b(x)$ is a holomorphic function in the neighborhood of 0. In a sector of bounded angle in the variable z we have that $\psi_{\beta,E}^{0} z^{\nu(\beta)}$ is bounded both by above and by below.

We define $K(x,z) = \psi_{\beta,E}(x,z) - \psi_{\beta,E}^{0}(x,z)$. Consider the function $J = x$ if $N(\beta) = 1$ and $J = x/z$ if $N(\beta) > 1$. We have

$$v(0,z)z^{\nu(\beta)+1}\frac{\partial K}{\partial z} = \frac{v(0,z)z^{\nu(\beta)+1-s_1}}{v(x, z+\gamma_1(x)) \prod_{j=2}^{p}(z + \gamma_1(x) - \gamma_j(x))^{s_j}} - 1 = O(J).$$

Thus $\partial K/\partial z$ is a $O(J/z^{\nu(\beta)+1})$. Let $(x, re^{i\omega}) \in E_{\beta} \cap (|argz| \leq \theta)$. We obtain

$$|K(x, \eta e^{i\omega})| \leq |K(x,\eta)| + \left| \int_{\eta}^{\eta e^{i\omega}} \frac{\partial K}{\partial z} dz \right| = O(x) + O(x) = O(x) \ \forall \omega \in [-\theta, \theta].$$

Consider $\gamma : [0,1] \to \mathbb{C}^2$ defined by $\gamma(\upsilon) = (x, e^{i\omega}[(1-\upsilon)\eta + \upsilon r])$. We obtain

$$|K(x, re^{i\omega}) - K(x, \eta e^{i\omega})| \leq \left| \int_{\gamma} \frac{\partial K}{\partial z} dz \right| \leq \left| \int_{0}^{1} \frac{\partial K}{\partial z}(\gamma(\upsilon))\gamma'(\upsilon) d\upsilon \right|$$

We define $C_0 \equiv |x|$ if $N(\beta) = 1$ and $C_0 \equiv 1/\rho$ if $N(\beta) > 1$. We get

$$|K(x, re^{i\omega}) - K(x, \eta e^{i\omega})| \leq AC_0(x) \int_{0}^{1} \frac{\eta - r}{[(1-\upsilon)\eta + \upsilon r]^{\nu(\beta)+1}} d\upsilon \leq B \left| \frac{C_0(x)}{z^{\nu(\beta)}} \right|$$

for some $A, B > 0$ depending on θ. We obtain $|K(x,z)| = O(x) + O(C_0(x)/z^{\nu(\beta)})$ and then

$$\left| \frac{\psi_{X,\beta}}{\psi_{X,\beta}^{0}} - 1 \right| = \left| \frac{K}{\psi_{\beta,E}^{0}} \right| \leq D\,|C_0(x)|$$

in $E_{\beta} \cap [|\arg z| \leq \theta] \cap [x \in B(0, \delta(\zeta,\theta))]$ for some $D > 0$ depending on θ. □

Remark 7.3. *The previous lemma implies that $\psi_{X,\beta} \sim 1/(x^{d_{\beta}}(t - \gamma_1(x))^{\nu(\beta)})$ in a parabolic exterior set E_{β} for $|\arg(t - \gamma_1(x))|$ bounded.*

Proposition 7.3. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and let $E_\beta = [\eta \geq |t| \geq \rho|x|]$ be a parabolic exterior set associated to X. Consider $C \in SC_{\mu X}^{\beta;\eta}(r,\lambda)$ for $r\lambda$ in a neighborhood of 0 and $\mu \in \mathbb{S}^1$. Then C is contained in a sector centered at $t = \gamma_1(r\lambda)$ of angle lesser than θ for some $\theta > 0$ independent of r, λ, C and μ.*

Proof. We use the notations in lemma 7.5. We have that the extrema of a connected component of $\partial B(0,\eta) \setminus TE_{\mu X}^{\beta,\eta}(r,\lambda)$ lie in an angle centered at $z = 0$ of angle similar to $\pi/\nu(\beta)$. Then it is enough to prove that $\Gamma = \Gamma(\mu\lambda^{d_\beta} X_{\beta,E}, (r,\lambda,t_0), E_\beta)$ lies in a sector of bounded angle for $t_0 \in TE_{\mu X}^{\beta,\eta}(r,\lambda)$.

Denote $\psi^0 = -1/(\nu(\beta)v(0,0)z^{\nu(\beta)})$. We have $\lim_{z\to 0} \psi_{\beta,E}/\psi^0 = 1$ in big sectors; we can suppose that $|\psi_{\beta,E}/\psi^0 - 1| < \zeta$ for arbitrary $\zeta > 0$ by taking $0 < \eta << 1$. Since the set $(\psi_{\beta,E}/\mu\lambda^{d_\beta})(\Gamma)$ is contained in $(\psi_{\beta,E}/\mu\lambda^{d_\beta})(r,\lambda,t_0) + \mathbb{R}$ then it lies in a sector of angle similar to π. Since $\psi_{\beta,E}/\psi^0 \sim 1$ then Γ lies in a sector of center $t = \gamma_1(r,\lambda)$ and angle close to $\pi/\nu(\beta)$. □

Remark 7.4. *We have that $\psi_{X,\beta} \sim 1/(x^{d_\beta}(t-\gamma_1(x))^{\nu(\beta)})$ in $E_\beta \cap \overline{C}$ for a parabolic exterior set E_β and all $C \in SC_{\mu X}^{\beta,\eta}$*

7.3. **Nature of the polynomial vector fields.** The study of polynomial vector fields related to stability properties of unfoldings of elements $h \in \mathrm{Diff}_1(\mathbb{C},0)$ has been introduced in [6]. Their choices are associated with the elements in the deformation whereas ours depend on the infinitesimal properties of the unfolding.

7.3.1. *Directions of unstability.* Let M_β be a magnifying glass set associated to a vector field $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. We consider

$$X_\beta(\lambda) = \lambda^{m_\beta} C(w-w_1)^{s_1} \dots (w-w_p)^{s_p} \partial/\partial w$$

where $C \in \mathbb{C}^*$ and $w_j \in \mathbb{C}$ for all $j \in \{1,\dots,p\}$. Denote $r_\beta^j(X) = Res(X_\beta(1), w_j)$ for $1 \leq j \leq p$. Consider the set $sum_\beta(X)$ whose elements are the non-vanishing sums of the form $\sum_{j\in E} r_\beta^j$ for any $E \subset \{1,\dots,p\}$. We define

$$B_\beta(X) = \{(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : sum_\beta \cap \lambda^{m_\beta}\mu i\mathbb{R} \neq \emptyset\}.$$

We denote $\mathbb{S}^1/\sim$ the quotient of $\mathbb{S}^1$ by the equivalence relation identifying μ and $-\mu$. We denote by $\tilde{B}_\beta(X) \subset \mathbb{S}^1/\sim \times \mathbb{S}^1/\sim$ the quotient of $B_\beta(X)$. Now we define

$$B_{\beta,\lambda}(X) = \{\mu \in \mathbb{S}^1 : (\lambda,\mu) \in B_\beta(X)\} \ \text{ and } \ B_\beta^\mu(X) = \{\lambda \in \mathbb{S}^1 : (\lambda,\mu) \in B_\beta(X)\}.$$

In an analogous way we can define $\tilde{B}_{\beta,\lambda}(X) \subset \mathbb{S}^1/\sim$ and $\tilde{B}_\beta^\mu(X) \subset \mathbb{S}^1/\sim$ for $\lambda,\mu \in \mathbb{S}^1/\sim$. Roughly speaking we claim that $Re(\mu X)$ has a stable behavior in I_β at the direction $x \in \mathbb{R}^+\lambda$ for $(\lambda,\mu) \notin B_\beta(X)$. We define B_X as the union of $B_\beta(X)$ for every magnifying glass set M_β associated to X. Analogously we can define $B_{X,\lambda}$, B_X^μ, $\tilde{B}_{X,\lambda}$ and $\tilde{B}_X^\mu$. The sets $B_{X,\lambda}$ and B_X^μ are finite for all $\lambda,\mu \in \mathbb{S}^1$. Moreover we have $B_{X,\lambda'} \cap B_{X,\lambda} = \emptyset$ and $B_X^{\lambda'} \cap B_X^\lambda = \emptyset$ for all $\lambda' \in \mathbb{S}^1$ in a pointed neighborhood of λ.

7.3.2. *Non-parabolic exterior sets.* Let $E_{\beta w_1}$ be a non-parabolic exterior set where $w_1 \in \mathbb{C}$. Thus we have

$$X = x^{m_\beta} h(x,w)(w-w_1(x))(w-w_2(x))^{s_2} \dots (w-w_p(x))^{s_p} \partial/\partial w$$

in M_β where $w_1(0) = w_1$ and $h(x,w) - h(0,0) \in (x)$. This expression implies

$$X = x^{m_\beta} \frac{1}{r_\beta^1}(w - w_1(x))(1 + H(x,w))\frac{\partial}{\partial w}$$

in $E_{\beta w_1}$ for some $H \in (x, w - w_1) = (x, w - w_1(x))$.

Fix $\mu \in \mathbb{S}^1$ and a compact set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. By definition of B_X^μ we obtain that $\lambda^{m_\beta}\mu/r_\beta^1 \notin i\mathbb{R}$ for all $\lambda \in K_X^\mu$. This implies $\lambda^{m_\beta}\mu(r_\beta^1)^{-1}(1 + H(r\lambda, w_1(r\lambda))) \notin i\mathbb{R}$ for $(r,\lambda) \in [0, r_0) \times K_X^\mu$ for some $r_0 > 0$ since K_X^μ is compact and $H(x, w_1(x)) \in (x)$. We deduce that the singular point $w = w_1(x_0)$ of $Re(\mu X)_{|x=x_0}$ is not a center for $x_0 \in (0, r_0)K_X^\mu$. Hence, it is either an attracting or a repulsing point.

The set $E_{\beta w_1}$ is of the form $|w - w_1| < c$ for some $0 < c << 1$. The vector field $Re(\mu X)_{|x=r\lambda}$ and the set $\partial E_{\beta w_1}$ are tangent at the set

$$TE_{\mu X}^{\beta w_1, c}(r,\lambda) = \left[\frac{\lambda^{m_\beta}\mu}{r_\beta^1}\frac{w - w_1(r\lambda)}{w - w_1}(1 + H(r\lambda, w)) \in i\mathbb{R}\right] \cap [|w - w_1(0)| = c].$$

The function $(w - w_1(r\lambda))/(w - w_1)$ tends to 1 when $r \to 0$ in $|w - w_1| = c$. Moreover since $H \in (x, w - w_1)$ we obtain that $TE_{\mu X}^{\beta w_1, c}(r,\lambda) = \emptyset$ for $r \in [0, r_0(c))$ and $\lambda \in K_X^\mu$. Then $Re(s\mu X)$ points towards $\dot{E}_{\beta w_1}$ for all $x \in (0, r_0)K_X^\mu$ and either $s = -1$ or $s = 1$. As a consequence $E_{\beta w_1} \cap [x = x_0]$ is in the basin of attraction of $(x_0, w_1(x_0))$ by $Re(s\mu X)$ for $x_0 \in (0, r_0)K_X^\mu$.

7.3.3. *Connexions at ∞.* We already described the dynamics of $Re(\mu X)$ in the exterior sets for $\mu \in \mathbb{S}^1$ and $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Next we analyze the dynamics of $Re(\mu X)$ in the intermediate sets.

Let $Y = C(w - w_1)^{s_1} \dots (w - w_p)^{s_p}\partial/\partial w$ be a polynomial vector field such that $\nu(Y) = s_1 + \ldots + s_p - 1 \geq 1$. Every vector field $X_\beta(\lambda)$ associated to a magnifying glass set is of this form. We want to characterize the behavior of Y in the neighborhood of ∞. We define the set $Tr_{\to\infty}(Y)$ of trajectories $\gamma : (c,d) \to \mathbb{C}$ of $Re(Y)$ such that $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R}$ and $\lim_{\zeta \to d}\gamma(\zeta) = \infty$. In an analogous way we define $Tr_{\leftarrow\infty}(Y) = Tr_{\to\infty}(-Y)$. We define $Tr_\infty(Y) = Tr_{\leftarrow\infty}(Y) \cup Tr_{\to\infty}(Y)$.

We consider a change of coordinates $z = 1/w$. The meromorphic vector field

$$Y = \frac{-C(1 - w_1 z)^{s_1} \dots (1 - w_p z)^{s_p}}{z^{\nu(Y)-1}}\frac{\partial}{\partial z}$$

is analytically conjugated to $1/(\nu(Y)z^{\nu(Y)-1})\partial/\partial z = (z^{\nu(Y)})^*(\partial/\partial z)$ in a neighborhood of ∞. We have $Tr_{\to\infty}(\partial/\partial z) = \mathbb{R}^-$ and $Tr_{\leftarrow\infty}(\partial/\partial z) = \mathbb{R}^+$. As a consequence the set $Tr_{\to\infty}(Y)$ has $\nu(Y)$ trajectories and there is exactly one of them which is tangent to the line $\arg(w) = -\arg(C)/\nu(Y) + 2\pi k/\nu(Y)$ for all $k \in \{0, \ldots, \nu(Y) - 1\}$. Analogously $Tr_{\leftarrow\infty}(Y)$ contains $\nu(Y)$ trajectories of $Re(Y)$ which are tangent to the lines $\arg(w) = -\arg(C)/\nu(Y) + \pi/\nu(Y) + 2\pi k/\nu(Y)$ for $k \in \{0, \ldots, \nu(Y) - 1\}$.

The complementary of the set $Tr_\infty(Y) \cup \{\infty\}$ has $2\nu(Y)$ connected components in the neighborhood of $w = \infty$. Each of these components is called an *angle*, the boundary of an angle contains exactly one $\to\infty$-trajectory and one $\leftarrow\infty$-trajectory.

We say that $Re(Y)$ has ∞-connections if there exists $P \in \mathbb{C}$ contained in $Tr_{\to\infty}(Y) \cap Tr_{\leftarrow\infty}(Y)$. In other words there exists a trajectory $\gamma : (c_{-1}, c_1) \to \mathbb{C}$ of $Re(Y)$ such that $c_{-1}, c_1 \in \mathbb{R}$ and $\lim_{\zeta \to c_s}\gamma(\zeta) = \infty$ for all $s \in \{-1, 1\}$. The notion of connexion at ∞ has been introduced in [6] for the study of deformations of elements of $\text{Diff}_1(\mathbb{C}, 0)$.

We define the α and ω limits $\alpha^Y(P)$ and $\omega^Y(P)$ respectively of a point $P \in \mathbb{C}$ by the vector field $Re(Y)$. If $P \in Tr_{\to\infty}(Y)$ we denote $\omega^Y(P) = \{\infty\}$ whereas if $P \in Tr_{\leftarrow\infty}(Y)$ we denote $\alpha^Y(P) = \{\infty\}$.

Lemma 7.6. *Let $Y \in \mathcal{X}(\mathbb{C}, 0)$ be a polynomial vector field such that $\nu(Y) \geq 1$. Then $\omega^Y(w_0) = \{\infty\}$ is equivalent to $w_0 \in Tr_{\to\infty}(Y)$. Analogously $\alpha^Y(w_0) = \{\infty\}$ is equivalent to $w_0 \in Tr_{\leftarrow\infty}(Y)$*

Proof. The vector field Y is a ramification of a regular vector field in a neighborhood of ∞. Thus there exists an open neighborhood V of ∞ and $c \in \mathbb{R}^+$ such that

$$\exp(cY)(V \setminus Tr_{\to\infty}(Y)) \cap V = \emptyset \ \text{ and } \ \exp(-cY)(V \setminus Tr_{\leftarrow\infty}(Y)) \cap V = \emptyset.$$

We are done since $w_0 \notin Tr_{\to\infty}(Y)$ implies $\omega^Y(w_0) \cap (\mathbb{P}^1(\mathbb{C}) \setminus V) \neq \emptyset$. □

We denote by $\mathcal{X}_\infty(\mathbb{C}, 0)$ the set of polynomial vector fields in $\mathcal{X}(\mathbb{C}, 0)$ such that $\nu(Y) \geq 1$ and $2\pi i \sum_{P \in S} Res(Y, P) \notin \mathbb{R} \setminus \{0\}$ for all subset S of $SingY$.

Lemma 7.7. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Then*

- *$Re(Y)$ has no ∞-connections.*
- *$\omega^Y(w_0) \neq \{\infty\}$ implies that $\sharp\omega^Y(w_0) = 1$ and $\omega^Y(w_0) \cap SingY \neq \emptyset$.*

Proof. Let Ω the unique meromorphic 1-form defined by $\Omega(Y) = 1$. Suppose that $\gamma : (c_{-1}, c_1) \to \mathbb{C}$ is an ∞-connexion of $Re(Y)$. Consider the connected component U of $\mathbb{P}^1(\mathbb{C}) \setminus (\gamma(c_{-1}, c_1) \cup \{\infty\})$ such that $Re(iY)$ points towards U at γ.

There exists a holomorphic integral ψ of the time form of Y in a neighborhood of $w = \infty$ such that $\psi \sim 1/w^{\nu(Y)}$. Let $h_n : [d_n, e_n] \to \mathbb{C}$ be a path (but not a trajectory of $Re(Y)$) such that $h_n(d_n) = \gamma(c_{-1} + 1/n)$ and $h_n(e_n) = \gamma(c_1 - 1/n)$ whereas $h_n(d_n, e_n) \subset U$. Moreover we can suppose that $\inf_{\zeta \in [d_n, e_n]} |h_n(\zeta)|$ tends to ∞ when $n \to \infty$. The theorem of the residues and the asymptotics of ψ in the neighborhood of ∞ imply that

$$2\pi i \sum_{P \in SingY \cap U} Res(Y, P) = \lim_{n \to \infty} \int_{h_n} \Omega = \int_\gamma \Omega = c_1 - c_{-1} \in \mathbb{R}^+.$$

This is a contradiction.

It is enough to prove that $\omega^Y(w_0) \cap (\mathbb{C} \setminus SingY) = \emptyset$ since $\omega^Y(w_0)$ is connected. Suppose $P \in \omega^Y(w_0) \cap (\mathbb{C} \setminus SingY)$. Denote $\gamma : [0, \infty) \to \mathbb{C}$ the trajectory of $Re(Y)$ passing through w_0. Consider a germ of transversal h to the vector field $Re(Y)$ passing through P. There exists some $\eta > 0$ such that $\exp((0, \eta]Y)(h) \cap h = \emptyset$. There also exists an increasing sequence of positive real numbers $j_n \to \infty$ such that $\gamma(j_n) \in h$ and $\lim_{n \to \infty} \gamma(j_n) = P$. We can suppose that $\gamma(j_n, j_{n+1}) \cap h = \emptyset$ for all $n \in \mathbb{N}$ by twisting a little bit the sequence.

Consider a holomorphicintegral ψ of the time form of Y defined in the neighborhood of P. Let L_n be the segment of h whose boundary is $\{\gamma(j_n), \gamma(j_{n+1})\}$. Denote by V_n the bounded component of $\mathbb{C} \setminus (\gamma[j_n, j_{n+1}] \cup L_n)$. By the theorem of the residues we obtain

$$\int_{\gamma[j_n, j_{n+1}]} \Omega + (\psi(\gamma(j_n)) - \psi(\gamma(j_{n+1}))) = \pm 2\pi i \sum_{P \in V_n \cap SingY} Res(Y, P)$$

By making n to tend to ∞ we deduce that there exists a subset S of $SingY$ such that $\pm 2\pi i \sum_{P \in S} Res(Y, P) \in [\eta, \infty)$. That is a contradiction. □

Corollary 7.2. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. Consider a magnifying glass set M_β associated to X. Then*

- *$Re(\mu X_\beta(\lambda))$ has no ∞-connections.*
- *$\omega^{\mu X_\beta(\lambda)}(w_0) \neq \infty \Rightarrow \sharp\omega^{\mu X_\beta(\lambda)}(w_0) = 1$ and $\omega^{\mu X_\beta(\lambda)}(w_0) \cap SingX_\beta(\lambda) \neq \emptyset$.*

for all $(\lambda,\mu) \notin B_\beta(X)$.

7.3.4. *The graph.* In this subsection we associate an oriented graph to every vector field $\mu X_\beta(\lambda)$ for $(\lambda,\mu) \notin B_\beta(X)$.

Lemma 7.8. *Let $Y \in \mathcal{X}_\infty(\mathbb{C},0)$. Then the mapping*

$$\omega^Y : \mathbb{C} \setminus (Tr_{\to\infty}(Y) \cup SingY) \to SingY$$

is locally constant.

Proof. Let $P \in \mathbb{C} \setminus (Tr_{\to\infty}(Y) \cup SingY)$. Denote $Q = \omega^Y(P)$. The singular point Q is not a center since then $Re(Y)$ would support cycles (lemma 7.7). If Q is an attracting singular point there is nothing to prove. If Q is parabolic then $P \in \cup_{\lambda \in D_1(Y)} V^\lambda_{\exp(Y)}$. We are done since $\cup_{\lambda \in D_1(Y)} V^\lambda_{\exp(Y)}$ is open and $\omega^Y(\cup_{\lambda \in D_1(Y)} V^\lambda_{\exp(Y)}) = Q$. □

We call *regions* of $Re(Y)$ the connected components of $\mathbb{C} \setminus (Tr_\infty(Y) \cup SingY)$. We denote by $Reg(Y)$ the set of regions of $Re(Y)$. Every $H \in Reg(Y)$ satisfies that $\alpha^Y(H)$ and $\omega^Y(H)$ are points. We denote by $Reg_j(Y)$ the set of regions H of $Re(Y)$ such that $\sharp\{\alpha^Y(H), \omega^Y(H)\} = j$ for $j \in \{1,2\}$. We associate an oriented graph to $Re(Y)$ for $Y \in \mathcal{X}_\infty(\mathbb{C},0)$. The vertexes are the points in $SingY$, the edges are the regions of $Re(Y)$. We say that $H \in Reg(Y)$ joins the points $\alpha^Y(H)$ and $\omega^Y(H)$. We denote $\alpha^Y(H) \xrightarrow{H} \omega^Y(H)$. The graph obtained in this way is denoted by $\mathcal{G}_Y$. We denote by $\mathcal{N}G_Y$ the unoriented graph obtained from $\mathcal{G}_Y$ by removing the reflexive edges and the orientations of the edges.

An angle is always contained in a region of $Re(Y)$. Such a region is characterized by the angles that it contains. Let A be an angle of the polynomial vector field Y. We denote by $\gamma^A_{\to\infty}$ the trajectory of $Tr_{\to\infty}$ contained in the closure of A. The definition of $\gamma^A_{\leftarrow\infty}$ is analogous.

Lemma 7.9. *Let $Y \in \mathcal{X}_\infty(\mathbb{C},0)$. Consider $H \in Reg(Y)$. Then H contains an angle A. Moreover $\alpha^Y(\gamma^A_{\to\infty}) = \alpha^Y(H)$ and $\omega^Y(\gamma^A_{\leftarrow\infty}) = \omega^Y(H)$.*

Proof. Let $P \in (\mathbb{C} \setminus SingY) \cap \partial H$; such a point exists since $Tr_\infty(Y)$ is contained in the complementary of H. Since α^Y and ω^Y are locally constant then either $\alpha^Y(P) = \infty$ or $\omega^Y(P) = \infty$. We have that $P \in \overline{H}$, thus there are points of H in every neighborhood of ∞. As a consequence H contains at least an angle A. The relations $\alpha^Y(\gamma^A_{\to\infty}) = \alpha^Y(H)$ and $\omega^Y(\gamma^A_{\leftarrow\infty}) = \omega^Y(H)$ can be deduced of the locally constant character of α^Y and ω^Y. □

Lemma 7.10. *Let $Y \in \mathcal{X}_\infty(\mathbb{C},0)$. Then we have $SingY \subset \overline{Tr_\infty(Y)}$.*

Proof. Let $P \in SingY$. Suppose that $V \cap Tr_\infty(Y) = \emptyset$ for some connected neighborhood V of P. Let H be the region of $Re(Y)$ containing $V \setminus \{P\}$. Since P is attracting, repulsing or parabolic then either $\alpha^Y(H) = P$ or $\omega^Y(H) = P$. Consider an angle $A \subset H$. We obtain $P \in \overline{\gamma^A_{\leftarrow\infty} \cup \gamma^A_{\to\infty}} \subset \overline{Tr_\infty(Y)}$. □

Lemma 7.11. *Let $Y \in \mathcal{X}_\infty(\mathbb{C},0)$. Consider $H \in Reg_1(Y)$. Then H contains exactly one angle.*

Proof. Let A be an angle contained in H. Denote $P = \alpha^Y(H) = \omega^Y(H)$. By lemma 7.9 we have that $\gamma = \{\infty\} \cup \gamma^A_{\to\infty} \cup \gamma^A_{\leftarrow\infty} \cup \{P\}$ is a closed simple curve. Let V the connected component of $\mathbb{P}^1(\mathbb{C}) \setminus \gamma$ containing A. The set $Tr_\infty(Y) \cap V$ is empty since A is the only angle contained in V. By lemma 7.10 we have that $V \cap SingY = \emptyset$. Hence H is equal to V and contains only one angle. □

Lemma 7.12. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Consider $H \in Reg_2(Y)$. Then H contains exactly two angles. Moreover $\mathbb{C} \setminus H$ has two connected components H_1 and H_2 such that $\alpha^Y(H) \in H_1$ and $\omega^Y(H) \in H_2$.*

Proof. Let A_1 be an angle contained in H. Fix a trajectory γ_0 of $Re(Y)$ contained in H. Denote
$$\gamma_1 = \gamma_0 \cup \gamma^{A_1}_{\to\infty} \cup \gamma^{A_1}_{\leftarrow\infty} \cup \{\alpha^Y(H), \omega^Y(H)\}.$$
Let V_1 the connected component of $\mathbb{C} \setminus \gamma_1$ containing A_1. Since V_1 contains only one angle then $V_1 \subset H$. By proceeding like in lemma 7.9 we can prove that there exists an angle A_2 contained in $H \setminus (V_1 \cup \gamma_0)$. Let V_2 be the connected component of $\mathbb{C} \setminus (\gamma_0 \cup \gamma^{A_2}_{\to\infty} \cup \gamma^{A_2}_{\leftarrow\infty} \cup \{\alpha^Y(H), \omega^Y(H)\})$ such that $A_2 \subset V_2$. Clearly we have $A_2 \neq A_1$ and $H = V_1 \cup \gamma_0 \cup V_2$. Now
$$\mathbb{C} \setminus (\gamma^{A_1}_{\to\infty} \cup \gamma^{A_1}_{\leftarrow\infty} \cup \gamma^{A_2}_{\to\infty} \cup \gamma^{A_2}_{\leftarrow\infty} \cup \{\alpha^Y(H), \omega^Y(H)\})$$
has three connected components H, J_1 and J_2 such that
$$\partial J_1 = \gamma^{A_1}_{\to\infty} \cup \gamma^{A_2}_{\to\infty} \cup \{\alpha^Y(H)\} \quad \text{and} \quad \partial J_2 = \gamma^{A_1}_{\leftarrow\infty} \cup \gamma^{A_2}_{\leftarrow\infty} \cup \{\omega^Y(H)\}.$$
Then $H_1 = J_1 \cup \partial J_1$ and $H_2 = J_2 \cup \partial J_2$ are the connected components of $\mathbb{C} \setminus H$. □

Corollary 7.3. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Then $\mathcal{N}G_Y$ has no cycles.*

Proof. Consider an edge $P \overset{H}{\to} Q$ of $\mathcal{G}_Y$ with $P \neq Q$. Consider the notations in the previous lemma. The fixed points are divided in two sets $H_1 \cap SingY$ and $H_2 \cap SingY$. The only edge of G_Y joining a vertex in the former set with a vertex in the latter set (or vice-versa) is $P \overset{H}{\to} Q$. Clearly $\mathcal{N}G_Y$ has no cycles. □

Proposition 7.4. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Then the graph $\mathcal{N}G_Y$ is connected.*

Proof. Let $G_1, \dots, G_l$ be the set of vertexes of the l connected components of $\mathcal{N}G_Y$. We define the open set $V_j = (\alpha^Y)^{-1}(G_j) \cup (\omega^Y)^{-1}(G_j)$ for all $j \in \{1, \dots, l\}$. The lack of ∞-connexions implies $\cup_{j=1}^l V_j = \mathbb{C}$. Moreover $V_j \cap V_k = \emptyset$ if $j \neq k$ since otherwise $G_j = G_k$. Clearly $l = 1$ since $\mathbb{C}$ is connected. □

Corollary 7.4. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Then $\sharp Reg_2(Y) = \sharp SingY - 1$.*

Let $Y \in \mathcal{X}(\mathbb{C}, 0)$. Consider $y_0 \in SingY$. We define $\nu_Y(y_0)$ as the only element of $\mathbb{N} \cup \{0\}$ such that $Y(y) \in (y - y_0)^{\nu_Y(y_0)+1} \setminus (y - y_0)^{\nu_Y(y_0)+2}$.

