Translative Sets and Functions and their Applications to Risk Measure Theory and Nonlinear Separation

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Abstract

Recently defined concepts such as nonlinear separation functionals due to Tammer and Weidner, coherent risk measures due to Delbaen et al., topical functions due to Gunawardena and Keane as well as isotonic Banach functionals coincide and can be traced back to books by Day from 1957 and Krasnosel'skij's from 1964. This paper is to get out the common background of these concepts and to give an extension to set–valued functions. Translativity with respect to one or a finite collection of elements turns out to be the key property of sets and functions considered in this note.

Keywords and phrases. nonlinear separation, coherent risk measure, isotonic Banach functional, topical function, set–valued convex function, translative function

1 Introduction

1.1 General remarks

Let X be a topological linear space, $D \subseteq X$ be a closed convex cone and $k \in D\setminus(-D)$. Define a functional $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ by

$$\varphi(x) := \inf\{t \in \mathrm{I\!R} : \ -x + tk \in D\} = \inf\{t \in \mathrm{I\!R} : \ x \leq_D tk\} \tag{1}$$

where $\leq_D$ denotes the order relation generated by D ($x \leq_D x'$ iff $x' - x \in D$). One can easily see (compare e.g. Lemma 7 of [22]) that φ is positively homogeneous, subadditive and enjoys the following translation property:

$$\forall x \in X, \ \forall s \in \mathrm{I\!R} : \ \varphi(x + sk) = \varphi(x) + s. \tag{2}$$

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It implies the linearity of φ on the one dimensional subspace of X generated by k. Moreover, φ is monotone nondecreasing with respect to $\leq_D$ ($x \leq_D x'$ implies $\varphi(x) \leq \varphi(x')$) and $-D$ is the sublevel set of φ at level 0.

Sublinear functionals of this type have been used in theoretical investigations within the framework of ordered linear spaces. For example, they appear in the book [6] by M. M. Day as an elegant tool for a proof of the fact that the Hahn-Banach extension and a linear closure property imply the interpolation property. M. A. Krasnosel'skij used them in order to establish necessary and sufficient conditions for a cone to be normal. Compare [36], Lemmata 1.1 – 1.3 and Theorem 1.1. M. M. Fel'dman and A. M. Rubinov in [13] and [48], respectively, investigated dual properties of such kind of functionals, namely their so-called support sets.

Independently, they appear as nonlinear separation and scalarization functionals for vector optimization problems in [45] (X finite dimensional with precursor [47]) and, in more generality, in papers by C. Tammer, P. Weidner and collaborators in [17], [18], [19], [59]. Dinh The Luc [40] and C. Zălinescu [62] also gave early contributions to this topic. In [55], functionals of type (1) have been applied in order to obtain vector valued variants of Ekeland's variational principle. For this topic, compare also [22] and [25]. Note that the originality of the approach in [19], [59], [55] relies on the fact that the set D defining a functional via (1) was assumed neither to be a cone nor convex. This can be of importance in vector optimization, see [60].

More recently, in their 1998 landmark paper [2], [3] Artzner et al. introduced the concept of *coherent risk measures* and their *acceptance sets*. Delbaen [7] extended the definition to general probability spaces. Rockafellar et al. [46] gave a new approach to coherent risk measures on L^2 via deviation measures.

Let Ω be a nonempty set. A functional $\rho : X \to \mathbb{R}$, where X is a linear space of random variables $x : \Omega \to \mathbb{R}$, is called a *coherent risk measure* iff it is sublinear, monotone nonincreasing with respect to the pointwise order (or almost everywhere pointwise) and satisfies the translation property

$$\forall x \in X, \ \forall s \in \mathbb{R} : \ \rho(x + se) = \rho(x) - s \tag{3}$$

where e is the random variable having (almost) everywhere the value 1. The sublevel set $S_\rho(0) := A$ of ρ at level 0 is a convex cone. It is called the acceptance set of ρ. It can be shown that a coherent risk measure admits a representation as

$$\rho(x) = \inf \{t \in \mathbb{R} : \ x + te \in A\}.$$

It turns out that a coherent risk measure can be identified with a functional $\varphi(\cdot)$ of Tammer-Weidner type by $\varphi(x) = \rho(-x)$. Föllmer and Schied [15] extended the notion of coherent risk measures to monetary measures of risk. These are real valued functions ρ on linear spaces of random variables enjoying the translation property above and monotonicity with respect to a pointwise order. Special cases are coherent and convex measures of risk with widely spread applications in modern financial mathematics. See e.g. [14], [32], [42], [53], [46].

Gunawardena and Keane [24] introduced the notion of topical functions in order to model the dynamic behaviour of discrete event systems. A motivation for this approach can be found in the introduction of [23]. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *topical* iff $F(y) - F(x) \in \mathbb{R}^n_+$ whenever $y - x \in \mathbb{R}^n_+$ and

$$\forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^n:\ F(x + te) = F(x) + te.$$

where $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$. It turns out that F is topical iff each of its components satisfies (2) and is $-\mathbb{R}^n_+$-monotone in the sense of Definition 7 below. In [23], many results on and examples of topical functions can be found. Rubinov and Singer [51] (compare also [43]) investigated this notion and they introduced the plus-Minkowski gauge of a set G coinciding with (1) if D is replaced by G. In a series of papers, this approach has been linked with concepts from the theory of abstract convexity and so–called monotonic analysis, see [50], [10], [11]. Especially, a characterization theorem for IPH functions (see [49]) on locally convex spaces in terms of functions of type (1) is given in [10] (Theorem 3.3).

Let us mention that there are still more similar concepts. For example, an isotonic Banach functional [9] is nothing else than a functional of type (1).

It might be observed that in different fields of applications different sets of properties of functions of type (1) are in the centre of consideration. For example, [13], [48] are focused on dual constructions, namely the support set and the subdifferential of sublinear operators, while in [19], [59], [55] and [22] a dual description is not used. In the theory of coherent risk measures, a dual representation theorem is essential, but this theory is restricted to spaces of random variables. On the other hand, there are several results which are strongly related or do even coincide. For example, Proposition 4.8 of [51] has a counterpart within the context of coherent risk measures. It states a one-to-one correspondence between coherent risk measures and radially closed acceptance sets, compare [3], [32], for example.

Observing the fact that there are at least three almost coincidental concepts with many applications in different fields of mathematics, it might be worthwhile to give a unifying approach.

The translation property (2), (3) (see Definiton 6 below) is the lynchpin of this paper. It turns out that there is an intimate relationship between translative functions and their zero sublevel sets being translatives of each other and enjoying a translation property theirself as well as a weak kind of algebraic closedness called radial closedness with respect to certain elements.

Therefore, the second section is devoted to the study of translative and radially closed sets

In the third section, we investigate algebraic and topological properties as well as duality features of translative functions. In contrast to most other references, we allow values of the function in $\mathbb{R} \cup \{\pm\infty\}$. We shall give several equivalent characterizations of translativity and the most general bijection theorem for translative, extended real–valued functions and their zero sublevel sets. Then, we study the interrelations of this property with other like positive homogenity, subadditivity, convexity and monotonicity. Necessary and sufficient conditions for finite valuedness and continuity are given. In locally convex spaces, a duality theorem is proven from which all known dual representation theorems can be derived.

The fourth section is devoted to set–valued extensions of translative functions. This is mainly motivated by the paper [34] where set–valued coherent risk measures are introduced. It turns out that it is more appropriate to consider set–valued extensions rather than vector–valued ones and that most results carry over from the real–valued case, sometimes in an even more natural way (compare e.g. Proposition 3 vs. Proposition 14).

Finally, we mention that there are at least two other types of extensions which are not within the scope of this work: One [1] admits values of the functions in question in lattice ordered groups, the other one [28], [29] is concerned with extended real–valued functions of type (1) defined on the power set of a linear space.

1.2 Some notation

By $\mathrm{I\!N}$ we denote the set of natural numbers (including zero), by $\mathrm{I\!R}$ and $\mathrm{I\!R}_+$ the set of real numbers and non–negative real numbers, respectively. For $m \in \mathrm{I\!N}$, $m \geq 1$, $\mathrm{I\!R}^m$ denotes an m–dimensional linear space. If $v \in \mathrm{I\!R}^m$, we write v_i, $i = 1, \ldots, m$ for the components of v with respect to the canonical base $\{e^1, \ldots, e^m\}$ with $e^i_j = 1$ if $i = j$ and $= 0$ otherwise. We write $\mathrm{I\!R}^m_+$ for the set of all $v \in \mathrm{I\!R}^m$ with $v_i \geq 0$ for all $i \in \{1, \ldots m\}$.

Let X be a real linear space. For subsets $A, B \subseteq X$ we define by

$$A \oplus B := \{a + b : \; a \in A, \, b \in B\}$$

the Minkowski sum of two subsets of X with $A \oplus \emptyset = \emptyset \oplus A = \emptyset$. If $T \subseteq \mathrm{I\!R}$ we write $TA = \{ta : \; t \in T, \, a \in A\}$ and simply tA for $\{t\} A$ (with $t \in \mathrm{I\!R}$). A set $A \subseteq X$ is said to be a *cone* iff $t > 0$ implies $tA \subseteq A$ and *closed under addition* iff $A \oplus A \subseteq A$. It is called *convex* iff $t \in [0, 1]$ implies $tA \oplus (1 - t) A \subseteq A$. It is well–known that a cone is closed under addition if and only if it is convex.

A convex cone $D \subseteq X$ containing $0 \in X$ generates a reflexive and transitive relation $\leq_D$ on X that is compatible with the linear structure by means of the definition

$$x \leq_D x' \quad :\Longleftrightarrow \quad x' - x \in D.$$

Such an order relation can be extended in the following way to the power set of X. Let us denote by $\mathcal{P}(X)$ the set of all nonempty subsets of X and by $\widehat{\mathcal{P}}(X) := \mathcal{P}(X) \cup \{\emptyset\}$. Define for $A, B \in \widehat{\mathcal{P}}(X)$

$$A \preccurlyeq_D B \quad :\Longleftrightarrow \quad B \subseteq A \oplus D. \tag{4}$$

It is easy to see that $\preccurlyeq_D$ is reflexive and transitive. Observe that $B \subseteq A \oplus D$ if and only if $B \oplus D \subseteq A \oplus D$. Moreover, we have $X \preccurlyeq_D \emptyset$ and $X \preccurlyeq_D A$ for all $A \in \widehat{\mathcal{P}}(X)$.

Thus, $\emptyset$ plays the role of $+\infty$ and the whole space X the role of $-\infty$. Note that for $D = \{0\}$, the relation $\preccurlyeq_D$ coincides with $\supseteq$.

Let $\mathcal{A} \subseteq \widehat{\mathcal{P}}(X)$ be given. The infimum and the supremum of $\mathcal{A}$ with respect to $\preccurlyeq_D$ can be expressed by

$$\inf \{\mathcal{A}, \preccurlyeq_D\} = \bigcup_{A \in \mathcal{A}} A \oplus D, \quad \sup \{\mathcal{A}, \preccurlyeq_D\} = \bigcap_{A \in \mathcal{A}} (A \oplus D), \tag{5}$$

respectively, see [26] for a (not very difficult) proof. Note that there is a second (canonical) extension $\preccurlyeq_D$ of $\leq_D$ to $\widehat{\mathcal{P}}(X)$ that is defined by $A \preccurlyeq_D B$ iff $A \subseteq B \oplus (-D)$. Such relations are widely used in theoretical computer sciences [5] and have recently been used in an optimization framework [38], [37] and also in [28], [29]. A more detailed account is given in [26].

2 Translative and radially closed sets

Let X be a linear space. In the following, we assume that we are given a collection $K := \{k^1, k^2, \dots, k^m\} \subset X$ of $m \geq 1$ linearly independent elements of X and a convex cone $C \subseteq \mathbb{R}^m$ containing $0 \in \mathbb{R}^m$. The set

$$\Gamma_K(C) := \left\{ \sum_{i=1}^{m} v_i k^i : \ v \in C \right\} \subseteq X.$$

is a convex cone containing $0 \in X$. Everything takes place with respect to these data.

2.1 Translative sets

Definition 1 *A set $A \subseteq X$ is said to be* translative *with respect to K and C iff*

$$\forall x \in A, \ \forall v \in C : \ x + \sum_{i=1}^{m} v_i k^i \in A. \tag{6}$$

Obviously, a set $A \subseteq X$ is translative with respect to K and C if and only if $A \oplus \Gamma_K(C) \subseteq A$. Moreover, for an arbitrary set $A \subseteq X$, the set $A \oplus \Gamma_K(C)$ is the smallest set (with respect to inclusion) that contains A and is translative with respect to K and C. This is, since if $B \subseteq X$ is translative with respect to K and C such that $A \subseteq B$, then $A \oplus \Gamma_K(C) \subseteq B \oplus \Gamma_K(C) \subseteq B$. This observation justifies the following definition.

Definition 2 *For a set $A \subseteq X$, the intersection of all translative sets containing A is called the* translative hull *of A and is denoted by* $\operatorname{tr} A$.

Hence, $\operatorname{tr} A = A \oplus \Gamma_K(C)$. Of course, $\operatorname{tr} A$ highly depends on C and K, but we do not refer to this dependence in the symbol $\operatorname{tr} A$. Similarly, speaking of a translative set we always mean a set being translative with respect to the fixed given K and C.

Lemma 1 *It holds* $A \subseteq \operatorname{tr} A$*,* $\operatorname{tr} A$ *is translative and* $\operatorname{tr}(\operatorname{tr} A) = \operatorname{tr} A$*.*

Proof. Take $\operatorname{tr} A = A \oplus \Gamma_K(C)$, $0 \in \Gamma_K(C)$ and $\Gamma_K(C) \oplus \Gamma_K(C) \subseteq \Gamma_K(C)$ into account. ■

Observation 1. Let $A \subseteq X$ be closed under addition such that $\Gamma_K(C) \subseteq A$. Then A is translative with respect to K and C. If A is translative and $0 \in A$, then $\Gamma_K(C) \subseteq A$. Especially, a convex cone $A \subseteq X$ containing $0 \in X$ is translative if and only if $\Gamma_K(C) \subseteq A$.

Remark 1 *The cone* $\Gamma := \Gamma_K(C)$ *generates an order relation* $\preccurlyeq_\Gamma$ *on* $\widehat{\mathcal{P}}(X)$ *of type (4), defined by*

$$A \preccurlyeq_\Gamma B \quad :\Longleftrightarrow \quad B \subseteq A \oplus \Gamma_K(C).$$

Then, $\preccurlyeq_\Gamma$ *is reflexive and transitive and* $A \preccurlyeq_\Gamma B$ *if and only if* $\operatorname{tr} A \supseteq \operatorname{tr} B$*. Moreover,* $A \preccurlyeq_\Gamma B$ *and* $B \preccurlyeq_\Gamma A$ *holds if and only if* $\operatorname{tr} A = \operatorname{tr} B$*, i.e., the set* $T(K,C) = \left\{A \in \widehat{\mathcal{P}}(X) :\ A = \operatorname{tr} A\right\}$ *can be identified with the set of equivalence classes on* $\widehat{\mathcal{P}}(X)$ *generated by the quasiorder* $\preccurlyeq_\Gamma$*. On* $T(K,C)$*, the relation* $\preccurlyeq_\Gamma$ *is a partial order and we have* $A' \preccurlyeq_\Gamma B'$ *if and only if* $A' \supseteq B'$ *for* $A', B' \in T(K,C)$*.*

Consider the case $m = 1$, $C = \mathbb{R}_+$ and $k^1 = k \in X \backslash \{0\}$ that is the mostly used one, compare Section 2.3 of [21] and the references therein, [15] and the references therein with respect to (financial) risk measures and [51] with respect to topical functions. In this case, (6) shrinks down to

$$\forall x \in A,\ \forall s \geq 0 :\ x + sk \in A. \tag{7}$$

and $\operatorname{tr} A = A \oplus \mathbb{R}_+ \{k\} = A \oplus [0, +\infty) \{k\}$. In the one dimensional case, we simply say that A is translative with respect to k. Moreover, if $A \subseteq X$ is a convex cone containing $0 \in X$ then it is translative with respect to k if and only if $k \in A$. See Observation 1 and [32].

2.2 Radially closed sets

Definition 3 *A set* $A \subseteq X$ *is said to be* radially closed *with respect to* K *iff* $x \in A$*,* $\{v^n\}_{n \in \mathbb{N}} \subset \mathbb{R}^m$*,* $\lim_{n \to \infty} v^n = v$ *and* $x + \sum_{i=1}^m v_i^n k^i \in A$ *implies* $x + \sum_{i=1}^m v_i k^i \in A$*.*

Note that only topological properties of $\mathbb{R}^m$ enter into this definition, not of X.

In the following, speaking of a set being radially closed we always mean a set being radially closed with respect to a given fixed collection K.

Note that radial closedness with respect to K is weaker in general than the property of being algebraically closed. For example, the set $A = \left\{x \in \mathbb{R}^2 :\ x_1 > 0\right\}$ is radially closed with respect to $K = \left\{k = (0,1)^T\right\}$, but not algebraically closed. Therefore, it is not a "coherent acceptance set" in the sense of [32].

