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Rational Ergodicity for Skew Products Cylinder Maps

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“Esta manhã, casais de borboletas brancas, douradas, azuis, passam inúmeras contra o fundo de bambus e samambaias da montanha. É um prazer para mim vê-las voar, não o seria, porém, apanhá-las, pregá-las em um quadro... Eu não quisera guardar delas senão a impressão viva, o frêmito de alegria da natureza, quando elas cruzam o ar, agitando as flores. Em uma coleção, é certo, eu as teria sempre diante da vista, mortas, porém, como uma poeira conservada junta pelas cores sem vida... O modo único para mim de guardar essas borboletas eternamente as mesmas, seria fixar o seu vôo instantâneo pela minha nota íntima equivalente... Como com as borboletas, com os vagalumes e com todos os outros deslumbramentos da vida... De nada nos serve recolher o despojo; o que importa é só o raio interior que nos feriu, o nosso contato com eles... e este como que eles também o levam embora consigo.”

Joaquim Nabuco, em *Minha Formação.*

eee... arrancando!

Abstract

In this thesis we study the asymptotic behavior of the ergodic Birkhoff Sums for cylinder skew products over irrational rotation preserving a σ-finite measure. We prove that such maps are ergodic, rationally ergodic and weakly homogeneous, calculating explicitly the Ergodic Sums for an increasing sequence of time and identifying the return sequence. From that, it is possible to obtain a second order ergodic theorem, which asserts that the double average renormalized by the return sequence converges to the integral of the observable function almost everywhere. We recall that the classical Birkhoff Theorem does not hold when the invariant measure is infinite.

Keywords: infinite ergodic theory, cylinder skew product, irrational rotation, ergodicity, rationally ergodic, weakly homogeneous.

Resumo

Nesta tese estudamos o comportamento assintótico das somas ergódicas de Birkhoff para sistemas dinâmicos do tipo skew products do cilindro preservando uma medida σ-finita. Provamos que tais aplicações são racionalmente ergódicas, calculando explicitamente as somas ergódicas para uma subsequência crescente de tempos e identificando as sequências de retorno. Com isto é possível obter um teorema de Birkhoff de segunda ordem, que afirma que quase certamente as médias ergódicas duplas, renormalizadas pela sequência de retorno, convergem para a integral do observável. Vale ressaltar que o teorema ergódico de Birkhoff clássico não é válido quando a medida invariante é infinita.

Palavras-chave: teoria ergódica infnita, skew products do cilindro, rotação irracional, ergodicidade, racionalmente ergódico, fracamente homogêneo.

Contents

1 Introduction

Ergodic theory is the quantitative theory of dynamical systems that deals with measure preserving transformations on a measure space. Usually the space where the map acts is assumed to be finite; however, there exist interesting systems that have an infinite invariant measure.

It is well known that for a measure preserving transformation in a probability measure space, almost all points are recurrent. Formally,

Theorem 1.1 (Poincaré's Recurrence Theorem)**.** *Let $F : X \to X$ be a measure preserving transformation on a probability space $(X, \mathfrak{B}, \mu)$. Let $E \in \mathfrak{B}$ with $\mu(E) > 0$. Then almost every point of E returns infinitely often to E by iterations of F.*

This is not the case if μ is infinite. In fact, the translation map $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x+1$ preserves the Lebesgue measure, but it is clear that there are no recurrent points. Indeed, every interval $(x, x+1)$ wanders away by the action of T.

Since we are interested in understanding how often a set will be visited by typical orbits, we would like to exclude systems possessing sets that many points do not return.

Definition 1.2. A measure preserving transformation F on $(X, \mathfrak{B}, \mu)$ is called *recurrent* if given any positive measure set B, almost every point will eventually return to B, i.e, $B \subseteq \bigcup_{n\geq 1} F^{-n}B \quad \mu - a.e.$

Definition 1.3. A measure preserving transformation F on a space $(X, \mathfrak{B}, \mu)$ is said to be *ergodic* if all its invariant sets, $A = F^{-1}(A)$, are such that $\mu(A) = 0$ or $\mu(A^c) = 0$.

An example of an ergodic transformation which is invariant by an infinite measure is the Boole map,

$$\begin{array}{rccc} F: & \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ & x & \mapsto & x - \frac{1}{x} \end{array}$$

The invariant measure for the Boole map is the Lebesgue measure in the real line. It was first studied in [11] and its ergodicity was shown later in [5].

There is also a two dimensional version of the Boole's map, $T : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$, $T(x, y) = (x - \frac{1}{y}, x + y - \frac{1}{y})$, where the invariant measure is the Lebesgue measure in the whole two dimensional plane [17].

It is important to note that the fact of preserving an infinite measure is not intrinsically connected with the circumstance of the domain of the transformation be not compact. For example, the Pomeau-Manneville maps, $T : [0,1] \to [0,1]$, $T(x) = x + cx^p mod(1)$, in which zero is a parabolic fixed point (i.e. $T'(0) = 1$), and in case that $p \geq 2$ ($c > 0$) the invariant measure has support in $[0,1]$ but gives infinite measure to the interval as shown in [31] and [15].

Polynomial and rational maps on $\mathbb{C}$ (quotient of polynomials acting on $\mathbb{C}$) with parabolic fixed points (points where the derivative has modulus one) in the Julia set and no critical points they also preserve an infinite measure which is a $h-$conformal measure concentrated in the Julia set and h is the Hausdorff dimension of the Julia set. This class of examples were studied in [4].