Proposition 7.5. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Consider $y_0 \in SingY$. Then there exist exactly $2\nu_Y(y_0)$ regions of $Re(Y)$ contained in $(\alpha^Y, \omega^Y)^{-1}(y_0, y_0)$.*

Proof. If y_0 is attracting or repulsing the result is obvious since on the one hand $\nu_Y(y_0) = 0$ and on the other hand $(\alpha^Y, \omega^Y)^{-1}(y_0, y_0) = \{y_0\}$. We can suppose that y_0 is a parabolic point.

Let Y_0 be the germ of Y in the neighborhood of y_0, we have $\nu(Y_0) = \nu_Y(y_0)$. Consider the strict transform $\tilde{Y}$ of $Re(Y)$ by the real blow-up $\pi(r, \lambda) = y_0 + r\lambda$. By the discussion in section 4.2 there exists a unique region of $Re(Y)$ adhering to

$[(r,\lambda) \in \{0\} \times [\lambda_0, \lambda_0 e^{i\pi/\nu(Y_0)}]]$ for all $\lambda_0 \in D(Y_0)$. In this way we find $2\nu_Y(y_0)$ regions of $Re(Y)$ contained in $(\alpha^Y, \omega^Y)^{-1}(y_0, y_0)$. Any other region would adhere to a single point in $D(Y_0)$. Such a point would be both attracting and repelling for $\tilde{Y}$; that is impossible. □

Corollary 7.5. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Then $\sharp Reg(Y) = 2\nu(Y) - \sharp(SingY) + 1$.*

Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Consider a trajectory γ_H for every region $H \in Reg_2(Y)$. There exists $\rho_0 > 0$ such that

$$\left\{ \begin{array}{l} SingY \subset B(0,\rho_0) \text{ and } \sharp T_Y^\rho = 2\nu(Y) \text{ for all } \rho \geq \rho_0. \\ \gamma_H \subset B(0,\rho_0) \text{ for all } H \in Reg_2(Y). \end{array} \right. \tag{7}$$

Let $P \in \overline{B}(0,\rho)$. We define $\omega_\rho^Y(P) = \infty$ if $It(Y, P, \overline{B}(0,\rho))$ does not contain $(0,\infty)$. Otherwise we define $\omega_\rho^Y(P) = \omega^Y(P)$. We define α_ρ^Y in an analogous way. Denote by $Reg(Y,\rho)$ the set of connected components of

$$B(0,\rho) \setminus ((\alpha_\rho^Y)^{-1}(\infty) \cup (\omega_\rho^Y)^{-1}(\infty) \cup SingY).$$

Denote

$$Reg_j(Y,\rho) = \{H \in Reg(Y,\rho) : \sharp\{\alpha_\rho^Y(H), \omega_\rho^Y(H)\} = j\}$$

for $j \in \{1,2\}$. The set of connected components of $B(0,\rho) \setminus (SingY \cup \cup_{H \in Reg(Y,\rho)} \overline{H})$ will be called $Reg_\infty(Y,\rho)$. The dynamics of $Re(Y)$ in $\mathbb{C}$ and $B(0,\rho_0)$ is analogous.

Proposition 7.6. *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Consider $\rho >> 0$. There exist bijections $F : Reg(Y,\rho) \to Reg(Y)$ and $G : Reg_\infty(Y,\rho) \to Tr_\infty(Y)$ such that*

- *$H \subset F(H)$ for all $H \in Reg(Y,\rho)$*
- *$\sharp(\partial H \cap T_Y^\rho) = j$ for all $H \in Reg_j(Y,\rho)$ and $j \in \{1,2\}$.*
- *$\sharp(\partial J \cap T_Y^\rho) = 1$ for each connected component J of $H \setminus \gamma_H$ and $H \in Reg_2(Y,\rho)$.*
- *$G(K) \cap B(0,\rho) \subset K$ for all $K \in Reg_\infty(Y,\rho)$.*

Proof. We define $F_1(H)$ as the element of $Reg(Y,\rho)$ containing γ_H for $H \in Reg(Y)$. Every $H \in Reg(Y,\rho)$ is contained in a unique $F(H) \in Reg(Y)$. It is clear that $F \circ F_1 \equiv Id$. This implies $\sharp(Reg_j(Y,\rho)) \geq \sharp(Reg_j(Y))$ for $j \in \{1,2\}$.

Let $H \in Reg(Y,\rho)$. We have $\partial H \cap \partial B(0,\rho) = \partial H \cap T_Y^\rho$. Thus we obtain $\sharp(\partial H \cap T_Y^\rho) \geq 1$. Let $H \in Reg_2(Y,\rho)$. Every connected component of $F(H) \setminus \gamma_H$ contains at least a point in $\partial H \cap T_Y^\rho$ and then $\sharp(\partial H \cap T_Y^\rho) \geq 2$. We have

$$2\nu(Y) = \sharp T_Y^\rho \geq \sharp Reg_1(Y,\rho) + 2\sharp Reg_2(Y,\rho) \geq \sharp Reg_1(Y) + 2\sharp Reg_2(Y) = 2\nu(Y).$$

Hence all the inequalities are indeed equalities. We obtain $\sharp Reg_j(Y,\rho) = \sharp Reg_j(Y)$ and $\sharp(\partial H \cap T_Y^\rho) = j$ for all $j \in \{1,2\}$ and $H \in Reg_j(Y,\rho)$. We deduce that $F_1 = F^{\circ(-1)}$ and that $\{\alpha_\rho^Y(Q), \omega_\rho^Y(Q)\} \subset SingY$ for all $Q \in T_Y^\rho$.

Let l be a connected component of $\partial B(0,\rho) \setminus T_Y^\rho$ such that $Re(sY)$ points towards $B(0,\rho)$ for some $s \in \{-1,1\}$. We claim that $\exp(s(0,\infty)Y)(l)$ is a connected component of $Reg_\infty(Y,\rho)$. Suppose $s = 1$ without lack of generality. Since $\omega_\rho^Y(\partial l) \subset SingY$ and $\mathcal{N}G_Y$ is connected then $\omega_\rho^Y(l) = \omega_\rho^Y(\partial l)$ is a singleton contained in $SingY$. The claim is proved, it implies $\sharp Reg_\infty(Y,\rho) = 2\nu(Y)$. There exists a unique $\gamma(l) \in Tr_\infty(Y)$ such that $\gamma(l) \cap l \neq \emptyset$. The mapping $G(K) = \gamma(\partial K \cap (\partial B(0,\rho) \setminus T_Y^\rho))$ is the one we are looking for. □

7.3.5. *Dynamical description in the intermediate sets.* Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a magnifying glass set $M_\beta = [|w| \leq \rho]$ associated to X. We have

$$X = x^{m_\beta} h(x,w)(w - w_1(x))^{s_1} \dots (w - w_p(x))^{s_p} \partial/\partial w$$

where $h(0,w) \equiv h(0,0)$. Fix $\mu \in \mathbb{S}^1$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_\beta^\mu(X)$. A value $\rho_0 > 0$ satisfying the conditions 7 for $\mu X_\beta(\lambda_0)$ and some $\lambda_0 \in K_X^\mu$ it also satisfies 7 for $\mu X_\beta(\lambda)$ and all $\lambda \in \mathbb{S}^1$ in a neighborhood of λ_0. Since K_X^μ is compact we choose $\rho_0 > 0$ satisfying the conditions 7 for $\mu X_\beta(\lambda)$ and all $\lambda \in K_X^\mu$. Consider the intermediate set $I_\beta = (|w| \leq \rho) \setminus \cup_{\zeta \in S_\beta} (|w - \zeta| < r(\zeta))$ where $0 < r(\zeta) << 1$ for all $\zeta \in S_\beta$ and $\rho \geq \rho_0$. Given $\lambda \in K_X^\mu$ and $P \in T_{\mu X_\beta(\lambda)}^\rho$ the interval $It = It(\mu X_\beta(\lambda), P, I_\beta)$ is compact. The set $\exp(\partial It \mu X_\beta(\lambda))(P)$ is contained in $\cup_{\zeta \in S_\beta} (|w - \zeta| = r(\zeta))$. Since the tangent points are convex then $\exp(\partial It \mu X_\beta(\lambda))(P)$ does not contain points of $\cup_{\zeta \in S_\beta} TE_{\mu X}^{\beta\zeta, r(\zeta)}(0,\lambda)$. The interval $It(Q) = It(\mu \lambda^{m_\beta} X_{\beta,M}, Q, I_\beta)$ is compact and depends on $Q \in TI_{\mu X}^{\beta;\rho} \cap [\lambda \in K_X^\mu]$ continuously. We obtain $\exp(\partial It(Q) \mu \lambda^{m_\beta} X_{\beta,M})(Q) \subset \cup_{\zeta \in S_\beta} (|w - \zeta| = r(\zeta))$.

The dynamics of $Re(\mu X)$ restricted to I_β is a topological product in $x \in (0, \delta_0) K_X^\mu$ (indeed this is an abuse of notation, it could be necessary to consider a smaller $\delta_0 > 0$). We denote by $Reg^*(\mu X, \beta, K_X^\mu)$ the set of connected components of

$$(\dot{I}_\beta \cap [(r,\lambda) \in [0,\delta_0) \times K_X^\mu]) \setminus \cup_{Q \in TI_{\mu X}^{\beta;\rho}} \exp\left(It(Q) \mu \lambda^{m_\beta} X_{\beta,M}\right)(Q).$$

An element $H \in Reg^*(\mu X, \beta, K_X^\mu)$ is open in $(r,\lambda) \in [0,\delta_0) \times K_X^\mu$. Fix $\lambda_1 \in K_X^\mu$; by definition $H \in Reg^*(\mu X, \beta, K_X^\mu)$ belongs to $Reg_j(\mu X, \beta, K_X^\mu)$ for $j \in \{1, 2, \infty\}$ if there exists $J \in Reg_j(\mu X_\beta(\lambda_1), \rho)$ such that $H(0, \lambda_1) \subset J$. Since H depends continuously on $(r,\lambda) \in [0,\delta_0) \times K_X^\mu$ the definition does not depend on the choice of λ_1. We define

$$Reg(\mu X, \beta, K_X^\mu) = Reg_1(\mu X, \beta, K_X^\mu) \cup Reg_2(\mu X, \beta, K_X^\mu).$$

Let $Y = \mu X_\beta(\lambda_1)$. We define $\omega_\beta^{\mu X}(H) = \omega_\rho^Y(H(0,\lambda_1))$ for $H \in Reg^*(\mu X, \beta, K_X^\mu)$. The definition of $\alpha_\beta^{\mu X}(H)$ is analogous. A *fundamental arc* is a connected component of

$$\partial H \cap ((|w| = \rho) \cup_{\zeta \in S_\beta} (|w - \zeta| = r(\zeta)))$$

for some $H \in Reg^*(\mu X, \beta, K_X^\mu)$. We denote by $arc(\mu X, \beta, K_X^\mu)$ the set of fundamental arcs. Moreover since given $ac \in arc(\mu X, \beta, K_X^\mu)$ there exists a unique $H_{ac} \in Reg^*(\mu X, \beta, K_X^\mu)$ such that $ac \subset \overline{H_{ac}}$ then we define

$$arc_j(\mu X, \beta, K_X^\mu) = \{ac \in arc(\mu X, \beta, K_X^\mu) : H_{ac} \in Reg_j(\mu X, \beta, K_X^\mu)\}$$

for $j \in \{1, 2, \infty\}$.

We sketch next how to build for all $(r_0, \lambda_0) \in [0,\delta_0) \times K_X^\mu$ a homeomorphism

$$F_{(r_0,\lambda_0)} : I_\beta(r_0,\lambda_0) \to I_\beta(0,\lambda_1)$$

such that F is continuous in $I_\beta \cap [(r,\lambda) \in [0,\delta_0) \times K_X^\mu]$, it conjugates orbits of $Re(\mu \lambda_0^{m_\beta} X_{\beta,M})_{|(r,\lambda)=(r_0,\lambda_0)}$ and $Re(\mu \lambda_1^{m_\beta} X_{\beta,M})_{|(r,\lambda)=(0,\lambda_1)}$ and $F_{(0,\lambda_1)} \equiv Id$.

It is straightforward to construct F once we know its restriction to ∂I_β. We can choose for $F_{|w|=\rho}$ any homeomorphism $G_{(r_0,\lambda_0)} : |w| = \rho \to |w| = \rho$ such that $G_{(r_0,\lambda_0)}(TI_{\mu X}^{\beta;\rho}(r_0,\lambda_0)) = TI_{\mu X}^{\beta;\rho}(0,\lambda_1)$. The choice of $F_{|w|=\rho}$ determines $F_{|ac}$ for every $ac \in arc_\infty(\mu X, \beta, K_X^\mu)$.

Given $H \in Reg_j(\mu X, \beta, K_X^\mu)$ with $j \in \{1,2\}$ there exist two fundamental arcs $ac(\alpha)$ and $ac(\omega)$ such that $ac(\upsilon) \subset [|w - v_\beta^{\mu X}(H)| = r(v_\beta^{\mu X}(H))]$ for all $\upsilon \in \{\alpha, \omega\}$. Suppose $j = 2$. The flow of $Re(\mu X)$ establishes a homeomorphism from $ac(\alpha)$ to $ac(\omega)$. We already defined $F_{|\partial ac(\alpha)}$ and we can extend F to $ac(\alpha)$ in any way such that the restriction of $F_{(r_0,\lambda_0)}$ to $ac(\alpha)(r_0, \lambda_0)$ is a homeomorphism varying continuously on (r_0, λ_0). The value of $F_{|ac(\alpha)}$ determines $F_{|ac(\omega)}$. Suppose $j = 1$. Denote $\zeta = \omega_\beta^{\mu X}(H)$. We obtain $ac(\alpha) = ac(\omega) \subset [|w - \zeta| = r(\zeta)]$. The flow of $Re(\mu X)$ establishes an involution of $ac(\omega)$ which has exactly one fixed point for $(r, \lambda) = (r_0, \lambda_0)$. Moreover such fixed point belongs to $TE_{\mu X}^{\beta\zeta, r(\zeta)}(r_0, \lambda_0)$. We can extend F to $ac(\omega)$ in such a way that F commutes with the involution. In this way we defined $F_{|\partial I_\beta}$ since we considered all the fundamental arcs throughout the process. Since $\cup_{\zeta \in S_\beta} TE_{\mu,X}^{\beta\zeta, r(\zeta)}$ is contained in the union of the arcs in $arc_1(\mu X, \beta, K_X^\mu)$ then the restriction of F to $\cup_{\zeta \in S_\beta} TE_{\mu,X}^{\beta\zeta, r(\zeta)}$ is the identity.

7.4. **Assembling the dynamics of the polynomial vector fields.** We already proved that the dynamics of $Re(\mu X)$ ($X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and $\mu \in \mathbb{S}^1$) is a topological product in the exterior sets whereas such a result is true for the intermediate sets when we avoid the directions in B_X^μ. We want to assemble the information attached to the exterior and intermediate sets to describe the behavior of $Re(\mu X)$ in $|y| \leq \epsilon$.

Throughout this section K_X^μ is some compact connected set contained in $\mathbb{S}^1 \setminus B_X^\mu$.

Lemma 7.13. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Fix $\mu \in \mathbb{S}^1$. Let $P_0 \in [0, \delta_0) \times K_X^\mu \times \partial B(0, \epsilon)$ such that $Re(\mu X)$ does not point towards $\mathbb{C} \setminus \overline{B}(0, \epsilon)$ at P. Then the interval $[0, \infty)$ is contained in $It(\mu X, P_0, \overline{B}(0, \epsilon))$ and $\lim_{\zeta \to \infty} \exp(\zeta \mu X)(P_0) \in SingX$.*

Proof. Denote $P_0 = (r_0, \lambda_0, y_0)$. The result for $r_0 = 0$ is a consequence of corollary 7.1 since $\{0\} \times \overline{B}(0, \epsilon) = E_0$.

Suppose $r_0 \neq 0$. Since $E_0(r_0, \lambda_0) \cap SingX = \emptyset$ then the value $c(\mu, P_0)$ provided by corollary 7.1 belongs to $\mathbb{R}^+$. Denote $Q_0 = \exp(c(\mu, P_0)\mu\lambda_0^{d_0} X_{0,E})(P_0)$. We have that $Q_0 \in \partial M_0$ and $Re(\mu X)$ points towards $\dot{I}_0$ at Q_0. The point Q_0 is contained in a fundamental arc $ac \in arc(\mu X, 0, K_X^\mu)$ contained in the closure of a unique $H \in Reg_\infty(\mu X, 0, K_X^\mu)$. There exists $\zeta \in \mathbb{C}$ such that $\omega_0^{\mu X}(H) = \zeta$ since $\mathcal{G}_{\mu X_0(\lambda_0)}$ is connected. We have that $d = \sup(It(\mu X, Q, I_0))$ belongs to $\mathbb{R}^+$ and that $\exp(d\mu X)(Q) \in \partial M_0 \cap \partial E_{0\zeta}$. Moreover $Re(\mu X)$ points towards $\dot{E}_{0\zeta}$ at $\exp(d\mu X)(Q)$. Denote $\beta(0) = 0$ and $\beta(1) = 0\zeta$. By proceeding analogously we obtain a sequence of points $P_0,\ Q_0,\ P_1,\ Q_1,\ \ldots,\ P_k$ ($k \geq 1$) contained in $\Gamma(\mu X, P, |y| \leq \epsilon)$ and such that

$$Q_j \in \partial E_{\beta(j)} \cap \partial M_{\beta(j)} \quad \text{and} \quad P_j \in \partial M_{\beta(j-1)} \cap \partial E_{\beta(j)} \quad \forall j \in \{1, \ldots, k\}$$

and $E_{\beta(k)} = T_{\beta(k)}$. By corollary 7.1 and the discussion in subsection 7.3.2 then $Re(\mu X)$ points towards $\dot{E}_{\beta(k)}$ at $\partial E_{\beta(k)}$ and P_k is in the basin of attraction of $SingX \cap E_{\beta(k)}$. Thus we obtain $\lim_{z \to \infty} \exp(z\mu X)(P) \in SingX \cap E_{\beta(k)}$. $\square$

We define $Reg^*(\epsilon, \mu X, K_X^\mu)$ the set of connected components of

$$[[|y| < \epsilon] \cap [x \in [0, \delta_0) K_X^\mu]] \setminus (SingX \cup_{x \in [0,\delta_0) K_X^\mu} \cup_{P \in T_{\mu X}^\epsilon(x)} \Gamma(\mu X, P, |y| \leq \epsilon)).$$

We define $\alpha^{\mu X}(P) = \lim_{z \to -\infty} \exp(z\mu X)(P)$ for all $P \in [|y| \leq \epsilon]$ such that $It(\mu X, P, |y| \leq \epsilon)$ contains $(-\infty, 0)$. Otherwise we define $\alpha^{\mu X}(P) = \infty$. We define $\omega^{\mu X}(P)$ in an analogous way.

Given $H \in Reg^{*}(\epsilon, \mu X, K_X^{\mu})$ the functions $(\alpha^{\mu X})_{|H}$ and $(\omega^{\mu X})_{|H}$ satisfy that either they are identically ∞ or their value is never ∞. Since the basins of attraction and repulsion of the curves in $Sing_V \mu X$ in $x \in [0, \delta_0) K_X^{\mu}$ are open sets then $(\alpha^{\mu X})_{|H}$ and $(\omega^{\mu X})_{|H}$ are continuous. Thus $(\alpha^{\mu X})_{|H(x)}$ and $(\omega^{\mu X})_{|H(x)}$ are constant for all $x \in [0, \delta_0) K_X^{\mu}$. Indeed we can interpret $\alpha^{\mu X}(H)$ either as ∞ if $(\alpha^{\mu X})_{|H} \equiv \infty$ or as the element of $Sing_V X$ that contains $\alpha^{\mu X}(H)$ otherwise. Denote

$$Reg_{\infty}(\epsilon, \mu X, K_X^{\mu}) = Reg^{*}(\epsilon, \mu X, K_X^{\mu}) \cap ((\alpha^{\mu X})^{-1}(\infty) \cup (\omega^{\mu X})^{-1}(\infty))$$

and $Reg(\epsilon, \mu X, K_X^{\mu}) = Reg^{*}(\epsilon, \mu X, K_X^{\mu}) \setminus Reg_{\infty}(\epsilon, \mu X, K_X^{\mu})$. We define

$$Reg_j(\epsilon, \mu X, K_X^{\mu}) = \{H \in Reg(\epsilon, \mu X, K_X^{\mu}) : \sharp\{\alpha^{\mu X}(H), \omega^{\mu X}(H)\} = j\}$$

for $j \in \{1, 2\}$. We have that the set $H(x)$ is connected for $H \in Reg(\epsilon, \mu X, K_X^{\mu})$ and $x \in (0, \delta_0) K_X^{\mu}$. The set $H(0)$ is connected for $H \notin Reg_2(\epsilon, \mu X, K_X^{\mu})$ whereas otherwise $H(0)$ has two connected components.

We define an oriented graph $\mathcal{G}(\mu X, K_X^{\mu})$. The set of vertexes is $Sing_V X$ whereas the edges are the elements of $Reg(\epsilon, \mu X, K_X^{\mu})$. The edge $H \in Reg(\epsilon, \mu X, K_X^{\mu})$ joins the vertexes $\alpha^{\mu X}(H)$ and $\omega^{\mu X}(H)$, we denote $\alpha^{\mu X}(H) \xrightarrow{H} \omega^{\mu X}(H)$. The graph $\mathcal{N}G(\mu X, K_X^{\mu})$ is obtained from $\mathcal{G}(\mu X, K_X^{\mu})$ by removing the reflexive edges and the orientation of edges.

Proposition 7.7. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Fix $\mu \in \mathbb{S}^1$ and a compact connected set $K_X^{\mu} \subset \mathbb{S}^1 \setminus B_X^{\mu}$. Then the graph $\mathcal{N}G(\mu X, K_X^{\mu})$ is acyclic and connected.*

We say that a exterior set E_{β} has depth 0 if $N(\beta) = 1$. In general given E_{β} such that $N(\beta) > 1$ we define $depth(E_{\beta}) = 1 + \sup_{\zeta \in S_{\beta}} depth(E_{\beta\zeta})$.

Proof. A exterior set $E_{\beta} = [\eta \geq |t| \geq \rho |x|]$ is contained in $T_{\beta} = [\eta \geq |t|]$. We can associate graphs $\mathcal{G}_{\beta}(\mu X, K_X^{\mu})$ and $\mathcal{N}G_{\beta}(\mu X, K_X^{\mu})$ to the vector field $Re(\mu \lambda^{d_{\beta}} X_{\beta, E})$ defined in T_{β}.

Consider an exterior set E_{β} such that $depth(E_{\beta}) = 0$. The graph $\mathcal{N}G_{\beta}(\mu X, K_X^{\mu})$ has only one vertex and no edges, therefore it is connected and acyclic.

Suppose that $\mathcal{N}G_{\beta}(\mu X, K_X^{\mu})$ is connected and acyclic for all exterior set E_{β} such that $depth(E_{\beta}) \leq k$. It is enough to prove that the result is true for every exterior set E_{β} such that $depth(E_{\beta}) = k + 1$.