Observation 2. A set $A \subseteq X$ is radially closed if and only if $y \in X$, $\{w^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$, $\lim_{n\to\infty} w^n = 0 \in \mathbb{R}^m$ and $y + \sum_{i=1}^m w_i^n k^i \in A$ implies $y \in A$. One implication can be seen replacing y by $x + \sum_{i=1}^m v_i k^i$ and setting $w^n = v^n - v$, the other one replacing x by $y + \sum_{i=1}^m w_i^0 k^i \in A$ and setting $v^n = w^n - w^0$. Using this, it is not hard to see that the radial closure of a convex set and of a cone is again convex and a cone, respectively.

Definition 4 *The intersection of all subsets of X being radially closed and containing $A \subseteq X$ is called the* radially closed hull *of A with respect to K. It is denoted by* $\operatorname{ra} A$.

Lemma 2 *It holds $A \subseteq \operatorname{ra} A$, the set $\operatorname{ra} A$ is radially closed and $\operatorname{ra}(\operatorname{ra} A) = \operatorname{ra} A$.*

Proof. By definition, $A \subseteq \operatorname{ra} A$. We shall show that $\operatorname{ra} A$ is radially closed. Take $x \in \operatorname{ra} A$, $v^n \to v$ in $\mathbb{R}^m$ such that $x + \sum_{i=1}^m v_i^n k^i \in \operatorname{ra} A$ for each $n \in \mathbb{N}$. Then, by definition of $\operatorname{ra} A$, $x + \sum_{i=1}^m v_i^n k^i \in B$ whenever $A \subseteq B \subseteq X$ and B is radially closed. Then $x + \sum_{i=1}^m v_i k^i \in B$ for each such B. Hence $x + \sum_{i=1}^m v_i k^i \in \operatorname{ra} A$. Now, $\operatorname{ra}(\operatorname{ra} A) = \operatorname{ra} A$ is obvious. ■

Lemma 3 *For any set $A \subseteq X$,*

$$\operatorname{ra} A = \left\{ y \in X : \exists \{w^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m : \lim_{n\to\infty} w^n = 0,\ \forall n \in \mathbb{N} : y + \sum_{i=1}^m w_i^n k^i \in A \right\}$$

$$= \left\{ x + \sum_{i=1}^m v_i k^i : x \in A,\ \exists \{v^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m : \lim_{n\to\infty} v^n = v,\ \forall n \in \mathbb{N} : x + \sum_{i=1}^m v_i^n k^i \in A \right\}$$

Proof. Straightforward using Definition 3 and Observation 2. ■

The sum of two radially closed sets is not radially closed in general. Examples that the sum of two closed sets is not closed in general work also for this case. However, the following relationships hold true.

Lemma 4 *Let $A_1, A_2, A_3 \subseteq X$. Then*

$$\operatorname{ra} A_1 \oplus \operatorname{ra} A_2 \subseteq \operatorname{ra}(A_1 \oplus A_2), \tag{8}$$

$$\operatorname{ra}(\operatorname{ra} A_1 \oplus \operatorname{ra} A_2) = \operatorname{ra}(A_1 \oplus A_2), \tag{9}$$

$$\operatorname{ra} A_1 \oplus A_2 \subseteq \operatorname{ra}(A_1 \oplus A_2), \tag{10}$$

$$\operatorname{ra}(A_1 \oplus A_2 \oplus A_3) = \operatorname{ra}(\operatorname{ra}(A_1 \oplus A_2) \oplus A_3) = \operatorname{ra}(A_1 \oplus \operatorname{ra}(A_2 \oplus A_3)). \tag{11}$$

Proof. To prove (8), take $x \in \operatorname{ra} A_1$, $y \in \operatorname{ra} A_2$. In virtue of Lemma 3, there are sequences $\{v^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$, $\{w^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ both converging to $0 \in \mathbb{R}^m$ such that

$$\forall n \in \mathbb{N} : \ x + \sum_{i=1}^m v_i^n k^i \in A_1;\ y + \sum_{i=1}^m w_i^n k^i \in A_2.$$

Hence $x + y + \sum_{i=1}^{m} (v_i^n + w_i^n)\, k^i \in A_1 \oplus A_2$. Since $v^n + w^n \to 0$, it follows $x + y \in$ ra $(A_1 \oplus A_2)$.

From (8) it follows that ra $(\mathrm{ra}\, A_1 \oplus \mathrm{ra}\, A_2) \subseteq$ ra $(A_1 \oplus A_2)$. The converse inclusion is obvious, hence (9) holds true.

The inclusion (10) follows from (8) and $A_2 \subseteq \mathrm{ra}\, A_2$ and (11) follows by repeated applications of (9) using the associativity of $\oplus$. ■

Finally, note that in the most popular case $m = 1$, $C = \mathrm{IR}_+$ and $k^1 = k \in X \setminus \{0\}$, the condition in Definition 3 shrinks down to

$$x \in A,\ \{s_n\}_{n \in \mathrm{IN}} \subset \mathrm{IR},\ \lim_{n \to \infty} s_n = s \in \mathrm{IR},\ x + s_n k \in A \quad \Longrightarrow \quad x + sk \in A. \tag{12}$$

2.3 Radially closed and translative sets

The problem is, for a given set $A \subseteq X$, to find the smallest set containing A and being radially closed and tranlative at the same time.

Definition 5 *For a set $A \subseteq X$, the intersection of all sets being radially closed, translative and containing A is called the* radially closed, translative hull *of A. It is denoted by* rt A.

Lemma 5 *It holds $A \subseteq \mathrm{rt}\, A$, the set* rt A *is radially closed, translative and* rt $(\mathrm{rt}\, A) = \mathrm{rt}\, A$ *holds true. Moreover,* $\mathrm{rt}\, A = \mathrm{ra}\, (A \oplus \Gamma_K(C)) = \mathrm{ra}\, (\mathrm{tr}\, A)$.

Proof. By definition, $A \subseteq \mathrm{rt}\, A$.

We shall show that rt A is radially closed. Take $x \in \mathrm{rt}\, A$, $v^n \to v$ in IR^m such that $x + \sum_{i=1}^{m} v_i^n k^i \in \mathrm{rt}\, A$ for each $n \in \mathrm{IN}$. Then, by definition of rt A, $x + \sum_{i=1}^{m} v_i^n k^i \in B$ whenever $A \subseteq B \subseteq X$ and B is radially closed and translative. Hence $x + \sum_{i=1}^{m} v_i k^i \in B$ for each such B and therefore $x + \sum_{i=1}^{m} v_i k^i \in \mathrm{rt}\, A$.

In order to show that rt A is translative, take a radially closed, translative $B \subseteq X$ such that $\mathrm{rt}\, A \subseteq B$. Then $\mathrm{rt}\, A \oplus \Gamma_K(C) \subseteq B \oplus \Gamma_K(C) \subseteq B$, hence $\mathrm{rt}\, A \oplus \Gamma_K(C)$ is contained in the intersections of all such B's and therefore in rt A.

Since, by definition, $\mathrm{rt}\, (\mathrm{rt}\, A) \subseteq B$ whenever $\mathrm{rt}\, A \subseteq B \subseteq X$ and B is radially closed and translative, we may choose $B = \mathrm{rt}\, A$ obtaining $\mathrm{rt}\, (\mathrm{rt}\, A) \subseteq \mathrm{rt}\, A$. The converse inclusion is obvious.

Next, we claim that $\mathrm{ra}\, (\mathrm{tr}\, A) \subseteq \mathrm{rt}\, A$. Again, take a radially closed, translative $B \subseteq X$ such that $\mathrm{rt}\, A \subseteq B$. Then all the more $A \subseteq B$, hence $A \oplus \Gamma_K(C) \subseteq B \oplus \Gamma_K(C) \subseteq B$. This implies $\mathrm{ra}\, (A \oplus \Gamma_K(C)) \subseteq B$ since $B = \mathrm{ra}\, B$, and the claim follows.

In order to show the converse inclusion we note that ra (tr A) is radially closed and contains A. The desired result follows if we can prove that ra (tr A) is also translative. Applying (10) with A replaced by $A \oplus \Gamma_K(C)$ and B by $\Gamma_K(C)$, we obtain

$$\mathrm{ra}\, (A \oplus \Gamma_K(C)) \oplus \Gamma_K(C) \subseteq \mathrm{ra}\, (A \oplus \Gamma_K(C))$$

as desired. This completes the proof of the lemma. ■

Note that tr (ra A) is not radially closed in general. Consider, for example, the set

$$A = \left\{ x \in \mathbb{R}^2 : \; x_1 > 0, \, x_2 \geq \frac{1}{x_1} \right\}$$

and let $\Gamma_K\left(\mathbb{R}^2_+\right)$ be generated by $K = \left\{ (-1,0)^T, (-1,1)^T \right\}$. Of course, $A = \operatorname{ra} A$, but $A \oplus \Gamma_K(C)$ is not radially closed with respect to K. Note that, in contrast, $A \oplus \mathbb{R}_+ \left\{ (-1,0)^T \right\}$ is radially closed with respect to $K = \left\{ (-1,0)^T \right\}$. This suggests that in case $m = 1$, a better behaviour might be expected. The following result is due to [54].

Lemma 6 *If* $m = 1$, $C = \mathbb{R}_+$, $k = k^1 \in X \backslash \{0\}$, *then*

$$\operatorname{rt} A = \operatorname{ra}\left(\operatorname{tr} A\right) = \operatorname{tr}\left(\operatorname{ra} A\right) = \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}.$$

PROOF. We shall show that $\operatorname{ra} A \oplus \mathbb{R}_+ \{k\}$ is radially closed. Take $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ with $\lim_{n \to \infty} = s \in \mathbb{R}$ and $x \in \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}$ such that $x + s_n k \in \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}$ for all $n \in \mathbb{N}$. We have to show that $x + sk \in \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}$.

If there would exist $n_0 \in \mathbb{N}$ such that $s_{n_0} \leq s$, then

$$x + sk = x + s_{n_0} k + \left(s - s_{n_0}\right) k \in \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}.$$

Hence, we may assume that $s < s_n$ and $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$. Assume that $x + sk \notin \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}$. Then, $x + sk \notin \operatorname{ra} A$ and

$$\exists n_0 \in \mathbb{N} : \; \forall r \leq s_{n_0} : \; x + rk \notin \operatorname{ra} A.$$

Otherwise,

$$\forall n \in \mathbb{N}, \; \exists r_n \leq s_n : \; x + r_n k \in \operatorname{ra} A$$

and this would imply $x + sk \in \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}$, either since there is some n with $r_n \leq s$ or since if $s \leq r_n \leq s_n \to s$, the radially closedness of $\operatorname{ra} A$ implies $x + sk \in \operatorname{ra} A \subseteq \operatorname{ra} A \oplus \mathbb{R}_+ \{k\}$.

Hence, the monotonicity of the sequence $\{s_n\}_{n \in \mathbb{N}}$ implies

$$\forall n \geq n_0 : \; x + s_n k \notin \operatorname{ra} A$$

which contradicts the assumption. ∎

Denote by

$$\mathcal{S}(K, C) = \left\{ A \in \widehat{\mathcal{P}}(X) : \; A = \operatorname{rt} A \right\}$$

the set of all subsets of X (including the empty set that is considered to be translative and radially closed by definition) that are radially closed and translative with respect to K and C. The set $\mathcal{S}(K, C)$ can be partially ordered e.g. by $\supseteq$.

Lemma 7 *The pair* $(\mathcal{S}(K,C), \supseteq)$ *is a partially ordered, complete lattice. For a subset* $\mathcal{A} \subseteq \mathcal{S}(K,C)$ *it holds*

$$\inf\{\mathcal{A}, \supseteq\} = \operatorname{rt} \bigcup_{A \in \mathcal{A}} A, \qquad \sup\{\mathcal{A}, \supseteq\} = \bigcap_{A \in \mathcal{A}} A$$

Proof. Of course, it holds $\operatorname{rt} \bigcup_{A \in \mathcal{A}} A \supseteq A'$ for each $A' \in \mathcal{A}$. The infimum property follows from the fact that if $B \in \mathcal{S}(K,C)$ with $B \supseteq A$ for each $A \in \mathcal{A}$, then $B \supseteq \operatorname{rt} \bigcup_{A \in \mathcal{A}} A$ by definition of the radially closed, translative hull operator.

Concerning the supremum we shall note that $\bigcap_{A \in \mathcal{A}} A \in \mathcal{S}(K,C)$ since on one hand $\bigcap_{A \in \mathcal{A}} A \subseteq \operatorname{rt} \bigcap_{A \in \mathcal{A}} A$ and on the other hand $\operatorname{rt} \bigcap_{A \in \mathcal{A}} A \subseteq B$ for each $B \in \mathcal{S}(K,C)$ with $\bigcap_{A \in \mathcal{A}} A \subseteq B$. Hence $\operatorname{rt} \bigcap_{A \in \mathcal{A}} A \subseteq A'$ for each $A' \in \mathcal{A}$, hence $\operatorname{rt} \bigcap_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} A$. The supremal property of the intersection with respect to $\supseteq$ is well-known and easy to check.

From these formula it follows that every subset of $\mathcal{S}(K,C)$ has an infimum and a supremum in $\mathcal{S}(K,C)$, possibly the empty set. This completes the proof. ■

Remark 2 *Denoting* $\mathcal{S}^{co}(K,C) = \{A \in \mathcal{S}(K,C) : \ A = \operatorname{co} A\}$ *and using Observation 2, we can establish a result parallel to Lemma 7. Only the infimum formula for a subset* $\mathcal{A} \subseteq \mathcal{S}^{co}(K,C)$ *has to be changed to*

$$\inf\{\mathcal{A}, \supseteq\} = \operatorname{rt}\left(\operatorname{co} \bigcup_{A \in \mathcal{A}} A\right).$$

Similarly, we can select the class of all convex cones in $\mathcal{S}(K,C)$*.*

Note that $\emptyset$ is the largest and X the smallest element of $\mathcal{S}(K,C)$ with respect to $\supseteq$. In the following sections, we shall establish order preserving bijections between $(\mathcal{S}(K,C), \supseteq)$ and classes of functions mapping into the power set of $\mathbb{R}^m$ and enjoying a certain translation as well as a closedness property. One might see that for translative functions radial closedness of their zero sublevel sets corresponds to topological closedness of their values as subsets of $\mathbb{R}^m$.

3 Translative extended real-valued functions

In this section, objects under consideration are extended real–valued functions $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$. As usual, we denote for such functions by

$$\operatorname{dom} \varphi := \{x \in X : \ \varphi(x) \neq +\infty\}$$

the *(effective) domain*, by

$$\operatorname{epi} \varphi := \{(x,r) \in X \times \mathbb{R} : \ f(x) \leq r\}, \quad \operatorname{hypo} \varphi := \{(x,r) \in X \times \mathbb{R} : \ r \leq f(x)\}$$

the *epigraph* and the *hypograph* of φ, respectively, and by

$$S_\varphi(r) := \{x \in X : \; \varphi(x) \le r\}, \; r \in \mathrm{IR},$$

the sublevel sets of φ. The function φ is called *proper* iff the set $\{x \in X : \; \varphi(x) = -\infty\} = \emptyset$ and $\operatorname{dom}\varphi \neq \emptyset$.

A function $\varphi : X \to \mathrm{IR} \cup \{\pm\infty\}$ is called *positively homogeneous* iff $\operatorname{epi}\varphi \subseteq X \times \mathrm{IR}$ is a cone. A function $\varphi : X \to \mathrm{IR} \cup \{\pm\infty\}$ is called *subadditive (superadditive)* iff $\operatorname{epi}\varphi \subseteq X \times \mathrm{IR}$ ($\operatorname{hypo}\varphi \subseteq X \times \mathrm{IR}$) is closed under addition. A function $\varphi : X \to \mathrm{IR} \cup \{\pm\infty\}$ is called *convex (concave)* iff $\operatorname{epi}\varphi \subseteq X \times \mathrm{IR}$ ($\operatorname{hypo}\varphi \subseteq X \times \mathrm{IR}$) is a convex set.

It is well-known that a positively homogeneous function $\varphi : X \to \mathrm{IR} \cup \{\pm\infty\}$ is convex (concave) if and only if it is subadditive (superadditive).

In view of Section 2, the setting of this section is $m = 1$, $C = \mathrm{IR}_+$, $k = k^1 \in X \backslash \{0\}$.

3.1 Algebraic features

In this subsection, we shall introduce the concept translativity for extended real–valued functions and investigate the relationships between translative functions and their zero sublevel sets on one hand and between translativity and other algebraic properties of functions on the other hand.

Definition 6 *A function $\varphi : X \to \mathrm{IR} \cup \{\pm\infty\}$ is called* translative *with respect to $k \in X \backslash \{0\}$ iff*

$$\forall x \in X, \; \forall s \in \mathrm{IR} : \; \varphi(x + sk) = \varphi(x) - s. \tag{13}$$

Note that (13) is in fact equivalent to

$$\forall x \in X, \; \forall s \in \mathrm{IR} : \; \varphi(x + sk) \le \varphi(x) - s.$$

Observation 3. A function φ that is translative with respect to $k \in X \backslash \{0\}$ has a nonempty domain if and only if the sublevel set $S_\varphi(0)$ is nonempty. This is trivial if $\varphi \equiv -\infty$. If $x \in X$ such that $-\infty < \varphi(x) < +\infty$, then, by (13), $\varphi(x + \varphi(x)k) = 0$, hence $x + \varphi(x)k \in S_\varphi(0)$. The converse implication is obvious.