Other examples that can be cited are some quadratic unimodal maps (or logistic type maps) where the invariant measure is absolute continuous and giving infinite measure to the domain, see references [23], [14] and [6].

A standard result in classical ergodic theory that also fails for infinite measure preserving systems asserts about ergodic sums $S_n(f)(x) := \sum_{j=0}^{n-1} f \circ F^j(x)$.

Theorem 1.4 (Classical Birkhoff's Pointwise Ergodic Theorem)**.** *Suppose $F : X \to X$ is an ergodic measure preserving transformation in a probability space $(X, \mathfrak{B}, \mu)$. Then for all $f \in L^1(\mu)$, $\frac{1}{n} S_n(f)(x) \xrightarrow{n\to\infty} \int_X f d\mu$ a.e. on X.*

Taking f as the characteristic function $\mathbb{1}_A$ of a set $A \in \mathfrak{B}$ this theorem tell us that the rates $S_n(A)(x)$ of occupations of x in A are asymptotically the same for almost every point $x \in X$ and depend only on the set A. Furthermore, it says that the pointwise rate is given proportional to n.

Definition 1.5. A measure space $(X, \mathfrak{B}, \mu)$ is said to be σ*-finite* if there exists a countable family $\{X_i\}_{i\in\mathbb{N}}$ with $X_i \subset X$ with $\mu(X_i) < \infty$ such that $X = \bigcup_{i=1}^{\infty} X_i$.

Whenever we refer to a σ-finite space, we will in addition consider that the measure of the hole space is $\mu(X) = \infty$.

While for a finite measure space the classical Birkhoff's theorem holds, when we are treating with infinite measure what happens is the following

Theorem 1.6. *Let F be a recurrent ergodic measure preserving transformation in a σ-finite measure space $(X, \mathfrak{B}, \mu)$. Then for all $f \in L^1(\mu)$,*

$$\frac{1}{n} S_n(f) \xrightarrow{n\to\infty} 0 \ a.e. \ on \ \ X.$$

This means that, independently on the observable f chosen, the result does not say so much. Just state that $S_n(f)$ has a sublinear growth, nothing else. Nor how can the asymptotic behavior of $S_n(A)(x)$ depends on x or A.

Remark 1.7. Theorem above tell us that for almost every point we could not expect positive Lyapunov exponents for smooth systems preserving an infinite measure.

An expectable question at this point is to ask if, despite not having information about Birkhoff's sums, could we find another "appropriated" rate of convergence, that is, a renormalizing sequence of constants (a_n) such that $\frac{1}{a_n}S_n(A) \to \mu(A)$ a.e.? The result of below comes to tell us that such sequence does not exist.

Theorem 1.8 (Aaronson's Ergodic Theorem)**.** *Suppose that T is a recurrent, ergodic measure preserving transformation of the σ-finite space $(X, \mathfrak{B}, \mu)$ and let $(a_n)_{n\geq 1}$ be any positive sequence. Then*

either $\liminf\limits_{n\to\infty} \frac{S_n(f)}{a_n} = 0$ *a.e.* $\forall f \in L^1_+(\mu)$,

or $\limsup\limits_{n\to\infty} \frac{S_n(f)}{a_n} = \infty$ *a.e.* $\forall f \in L^1_+(\mu)$.

This shows that the occupations times $S_n(A)$ are very complicated. The rates at which the sums $S_n(A)$ grow are not uniform and any attempt of normalization will underestimate or overestimate the behavior of Birkhoff sums.

Although the theorems above give negative answers to get convergence of ergodic sums, there are results that work well in the positive direction.

Theorem 1.9 (Hopf's Ratio Ergodic Theorem)**.** *Let F be a recurrent ergodic measure preserving transformation on a σ-finite space $(X, \mathfrak{B}, \mu)$. If $f, g \in L^1_+(\mu)$ and $\int_X g d\mu \neq 0$ then*

$$\frac{S_n(f)}{S_n(g)} \xrightarrow{n\to\infty} \frac{\int_X f d\mu}{\int_X g d\mu} \qquad \text{a.e. on } X.$$

Hopf's Ratio ergodic theorem states that, even substantially depending on the point and on the function $f \in L^1_+(\mu)$, there is a kind of proportion in the behavior of the ergodic sums.

Definition 1.10. A recurrent, ergodic measure preserving transformation F on $(X, \mathfrak{B}, \mu)$ is called *rationally ergodic* if there is a set $A \in \mathfrak{B}$, $0 < \mu(A) < \infty$, satisfying a *Renyi* *inequality*: $\exists M > 0$ such that

$$\int_A (S_{i_n}(A))^2 \, d\mu \; < \; M \; \left(\int_A S_{i_n}(A) d\mu \right)^2$$

for some strictly increasing sequence $(i_n)_{n \in \mathbb{N}}$ of natural numbers.

We note that the preceding definition lightly differs from that one presented in [1], since we are asking that the Renyi inequality does not hold for all natural numbers, but only for a subsequence $(i_n) \subset \mathbb{N}$. This adaptation is performed to prove our results for cylinder map.