Fix $\lambda_0 \in K_X^{\mu}$. The graph $\mathcal{N}G_{\mu X_{\beta}(\lambda_0)}$ is connected and acyclic by corollary 7.3 and proposition 7.4. Consider an edge $J_0 \in Reg(\mu X_{\beta}(\lambda_0))$ of the graph $\mathcal{G}_{\mu X_{\beta}(\lambda_0)}$ joining the vertexes $\zeta(1)$ and $\zeta(2)$. We denote also by J_0 the component of $Reg(\mu X_{\beta}(\lambda_0), \rho)$ associated to J_0 by proposition 7.6 where $M_{\beta} = [|w| \leq \rho]$. Let J_1 be the element of $Reg(\mu X, \beta, K_X^{\mu})$ such that $J_1(0, \lambda_0) \subset J_0$. By lemma 7.13 applied to $Re(\mu \lambda^{d_{\beta\zeta(1)}} X_{\beta\zeta(1), E})$ in $T_{\beta\zeta(1)}$ we deduce that $\alpha^{\mu X}(J_1) \subset SingX$. By the open character of the singular points in $(r, \lambda) \in [0, \delta_0) \times K_X^{\mu}$ we obtain that $\alpha^{\mu X}(J_1)$ is contained in an irreducible component γ_1 of $SingX$. Analogously $\omega^{\mu X}(J_1)$ is contained in an irreducible component γ_2 of $SingX$. Denote by J_2 the edge of $\mathcal{N}G_{\beta}(\mu X, K_X^{\mu})$ joining γ_1 and γ_2.

The set $\mathbb{C} \setminus J_0$ has two connected components $H_1 \ni \zeta(1)$ and $H_2 \ni \zeta(2)$ (lemma 7.12). Denote $Sg_j = H_j \cap S_{\beta}$ for $j \in \{1, 2\}$. We obtain that there is no edge different than J_2 of $\mathcal{G}_{\beta}(\mu X, K_X^{\mu})$ joining a vertex of $\mathcal{N}G_{\beta\upsilon}(\mu X, K_X^{\mu})$ and a vertex of $\mathcal{N}G_{\beta\kappa}(\mu X, K_X^{\mu})$ for $\upsilon \in Sg_1$ and $\kappa \in Sg_2$. Moreover the restriction of $\mathcal{G}_{\beta}(\mu X, K_X^{\mu})$ to $Sing_V X_{\beta\upsilon, E}$ is $\mathcal{G}_{\beta\upsilon}(\mu X, K_X^{\mu})$ for all $\upsilon \in S_{\beta}$. Then the aciclicity of every $\mathcal{N}G_{\beta\upsilon}(\mu X, K_X^{\mu})$ for all $\upsilon \in S_{\beta}$ imply that $\mathcal{N}G_{\beta}(\mu X, K_X^{\mu})$ is acyclic.

Finally, since $\mathcal{N}G_{\mu X_{\beta(\lambda_0)}}$ and $\mathcal{N}G_{\beta\upsilon}(\mu X, K_X^\mu)$ are connected for all $\upsilon \in S_\beta$ then $\mathcal{N}G_\beta(\mu X, K_X^\mu)$ is connected. $\square$

The properties of $\mathcal{G}(\mu X, K_X^\mu)$ are inherited of the properties of the polynomial vector fields associated to X.

Proposition 7.8. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Fix $\mu \in \mathbb{S}^1$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Then we have*

$$\sharp(Reg(\epsilon, \mu X, K_X^\mu) \cap (\alpha^{\mu X}, \omega^{\mu X})^{-1}(\gamma, \gamma)) = 2\nu_X(\gamma)$$

for all $\gamma \in Sing_V X$. Moreover we have $\sharp Reg(\epsilon, \mu X, K_X^\mu) = 2\nu(X) - N(X) + 1$.

7.5. **Analyzing the regions.** Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Fix $\mu \in \mathbb{S}^1$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Consider a region $H \in Reg_1(\epsilon, \mu X, K_X^\mu)$. We denote by $T_{\mu X,H}^\epsilon(x)$ the unique tangent point in $T_X^\epsilon(x) \cap \overline{H(x)}$ for all $x \in [0, \delta_0)K_X^\mu$. Let ψ be an integral of the time form of X defined in a neighborhood of $T_{\mu X,H}^\epsilon(0)$. By analytic continuation we obtain an integral of the time form $\psi_{H,L}^X = \psi_{H,R}^X$ of X in $H = H_L = H_R$ such that it is holomorphic in $H \setminus [x = 0]$ and continuous in H. Moreover $(x, \psi_{H,L}^X) = (x, \psi_{H,R}^X)$ is injective in H since $\psi_{H,L}^X(H(x))$ is simply connected for all $x \in [0, \delta_0)K_X^\mu$.

Let $H \in Reg_2(\epsilon, \mu X, K_X^\mu)$. Denote by $L_{\mu X,H}^\epsilon(x)$ the unique point $T_X^\epsilon(x) \cap \overline{H(x)}$ such that $Re(-\mu i X)$ points towards H for all $x \in [0, \delta_0)K_X^\mu$. We define H_L as the union of $H \setminus [x = 0]$ and the connected component $H_L(0)$ of $H(0)$ such that $L_{\mu X,H}^\epsilon(0) \in \overline{H_L(0)}$. We denote by $R_{\mu X,H}^\epsilon(x)$ the other point in $T_X^\epsilon(x) \cap \overline{H(x)}$ for $x \in [0, \delta_0)K_X^\mu$. We define $H_R = H \setminus H_L(0)$. Let ψ_κ be a holomorphic integral of the time form of X defined in a neighborhood of $\kappa_{\mu X,H}^\epsilon(0)$ for $\kappa \in \{L, R\}$. We obtain an integral $\psi_{H,\kappa}^X$ of the time form of X in H_κ obtained by analytic continuation of ψ_κ for $\kappa \in \{L, R\}$. The function $\psi_{H,\kappa}^X$ is holomorphic in $H \setminus [x = 0]$ and continuous in H_κ for $\kappa \in \{L, R\}$. Moreover $(x, \psi_{H,L}^X)$ and $(x, \psi_{H,R}^X)$ are injective in H_L and H_R respectively. The theorem of the residues implies that

$$\psi_{H,L}^X(x, y) - \psi_{H,R}^X(x, y) - 2\pi i \sum_{P \in J(x)} Res(X, P)$$

is bounded in $H \setminus [x = 0]$ where $J(x)$ is the subset of $(Sing X)(x)$ of points contained in the same connected component of $B(0, \epsilon) \setminus H(x)$ than $\omega^{\mu X}(H(x))$. Since $H(0)$ is disconnected the function $x \to \sum_{P \in J(x)} Res(X, P)$ is not bounded in $x \in (0, \delta_0)K_X^\mu$. Indeed $x \to \sum_{P \in J(x)} Res(X, P)$ can be extended to a pure meromorphic function defined in a neighborhood of $x = 0$.

We call subregion of a region $H \in Reg(\epsilon, \mu X, K_X^\mu)$ to every set of the form $H \cap E_\beta$ or $H \cap I_\beta$ where E_β is an exterior set and I_β is an intermediate set. We say that all the subregions of $H \in Reg_1(\epsilon, \mu X, K_X^\mu)$ are both L-subregions and R-subregions. Consider $H \in Reg_2(\epsilon, \mu X, K_X^\mu)$. There exists a magnifying glass set $M_{\beta(0)}$ such that the curves $\alpha^{\mu X}(H)$ and $\omega^{\mu X}(H)$ are contained in $M_{\beta(0)}$ but they are in differented connected components of $M_{\beta(0)} \setminus I_{\beta(0)}$. A subregion of H contained in $M_{\beta(0)}$ is both a L-subregion and an R-subregion. A subregion in the same connected component of $\overline{H \setminus M_{\beta(0)}}$ than $L_{\mu X,H}^\epsilon$ is called a L-subregion. A subregion of H in the same connected component of $\overline{H \setminus M_{\beta(0)}}$ than $R_{\mu X,H}^\epsilon$ is called a R-subregion. We define H^L the union of the L-subregions of H whereas H^R is the union of the R-subregions of H. Clearly we have $H = H^L \cup H^R$.

Remark 7.5. *The dynamical splitting can be generalized to study any germ of vector field of the form $x^m Y$ for some $Y \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$. There is no problem in dealing with singular vector fields at $x = 0$ if $Y \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Indeed we already dit it since all the transforms of X outside of the first exterior set are singular at $x = 0$. The only difference is that the regions do not depend on x but on (r, λ). If $Y \notin \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ we just consider a ramification (x^k, y) such that $(x^k, y)^* Y$ belongs to $\mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. The expression for the sets of the splitting is a little bit different, for instance the first exterior set is of the form $\rho|x|^{1/k} \leq |y| \leq \eta$.*

8. Extension of the Fatou coordinates

A diffeomorphism $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ is a small deformation of its convergent normal form $\exp(X)$ in suitable domains. The dynamical splitting associated to X provides information about the dynamics of φ. That is going to lead us to define the analogue of the Ecalle-Voronin invariants. For such a purpose we need to mesure the "distance" from $\exp(X)$ to φ.

8.1. Comparing $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ and a convergent normal form.

Let $\exp(X)$ be a convergent normal form of φ. We consider

$$\sigma_z(x, y) = (x, (1 - z)(y \circ \exp(X)) + z(y \circ \varphi))$$

for $z \in B(0, 2)$. Let ψ be an integral of the time form of X, i.e. $X(\psi) = 1$. We define $\Delta_\varphi = \psi \circ \sigma_1(P) - (\psi(P) + 1)$ for $P \notin Fix\varphi$ in a neighborhood of $(0, 0)$ as follows: we choose a determination $\psi_{|x=x(P)}$ in the neighborhood of P, we define $\psi \circ \sigma_1(P)$ as the evaluation at $\sigma_1(P)$ of the analytic continuation of $\psi_{|x=x(P)}$ along the path $\gamma : [0, 1] \to [x = x(P)]$ given by $\gamma(z) = \sigma_z(P)$. The value of Δ_φ does not depend on the determination of ψ that we chose. Clearly Δ_φ is holomorphic in $U \setminus Fix\varphi$ for some neighborhood U of $(0, 0)$. Indeed we have:

Lemma 8.1. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ (with fixed convergent normal form). Then the function Δ_φ belongs to the ideal $(y \circ \varphi - y)$ of the ring $\mathbb{C}\{x, y\}$.*

The result is a consequence of Taylor's formula applied to

$$\Delta_\varphi = \psi \circ \varphi - \psi \circ \exp(X) \sim \frac{\partial \psi}{\partial y} \circ \exp(X)(y \circ \varphi - y \circ \exp(X)) = O(y \circ \varphi - y).$$

Proposition 8.1. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in \mathbb{S}^1$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1$. Consider $H \in Reg(\epsilon, \mu X, K_X^\mu)$. Then we have*

$$\Delta_\varphi = O(X(y)) = O\left(\frac{1}{(1 + |\psi_{H,\kappa}^X|)^{1+1/\nu(\varphi)}}\right)$$

for every κ-subregion of H and all $\kappa \in \{L, R\}$.

Proof. Denote $f = X(y)$. Let us prove the result for a L-subregion J without lack of generality. There exists a sequence $B(0), \ldots, B(k) = J$ of L-subregions of H such that

- $B(2j) \subset E_{\beta(2j)}$ for all $0 \leq 2j \leq k$.
- $B(2j + 1) \subset I_{\beta(2j)}$ for all $0 \leq 2j + 1 \leq k$.
- $\beta(0) = 0$ and $\beta(2j+2) = \beta(2j)\upsilon(j)$ for some $\upsilon(j) \in \mathbb{C}$ and all $0 \leq 2j+2 \leq k$.

Denote $K(2j) = E_{\beta(2j)}$, $h(2j) = d_{\beta(2j)}$, $K(2j+1) = I_{\beta(2j)}$ and $h(2j+1) = m_{\beta(2j)}$. Denote $\partial_e B(0) = [|y| \leq \epsilon] \cap B(0)$ and $\partial_e B(j) = \overline{B(j)} \cap \partial K(j-1)$ for $j \geq 1$. We define the property $Pr(j)$ as

$$Pr(j) : \left\{ \begin{array}{l} \sup |(\psi_{H,L}^X)_{|\partial_e B(j)}| \leq M_j/|x|^{h(j)} \text{ for some } M_j \in \mathbb{R}^+ \text{ if } j \leq k \\ f = O(1/(1+|\psi_{H,L}^X|)^{1+1/\nu(\varphi)}) \text{ in } B(0) \cup \ldots \cup B(j-1). \end{array} \right.$$

We have that $Pr(k+1)$ implies the result in the proposition for J. The result is true for $j=0$. It is enough to prove that $Pr(j) \implies Pr(j+1)$ for all $0 \leq j \leq k$.

From the construction of the splitting we obtain that $f \in (x^{h(j)+[(j+1)/2]})$ in $K(j)$ for all $0 \leq j \leq k$ (let us remark that $[(j+1)/2]$ is the integer part of $(j+1)/2$). Denote $Y = (X/x^{h(j)})_{|K(j)}$. There exists a holomorphic integral ψ_j of the time form of Y in a neighborhood of the simply connected set $\overline{B(j)}$ such that $|\psi_j| \leq M_j'$ in $\partial_e B(j)$ for some $M_j' > 0$. Suppose that $K(j) = [\eta \geq |t| \geq \rho|x|]$ is a parabolic exterior set, since $\nu(Y) \leq \nu(X)$ we obtain

$$f = O\left(\frac{x^{h(j)+[(j+1)/2]}}{(1+|\psi_j|)^{1+1/\nu(Y)}}\right) = O\left(\frac{x^{h(j)+[(j+1)/2]}}{(1+|\psi_j|)^{1+1/\nu(X)}}\right)$$

by remark 7.4. The inequality $|x^{h(j)}\psi_{H,L}^X - \psi_j| \leq M_j + M_j'$ implies

$$f = O\left(\frac{x^{h(j)+[(j+1)/2]-h(j)(1+1/\nu(X))}}{(1+|\psi_{H,L}^X|)^{1+1/\nu(X)}}\right) = O\left(\frac{1}{(1+|\psi_{H,L}^X|)^{1+1/\nu(X)}}\right)$$

in $B(j)$ since $h(j) \leq [(j+1)/2]\nu(X)$ by construction. Moreover if $j < k$ then we have $|\psi_j| = O(1/|x|^{\nu(Y)})$ in $\partial_e B(j+1)$. We deduce that there exists $M_{j+1} > 0$ such that $|\psi_{H,L}^X| \leq M_{j+1}/|x|^{h(j+1)}$ in $\partial_e B(j+1)$ since $h(j+1) = h(j) + \nu(Y)$,

Suppose that $K(j) = [\eta \geq |t|]$ is a non-parabolic exterior set, this implies $j = k$. We have that $\psi_j(r,\lambda,t)\lambda^{-d_\beta}\mu^{-1} - C(r,\lambda)\ln(t-\gamma(x))$ is bounded in $B(j)$ where $t = \gamma(x)$ is the only irreducible component of $Sing X_{\beta(j),E}$ by the discussion in subsection 7.3.2. There exists $\upsilon > 0$ such that $\arg(C(r,\lambda))$ in $(-\pi/2+\upsilon, \pi/2-\upsilon)$ for all $(r,\lambda) \in [0,\delta_0) \times K_X^\mu$ if $B(j)$ is a basin of repulsion, otherwise we have that $\arg(C(r,\lambda)) \in (\pi/2+\upsilon, 3\pi/2-\upsilon)$ for all $(r,\lambda) \in [0,\delta_0) \times K_X^\mu$. We deduce that

$$f = O(x^{h(j)+[(j+1)/2]}(t-\gamma(x))) = O(x^{h(j)+[(j+1)/2]}e^{-K|\psi_j|})$$

in $B(j)$ for some $K > 0$ and then

$$f = O\left(\frac{x^{h(j)+[(j+1)/2]}}{(1+|\psi_j|)^{1+1/\nu(X)}}\right) = O\left(\frac{1}{(1+|\psi_{H,L}^X|)^{1+1/\nu(X)}}\right)$$

in $B(j)$.

Finally suppose that $K(j)$ is an intermediate set. We have that ψ_j is bounded in $B(j)$. Thus there exists $M_{j+1} > 0$ such that $|\psi_{H,L}^X| \leq M_{j+1}/|x|^{h(j)} = M_{j+1}/|x|^{h(j+1)}$ in $\overline{B(j)}$ and then in $\partial_e(B(j+1))$. We obtain

$$f = O(x^{h(j)+[(j+1)/2]}) = O\left(\frac{x^{h(j)+[(j+1)/2]}}{(1+|\psi_j|)^{1+1/\nu(X)}}\right) = O\left(\frac{1}{(1+|\psi_{H,L}^X|)^{1+1/\nu(X)}}\right)$$

in $B(j)$. $\square$

8.2. **Constructing Fatou coordinates.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with convergent normal form $\alpha = \exp(X)$. Fix $\mu = ie^{i\theta}$ with $\theta \in (-\pi/2, \pi/2)$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Consider $H \in Reg(\epsilon, \mu X, K_X^\mu)$. Let $P \in H$, suppose $P \in H^L$ without lack of generality. The trajectory $\Gamma = \Gamma(\mu X, P, |y| \leq \epsilon)$ is contained in H^L. Let $B(P)$ be the strip $\exp([0,1]X)(\Gamma)$ and $\Gamma' = \alpha(\Gamma)$. The distance between the lines $\psi_{H,L}^X(\Gamma)$ and $\psi_{H,L}^X(\Gamma')$ is $\cos\theta$. Since $\psi_{H,L}^X \circ \varphi = \psi_{H,L}^X \circ \alpha + \Delta_\varphi$ then Γ and $\varphi(\Gamma)$ enclose a strip $B_1(P)$ whenever $sup_{B(0,\delta_0) \times B(0,\epsilon)} |\Delta_\varphi| < (\cos\theta)/3$. Since $\Delta_\varphi(0,0) = 0$ this condition is fulfilled by taking μ away from -1 and 1 and a small neighborhood $B(0,\delta_0) \times B(0,\epsilon)$ of $(0,0)$.

Let $\tilde{B}(P)$ be the complex space obtained from $B(P)$ by identifying Γ and Γ'. Let $\tilde{B}_1(P)$ be the complex space obtained from $B_1(P)$ by identifying Γ and $\varphi(\Gamma)$. The space $\tilde{B}(P)$ is biholomorphic to $\mathbb{C}^*$ by $e^{2\pi i z} \circ \psi_{H,L}^X$. A natural compactification $\overline{B}(P)$ is obtained by adding $0 \sim \omega^{\mu X}(P)$ and $\infty \sim \alpha^{\mu X}(P)$. Analogously we will obtain a biholomorphism from $\overline{B}_1(P)$ to $\mathbb{P}^1(\mathbb{C})$. The space of orbits of $\varphi_{|H_L(x(P))}$ is then rigid, that will allow us to define analytic invariants of φ. Let us remark that $\tilde{B}_1(P)$ is the restriction of the space of orbits of φ to $H_L(x(P))$ for all choices of H and P if and only if $\nu_X(\gamma) \geq 1$ for all $\gamma \in Sing_V X$ [22]. In general the complete space of orbits is messier, we obtain further identifications via return maps.

We consider the coordinates given by $\psi_{H,L}^X$. We define

$$\sigma_0(z) = z + \eta(\cos\theta Re((z - \psi_{H,L}^X(P))e^{-i\theta}))\Delta_\varphi \circ \alpha^{\circ(-1)} \circ (\psi_{H,L}^X)^{\circ(-1)}(z)$$

where $\eta : \mathbb{R} \to [0,1]$ is a C^∞ function such that $\eta(b) = 0$ for all $b \leq 1/3$ and $\eta(b) = 1$ for all $b \geq 2/3$. This definition implies that $\sigma = (\psi_{H,L}^X)^{\circ(-1)} \circ \sigma_0 \circ \psi_{H,L}^X$ satisfies

$$\sigma_{\exp([0,1/3]X)(\Gamma)} \equiv Id \quad \text{and} \quad \sigma_{\exp([-1/3,0]X)(\Gamma')} \equiv \varphi \circ \alpha^{\circ(-1)}.$$

The mappings σ_0 and σ depend on the choice of the base point P. The function $\Delta_\varphi \circ \alpha^{\circ(-1)} \circ (\psi_{H,L}^X)^{\circ(-1)}$ is holomorphic. By Cauchy's integral formula we obtain

$$\frac{\partial(\Delta_\varphi \circ \alpha^{\circ(-1)} \circ (\psi_{H,L}^X)^{\circ(-1)})}{\partial z}(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=1} \frac{\Delta_\varphi \circ \alpha^{\circ(-1)} \circ (\psi_{H,L}^X)^{\circ(-1)}}{(z-z_0)^2} dz.$$

By proposition 8.1 there exists $C > 1$ such that

$$\left| \frac{\partial(\Delta_\varphi \circ \alpha^{\circ(-1)} \circ (\psi_{H,L}^X)^{\circ(-1)})}{\partial z} \right|(z) \leq C \min\left(\frac{1}{(1+|z|)^{1+1/\nu(\varphi)}}, \sup_{B(P)} |\Delta_\varphi| \right) \tag{8}$$

for all $z \in \psi_{H,L}^X(B(P))$. The jacobien matrix $\mathcal{J}\sigma_0$ of σ_0 is a 2×2 real matrix. The coefficients of $\mathcal{J}\sigma_0 - Id$ are bounded by an expression like the one in the right hand side of equation 8, maybe for a bigger $C > 1$. We obtain that $\sup_{B(0,\delta)\times B(0,\epsilon)} |\Delta_\varphi|$ small implies $\mathcal{J}\sigma_0 \sim Id$ and then σ is a C^∞ diffeomorphism from $\tilde{B}(P)$ onto $\tilde{B}_1(P)$.

The mapping $\xi = e^{2\pi i z} \circ \psi_{H,L}^X \circ \sigma^{\circ(-1)}$ is a C^∞ diffeomorphism from $\tilde{B}_1(P)$ onto $\mathbb{C}^*$. The function $\psi_{H,L}^X \circ \sigma^{\circ(-1)}$ is a Fatou coordinate, even if not holomorphic in general, of φ in $B_1(P)$. The complex dilatation χ_{σ_0} of σ_0 satisfies

$$|\chi_{\sigma_0}|(z) = \frac{\left|\frac{\partial \sigma_0}{\partial \bar{z}}\right|}{\left|\frac{\partial \sigma_0}{\partial z}\right|}(z) \leq K(H) \min\left(\frac{1}{(1+|z|)^{1+1/\nu(\varphi)}}, \sup_{H(x(P))} |\Delta_\varphi| \right)$$

for all $z \in \psi_{H,L}^X(B(P))$ and some $K(H) > 1$ independent of $P \in H$. Since $\xi^{\circ(-1)}$ is equal to $(\psi_{H,L}^X)^{\circ(-1)} \circ \sigma_0 \circ ((1/2\pi i)\ln z)$ then

Lemma 8.2. $|\chi_{\xi^{\circ(-1)}}|(z) \leq K(H)\min(1/(1+|\ln z|/(2\pi))^{1+1/\nu(\varphi)}, \sup_{H(x(P))}|\Delta_\varphi|)$ *for all* $z \in e^{2\pi i w} \circ \psi_{H,L}^X(B(P))$.

The mapping ξ and then $\chi_{\xi^{\circ(-1)}}$ depend on the base point P. We look for a quasi-conformal mapping $\tilde{\rho} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\chi_{\tilde{\rho}} = \chi_{\xi^{\circ(-1)}}$. Since we can suppose $||\chi_{\xi^{\circ(-1)}}||_\infty = \sup_{\mathbb{C}^*}|\chi_{\xi^{\circ(-1)}}| < 1/2 < 1$ then such a mapping exists by the Ahlfors-Bers theorem. The choice of $\tilde{\rho}$ is unique if $\tilde{\rho}$ fulfills $\tilde{\rho}(0) = 0$, $\tilde{\rho}(1) = 1$ and $\tilde{\rho}(\infty) = \infty$. By construction $\tilde{\rho} \circ \xi$ is a biholomorphism from $\tilde{B}_1(P)$ to $\mathbb{C}^*$.

We define

$$J(r) = \frac{2}{\pi}\int_{|z|<r} \frac{K(H)}{(1+2^{-1}\pi^{-1}|\ln|z||)^{1+1/\nu(\varphi)}}\frac{1}{|z|^2}d\sigma$$

for $r \in \mathbb{R}^+$. We have that $J(r) < \infty$ for all $r \in \mathbb{R}^+$.

Lemma 8.3. *The mapping $\tilde{\rho}$ is conformal at* 0 *and at* ∞. *Moreover we have*

$$\left|\frac{\tilde{\rho}(z)}{z} - \frac{\partial\tilde{\rho}}{\partial z}(0)\right| \leq \left|\frac{\partial\tilde{\rho}}{\partial z}(0)\right|\iota(|z|) \text{ and } \left|\frac{z}{\tilde{\rho}(z)} - \frac{\partial\tilde{\rho}}{\partial z}(\infty)^{-1}\right| \leq \left|\frac{\partial\tilde{\rho}}{\partial z}(\infty)\right|^{-1}\iota(1/|z|)$$

where ι depends on $K(H)$, it satisfies $\lim_{|z|\to 0}\iota(|z|) = 0$. *We have*

$$min_{|z|=1}|\tilde{\rho}(z)|e^{-J(1)} \leq |\partial\tilde{\rho}/\partial z|(0), |\partial\tilde{\rho}/\partial z|(\infty) \leq max_{|z|=1}|\tilde{\rho}(z)|e^{J(1)}.$$

Proof. We define

$$I(r) = \frac{1}{\pi}\int_{|z|<r}\frac{1}{1-|\chi_{\tilde{\rho}}|}\frac{|\chi_{\tilde{\rho}}(z)|}{|z|^2}d\sigma$$

for all $r \in \mathbb{R}^+$. We have $I(r) \leq J(r)$ for all $r \in \mathbb{R}^+$. To get the conformality of $\tilde{\rho}$ at $z = 0$ it is enough to prove that $I(r) < \infty$ for all $r \in \mathbb{R}^+$ (theorem 6.1 in page 232 of [14]). This is clear since $J(r) < \infty$ for all $r \in \mathbb{R}^+$. The inequality is obtained for a function ι such that $\lim_{|z|\to 0}\iota(|z|) = 0$, it depends on J and then on $K(H)$. The proof for $z = \infty$ is obtained by applying the result in [14] to $1/\tilde{\rho}(1/z)$. □

We denote by $[z_0, z_1]$ the spherical distance for $z_0, z_1 \in \mathbb{P}^1(\mathbb{C})$.

Lemma 8.4. *(*[1]*, lemma 17, page 398). Let χ be a measurable complex-valued function in $\mathbb{P}^1(\mathbb{C})$. Suppose $||\chi||_\infty < 1$. Then there exists a unique quasi-conformal mapping $\upsilon : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\chi_\upsilon = \chi$, $\upsilon(0) = 0$, $\upsilon(1) = 1$, $\upsilon(\infty) = \infty$ and*

$$[\upsilon(z), z] \leq C_0||\chi||_\infty$$

for all $z \in \mathbb{P}^1(\mathbb{C})$ and some $C_0 > 0$ not depending on χ.