Obviously, the functions $\varphi(x) \equiv -\infty$ and $\varphi(x) \equiv +\infty$ are the only functions that can satisfy (13) with respect to $k = 0 \in X$. Therefore, we do not consider the case $k = 0$.

The next theorem shows that all sublevel sets of functions satisfying (13) are translates of the zero sublevel set. This justifies the term *translative* for (13). Therefore, we prefer using this term rather than *plus-homogenity* [51] or *translation invariance* [2] or *translation equivariance* in [53]. For (iv) of the theorem below, compare [62], Lemma 3, (i) and, in a rather finite dimensional setting, [51], Lemma 4.1.

Theorem 1 *The following conditions are equivalent for a function $\varphi : X \to \mathrm{IR} \cup \{\pm\infty\}$:*

(i) $\operatorname{epi}\varphi \subseteq X \times \mathrm{I\!R}$ *is translative with respect to* $(k,-1)$ *and* $(-k,1)$*;*
(ii) φ *is translative with respect to* $k \in X \setminus \{0\}$*;*
(iii) $\operatorname{epi}\varphi = \{(x,s) \in X \times \mathrm{I\!R} : \ x + sk \in S_\varphi(0)\}$*;*
(iv) $\forall s \in \mathrm{I\!R}$*:* $S_\varphi(s) = S_\varphi(0) \oplus \{-sk\}$*.*

Proof. (i) $\Rightarrow$ (ii): First, consider the case $\varphi(x) = +\infty$. Take $s \in \mathrm{I\!R}$ and assume $\varphi(x+sk) < +\infty$. Then, there is $s' \in \mathrm{I\!R}$ such that $(x+sk, s') \in \operatorname{epi}\varphi$. Then, by (i),

$$\forall r \in \mathrm{I\!R} : \ (x + sk + rk, s' - r) \in \operatorname{epi}\varphi,$$

which gives for $r = -s$ especially $(x, s'+s) \in \operatorname{epi}\varphi$ contradicting $\varphi(x) = +\infty$. Next, assume $\varphi(x) \in \mathrm{I\!R}$ and take $s \in \mathrm{I\!R}$. Then $(x+sk, \varphi(x) - s) \in \operatorname{epi}\varphi$, hence $\varphi(x+sk) \leq \varphi(x) - s$ which is enough for translativity. Finally, if $\varphi(x) = -\infty$ and $s \in \mathrm{I\!R}$, then $(x+sk, r-s) \in \operatorname{epi}\varphi$ for all $r \in \mathrm{I\!R}$, hence $\varphi(x+sk) \leq r - s$ for all $r \in \mathrm{I\!R}$. This forces $\varphi(x+sk) = -\infty$.

(ii) $\Rightarrow$ (i): Take $(x,r) \in \operatorname{epi}\varphi$ and $s \in \mathrm{I\!R}$. Then, by (13), $\varphi(x+sk) \leq \varphi(x) - s \leq r - s$, hence $(x,r) + s(k,-1) = (x+sk, r-s) \in \operatorname{epi}\varphi$. Since $s \in \mathrm{I\!R}$ is arbitrary, this proves (i).

(ii) $\Rightarrow$ (iii): Translativity yields that we have $(x,s) \in \operatorname{epi}\varphi$ if and only if $\varphi(x+sk) = \varphi(x) - s \leq 0$ and further if and only if $x + sk \in S_\varphi(0)$.

(iii) $\Rightarrow$ (iv): We have $x \in S_\varphi(s)$ if and only if $(x,s) \in \operatorname{epi}\varphi$ and this by (iii) if and only if $x + sk \in S_\varphi(0)$.

(iv) $\Rightarrow$ (ii): First, consider the case $\varphi(x) = +\infty$. Take $s \in \mathrm{I\!R}$ and assume $\varphi(x+sk) < +\infty$. Then, there is $r \in \mathrm{I\!R}$ such that $x + sk \in S_\varphi(r)$, hence $x + (s+r)k \in S_\varphi(0) = S_\varphi(s+r) \oplus \{(s+r)k\}$. Therefore, $x = x + (s+r)k - (s+r)k \in S_\varphi(s+r)$, a contradiction. Next, assume $\varphi(x) \in \mathrm{I\!R}$, i.e., $x \in S_\varphi(\varphi(x)) = S_\varphi(0) \oplus \{-\varphi(x)k\}$. Then, for $s \in \mathrm{I\!R}$, $x + sk \in S_\varphi(0) \oplus \{s - \varphi(x)k\} = S_\varphi(\varphi(x) - s)$. This gives $\varphi(x+sk) \leq \varphi(x) - s$ which is enough for translativity. Thirdly, if $\varphi(x) = -\infty$ and $s \in \mathrm{I\!R}$, then $x + sk \in S_\varphi(r) \oplus \{sk\} = S_\varphi(r-s)$ for all $r \in \mathrm{I\!R}$. This forces $\varphi(x+sk) = -\infty$ and completes the proof of the theorem. ■

Condition (i) in Theorem 1 means that if $(x,r) \in \operatorname{epi}\varphi$, then the whole straight line starting at (x,r) in direction $(k,-1)$ (and $(-k,1)$) is contained in $\operatorname{epi}\varphi$. Compare the discussion in Section 3 of [51] for the finite dimensional case and a special k.

Proposition 1 *Let* $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ *be translative with respect to* $k \in \operatorname{dom}\varphi$, $k \neq 0$*. If* $\varphi(0) = 0$*, then* $\varphi(k) = -\varphi(-k) = -1$*. Moreover,* φ *is linear on the one dimensional subspace* $L(k)$ *spanned by* k*.*

Proof. Straightforward. ■

There is a reverse of the last proposition for subadditive functions.

Proposition 2 *Let* $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ *be a subadditive function such that* $0 \neq y \in \operatorname{dom}\varphi$ *and* $\varphi(y) \neq 0$*. If* φ *is linear on the one dimensional subspace* $L(y)$ *spanned by* y*, then* φ *is translative with respect to*

$$k := -\frac{1}{\varphi(y)} y.$$

PROOF. Take $s \in \mathbb{R}$. Since φ is linear on $L(y)$, we have $\varphi(k) = -1$ and $\varphi(sk) = s\varphi(k) = -s$. This implies

$$\forall x \in X: \ \varphi(x + sk) \leq \varphi(x) + \varphi(sk) = \varphi(x) - s$$

since φ is subadditive. As remarked, the latter inequality is sufficient for (13). ■

Observation 4. Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ be a subadditive function such that $\varphi(0) = 0$. Then φ is translative with respect to $k \in X \backslash \{0\}$ if and only if $\varphi(k) = -1$ and φ is linear on the one dimensional subspace spanned by k. A special case for this situation is the directional derivative of a sublinear function: Let $p : X \to \mathbb{R} \cup \{+\infty\}$ be sublinear and x_0 a point of the algebraic interior of the domain of p. Then, the function

$$\varphi(x) := \inf_{t>0} \frac{1}{t} \left(p(x_0 + tx) - p(x_0) \right)$$

is everywhere finite, sublinear and linear on the one dimensional linear subspace *span* $\{x_0\}$. If $x_0 \neq 0$ and $p(x_0) > 0$, then φ is translative with respect to $-\frac{1}{p(x_0)} x_0$.

In the following, we investigate the relationships of translative functions and their zero sublevel sets. Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Define the set $A_\varphi \subseteq X$ by

$$A_\varphi := S_\varphi(0) := \{x \in X : \ \varphi(x) \leq 0\}. \tag{14}$$

Let $A \subseteq X$ be a set and $k \in X \backslash \{0\}$. Define the function $\varphi_A : X \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\varphi_A(x) := \inf \{t \in \mathbb{R} : \ x + tk \in A\} \tag{15}$$

agreeing on $\inf \emptyset = +\infty$, $\inf \mathbb{R} = -\infty$.

Observation 5. Let $x' \in X'$ be a linear function on X and let us consider the set $A := \{x \in X : \ x'(x) \leq 0\}$. It is not hard to see that the following formula holds true iff $x'(k) < 0$:

$$\forall x \in X : \ x'(x) = -x'(k) \varphi_A(x),$$

i.e., φ_A is linear if A is a halfspace with k in its algebraic interior. If $x'(k) \geq 0$, then φ_A takes only values in $\{-\infty, +\infty\}$.

For the following results, recall (12) as the condition for a set to be radially closed with respect to $k \in X \backslash \{0\}$, compare also Definition 3 with $K = \{k\}$.

Proposition 3 *(i) For $A \subseteq X$, φ_A is translative with respect to $k \in X \backslash \{0\}$ and $A \subseteq A_{\varphi_A}$. If A is radially closed and translative with respect to k, then $A = A_{\varphi_A}$. (ii) Let φ be translative with respect to $k \in X \backslash \{0\}$. Then A_φ is translative, radially closed with respect to k and $\varphi = \varphi_{A_\varphi}$.*

PROOF. (i) Take $s \in \mathbb{R}$, $x \in X$. From the definition of φ_A it follows

$$\begin{aligned} \varphi_A(x + sk) &= \inf \{t \in \mathbb{R} : \ x + (t + s) k \in A\} \\ &= \inf \{t + s \in \mathbb{R} : \ x + (t + s) k \in A\} - s \\ &= \varphi_A(x) - s \end{aligned}$$

with $\varphi_A(x+sk) = +\infty$ for all $s \in \mathbb{R}$ if and only if $\varphi_A(x) = +\infty$ and $\varphi_A(x+sk) = -\infty$ for all $s \in \mathbb{R}$ if and only if $\varphi_A(x) = -\infty$.

Obviously, $A \subseteq A_{\varphi_A}$. To show the converse, take $x \in A_{\varphi_A}$, i.e., $\varphi_A(x) \leq 0$. Set $P_A(x) := \{t \in \mathbb{R} : \; x + tk \in A\}$. Let $\bar{t} \in P_A(x)$. Then $t \in P_A(x)$ for all $t \geq \bar{t}$ since

$$x + tk = x + \bar{t}k + (t - \bar{t})\, k \in A$$

due to (7) and $t - \bar{t} \geq 0$. Hence $P_A(x)$ is either $\emptyset$, $\mathbb{R}$ or of the form $[\bar{t}, +\infty)$, $(\bar{t}, +\infty)$. (For this discussion, compare [36], Subsection 1.1.4 in the sublinear case.) The latter case can not occure since A is radially closed with respect to k. Hence either $\varphi_A(x) = +\infty$, $\varphi_A(x) = -\infty$ or the infimum in (15) is attained. Since $\varphi_A(x) \leq 0$, the first case is not possible. In the other cases, there is $s \geq 0$ such that $x - sk \in A$. The translativity of A implies $\{x - sk\} \oplus \mathbb{R}_+ \{k\} \subseteq A$ and therefore $x \in A$. This proves the first part of the proposition.

(ii) Take $x \in A_\varphi$, $s \geq 0$. Then, by (13),

$$\varphi(x + sk) = \varphi(x) - s \leq -s \leq 0,$$

hence $x + sk \in A_\varphi$. This means, A_φ satisfies (7).

Take $x \in A_\varphi$ and a sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ converging to $s \in \mathbb{R}$ such that $x + s_n k \in A_\varphi$. Then

$$\forall n \in \mathbb{N} : \; \varphi(x + s_n k) = \varphi(x) - s_n \leq 0,$$

implying $\varphi(x + sk) = \varphi(x) - s \leq 0$. This means, $x + sk \in A_\varphi$, i.e., A_φ is radially closed.

With the help of (13), we may obtain

$$\begin{aligned} p_{A_\varphi}(x) &= \inf\{t \in \mathbb{R} : \; x + tk \in A_\varphi\} \\ &= \inf\{t \in \mathbb{R} : \; \varphi(x + tk) \leq 0\} \\ &= \inf\{t \in \mathbb{R} : \; \varphi(x) \leq t\} = \varphi(x) \end{aligned}$$

finishing the proof of the proposition. ∎

Proposition 3 tells us that there is a one-to-one correspondence between radially closed, translative sets $A \subseteq X$ and translative functions. This observation is well–known in coherent risk measure theory, compare e.g. [2], Propositions 2.1, 2.2 and [32], Corollary 1. Concerning topical functions (the finite dimensional case of Proposition 3 above with $k = (-1, \ldots, -1)^T \in \mathbb{R}^n$, $A = G \subseteq \mathbb{R}^n$), compare Propositions 4.7 and 4.8 of [51]. Therein, the property of a set being radially closed is called *closed along diagonal lines* and the translation property (7) is called *plus-radiant*.

The observation (see (i) of Proposition 3) that φ_A is translative with respect to k wether or not A is translative and radially closed and that $A \subseteq A_{\varphi_A}$ is always true, gives rise to ask for the relationships between A and $B := A_{\varphi_A}$ on one hand as well as between φ_A and φ_B on the other hand. The result reads as follows (compare [54]).

Proposition 4 *Let $A \subseteq X$ be a nonempty set and $k \in X \backslash \{0\}$. Then $A_{\varphi_A} = \operatorname{rt} A$ and $\varphi_A = \varphi_{\operatorname{rt} A}$.*

PROOF. By Proposition 3, (ii), A_{φ_A} is radially closed and translative. Obviously, $A \subseteq A_{\varphi_A}$. Hence $\mathrm{rt}\, A \subseteq A_{\varphi_A}$. To see the converse inclusion, observe that for $B \subseteq X$ being radially closed and translative with $A \subseteq B$ it holds

$$\begin{aligned} A_{\varphi_A} &= \{x \in X : \varphi_A(x) \leq 0\} \\ &= \{x \in X : \inf\{t \in \mathbb{R} : x + tk \in A\} \leq 0\} \\ &\subseteq \{x \in X : \inf\{t \in \mathbb{R} : x + tk \in B\} \leq 0\} \\ &= B_{\varphi_B} = B. \end{aligned}$$

The last equation in this chain is a consequence of Proposition 3, (i). The equation $\varphi_A = \varphi_{\mathrm{rt}\, A}$ is a consequence of $A_{\varphi_A} = \mathrm{rt}\, A$ and (ii) of Proposition 3. ■

Corollary 1 *Let $A \subseteq X$ be a nonempty set and $k \in X \setminus \{0\}$. Then*

$$\mathrm{epi}\, \varphi_A = \{(x, s) \in X \times \mathbb{R} : x + sk \in \mathrm{rt}\, A\}.$$

PROOF. Invoke Theorem 1, (iii) and Proposition 4. ■

Corollary 2 *Let $A \subseteq X$ be a nonempty set and $k \in X \setminus \{0\}$. Then $\varphi_A(0) \leq 0$ if and only if $0 \in \mathrm{rt}\, A$.*

PROOF. Proposition 4 tells us that $\varphi_A(0) = \varphi_{\mathrm{rt}\, A}(0)$. ■

As a consequence of Proposition 4, we have $\mathrm{rt}\, A = \mathrm{rt}\, B$ for $A, B \subseteq X$ if and only if $\varphi_A = \varphi_B$. This has been observed in [54].

Let us denote by $\mathcal{F}(k)$ the set of all functions $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ that are translative with respect to $k \in X \setminus \{0\}$ and define a partial order on $\mathcal{F}(k)$ by

$$\varphi \preceq \psi \quad :\Longleftrightarrow \quad \forall x \in X : \varphi(x) \leq \psi(x)$$

for $\varphi, \psi \in \mathcal{F}(k)$. For the sake of abbreviation, we write $\mathcal{S}(k)$ for $\mathcal{S}(\{k\}, \mathbb{R}_+)$, the set of all subsets of X that are translative and radially closed with respect to $K = \{k\}$. From Lemma 7 we know that $(\mathcal{S}(k), \supseteq)$ is a partially ordered, complete lattice.

Theorem 2 *Let X be a linear space and $k \in X \setminus \{0\}$.*
(a) For $A, B \in \mathcal{S}(k)$ it holds $A \supseteq B$ if and only if $\varphi_A \preceq \varphi_B$; for $\varphi, \psi \in \mathcal{F}(k)$ it holds $\varphi \preceq \psi$ if and only if $A_\varphi \supseteq A_\psi$;
(b) The relationships (14) and (15) define an order preserving bijection between $(\mathcal{F}(k), \preceq)$ and $(\mathcal{S}(k), \supseteq)$;
(c) $(\mathcal{F}(k), \preceq)$ is a partially ordered, complete lattices.

PROOF. (a) If $B \subseteq A$, then $\varphi_A(x) \preceq \varphi_B(x)$ for all $x \in X$ by (15). If $\varphi_A \preceq \varphi_B$ and $x \in B$, then

$$\varphi_A(x) = \inf\{t \in \mathbb{R} : x + tk \in A\} \leq \inf\{t \in \mathbb{R} : x + tk \in B\} = \varphi_B(x) \leq 0.$$

Hence $\varphi_A(x) \leq 0$ and this implies $x \in A$ since $A = \operatorname{tr} A$.

If $\varphi \preceq \psi$, then $\psi(x) \leq 0$ implies $\varphi(x) \leq 0$, hence $A_\varphi \supseteq A_\psi$. Conversely, take an arbitrary $x \in X$. If $\psi(x) = +\infty$, there is nothing to prove. If $\psi(x) \in \mathrm{I\!R}$, then $x + \psi(x) k \in A_\psi \subseteq A_\varphi$, hence $\varphi(x + \psi(x) k) = \varphi(x) - \psi(x) \leq 0$ by translativity of φ. Finally, if $\psi(x) = -\infty$, then $x + tk \in A_\psi \subseteq A_\varphi$ for all $t \in \mathrm{I\!R}$, hence $\varphi(x) = -\infty$.