Theorem 1.11 (Aaronson [2]). *If F is rationally ergodic, satisfying the Renyi inequality along the sequence (i_n) then there is a sequence of constants $a_n \uparrow \infty$, unique up to asymptotic behavior, such that for all $(m_l) \subset (i_n)$ with $m_l \uparrow \infty$, there is a subsequence $n_k = m_{l_k}$ for which the following is true:*

$$\frac{1}{N} \sum_{k=1}^{N} \frac{1}{a_{n_k}} \sum_{j=0}^{n_k - 1} f \circ F^j \xrightarrow{N \to \infty} \int_X f d\mu \qquad a.e. \forall f \in L^1(\mu). \tag{1.1}$$

Remark 1.12. Two sequences a_n and a'_n of positive real numbers are of the same asymptotic type if the limit $\lim\limits_{n \to \infty} \frac{a_n}{a'_n}$ exists.

The constants a_n given by the theorem above are explicitly determined by the set A which satisfies the Renyi inequality. In fact,

$$a_n = a_n(A) := \frac{1}{\mu(A)^2} \sum_{k=0}^{n-1} \mu(A \cap F^{-k} A).$$

Definition 1.13. A recurrent, ergodic, measure preserving dynamical system satisfying (1.1) with respect to some sequence of constants (a_n) is called *weakly homogeneous*. The sequence (a_n) is called the *return sequence* of F and it is unique up to asymptotic type, see [1].

Theorem 1.11 establishes that, in the rationally ergodic context, there is a sort of second order Birkhoff's theorem. That is, the double average of the ergodic sums converges to the integral of the observable function $f \in L^1(\mu)$.

The proof of the rationally ergodic theorem when the Renyi inequality hold just for a subsequence of $S_{i_n}(A)$ follows in the same way of that one presented at Section 3.3 of [1], since at the very beginning of the proof a subsequence $\frac{S_{\nu_j}}{a_{\nu_j}}$ of $\frac{S_n}{a_n}$ is taken and used during all the rest of the proof.

1.1 Previous examples of rationally ergodic systems

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the circle, parameterized by $[0, 1)$. Examples of rationally ergodic systems on the cylinder $\mathbb{T} \times \mathbb{Z}$ were studied by Aaronson and Keane in [3]. They analyze ergodic properties for the map

$$\begin{array}{rccc} T_\alpha : & \mathbb{T} \times \mathbb{Z} & \to & \mathbb{T} \times \mathbb{Z} \\ & (x, z) & \mapsto & (x + \alpha, z + T(x)) \end{array}$$

which preserves the measure μ on the cylinder. Here T is the same as defined in (1.2) below, but they use only T in the skew product, without composing it with the expanding maps q_j as we will do. They prove (bounded) rational ergodicity for the map T_α when α is a quadratic surd, that is, α is a root of a quadratic polynomial with integer coefficients. We recall that this set of irrational numbers has zero Hausdorff dimension.

The ergodicity of maps of type T_α was previously shown by Conze and Keane in [16] for every irrational number α.

Others examples of rationally ergodic systems were described by Ledrappier and Sarig in [26]. They shown that the horocycle flow on the unit tangent bundle of a $\mathbb{Z}^d$-cover of a hyperbolic surface of finite area, equipped with the volume measure are rationally ergodic. And for semi-dispersing billiards with an infinite cusp, in [27] it is proved by Lenci that those billiards exhibit an infinite invariant measure and are rationally ergodic.

1.2 Statement of results

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the circle parameterized by $[0, 1)$ fitted with the Lebesgue measure. In order to introduce the results, we consider the function $T : \mathbb{T} \to \{-1, 1\}$ defined by

$$T(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in [\frac{1}{2}, 1). \end{cases} \tag{1.2}$$

Let us denote by α an irrational number and $\frac{p_j}{q_j}$ be a subsequence [1] of convergents of the continued fractional expansion of α.

Now, given α and a subsequence of convergents, the objects we study are skew product maps on the cylinder:

[1]Conditions for that subsequence will be specified in section 2.2.1.

$$
\begin{array}{rccc}
F = F_{\alpha,(q_j)_j}: & \mathbb{T} \times 2\mathbb{Z} & \longrightarrow & \mathbb{T} \times 2\mathbb{Z} \\
 & (x, z) & \longmapsto & (x + \alpha, z + \phi(x)),
\end{array}
$$

where

$$
\phi(x) = \phi(\alpha, (q_j)_j)(x) := \sum_{j=1}^{\infty} T(q_j x + q_j \alpha) - T(q_j x).
$$

There exists a set $\Lambda_\infty \subset \mathbb{T}$, with $\mathrm{Leb}(\Lambda_\infty) = 1$ such that $|\phi(x)| < \infty$ for all $x \in \Lambda_\infty$, see Section 2.3 for details.

Let μ be the measure on the cylinder $\mathbb{T} \times 2\mathbb{Z}$ given by the sum of the Lebesgue measures on the fibers, which is invariant by the map F.