Corollary 8.1. $[\tilde{\rho}(z), z] \leq C_0||\chi_{\xi^{\circ(-1)}}||_\infty$ *for all* $z \in \mathbb{P}^1(\mathbb{C})$

We define $\rho = \tilde{\rho}/(\partial\tilde{\rho}/\partial z)(0)$. The quasi-conformal mapping ρ is the only solution of $\chi_\rho = \chi_{\xi^{\circ(-1)}}$ such that $\rho(0) = 0$, $\rho(\infty) = \infty$ and $(\partial\rho/\partial z)(0) = 1$.

Lemma 8.5. $\lim_{||\chi_{\xi^{\circ(-1)}}||_\infty \to 0}(\partial\tilde{\rho}/\partial z)(z_0) = 1$ *for* $z_0 \in \{0, \infty\}$. *In particular we have* $\lim_{||\chi_{\xi^{\circ(-1)}}||_\infty \to 0}(\partial\rho/\partial z)(\infty) = 1$.

Proof. Denote $\chi = \chi_{\xi^{\circ(-1)}}$. For $||\chi||_\infty$ small enough there exists $C_1 > 0$ such that $|\tilde{\rho}(z) - z| \leq C_1||\chi||_\infty$ for all $z \in \overline{B}(0,1)$ (lemma 8.4). This leads us to

$$\left|\frac{\partial\tilde{\rho}}{\partial z}(0) - 1\right| \leq (1+C_1)e^{J(1)}\iota(|z|) + \frac{C_1}{|z|}||\chi||_\infty$$

for all $z \in \overline{B}(0,1) \setminus \{0\}$ (lemma 8.3 and corollary 8.1). By evaluating at $z = \sqrt{||\chi||_\infty}$ we obtain that $\lim_{||\chi_{\xi^{\circ(-1)}}||_\infty \to 0} (\partial \tilde{\rho}/\partial z)(0) = 1$. In an analogous way we can prove that $\lim_{||\chi_{\xi^{\circ(-1)}}||_\infty \to 0} (\partial \tilde{\rho}/\partial z)(\infty) = 1$. Since

$$(\partial \rho/\partial z)(\infty) = (\partial \tilde{\rho}/\partial z)(\infty)/(\partial \tilde{\rho}/\partial z)(0)$$

then we obtain that $\lim_{||\chi_{\xi^{\circ(-1)}}||_\infty \to 0} (\partial \rho/\partial z)(\infty) = 1$. □

Lemma 8.6. $\lim_{||\chi_{\xi^{\circ(-1)}}||_\infty \to 0} \sup_{z \in \mathbb{P}^1(\mathbb{C})} |\rho(z)/z - 1| = 0.$

Proof. Denote $\chi = \chi_{\xi^{\circ(-1)}}$. Let $b > 0$. By lemma 8.3 there exists $r_0 \in \mathbb{R}^+$ such that $|\rho(z)/z - 1| < b$ for all $z \in B(0, r_0)$. We also obtain

$$\left| \frac{z}{\rho(z)} - \frac{\partial \rho}{\partial z}(\infty)^{-1} \right| \leq \left| \frac{\partial \rho}{\partial z}(\infty)^{-1} \right| \iota(1/|z|).$$

Since $\lim_{||\chi||_\infty \to 0} (\partial \rho/\partial z)(\infty) = 1$ then there exist $a_0 > 0$ and $r_1 > 0$ such that $|\rho(z)/z - 1| < b$ for all $z \in \mathbb{C} \setminus B(0, r_1)$ if $||\chi||_\infty < a_0$. There exists $a_1 > 0$ and $C_1 > 0$ such that $|\tilde{\rho}(z) - z| < C_1 ||\chi||_\infty$ for all $z \in \overline{B}(0, r_1) \setminus B(0, r_0)$ if $||\chi||_\infty < a_1$. We deduce that

$$\left| \frac{\rho(z)}{z} - 1 \right| \leq |1 - 1/(\partial \tilde{\rho}/\partial z)(0)| + \frac{C_1 ||\chi||_\infty}{|\partial \tilde{\rho}/\partial z|(0)|z|}$$

for all $z \in \overline{B}(0, r_1) \setminus B(0, r_0)$ and $||\chi||_\infty < a_1$. By lemma 8.5 there exists $a \in \mathbb{R}^+$ such that $|\rho(z)/z - 1| < b$ for all $z \in \mathbb{P}^1(\mathbb{C})$ if $||\chi||_\infty < a$. □

Now we can define the function

$$\psi^\varphi_{H,L,P} = \frac{1}{2\pi i} \ln z \circ \rho \circ e^{2\pi i z} \circ \psi^X_{H,L} \circ \sigma^{\circ(-1)}.$$

It is an injective Fatou coordinate of φ in the neighborhood of $B_1(P)$. By using $\psi^\varphi_{H,L,P} \circ \varphi = \psi^\varphi_{H,L,P} + 1$ we can extend $\psi^\varphi_{H,L,P}$ to $H_L(x(P))$.

It looks like $\psi^\varphi_{H,L,P}$ depends on the choice of the base point $P \in H^L$. Nevertheless the functions $\psi^\varphi_{H,L,P}$ paste together to provide a Fatou coordinate $\psi^\varphi_{H,L}$, it is continuous in H_L and holomorphic in $\dot{H}$.

Lemma 8.7. *Denote $\xi_0 = e^{2\pi i z} \circ \psi^X_{H,L}$. There exists $C > 0$ independent of $P \in H^L$ such that*

$$\left| \psi^\varphi_{H,L,P} - \psi^X_{H,L} \right| \leq \frac{1}{\pi} \left|\left| \frac{\rho}{z} - 1 \right|\right|_\infty + \frac{C}{(1 + |\psi^X_{H,L}|)^{1+1/\nu(\varphi)}}$$

in $B_1(P)$. Moreover we have

$$\lim_{Z \in B_1(Q), \xi_0(Z) \to z_0} \psi^\varphi_{H,L,P}(Z) - \psi^X_{H,L}(Z) = \frac{1}{2\pi i} \ln \frac{\partial \rho}{\partial z}(z_0)$$

for all $z_0 \in \{0, \infty\}$ and all $Q \in H_L(x(P))$.

Proof. Denote $\chi = \chi_{\xi^{\circ(-1)}}$ and $\kappa = \rho/z - 1$. We have $\lim_{||\chi||_\infty \to 0} ||\kappa||_\infty = 0$ (lemma 8.6). Thus we obtain

$$\left| \psi^\varphi_{H,L,P} - \psi^X_{H,L} \circ \sigma^{\circ(-1)} \right| = \frac{1}{2\pi} \left| \ln(1 + \kappa(z)) \circ e^{2\pi i z} \circ \psi^X_{H,L} \circ \sigma^{\circ(-1)} \right| \leq \frac{||\kappa||_\infty}{\pi}$$

for $||\kappa||_\infty$ small enough. On the other hand we get

$$\left|\psi_{H,L}^X \circ \sigma^{\circ(-1)} - \psi_{H,L}^X\right| = \left|\sigma_0^{\circ(-1)} \circ \psi_{H,L}^X - \psi_{H,L}^X\right| \leq \frac{C}{(1+|\psi_{H,L}^X|)^{1+1/\nu(\varphi)}}$$

for some $C>0$ and all $z \in B_1(P)$ (prop. 8.1). Analogously we obtain

$$\lim_{Z \in B_1(P), \xi_0(Z) \to z_0} \psi_{H,L,P}^{\varphi}(Z) - \psi_{H,L}^X(Z) = \frac{1}{2\pi i} \ln \frac{\partial \rho}{\partial z}(z_0)$$

for $z_0 \in \{0,\infty\}$. We can suppose $\sup_{B(0,\delta_0)\times B(0,\epsilon)} |\Delta_\varphi| < 1/2$. As a consequence given $Q \in H_L(x(P))$ there exists $k(Q) \in \mathbb{N}$ such that every $Z \in B_1(Q)$ is of the form $\varphi^{\circ(j(Z))}(P')$ for some $P' \in B_1(P)$ and $j(Z) \in [-k(Q), k(Q)]$. Moreover if $j(Z) \geq 0$ then $\varphi^{\circ(l)}(P') \in H_L(x(P))$ for $0 \leq l < j(Z)$ whereas for $j(Z) < 0$ we have that $\varphi^{\circ(-l)}(P') \in H_L(x(P))$ for $0 \leq l < -j(Z)$.

Fix $Q \in H_L(x(P))$. Consider $Z \in B_1(Q)$, we can suppose $j(Z) > 0$ without lack of generality. This leads us to

$$\psi_{H,L,P}^{\varphi}(Z) - \psi_{H,L}^X(Z) = (\psi_{H,L,P}^{\varphi}(P') - \psi_{H,L}^X(P')) - \sum_{l=0}^{j(Z)-1} \Delta_\varphi \circ \varphi^{\circ(l)}(P').$$

Since $|\psi_{H,L}^X(\varphi^{\circ(l)}(P')) - \psi_{H,L}^X(Z) + j(Z) - l| < k(Q)/2$ for all $0 \leq l \leq j(Z)$ then

$$\left|\sum_{l=0}^{j(Z)-1} \Delta_\varphi \circ \varphi^{\circ(l)}(P')\right| \leq \frac{k(Q)C}{(1 - k(Q)/2 + |Img(\psi_{H,L}^X(Z))|)^{1+1/\nu(\varphi)}}$$

Now $\xi_0(Z) \to 0,\infty$ implies $|Img(\psi_{H,L}^X(Z))| \to \infty$. Thus we obtain

$$\lim_{Z \in B_1(P), \xi_0(Z) \to z_0} \psi_{H,L,P}^{\varphi}(Z) - \psi_{H,L}^X(Z) = \lim_{Z \in B_1(Q), \xi_0(Z) \to z_0} \psi_{H,L,P}^{\varphi}(Z) - \psi_{H,L}^X(Z)$$

for $z_0 \in \{0,\infty\}$. $\square$

We prove next that $\psi_{H,L,P}$ depends only on $x(P)$.

Lemma 8.8. *Let $x_0 \in [0,\delta_0)K_X^{\mu}$. We have $\psi_{H,L,P}^{\varphi} \equiv \psi_{H,L,Q}^{\varphi}$ in $H_L(x_0)$ for all $P, Q \in H^L(x_0)$. We also have $\psi_{H,L,P}^{\varphi} - \psi_{H,L}^X \equiv \psi_{H,R,Q}^{\varphi} - \psi_{H,R}^X$ if $x_0 \neq 0$ and $(P,Q) \in H^L(x_0) \times H^R(x_0)$. Then $(\partial\rho/\partial z)(\infty)$ depends only on H and $x(P)$.*

Proof. Let $P, Q \in H^L(x_0)$. We have $\psi_{H,L,P}^{\varphi} - \psi_{H,L,Q}^{\varphi} \in \vartheta(\tilde{B}_1(P))$ since

$$(\psi_{H,L,P}^{\varphi} - \psi_{H,L,Q}^{\varphi}) \circ \varphi \equiv \psi_{H,L,P}^{\varphi} - \psi_{H,L,Q}^{\varphi}.$$

We define $h = (\psi_{H,L,P}^{\varphi} - \psi_{H,L,Q}^{\varphi}) \circ (\psi_{H,L,P}^{\varphi})^{\circ(-1)} \circ 1/(2\pi i) \ln z$ in $\mathbb{C}^*$. The function extends to a holomorphic function in $\mathbb{P}^1(\mathbb{C})$ such that $h(0) = 0$ by lemma 8.7. Therefore we obtain $h \equiv 0$ and then $\psi_{H,L,P}^{\varphi} \equiv \psi_{H,L,Q}^{\varphi}$.

We have $(\psi_{H,L}^X - \psi_{H,R}^X)(x_0, y) \equiv b(x_0)$ in $H(x_0)$ for some $b(x_0) \in \mathbb{C}$. We define $g = (\psi_{H,L,P}^{\varphi} - \psi_{H,R,Q}^{\varphi}) \circ (\psi_{H,L,P}^{\varphi})^{\circ(-1)} \circ 1/(2\pi i) \ln z$ in $\mathbb{C}^* = (e^{2\pi i z} \circ \psi_{H,L,P})(H(x_0))$. By lemma 8.7 the complex function g admits a continuous extension to $\mathbb{P}^1(\mathbb{C})$ such that $g(0) = b(x_0)$. We are done since then $g \equiv b(x_0)$. $\square$

Here it is important the choice $\rho(0) = 0$, $\rho(\infty) = \infty$, $\rho'(0) = 1$. By replacing ρ by the canonical choice $\tilde{\rho}(0) = 0$, $\tilde{\rho}(1) = 1$, $\tilde{\rho}(\infty) = \infty$ in the definition of $\psi_{H,L,P}^{\varphi}$ we would have $\psi_{H,L,P}^{\varphi} \not\equiv \psi_{H,L,Q}^{\varphi}$ in general.

Denote by $\psi_{H,L}^{\varphi}$ any of the functions $\psi_{H,L,P}^{\varphi}$ defined in H_L. The definition of $\psi_{H,R}^{\varphi}$ is analogous. We denote by $\psi_H^{\varphi} - \psi_H^X$ the function defined in H which is given by the expression $\psi_{H,l}^{\varphi} - \psi_{H,l}^X$ in H_l for $l \in \{L, R\}$. The definitions of $\psi_{H,L}^{\varphi}$, $\psi_{H,R}^{\varphi}$ and $\psi_H^{\varphi} - \psi_H^X$ allow to deduce asymptotic properties of those functions when approaching the fixed points without checking out that they are stable by iteration.

Proposition 8.2. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^{\mu} \subset \mathbb{S}^1 \backslash B_X^{\mu}$. Let $H \in Reg(\epsilon, \mu X, K_X^{\mu})$; the mappings $(x, \psi_{H,L}^{\varphi})$ and $(x, \psi_{H,R}^{\varphi})$ are holomorphic in $\dot{H}$ and continuous and injective in H_L and H_R respectively.*

Proof. Consider $P = (x_0, y_0) \in H^L$. The mapping $\sigma_0(x, z)$ depends holomorphically on x. There exists a continuous section $P(x_1) \in [x = x_1]$ for x_1 in a neighborhood V of x_0 in $[0, \delta_0)K_X^{\mu}$ such that $\psi_{H,L}^X(P(x_1)) = \psi_{H,L}^X(P)$ and $P(x_0) = P$. The mapping $\sigma = \psi_{H,L}^X \circ \sigma_0 \circ (\psi_{H,L}^X)^{\circ(-1)}$ maps $B(P(x))$ onto $B_1(P(x))$ and establishes a C^{∞} diffeomorphism from $\tilde{B}(P(x))$ onto $\tilde{B}_1(P(x))$ for all $x \in V$. The complex dilation $\chi_{\xi^{\circ(-1)}}$ depends holomorphically on $x \in \dot{V}$ and continuously on $x \in V$. Hence the dependance of the canonical solution $\tilde{\rho} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of $\chi_{\tilde{\rho}} = \chi_{\xi^{\circ(-1)}}$ with respect to x is continuous in V and holomorphic in $\dot{V}$. In particular the function $x \to (\partial\tilde{\rho}/\partial z)(x, 0)$ is holomorphic in $\dot{V}$ and continuous in V. We deduce that $\rho(x, z) = \tilde{\rho}(x, z)/(\partial\tilde{\rho}/\partial z)(x, 0)$ depends continuously on $x \in V$ and holomorphically on $x \in \dot{V}$. Then $\psi_{H,L}^{\varphi}$ is continuous in $\cup_{x \in V} B_1(P(x))$ and holomorphic in a neighborhood of $\cup_{x \in \dot{V}} B_1(P(x))$. Since P can be any point of H^L then $\psi_{H,L}^{\varphi}$ is holomorphic in $\dot{H}$ and continuous in H_L. Moreover $(x, \psi_{H,\upsilon}^{\varphi})$ is injective in H_{υ} for $\upsilon \in \{L, R\}$ since $\psi_{H,\upsilon}^{\varphi}$ is injective in the fundamental domains of type $B_1(P)$. □

Corollary 8.2. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^{\mu} \subset \mathbb{S}^1 \backslash B_X^{\mu}$. Let $H \in Reg(\epsilon, \mu X, K_X^{\mu})$. The function $x \to (\partial\rho/\partial z)(H, x, \infty)$ is well-defined and continuous in $[0, \delta_0)K_X^{\mu}$. It is holomorphic in $(0, \delta_0)\dot{K}_X^{\mu}$ and $(\partial\rho/\partial z)(H, 0, \infty) = 1$. Moreover we have $(\partial\rho/\partial z)(H, x, \infty) \equiv 1$ if $H \in Reg_1(\epsilon, \mu X, K_X^{\mu})$.*

Proof. By the proof of the previous proposition we have that $x \to (\partial\tilde{\rho}/\partial z)(x, 0)$ and $x \to (\partial\tilde{\rho}/\partial z)(x, \infty)$ are continuous in $[0, \delta_0)K_X^{\mu}$ and holomorphic in $(0, \delta_0)\dot{K}_X^{\mu}$. The same property is clearly fulfilled by $x \to (\partial\rho/\partial z)(x, \infty)$.

Consider $P = \exp(sX)(L_{\mu X}^H(0))$ if $H \in Reg_2(\epsilon, \mu X, K_X^{\mu})$ for all $s \in \mathbb{R}^+$. For $H \in Reg_1(\epsilon, \mu X, K_X^{\mu})$ consider $P = \exp(sX)(T_{\mu X}^H(0))$ for $s \in \mathbb{R}^+$ if $Re(-i\mu X)$ points towards H at $T_{\mu X}^H(0)$, otherwise we denote $P = \exp(-sX)(T_{\mu X}^H(0))$ for $s \in \mathbb{R}^+$. Then P is well-defined and belongs to $H^L(0) = H_L(0)$ for all $s \in \mathbb{R}^+$. Moreover $\inf_{Q \in B(P)} |\psi_{H,L}^X(Q)|$ tends to ∞ when $s \to \infty$. We obtain $||\chi_{\xi^{\circ(-1)}}||_{\infty} \to 0$ when $s \to 0$ by lemma 8.2. This implies $(\partial\rho/\partial z)(0, \infty) = 1$ by lemma 8.5. The prove of $(\partial\rho/\partial z)(x, \infty) \equiv 1$ in the case $H \in Reg_1(\epsilon, \mu X, K_X^{\mu})$ is analogous. □

Proposition 8.3. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^{\mu} \subset \mathbb{S}^1 \backslash B_X^{\mu}$. Let $H \in Reg(\epsilon, \mu X, K_X^{\mu})$; the function $\psi_H^{\varphi} - \psi_H^X$ is continuous in $H \cup [Fix\varphi \cap \partial H]$.*

Proof. The function $\psi_H^\varphi - \psi_H^X$ is clearly continuous in H. We define

$$(\psi_H^\varphi - \psi_H^X)(\alpha^{\mu X}(H(x))) = \frac{1}{2\pi i}\ln\frac{\partial\rho}{\partial z}(H,x,\infty) \quad \text{and} \quad (\psi_H^\varphi - \psi_H^X)(\omega^{\mu X}(H(x))) = 0$$

for all $x \in [0,\delta_0)K_X^\mu$ where $\ln 1 = 0$. The function $(\psi_H^\varphi - \psi_H^X)_{|Fix\varphi\cap\partial H}$ is continuous by corollary 8.2. Let $P \in H^L$. From

$$\psi_{H,L,P}^\varphi - \psi_{H,L}^X = (\psi_{H,L,P}^\varphi - \psi_{H,L}^X \circ \sigma^{\circ(-1)}) + (\psi_{H,L}^X \circ \sigma^{\circ(-1)} - \psi_{H,L}^X)$$

we deduce that

$$\left|\psi_H^\varphi - \psi_H^X - \frac{1}{2\pi i}\left(\ln\left(\frac{\rho}{z}\right)\circ e^{2\pi i z}\circ\psi_{H,L}^X\circ\sigma^{\circ(-1)}\right)\right| \leq \frac{C}{(1+|\psi_{H,L}^X|)^{1+1/\nu(\varphi)}}$$

in $B_1(P)$ for some $C>0$ independent of $P \in H^L$. We can suppose that the function ι provided by lemma 8.3 is increasing. By varying P we obtain

$$|\psi_H^\varphi - \psi_H^X| \leq \frac{1}{\pi}\iota\circ e^z(-\pi Img(\psi_{H,L}^X)) + \frac{C}{(1+|\psi_{H,L}^X|)^{1+1/\nu(\varphi)}}$$

in $H^L \cap [Img\psi_{H,L}^X > J_0]$ for some $J_0 > 0$. An analogous expression can be obtained in H^R by replacing $\psi_{H,L}^X$ with $\psi_{H,R}^X$. Lemma 8.3 implies the existence of an increasing ι' independent of $P \in H^L$ such that $\lim_{|z|\to 0}\iota'(|z|) = 0$ and $|\rho/z(\partial\rho/\partial z)(\infty)^{-1} - 1| \leq \iota'(1/|z|)$. We deduce that

$$\left|\psi_H^\varphi - \psi_H^X - \frac{1}{2\pi i}\ln\frac{\partial\rho}{\partial z}(x,\infty)\right| \leq \frac{1}{\pi}\iota'\circ e^z(\pi Img(\psi_{H,L}^X)) + \frac{C}{(1+|\psi_{H,L}^X|)^{1+1/\nu(\varphi)}}$$

in $H^L \cap [Img\psi_{H,L}^X < -J_1]$ for some $J_1 > 0$. Again an analogous expression can be obtained for H^R.

Consider $(x_0,y_0) \in Fix\varphi \cap \partial H$. Suppose $x_0 \neq 0$ and $H \in Reg_2(\epsilon,\mu X, K_X^\mu)$. Since $H = H^L \cup H^R$ and

$$\lim_{(x,y)\in H,\ (x,y)\to(x_0,y_0)} \min(|Img(\psi_{H,L}^X(x,y))|, |Img(\psi_{H,R}^X(x,y))|) = \infty$$

then the discussion in the previous paragraph implies that $\psi_H^\varphi - \psi_H^X$ is continuous at (x_0,y_0). Suppose now that $x_0 = 0$ or $H \in Reg_1(\epsilon,\mu X,K_X^\mu)$. We have

$$\lim_{(x,y)\in H_\kappa,\ (x,y)\to(x_0,y_0)} |\psi_{H,\kappa}^X(x,y)| = \infty$$

for $\kappa \in \{L,R\}$. It is enough to prove that $(\psi_H^\varphi - \psi_H^X)_{H^l\cup\{(x_0,y_0)\}}$ is continuous at (x_0,y_0) for $l \in \{L,R\}$. Suppose $l = L$ without lack of generality. Analogously to the previous case we obtain

$$\lim_{(x,y)\in H^L,\ |Img(\psi_{H,L}^X(x,y))|\to\infty,\ (x,y)\to(x_0,y_0)} (\psi_H^\varphi - \psi_H^X)(x,y) = 0.$$

There exists a function $\upsilon : \mathbb{R}^+ \to \mathbb{R}_{\geq 0}$ such that $\lim_{b\to\infty}\upsilon(b) = \infty$ and satisfying that given $P \in H^L$ the strip $B(P)$ is contained in $[|\psi_{H,L}^X| > \upsilon(|Re(\psi_{H,L}^X(P))|)]$. The value $||\rho/z - 1||_\infty$ tends to 0 when $||\chi_{\xi^{\circ(-1)}}||_{|\infty}$ by lemma 8.6. Moreover we have $||\chi_{\xi^{\circ(-1)}}||_{|\infty} \leq K(H)/(1+\upsilon(|Re(\psi_{H,L}^X(P))|))^{1+1/\nu(\varphi)}$ by lemma 8.2. The lemma 8.7 implies that

$$\lim_{(x,y)\in H^L,\ |Re(\psi_{H,L}^X(x,y))|\to\infty,\ (x,y)\to(x_0,y_0)} (\psi_H^\varphi - \psi_H^X)(x,y) = 0$$

and then the result is proved. □

The previous proposition implies that by considering a smaller domain of definition $|y| \leq \epsilon$ we can suppose that $sup_{Q \in H}|\psi_H^{\varphi} - \psi_H^X|(Q)$ is as small as desired for all $H \in Reg(\epsilon, \mu X, K_X^{\mu})$ since $(\psi_H^{\varphi} - \psi_H^X)(0,0) = 0$.

Remark 8.1. *The construction of $\psi_{H,L,P}^{\varphi}$ or $\psi_{H,R,P}^{\varphi}$ in $B_1(P)$ depends only on getting small values of $||\chi_{\xi^{\circ(-1)}}||_{\infty}$. This condition is automatically fulfilled for* $\sup_{B(0,\delta)\times B(0,\epsilon)} |\Delta_{\varphi}|$ *small enough (lemma 8.2).*

Remark 8.2. *We can do the same process in this section for every $\varphi \in \mathrm{Diff}_p(\mathbb{C}^2, 0)$ with convergent normal form $\exp(x^m Y)$ for some $Y \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$. The functions $\psi_H^{\varphi} - \psi_H^X$ are continuous in $H \cup (\overline{H} \cap Fix\varphi) \cup (\{0\} \times B(0,\epsilon))$ and holomorphic in $\dot{H}$ even if ψ_H^X has a pole of order m at $x = 0$. The calculations are basically the same. We omit them for the sake of simplicity.*

9. Defining the analytic invariants

Now we define an extension of the Ecalle-Voronin invariants for $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$. It is the key to prove the main theorems in this paper.