(b) The relationships (14) and (15) define a bijection since for $A, B \in \mathcal{S}(k)$, we have $\varphi_A = \varphi_B$ if and only if $A = B$ and for $\varphi, \psi \in \mathcal{F}(k)$ we have holds $\varphi = \psi$ if and only if $A_\varphi = A_\psi$. The bijection preserves order by (a).

(c) Follows from (a), (b) and Lemma 7. ∎

Corollary 3 *For $\mathcal{G} \subseteq \mathcal{F}(k)$, it holds* $\inf\{\mathcal{G}, \preceq\} = \varphi_I$ *and* $\sup\{\mathcal{G}, \preceq\} = \varphi_S$ *with*

$$\forall x \in X: \ \varphi_I(x) = \inf\{\varphi(x) : \ \varphi \in \mathcal{G}\}, \ \varphi_S(x) = \sup\{\varphi(x) : \ \varphi \in \mathcal{G}\}$$

where

$$I = \operatorname{rt} \bigcup_{\varphi \in \mathcal{G}} A_\varphi \qquad S = \bigcap_{\varphi \in \mathcal{G}} A_\varphi.$$

PROOF. The result follows from Lemma 7 and Theorem 2 with the help of (15). ∎

The next results deal with further algebraic properties of functions and sets being in relation via (14) and (15). There are still more properties (linearity and being a half space, superadditivity and closedness under addition of the hypograph etc.) for which similar assertions hold true. Compare the comprehensive investigation [59] for further examples. One point of view to these results is that important properties of functions of type (15) can be expressed as properties of their zero sublevel set.

Proposition 5 *(i) Let $A \subseteq X$ be a cone. Then φ_A is positively homogenous. (ii) Let $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ be positively homogenous. Then A_φ is a cone.*

Proposition 6 *(i) Let $A \subseteq X$ be convex. Then φ_A is convex. (ii) Let $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ be convex. Then A_φ is convex.*

Proposition 7 *(i) Let $A \subseteq X$ be closed under addition. Then φ_A is subadditive. (ii) Let $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ be subadditive. Then A_φ is closed under addition.*

The proofs of Proposition 5, 6 and 7 are straightforward via (14) and (15) and therefore omitted.

Definition 7 *Let $D \subseteq X$ be a nonempty subset of X.*

*(i) A function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is called D–*monotone *iff:*

$$x_2 - x_1 \in D \quad \Longrightarrow \quad \varphi(x_2) \leq \varphi(x_1).$$

*(ii) A set $A \subseteq X$ is called D–*upward *iff $A \oplus D \subseteq A$.*

Compare e.g. [19], [59], [21] and the references therein with respect to this monotonicity concept. In [43] (Definition 1), a set $A \subseteq \mathbb{R}^n$ is called *downward* iff $x \in A$ and $x' \leq_{\mathbb{R}^n_+} x$ implies $x' \in A$. This is equivalent to $A \oplus \left(-\mathbb{R}^n_+\right) \subseteq A$. This explains our term "upward" for the property in (ii) of the above definition which is also used for $\mathbb{R}^n_+$–upward sets in [51].

Proposition 8 *(i) If $A \subseteq X$ is D–upward, then φ_A is D–monotone. (ii) If $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ is D–monotone, then A_φ is D–upward.*

PROOF. (i) Take $x_1, x_2 \in X$ such that $x_2 - x_1 \in D$. Since then $A \oplus \{x_2 - x_1\} \subseteq A \oplus D \subseteq A$, we have the following relationsships:

$$\begin{aligned} \varphi_A(x_1) &= \inf\{t \in \mathbb{R} : \ x_1 + tk \in A\} \\ &= \inf\{t \in \mathbb{R} : \ x_2 + tk \in A \oplus \{x_2 - x_1\}\} \\ &\geq \inf\{t \in \mathbb{R} : \ x_2 + tk \in A \oplus D\} \\ &\geq \inf\{t \in \mathbb{R} : \ x_2 + tk \in A\} = \varphi_A(x_2). \end{aligned}$$

(ii) Take $x_1 \in A_\varphi$, $x_2 \in D$. Then $x_1 + x_2 \in \{x_1\} \oplus D$. Hence, by assumption, $\varphi(x_1 + x_2) \leq \varphi(x_1) \leq 0$ which gives $x_1 + x_2 \in A_\varphi$. ■

Note that if A is downward in the sense of [43], then φ_A is $\left(-\mathbb{R}^n_+\right)$–monotone.

Remark 3 *Using the properties of Propositions 5, 6, 7, 8 one might select subclasses of $\mathcal{S}(k)$ and $\mathcal{F}(k)$ in order to get bijection theorems parallel to Theorem 2, e. g. the classes $\mathcal{S}^{co}(k)$ and $\mathcal{F}^{co}(k)$ of convex elements of $\mathcal{S}(k)$ and $\mathcal{F}(k)$, respectively. See Remark 2. Corollary 4 below is a similar result and may serve as a blueprint for various one-to-one-correspondence results in different fields of applications.*

Corollary 4 *Let $k \in X \backslash \{0\}$. (i) If $A \subseteq X$ is a convex set with $\mathbb{R}_+\{k\} \bigcap (-\mathrm{rt}\, A) = \{0\}$, then φ_A is convex, translative with respect to k such that $\varphi_A(0) = 0$. If A is additionally a cone, the φ_A is sublinear. (ii) If $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ is convex, translative with respect to k and $\varphi(0) = 0$, then A_φ is convex, $\mathrm{rt}\, A_\varphi = A_\varphi$ and $\mathbb{R}_+\{k\} \bigcap (-A_\varphi) = \{0\}$ holds true. If φ is additionally positively homogenous, then A_φ is a convex cone.*

PROOF. Combine Propositions 3, 5, 6, 7 and Corollary 2. Observe also that $\varphi_A(0) \geq 0$ if and only if $\mathbb{R}_+\{k\} \bigcap (-A) \subseteq \{0\}$ is true and $\varphi(0) \geq 0$ if and only if $\mathbb{R}_+\{k\} \bigcap (-A_\varphi) \subseteq \{0\}$. ■

A monotonicity condition as in Proposition 8 can be added in (i) and (ii) of Corollary 4. It is an abstract version of corresponding results for convex and coherent risk measures, see below. Compare also Section 5 of [43] and Proposition 4.8 of [51].

Up to now, the trivial cases $A = X, \emptyset$ and $\varphi \equiv -\infty, +\infty$ are not excluded. We have $\mathrm{tr}\, A = X$ if and only if $\varphi_A \equiv -\infty$ and $A = \emptyset$ if and only if $\varphi_A \equiv +\infty$. If φ is translative with respect to $k \in X \backslash \{0\}$, then $\varphi \equiv -\infty$ if and only if $A_\varphi = X$ and $\varphi \equiv +\infty$ if and only if $A_\varphi = \emptyset$. Moreover, we have the following result.

Proposition 9 *Let $k \in X \backslash \{0\}$. (i) If $A \subseteq X$ is nonempty such that*

$$\forall x \in X, \; \exists t \in \mathrm{I\!R}: \; x + tk \notin \operatorname{tr} A, \tag{16}$$

then φ_A is proper. If (16) holds true and $X = A \oplus \mathrm{I\!R}\{k\}$, then $\varphi_A(X) \subseteq \mathrm{I\!R}$. (ii) If $\varphi : X \to \mathrm{I\!R} \cup \{+\infty\}$ is proper and translative with respect to k, then A_φ is nonempty and

$$\forall x \in X, \; \exists t \in \mathrm{I\!R}: \; x + tk \notin A_\varphi. \tag{17}$$

If φ is translative with respect to k and $\varphi(X) \subseteq \mathrm{I\!R}$, then (17) holds true and $X = A_\varphi \oplus \mathrm{I\!R}\{k\}$.

Proof. (i) By definition of φ_A, $A \neq \emptyset$ implies $\operatorname{dom} \varphi_A \neq \emptyset$. If $x + tk \notin \operatorname{tr} A$, then $x + t'k \notin \operatorname{tr} A = A \oplus \mathrm{I\!R}_+\{k\}$ for all $t' \leq t$, hence $t \leq \varphi_A(x) = \varphi_{\operatorname{tr} A}(x)$, i.e., $\varphi_A(x) = -\infty$ is not possible. On the other hand, if $X = A \oplus \mathrm{I\!R}\{k\}$, then $\varphi_A(x) = +\infty$ is not possible. This proves (i).

(ii) It suffices to note that $x \in \operatorname{dom} \varphi$ implies $x + \varphi(x)k \in A_\varphi$. ■

Usually, coherent (and convex) risk measures (see [3], [7], [14], even Definition 2.1 in [42]) as well as nonlinear separation functionals (see [19]) are assumed to be real–valued. In [53], functions with values in $\mathrm{I\!R} \cup \{\pm\infty\}$ are considered, but conditions for finite valuedness are not given. In [32], conditions for the properness of φ_A for A being a convex cone are given. In [19], necessary and sufficient conditions for $\varphi_A(X) \subseteq \mathrm{I\!R}$ close to those of Proposition 9 can be found. Compare also Theorem 2.3.1 (b), (c) in [21] within a topological setting. In [43], Remark 2, a similar property for the special case $A \subseteq \mathrm{I\!R}^n$ (and downward) can be found, but without any reference to previous works. Another algebraic characterization is the following result due to [54].

Corollary 5 *(i) Let $A \subseteq X$ be translative with respect to $k \in X \backslash \{0\}$. Then $\varphi_A(X) \subseteq \mathrm{I\!R}$ if and only if $A \oplus \mathrm{I\!R}\{k\} = X \backslash A \oplus \mathrm{I\!R}\{k\} = X$. (ii) Let $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ be translative with respect to $k \in X \backslash \{0\}$. Then $\varphi(X) \subseteq \mathrm{I\!R}$ if and only if $A_\varphi \oplus \mathrm{I\!R}\{k\} = X \backslash A_\varphi \oplus \mathrm{I\!R}\{k\} = X$.*

Proof. (i) Note that (16) can be re-written as $X = X \backslash A \oplus \mathrm{I\!R}\{k\}$ if A is translative. Now, the assertions follows from (i) of Proposition 9.

(ii) Is a consequence of part (i) since $\varphi = \varphi_{A_\varphi}$ by Proposition 3, (ii). ■

The importance of Proposition 9 and Corollary 5 is illustrated by the following result.

Corollary 6 *Let $D \subseteq X$ be a pointed (i.e., $D \cap -D = \{0\}$) convex cone containing $0 \in X$ and $k \in D \backslash \{0\}$. Then, $\varphi_D(X) \subseteq \mathrm{I\!R}$ if and only if k is an order unit for the partial order $\leq_D$ generated by D.*

Proof. If φ_D is everywhere finite, then translativity implies $\varphi_D(x + \varphi_D(x)k) = 0$, hence $x + \varphi_D(x)k \in K$ on one hand and $\varphi_D(-x + \varphi_D(-x)k) = 0$, hence $-x +$

$\varphi_D(-x)\, k \in K$. Therefore, for each $x \in X$ there is $s \in \mathrm{I\!R}$ such that $-sk \leq_D x \leq_D sk$, i.e., k is an order unit. Conversely, assuming $-sk \leq_D x \leq_D sk$ for $x \in X$, $s \in \mathrm{I\!R}$, we may conclude that $x + sk \in K$, hence $\varphi_D(x) < +\infty$ on one hand and $x - sk \in -K$ on the other hand which either implies $\varphi_D(x) = \varphi_D(sk) = -s$ or $x - sk \notin K$, hence $-\infty < \varphi_D(x)$ in each case. ■

If φ_D is everywhere finite, then the function

$$x \to \inf\{t \in \mathrm{I\!R} : \ x + tk \in K, \ -x + tk \in K\} = \max\{\varphi_D(x), \varphi_D(-x)\}$$

is a norm on X and, by definition, φ_D is globally Lipschitz continuous with constant 1 (non-expansive) with respect to this norm. This property is of theoretical (see e.g. Section 4 of [44] and Chapter 1 of [36]) and practical relevance (see e.g. [23] and the references therein).

3.2 Topological features

Let X be a topological linear space and consider $\mathrm{I\!R}$ to be supplied with the usual topology. Then $X \times \mathrm{I\!R}$ supplied with the corresponding product topology is a topological linear space as well. A function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is called *lower (upper) semicontinuous* iff the set $\operatorname{epi}\varphi \subseteq X \times \mathrm{I\!R}$ ($\operatorname{hypo}\varphi \subseteq X \times \mathrm{I\!R}$) is closed.

There are several equivalent characterizations of lower (upper) semicontinuity in infinite dimensional spaces, compare [12] or [56], 5.2 and 5.7. For instance, φ is lower semicontinuous if and only if the sublevel set $S_\varphi(r) \subseteq X$ for each $r \in \mathrm{I\!R}$ is closed.

Theorem 1 has an important consequence concerning lower semicontinuity.

Corollary 7 *Let X be a topological linear space. (i) If $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is translative with respect to $k \in X\backslash\{0\}$, then φ is lower semicontinuous if and only if A_φ is closed. (ii) If $A \subseteq X$ is radially closed and translative with respect to $k \in X\backslash\{0\}$, then A is closed if and only if φ_A is lower semicontinuous.*

Proof. A function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is lower semicontinuous if and only if each of its sublevel sets is closed. Invoking Theorem 1, we get the results. ■

Note that if a set $A \subseteq X$ is closed, then it is all the more radially closed with respect to any $k \in X\backslash\{0\}$. Therefore, the following corollary is a consequence of Proposition 3 and Corollary 7.

Corollary 8 *Let X be a topological linear space. (i) Let $A \subseteq X$ be closed and translative with respect to $k \in X\backslash\{0\}$. Then φ_A is lower semicontinuous, translative with respect to k and $A = A_{\varphi_A}$ holds true. (ii) Let φ be lower semicontinuous and translative with respect to $k \in X\backslash\{0\}$. Then A_φ is closed, translative with respect to k and $\varphi = \varphi_{A_\varphi}$ holds true. (iii) There is a one-to-one-correspondence between lower semicontinuous translative functions $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ and closed translative sets $A \subseteq X$ via (14) and (15).*

Corresponding results within the framework of coherent risk measures are Proposition 2.1, 2.2 in [2] and [3] and, more detailed, Corollary 3 in [32]. With respect to topical functions, compare Proposition 4.7, 4.8 in [51].

Next, we ask for conditions characterizing the continuity of φ_A. A function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is said to be *continuous* iff φ as well as $-\varphi$ are lower semicontinuous using the convention $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$. Note that φ can still have $+\infty$ and $-\infty$ among its values. Therefore, this concept does not coincide with the usual concept of continuity. For example, define a function $g : \mathrm{I\!R} \to \mathrm{I\!R} \cup \{\pm\infty\}$ by $g(t) = -\infty$ whenever $t \leq -\frac{\pi}{2}$, $g(t) = \tan t$ whenever $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $g(t) = +\infty$ whenever $t \geq \frac{\pi}{2}$. Consider $A := \left\{(x_1, x_2) \in \mathrm{I\!R}^2 : \; g(x_1) \leq x_2\right\}$ and $k = (0,1)^T$. Then $\varphi_A : \mathrm{I\!R}^2 \to \mathrm{I\!R} \cup \{\pm\infty\}$ is continuous in the sense above.

However, it is also possible to extend the topology of $\mathrm{I\!R}$ to $\mathrm{I\!R} \cup \{\pm\infty\}$ in order to get a topology based characterization of this type of continuity property.

Corollary 9 *Let X be a topological linear space and $k \in X \backslash \{0\}$.*
(i) If $A \subseteq X$ is a closed set such that $A \oplus (0, +\infty)\{k\} \subseteq \operatorname{int} A$, then

(a) for each $t \in \mathrm{I\!R}$, $\{x \in X : \; \varphi_A(x) < t\} = \{-tk\} \oplus \operatorname{int} A$ and $\varphi_A(x) = t$ if and only if $x \in \{-tk\} \oplus \operatorname{bd} A$;

(b) φ_A is continuous and it holds $\operatorname{int}(\operatorname{dom}\varphi_A) = \bigcup_{t \in \mathrm{I\!R}} \{-tk\} \oplus \operatorname{int} A$.

(ii) If $\varphi : X \to \mathrm{I\!R}$ is translative with respect to k and continuous, then

(c) for each $t \in \mathrm{I\!R}$, $\{x \in X : \; \varphi(x) < t\} = \{-tk\} \oplus \operatorname{int} A_\varphi$ and $\varphi(x) = t$ if and only if $x \in \{-tk\} \oplus \operatorname{bd} A_\varphi$;

(d) A_φ is closed and it holds $\operatorname{int} A_\varphi \neq \emptyset$, $A_\varphi \oplus (0, +\infty)\{k\} \subseteq \operatorname{int} A_\varphi$ as well as $\operatorname{int}(\operatorname{dom}\varphi) = \bigcup_{t \in \mathrm{I\!R}} \{-tk\} \oplus \operatorname{int} A_\varphi$.