Our purpose is to investigate ergodic properties of the infinite measure preserving system given by the skew product F above. Precisely, we prove the following

Theorem A [Cirilo, Lima, Pujals] *There is a set $H_1 \subset [0,1] \setminus \mathbb{Q}$ with $\mathrm{Leb}(H_1) = 1$ such that if $\alpha \in H_1$ then there exists $F = F(\alpha, (q_j)_j)$ verifying*

1. *$F \in L^P(\mu)$, for every $P \geq 1$, and*
2. *F is recurrent and ergodic.*

Theorem B [Cirilo, Lima, Pujals] *There is a set $H_2 \subset H_1$ with $HD(H_2) > 0$ such that if $\alpha \in H_2$ then there exists $F = F_{\alpha,(q_j)_j}$ verifying the thesis of theorem above and in addition*

1. *F is rationally ergodic (and so weakly homogeneous);*
2. *along the subsequence q_{n+1}, the constants $a_{q_{n+1}}$ appearing in the double average of weakly homogeneity are of type $a_{q_{n+1}} \simeq \dfrac{q_{n+1}}{\sqrt{\pi n}}$;*
3. *for x in a set of measure $1 - \varepsilon_n$, with $\varepsilon_n \xrightarrow{n\to\infty} 0$, we can estimate*

$$
\frac{S_{q_{n+1}}(A)(x,y)}{a_{q_{n+1}}} \simeq \binom{n}{\frac{n+m_n(x)}{2}} \frac{\sqrt{\pi n}}{2^n},
$$

where $m_n(x) = \sum_{j=1}^{n} T(q_j x)$ and A is a fiber $\mathbb{T} \times \{z\}$ of the cylinder.

Here, when saying that $a_n \simeq b_n$ we mean that $\frac{a_n}{b_n} \longrightarrow 1$ exponentially fast and out of a set of Lebesgue measure ϵ_n, with $\epsilon_n \xrightarrow{n\to\infty} 0$.

This thesis is organized as follows. In Section 2 we present standard background that we will use, as much as continued fractions and random walk properties. Still in Section 2, we introduce specific tools that we will use to attack our problem. The proof of Theorem A is presented in Sections 3 and 4, while the last two sections are devoted to conclude the proof of Theorem B.

2 Preliminaries and Preparations

This section is devoted to present standard backgrounds and to introduce some tools we will use in during the text.

2.1 Continued fractions

Proofs of properties of continued fraction cited in this section can be founded in [24], [28] and [29].

Our skew product on the cylinder will depend on the continued fraction expansion of the irrational number.

Let α be a real number and consider the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} =: [a_0, a_1, a_2, \ldots]$$

with *convergents* $\alpha_n = \frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n]$ which give the best rational approximation to α.

The speed of approximation of α by rational numbers is related to the growth rate of (q_n), also called *continuants* q_n. A more quantitative way of characterizing this is by the *order* of α, which is the best exponent one can have in the approximations. More specifically, for a real number $\tau \geq 2$, let

$$W(\tau) = \left\{ \alpha \in \mathbb{R}\,;\ \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\tau} \text{ for infinitely many rational numbers } \frac{p}{q} \right\}.$$

and

$$\operatorname{ord}(\alpha) := \sup\{\tau > 0; \|q\alpha\|_{\mathbb{Z}} < q^{1-\tau} \text{ for infinitely many } q \in \mathbb{Z}\}.$$

If $\|r\|_Z$ denotes the distance from a real number r to the nearest integer, we have

$$\|q_n\alpha\|_{\mathbb{Z}} = q_n \left|\alpha - \frac{p_n}{q_n}\right|$$

and

$$\|q_n\alpha\|_{\mathbb{Z}} = \min\{\|q\alpha\|_{\mathbb{Z}}; 0 < q < q_{n+1}, q \in \mathbb{Z}\}.$$

Recall that for almost every irrational α, the approximation condition $\|\alpha - \frac{p}{q}\| < \frac{1}{q^2 \log q}$ has an infinite number of rational solutions $\frac{p}{q}$. In particular, if $(q_n)_n$ is the sequence of continuants of the expansion in continued fractions, we can chose a subsequence that $\|q_j\alpha\|_Z < \frac{1}{q_j \log q_j} < \frac{1}{q_j}$.

Two others useful properties of continued factions for our work are

$$\frac{1}{2} < q_{n+1}\|q_n\alpha\|_Z \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}.$$

In addition, if $q_j = q_{n_j}$ is a subsequence of the consecutive continuants q_n, for which $2q_j < q_{j+1}$, using the properties above we get

$$\sum_{j>n} \|q_j\alpha\| < \sum_{j>n} \frac{1}{q_j} < \frac{1}{q_{n+1}} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \leq 2\frac{1}{q_{n+1}} \leq 4\|q_n\alpha\|_Z. \tag{2.1}$$

For our methods we will be interested in a class of irrational numbers having the following divisibility property:

Definition 2.1. An irrational number α will be called *divisible* if it admits a sequence $(q_{n_j})_{j\geq 1}$ of continuants such that $2q_{n_j}$ divides $q_{n_{j+1}}$.

Divisible numbers satisfies the property (2.1) and these numbers have full Hausdorff dimension inside each class $W(\tau)$, according to [30].

For Theorem B, we will be interested in irrational numbers which are divisible and have order greater then four.