9.1. **Normalizing the Fatou coordinates.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set K_X^{μ} contained in $\mathbb{S}^1 \setminus B_X^{\mu}$. There are $2\nu(\varphi)$ continuous sections $T_X^{\epsilon,1}$, …, $T_X^{\epsilon,2\nu(\varphi)}$ of the set T_X^{ϵ}. We will always suppose that $T_X^{\epsilon,1}$, …, $T_X^{\epsilon,2\nu(\varphi)}$, $T_X^{\epsilon,2\nu(\varphi)+1} = T_X^{\epsilon,1}$ are ordered in counter clock-wise sense. For all $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$ there exists a function $\theta_j : B(0,\delta) \to \mathbb{R}^+$ such that

$$T_X^{\epsilon,j+1}(x) = T_X^{\epsilon,j}(x)e^{i\theta_j(x)} \text{ and } T_X^{\epsilon,j}(x)e^{i(0,\theta_j(x))} \cap T_X^{\epsilon}(x) = \emptyset \ \ \forall x \in B(0,\delta).$$

There exists a unique $T_{\mu X}^{\epsilon,j}(x)$ in $T_X^{\epsilon,j}(x)e^{i(0,\theta_j(x))}$. Denote by $\upsilon_j(x)$ the only value in $(0, 2\pi)$ such that $T_{\mu X}^{\epsilon,j+1}(x) = T_{\mu X}^{\epsilon,j}(x)e^{i\upsilon_j(x)}$. We define $H(j)$ as the element of $Reg(\epsilon, \mu X, K_X^{\mu})$ such that $T_{\mu X}^{\epsilon,j}(x) \in \partial H(j)(x)$ for all $x \in [0,\delta_0)K_X^{\mu}$. The region $H \in Reg_k(\epsilon, \mu X, K_X^{\mu})$ appears k times in the sequence $H(1)$, …, $H(2\nu(\varphi))$. We denote by $H_{\infty}(j)$ the element of $Reg_{\infty}(\epsilon, \mu X, K_X^{\mu})$ such that $T_X^{\epsilon,j+1}(x)$ belongs to $\partial(H_{\infty}(j)(x))$ for all $x \in [0,\delta_0)K_X^{\mu}$. Consider $BH(j) \subset [x \in [0,\delta_0)K_X^{\mu}]$ such that

$$BH(j)(x) = (H(j)(x) \cup H(j+1)(x) \cup \overline{H_{\infty}(j)(x)}) \cap ([|y| < \epsilon] \setminus Fix\varphi)$$

for $x \in [0,\delta_0)K_X^{\mu}$. It is a simply connected open set for every point $x \in (0,\delta_0)K_X^{\mu}$. The set $BH(j)(0)$ can have 1, 2 or 3 connected components. Denote by $GH(j)(0)$ the connected component of $BH(j)(0)$ containing $T_{\mu X}^{\epsilon,j}(0)$ in its closure. We define

$$GH(j) = (BH(j) \setminus [x=0]) \cup GH(j)(0).$$

It is connected and simply connected for all $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$ and $x \in [0,\delta_0)K_X^{\mu}$.

We define the function $\zeta_{\varphi}(x) = -\pi i \nu(\varphi)^{-1} \sum_{P \in (Fix\varphi)(x)} Res(\varphi, P)$. It is holomorphic in a neighborhood of 0. Fix $j_0 \in \{1, \ldots, 2\nu(\varphi)\}$. Consider an integral $\psi_{j_0}^X$ of the time form of X defined in the neighborhood of $T_{\mu X}^{\epsilon,j_0}(0)$. We can extend it to $GH(j_0)$ by analytic continuation. In an analogous way we can define $\psi_{j_0+k}^X$ in $GH(j_0+k)$ for all $k \in \mathbb{Z}$; we choose $\psi_{j_0+k}^X(T_{\mu X}^{\epsilon,j_0+k}(0))$ to be the result of evaluating the analytic extension of $\psi_{j_0}^X + k\zeta_{\varphi}$ along the curve $t \to T_{\mu X}^{\epsilon,j_0}(0)e^{it\kappa}$ for $t \in [0,1]$ where $\kappa = \sum_{l=0}^{k-1} \upsilon_{j_0+l}(0)$ if $k > 0$ and $\kappa = -\sum_{l=1}^{-k} \upsilon_{j_0-l}(0)$ for $k < 0$. If $Re(-i\mu X)$

points towards $H(j)$ at $T_{\mu X}^{\epsilon,j}(0)$ then we define $\psi_{H(j),L} = \psi_{H(j+1),R} = \psi_j^X$, otherwise we define $\psi_{H(j),R} = \psi_{H(j+1),L} = \psi_j^X$. We obtain $\psi_{j+2\nu(\varphi)}^X \equiv \psi_j^X$ for $j \in \mathbb{Z}$.

We choose an element $\gamma_1 \equiv (y = \alpha_1(x))$ of $Sing_V X$, we call γ_1 the *privileged curve* associated to X (or φ). We have $X = u(x,y)\prod_{j=1}^{N}(y-\alpha_j(x))^{n_j}\partial/\partial y$ for some unit $u \in \mathbb{C}\{x,y\}$. Denote by γ_j the curve $y = \alpha_j(x)$ for $2 \leq j \leq N(\varphi)$. We look for functions $c_1, \ldots, c_N$ contained in $C^0([0,\delta_0)K_X^\mu) \cap \vartheta((0,\delta_0)\dot{K}_X^\mu)$ such that

- $c_1 \equiv 0$
- Given $\gamma_j \xrightarrow{H} \gamma_k$ of $\mathcal{G}(\mu X, K_X^\mu)$ then $(c_j - c_k)(x) \equiv 1/(2\pi i)\ln(\partial\rho/\partial z)(H, x, \infty)$.

By corollary 8.2 the reflexive edges of $\mathcal{G}(\mu X, K_X^\mu)$ do not impose any restriction. There is a unique solution $c_1, \ldots, c_N$ since $\mathcal{N}G(\mu X, K_X^\mu)$ is connected. We say that $c_1, \ldots, c_N$ is a sequence of *privileged functions* associated to $(X, \varphi, K_X^\mu, \gamma_1)$.

Denote $\gamma_{k(j)} = \omega^{\mu X}(H(j))$. We define a Fatou coordinate ψ_j^φ of φ in the set $H(j) \cap GH(j)$ such that

$$\psi_j^\varphi(x,y) = \psi_{H(j),L}^\varphi(x,y) + c_{k(j)}(x) \quad \text{or} \quad \psi_j^\varphi(x,y) = \psi_{H(j),R}^\varphi(x,y) + c_{k(j)}(x).$$

depending on whether $H(j) \cap GH(j)$ is equal to $H(j)_L$ or $H(j)_R$ respectively. We obtain that $(\psi_j^\varphi - \psi_j^X)_{|\gamma_k} \equiv c_k$ for $\gamma_k \in \{\alpha^{\mu X}(H(j)), \omega^{\mu X}(H(j))\}$. Since given ψ_j^φ the function $\psi_j^\varphi + c(x)$ is also a Fatou coordinate we normalize by fixing a privileged curve and the sequence of privileged functions attached to such a choice.

9.2. **Defining the changes of charts.** Our aim is to define

$$\xi_{\varphi,K_X^\mu}^j(x,z) = \psi_{j+1}^\varphi \circ (x, \psi_j^\varphi)^{\circ(-1)}$$

for $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$. A priori it seems that this does not make any sense since the domains of definition of ψ_j^φ and ψ_{j+1}^φ are disjoint. Nevertheless we can extend those domains, the function $\xi_{\varphi,K_X^\mu}^j$ will be defined in a strip.

We denote $D(\varphi) = \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$. We define

$$D_1(\varphi) = \{j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z}) : Re(X) \text{ points at } T_{\mu X}^{\epsilon,j}(0) \text{ towards } H(j)\}.$$

The condition $j \in D_1(\varphi)$ is equivalent to $Re(-\mu X)$ pointing towards $|y| < \epsilon$ at $(\partial H_\infty(j) \cap [|y| = \epsilon]) \setminus T_{\mu X}^\epsilon$. We denote $D_{-1}(\varphi) = D(\varphi) \setminus D_1(\varphi)$.

Suppose without lack of generality that $j \in D_{-1}(\varphi)$. There exists a constant $W \in \mathbb{R}^+$ such that $|Re(\psi_j^X(B) - \psi_j^X(A))| < W$ for all $A, B \in H_\infty(j)(x)$ and all $x \in [0,\delta_0)K_X^\mu$. Denote $Im(x) = Img(\psi_j^X(T_X^{\epsilon,j+1}(x)))$. We obtain that every $Q \in H_\infty(j) \cap [Img\psi_j^X > Im]$ fulfills $[-W, W] \subset It(X, Q, |y| < \epsilon)$; we obtain

$$\exp((0,W)X)(Q) \cap H(j+1) \neq \emptyset \text{ and } \exp((-W,0)X)(Q) \cap H(j) \neq \emptyset.$$

Denote $\Gamma_l(x) = \Gamma(\mu X, T_{\mu X}^{\epsilon,l}(x), |y| \leq \epsilon)$. We define the strip $St_j(x)$ enclosed by Γ_j and $\varphi^{\circ(-1)}(St_j(x))$ whereas $St_{j+1}(x)$ is the strip enclosed by $\Gamma_{j+1}(x)$ and $\varphi(\Gamma_{j+1}(x))$ for all $x \in [0,\delta_0)K_X^\mu$.

The functions $\psi_l^\varphi - \psi_l^X$ are bounded in $H(l) \cap GH(l)$ and continuous at the curve $\omega^{\mu X}(BH(j))$ for $l \in \{j, j+1\}$ (prop. 8.3). Suppose that $\sup_{B(0,\delta)\times B(0,\epsilon)}|\Delta| < 1/2$. It is easy to see that ψ_j^φ can be defined by iteration in the set E_j given by

$$E_j(x) = ([St_{j+1}(x) \cup \overline{H_\infty(j)(x))}] \setminus Fix\varphi) \cap [Img(\psi_j^X) > Im(x) + 1 + W]$$

for $x \in [0,\delta_0)K_X^\mu$. The function $\psi_j^\varphi(x,.)$ is injective in the simply connected set $(H(j)(x) \cap GH(j)(x)) \cup E_j(x)$. Moreover since we only need a finite number of iterations the function $\psi_j^{\varphi} - \psi_j^X$ is still bounded in E_j and continuous at the curve $\omega^{\mu X}(BH(j))$. There exists $I \in \mathbb{R}^+$ such that $\xi_{\varphi,K_X^\mu}^j$ is defined in

$$[\cup_{x\in[0,\delta_0)K_X^\mu}\{x\} \times \psi_j^{\varphi}(St_{j+1}(x))] \cap [Img(z) > I].$$

Since we have $\xi_{\varphi,K_X^\mu}^j(x,z+1) = \xi_{\varphi,K_X^\mu}^j(x,z)+1$ then $\xi_{\varphi,K_X^\mu}^j$ is defined in $Imgz > I$. The value of

$$\psi_{j+1}^{\varphi} - \psi_j^{\varphi} = (\psi_{j+1}^{\varphi} - \psi_{j+1}^X) - (\psi_j^{\varphi} - \psi_j^X) + (\psi_{j+1}^X - \psi_j^X)$$

at the curve $\gamma_{k(j)} = \omega^{\mu X}(BH(j))$ is $c_{k(j)} - c_{k(j)} + \zeta_\varphi \equiv \zeta_\varphi$, thus $\xi_{\varphi,K_X^\mu}^j$ admits a expression of the type $\xi_{\varphi,K_X^\mu}^j(x,z) = z+\zeta_\varphi(x)+\sum_{l=1}^{\infty} a_{j,l,K_X^\mu}^{\varphi}(x)e^{2\pi i l z}$. In particular the function $a_{j,l,K_X^\mu}^{\varphi}$ is continuous in $[0,\delta_0)K_X^\mu$ and holomorphic in $(0,\delta_0)\dot{K}_X^\mu$ for all $l \in \mathbb{N}$. The case $j \in D_1(\varphi)$ is analogous. The previous discussion implies:

Proposition 9.1. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \backslash B_X^\mu$. Then there exists $I \in \mathbb{R}^+$ such that for all $s \in \{-1,1\}$ and $j \in D_s(\varphi)$ we have*

- $\xi_{\varphi,K_X^\mu}^j \circ (z+1) \equiv (z+1) \circ \xi_{\varphi,K_X^\mu}^j$.
- $\xi_{\varphi,K_X^\mu}^j \in C^0([0,\delta_0)K_X^\mu \times [sImgz < -I]) \cap \vartheta((0,\delta_0)\dot{K}_X^\mu \times [sImgz < -I])$.
- $\lim_{|Img(z)|\to\infty} \xi_{\varphi,K_X^\mu}^j(x,z) - (z+\zeta_\varphi(x)) = 0$.
- $\xi_{\varphi,K_X^\mu}^j$ *is of the form* $z+\zeta_\varphi(x)+\sum_{l=1}^{\infty} a_{j,l,K_X^\mu}^{\varphi}(x)e^{-2\pi i s l z}$.

Let $orb_{H,j}(\varphi)$ be the space of orbits of $\varphi_{|H(j)\cap GH(j)}$ for $H(j) \in Reg(\epsilon,\mu X,K_X^\mu)$. The mapping $\Theta_j : orb_{H,j}(\varphi) \to [0,\delta_0) \times \mathbb{P}^1(\mathbb{C})$ given by $\Theta_j \equiv (x, e^{2\pi i z} \circ \psi_j^{\varphi})$ is continuous everywhere and holomorphic outside of $x=0$. We define the *μ-space of orbits* of φ at K_X^μ as the variety obtained by considering an atlas composed of $2\nu(\varphi)$ charts $W_j \sim [0,\delta_0) \times \mathbb{P}^1(\mathbb{C})$ for $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$ and the $2\nu(\varphi)$ changes of charts $\Theta_{j+1} \circ \Theta_j^{\circ(-1)}$ identifying subsets of $orb_{H,j}(\varphi)$ and $orb_{H,j+1}(\varphi)$ for all $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$.

Let $j \in D_s(\varphi)$. The trajectory $t \to \exp(stX)(T_{\mu X}^{\epsilon,j}(0))$ (for $t \in \mathbb{R}^+$) adheres to a direction $\Lambda(\varphi,j) \in D_s(\varphi_{|x=0})$ when $t \to \infty$. The mapping $\Lambda(\varphi)$ is a bijection from $\mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$ to $D(\varphi_{|x=0})$. The restriction of the changes of charts to $x=0$ provide the Ecalle-Voronin invariants of $\varphi_{|x=0}$.

Corollary 9.1. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Then the functions $\xi_{\varphi,K_X^\mu}^j(0,z)$ ($j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$) are the changes of charts of $\varphi_{|x=0}$. Indeed we have $\xi_{\varphi,K_X^\mu}^j(0,z) \equiv \xi_{\varphi_{|x=0}}^{\Lambda(\varphi,j)}(z)$ for all $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$.*

We have extended the Ecalle-Voronin invariants to all the lines $x = cte$ in a neighborhood of $x=0$ even if in general they do not support elements of $\mathrm{Diff}_1(\mathbb{C},0)$.

Remark 9.1. *We can define $\xi_{\varphi,K}^j$ for every $\varphi \in \mathrm{Diff}_p(\mathbb{C}^2,0)$ whith convergent normal form of the form $\exp(x^m Y)$ for some $Y \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ (see remark 8.2).*

9.3. **Nature of the invariants.** Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. In our sectors $[0, \delta_0)K_X^\mu$ the direction $\mu \in \mathbb{S}^1$ providing the real flow $Re(\mu X)$ is fixed. The analogue in [16] is allowed to vary continuously. Such a thing is also possible with our approach.

More precisely we want to find connected sets $E \subset \mathbb{S}^1$ and a continuous function $\mu : E \to e^{i(0,\pi)}$ such that $\mu(\lambda) \notin B_{X,\lambda}$ (see subsection 7.3.1) for all $\lambda \in E$. A maximal set with respect to the previous property will be called a *maximal sector*. The idea is that for every compact connected set K contained in a maximal sector there exists $\delta_0(K) > 0$ such that $Re(\mu(\lambda)X)_{(r,\lambda,y)\in[0,\delta_0(K))\times K\times B(0,\epsilon)}$ has a simple stable behavior. Thus the maximal sectors provide sectorial domains of stability.

Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with convergent normal form $\exp(X)$. Given $x \in \lambda\mathbb{R}^+$ and μ_0, μ_1 in the same connected component of $e^{i(0,\pi)} \setminus B_{X,\lambda}$ we claim that there exists a compact connected neighborhood $K = K_X^{\mu_0} = K_X^{\mu_1}$ of λ in $\mathbb{S}^1$ such that $\xi^j_{\varphi,K_X^{\mu_0}} \equiv \xi^j_{\varphi,K_X^{\mu_1}}$ for all $j \in \mathbb{Z}$. Then we can define changes of charts $\xi^j_{\varphi,K}(x, z)$ which are continuous in $[x \in [0, \delta_0(K))K] \cap [sImgz < -I]$ and holomorphic in its interior for $j \in D_s(\varphi)$. We do not include a rigorous proof of the claim but only a sketch. Consider a path $e^{i[\theta_0,\theta_1]} \subset e^{i(0,\pi)} \setminus B_{X,\lambda}$ joining μ_0 and μ_1. We have

1. The elements $H(j, \mu) \in Reg(\epsilon, \mu X, K)$ depend continuously on $\mu \in e^{i[\theta_0,\theta_1]}$.
2. $\alpha^{\mu X}(H(j, \mu))$ and $\omega^{\mu X}(H(j, \mu))$ do not depend on $\mu \in e^{i[\theta_0,\theta_1]}$ for $H(j, \mu)$ in $Reg(\epsilon, \mu X, K)$.
3. ψ_j^φ is continuous in $H(j, \mu) \cap GH(j, \mu)$ and holomorphic in the interior $\forall j \in \mathbb{Z}$ and $\mu \in e^{i[\theta_0,\theta_1]}$.

The first property is a consequence of the continuous dependance of $T^\epsilon_{\mu X}$ and $Re(\mu X)$ with respect to $\mu \in \mathbb{S}^1$. The open character of the set points implies the second property since $e^{[\theta_0,\theta_1]}$ is connected.

Regarding the third property we can define ψ_j^φ in $H(j, e^{i\theta_0}) \cap GH(j, e^{i\theta_0})$ and then to extend it to $\cup_{\mu\in e^{i[\theta_0,\theta_1]}} H(j, \mu) \cap GH(j, \mu)$ by using $\psi_j^\varphi \circ \varphi = \psi_j^\varphi + 1$. The trickiest part of the proof is showing that $\psi_j^\varphi - \psi_j^X$ is continuous in

$$\partial(H(j, \mu) \cap GH(j, \mu)) \cap SingX \quad \forall \mu \in e^{i[\theta_0,\theta_1]}.$$

Since $(\psi_j^\varphi - \psi_j^X) \circ \varphi^{\circ(k)} = (\psi_j^\varphi - \psi_j^X) - \sum_{b=0}^{k-1} \Delta_\varphi \circ \varphi^{\circ(b)}$ the desired property is a consequence of $\Delta_\varphi \circ \varphi^{\circ(k)} = O(1/(k + \psi^X)^{1+1/\nu(\varphi)})$ and

$$\lim_{|Imgz|\to\infty} \sum_{k\in -z+\mathbb{R}e^{i[\theta_0-\upsilon,\theta_1+\upsilon]}} \frac{1}{|z + k|^{1+1/\nu(\varphi)}} = 0.$$

The previous discussion implies that given $x \in \lambda\mathbb{R}^+$ the choices of μ-spaces of orbits of φ at $\{\lambda\}$ are at most the number of connected components of $e^{i(0,\pi)} \setminus B_{X,\lambda}$.

It is remarkable that the dependance of $B_{X,\lambda}$ with respect to λ is not product-like. For instance $B_{\beta,\lambda e^{i\theta}}(X) = e^{-im_\beta\theta} B_{\beta,\lambda}(x)$ for a magnifying glass M_β associated to X. Hence the points of $B_{X,\lambda}$ turn at different speeds.

There are much simpler cases. Suppose $N(\varphi) = 2$. Let p be the order of contact between the two curves of fixed points. We have that

$$(y \circ \varphi - y)(x, y) = u(x, y)(y - \gamma_1(x))^{n_1}(y - \gamma_2(x))^{n_2}$$

for some unit $u \in \mathbb{C}\{x, y\}$. The order p is equal to $\nu(\gamma_1(x)-\gamma_2(x))$. The magnifying glasses associated to X are organised in a sequence M_0, M_{0b_1}, …, $M_{0b_1\ldots b_{p-1}}$. Denote $\beta(q) = 0b_1 \ldots b_{q-1}$ for $q \in \{1, \ldots, p\}$. Since the vector field $X_{\beta(q)}(1)$ has a unique singular point for $q < p$ then $B_{\beta(q)}(X) = \emptyset$ for $q < p$. Moreover $sum_{\beta(p)}$

(see subsection 7.3.1 for definition) is composed by two opposite different points. Hence we have $\sharp \tilde{B}_{\beta(p),\lambda}(X) = 1$ and then $\sharp \tilde{B}_{X,\lambda} = 1$ for all $\lambda \in \mathbb{S}^1$. The number of connected components of $e^{i(0,\pi)} \setminus B_{X,\lambda}$ being bounded by 2, we have that in general there are two choices of μ-space of orbits. The situation is analogous to the one described in [16].

We have $m_{\beta(p)} = p(n_1 + n_2 - 1)$ and

$$X_{\beta(p)}(\lambda) = \lambda^{p(n_1+n_2-1)} C(w - w_1)^{n_1} (w - w_2)^{n_2} \frac{\partial}{\partial w}$$

for some $C \in \mathbb{C}^*$ and $w_1, w_2 \in \mathbb{C}$ with $w_1 \neq w_2$. The maximal sectors are of the form $\lambda_0 e^{i(0,2\pi/m(\beta(p)))}$ for $\lambda_0 \in B_X^1$. Since $\sharp B_X^1 = 2m(\beta(p))$ then there are $2p(n_1+n_2-1)$ maximal sectors whose union is $\mathbb{S}^1$. Each of them supports a sectorial domain of angle $2\pi/(p(n_1 + n_2 - 1))$.

Suppose $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0) \setminus \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with $N = 2$. Then φ is of the form

$$\varphi(x, y) = (x, y + f(x, y)^n)$$

where $\nu(f(0, y)) = 2$. Denote by $p \in \mathbb{N}/2 \setminus \mathbb{N}$ the order of contact between the two branches of $Fix\varphi$. We can find an extension of the Fatou coordinates and then of the Ecalle-Voronin invariants of φ by studying the diffeomorphism

$$\tilde{\varphi} = (x^{1/2}, y) \circ \varphi \circ (x^2, y) \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$$

and then undoing the ramification. The order of contact between the the two irreducible components of $\tilde{\varphi}$ is $2p$. The constants n_1 and n_2 representing the multiplicities of the irreducible components are both equal to n. Thus the maximal sectors support sectorial domains of angle

$$2\frac{2\pi}{(2p)(n_1 + n_2 - 1)} = \frac{2\pi}{p(n_1 + n_2 - 1)} = \frac{2\pi}{p(2n - 1)}.$$

There are $2p(n_1 + n_2 - 1) = 2p(2n - 1)$ maximal sectors.

The situation described is analogous to that in [16]. They work with diffeomorphisms of the form $\varphi(x, y) = (x, y - x + c_1(x)y^2 + O(y^3))$ where $c_1(0) \neq 0$. They consider also its ramified version $\tilde{\varphi} = (w^{1/2}, y) \circ \varphi \circ (w^2, y)$. In the w coordinate we have $p = 1$ and $n_1 = n_2 = 1$. Then $2 = 2p(n_1 + n_2 - 1)$ sectors are required to cover $\mathbb{S}^1$ describing angles as close to 2π as desired. In the x coordinate we have $p = 1/2$ and $n_1 = n_2 = 1$. Only one sector is require to cover $\mathbb{S}^1$, it describes an angle as close to 4π as desired. We obtain the same division in the parameter space; nevertheless our techniques can be applied to every unfolding of tangent to the identity germs and not only to the generic ones.

9.4. **Embedding in a flow.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with fixed convergent normal form $\exp(X)$. We say that a sequence $K_X^{\mu_1}$, …, $K_X^{\mu_l}$ of compact connected subsets of $\mathbb{S}^1$ is a EV-covering if

- $\mu_j \in e^{i(0,\pi)}$ and $K_X^{\mu_j} \subset \mathbb{S}^1 \setminus B_X^{\mu_j}$ for all $j \in \{1, \dots, l\}$.
- $\cup_{j=1}^{l} \dot{K}_X^{\mu_j} = \mathbb{S}^1$.

Such a covering exists. We have $B_X^i \cap B_X^\kappa = \emptyset$ for $\kappa \in \mathbb{S}^1$ in the neighborhood of i. Fix such κ, then we can choose a EV-covering such that $\{\mu_1, \dots, \mu_l\} \subset \{i, \kappa\}$. This construction is a generalization of the trivial type case. In that context we can choose $K_X^i = \mathbb{S}^1$ as the only element of the EV-covering.

Remark 9.2. *The definition of EV-covering does not depend on the choice of the convergent normal form but on $Fix\varphi$ and $Res(\varphi)$ (remark 7.2).*

Proposition 9.2. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with fixed convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Suppose that φ is not analytically trivial. Then there exists $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$ such that $\xi_{\varphi,K_X^\mu}^j \not\equiv z + \zeta_\varphi$.*

The reciprocal is obvious, i.e. $\log \varphi \in \mathcal{X}(\mathbb{C}^2, 0)$ implies that $\xi_{\varphi,K_X^\mu}^j \equiv z + \zeta_\varphi$ for all the choices of K_X^μ and j.

Proof. Suppose it is not true. The functions $\psi_H^\varphi - \psi_H^X$ for $H \in Reg(\epsilon, \mu X, K_X^\mu)$ paste together in a function J defined in $([0, \delta_0)K_X^\mu \times B(0, \epsilon')) \setminus Fix\varphi$ and analytic in its interior for some $0 < \epsilon' < \epsilon$. Moreover J is continuous in $[0, \delta_0)K_X^\mu \times B(0, \epsilon')$ and analytic in its interior (prop. 8.3) and satisfies $J - J \circ \varphi = \Delta_\varphi$. By Cauchy's integral formula we obtain $|\partial J/\partial y| \leq M$ in $|y| < \epsilon'/2$ for some $M > 0$. We get

$$\frac{\partial \psi^\varphi}{\partial y} = \frac{\partial \psi^X}{\partial y} + \frac{\partial J}{\partial y} = \frac{1}{X(y)} + \frac{\partial J}{\partial y} = \left(\frac{X(y)}{1 + X(y)\partial J/\partial y} \right)^{-1}.$$

We define the vector field

$$X(K_X^\mu) = \frac{X(y)}{1 + X(y)\partial J/\partial y} \frac{\partial}{\partial y}.$$

Since $X(K_X^\mu)(\psi^\varphi) = 1$ then $\varphi = \exp(X(K_X^\mu))$ in $[0, \delta_0)K_X^\mu \times B(0, \epsilon_1)$ for some $0 < \epsilon_1 < \epsilon'/2$. Moreover by choosing ϵ_1 properly we obtain that $X(K_X^\mu)$ is of the form $X(y)(1 + X(y)A_1)\partial/\partial y$ for some $A_1 \in C^0([0, \delta_0)K_X^\mu \times B(0, \epsilon_1))$.