Proof. (i) First, assume that $t \in \mathrm{I\!R}$ and $x + tk \in \operatorname{int} A$. Then, there is $\varepsilon > 0$ such that $x + tk - \varepsilon k \in A$, hence $\varphi_A(x) \leq t - \varepsilon < t$. Conversely, assume that $\varphi_A(x) < t$ for $x \in X$ and $t \in \mathrm{I\!R}$. Then, there is $s < t$ such that $x + sk \in A$ by definition of φ_A. This implies

$$x \in \{-sk\} \oplus A = \{-t + (t-s)k\} \oplus A \subseteq \{-tk\} \oplus \operatorname{int} A$$

by assumption.

Next, if $\varphi_A(x) = t$, then for all $s > t$ we have $x + sk \in A$. Since A is closed, this implies $x + tk \in A$. On the other hand, for all $s < t$ we have $x + sk \notin A$, hence $x + tk \notin \operatorname{int} A$. This gives $x + tk \in A \backslash \operatorname{int} A = \operatorname{bd} A$.

Conversely, if $x + tk \in \operatorname{bd} A$, then $x + tk \in A$ since A is closed. Hence $\varphi(x) \leq t$. On the other hand, if $x + sk \in A$ for some $s < t$, then by assumption $x + tk = x + sk + (t-s)k \in A \oplus (0, +\infty)\{k\} \subseteq \operatorname{int} A$ which is a contradiction since $x + tk \in A \backslash \operatorname{int} A$. This proves $\varphi(x) \geq t$ and therefore equality. The proof of (a) is complete.

Since A is closed, φ_A is lower semicontinuous. But $-\varphi_A$ is also lower semicontinuous since its sublevel sets are the complements of $\{x \in X : \; \varphi_A(x) < t\}$ and therefore closed. Hence φ_A is continuous.

The formula for the interior of the domain of φ_A follows immediately from (a). This completes the proof of (i).

(ii) Since φ is especially lower semicontinuous, A_φ is closed. From Proposition 3, (ii) we get that $\varphi = \varphi_{A_\varphi}$ since φ is translative. We shall show that $\varphi(x) < t$ implies that $x + tk \in \operatorname{int} A_\varphi$. Indeed, since φ is continuous, there is a neighborhood N of $0 \in X$ such that $\varphi(x') < t$ whenever $x' \in \{x\} \oplus N$. Hence $\varphi(x' + tk) = \varphi(x') - t < 0$, i.e., $\{x + tk\} \oplus N \subseteq A_\varphi$ and $x + tk \in \operatorname{int} A_\varphi$. The remaining part follows from part (i) applied to φ_{A_φ}. ■

Part (i) of Corollary 9 is a refinement of part (f) of Theorem 2.3.1. in [21], compare also (ii) of Lemma 3 in [62]. It is a well-known fact from Convex Analysis that a convex function is continuous on the interior of its domain. Corollary 9 might be considered as the counterpart of this result for translative functions.

Application: Nonlinear Separation. In [45] and [18], nonlinear separation functionals of type (15) in a rather general setting were used for the first time in order to scalarize vector optimization problems. See also [62], [19]. In [59] and the subsequent papers [58], [57], [61] many properties of functionals of type (15) can be found. Theorem 2.3.6. of [21] contains the separation theorems of [18] and [19] as special cases. We shall state a similar result in order to show the main idea. More general results in a merely algebraic setting are in [54] and the forthcoming paper [30].

Theorem 3 *Let X be a linear space, $A \subseteq X$ a proper subset, radially closed and translative with respect to $k \in X \backslash \{0\}$ and $B \subseteq X$ such that $A \cap B = \emptyset$. Then*

$$\forall a \in A,\ b \in B:\ \varphi_A(a) \leq 0 < \varphi_A(b).$$

If, additionally, $A \oplus \mathbb{R}\{k\} = X \backslash A \oplus \mathbb{R}\{k\} = X$, then φ_A is finite valued. If, additionally, X is a topological linear space and $A \oplus (0, +\infty)\{k\} \subseteq \operatorname{int} A$, then φ_A is continuous.

Proof. The first assertion is a consequence of the fact that $A = A_{\varphi_A}$ is the sublevel set of φ for the level $0 \in \mathbb{R}$. The second and the third assertion follow from Proposition 5, (i) and Corollary 9, (i), respectively. ■

For applications of these ideas to scalarization of optimization problems with vectorvalued objective compare [62], [41], [19] and the book [21].

3.3 Duality features

In this section, let X be a separated, locally convex, topological linear space. See [35] for a definition and basic results. The topological dual of X is denoted by X^* and by $x^*(x)$ we denote the value of the continuous linear functional $x^* \in X^*$ at $x \in X$.

We focus on the convex conjugate and the so–called dual representation of convex functions being translative with respect to $k \in X \backslash \{0\}$. The convex conjugate (polar function, Fenchel conjugate) of a function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ is the function $\varphi^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi^*(x^*) = \sup_{x \in X} \{x^*(x) - \varphi(x)\}$$

and its biconjugate (bipolar) is the function $\varphi^{**} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ defined by

$$\varphi^{**}(x) = \sup_{x^* \in X^*} \{x^*(x) - \varphi^*(x^*)\}.$$

The central result about convex conjugates is the biconjugation theorem which tells us that $\varphi = \varphi^{**}$ if and only if φ is proper, lower semicontinuous and convex or φ is identically $+\infty$ or $-\infty$. See [12], Proposition 4.1, [56], 6.18.

The following theorem contains the essentials of the duality features for convex translative functions. It shows the strong relationsship between a convex translative function $\varphi : X \to \mathrm{I\!R} \cup \{+\infty\}$ and the support function of its zero sublevel set A_φ which is defined by

$$\sigma_{A_\varphi}(x^*) := \sup_{x \in A_\varphi} x^*(x).$$

Theorem 4 *Let X be a separated, locally convex, topological linear space. (i) Let $\varphi : X \to \mathrm{I\!R} \cup \{+\infty\}$ be proper, lower semicontinuous, convex and translative with respect to $k \in X \backslash \{0\}$. Then φ^* is proper and*
(a) with $M_k^* := \{x^* \in X^* : \; x^*(k) = -1\}$ it holds

$$\forall x^* \in M_k^* : \; \varphi^*(x^*) = \sigma_{A_\varphi}(x^*);$$

(b) $\operatorname{dom} \varphi^* = M_k^* \cap \operatorname{dom} \sigma_{A_\varphi}$*;*
(c) the following representation formula holds true:

$$\forall x \in X : \; \varphi(x) = \sup_{x^* \in M_k^*} \left\{x^*(x) - \sigma_{A_\varphi}(x^*)\right\}.$$

(ii) Let $A \subseteq X$ be a nonempty, closed, convex set that is translative with respect to $k \in X \backslash \{0\}$ and satisfies (16) of Proposition 9. Then φ_A is proper, lower semicontinuous, convex, translative with respect to $k \in X \backslash \{0\}$ and
(d) it holds

$$\forall x^* \in M_k^* : \; (\varphi_A)^*(x^*) = \sigma_A(x^*);$$

(e) $\operatorname{dom} (\varphi_A)^* = M_k^* \cap \operatorname{dom} \sigma_A$*;*
(f) the following representation formula holds true:

$$\forall x \in X : \; \varphi_A(x) = \sup_{x^* \in M_k^*} \{x^*(x) - \sigma_A(x^*)\}.$$

Proof. (i) Since φ is proper, convex and lower semicontinuous, there is an affine minorant of φ. Hence $\operatorname{dom} \varphi^*$ is nonempty and φ^* is proper since φ is.

(a) The definitions of φ^* and A_φ yield for all $x^* \in X^*$

$$\varphi^*(x^*) = \sup_{x \in X} \{x^*(x) - \varphi(x)\} \geq \sup_{x \in A_\varphi} \{x^*(x) - \varphi(x)\} \geq \sigma_{A_\varphi}(x^*). \tag{18}$$

On the other hand, for $x^* \in M_k^*$ and $x \in \operatorname{dom} \varphi$ we have $x + \varphi(x) k \in A_\varphi$ and

$$\sigma_{A_\varphi}(x^*) \geq x^*(x + \varphi(x) k) = x^*(x) - \varphi(x)$$

since $x^*(k) = -1$. Hence

$$\forall x \in X: \ \sigma_{A_\varphi}(x^*) \geq x^*(x) - \varphi(x),$$

since this inequality is all the more true if $x \notin \operatorname{dom}\varphi$. Taking the supremum over $x \in X$ on the right hand side, we obtain

$$\forall x^* \in M_k^*: \ \sigma_{A_\varphi}(x^*) \geq \varphi^*(x^*). \tag{19}$$

Together with (18), this proves (a).

(b) It follows also from (18) that $\operatorname{dom}\varphi^* \subseteq \operatorname{dom}\sigma_{A_\varphi}$. The definition of φ^*, the fact that $y + k$ with $k \neq 0$ runs through all of X if y runs through all of X and the translativity of φ yield the following equations

$$\begin{aligned}
\varphi^*(x^*) &= \sup_{x \in X}\{x^*(x) - \varphi(x)\} \\
&= \sup_{y \in X}\{x^*(y+k) - \varphi(y+k)\} \\
&= \sup_{y \in X}\{x^*(y) - \varphi(y)\} + x^*(k) + 1 \\
&= \varphi^*(x^*) + x^*(k) + 1.
\end{aligned}$$

This means that $\varphi^*(x^*) = +\infty$ if $x^*(k) \neq -1$, i.e., $\operatorname{dom}\varphi^* \subseteq M^*$. Hence $\operatorname{dom}\varphi^* \subseteq M_k^* \cap \operatorname{dom}\sigma_{A_\varphi}$ is established.

On the other hand, the inclusion $M_k^* \cap \operatorname{dom}\sigma_{A_\varphi} \subseteq \operatorname{dom}\varphi^*$ follows from (19). This proves $\operatorname{dom}\varphi^* = M_k^* \cap \operatorname{dom}\sigma_{A_\varphi}$.

(c) Since φ is proper, convex and lower semicontinuous, the biconjugation theorem yields

$$\forall x \in X: \ \varphi(x) = \sup_{x^* \in X^*}\{x^*(x) - \varphi^*(x^*)\} = \varphi^{**}(x).$$

Since $\operatorname{dom}\varphi^* \subseteq M_k^*$ and $\varphi^* = \sigma_{A_\varphi}$ on M_k^* we have

$$\sup_{x^* \in X^*}\{x^*(x) - \varphi^*(x^*)\} = \sup_{x^* \in M_k^*}\{x^*(x) - \varphi^*(x^*)\} = \sup_{x^* \in M_k^*}\left\{x^*(x) - \sigma_{A_\varphi}(x^*)\right\}.$$

(ii) It suffices to note that φ_A is proper, lower semicontinuous, convex, translative with respect to $k \in X\backslash\{0\}$ and that $A = A_{\varphi_A}$. Hence $\sigma_A = \sigma_{A_{\varphi_A}}$ and (d), (e), (f) follow from (a), (b), (c). This completes the proof of the theorem. ∎

Remark 4 *Let $M^* \subseteq M_k^*$ and a function $\psi^*: M^* \to \mathrm{I\!R} \cup \{\pm\infty\}$ be given. Then, the function*

$$\varphi(x) := \sup_{x^* \in M^*}\{x^*(x) - \psi^*(x^*)\} \tag{20}$$

is convex and lower semicontinuous since it is the supremum of continuous affine functions. It is also translative with respect to k. The definition of φ implies

$$\forall x \in X, \ \forall x^* \in M^*: \ \varphi(x) \geq x^*(x) - \psi^*(x^*).$$

Rearranging the terms and taking a supremum with respect to x, we may obtain that $\psi^*(x^*) \geq \varphi^*(x^*)$ for all $x^* \in M^*$ and $\operatorname{dom}\psi^* \subseteq \operatorname{dom}\varphi^*$. It follows that φ^* is the (pointwise) smallest function with the largest (with respect to inclusion) domain that can be used as a penalty function ψ such that (20) holds true. The expression "penalty function" is due to Foellmer and Schied, compare Proposition 9 of [14] and [15]. This construction is especially useful if X is non-reflexive. For example, if $X = L^\infty$ and φ is weakly lower semicontinuous, then it has a dual representation of the form (20) with a subset $M^* \subseteq L^1$ rather than $M^* \subseteq (L^\infty)^*$.*

Observation 6. If $\varphi_A^*(x^*) > \sigma_A(x^*)$, then $(\varphi_A)^*(x^*) = +\infty$ and $\sigma_A(x^*) \in \mathbb{R}$. This case may happen as the following example due to C. Schrage shows: Fix $x^* \in X^*$ and set $A := \{x \in X : \ x^*(x) \leq 0\}$. Choose $k \in X\backslash\{0\}$ such that $x^*(k) < -1$. Then $(\varphi_A)^*(x^*) = +\infty$, but $\sigma_A(x^*) = 0$.

Observation 7. Let $\psi^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$ be a function satisfying

$$\forall x^* \in X^* : \ \sigma_{A_\varphi}(x^*) \leq \psi^*(x^*) \leq \varphi^*(x^*).$$

Then, for all $x \in X$

$$\sup_{x^* \in M_k^*} \{x^*(x) - \varphi^*(x^*)\} \leq \sup_{x^* \in M_k^*} \{x^*(x) - \psi^*(x^*)\} \leq \sup_{x^* \in M_k^*} \left\{x^*(x) - \sigma_{A_\varphi}(x^*)\right\}$$

is true and the left as well as the right hand side are equal to $\varphi(x)$. It follows that

$$\varphi(x) = \sup_{x^* \in M_k^*} \{x^*(x) - \psi^*(x^*)\}.$$

Observation 8. If $0 \in \operatorname{dom}\varphi$, then one may assume $\varphi(0) = 0$ without loss of generality: If $\varphi(0) \in \mathbb{R}\backslash\{0\}$, one may replace φ by ψ defined by $\psi(x) := \varphi(x) - \varphi(0)$. If φ is proper, convex, lower semicontinuous and translative with respect to k, so is ψ. This process is called *normalization* in [15].

Observation 9. If $K \subseteq A_\varphi$ for some convex cone K, then $\operatorname{dom}\varphi^* \subseteq K^*$ where $K^* := \{x^* \in X^* : \ \forall x \in K : \ x^*(x) \leq 0\}$ denotes the negative dual cone of K. In fact, if $x^* \in \operatorname{dom}\varphi^*$ we have

$$\forall x \in X : \ x^*(x) - \varphi^*(x^*) \leq \varphi(x),$$

hence $x^*(x) - \varphi^*(x^*) \leq 0$ for all $x \in K$. Since $tx \in K$ if $x \in K$ and $t > 0$, this implies

$$\forall t > 0, \ x \in K : \ tx^*(x) - \varphi^*(x^*) \leq 0.$$

This is not possible if $x^*(x) > 0$.

We formulate the special case of a sublinear and translative function φ. In this case, A_φ is a convex cone, φ is the support function of $M_k^* \bigcap A_\varphi^*$ and φ^* is the indicator function of A_φ^*.

Corollary 10 *Let X be a separated, locally convex, topological linear space and $k \in X \backslash \{0\}$. (i) If the function $\varphi : X \to \mathrm{I\!R} \cup \{+\infty\}$ is lower semicontinuous, sublinear, translative with respect to k and satisfies $\varphi(0) = 0$, then*

$$\varphi^*(x^*) = \begin{cases} 0 & : \; x^*(k) = -1, \; x^* \in A_\varphi^* \\ +\infty & : \quad \text{else} \end{cases}$$

where A_φ^* is the negative dual cone of the convex cone A_φ. Moreover, it holds

$$\varphi(x) = \sup\left\{x^*(x) : \; x^*(k) = -1, \; x^* \in A_\varphi^*\right\}. \tag{21}$$

(ii) If $A \subseteq X$ is a convex cone, then the function

$$\varphi_A(x) = \sup\{x^*(x) : \; x^*(k) = -1, \; x^* \in A^*\} \tag{22}$$

is lower semicontinuous, sublinear, translative with respect to k and satisfies $\varphi_A(0) = 0$.

Proof. (i) It suffices to note that

$$\sigma_{A_\varphi}(x^*) = \sup_{x \in A_\varphi} x^*(x) = \begin{cases} 0 & : \; x^* \in A_\varphi^* \\ +\infty & : \; x^* \notin A_\varphi^*. \end{cases}$$

Hence $\operatorname{dom} \varphi^* = M_k^* \cap A_\varphi^*$ and therefore,

$$\varphi^*(x^*) = \begin{cases} 0 & : \; x^* \in M_k^* \cap A_\varphi^* \\ +\infty & : \; x^* \notin M_k^* \cap A_\varphi^* \end{cases}$$

which completes the proof of part (i). Part (ii) is obvious. ■

Remark 5 *Let $M^* \subseteq M_k^*$ be given. Then, the function*

$$\varphi(x) := \sup_{x^* \in M^*} x^*(x) = \sigma_{M^*}(x)$$

is lower semicontinuous, sublinear, translative with respect to k and satisfies $\varphi(0) = 0$. Starting with such a set M^* is a third possibility to get a coherent measure of risk. Compare Definition 3.1 in [3] and [46].

Observation 10. Formula (21), that is

$$\varphi(x) = \inf\{t \in \mathrm{I\!R} : \; x + tk \in A_\varphi\} = \sup\left\{x^*(x) : \; x^*(k) = -1, \; x^* \in A_\varphi^*\right\},$$

can be given another interpretation: The value of φ at $x \in X$ is the optimal value of a linear optimization problem in infinite dimensions. The constraint is an inequality with respect to the order relation in X generated by the convex cone A_φ. The dual problem has one equality constraint and non–negativity conditions. Formula (21) states that strong duality holds for the two problems – which is not the case for linear optimization problems in infinite dimensions in general. See [20], Section 1.4 for a counterexample.