2.2 Introducing our map

In general, when treating skew products of type

$$\begin{array}{rccc} F: & \mathbb{T} \times 2\mathbb{Z} & \to & \mathbb{T} \times 2\mathbb{Z} \\ & (x, z) & \mapsto & (x + \alpha, z + \phi(x)) \end{array}$$

the dynamics of F is intimately connected to the cocycle $S(\alpha, \phi) : \mathbb{T} \times \mathbb{Z} \to \mathbb{R}$ defined as the Birkhoff sums of ϕ with respect to the rotation $x \mapsto x + \alpha$:

$$S(\alpha, \phi)(x, n) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \phi(x + k\alpha) & , \text{if } n \geq 1 \\ 0 & , \text{if } n = 0 \\ \displaystyle -\sum_{k=1}^{-n} \phi(x - k\alpha) & , \text{if } n < 0. \end{cases}$$

The function $S(\alpha,\phi)(\cdot,n):\mathbb{T}\to\mathbb{R}$ used to be denoted by $S_n(\alpha,\phi)$ for simplicity. By Birkhoff theorem,

$$\frac{S_n(\alpha,\phi)(x)}{n}\xrightarrow{n\to+\infty}\int_{\mathbb{T}}\phi d\,\mathrm{Leb}$$

for Leb-almost every $x\in\mathbb{T}$. Then, if $\int\phi d\,\mathrm{Leb}\neq 0$, almost every point diverges on the second direction, which does not allow any kind of recurrence. For our intend we are interested in maps ϕ that has zero-mean. In this situation, the Birkhoff sums have at most a sublinear growth.

Given a divisible irrational number α, and any subsequence of divisible continuants $(q_j)_{j\geq 1}$, define $\phi:\mathbb{T}\to 2\mathbb{Z}$ by

$$\phi(x)=\sum_{j=1}^{\infty}T(q_jx+q_j\alpha)-T(q_jx). \tag{2.2}$$

As justified in Section 2.3, the map ϕ is well defined in a set of full measure $\Lambda_\infty\subset\mathbb{T}$.

Consider now the skew product determined by α and ϕ,

$$\begin{array}{rccc} F=F(\alpha,\phi)=F_{\alpha,(q_j)_j}: & \Lambda_\infty\times 2\mathbb{Z} & \to & \Lambda_\infty\times 2\mathbb{Z} \\ & (x,z) & \mapsto & (x+\alpha,z+\phi(x)). \end{array} \tag{2.3}$$

Let μ be a measure on the cylinder $\mathbb{T}\times 2\mathbb{Z}$ given by the sum of the Lebesgue measures on the fibers. Since the skew product F defined above is over a rigid rotation on the basis $\mathbb{T}$, this measure μ will be invariant by the map F.

Our purpose is to investigate ergodic properties of F with respect to the measure μ, study the almost sure asymptotic behavior of its ergodic sums and identify the return sequence of the weakly homogeneity.

For that we will use the auxiliary skew products F_n defined below. Recall that we are denoting by $\alpha_n=\frac{p_n}{q_n}$ the convergents of the continued fraction expansion of α. Let

$$\phi_n(x):=\sum_{j=1}^{n}T(q_jx+q_j\alpha_{n+1})-T(q_jx)$$

and

$$\begin{array}{rccc} F_n=F_n(\alpha_{n+1},\phi_n): & \mathbb{T}\times 2\mathbb{Z} & \to & \mathbb{T}\times 2\mathbb{Z} \\ & (x,z) & \mapsto & (x+\alpha_{n+1},z+\phi(x)) \end{array} \tag{2.4}$$

In Section 3 we will show that F_n converges to F in the $L^P(\mu)$ space for all $P\geq 1$. And we will explicitly use the skew products F_n to the return times counting procedure in Section 5.

Again, the dynamics of F is intimately connected to the cocycle $S(\alpha_{n+1}, \phi_n) : \mathbb{T}\times\mathbb{Z} \to \mathbb{R}$ defined as the Birkhoff sums of ϕ_n with respect to the rotation $x \mapsto x + \alpha_{n+1}$. More detailed,

$$\begin{aligned} F_n^k(x,z) &= (x + k\alpha_{n+1}, z + \phi_n(x) + \phi_n(x+\alpha_{n+1}) + \cdots + \phi_n(x + (k-1)\alpha_{n+1})) \\ &= (x + k\alpha_{n+1}, z + S_k(\alpha_{n+1}, \phi_n)(x)) \\ &= (x + k\alpha_{n+1}, z + \textstyle\sum_{i=0}^{k-1} \phi_n \circ R^i_{\alpha_{n+1}}(x)) \\ &= (x + k\alpha_{n+1}, z + \textstyle\sum_{j=0}^{n} T(q_j x + k q_j \alpha_{n+1}) - T(q_j x)) \\ &=: (x + k\alpha_{n+1}, z + \phi_n^k(x)). \end{aligned}$$

Analogously, we can do the same for the iterates of the skew product F when the point x is in the security region (see Section 2.3) and the function ϕ can be truncated as $\sum_{j=1}^{n} T(q_j x + q_j\alpha) - T(q_j x)$.

We will denote the sum appearing in the last expression of the k-th iterated of F_n (and F, respectively) by

$$\begin{aligned} \phi_n^k(x) &:= \sum_{j=0}^{n} T(q_j x + k q_j \alpha_{n+1}) - T(q_j x), \\ \phi^k(x) &:= \sum_{j=0}^{\infty} T(q_j x + k q_j \alpha) - T(q_j x). \end{aligned} \tag{2.5}$$

2.2.1 The sets H_1 and H_2

Here we will define the sets H_1 and H_2 in Theorems A and B respectively.