Consider a minimal EV-covering $K_1 = K_X^\mu$, $K_2 = K_X^{\mu_2}$, …, $K_l = K_X^{\mu_l}$. Consider K_b such that $\dot{K}_1 \cap \dot{K}_b \neq \emptyset$. We define $\psi_{H,L}^\varphi = \psi_{H,L}^X + J$ and $\psi_{H,R}^\varphi = \psi_{H,R}^X + J$ in $[0, \delta_0)(K_1 \cap K_b) \times B(0, \epsilon_1)$ for all $H \in Reg(\epsilon, \mu_b X, K_b)$. Since $J - J \circ \varphi = \Delta_\varphi$ then $\psi_{H,L}^\varphi$ and $\psi_{H,R}^\varphi$ are Fatou coordinates of φ for $H \in Reg(\epsilon, \mu_b X, K_b)$. We obtain $\xi_{\varphi,K_b}^j \equiv z + \zeta_\varphi$ for $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$ in $x \in \dot{K}_1 \cap \dot{K}_b$ and then in $x \in K_b$ by analytic continuation. Analogously to $X(K_1)$ we can construct a vector field $X(K_b)$ such that $\varphi = \exp(X(K_b))$ in $[0, \delta_0)K_b \times B(0, \epsilon_b)$ for some $\epsilon_b > 0$. Moreover the construction implies that $X(K_1) \equiv X(K_b)$ in $[0, \delta_0)(\dot{K}_1 \cap \dot{K}_b) \times B(0, \min(\epsilon_1, \epsilon_b))$. Finally we obtain $Y \in \mathcal{X}(\mathbb{C}^2, 0)$ of the form $Y = X(y)(1 + X(y)A)\partial/\partial y$ for some $A \in \mathbb{C}\{x, y\}$ such that $\varphi = \exp(Y)$. Since Y is nilpotent then $\log \varphi = Y$. $\square$

10. Applications

In this section we complete the task of classifying analytically the elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Moreover given $\varphi_1 \overset{sp}{\sim} \varphi_2$ we provide the formal power series developments of the conjugating diffeomorphisms. We also relate the analytic class of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ and the analytic classes of the elements of $\{\varphi_{|x=x_0}\}_{x_0 \in B(0,\delta_0)}$.

10.1. **Moderated conjugations.** We want to identify how an analytic conjugation between elements of $\mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ acts on the changes of charts. We remind the reader that $N(X)$ is the number of points in $(SingX)(x_0)$ for x_0 generic in a neighborhood of 0. Given $X \in \mathcal{X}(\mathbb{C}^2, 0)$ we say that a mapping κ defined in a neighborhood of $(SingX)(x_0)$ in $\mathbb{C}$ is x_0-*special* if $\kappa_{|(SingX)(x_0)} \equiv Id$. We just say that κ is special if the value of x_0 is implicit.

Lemma 10.1. *Let $X \in \mathcal{X}(\mathbb{C}^2,0)$ with $N(X) \geq 2$. Fix $r \geq 0$. There exists a function $R:(0,r) \to \mathbb{R}^+$ with $\lim_{b\to 0} R(b)=0$ such that all x_0-special r-moderated mapping κ is $r_1 R(r_1)$-moderated for all x_0 in a pointed neighborhood $V(r_1)$ of 0.*

Proof. Let $\gamma_1(x_0)$ and $\gamma_2(x_0)$ be two different points of $(SingX)(x_0)$. We define

$$\kappa_1(y) = \frac{\kappa((r-|\gamma_1(x_0)|)y + \gamma_1(x_0)) - \gamma_1(x_0)}{(r-|\gamma_1(x_0)|)(\partial\kappa/\partial y)(\gamma_1(x_0))}.$$

By construction κ_1 is a Schlicht function, i.e. it is univalent in $B(0,1)$, $\kappa_1(0)=0$ and $(\partial\kappa_1/\partial y)(0)=1$. Denote $\upsilon(x_0) = (\gamma_2(x_0)-\gamma_1(x_0))/(r-|\gamma_1(x_0)|)$. We have $\kappa_1(\upsilon(x_0)) = \upsilon(x_0)/(\partial\kappa/\partial y)(\gamma_1(x_0))$. This implies

$$(1-|\upsilon(x_0)|)^2 \leq \left|\frac{\partial\kappa}{\partial y}(\gamma_1(x_0))\right| \leq (1+|\upsilon(x_0)|)^2$$

by Koebe's distortion theorem (see [5], page 65). This leads us to

$$\sup_{y\in B(0,r_1)} |\kappa(y)| \leq (r-|\gamma_1(x_0)|)(\partial\kappa/\partial y)(\gamma_1(x_0)) \sup_{y\in B(0,A(r_1))} |\kappa_1(y)| + |\gamma_1(x_0)|$$

where $A(r_1) = (r_1+|\gamma_1(x_0)|)/(r-|\gamma_1(x_0)|)$. We obtain

$$\sup_{y\in B(0,r_1)} |\kappa(y)| \leq (r-|\gamma_1(x_0)|)(1+|\upsilon(x_0)|)^2 \frac{A(r_1)}{(1-A(r_1))^2} + |\gamma_1(x_0)|$$

again by Koebe's distortion theorem. The value $R(r_1)$ can be chosen as close to $r_1/(1-r_1/r)^2$ as desired. $\square$

The last lemma implies that in our context the existence of r-moderated and rR-moderated conjugations are equivalent concepts. The next result is the analogue of lemma 5.4 in the moderated setting.

Lemma 10.2. *Let $\varphi_1,\varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with common convergent normal form $\exp(X)$. There exist a neighborhood $V \subset \mathbb{C}$ of 0 and $D(r,R) \in \mathbb{R}^+$ such that a special rR-moderated mapping κ conjugating $(\varphi_1)_{|x=x_0}$, $(\varphi_2)_{|x=x_0}$ is of the form $y + X(y)(x_0,y)J_\kappa(y)$ where $\sup_{B(0,r)}|J_\kappa| < D(r,R)$ for all $x_0 \in V\setminus\{0\}$.*

Proof. Denote $X(y) = u(x,y)(y-\gamma_1(x))^{n_1}\dots(y-\gamma_N(x))^{n_N}$ where $u \in \mathbb{C}\{x,y\}$ is a unit. By hypothesis we have $\kappa = y + (y-\gamma_1(x_0))\dots(y-\gamma_N(x_0))A(y)$ for some $A \in \vartheta(B(0,r))$. By the modulus maximum principle we obtain

$$\sup_{B(0,r)}|A| = \lim_{s\to r} \sup_{y\in B(0,s)} \frac{|\kappa(y)-y|}{|(y-\gamma_1(x_0))\dots(y-\gamma_N(x_0))|} \leq \frac{r+R}{(r/2)^N}$$

for all x_0 in a pointed neighborhood of 0. We have that

$$\left|\frac{\partial\kappa}{\partial y}(\gamma_j(x_0)) - 1\right| \leq \frac{2^N(r+R)}{r^N} \prod_{k\in\{1,\dots,N\}\setminus\{j\}} |\gamma_j(x_0)-\gamma_k(x_0)|.$$

Fix $j \in \{1,\dots,N\}$. We claim that $(y-\gamma_j(x_0))^{n_j}$ divides κ. We can suppose $n_j>1$. Denote by ζ_1, ζ_2 and υ the germs of diffeomorphism induced by $(\varphi_1)_{|x=x_0}$, $(\varphi_2)_{|x=x_0}$ and κ respectively in the neighborhood of x_0. We have $\upsilon = Z_{\zeta_2}^{\lambda,t} \circ \hat{\sigma}(\zeta_1,\zeta_2)$ for some $t \in \mathbb{C}$ and $\lambda = (\partial\kappa/\partial y)(\gamma_j(x_0)) \in < e^{2\pi i/(n_j-1)} >$ (prop. 4.2). This implies $\lambda = 1$ for x_0 in a neighborhood of 0 since $N \geq 2$. We have that $y\circ\upsilon - y - t(\log\zeta_2)(y)$ belongs to $(y-\gamma_j(x_0))^{n_j+1}$. Thus $y\circ\kappa - y$ belongs to $(y-\gamma_j(x_0))^{n_j}$. Denote

$J_\kappa = (\kappa - y)/X(y)_{|x=x_0}$, it belongs to $\vartheta(B(0,r))$. Analogously than for A we obtain

$$\sup_{B(0,r)} |J_\kappa| \leq D(r,R) = \frac{(r+R)}{(r/2)^{\nu(X)+1}} \frac{1}{\inf_{B(0,r)} |u|}$$

for all x_0 in a pointed neighborhood of 0. $\square$

Lemma 10.3. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Fix r, R in $\mathbb{R}^+$ and $0 < r_1 < r$. There exist $M(r,R,r_1) \in \mathbb{R}^+$ and a neighborhood $V \subset \mathbb{C}$ of 0 such that a special rR-moderated mapping κ conjugating $\varphi_1(x_0, y)$ and $\varphi_2(x_0, y)$ satisfies*

$$\sup_{B(0,r_1)} \left| \frac{\partial \kappa}{\partial y} - 1 \right| \leq M(r,R,r_1)$$

for all $x_0 \in V \setminus \{0\}$. Moreover we have $\lim_{r_1 \to 0} M(r,R,r_1) = 0$.

Proof. Denote $X(y) = u(x,y) \prod_{j=1}^{N} (y - \gamma_j(x))^{n_j}$ where $u \in \mathbb{C}\{x,y\}$ is a unit. By lemma 10.2 we have that κ is of the form $y + A(y) \prod_{j=1}^{N} (y - \gamma_j(x))^{n_j}$ for some $A \in \vartheta(B(0,r))$. We have $\sup_{B(0,r)} |A| \leq H(r,R)$ for some $H(r,R) \in \mathbb{R}^+$ and all x_0 in a pointed neighborhood of 0. Fix $0 < r_1 < r$. We obtain

$$\left| \frac{\partial \kappa}{\partial y}(y) - 1 \right| \leq (\nu(X)+1)|A(y)|(2r_1)^{\nu(X)} + \left| \frac{\partial A}{\partial y} \right| (2r_1)^{\nu(X)+1}$$

for all $y \in B(0,r_1)$ and all x_0 in a pointed neighborhood $V(r_1)$ of 0. Cauchy's integral formula implies

$$\left| \frac{\partial \kappa}{\partial y}(y) - 1 \right| \leq H(r,R)(\nu(X)+1)(2r_1)^{\nu(X)} + \frac{H(r,R)}{r - r_1} (2r_1)^{\nu(X)+1}$$

for $y \in B(0,r_1)$. We define $M(r,R,r_1)$ as the right hand side of the previous formula. Clearly we have $\lim_{r_1 \to 0} M(r,R,r_1) = 0$. $\square$

Last lemma implies that given a special rR-moderated conjugation κ we can suppose that $\sup_{B(0,r)} |\partial \kappa / \partial y - 1|$ is as small as desired just by considering a smaller $r > 0$. We will make this kind of assumption without stressing it every time. We define $\kappa_t(y) = y + t(\kappa(y) - y)$ for $y \in B(0,r)$ and $t \in \mathbb{C}$.

Lemma 10.4. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Fix $r, R \in \mathbb{R}^+$. There exist $0 < r_1 < r$ and an open set $0 \in V \subset \mathbb{C}$ such that for all special rR-moderated mapping κ conjugating $\varphi_1(x_0, y)$ and $\varphi_2(x_0, y)$ and all $x_0 \in V \setminus \{0\}$ we have that κ_t is a $r_1 R$-moderated mapping for all $t \in B(0,2)$.*

Proof. We can choose $0 < r_1 < \min(r, R/7)$ such that $\sup_{B(0,r_1)} |\partial \kappa / \partial y - 1| \leq 1/4$ by lemma 10.3. Therefore we obtain $\sup_{B(0,r_1)} |\kappa| \leq 2r_1$ for all x_0 in a pointed neighborhood $V(r_1)$ of 0. This implies $\sup_{B(0,r_1)} |\kappa_t| \leq 7r_1 < R$ for all $t \in B(0,2)$. Moreover since $\sup_{B(0,r_1)} |\partial \kappa_t / \partial y - 1| \leq 1/2$ then κ_t is injective and hence a $r_1 R$-moderated mapping for all $t \in B(0,2)$. $\square$

Let ψ^X be a holomorphic integral of the time form of X. We can define the function $\psi^X \circ \kappa(x,y) - \psi^X(x,y)$ in an analogous way than Δ_φ. The continuous path that we use to extend ψ^X is parameterized by $t \to \kappa_t(x,y)$ for $t \in [0,1]$. The function $\psi^X \circ \kappa - \psi^X$ is well-defined and holomorphic in $B(0,r) \setminus SingX$.

Lemma 10.5. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Fix r, R in $\mathbb{R}^+$. There exist $0 < r_1 < r$ and $C(r, R) > 0$ such that for all special rR-moderated mapping κ conjugating $\varphi_1(x_0, y)$ and $\varphi_2(x_0, y)$ we have that $\sup_{B(0,r_1)} |\psi^X \circ \kappa - \psi^X| \leq C(r, R)$ for all x_0 in a pointed neighborhood of 0. In particular we obtain that $\psi^X \circ \kappa - \psi^X$ belongs to $\vartheta(B(0, r_1))$.*

Proof. Denote $X(y) = u(x, y)(y - \gamma_1(x))^{n_1} \dots (y - \gamma_N(x))^{n_N}$ where $u \in \mathbb{C}\{x, y\}$ is a unit. Consider a positive real number $H(r, R)$ such that

$$\sup_{B(0,r)} \left| \frac{\kappa - y}{\prod_{j=1}^{N} (y - \gamma_j(x_0))^{n_j}} \right| \leq H(r, R).$$

Therefore we obtain

$$\left| \frac{\partial \kappa_t}{\partial t}(y) \right| \leq \frac{H(r, R)}{|u \circ \kappa_t(y)|} |X(y) \circ \kappa_t(y)| \left| \frac{\prod_{j=1}^{N} (y - \gamma_j(x_0))^{n_j}}{\prod_{j=1}^{N} (y - \gamma_j(x_0))^{n_j} \circ \kappa_t(y)} \right|$$

for all $y \in B(0, r) \setminus (Sing X)(x_0)$. Denote $C(r, R) = 2^{\nu(X)+1} H(r, R) / \inf_{B(0,R)} |u|$. Since $\nu(X) \geq 1$ there exists $0 < r_1 < r_2 < r$ and a neighborhood V of 0 such that $\exp(B(0, C(r, R))X)(V \times B(0, r_1)) \subset V \times B(0, r_2)$ and

$$\frac{1}{2} \leq \sup_{B(0,r_2)} \left| \frac{y - \gamma_j(x_0)}{(y - \gamma_j(x_0)) \circ \kappa_t} \right| \leq 2.$$

for all $t \in [0, 1]$ and $j \in \{1, \dots, N\}$. The previous discussion implies

$$\left| \frac{\partial \kappa_t}{\partial t}(y) \right| \leq C(r, R) |X(y) \circ \kappa_t(y)|$$

for all $y \in B(0, r_2) \setminus (Sing X)(x_0)$ and $t \in [0, 1]$. As a consequence we obtain $|\psi^X \circ \kappa - \psi^X|(y) \leq C(r, R)$ for all $y \in B(0, r_1) \setminus (Sing X)(x_0)$. By Riemann's theorem $\psi^X \circ \kappa - \psi^X$ belongs to $\vartheta(B(0, r_1))$. □

The nexts results are important. Later on they will allow us to establish the connection between the formal and analytic conjugations.

Lemma 10.6. *Let $Y \in \mathcal{X}(\mathbb{C}, 0)$. Consider an integral of the time form ψ of Y. Suppose that $\kappa \in \text{Diff}(\mathbb{C}, 0)$ satisfies that $\psi \circ \kappa - \psi$ belongs to $\mathbb{C}\{y\}$. Then we have*

$$\frac{\partial \kappa}{\partial y}(0) = e^{(\psi \circ \kappa - \psi)(0) \frac{\partial Y(y)}{\partial y}(0)}.$$

Supposed $(\partial Y(y)/\partial y)(0) = 0$ we also obtain

$$\frac{\partial^{\nu(Y)+1} \kappa}{\partial y^{\nu(Y)+1}}(0) = (\psi \circ \kappa - \psi)(0) \frac{\partial^{\nu(Y)+1} Y(y)}{\partial y^{\nu(Y)+1}}(0)$$

and $(\partial^j \kappa / \partial y^j)(0) = 0$ for all $2 \leq j \leq \nu(Y)$.

Proof. Denote $\lambda = (\partial Y(y)/\partial y)(0)$. We have that $\psi \circ \kappa - \psi$ is of the form $d + L(y)$ for some $d \in \mathbb{C}$ and $L \in (y)$. Suppose $\lambda \neq 0$. Then ψ is of the form $(\ln y)/\lambda + B(y)$ in the neighborhood of 0 where $B \in \mathbb{C}\{y\}$. Therefore we obtain $d = (\ln(\partial \kappa / \partial y)(0))/\lambda$. Suppose $\lambda = 0$. We obtain $\kappa(y) = \exp((d + t)Y(y)\partial/\partial y)(y, L(y))$. This implies $\kappa(y) = y + dY(y) + O(y^{\nu(Y)+2})$. The result is a consequence of last formula. □

Every $\phi \in \text{Diff}(\mathbb{C}, 0)$ such that $(\partial \phi / \partial y)(0)$ is not in $e^{2\pi i \mathbb{Q}} \setminus \{1\}$ has a convergent normal form. If the linear part is the identity is a consequence of proposition 3.3. Otherwise it is clear since ϕ is formally linearizable.

Corollary 10.1. *Let $\phi \in \text{Diff}\,(\mathbb{C},0) \setminus \{Id\}$ such that $(\partial\phi/\partial y)(0) \notin e^{2\pi i \mathbb{Q}} \setminus \{1\}$. Consider a convergent normal form $\exp(Y)$ of ϕ. Let ψ a holomorphic integral of the time form of Y. Suppose that $\psi \circ \upsilon - \psi$ belongs to $\mathbb{C}\{y\} \cap (y)$ for some $\upsilon \in Z(\phi)$. Then we have $\upsilon \equiv Id$.*

Proof. By lemma 10.6 we have $j^1\upsilon \equiv Id$. Moreover, if $(\partial\phi/\partial y)(0) \neq 1$ then $\upsilon \equiv Id$ (prop. 4.2). Suppose $(\partial\phi/\partial y)(0) = 1$, then we have $y \circ \upsilon - y \in (y^{\nu(Y)+2})$ (lemma 10.6). We obtain $\upsilon = \hat{\sigma}(\phi,\phi) = Id$. □

Lemma 10.7. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2,0)$ with common normal form $\exp(X)$. Fix $\gamma \equiv (y = \gamma_1(x)) \in Sing_V X$ and $\hat{c} \in \mathbb{C}[[x]]$. Then we have*

$$\frac{\partial(\exp(\hat{c}\log\varphi_2) \circ \hat{\sigma}(\varphi_1,\varphi_2,\gamma))}{\partial y}(x,\gamma_1(x)) \equiv e^{\hat{c}(x)\frac{\partial X(y)}{\partial y}(x,\gamma_1(x))}.$$

Supposed $(\partial X(y)/\partial y)(x,\gamma_1(x)) \equiv 0$ we also obtain

$$\frac{\partial^{\nu_X(\gamma)+1}(\exp(\hat{c}\log\varphi_2) \circ \hat{\sigma}(\varphi_1,\varphi_2,\gamma))}{\partial y^{\nu_X(\gamma)+1}}(x,\gamma_1(x)) \equiv \hat{c}(x)\frac{\partial^{\nu_X(\gamma)+1}X(y)}{\partial y^{\nu_X(\gamma)+1}}(x,\gamma_1(x))$$

and $(\partial^j(\exp(\hat{c}\log\varphi_2) \circ \hat{\sigma}(\varphi_1,\varphi_2,\gamma))/\partial y^j)(x,\gamma_1(x)) \equiv 0$ for all $2 \leq j \leq \nu_X(\gamma)$.

Proof. Since $y \circ \hat{\sigma}(\varphi_1,\varphi_2,\gamma) - y \in I(\gamma)^{\nu_X(\gamma)+2}$ and $\hat{\sigma}(\varphi_1,\varphi_2,\gamma)$ and $\exp(\hat{c}(x)\log\varphi_2)$ are special then it is enough to prove the result for $\exp(\hat{c}(x)\log\varphi_2)$. We denote $\hat{X} = \log\varphi_2$, the equation

$$\sum_{j=0}^{\infty}\frac{\hat{c}(x)^j}{j!}\frac{\partial\hat{X}^{\circ(j)}(y)}{\partial y}(x,\gamma_1(x)) \equiv \sum_{j=0}^{\infty}\frac{\hat{c}(x)^j}{j!}\frac{\partial X(y)}{\partial y}(x,\gamma_1(x))^j \equiv e^{\hat{c}(x)\frac{\partial X(y)}{\partial y}(x,\gamma_1(x))}$$

implies the first part of the lemma. Suppose $(\partial X(y)/\partial y)(x,\gamma_1(x)) \equiv 0$. Since $\hat{X}(y) - X(y) \in (y \circ \varphi_2 - y)^2 \subset (y - \gamma_1(x))^{2\nu_X(\gamma)+2}$ then we obtain

$$y \circ \exp(\hat{c}(x)\log\varphi_2) - y = \hat{c}(x)X(y) + O((y-\gamma_1(x))^{\nu_X(\gamma)+2}).$$

The rest of the proof is trivial. □

10.2. **Analytic classification and centralizer.** Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2,0)$ with common convergent normal form. Given a special $\hat{\eta} \in \widehat{\text{Diff}}\,(\mathbb{C}^2,0)$ conjugating them we express the condition $\hat{\eta} \in \text{Diff}\,(\mathbb{C}^2,0)$ in terms of the changes of charts.

Proposition 10.1. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2,0)$ with common convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Fix a privileged curve $y = \gamma_1(x)$ associated to X. Consider a special r-moderated mapping κ conjugating $(\varphi_1)_{|x=x_0}$ and $(\varphi_2)_{|x=x_0}$. Then we have*

$$\xi^j_{\varphi_2,K_X^\mu}(x_0,z) = (z + c(x_0)) \circ \xi^j_{\varphi_1,K_X^\mu}(x_0,z) \circ (z - c(x_0)) \quad \forall j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})$$

for all $x_0 \in (0,\delta_0)K_X^\mu$ where $c(x_0) = (\psi^X \circ \kappa - \psi^X)(x_0,\gamma_1(x_0))$.

Proof. Suppose that κ is rR-moderated by considering a smaller $0 < r < \epsilon$ if necessary (lemma 10.1). Denote $X = u(x,y)(y-\gamma_1(x))^{n_1}\ldots(y-\gamma_N(x))^{n_N}\partial/\partial y$ where $u \in \mathbb{C}\{x,y\}$ is a unit. Let $c_1^l, \ldots, c_N^l$ be the privileged functions associated to the 4-uple $(X,\varphi_l,K_X^\mu,\gamma_1)$ for $l \in \{1,2\}$. Consider the sections $T_{\mu X}^{\epsilon,1}, \ldots, T_{\mu X}^{\epsilon,2\nu(X)}$. Denote by $H(j)$ the unique element of $Reg(\epsilon,\mu X,K_X^\mu)$ such that $T_{\mu X}^{\epsilon,j}(x) \in \partial H(j)(x)$ for all $x \in [0,\delta_0)K_X^\mu$. Let $0 < r_1 < r$ be the constant provided by lemma 10.5. We choose

r_1 such that $\exp(B(0,C(r,R))X)(|y|<r_1) \subset (|y|<\epsilon)$, we obtain $\kappa(H(j)') \subset H(j)$ for all $j \in \mathbb{Z}$ where $H(j)'$ is the element of $Reg(r_1, \mu X, K_X^{\mu})$ contained in $H(j)$.

We define $\phi_j^{\varphi_1} = \psi_j^{\varphi_2} \circ \kappa$ for $j \in \mathbb{Z}$. Since

$$\phi_j^{\varphi_1} - \psi_j^X = (\psi_j^{\varphi_2} - \psi_j^X) \circ \kappa + (\psi_j^X \circ \kappa - \psi_j^X)$$

then $\phi_j^{\varphi_1} - \psi_j^X$ is continuous in $H(j)'(x_0) \cup (\partial H(j)'(x_0) \cap SingX)$ by proposition 8.3 and lemma 10.5. Therefore $(\phi_j^{\varphi_1} - \psi_j^{\varphi_1})(x_0,y)$ is continuous in $\partial H(j)'(x_0) \cap SingX$ and then constant. Clearly $\phi_j^{\varphi_1}$ can be extended by iteration to a Fatou coordinate of φ_1 in $H(j)(x_0)$. We have that $\alpha^{\mu X}(H(j))$ and $\omega^{\mu X}(H(j))$ are equal to curves $y=\gamma_{k(j,\alpha)}(x)$ and $y=\gamma_{k(j,\omega)}(x)$ respectively. We obtain

$$\lim_{y \to \gamma_k(x_0)} (\phi_j^{\varphi_1} - \psi_j^X)(x_0,y) = c_k^2(x_0) + (\psi^X \circ \kappa - \psi^X)(x_0, \gamma_k(x_0))$$

where $k \in \{k(j,\alpha), k(j,\omega)\}$. We deduce that

$$\lim_{y \to \gamma_k(x_0)} (\phi_j^{\varphi_1} - \psi_j^{\varphi_1})(x_0,y) = c_k^2(x_0) - c_k^1(x_0) + (\psi^X \circ \kappa - \psi^X)(x_0, \gamma_k(x_0))$$

for $k \in \{k(j,\alpha), k(j,\omega)\}$. Since $(\phi_j^{\varphi_1} - \psi_j^{\varphi_1})(x_0,y)$ is constant then

$$c_{k(j,\upsilon)}^2(x_0) - c_{k(j,\upsilon)}^1(x_0) + (\psi^X \circ \kappa - \psi^X)(x_0, \gamma_{k(j,\upsilon)}(x_0))$$

does not depend on $\upsilon \in \{\alpha, \omega\}$. The graph $\mathcal{G}(\mu X, K_X^{\mu})$ is connected (prop. 7.7), hence $c_k^2(x_0) - c_k^1(x_0) + (\psi^X \circ \kappa - \psi^X)(x_0, \gamma_k(x_0))$ does not depend on $k \in \{1, \ldots, N\}$. In particular we obtain that $(\phi_j^{\varphi_1} - \psi_j^{\varphi_1})(x_0,y)$ is equal to the constant function $c(x_0)$ for all $j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})$. By construction we get

$$\xi_{\varphi_2, K_X^{\mu}}^j(x_0,z) = \phi_{j+1}^{\varphi_1} \circ \left(\phi_j^{\varphi_1}\right)^{\circ(-1)}(x_0,z) = (z + c(x_0)) \circ \xi_{\varphi_1, K_X^{\mu}}^j(x_0,z) \circ (z - c(x_0))$$

for all $j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})$ as we wanted to prove. $\square$

Proposition 10.2. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with common convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^{\mu} \subset \mathbb{S}^1 \setminus B_X^{\mu}$. Fix a privileged curve $y=\gamma_1(x)$ associated to X and a constant $M>0$. Suppose that*

$$\xi_{\varphi_2, K_X^{\mu}}^j(x_0,z) = (z + c(x_0)) \circ \xi_{\varphi_1, K_X^{\mu}}^j(x_0,z) \circ (z - c(x_0)) \quad \forall j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})$$

for some $x_0 \in [0,\delta_0)K_X^{\mu}$ and $c(x_0) \in B(0,M)$. Then there exists a special r-moderated mapping κ such that $\kappa \circ (\varphi_1)_{|x=x_0} = (\varphi_2)_{|x=x_0} \circ \kappa$. The constant $r \in \mathbb{R}^+$ does not depend on x_0. Moreover we get $(\psi^X \circ \kappa - \psi^X)(x_0, \gamma_1(x_0)) = c(x_0)$.