In [42], this linear optimization duality is used to compute the values of a coherent risk measure via its dual representation (21).

Observation 11. Within the setting of Corollary 10 it is easy to determine the subdifferential of φ at $0 \in X$: $\partial\varphi(0) = \left\{x^* \in X^* : \ x^*(k) = -1, \ x^* \in A^*_\varphi\right\}$. In [48] this has been called the *support* of the sublinear function φ. Compare also Proposition 3.7 of [10].

Application: Convex and coherent risk measures. Föllmer and Schied introduced the notion of a *convex measure of risk* defined on certain spaces of measurable functions, see Definition 1 in [14] and compare also the book [15]. See also [16] for a similar approach in which translativity is not the central concept. We shall describe the notion in the following on L^p–spaces, $p \in [1, +\infty)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, i.e., Ω is a nonempty set and $\mathcal{F}$ a σ–field of subsets of Ω and P a probability measure. Let $p \in [1, +\infty)$ and denote by $X = L^p(\Omega, \mathcal{F}, P)$ the set of equivalence classes (w.r.t. sets of zero P-measure) of functions $x : \Omega \to \mathrm{I\!R}$ with

$$\int_\Omega |x(\omega)|^p \, dP < +\infty$$

and by L^p_+ the closed convex (and pointed) cone of all $x \in L^p(\Omega, \mathcal{F}, P)$ with

$$P(\{\omega \in \Omega : \ x(\omega) < 0\}) = 0.$$

By e we denote the element of $L^p(\Omega, \mathcal{F}, P)$ with

$$P(\{\omega \in \Omega : \ e(\omega) \neq 1\}) = 0.$$

A function $\varrho : L^p(\Omega, \mathcal{F}, P) \to \mathrm{I\!R} \cup \{\pm\infty\}$ is called a *convex risk function* iff $\varrho(0) = 0$ and it is convex, L^p_+–monotone and translative with respect to e. This means, ϱ especially matches the conditions of Corollary 4.

$L^p(\Omega, \mathcal{F}, P)$ is a reflexive Banach space for $p \in [1, +\infty)$ and its topological dual can be identified with $L^q(\Omega, \mathcal{F}, P)$ with $\frac{1}{p} + \frac{1}{q} = 1$. The negative dual cone of L^p_+ is $(L^p_+)^* = -L^q_+$.

According to Theorem 4 and Observation 8 with $K = L^p_+$, a convex measure of risk admits representations as

$$\varrho(x) = \inf\{t \in \mathrm{I\!R} : \ x + te \in A_\varrho\} = \sup_{x^* \in M^*_e \cap \left(-L^q_+\right)} \left\{x^*(x) - \sigma_{A_\varrho}(x^*)\right\}.$$

Since $-x^*$ for $x^* \in M^*_e \cap (-L^q_+)$ generates a probability measure Q by

$$Q(F) = \int_F -x^*(\omega) \, dP, \quad F \in \mathcal{F}$$

(especially, $\int_\Omega -x^*(\omega)\,dP = 1$ and $-x^* \in L^q_+$ and, moreover, a set of zero P-measure is also a set of zero Q-measure), there exists a set $\mathcal{Q}$ of probability measures such that

$$\varrho(x) = \sup_{Q\in\mathcal{Q}} \left\{ E^Q[-x] - \sup_{y\in A_\varrho} E^Q[-y] \right\} = \sup_{Q\in\mathcal{Q}} \left\{ E^Q[-x] + \inf_{y\in A_\varrho} E^Q[y] \right\}$$

where $E^Q[u]$ denotes the expectation of $u \in L^p(\Omega, \mathcal{F}, P)$ with respect to Q.

If ϱ is additionally positively homogenous, i.e., a coherent measure of risk, the representation

$$\varrho(x) = \sup_{Q\in\mathcal{Q}} \left\{ E^Q[-x] \right\}$$

holds true where $\mathcal{Q}$ is a closed convex set of probability measures generated by functions of $L^q(\Omega, \mathcal{F}, P)$. In this case, the set $\mathcal{Q}$ has been called set of *risk enveloppes* in [46].

The notion of convex risk measures has found far–reaching applications in financial mathematics. There are several subsequent paper on convex risk functions. We mention [53], [42], since they have explicitely relationsships to Convex Analysis in view. For representation theorems, compare Theorem 5, 6 in [14], Proposition 4.14 and Theorem 4.15 in [15], Theorem 2 and its Corollar 1 in [53] and Theorem 2.4 and its Corollary 2.5 in [42]). The case of coherent risk measures has been treated e.g. in [3], [8] and [32], Theorem 2, [46] (L^2 case). Finally, let us note that the "non–reflexive" case $X = L^\infty(\Omega, \mathcal{F}, P)$ requires a more sophisticated analysis. The main question in this case is under what condition the set Q is a subset of $L^1(\Omega, \mathcal{F}, P)$ rather than of $(L^\infty(\Omega, \mathcal{F}, P))^*$. This is essentially weak* lower semicontinuity, compare [8], [14], [15] and [52].

4 Translative set–valued functions

In this section, we extend the results of Section 3 to functions with values in $\widehat{\mathcal{P}}(\mathbb{R}^m)$. It turns out that almost all results have their counterparts in the set–valued setting.

We are given a natural number $m \geq 2$ and a collection $K := \{k^1, k^2, \ldots, k^m\} \subset X$ of linearly independent elements of X. Further, let $C \subseteq \mathbb{R}^m$ be a convex cone containing $0 \in \mathbb{R}^m$. In order to compare the values of a function $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ we shall use the order relation $\preccurlyeq_C$ on $\widehat{\mathcal{P}}(X)$ introduced in Section 1 (see (4)).

Considering a function $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$, we set

$$\begin{aligned} \operatorname{epi}\Phi &:= \{(x,v) \in X \times \mathbb{R}^m : \ v \in \Phi(x) \oplus C\}, \\ \operatorname{EPI}\Phi &:= \left\{(x,V) \in X \times \widehat{\mathcal{P}}(\mathbb{R}^m) : \ \Phi(x) \preccurlyeq_C V\right\}, \\ \operatorname{dom}\Phi &:= \{x \in X : \ \Phi(x) \neq \emptyset\}. \end{aligned}$$

For $V \subseteq \mathbb{R}^m$, the sublevel set of Φ at level V is defined by

$$S_\Phi(V) := \{x \in X : \ \Phi(x) \preccurlyeq_C V\} = \{x \in X : \ V \subseteq \Phi(x) \oplus C\}.$$

We set $S_\Phi(v) := S_\Phi(\{v\})$ for $v \in \mathrm{I\!R}^m$. Then

$$S_\Phi(V) = \bigcap_{v \in V} S_\Phi(v)$$

and, especially,

$$S_\Phi(0) = S_\Phi(\{0\}) = \{x \in X : \ 0 \in \Phi(x) \oplus C\} = \{x \in X : \ C \subseteq \Phi(x) \oplus C\}$$

where 0 denotes the m-dimensional zero vector.

We shall investigate translative functions from X into $\widehat{\mathcal{P}}(\mathrm{I\!R}^m)$. The dimension m coincides with the number of elements with respect to which translativity is satisfied.

Definition 8 *A function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ is called* translative *with respect to K iff*

$$\forall x \in X, \ \forall v \in \mathrm{I\!R}^m : \ \Phi\left(x + \sum_{i=1}^{m} v_i k^i\right) = \Phi(x) \oplus \{-v\}. \tag{23}$$

Translativity of set–valued functions in this section always means translativity with respect to the given collection K.

In the one dimensional case $m = 1$, $k^1 = k \in X \backslash \{0\}$ and $C = \mathrm{I\!R}_+$, the set–valued function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R})$ defined by $\Phi(x) = [\varphi(x), +\infty)$ is translative with respect to $K = \{k\}$ in the sense of Definition 8 if and only if $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is translative with respect to k in the sense of Definition 6.

The next result shows that the translation of zero sublevel sets remains valid in a certain sense. Compare Theorem 1.

Theorem 5 *For a function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ the following conditions are equivalent:*
(i) epi Φ *has the property*

$$\forall (x, v) \in \operatorname{epi}\Phi, \ \forall w \in \mathrm{I\!R}^m : \ (x, v) + \left(\sum_{i=1}^{m} w_i k^i, -w\right) \in \operatorname{epi}\Phi;$$

(ii) Φ is translative with respect to K;
(iii) $\operatorname{epi}\Phi = \{(x, v) \in X \times \mathrm{I\!R}^m : x + \sum_{i=1}^{m} v_i k^i \in S_\Phi(0)\}$*;*
(iv) it holds

$$\forall v \in \mathrm{I\!R}^m : \ S_\Phi(v) = S_\Phi(0) \oplus \left\{-\sum_{i=1}^{m} v_i k^i\right\}$$

and, equivalently,

$$\forall V \in \mathcal{P}(\mathrm{I\!R}^m) : \ S_\Phi(0) = \bigcap_{v \in V}\left[S_\Phi(\{v\}) \oplus \left\{\sum_{i=1}^{m} v_i k^i\right\}\right].$$

PROOF. (i) $\Rightarrow$ (ii): Take $x \in X$, $v \in \mathrm{IR}^m$. First, assume $\Phi(x) = \emptyset$. If $\Phi\left(x + \sum_{i=1}^m v_i k^i\right) \neq \emptyset$, then there is $w \in \mathrm{IR}^m$ with $\left(x + \sum_{i=1}^m v_i k^i, w\right) \in \operatorname{epi}\Phi$. Then, (i) implies

$$\left(x + \sum_{i=1}^m v_i k^i, w\right) + \left(-\sum_{i=1}^m v_i k^i, v\right) = (x, v + w) \in \operatorname{epi}\Phi,$$

a contradiction. Hence $\Phi\left(x + \sum_{i=1}^m v_i k^i\right) = \emptyset$. Next, if $\Phi(x) \neq \emptyset$, for $(x, w) \in \operatorname{epi}\Phi$ we have by (i) for each $v \in \mathrm{IR}^m$

$$(x, w) + \left(\sum_{i=1}^m v_i k^i, -v\right) = \left(x + \sum_{i=1}^m v_i k^i, w - v\right) \in \operatorname{epi}\Phi$$

which means $w - v \in \Phi\left(x + \sum_{i=1}^m v_i k^i\right)$. This proves $\Phi(x) \oplus \{-v\} \subseteq \Phi\left(x + \sum_{i=1}^m v_i k^i\right)$. Especially, $\Phi\left(x + \sum_{i=1}^m v_i k^i\right) \neq \emptyset$. Conversely, take $u \in \Phi\left(x + \sum_{i=1}^m v_i k^i\right)$. We have by (i) that $\left(x + \sum_{i=1}^m v_i, u\right) \in \operatorname{epi}\Phi$ implies $(x, u + v) \in \operatorname{epi}\Phi$ for each $v \in \mathrm{IR}^m$. This proves $\Phi\left(x + \sum_{i=1}^m v_i k^i\right) \subseteq \Phi(x) \oplus \{-v\}$.

(ii) $\Rightarrow$ (iii): It suffices to note that $x + \sum_{i=1}^m v_i k^i \in S_\Phi(0)$ if and only if $0 \in \Phi\left(x + \sum_{i=1}^m v_i k^i\right) = \Phi(x) \oplus \{-v\}$, the latter equation is (23).

(iii) $\Rightarrow$ (iv): Of course, $x \in S_\Phi(v)$ if and only if $v \in \Phi(x)$. According to (iii), this is equivalent to $x \in S_\Phi(0) \oplus \left\{\sum_{i=1}^m v_i k^i\right\}$.

(iv) $\Rightarrow$ (i): Take $(x, v) \in \operatorname{epi}\Phi$, $w \in \mathrm{IR}^m$. Then

$$x + \sum_{i=1}^m w_i k^i \in S_\Phi(v) \oplus \left\{\sum_{i=1}^m w_i k^i\right\} = S_\Phi(0) \oplus \left\{\sum_{i=1}^m (w_i - v_i) k^i\right\} = S_\Phi(v - w).$$

This gives $(x, v) + \left(\sum_{i=1}^m w_i k^i, -w\right) \in \operatorname{epi}\Phi$.

The equivalent formulation of (iv) is obvious. ∎

As in the real–valued case one may conclude that $\operatorname{dom}\Phi$ is nonempty if and only if $S_\Phi(0)$ is nonempty.

Of course, a function with values in $\widehat{\mathcal{P}}(\mathrm{IR}^m)$ can not be linear in the usual sense, since $\widehat{\mathcal{P}}(\mathrm{IR}^m)$ is not a linear space. We still have the following analogy to Propositions 1 and 2.

Denote by $L(K) = span\ \{k^1, k^2, \ldots, k^m\}$ the linear subspace of X that is spanned by K.

Proposition 10 *Let $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{IR}^m)$ be translative with respect to K. Then*

$$\forall i \in \{1, \ldots, m\} : \ \Phi(0) = \Phi\left(k^i\right) \oplus \left\{e^i\right\} = \Phi\left(-k^i\right) \oplus \left\{-e^i\right\}.$$

If, additionally, $0 \in \Phi(0)$ and $\Phi(0)$ is closed under addition then

$$\forall x, y \in L(K) : \quad \Phi(x + y) = \Phi(x) \oplus \Phi(y).$$

If, additionally, $0 \in \Phi(0)$ and $\Phi(0)$ is a cone then

$$\forall s > 0,\ \forall x \in L(K) : \ \Phi(sx) = s\Phi(x).$$

PROOF. The first assertion is immediate from (23). To show the second one, take two elements $x = \sum_{i=1}^m v_i k^i$, $y = \sum_{i=1}^m w_i k^i$ of $spanK$. Then

$$\Phi(x+y) = \Phi\left(\sum_{i=1}^m (v_i + w_i) k^i\right) = \Phi(0) \oplus \{-v-w\}.$$

Since $\Phi(0) \oplus \Phi(0) = \Phi(0)$ we obtain $\Phi(x+y) = \Phi(x) \oplus \Phi(y)$.

Finally, with $s > 0$ and $x = \sum_{i=1}^m v_i k^i$, (23) and the cone property of $\Phi(0)$ imply

$$\Phi(sx) = \Phi(0) \oplus (-sv) = s\left[\Phi(0) \oplus (-v)\right]$$

which completes the proof of the proposition. ■

In the following, we assign to each set $A \subseteq X$ a function $\Phi_A : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ and vice versa, to a function $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ a set $A_\Phi \subseteq X$. This is done by

$$A_\Phi := S_\Phi(0) = \{x \in X : \ \Phi(x) \preccurlyeq_C \{0\}\} = \{x \in X : \ C \subseteq \Phi(x) \oplus C\}. \tag{24}$$

and

$$\Phi_A(x) := \left\{v \in \mathbb{R}^m : \ x + \sum_{i=1}^m v_i k^i \in A\right\}, \tag{25}$$

respectively.

Observation 12. In view of (5), one may write (25) as

$$\Phi_A(x) = \inf\left\{\left\{\{v\} : \ v \in \mathbb{R}^m, \ x + \sum_{i=1}^m v_i k^i \in A\right\}, \preccurlyeq_C\right\}.$$

This means, in complete analogy to (15), $\Phi_A(x)$ is the infimum of a (very special) subset of $\widehat{\mathcal{P}}(\mathbb{R}^m)$.

Immediately from (24) and (25) we have the following result. Note that a closedness property is not necessary.

Proposition 11 *(i) For $A \subseteq X$, Φ_A is translative with respect to K. (ii) If $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ is translative with respect to K, then A_Φ is translative with respect to K.*

PROOF. (i) Take $x \in X$, $v \in \mathbb{R}^m$. Then

$$\begin{aligned}
\Phi_A\left(x + \sum_{i=1}^m v_i k^i\right) &= \left\{w \in \mathbb{R}^m : \ x + \sum_{i=1}^m v_i k^i + \sum_{i=1}^m w_i k^i \in A\right\} \\
&= \left\{w + v \in \mathbb{R}^m : \ x + \sum_{i=1}^m (w_i + v_i) k^i \in A\right\} \oplus \{-v\} \\
&= \Phi_A(x) \oplus \{-v\}.
\end{aligned}$$

(ii) Take $x \in A_\Phi$, i.e., $C \subseteq \Phi(x) \oplus C$, and $v \in C$. Then, by (23),

$$\Phi\left(x + \sum_{i=1}^{m} v_i k^i\right) \oplus C = \Phi(x) \oplus C \oplus \{-v\}.$$

Hence $C \subseteq C \oplus \{-v\} \subseteq \Phi\left(x + \sum_{i=1}^{m} v_i k^i\right) \oplus C$ implying $x + \sum_{i=1}^{m} v_i k^i \in A_\Phi$. ■

Lemma 8 *If A is translative, then $\Phi_A(x)$ is C–upward for all $x \in X$, i.e.,*

$$\forall x \in X:\ \Phi_A(x) = \Phi_A(x) \oplus C.$$

Proof. Since $0 \in C$ we have $\Phi_A(x) \subseteq \Phi_A(x) \oplus C$. Conversely, take $v \in \Phi_A(x)$ and $w \in C$. Then

$$x + \sum_{i=1}^{m} (v_i + w_i) k^i = x + \sum_{i=1}^{m} v_i k^i + \sum_{i=1}^{m} w_i k^i \in A$$

by (6) since $x + \sum_{i=1}^{m} v_i k^i \in A$. Hence $v + w \in \Phi_A(x)$ for each $v \in \Phi_A(x)$ and $w \in C$ implying $\Phi_A(x) \oplus C \subseteq \Phi_A(x)$. ■

Lemma 8 tells us that in order to have $\Phi = \Phi_{A_\Phi}$, we must assume that Φ has C–upward values. The result reads as follows.