Let H_1 be the set of irrational numbers $\alpha \in [0,1)$ satisfying:

1. $\left|\alpha - \dfrac{p_n}{q_n}\right| < \dfrac{1}{q_n^2 \log q_n}$, for infinitily many convergents $\dfrac{p_n}{q_n}$,
2. there exist a subsequence q_{n_j} satisfying the condition above and such that
   (a) $\sum_j \frac{1}{\log q_{n_j}} < \infty$,
   (b) $2q_{n_j}$ divides $q_{n_{j+1}}$.

Remark 2.2. The set H_1 has total Lebesgue measure. In fact, the first condition holds for almost every irrational number. The second one is true for that set since the continuants q_j growth exponentially. And still with total Lebesgue measure we can extract a subsequence of that continuants which satisfies $2q_j \mid q_{j+1}$, see [30] for details.

Let H_2 is given by the set of irrational numbers $\alpha \in [0,1)$ contained in H_1 and satisfying

1. $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^{4+\epsilon}}$, for infinitely many rational $\frac{p_n}{q_n}$
2. there exist a subsequence q_{n_j} satisfying the condition above and such that $2q_{n_j}$ divides $q_{n_{j+1}}$

Lemma 2.3. *The set H_2 has positive Hausdorff dimension, in fact, it is equal to $\frac{1}{2}$.*

Proof. The positiveness of the Hausdorff dimension of the set H_2 follows from the Borosh-Fraenkel's theorem [12]. While the calculation of the exact Hausdorff dimension is a variant form of the Bugeaud-Moreira's theorem, see [13] and [30]. □

As an explicit example of an irrational number belonging to the set H_2, we can take $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} + \frac{1}{l_n}$, with l_n even to guarantee the divisibility property and l_n large enough to ensure the approximation condition, for example choose $l_n = (2q_1)^{n!}$. Then $\alpha = \frac{p_1}{q_1} + \sum_{n>1} \frac{1}{l_n}$ satisfies $q_{n+1} = (2q_1)^{n!} q_n$ and

$$\begin{aligned}|\alpha - \frac{p_n}{q_n}| &\leq \sum_{j>n} \frac{1}{l_j} = \sum_{j>n} \frac{1}{(2q_1)^{n!}} \\ &= \frac{1}{(2q_1)^{(n+1)!}} \sum_{j>n} \frac{1}{(2q_1)^{j!-(n+1)!}} \leq \frac{2}{(2q_1)^{(n+1)!}} \\ &< \frac{1}{(2q_1)^{(4+\epsilon)[(n-1)!+(n-2)!+\cdots+1]}} = \frac{1}{q_n^{(4+\epsilon)}}\end{aligned}$$

where the last inequality holds since

$$\begin{aligned}&\tfrac{(4+\epsilon)[(n-1)!+(n-2)+\cdots+1]}{(n+1)!} = \tfrac{(4+\epsilon)}{(n+1)n}\left[1 + \tfrac{(n-2)!}{(n-1)!} + \cdots + \tfrac{1}{(n-1)!}\right] \\ &= \tfrac{(4+\epsilon)}{(n+1)n}\left[1 + \tfrac{1}{(n-1)} + \tfrac{1}{(n-1)(n-2)} + \cdots + \tfrac{1}{(n-1)!}\right] \\ &\xrightarrow{n\to\infty} 0.\end{aligned}$$

2.3 Security regions

To understand the dynamics of the skew products (1.2) it is important to comprehend the behavior of the function ϕ. A good way to realizes that is to infer information about the truncations $\sum_{j=1}^{n} T(q_jx+q_j\alpha)-T(q_jx)$ of ϕ. We will present conditions guaranteeing that ϕ and its truncation coincides.

Let C denotes one of the two discontinuities of the map T, defined in (1.2). Unless we really need to specify which one of the discontinuities we are using, we will refer to a discontinuity of T simply as C.

Definition 2.4. We define the set $\Gamma_n := \left\{ x \in \mathbb{T} : d(q_jx, C) > \frac{1}{\log q_j}, \text{forall} j > n \right\}$.

The sets Γ_n will be the security regions that we will use to prove the ergodicity in item (2) of Theorem A.

We note that $\mathrm{Leb}(\Gamma_n) \to 1$ as n goes to infinity. In fact, since the denominators q_j satisfies $\sum_j \frac{1}{\log q_j} < \infty$, then

$$\begin{aligned} \mathrm{Leb}(\Gamma_n^c) &\leq 2 \sum_{j>n} \frac{1}{\log q_j} \\ &\xrightarrow{n\to\infty} 0 \end{aligned}$$

Remark 2.5. If x belongs to Γ_n and $k \leq q_n$ then q_jx and $q_jx + kq_j\alpha$ are in the same *plateau* for all $j \geq n$ (see Section 2.4 for precise definition of plateau). Therefore,

$$x \in \Gamma_n \quad \Rightarrow \quad \phi^k(x) = \sum_{j=1}^{n} T(q_jx + q_j\alpha) - T(q_jx) \qquad \forall k \leq q_n.$$

In fact observe that it is enough to require that $||q_jk\alpha||_{\mathbb{Z}} < d(q_jx, C)$, but in fact