Proof. Consider the notations in proposition 10.1. We want to define

$$\kappa(y) = \left(\psi_j^{\varphi_2}\right)^{\circ(-1)} \circ (x_0, z + c(x_0)) \circ \psi_j^{\varphi_1}(x_0,y)$$

for $j \in \mathbb{Z}$. There exists $A \in \mathbb{R}^+$ such that $\sup_{H(j)} |\psi_j^{\varphi_l} - \psi_j^X| \leq A$ for $l \in \{1,2\}$ (prop. 8.3). We have $\exp(B(2A+M)X)(|y|<R) \subset (|y|<\epsilon)$ for some $R \in \mathbb{R}^+$ Let E be the union of the elements of $Reg(R, \mu X, K_X^{\mu})$. We deduce that κ is well-defined in $E(x_0)$ and satisfies $\sup_{E(x_0)} |\psi^X \circ \kappa - \psi^X| < 2A+M$, in particular we have $\kappa(E(x_0)) \subset B(0,\epsilon)$. Denote $D = \max_{l \in \{1,2\}, s \in \{-1,1\}} \sup_{B(0,R)} |\Delta_{\varphi_l^{\circ(s)}}|$. There exist $0<r<R$ and $B \in \mathbb{N}$ such that for all $J \in Reg_{\infty}(r, \mu X, K_X^{\mu})$ we have

- $\cup_{k \in \{-B,\ldots,B\}} \{\varphi_1^{\circ(k)}(P)\} \subset (|y|<R)$ for all $P \in \overline{J} \setminus SingX$.
- $\exists 0 \leq k_0, k_1 \leq B$ such that $\{\varphi_1^{\circ(-k_0)}(P), \varphi_1^{\circ(k_1)}(P)\} \subset E \ \ \forall P \in \overline{J} \setminus SingX$.
- $\exp((2A+M+2BD)X)(|y|<r) \subset (|y|<R)$.

We can define κ in $\overline{J}(x_0) \setminus SingX$ as either $\varphi_2^{\circ(k_0)} \circ \kappa \circ \varphi_1^{\circ(-k_0)}$ or $\varphi_2^{\circ(-k_1)} \circ \kappa \circ \varphi_1^{\circ(k_1)}$. By the construction and the hypothesis κ is a well-defined holomorphic mapping in $B(0,r) \setminus (SingX)(x_0)$ conjugating $(\varphi_1)_{|x=x_0}$ and $(\varphi_2)_{|x=x_0}$. Moreover, we have $sup_{B(0,r)}|\psi^X \circ \kappa - \psi^X| < 2A + M + 2BD$. As a consequence we can extend κ to $B(0,r)$ in a continuous (and then holomorphic) way by defining $\kappa_{|(SingX)(x_0)} \equiv Id$. The mapping κ satisfies $\kappa(B(0,r)) \subset B(0,R)$. Analogously by defining

$$\kappa^{\circ(-1)}(y) = \left(\psi_j^{\varphi_1}\right)^{\circ(-1)} \circ (x_0, z - c(x_0)) \circ \psi_j^{\varphi_2}(x_0, y)$$

for $j \in \mathbb{Z}$ we obtain a mapping $\kappa^{\circ(-1)} : B(0,r') \to B(0,R')$ conjugating $(\varphi_2)_{|x=x_0}$ and $(\varphi_1)_{|x=x_0}$. By taking $R \leq r'$ in the construction of κ we obtain that κ is a rR-moderated mapping. □

The next theorem is the analogue of proposition 6.1 in the non-trivial type case.

Theorem 10.1. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Consider a privileged curve $\gamma \equiv (y = \gamma_1(x))$ in $Sing_V X$. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if there exists $d \in \mathbb{C}\{x\}$ such that*

$$\xi^j_{\varphi_2, K_X^\mu}(x,z) \equiv (z + d(x)) \circ \xi^j_{\varphi_1, K_X^\mu} \circ (x, z - d(x)) \quad \forall j \in \mathbb{Z}/(2\nu(X)\mathbb{Z}).$$

The previous equation is equivalent to $\exp(d(x)\log\varphi_2) \circ \hat{\sigma}(\varphi_1, \varphi_2, \gamma) \in \text{Diff}\,(\mathbb{C}^2, 0)$

Proof. Implication $\Rightarrow$. Let σ be a special mapping conjugating φ_1 and φ_2. Denote $c(x) \equiv (\psi^X \circ \sigma - \psi^X)(x, \gamma_1(x))$, we have $c \in \mathbb{C}\{x\}$ (lemma 10.5). We deduce that

$$\xi^j_{\varphi_2, K_X^\mu}(x,z) \equiv (z + c(x)) \circ \xi^j_{\varphi_1, K_X^\mu}(x,z) \circ (x, z - c(x)) \quad \forall j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})$$

by proposition 10.1. The mapping σ is of the form $\exp(\hat{c}(x)\log\varphi_2) \circ \hat{\sigma}(\varphi_1, \varphi_2, \gamma)$ (lemma 5.2). Lemmas 10.6 and 10.7 imply $\hat{c} \equiv c$.

Implication $\Leftarrow$. Fix an EV-covering $K_X^{\mu_1} = K_X^\mu$, $K_X^{\mu_2}$, …, $K_X^{\mu_l}$. Supposed

$$\xi^j_{\varphi_2, K_X^{\mu_p}}(x,z) \equiv (z + d(x)) \circ \xi^j_{\varphi_1, K_X^{\mu_p}}(x,z) \circ (x, z - d(x)) \quad \forall j \in \mathbb{Z}/(2\nu(X)\mathbb{Z}) \tag{9}$$

for some $p \in \{1, \ldots, l\}$ we can define a continuous special mapping $\sigma_p(x,y)$ in the set $[0,\delta_0)K_X^\mu \times B(0,r)$ such that it is holomorphic in $(0,\delta_0)\dot{K}_X^\mu \times B(0,r)$, it conjugates φ_1 and φ_2 and $\sigma_p(x_0, y)$ is rR-moderated for all $x_0 \in [0,\delta_0)K_X^\mu$ and some $r, R \in \mathbb{R}^+$ (see proof of prop. 10.2). Moreover we obtain $(\psi^X \circ \sigma_p - \psi^X)(x, \gamma_1(x)) \equiv d(x)$.

The existence of σ_1 and proposition 10.1 imply that

$$\xi^j_{\varphi_2, K_X^{\mu_q}}(x_0,z) \equiv (z + d(x_0)) \circ \xi^j_{\varphi_1, K_X^{\mu_q}}(x_0,z) \circ (z - d(x_0))$$

for all $j \in \mathbb{Z}$ and for all $x_0 \in (0,\delta_0)(\dot{K}_X^{\mu_1} \cap \dot{K}_X^{\mu_q})$. By analytic continuation we obtain the same result for $x_0 \in [0,\delta_0)\dot{K}_X^{\mu_q}$ if $\dot{K}_X^{\mu_1} \cap \dot{K}_X^{\mu_q} \neq \emptyset$. The iteration of this process shows that the equation 9 is fulfilled for all $q \in \{1, \ldots, l\}$ and $x_0 \in [0,\delta_0)K_X^{\mu_q}$.

Suppose $\dot{K}_X^{\mu_p} \cap \dot{K}_X^{\mu_q} \neq \emptyset$ for $p, q \in \{1, \ldots, l\}$. Denote $h = (\sigma_q)^{\circ(-1)} \circ \sigma_p$. We obtain $h \circ \varphi_1 = \varphi_1 \circ h$ in $x \in [0,\delta_0)(\dot{K}_X^{\mu_p} \cap \dot{K}_X^{\mu_q})$ and $(\psi^X \circ h - \psi^X)(x, \gamma_1(x)) \equiv 0$. Corollary 10.1 implies $h(x,y) \equiv Id$ and then $\sigma_p \equiv \sigma_q$ in $[0,\delta_0)(\dot{K}_X^{\mu_p} \cap \dot{K}_X^{\mu_q}) \times B(0,r)$. Thus all the σ_b ($b \in \{1, \ldots, l\}$) paste together in a mapping σ such that it is continuous in $B(0,\delta_0) \times B(0,r)$ and holomorphic in $(B(0,\delta_0) \setminus \{0\}) \times B(0,r)$. By Riemann's theorem σ is a special element of $\text{Diff}\,(\mathbb{C}^2, 0)$ conjugating φ_1 and φ_2. Moreover we have $\sigma = \exp(d(x)\log\varphi_2) \circ \hat{\sigma}(\varphi_1, \varphi_2, \gamma)$ by the first part of the proof. □

Remark 10.1. *The previous theorem is fulfilled also if* $\varphi_1, \varphi_2 \in \mathrm{Diff}_{up}(\mathbb{C}^2, 0)$ *have convergent normal form* $\exp(x^m Y)$ *for some* $Y \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$*. The only difference is that the condition* $d \in \mathbb{C}\{x\}$ *should be replaced with* $x^m d \in \mathbb{C}\{x\}$.

Proposition 10.3. *Let* $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ *such that* $\log \varphi \notin \mathcal{X}_{p1}(\mathbb{C}^2, 0)$ *and* $Fix\varphi$ *is not of trivial type. Then there exists* $q \in \mathbb{N}$ *such that* $Z_{sp}(\varphi) =< \exp(q^{-1} \log \varphi) >$.

Proof. We can suppose $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ up to a ramification (x^k, y). Let $\exp(X)$ be a convergent normal form of φ. A diffeomorphism $\eta \in Z_{sp}(\varphi)$ is of the form $\exp(c(x) \log \varphi)$ by lemma 5.2. Consider a privileged $y = \gamma_1(x)$ in $Sing_V X$. We have $(\psi^X \circ \eta - \psi^X)(x, \gamma_1(x)) \equiv c(x)$ by lemmas 10.6 and 10.7. Fix $\mu \in e^{i(0,\pi)}$ and a compact connected set $K_X^\mu \subset \mathbb{S}^1 \setminus B_X^\mu$. Denote $E = \{l \in \mathbb{N} : \exists j \in \mathbb{Z}\, s.t.\, a^{\varphi}_{j,l,K_X^\mu} \not\equiv 0\}$. The set E is not empty (prop. 9.2). Denote $q = \gcd E$. The continuous functions $c(x)$ satisfying the equation

$$\xi^j_{\varphi, K_X^\mu}(x, z) = (z + c(x)) \circ \xi^j_{\varphi, K_X^\mu}(x, z) \circ (x, z - c(x)) \ \ \forall j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z}).$$

are the constant functions of the form p/q for some $p \in \mathbb{Z}$. Thus the result is a consequence of theorem 10.1. $\square$

10.3. **Complete system of analytic invariants.** We can introduce a complete system of analytic invariants for elements $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. The presentation is slightly simpler if $\varphi_{|x=0}$ is not analytically trivial. In such a case we obtain the generalization of Mardesic-Roussarie-Rousseau's invariants.

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix\varphi_1 = Fix\varphi_2$ and $Res(\varphi_1) \equiv Res(\varphi_2)$. Suppose that $Fix\varphi_1$ is not of trivial type. Let $\exp(X)$ be a convergent normal form of φ_1. There exists $k \in \mathbb{N}$ such that $Y = (x^k, y)^* X$ belongs to $\mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Fix a privileged curve $\gamma \in Sing_V Y$. Consider an EV-covering $K_1 = K_Y^{\mu_1}, \ldots, K_l = K_Y^{\mu_l}$. We say that $m_{\varphi_1}(x_0) = m_{\varphi_2}(x_0)$ for x_0 in $B(0, \delta_0) \setminus \{0\}$ if there exist $c(x_0) \in \mathbb{C}$ and $b(x_0) \in \{1, \ldots, l\}$ such that $x_0 \in \mathbb{R}^+ \dot{K}_{b(x_0)}$ and

$$\xi^j_{\varphi_2, K_{b(x_0)}}(x_0, z) \equiv (z + c(x_0)) \circ \xi^j_{\varphi_1, K_{b(x_0)}}(x_0, z) \circ (x_0, z - c(x_0)) \ \ \forall j \in \mathbb{Z}. \tag{10}$$

The definition makes sense since an EV-covering depends only on $Fix\varphi$ and $Res(\varphi)$ for $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ by remark 9.2. We denote $m_{\varphi_1}(0) = m_{\varphi_2}(0)$ if we have $(\varphi_1)_{|x=0} \sim (\varphi_2)_{|x=0}$. We say that $Inv(\varphi_1) \sim Inv(\varphi_2)$ if $m_{\varphi_1}(x_0) = m_{\varphi_2}(x_0)$ for all x_0 in a pointed neighborhood of 0 and we can choose $c : B(0, \delta_0) \setminus \{0\} \to \mathbb{C}$ such that $Img(c)$ is bounded. Both invariants m_φ and $Inv(\varphi)$ can be expressed in terms of μ-spaces of orbits. In this section we prove that $\varphi_1 \overset{sp}{\sim} \varphi_2$ is equivalent to $Inv(\varphi_1) \sim Inv(\varphi_2)$.

Lemma 10.8. *Let* $f(x)$ *be a multi-valuated holomorphic function of* $B(0, \delta) \setminus \{0\}$ *such that* $f(e^{2\pi i} x) - f(x) \equiv C$ *for some* $C \in \mathbb{R}$*. Suppose that* $|Imgf(x)|$ *is bounded in a neighborhood of* 0*. Then* f *belongs to* $\vartheta(B(0, \delta))$.

Proof. We define $F = f(x) - (C/2\pi i) \ln x$, we obtain $F \in \vartheta(B(0, \delta) \setminus \{0\})$. Moreover we have $ImgF = Imgf + (C/2\pi) \ln |x|$. Suppose $C = 0$, then f has a removable singularity at $x = 0$ since $Imgf$ is bounded.

Suppose $C \neq 0$. Since $\lim_{x \to 0} ImgF \in \{-\infty, +\infty\}$ then F does not have an essential singularity. We claim that F does not have a pole at $x = 0$. Otherwise F is of the form $Ae^{i\theta}/x^l + O(1/x^{l-1})$ for some $(l, A, \theta) \in \mathbb{N} \times \mathbb{R}^+ \times \mathbb{R}$. Since

$$\lim_{r \to 0} Imgf(re^{\frac{i(\theta - \pi/2)}{l}}) = \lim_{r \to 0} \frac{A}{r^l} - \frac{C}{2\pi} \ln r + O\left(\frac{1}{r^{l-1}}\right) = \infty$$

we obtain a contradiction with the boundness of $Imgf$. The equation

$$\lim_{x\to 0} Imgf(x) = ImgF(0) - (C/2\pi)\lim_{x\to 0}\ln|x|$$

implies that $C=0$. Hence f belongs to $\vartheta(B(0,\delta))$ by the first part of the proof. □

All the elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ can be interpreted as elements of $\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ up to a ramification (x^m,y). The ramification preserves the analytic classes of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.

Lemma 10.9. *Let $\varphi_1,\varphi_2\in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with $Fix\varphi_1 = Fix\varphi_2$. Consider $m\in\mathbb{N}$. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if $(x^{1/m},y)\circ\varphi_1\circ(x^m,y) \overset{sp}{\sim} (x^{1/m},y)\circ\varphi_2\circ(x^m,y)$.*

Proof. The sufficient condition is obvious.

Denote $\tilde{\varphi}_j = (x^{1/m},y)\circ\varphi_j\circ(x^m,y)$ for $j\in\{1,2\}$. We have $Fix\varphi_1 = Fix\varphi_2$ by hypothesis and $Res(\varphi_1)\equiv Res(\varphi_2)$ since the residues are analytic invariants. We can suppose that φ_1 and φ_2 are not analytically trivial. Otherwise both $\log\varphi_1$ or $\log\varphi_2$ belong to $\mathcal{X}(\mathbb{C}^2,0)$, we obtain $\varphi_1 \overset{sp}{\sim} \varphi_2$ by proposition 5.2.

Let σ_0 be a special diffeomorphism conjugating $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$. Since we have $(e^{-2\pi i/m}x,y)\circ\tilde{\varphi}_j\circ(e^{2\pi i/m}x,y) = \tilde{\varphi}_j$ for $j\in\{1,2\}$ then

$$\sigma_k = (e^{-2\pi ik/m}x,y)\circ\sigma_0\circ(e^{2\pi ik/m}x,y)$$

conjugates $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ for $k\in\{0,\ldots,m\}$. The diffeomorphism $\sigma_0^{\circ(-1)}\circ\sigma_1$ belongs to $Z_{up}(\tilde{\varphi}_1)$, hence it is of the form $\exp(C\log\tilde{\varphi}_1)$ for some $C\in\mathbb{Q}$ by proposition 10.3. The diffeomorphism $\sigma_k^{\circ(-1)}\circ\sigma_{k+1}$ is equal to

$$(e^{-2\pi ik/m}x,y)\circ\exp(C\log\tilde{\varphi}_1)\circ(e^{2\pi ik/m}x,y) = \exp(C\log\tilde{\varphi}_1).$$

This implies

$$Id = (\sigma_0^{\circ(-1)}\circ\sigma_1)\circ(\sigma_1^{\circ(-1)}\circ\sigma_2)\circ\ldots\circ(\sigma_{m-1}^{\circ(-1)}\circ\sigma_m) = \exp(Cm\log\tilde{\varphi}_1).$$

We obtain $C=0$ by uniqueness of the infinitesimal generator. Since σ_0 and $(e^{2\pi i/m}x,y)$ commute we deduce that $\sigma=(x^m,y)\circ\sigma_0\circ(x^{1/m},y)$ is a special element of $\mathrm{Diff}(\mathbb{C}^2,0)$ conjugating φ_1 and φ_2. □

We can prove now that Inv provides a complete system of analytic invariants.

Theorem 10.2. *Let $\varphi_1,\varphi_2\in\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Suppose that $Fix\varphi_1 = Fix\varphi_2$ and $Res\varphi_1\equiv Res\varphi_2$. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ is equivalent to $Inv(\varphi_1)\sim Inv(\varphi_2)$.*

Proof. We can suppose that $Fix\varphi_1$ is not of trivial type by proposition 6.4.

We consider the notations at the beginning of this section. We can suppose that $\log\varphi_1$ and $\log\varphi_2$ are divergent, otherwise we have that $\varphi_1 \overset{sp}{\sim} \varphi_2$ (prop. 5.2) and we can choose $c\equiv 0$. Let α_j be a convergent normal form of φ_j for $j\in\{1,2\}$. There exists a special mapping σ_0 conjugating α_1 and α_2 (prop. 5.2). Up to replace φ_2 with $\sigma_0^{\circ(-1)}\circ\varphi_2\circ\sigma_0$ and $\xi^j_{\varphi_2,K_b}$ with $(z-d(x))\circ\xi^j_{\varphi_2,K_b}\circ(x,z+d(x))$ for all $(b,j)\in\{1,\ldots,l\}\times\mathbb{Z}$ and some $d\in\mathbb{C}\{x\}$ we can suppose that φ_1 and φ_2 have common convergent normal form. Finally we can suppose that φ_1 and φ_2 belong to $\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ by lemma 10.9.

The sufficient condition is a consequence of theorem 10.1. Since change of charts commute with $z\to z+1$ we can suppose that c is bounded by replacing $c(x)$ with $c(x)-[Re(c(x))]$ where [] is the integer part. There exists a special r-moderated mapping conjugating $\varphi_1(x_0, y)$ and $\varphi_2(x_0, y)$ for all x_0 in a pointed neighborhood of 0 and some $r \in \mathbb{R}^+$ by proposition 10.2. We obtain

$$\xi^j_{\varphi_2, K_b}(x_0, z) \equiv (z + c(x_0)) \circ \xi^j_{\varphi_1, K_b}(x_0, z) \circ (z - c(x_0)) \quad \forall j \in \mathbb{Z} \quad \forall b \in \{1, \ldots, l\}$$

for all $x_0 \in (0, \delta_0)\dot{K}_b$ by proposition 10.1.

Suppose $\sup_{B(0,\delta_0)\setminus\{0\}} |Img\ c| < M$. Fix $p \in \{1, \ldots, l\}$. Consider the set

$$E^p_s(\varphi_1) = \{(j, m) \in D_s(\varphi_1) \times \mathbb{N} : a^{\varphi_1}_{j,m,K_p} \not\equiv 0\}.$$

We define $E^p(\varphi_1) = E^p_{-1}(\varphi_1) \cup E^p_1(\varphi_1)$. We have $E^p(\varphi_1) \neq \emptyset$ by proposition 9.2. Let $x_1 \in (0, \delta_0)\dot{K}^{\mu_p}_X$ such that $(j, m) \in E^p(\varphi_1)$ implies $a^{\varphi_1}_{j,m,K_p}(x_1) \neq 0$. We define

$$d_{j,m} = \frac{1}{2\pi i m s} \ln \frac{a^{\varphi_2}_{j,m,K_p}}{a^{\varphi_1}_{j,m,K_p}}$$

for all $(j, m) \in E^p_s(\varphi_1)$ where we choose $d_{j,m}(x_1) = c(x_1)$. Since

$$e^{-2\pi m M} \leq \left| \frac{a^{\varphi_2}_{j,m,K_p}}{a^{\varphi_1}_{j,m,K_p}} \right| \leq e^{2\pi m M}$$

in $(0, \delta_0)K_p$ we deduce that $d_{j,m} \in \vartheta((0, \delta_0)\dot{K}_p)$ for all $(j, m) \in E^p(\varphi_1)$. We have that $d_{j,m}(x_0) - c(x_0) \in \mathbb{Z}/m$ for $(j, m) \in E^p(\varphi_1)$ and $a^{\varphi_1}_{j,m,K_p}(x_0) \neq 0$. Thus the image of $d_{j,m} - d_{j',m'}$ is contained in $\mathbb{Z}/m + \mathbb{Z}/m'$; since $d_{j,m}(x_1) = d_{j',m'}(x_1)$ we deduce that $d_{j,m} \equiv d_{j',m'}$ for $(j, m), (j', m') \in E^p(\varphi_1)$. Denote by d_p any of the functions $d_{j,m}$ for $(j, m) \in E^p(\varphi_1)$. By construction we obtain

$$\xi^j_{\varphi_2, K^{\mu_p}_X}(x_0, z) \equiv (z + d_p(x_0)) \circ \xi^j_{\varphi_1, K^{\mu_p}_X}(x_0, z) \circ (z - d_p(x_0))$$

for all $j \in \mathbb{Z}$ and all $x_0 \in (0, \delta_0)\dot{K}_p$. We also get $|Img(d_p)| \leq M$ in $(0, \delta_0)\dot{K}_p$.

Consider $p, q \in \{1, \ldots, l\}$ such that $\dot{K}_p \cap \dot{K}_p \neq \emptyset$. Consider $(j, m) \in E^p(\varphi_1)$ and $(j', m') \in E^q(\varphi_1)$. We have $d_p(x_0) - c(x_0) \in \mathbb{Z}/m$ and $d_q(x_0) - c(x_0) \in \mathbb{Z}/m'$ for all $x_0 \in (0, \delta_0)(\dot{K}_p \cap \dot{K}_q)$ such that $(a^{\varphi_1}_{j,m,K_p} a^{\varphi_1}_{j',m',K_q})(x_0) \neq 0$. We deduce that $d_p - d_q$ is a constant function, moreover $d_p - d_q \in \mathbb{Q}$. Then we can extend d_p to $(0, \delta_0)(\dot{K}_p \cup \dot{K}_q)$. We get that d_1 is a multi-valuated function in $B(0, \delta_0) \setminus \{0\}$ such that $d_1(e^{2\pi i} x) - d_1(x) \equiv C$ for some $C \in \mathbb{Q}$. We also have $|Img(d_1)| \leq M$ in $B(0, \delta_0) \setminus \{0\}$ and then $d_1 \in \vartheta(B(0, \delta_0))$ by lemma 10.8. Then φ_1 and φ_2 are conjugated by a special element of $\text{Diff}\,(\mathbb{C}^2, 0)$ by theorem 10.1. □

We give now a geometrical interpretation of our complete system of analytic invariants. Roughly speaking, given $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ the next theorem claims that the analytic classes of $\varphi_{|x=x_0}$ for $x_0 \in B(0, \delta_0) \setminus \{0\}$ characterize the analytic class of φ whenever we exclude singularities of the conjugating mappings at $x_0 = 0$. The result is the analogue of proposition 6.3 in the non-trivial type case.

Theorem 10.3. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix\varphi_1 = Fix\varphi_2$. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if $(\varphi_1)_{|x=x_0}$ and $(\varphi_2)_{|x=x_0}$ are conjugated by a special r-moderated mapping for some $r \in \mathbb{R}^+$ and all x_0 in a pointed neighborhood of 0.*

Proof. By proposition 6.3 we can suppose that $Fix\varphi_1$ is not of trivial type.