Proposition 12 *(i) Let $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ be a function that is translative with respect to K and has C–upward values. Then A_Φ is translative with respect to K and $\Phi = \Phi_{A_\Phi}$ holds true. (ii) Let $A \subseteq X$ be translative with respect to K. Then Φ_A is translative with respect to K, has C–upward values and $A = A_{\Phi_A}$ holds true.*

Proof. (i) A_Φ is translative with respect to K by Proposition 11. With the help of (23) one may see

$$\begin{aligned}
\Phi_{A_\Phi}(x) &= \left\{v \in \mathbb{R}^m :\ x + \sum_{i=1}^{m} v_i k^i \in A_\Phi\right\} \\
&= \left\{v \in \mathbb{R}^m :\ \Phi\left(x + \sum_{i=1}^{m} v_i k^i\right) \preccurlyeq_C \{0\}\right\} \\
&= \{v \in \mathbb{R}^m :\ 0 \in \Phi(x) \oplus C \oplus \{-v\}\} = \Phi(x) \oplus C = \Phi(x).
\end{aligned}$$

(ii) Φ_A satisfies (23) by Proposition 11. It remains to show $A = A_{\Phi_A}$. Since

$$A_{\Phi_A} = \{x \in X :\ \Phi_A(x) \preccurlyeq_C \{0\}\} = \left\{x \in X :\ 0 \in \left\{v \in \mathbb{R}^m :\ x + \sum_{i=1}^{m} v_i k^i \in A\right\} \oplus C\right\}$$

we have $A \subseteq A_{\Phi_A}$. To show the opposite inclusion, take $\bar{x} \in A_{\Phi_A}$, i.e., $\Phi_A(\bar{x}) \preccurlyeq_C \{0\}$. In view of Lemma 8, this means

$$0 \in \Phi_A(\bar{x}) \oplus C = \Phi_A(\bar{x}) = \left\{v \in \mathbb{R}^m :\ \bar{x} + \sum_{i=1}^{m} v_i k^i \in A\right\}.$$

Hence $\bar{x} \in A$. ■

As in the real–valued case, according to Proposition 11 (i), Φ_A is translative whether or not A is. One may ask for the relationship of A and A_{Φ_A} in the general case.

Proposition 13 *Let $A \subseteq X$ be a nonempty set. Then $A_{\Phi_A} = \operatorname{tr} A$ and $\Phi_A = \Phi_{\operatorname{tr} A}$.*

PROOF. Since $A \subseteq A_{\Phi_A}$ and A_{Φ_A} is translative according to Proposition 11, (ii), we have $\operatorname{tr} A \subseteq A_{\Phi_A}$. On the other hand,

$$\begin{aligned} A_{\Phi_A} &= \{x \in X : \ 0 \in \Phi_A(x) \oplus C\} \\ &= \left\{x \in X : \ 0 \in \left\{v \in \mathbb{R}^m : \ x + \sum_{i=1}^{m} v_i k^i \in A\right\} \oplus C\right\} \\ &\subseteq \left\{x \in X : \ 0 \in \left\{v \in \mathbb{R}^m : \ x + \sum_{i=1}^{m} v_i k^i \in B\right\} \oplus C\right\} \\ &= B_{\Phi_B} = B \end{aligned}$$

for each $B \subseteq X$ being translative and containing A. The last equation in this chain is a consequence of Proposition 12, (ii). ■

We shall extend Proposition 3 to the setting of this section.

Proposition 14 *(i) For $A \subseteq X$, Φ_A is translative with respect to K and $A \subseteq A_{\Phi_A}$. If A is radially closed and translative with respect to K, then Φ_A is translative, has C–upward, closed values and $A = A_{\Phi_A}$ holds true. (ii) Let $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ be a function that is translative with respect to K and has C–upward, closed values. Then A_Φ is radially closed, translative and $\Phi = \Phi_{A_\Phi}$ holds true.*

PROOF. (i) In view of Proposition 12 and Lemma 8, it suffices to show that Φ_A has closed values. Take a sequence $\{v^n\}_{n\in\mathbb{N}} \subset \Phi_A(x)$ such that $v^n \to v$. Then $x + \sum_{i=1}^{m} v_i^n k^i \in A$ for all $n \in \mathbb{N}$. Since A is radially closed, this implies $x + \sum_{i=1}^{m} v_i k^i \in A$. This is $v \in \Phi_A(x)$.

(ii) In view of Proposition 12, it suffices to show the radial closedness of A_Φ. Take $x \in X$ and a sequence $\{v^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ such that $v^k \to v$ and $x + \sum_{i=1}^{m} v_i^n k^i \in A_\Phi$. Then, by definition of A_Φ and (23),

$$0 \in \Phi\left(x + \sum_{i=1}^{m} v_i^n k^i\right) \oplus C = \Phi(x) \oplus C \oplus \{-v^n\}$$

for each $n \in \mathbb{N}$. Since Φ is C-closed, this implies $0 \in \Phi(x) \oplus C \oplus \{-v\}$. (23) yields $0 \in \Phi\left(x + \sum_{i=1}^{m} v_i k^i\right) \oplus C$, hence $x + \sum_{i=1}^{m} v_i k^i \in A_\Phi$. ■

If $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ is an extended real–valued function, the sets $\{\varphi(x)\}$ and $\{\varphi(x)\} \oplus \mathbb{R}_+$ are automatically closed. Therefore, there is no special closedness assumption in Proposition 3 with respect to φ.

Let $A \subseteq X$ be given. Define a function $\Psi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ by

$$\Psi_A(x) := \mathrm{cl}\,(\Phi_A(x) \oplus C), \; x \in X. \tag{26}$$

Parallel to Proposition 4, we have the following result.

Proposition 15 *Let $A \subseteq X$ be a nonempty set. Then $A_{\Psi_A} = \mathrm{rt}\, A$ and $\Psi_A = \Phi_{\mathrm{rt}\, A}$.*

PROOF. The function Ψ_A is translative since

$$\begin{aligned}
\Psi_A\left(x + \sum_{i=1}^m v_i k^i\right) &= \mathrm{cl}\left\{u + w \in \mathrm{I\!R}^m : \; x + \sum_{i=1}^m (u_i + v_i)\, k^i \in A, \; w \in C\right\} \\
&= \mathrm{cl}\left\{u + v + w \in \mathrm{I\!R}^m : \; x + \sum_{i=1}^m (u_i + v_i)\, k^i \in A, \; w \in C\right\} \oplus \{-v\} \\
&= \mathrm{cl}\,(\Phi_A(x) \oplus C) \oplus \{-v\} = \Psi_A(x) \oplus \{-v\}\,.
\end{aligned}$$

Moreover, Ψ_A has closed values by definition and C–upward values since $\mathrm{cl}\,(\Phi_A(x) \oplus C) \oplus C \subseteq \mathrm{cl}\,(\Phi_A(x) \oplus C)$. Applying Proposition 14, (i) we get that A_{Ψ_A} is radially closed and translative. Since $A \subseteq A_{\Psi_A}$, this implies $\mathrm{rt}\, A \subseteq A_{\Psi_A}$.

On the other hand, let $A \subseteq B$ with $B \subseteq X$ being radially closed and translative. Then, since $\Psi_A(x) \oplus C \subseteq \Psi_A(x)$ and $\Phi_B(x) = \Psi_B(x)$,

$$\begin{aligned}
A_{\Psi_A} &= \{x \in X : \; 0 \in \Psi_A(x) \oplus C\} \\
&= \{x \in X : \; 0 \in \Psi_A(x)\} \subseteq \{x \in X : \; 0 \in \Psi_B(x)\} \\
&= \{x \in X : \; 0 \in \Phi_B(x)\} = B_{\Phi_B}.
\end{aligned}$$

Since B is radially closed and translative, from Theorem 14, (ii) it follows $B = B_{\Phi_B}$ and hence $A_{\Psi_A} \subseteq B$ for each radially closed and translative set B with $A \subseteq B$. Hence $A_{\Psi_A} \subseteq \mathrm{rt}\, A$ and therefore, $A_{\Psi_A} = \mathrm{rt}\, A$. Then, the equation $\Psi_A = \Phi_{\mathrm{rt}\, A}$ is a consequence of Theorem 14, (i) applied to Ψ_A. ■

Corollary 11 *Let $A \subseteq X$ be a nonempty set. Then*

$$\mathrm{epi}\,\Psi_A = \left\{(x, v) \in X \times \mathrm{I\!R}^m : \; x + \sum_{i=1}^m v_i k^i \in \mathrm{rt}\, A\right\}.$$

Moreover, $(x, V) \in \mathrm{EPI}\,\Psi_A$ if and only if $\{x\} \oplus \bigcup \{v_i k^i : \; v \in V\} \subseteq \mathrm{rt}\, A$.

PROOF. The first assertion is a consequence of Theorem 5, (iii) and Proposition 15. The second assertion follows from the first one and the fact that $(x, V) \in \mathrm{EPI}\,\Psi_A$ if and only if for all $v \in V$ it holds $(x, v) \in \mathrm{epi}\,\Psi_A$. ■

From Proposition 15 we may learn that $\mathrm{rt}\, A = \mathrm{rt}\, B$ for $A, B \subseteq X$ if and only if $\Psi_A = \Psi_B$. Let us denote by $\mathcal{F}(K, C)$ the set of all functions $\Phi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ that are translative with respect to K and have C–upward, closed values. Define a partial order on $\mathcal{F}(K,C)$ by

$$\Phi \preceq \Psi \quad :\Longleftrightarrow \quad \forall x \in X: \ \Phi(x) \supseteq \Psi(x)$$

for $\Phi, \Psi \in \mathcal{F}(k)$.

Recall that $\mathcal{S}(K,C)$ denotes the set of all $A \subseteq X$ with $A = \operatorname{rt} A$. From Lemma 7 we know that $(\mathcal{S}(K,C), \supseteq)$ is a partially ordered, complete lattice.

Theorem 6 *Let X be a linear space, $K := \{k^1, k^2, \dots, k^m\} \subset X$ a collection of m linearly independent elements of X and $C \subseteq \mathbb{R}^m$ a convex cone containing $0 \in \mathbb{R}^m$.*
(a) For $A, B \in \mathcal{S}(K,C)$ it holds $A \supseteq B$ if and only if $\Psi_A \preceq \Psi_B$; for $\Phi, \Psi \in \mathcal{F}(K,C)$ it holds $\Phi \preceq \Psi$ if and only if $A_\Phi \supseteq A_\Psi$;
(b) The relationships (14) and (15) define an order preserving bijection between $(\mathcal{F}(K,C), \preceq)$ and $(\mathcal{S}(K,C), \supseteq)$;
(c) $(\mathcal{F}(K,C), \preceq)$ is a partially ordered, complete lattices.

PROOF. (a) If $A \supseteq B$, then $\Phi_A(x) \supseteq \Phi_B(x)$ due to (25). Conversely, let $x \in B$. Then $0 \in \Phi_B(x) \subseteq \Phi_A(x)$, hence $x \in A$. Thus, $A \supseteq B$ if and only if $\Phi_A \preceq \Phi_B$. If $\Phi \preceq \Psi$, then $A_\Psi = \{x \in X : \ 0 \in \Psi(x)\} \subseteq \{x \in X : \ 0 \in \Phi(x)\} = A_\Phi$. Conversely, if $\Psi(x) = \emptyset$, then there is nothing to prove. If $v \in \Psi(x)$, then $0 \in \Psi\left(x + \sum_{i=1}^m v_i k^i\right)$, hence $x + \sum_{i=1}^m v_i k^i \in A_\Psi \subseteq A_\Phi$. This implies $0 \in \Phi\left(x + \sum_{i=1}^m v_i k^i\right)$ and, by translativity, $v \in \Phi(x)$. Hence, $\Phi \preceq \Psi$ if and only if $A_\Phi \supseteq A_\Psi$.

(b) It suffices to note that on one hand $A = B$ and (a) imply $\Phi_A \preceq \Phi_B$ as well as $\Phi_B \preceq \Phi_A$, hence $\Phi_A = \Phi_B$ and on the other hand, $\Phi = \Psi$ and (a) imply $A_\Phi \supseteq A_\Psi$ as well as $A_\Psi \supseteq A_\Phi$, hence $A_\Phi = A_\Psi$.

(c) Is a consequence of (a), (b) and Lemma 7. ■

Corollary 12 *For $\mathcal{G} \subseteq \mathcal{F}(K,C)$, it holds* $\inf\{\mathcal{G}, \preceq\} = \Phi_I$ *and* $\sup\{\mathcal{G}, \preceq\} = \Phi_S$ *where*

$$I = \operatorname{rt} \bigcup_{\Phi \in \mathcal{G}} A_\Phi, \qquad S = \bigcap_{\Phi \in \mathcal{G}} A_\Phi.$$

Moreover,

$$\Phi_I(x) = \operatorname{cl} \bigcup_{\Phi \in \mathcal{G}} \Phi(x), \qquad \Phi_S(x) = \bigcap_{\Phi \in \mathcal{G}} \Phi(x).$$

PROOF. Applying the infimum formula of Lemma 7 to $\mathcal{A} = \{A_\Phi : \ \Phi \in \mathcal{G}\}$, we obtain that $\inf\{\mathcal{G}, \preceq\} = \Phi_I$ by the results of Theorem 6. Analogously, $\sup\{\mathcal{G}, \preceq\} = \Phi_S$ follows.

It remains to check the formulas for Φ_I and Φ_S. One way for doing this is to prove that $\Phi_I \in \mathcal{F}(K,C)$ and that it is the infimum of $\mathcal{G}$. In fact, since

$$\operatorname{cl}\left(\bigcup_{\Phi \in \mathcal{G}} \Phi(x)\right) \oplus C \subseteq \operatorname{cl} \bigcup_{\Phi \in \mathcal{G}} \left(\Phi(x) \oplus C\right) = \operatorname{cl} \bigcup_{\Phi \in \mathcal{G}} \Phi(x)$$

Φ_I has closed, C–upward values. Its translativity follows straightforward as well as its infimum property.

With similar arguments, the formula for Φ_S can be proven. ∎

Before continuing with further algebraic properties of translative set–valued functions we shall give definitions for properties of subsets of $X \times \widehat{\mathcal{P}}(\mathbb{R}^m)$. This space has a *conlinear* algebraic structure which we do not recall here. We refer the reader to [26].

Definition 9 *A set $\mathcal{M} \subseteq X \times \widehat{\mathcal{P}}(\mathbb{R}^m)$ is called*
(a) a cone *iff $s > 0$, $(x, V) \in \mathcal{M}$ implies $(sx, sV) \in \mathcal{M}$;*
(b) closed under addition *iff $(x, V), (x', V') \in \mathcal{M}$ implies $(x + x', V \oplus V') \in \mathcal{M}$;*
(c) convex *iff $s \in (0, 1)$, $(x, V), (x', V') \in \mathcal{M}$ implies $(sx + (1 - s)x', sV \oplus (1 - s)V') \in \mathcal{M}$.*

We shall begin with positive homogenity. A function $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ is said to be *positively homogeneous* iff

$$\forall s > 0, \forall x \in X : \quad \Phi(sx) \preccurlyeq_C s\Phi(x).$$

Lemma 9 *For a function $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$, the following conditions are equivalent:*
(i) Φ is positively homogeneous;
(ii) $\forall s > 0$, $\forall x \in X$: $s\Phi(x) \subseteq \Phi(sx) \oplus C$;
(iii) $\operatorname{epi}\Phi \subseteq X \times \mathbb{R}^m$ *is cone;*
(iv) $\operatorname{EPI}\Phi \subseteq X \times \mathcal{P}(\mathbb{R}^m)$ *is a cone.*

PROOF. (i) ⇒ (ii) Just the definition of $\preccurlyeq_C$. (ii) ⇒ (iii) Straighforward. (iii) ⇒ (iv) Take $s > 0$ and $(x, V) \in \operatorname{EPI}\Phi$. The latter is equivalent to: $(x, v) \in \operatorname{epi}\Phi$ for all $v \in V$. Then, (iv) implies $(sx, sv) \in \operatorname{epi}\Phi$ for all $v \in V$, hence $(sx, sV) \in \operatorname{EPI}\Phi$. (iv) ⇒ (i) Since $(x, \Phi(x)) \in \operatorname{EPI}\Phi$ for all $x \in X$, (iv) implies $(sx, s\Phi(x)) \in \operatorname{EPI}\Phi$ whenever $s > 0$. This is (i). ∎

Proposition 16 *(i) Let $A \subseteq X$ be a cone. Then Φ_A is positively homogenous. (ii) Let $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ be positively homogenous. Then A_Φ is a cone.*

PROOF. (i) Take $s > 0$. Then

$$\Phi_A(sx) = \left\{ v \in \mathbb{R}^m : \; sx + \sum_{i=1}^m v_i k^i \in A \right\} = s \left\{ \frac{1}{s} v : \; x + \sum_{i=1}^m \frac{v_i}{s} k^i \in A \right\} = s\Phi_A(x).$$

(ii) Straightforward. ∎

The next property is subadditivity. A function $\Phi : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ is said to be *subadditive* iff

$$\forall x, x' \in X : \; \Phi(x + x') \preccurlyeq_C \Phi(x) \oplus \Phi(x').$$

Lemma 10 *For a function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$, the following conditions are equivalent:*
(i) Φ is subadditive;
(ii) $\forall x, x' \in X$: $\Phi(x) \oplus \Phi(x') \subseteq \Phi(x + x') \oplus C$;
(iii) $\operatorname{epi}\Phi \subseteq X \times \mathrm{I\!R}^m$ *is closed under addition;*
(iv) $\operatorname{EPI}\Phi \subseteq X \times \mathcal{P}(\mathrm{I\!R}^m)$ *is closed under addition.*

PROOF. (i) $\Rightarrow$ (ii) Just the definition of $\preccurlyeq_C$. (ii) $\Rightarrow$ (iii) Since (ii) implies $\Phi(x) \oplus \Phi(x') \oplus C \subseteq \Phi(x + x') \oplus C$ the assertion is immediate. (iii) $\Rightarrow$ (iv) Straightforward. (iv) $\Rightarrow$ (i) Since $(x, \Phi(x)), (x', \Phi(x')) \in \operatorname{EPI}\Phi$ for all $x, x' \in X$, (iv) implies $(x + x', \Phi(x') \oplus \Phi(x)) \in \operatorname{EPI}\Phi$ and (i) follows. ■

Proposition 17 *(i) Let $A \subseteq X$ be closed under addition. Then Φ_A is subadditive. (ii) Let $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ be subadditive. Then A_Φ is closed under addition.*

PROOF. (i) Let $v \in \Phi_A(x)$ and $w \in \Phi_A(y)$, i.e., $x + \sum_{i=1}^m v_i k^i \in A$ and $y + \sum_{i=1}^m w_i k^i \in A$. Since A is closed under addition, it follows $x + y + \sum_{i=1}^m (v_i + w_i) k^i \in A$, hence $\Phi_A(x) \oplus \Phi_A(y) \subseteq \Phi_A(x + y)$.