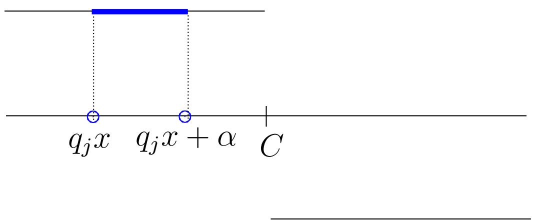
$$\begin{aligned} ||q_jk\alpha|| &\leq k\frac{1}{q_j \log q_j} \\ &\leq q_n \frac{1}{q_j \log q_j} \\ &\leq \frac{1}{\log q_j}, \ \forall j \geq n \\ &< d(q_jx, C), \ \text{because } x \in \Gamma_n \end{aligned}$$

Denoting by $\Gamma_\infty = \cup_{n\in\mathbb{N}}\Gamma_n$, we get a set in which $|\phi(x)| < \infty$ for all $x \in \Gamma_\infty$ and $\mathrm{Leb}(\Gamma_\infty) = 1$. Moreover, there exists a set $\widetilde{\Gamma}_\infty = \cap_{k\geq 0}(\Gamma_\infty + k\alpha)$, such that $|\phi(x)| < \infty$ and $|\phi(x + k\alpha)| < \infty$ for all iterated k and yet $\mathrm{Leb}(\widetilde{\Gamma}_\infty) = 1$.

For theorem B and for item 1 of theorem A, we will relax a little the security regions:

Definition 2.6. $\Lambda_n := \{x \in \mathbb{T} : d(q_jx, C) > \|q_j\alpha\|_{\mathbb{Z}}\ \forall j > n\}$

(Fig)



Since $\frac{1}{\log q_j} > \frac{1}{q_j} > \|q_j\alpha\|_{\mathbb{Z}}$, each set Λ_n contains the set Γ_n therefore we also have that $\mu(\Lambda_n) \to 1$ as n goes to infinity.

Analogously, we can define the sets $\Lambda_\infty = \cup_{n\in\mathbb{N}}\Lambda_n$, getting a set in which $|\phi(x)| < \infty$ for all $x \in \Lambda_\infty$ and $\mathrm{Leb}(\Lambda_\infty) = 1$. And still, $\widetilde{\Lambda}_\infty = \cap_{k\geq 0}(\Lambda_\infty + k\alpha)$, such that $|\phi(x)| < \infty$ and $|\phi(x + k\alpha)| < \infty$ for all iterated k and yet $\mathrm{Leb}(\widetilde{\Lambda}_\infty) = 1$.

2.4 Branches and Plateaus

In this section we will introduce the notion of branches and plateaus for the auxiliary functions T and $T \circ q_j$ used in the definition of ϕ. This will helps to obtain information about the behavior of ϕ.

2.4.1 Branches

Let $q_j : \mathbb{T} \to [0, 1)$ be the expanding map $x \mapsto q_jx(\mathrm{mod}\ 1)$ and denote by I_j^i the intervals where q_j is a diffeomorphism. Consider a decomposition of each I_j^i in two smaller intervals $I_j^{i,1} \cup I_j^{i,-1}$. Each of which has length $\frac{1}{2q_j}$ and corresponds respectively to the first and second half part of the interval I_j^i. The reason of this last decomposition will be more clear bellow, when we introduce the notion of plateaus.

The intervals $I_j^{(i,t_j)}$ are called *branches* of q_j. Sometimes we will omit an upper index and write the branches of q_j simply as $I_j^{i,t}$, where $t \in \{1, -1\}$.

For each j the circle can be decomposed in the union of branches

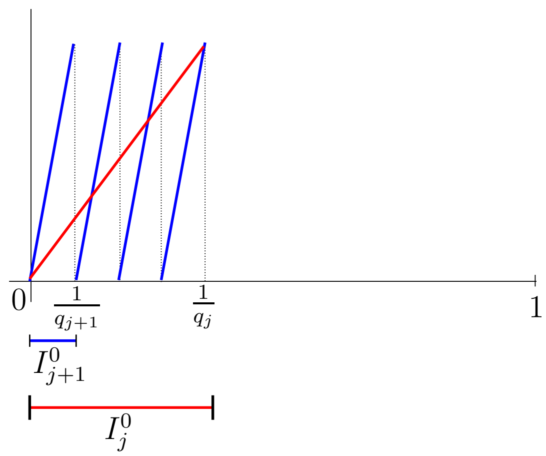
$$\mathbb{T} = [0,1) = \bigcup_{i=1}^{q_j} I_j^{(i,1)} \cup I_j^{(i,-1)}.$$

Remembering that $2q_j$ must divide q_{j+1}, we can take a further decomposition to write a branch on the level j as an union of branches of the next level $j+1$, for example,

$$I_j^{(1,1)} = \bigcup_{i=1}^{\frac{q_{j+1}}{2q_j}} I_{j+1}^{(i,1)} \cup I_{j+1}^{(i,-1)}.$$

We do not explicit the decomposition for every branch to avoid useless notation. What is important to our context is that all branches $I_j^{(i,t)}$ contains the same number of branches of the level $j+1$ in their decomposition, i.e., $\frac{q_{j+1}}{2q_j}$, as much as in the above example.