We have $Fix\varphi_1 = Fix\varphi_2$ by hypothesis and $Res(\varphi_1) \equiv Res(\varphi_2)$ since the residues are analytic invariants. Let α_j be a convergent normal form of φ_j for $j \in \{1, 2\}$. Then there exists a special $\zeta \in \text{Diff}\,(\mathbb{C}^2, 0)$ such that $\zeta \circ \alpha_1 = \alpha_2 \circ \zeta$

by proposition 5.2. By replacing φ_2 with $\zeta^{\circ(-1)} \circ \varphi_2 \circ \zeta$ we can suppose that φ_1 and φ_2 have a common normal form α_1. The mapping κ_{x_0} has to be replaced with $(\zeta^{\circ(-1)})_{|x=x_0} \circ \kappa_{x_0}$, it is still rR-moderated (maybe for a smaller $r \in \mathbb{R}^+$) by lemma 10.1 for all x_0 in a pointed neighborhood of 0.

There exists $m \in \mathbb{N}$ such that $(x^m, y)^* \log \alpha_1$ belongs to $\mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Fix a privileged $(y = \gamma_1(x)) \in Sing_V(x^m, y)^* \log \alpha_1$ and an EV-covering. Let us denote $c(x_0) = (\psi^X \circ \kappa_{x_0} - \psi^X)(x_0, \gamma_1(x_0))$. We are done since proposition 10.1 and lemma 10.5 assure that the hypothesis of theorem 10.2 is satisfied. $\square$

Remark 10.2. *Consider $\varphi_1, \varphi_2 \in \mathrm{Diff}_{up}(\mathbb{C}^2, 0)$ sharing a convergent normal form $\exp(x^m Y)$ for some $Y \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and $m \in \mathbb{N}$. The existence of r-moderated mappings conjugating $\varphi_1(x_0, y)$ and $\varphi_2(x_0, y)$ for all x_0 in a neighborhood of 0 does not imply $\varphi_1 \overset{sp}{\sim} \varphi_2$. The analogue of lemma 10.8 is fulfilled if and only if $C = 0$. The existence of moderated conjugations plus an extra monodromic invariant provide a complete system of analytic invariants.*

We are interested in knowing whether or not $m_{\varphi_1} \equiv m_{\varphi_2}$ is equivalent to $\varphi_1 \overset{sp}{\sim} \varphi_2$. Indeed we prove next that the moderated hypothesis in theorem 10.2 is generically superfluous (even if we will prove that it is necessary in general).

Theorem 10.4. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ satisfying that $Fix\varphi_1 = Fix\varphi_2$ and $Res(\varphi_1) \equiv Res(\varphi_2)$. Suppose that $(\varphi_1)_{|x=0} \in \mathrm{Diff}_1(\mathbb{C}, 0)$ is not analytically trivial. Then $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if $m_{\varphi_1} \equiv m_{\varphi_2}$.*

The analogue of this theorem for the generic case when $N(X) = 2$ is the main theorem in [16]. They do not impose any conditions on $(\varphi_1)_{|x=0}$. The next section provides counterexamples if $(\varphi_1)_{|x=0}$ is analytically trivial.

Proof. We can suppose that $Fix\varphi_1$ is not of trivial type by corollary 6.2. Moreover we can suppose that φ_1 and φ_2 have a common convergent normal form. Consider the notations at the beginning of this section.

We have $\xi_{\varphi, K_b}^{j}(0, z) = \xi_{\varphi(0,y)}^{\Lambda(j)}(z)$ for all $\varphi \in \{\varphi_1, \varphi_2\}$, $b \in \{1, \ldots, l\}$ and $j \in \mathbb{Z}$ where $\Lambda \equiv \Lambda(\varphi_1) \equiv \Lambda(\varphi_2)$ (cor. 9.1). Since $(\varphi_1)_{|x=0}$ is not analytically trivial then there exists $s(0) \in \{-1, 1\}$ and $(j(0), b(0), \beta) \in D_{s(0)}(\varphi_1) \times \mathbb{N} \times \mathbb{C} \setminus \{0\}$ such that $a_{j(0), b(0), K_p}^{\varphi_1}(0) = \beta$ for all $p \in \{1, \ldots, l\}$. Then $m_{\varphi_1}(0) = m_{\varphi_2}(0)$ implies that there exists $(j(1), \beta') \in D_{s(0)}(\varphi_1) \times \mathbb{C} \setminus \{0\}$ such that $a_{j(1), b(0), K_p}^{\varphi_2}(0) = \beta'$ for all $p \in \{1, \ldots, l\}$. Since $m_{\varphi_1} \equiv m_{\varphi_2}$ we have

$$\left\{ \begin{array}{l} a_{j(0), b(0), K_{b(x)}}^{\varphi_2}(x) = a_{j(0), b(0), K_{b(x)}}^{\varphi_1}(x) e^{2\pi i s(0) b(0) c(x)} \\ a_{j(1), b(0), K_{b(x)}}^{\varphi_2}(x) = a_{j(1), b(0), K_{b(x)}}^{\varphi_1}(x) e^{2\pi i s(0) b(0) c(x)}. \end{array} \right.$$

The first equation implies $-s(0) Img c(x) < K_1$ in a pointed neighborhood of 0 for some $K_1 \in \mathbb{R}$. We obtain $K_2 < -s(0) Img c(x)$ for $x \neq 0$ and some $K_2 \in \mathbb{R}$ from the second equation. This implies $|Img\ c(x)| \leq \max(|K_1|, |K_2|)$ for all $x \neq 0$ in a neighborhood of 0. Now $\varphi_1 \overset{sp}{\sim} \varphi_2$ is a consequence of theorem 10.2. $\square$

Remark 10.3. *The theorem can be easily improved; it remains true if we replace $m_{\varphi_1} \equiv m_{\varphi_2}$ with $m_{\varphi_1}(x_0) = m_{\varphi_2}(x_0)$ for all $x_0 \in E \cup \{0\}$ for some set $E \subset \mathbb{C}$ whose intersection with every neighborhood of 0 is uncountable.*

11. Optimality of the results

We introduce an example which proves that the hypothesis on the non-analytical triviality of $(\varphi_1)_{|x=0}$ in theorem 10.4 can not be dropped. It also shows that the moderated hypothesis in theorem 10.3 is essential. Denote $x = e^{2\pi i w}$, then w is a coordinate in the universal covering of $\mathbb{C}^*$.

Proposition 11.1. *Let $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$. There exist $\varphi_1, \varphi_2 \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with normal form $\exp(X)$ and such that $m_{\varphi_1} \equiv m_{\varphi_2}$ but $\varphi_1 \overset{sp}{\not\sim} \varphi_2$. Moreover there exists an analytic injective mapping σ conjugating φ_1 and φ_2 and defined in a domain $|y| < C_0 / \sqrt[\nu(X)]{|w|}$ for some $C_0 \in \mathbb{R}^+$.*

In particular we provide a counter-example to the main theorem in [16]. The size of the domain $|y| < C_0 / \sqrt[\nu(X)]{2\pi|w|}$ decays when x tends to 0. Theorem 10.3 made this property somehow expected. Anyway the decay is slower than algebraic.

Let $X = f(x,y)\partial/\partial y \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$. We consider vector fields of the form

$$X_v = \frac{f(x,y)}{1 + f(x,y)v(x,y,t)} \frac{\partial}{\partial y} + 2\pi i t \frac{\partial}{\partial t}$$

where v is defined in a domain of the form $B(0,\delta) \times B(0,\epsilon) \times B(0,2)$ in coordinates (x,y,t). The vector field X_v supports a dimension 1 foliation Ω_v preserving the hypersurfaces $x = cte$. Moreover since $X_v(t) = 2\pi i t$ then X_v is transversal to every hypersurface $t = cte$ except $t = 0$. As a consequence we can consider the holonomy mapping $hol_v(x,y,t_0,z_0)$ of the foliation given by X_v along a path $t \in e^{2\pi i [0,z_0]} t_0$, it maps the transversal $t = t_0$ to $t = t_0 e^{2\pi i z_0}$ for $t_0 \neq 0$. The restriction of $hol_v(x,y,t,z)$ to $(x,y) \in Sing X$ is the identity. Supposed that $v = v(x,y)$ we have

$$hol_v(x,y,t,z) = \left(\exp\left(z \frac{f(x,y)}{1 + f(x,y)v(x,y)} \right)(x,y), e^{2\pi i z} t \right).$$

The restriction $(\Omega_v)_{x=0}$ is a germ of saddle-node for $v \in \mathbb{C}\{x,y,t\}$. The holonomy $hol_v(0,y,t_0,1)$ at a transversal $t = t_0$ to the strong integral curve $y = 0$ is analytically trivial if and only if $(\Omega_v)_{|x=0}$ is analytically normalizable [17]. In particular $(\Omega_y)_{|x=0}$ is analytically normalizable. Every foliation in the same formal class than $(\Omega_y)_{|x=0}$ is analytically conjugated to some $(\Omega_v)_{|x=0}$, we just truncate the formal conjugation. Every formal class contains non-analytically normalizable elements, hence there exists $v^0 \in \mathbb{C}\{y,t\} \cap (y,t)$ such that the saddle-node

$$(X_{v^0})_{|x=0} = \frac{f(0,y)}{1 + f(0,y)v^0(y,t)} \frac{\partial}{\partial y} + 2\pi i t \frac{\partial}{\partial t}$$

is not analytically normalizable. Hence the holonomy $hol_{v^0}(0,y,t_0,1)$ is not analytically trivial for $t_0 \neq 0$. Moreover up to change of coordinates $(x,y,t) \to (x,y,\eta t)$ for some $\eta \in \mathbb{R}^+$ there exists $(\delta_0, \epsilon_0) \in \mathbb{R}^+$ such that

- $v^0 \in \vartheta(B(0,\epsilon_0) \times B(0,2))$ and $\sup_{B(0,\delta_0) \times B(0,2)} |v^0| < 1$.
- $\sup_{B(0,\delta_0) \times B(0,\epsilon_0)} |f| < C_0 < 1/16$.
- $1/2 < \sup_{B(0,\delta_0) \times B(0,\epsilon_0)} |f \circ \exp(zX)(x,y)| / |f(x,y)| < 2$ for all $z \in B(0,2)$.

The constant $C_0 > 0$ will be determined later on. There exists $k \in \mathbb{N}$ such that $(x^k, y)^* X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Denote $Y = (x^k, y)^* X$. Consider $U = B(0,\delta) \times B(0,\epsilon)$ such that there exists a EV-covering $K_1 = K_Y^{\mu_1}$, …, $K_l = K_Y^{\mu_l}$ fulfilling that $H(x)$ is well-defined for all $x \in [0,\delta)\dot{K}_p$, $H \in Reg(\epsilon, \mu_p X, K_p)$ and $p \in \{1, \ldots, l\}$. We can also suppose that there exists $C > 0$ such that

$$|f(x, y)| \leq \frac{C}{(1 + |\psi_{H,\kappa}^{X}(x, y)|)^{1+1/\nu(X)}}$$

for every κ-subregion of $H \in Reg(\epsilon, \mu_l X, K_l)$, every $(x, y) \in H_\kappa$ and $\kappa \in \{L, R\}$ by proposition 8.1. Finally we suppose that $\exp(B(0, 4)X)(U)$ is contained in $B(0, \delta_0) \times B(0, \epsilon_0)$.

Denote $V = B(0, \delta) \times B(0, \epsilon_0) \times B(0, 2)$. Let $v \in \vartheta(V)$ such that $\sup_V |v| < 2$. Consider an integral ψ of the time form of X. We have

$$X_v \left(\psi - \frac{1}{2\pi i} \ln t \right) = \frac{1}{1 + vf} - 1 = -\frac{vf}{1 + vf}.$$

We obtain

$$\psi \circ hol_v(x, y, t, z_0) = \psi(x, y) + z_0 - \int_0^{z_0} \frac{vf}{1 + vf} \circ hol_v(x, y, t, z) dz. \tag{11}$$

We claim that $hol_v(U \times B(0, 2) \setminus \{0\} \times [0, 1]) \subset V$. Otherwise there exist (x_0, y_0, t_0) in $U \times B(0, 2)$ and a minimum $z_0 \in [0, 1]$ such that $y \circ hol_v(x_0, y_0, t_0, z_0) \in \partial B(0, \epsilon_0)$. This leads us to

$$|\psi \circ hol_v(x_0, y_0, t_0, z_0) - \psi(x_0, y_0)| \leq |z_0| + |z_0| \frac{2C_0}{1 - 2C_0} \leq \frac{8}{7} |z_0| < 2$$

and that contradicts the choice of U. Denote $\Delta_v(x, y, t) = \psi \circ hol_v(x, y, t, 1) - (\psi + 1)$. We obtain

$$|\Delta_v(x, y, t)| \leq \frac{32}{7} |f(x, y)| < 5C_0 \quad \forall (x, y, t) \in U \times B(0, 2).$$

We define $\Delta_v^1(x, y) = \Delta_v(x, y, 1)$ and $\Delta_v^2(x, y) = \Delta_v(x, y, x)$. The function Δ_v^1 is holomorphic in U. The same property is true for Δ_v^2 since it is holomorphic in $U \setminus [x = 0]$ and bounded.

We define $\varphi_{1,v} = hol_v(x, y, 1, 1)$ and $\varphi_{2,v} = \exp(zX)(x, y, 1 + \Delta_v^2(x, y))$. Clearly $\varphi_{2,v}(x, y) = hol_v(x, y, x, 1)$ for $x \neq 0$.

Lemma 11.1. $\exp(X)$ *is a convergent normal form of* $\varphi_{1,v}$, $\varphi_{2,v}$ *for all* v *in* $\vartheta(V)$.

Proof. The equation 11 implies that Δ_v^1 and Δ_v^2 belong to (f). Since we have

$$y \circ \varphi = y + \sum_{j=1}^{\infty} (1 + \Delta_\varphi)^j \frac{X^{\circ(j)}(y)}{j!} = y \circ \exp(X) + O(f^2)$$

for $\varphi \in \{\varphi_{1,v}, \varphi_{2,v}\}$ then $\varphi_{1,v}$ and $\varphi_{2,v}$ have convergent normal form $\exp(X)$. □

Fix a privileged $\gamma \in Sing_V Y$. We choose $C_0 > 0$ such that there exists $I > 0$ holding that $\forall s \in \{-1, 1\}$ and $\forall j \in D_s(\exp(X))$ we have

$$\xi_{\varphi,K_p}^{j} \in C^0([0, \delta)\dot{K}_p \times [sImgz < -I]) \cap \vartheta((0, \delta)\dot{K}_p \times [sImgz < -I]) \ \forall 1 \leq p \leq l$$

whenever φ has convergent normal form $\exp(X)$ and $|\Delta_\varphi(x, y)| \leq 5 \min(C_0, |f(x, y)|)$ for all $(x, y) \in B(0, \delta) \times B(0, \epsilon)$ (remark 8.1).

By choice $(\varphi_{1,v^0})_{|x=0}$ is not analytically trivial. Thus there exists $(j(0), p(0), x_0)$ in $\mathbb{Z} \times \{1, \ldots, l\} \times (\delta/2, \delta) \times \dot{K}_{p(0)}$ such that $\xi_{\varphi_{1,v^0},K_{p(0)}}^{j(0)}(x_0, z) \not\equiv z + \zeta_{\varphi_{1,v^0}}(x_0)$. Denote $u = (x/x_0)v^0(y, t)$, we get $sup_V |u| < 2$. We define $\varphi_1 = \varphi_{1,u}$ and $\varphi_2 = \varphi_{2,u}$.

Lemma 11.2. φ_1 *is not analytically trivial.*

Proof. By construction $\xi^{j(0)}_{\varphi_1,K_{p(0)}}(x,z)$ is well-defined in $x \in [0,\delta) \times \dot{K}_{p(0)}$ and $\xi^{j(0)}_{\varphi_1,K_{p(0)}}(x_0,z) \not\equiv z + \zeta_{\varphi_1}(x_0)$. We deduce that φ_1 is not analytically trivial. $\square$

The next lemma is a consequence of $u(0,y,t) \equiv 0$.

Lemma 11.3. $(\varphi_1)_{|x=0} \equiv (\varphi_2)_{|x=0} \equiv \exp(X)_{|x=0}$. *In particular* $(\varphi_1)_{|x=0}$ *and* $(\varphi_2)_{|x=0}$ *are analytically trivial.*

Denote by $\sigma(x,y)$ the analytic mapping $hol_u(x,y,1,lnx/(2\pi i))$.

Lemma 11.4. *The mapping* $\sigma(x,y)$ *conjugates* φ_1 *and* φ_2 *in a domain of the form* $|y| < C_0/\sqrt[\nu(X)]{|\ln x|}$ *for some* $C_0 \in \mathbb{R}^+$. *Moreover* σ *is not univaluated since*

$$\sigma(e^{2\pi i}x,y) = hol_u\left(x,y,1,\frac{\ln x}{2\pi i}+1\right) = hol_u(x,y,x,1) \circ \sigma(x,y) = \varphi_2 \circ \sigma(x,y).$$

Proof. Consider a domain $W \subset B(0,\delta) \times B(0,\epsilon_0)$ such that

$$\exp\left(B\left(0,\frac{|\ln x|}{\pi}\right)X\right)(x,y) \in B(0,\delta) \times B(0,\epsilon_0) \ \ \forall (x,y) \in W.$$

Since $y \circ hol_u(x,y,1,s\ln x/(2\pi)) \subset \overline{B}(0,\epsilon_0)$ for all $s \in [0,s_0]$ and $s_0 \in [0,1]$ implies

$$\left|\psi \circ hol_u\left(x,y,1,s\frac{\ln x}{2\pi i}\right) - \psi(x,y)\right| \leq s_0\frac{|\ln x|}{2\pi} + \frac{s_0}{7}\frac{|\ln x|}{2\pi} < \frac{|\ln x|}{\pi} \tag{12}$$

by equation 11 then $hol_u(x,y,1,s\ln x/(2\pi i))$ is well-defined and belongs to V for all $(x,y,s) \in W \times [0,1]$. We have $\psi \sim 1/y^{\nu(X)}$ in the first exterior set by remark 7.4, we can deduce that W contains a domain of the form $|y| < C_0/\sqrt[\nu(X)]{|\ln x|}$ for some $C_0 \in \mathbb{R}^+$. $\square$

The domain $W_0 = [|y| < C_0/\sqrt[\nu(X)]{|\ln x|}]$ contains the germ of all the "algebraic" domains of the form $|y| < |x|^b$ for $b \in \mathbb{Q}^+$, in particular W_0 contains $SingX \setminus \{(0,0)\}$, every intermediate set and every exterior set except the first one.

Lemma 11.5. *We have*

$$\xi^j_{\varphi_2,K_p}(x_0,z) = \left(z + \frac{\ln x_0}{2\pi i}\right) \circ \xi^j_{\varphi_1,K_p}(x_0,z) \circ \left(z - \frac{\ln x_0}{2\pi i}\right)$$

for all $(j,p) \in \mathbb{Z} \times \{1,\ldots,l\}$ *and* $x_0 \in (0,\delta) \times \dot{K}_p$. *Then we get* $m_{\varphi_1} \equiv m_{\varphi_2}$ *and* $\varphi_1 \overset{sp}{\not\sim} \varphi_2$.

Proof. Let $(x_0,y_0) \in SingX \setminus \{(0,0)\}$. We remark that

$$\lim_{(x,y)\to(x_0,y_0)} \psi \circ hol_v\left(x,y,1,s\frac{\ln x}{2\pi i}\right) - \psi = s\frac{\ln x_0}{2\pi i}$$

for all $s \in [0,1]$ by equation 11. Basically the moderated hypothesis in proposition 10.1 is used to estimate $\psi \circ \kappa - \psi$ for a special r-moderated conjugation κ. Such an estimation is provided here by the inequality 12, hence we can proceed like in proposition 10.1 to obtain

$$\xi^j_{\varphi_2,K_p}(x_0,z) = \left(z + \frac{\ln x_0}{2\pi i}\right) \circ \xi^j_{\varphi_1,K_p}(x_0,z) \circ \left(z - \frac{\ln x_0}{2\pi i}\right)$$

for all $(j,p) \in \mathbb{Z} \times \{1,\ldots,l\}$ and $x_0 \in (0,\delta) \times \dot{K}_p$. We deduce $m_{\varphi_1} \equiv m_{\varphi_2}$ from the previous equation and $(\varphi_1)_{|x=0} \equiv (\varphi_2)_{|x=0}$.

We know that φ_1 and φ_2 are not analytically trivial. Thus we have $\varphi_1 \overset{sp}{\not\sim} \varphi_2$ since otherwise $|\ln|x||$ would be bounded in a neighborhood of 0 by theorem 10.2. $\square$

Remark 11.1. *The diffeomorphisms $\varphi_{1,v}$ and $\varphi_{2,v}$ are conjugated by a multivaluated transformation collapsing at $x=0$. Anyway since*

$$(\varphi_{2,v})_{|x=0} = \exp\left(\frac{f(0,y)}{1+f(0,y)v(0,y,0)}\frac{\partial}{\partial y}\right)$$

then $(\varphi_{2,v})_{|x=0}$ is always analytically trivial. Thus $m_{\varphi_{1,v}} \equiv m_{\varphi_{2,v}}$ forces $(\varphi_{1,v})_{|x=0}$ to be also analytically trivial.

Remark 11.2. *We do not characterize the diffeomorphisms $\varphi_1 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ such that there exists $\varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ satisfying the result in proposition 11.1. We already proved that the property $\log(\varphi_1)_{|x=0} \in \mathcal{X}(\mathbb{C},0)$ is necessary (theorem 10.4). It is unlikely that it is sufficient since in the trivial type case we need a condition on half of the changes of charts. Moreover, a careful look to the construction in this section makes clear that half of the changes of charts of $\varphi_{1,v}$ are trivial.*

References

[1] L. Ahlfors and L. Bers. Riemmann's mapping theorem for variable metrics. *Ann. of Math.*, 72:385–404, 1960.

[2] A. Bruno. Analytic form of differential equations i. *Trudy Moskov. Mat. Obshch.*, (25):119–262, 1971.

[3] A. Bruno. Analytic form of differential equations ii. *Trudy Moskov. Mat. Obshch.*, (26):199–239, 1972.

[4] Cesar Camacho. On the local structure of conformal mappings and holomorphic vector fields in $\mathbb{C}^2$. *Asterisque*, (59-60):83–94, 1978.

[5] J.B. Conway. *Functions of one complex variable II.* New York : Springer-Verlag, 1995.

[6] A. Douady, F. Estrada, and P. Sentenac. Champs de vecteurs polynômiaux sur $\mathbb{C}$. *To appear in the Proceedings of Boldifest.*

[7] J. Ecalle. Théorie des invariants holomorphes. *Public. Math. Orsay*, (67), 1974.

[8] J. Ecalle. Théorie itérative: introduction à la théorie des invariants holomorphes. *J. Math. pures et appl.*, (54):183–258, 1975.

[9] A. A. Glutsyuk. Confluence of singular points and the nonlinear stokes phenomenon. *Trans. Moscow Math. Soc.*, pages 49–95, 2001.

[10] Yu. S. Il'yashenko, S.M. Voronin, et al. *Nonlinear Stokes phenomena*, volume 14 of *Advances in soviet mathematics.* American mathematical society, 1992.

[11] V.P. Kostov. Versal deformations of differential forms of degree α on the line. *Funktsional. Anal. i Prilozhen*, 18(4):81–82, 1984.

[12] P. Lavaurs. *Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques.* PhD thesis, Universit de Paris-Sud, 1989.

[13] Leau. Étude sur les équations functionelles à une ou plusieurs variables. *Ann. Fac. Sci. Toulouse*, (11), 1897.

[14] O. Lehto and K.I. Virtanen. *Quasiconformal mappings in the plane.* New York, Heidelberg : Springer-Verlag, 1973.

[15] B. Malgrange. Travaux d'Écalle et de Martinet-Ramis sur les systèmes dynamiques. *Asterisque*, (92-93):59–73, 1982.

[16] P. Mardesic, R. Roussarie, and C. Rousseau. Modulus of analytic classification of unfoldings of generic parabolic diffeomorphisms. *Mosc. Math. J.*, 4(2):455–502, 2004.

[17] J. Martinet and J.-P. Ramis. Problèmes de modules pour des equations differentielles non lineaires du premier ordre. *Ins. Hautes Etudes Sci. Publ. Math.*, (55):63–164, 1982.

[18] J. Martinet and J.-P. Ramis. Classification analytique des équations differentielles non linéaires résonnantes du premier ordre. *Ann. Sci. Ecole Norm. Sup.*, 4(16):571–621, 1983.

[19] Jean Martinet. Remarques sur la bifurcation noeud-col dans le domaine complexe. singularités de équations différentielles (dijon 1985). *Asterisque*, (150-151):131–149, 1987.

[20] J.-F. Mattei and R. Moussu. Holonomie et intégrales premières. *Ann. Sci. Ecole Norm. Sup.*, (13):469–523, 1980.
[21] J. Ribón. Formal classification of unipotent parameterized diffeomorphisms. *math.math.DS/0511298*, 2005.
[22] J. Ribón. Topological classification of families of diffeomorphisms without small divisors. *http://www.preprint.impa.br/Shadows/SERIE_D/2005/10.html*, 2005.
[23] R. Oudkerk. *The parabolic implosion for* $f_0(z) = z + z^{\nu+1} + O(z^{\nu+2})$. PhD thesis, University of Warwick, 1999.
[24] R. Pérez-Marco. Fixed points and circle maps. *Acta Math.*, 2(179):243–294, 1997.
[25] J.-P. Ramis. Confluence et résurgence. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36(3):703–716, 1989.
[26] Jérome Rey. *Difféomorphismes résonnants de* $(\mathbb{C}, 0)$. Thesis. Université Paul Sabatier, 1996.
[27] Robert Roussarie. Modeles locaux de champs et de formes. *Asterisque*, (30):181 pp., 1975.
[28] C. Rousseau. Modulus of orbital analytic classification for a family unfolding a saddle-node. *Mosc. Math. J.*, 5(1):245–268, 2005.
[29] M. Shishikura. Bifurcation of parabolic fixed points. the mandelbrot set, theme and variations. *London Math. Soc. Leture Note Ser.*, 274:325–363, 2000.
[30] C. L. Siegel. Iterations of analytic functions. *Ann. of Math.*, 43:807–812, 1942.
[31] S.M. Voronin. Analytical classification of germs of conformal mappings $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with identity linear part. *Functional Anal. Appl.*, 1(15), 1981.
[32] J.-C. Yoccoz. Théorème de Siegel, polynômes quadratiques et nombres de Brjuno. *Asterisque*, (231), 1995.