(ii) Straightforward. ■

We turn to convexity. A function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ is said to be *convex* iff

$$\forall s \in [0,1], \ \forall x, x' \in X : \ \Phi(sx + (1-s)x') \preccurlyeq_C s\Phi(x) \oplus (1-s)\Phi(x').$$

Lemma 11 *For a function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$, the following conditions are equivalent:*
(i) Φ is convex;
(ii) $\forall s \in [0,1]$, $\forall x, x' \in X$: $s\Phi(x) \oplus (1-s)\Phi(x') \subseteq \Phi(sx + (1-s)x') \oplus C$;
(iii) $\operatorname{epi}\Phi \subseteq X \times \mathrm{I\!R}^m$ *is convex;*
(iv) $\operatorname{EPI}\Phi \subseteq X \times \mathcal{P}(\mathrm{I\!R}^m)$ *is convex.*

PROOF. Straightforward using arguments similar to those in the proofs of Lemma 9 and 10. ■

Proposition 18 *(i) Let $A \subseteq X$ be convex. Then Φ_A is convex. (ii) Let $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ be convex. Then A_Φ is convex.*

PROOF. (i) Take $t \in [0,1]$, $v \in \Phi_A(x)$ and $w \in \Phi_A(y)$, i.e., $x + \sum_{i=1}^m v_i k^i \in A$ and $y + \sum_{i=1}^m w_i k^i \in A$. The convexity of A yields $tx + (1-t)y \in \Phi_A(tx + (1-t)y)$, hence

$$t\Phi_A(x) \oplus (1-t)\Phi_A(y) \subseteq \Phi_A(tx + (1-t)y).$$

(ii) Straightforward. ■

Note that convexity of $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ with respect to $\preccurlyeq_C$ is equivalent to concavity of $-\Phi$ with respect to $\preccurlyeq^{C}$. This means, multiplying by -1 one not only have to change the sides of the inequality, but also the order relation. The same remark applies to subadditivity of Φ and superadditivity of $-\Phi$.

A positively homogeneous and subadditive function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ is called *sublinear*. One can show that a positively homogeneous function Φ is convex if and only if it is subadditive.

Note that the above definitions rely heavily on the order relation $\preccurlyeq_C$ in $\widehat{\mathcal{P}}(\mathrm{I\!R}^m)$. Well-known definitions for the convexity of set–valued maps (see e.g. [4], Definition 1.1 and, more recently, [21], [31] and the references therein) usually use this relation implicitely. The case $C = \{0\}$ is also possible, then $\preccurlyeq_C$ reduces to the partial order $\supseteq$. Further, note that using $\preccurlyeq_C$ we obtain formulations and results very close to the real–valued case.

We shall continue with monotonicity. Let $D \subseteq X$ be a nonempty subset of X. A function $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ is called *D–monotone* iff:

$$x' - x \in D \quad \Longrightarrow \quad \Phi(x') \preccurlyeq_C \Phi(x).$$

Parallel to the real–valued case we have the following result.

Proposition 19 *(i) If $A \subseteq X$ is D–upward, then Φ_A is D–monotone. (ii) If $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ is D–monotone, then A_Φ is D–upward.*

PROOF. (i) Take $x, x' \in X$ such that $x' \in x \oplus D$. Then $A \oplus \{x' - x\} \subseteq A \oplus D \subseteq A$ and we have the following relationsships:

$$\begin{aligned}
\Phi_A(x) &= \left\{ v \in \mathrm{I\!R}^m :\; x + \sum_{i=1}^m v_i k^i \in A \right\} \\
&= \left\{ v \in \mathrm{I\!R}^m :\; x' + \sum_{i=1}^m v_i k^i \in A \oplus \{x' - x\} \right\} \\
&\subseteq \left\{ v \in \mathrm{I\!R}^m :\; x' + \sum_{i=1}^m v_i k^i \in A \oplus D \right\} \\
&\subseteq \left\{ v \in \mathrm{I\!R}^m :\; x' + \sum_{i=1}^m v_i k^i \in A \right\} = \Phi_A(x').
\end{aligned}$$

This implies $\Phi_A(x) \subseteq \Phi_A(x') \oplus C$ as desired.

(ii) Take $x \in A_\Phi$, $x' \in D$. Then $x + x' \in \{x\} \oplus D$. Hence, by assumption, $\Phi(x + x') \preccurlyeq_C \Phi(x) \preccurlyeq_C \{0\}$ which gives $x + x' \in A_\Phi$. ■

Up to now, the trivial cases $\Phi(x) = \mathrm{I\!R}^m$ $\Phi \equiv \emptyset$ are not excluded. The next result is devoted to this question. Recall that $L(K) = \mathit{span}\ \{k^1, k^2, \ldots, k^m\}$ denotes the linear subspace of X that is spanned by K. For $m = 1$ and $k^1 = k \in X \setminus \{0\}$, $L(K)$ coincides with $\mathrm{I\!R}\{k\}$. Compare Propositions 9 and 5.

Proposition 20 *Let* $K = \{k^1, k^2, \ldots, k^m\}$ *be a collection of linearly independent elements of* X*. (i) If* $A \subseteq X$ *is nonempty such that*

$$\forall x \in X, \ \exists v \in \mathrm{I\!R}^m : \ x + \sum_{i=1}^{m} v_i k^i \notin \operatorname{tr} A, \tag{27}$$

then Φ_A *is proper. If (27) holds true and* $X = A \oplus L(K)$*, then* $\Phi_A(x) \neq \mathrm{I\!R}^m, \emptyset$ *for all* $x \in X$*. (ii) If* $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ *is proper and translative with respect to* K*, then* A_Φ *is nonempty and*

$$\forall x \in X, \ \exists v \in \mathrm{I\!R}^m : \ x + \sum_{i=1}^{m} v_i k^i \notin A_\Phi. \tag{28}$$

If Φ *is translative with respect to* K *and* $\Phi(x) \neq \mathrm{I\!R}^m, \emptyset$ *for all* $x \in X$*, then (28) holds true and* $X = A_\Phi \oplus L(K)$*.*

PROOF. (i) By definition of Φ_A, $A \neq \emptyset$ implies $\operatorname{dom} \Phi_A = \{x \in X : \ \Phi_A(x) \neq \emptyset\} \neq \emptyset$. Take $x \in X$ and $V \in \mathrm{I\!R}^m$ such that (28) is satisfied. The translativity of $\operatorname{tr} A$ and $A \subseteq \operatorname{tr} A$ imply that for all $w \in \{v\} \oplus (-C)$ it holds $x + \sum_{i=1}^{m} w_i k^i \notin A$. Hence $\Phi(x) \neq \mathrm{I\!R}^m$ for all $x \in X$. This proves (i).

(ii) Translativity implies that $x + \sum_{i=1}^{m} v_i k^i \in A_\Phi$ if and only if $v \in \Phi(x)$. Hence $A_\Phi \neq \emptyset$ and (28) holds true. ∎

Corollary 13 *(i) If* $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ *is translative with respect to* K *and* $\Phi(0) \neq \mathrm{I\!R}^m$*, then* $L(K) \nsubseteq A_\Phi$*. (ii) If* $L(K) \nsubseteq A$*, then* $\Phi_A(0) \neq \mathrm{I\!R}^m$*.*

PROOF. (i) Assume $L(K) \subseteq A_\Phi$. Then, for each $v \in \mathrm{I\!R}^m$, $\sum_{i=1}^{m} v_i k^i \in A_\Phi$. Therefore, $0 \in \Phi\left(\sum_{i=1}^{m} v_i k^i\right) = \Phi(0) \oplus \{-v\}$ for each $v \in \mathrm{I\!R}^m$, a contradiction.

(ii) By definition, we have $\Phi_A(0) = \mathrm{I\!R}^m$ if and only if $\sum_{i=1}^{m} v_i k^i \in A$ for all $v \in \mathrm{I\!R}^m$ which is true if and only if $L(K) \subseteq A$. ∎

Corollary 14 *(i) Let* $A \subseteq X$ *be translative with respect to* K*. Then* $\Phi(x) \neq \mathrm{I\!R}^m, \emptyset$ *for all* $x \in X$ *if and only if* $A \oplus L(K) = X \backslash A \oplus L(K) = X$*. (ii) Let* $\Phi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ *be translative with respect to* K*. Then* $\Phi(x) \neq \mathrm{I\!R}^m, \emptyset$ *for all* $x \in X$ *if and only if* $A_\Phi \oplus L(K) = X \backslash A_\Phi \oplus L(K) = X$*.*

PROOF. (i) It suffices to note that (27) can be re-written as $X = X \backslash A \oplus L(K)$ since $A = \operatorname{tr} A$. The assertion follows from Proposition 20.

(ii) Is a consequence of part (i) since A_Φ is translative and $\Phi = \Phi_{A_\Phi}$. ∎

Corollary 15 *(i) If* $A \subseteq X$ *is a convex set with* $L(K) \backslash \Gamma_K(\operatorname{cl} C) \bigcap \operatorname{rt} A = \emptyset$ *and* $0 \in \operatorname{rt} A$*, then* Ψ_A *is convex and translative such that* $\Psi_A(0) = \operatorname{cl} C$*. If* A *is additionally a cone, then* Ψ_A *is additionally positively homogenous.*

(ii) If $\Psi : X \to \widehat{\mathcal{P}}(\mathrm{I\!R}^m)$ *is convex, translative and has* C*–upward, closed values such that* $\Psi(0) = \operatorname{cl} C$*, then* A_Ψ *is convex,* $0 \in A_\Psi = \operatorname{rt} A_\Psi$ *and* $L(K) \backslash \Gamma_K(\operatorname{cl} C) \bigcap A_\Psi = \emptyset$ *holds true. If* Ψ *is additionally positively homogenous, then* A_Ψ *is a convex cone.*

Proof. (i) Recall that Ψ_A is defined by $\Psi_A(x) = \mathrm{cl}\,(\Phi_A \oplus C)$ for $x \in X$ and that $\Phi_{\mathrm{rt}\,A} = \Psi_A$ by Proposition 15.

Hence $0 \in \Psi_A(0)$ since $0 \in \mathrm{rt}\,A$ and $\mathrm{cl}\,C \subseteq \Psi_A(0)$ since Ψ_A has C–upward, closed values. On the other hand, if $v \notin \mathrm{cl}\,C$, then $\sum_{i=1}^m v_i k^i \in L(K) \backslash \Gamma_K(\mathrm{cl}\,C)$. Hence $\sum_{i=1}^m v_i k^i \notin \mathrm{rt}\,A$ by assumption and therefore $v \notin \Psi_A(0)$. This proves $\Psi_A(0) = \mathrm{cl}\,C$.

The remaining assertions follow from Propositions 14, 18 and 16.

(ii) We have $A_\Psi = \mathrm{rt}\,A_\Psi$ from Proposition 14, (ii). It holds $0 \in A_\Psi$ since $0 \in \Psi(0) \oplus C = \mathrm{cl}\,C \oplus C$ (mind that $0 \in C$). Assume there is $x \in L(K) \backslash \Gamma_K(\mathrm{cl}\,C) \bigcap A_\Psi$. Then on one hand $0 \in \Psi(x) \oplus C$ and on the other hand there is $v \in \mathbb{R}^m \backslash \mathrm{cl}\,C$ such that $x = \sum_{i=1}^m v_i k^i$. Translativity of Ψ implies $0 \in \Psi\left(\sum_{i=1}^m v_i k^i\right) \oplus C = \Psi(0) \oplus \{-v\} \oplus C$. Hence $v \in \mathrm{cl}\,C \oplus C \subseteq \mathrm{cl}\,C$ contradicting the assumption about v. Hence $L(K) \backslash \Gamma_K(\mathrm{cl}\,C) \bigcap A_\Psi = \emptyset$.

The remaining assertions again follow from Propositions 14, 18 and 16. ■

Application: Set–valued convex risk measures. In [34] (draft version [33]), Jouini et al. introduced set–valued coherent risk measures defined on $L_d^\infty(\Omega, \mathcal{F}, P)$, the space of (equivalence classes of) essentially bounded functions $x : \Omega \to \mathbb{R}^d$. Their constructions fit into the general framework of this section according to the following outline.

We consider $X = L_d^p(\Omega, \mathcal{F}, P)$ with $p \in [1, +\infty]$ and a convex cone $D \subseteq X$. Let m be a natural number with $1 \leq m \leq d$. Running j from 1 to m, define $k^j \in X$ by $k_i^j(\omega) = 0$ P-a.s. for $i \in \{1, 2, \dots, d\}$ with $i \neq j$ and $k_j^j(\omega) = 1$. This means, $k_j^j = e$ for $j = 1, \dots, m$ with e defined in the last paragraph of Section 3 and that the components k_i^j of k^j are the zero function for $i > m$. The case $m = d = 1$ is just the case discussed in Section 3. The following definition is apparantly new since in [34] only the sublinear (coherent) case has been considered, but straightforward.

Definition 10 *A function $R : X \to \widehat{\mathcal{P}}(\mathbb{R}^m)$ is called a* set-valued convex measure of risk *iff it is D–monotone, convex, translative and has closed values such that* $\mathbb{R}(0) \neq \mathbb{R}^m$. *A set-valued convex measure of risk is called* coherent *if it is additionally positively homogeneous.*

Observe that one can replace $R(x)$ by $\mathrm{cl}\,(R(x) \oplus C)$ which replaces the order relation $\preccurlyeq_C$ by $\supseteq$ and is essentially the transition from Φ_A to Ψ_A. Corollary 15 tells us that the cone C and $R(0)$ are strongly related. In [34], the cone C does not appear, but it is proven that in the coherent case, $R(0)$ is a closed convex cone, see Proposition 10 above and Property 3.1 in [34].

5 Conclusion

This note can be considered as an investigation of translative sets and functions in linear spaces.

In a natural way, to each set that is translative with respect to a finite collection of m elements corresponds a function with values in the power set of $\mathbb{R}^m$.

The real–valued case $m = 1$ has various applications in different fields of mathematics whereas the set–valued case is quite new. However, many constructions and properties can be extended from the one dimensional to higher dimensional cases. Thereby, a key tools are extensions of a partial order from a linear space to its power set.

Several questions arise naturally. We shall mention a few: (1) The images $\Phi_A(x)$ of a set–valued translative function constructed via a given set A are very large sets. If $m = 1$, then they are of the form $(r, +\infty)$ or $[r, +\infty)$ with $r \in (-\infty, +\infty)$. In this case, taking the "left boundary" of this interval one gets an extended real–valued function. Is a similar construction possible in the set–valued case? This means, can $\Phi_A(x)$ be replaced by the set $\min \{\Phi_A(x), \leq_C\}$ of minimal points of $\Phi_A(x)$ with respect to the partial order $\leq_C$? Is this a practical way dealing with the applications as set–valued convex risk measures? (2) Can a duality theory for convex set–valued functions be given such that dual representation theorems can be proven in a straightforward manner, such as Theorem 4 using the biconjugation theorem? Results in this direction are expected in the spirit of [39]. (3) There are many optimization problems in financial mathematics with an objective function that is a real–valued convex or coherent measures of risk, compare e.g. [52]. Also, solutions of vector optimization problems can be characterized as minimizers of real–valued monotone and translative functions. How shall we deal with such optimization problems in the set–valued case?

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