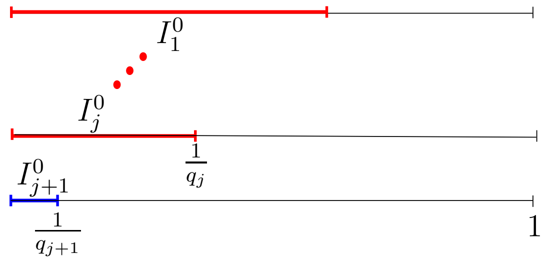
(Fig)



Given any point x on the circle, we denote by $I_j(x)$ the branch $I_j^{i,t}$ on the level j which contains the point x. This is not ambiguous since for each j, there is a unique i and a unique t such that $x \in I_j^{i,t}$.

Looking in the reverse way, if x belongs to $I_n(x)$, we can associate a (unique) sequence of pairs $(i_j, t_j)_{j \leq n}$ such that $x \in I_j^{(i_j,t_j)}$ for all $j = n, \cdots, 1$. And again using the condition $2q_j \mid q_{j+1}$, we obtain that

$$I_n^{(i_n,t_n)} \subset I_{n-1}^{(i_{n-1},t_{n-1})} \subset \cdots \subset I_1^{(i_1,t_1)}.$$



(Fig)

A simple but important observation is that for all $x' \in I_n(x)$, x' and x will belong to the same branches $I_j^{(i_j,t_j)}$ for all $j = n, \cdots, 1$.

2.4.2 q_j-intervals and plateaus

Let us remember the definition of the auxiliary function $T : \mathbb{T} \to \{-1, 1\}$

$$T(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in [\frac{1}{2}, 1); \end{cases} .$$

and the definition of function $\phi : \mathbb{T} \to \mathbb{Z}$,

$$\phi(x) = \sum_{j=1}^{\infty} T(q_j x + q_j \alpha) - T(q_j x).$$

Note that ϕ is defined by compositions of T with the expanding maps $x \mapsto q_j x$, for instance, $T_j(x) = T(q_j x)$. In each summand of ϕ we do the difference between T_j at the points $R_\alpha(x)$ and x.

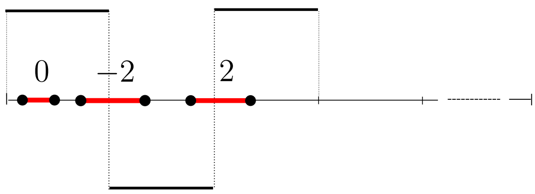
In this way, T_j is a function on the circle of period $\frac{1}{q_j}$

$$T_j(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2q_j}), \\ -1, & \text{if } x \in [\frac{1}{2q_j}, \frac{1}{q_j}) \end{cases}$$

which has discontinuities at the points $\{0, \frac{1}{2q_j}, \frac{1}{q_j}, \frac{3}{2q_j}, \frac{2}{q_j}, \cdots, \frac{q_j-1}{q_j}, \frac{2q_j-1}{2q_j}\}$. Unless we really need to specify which one of the discontinuity we are using, we will refer to a discontinuity of T or T_j simply as C.

We will call a q_j-*interval*, and denote it by $Q_j(x)$, the interval with extremes $q_j x$ and $q_j(x + \alpha)$ of length $q_j \alpha$ contained in $\mathbb{T}$. Observe that the value of each summand $T(q_j x + q_j \alpha) - T(q_j x)$ of ϕ can be $0, 2$ or -2. And what determines which one of the possible values will be taken is the position of the extremes of the q_j-interval $Q_j(x)$ relative to the discontinuities of T_j. Exactly, if that interval does not contain any discontinuity, so the corresponding summand is of null value. If the q_j-interval at the level j contains a discontinuity which is multiple of the period, then the related summand assume the value 2. The value -2 is designated when the q_j-interval contains a discontinuity that belongs to the middle of a fundamental domain of T_j.

(Fig)



To avoid ambiguity, we ask q_jx to be different from the discontinuities C of the level j. Note we have a finite number os discontinuities at each level and we are treating with enumerable levels. Then we are quitting an enumerable set of the circle which will not disturb our purpose in this work. More precisely, we are considering ϕ restricted to a set of full measure (indeed, this set is the circle minus an enumerable set). We will yet denote the map restricted to this set as ϕ, and we will not mention this again in the text.

We will call *plateaus* and denote by L_{s_j} the intervals of the circle where T_j is constant equals s_j. Therefore we are allowing $s_j = 1$ or $s_j = -1$. Note that for each j the circle is divided in $2q_j$ plateaus. Sometimes when we refer to a plateau at the level j we are meaning the union of the plateaus in which T_j has the same value being 1 or -1.

2.5 Random walks

Here we will remember some properties of random walks that will be used to prove the ergodicity. More details can be found in [18].

For almost every point x in $\mathbb{T}$ (excluding the discontinuities, where T_j are not well defined)

$$n \mapsto m_n(x) = \sum_{j=1}^{n} T_j(x) = \sum_{j=1}^{n} T(q_jx) \tag{2.6}$$

defines a simple random walk on $\mathbb{Z}$. The expectation of that random walk is zero.

To see that this is a random walk, observe that $x \mapsto T_j(x) = T(q_j x)$ can be interpreted as random variables with domain $\mathbb{T}$ and image in $\{-1, 1\}$ such that the probabilities $\mathbb{P}(T_j = 1) = \mathbb{P}(T_j = -1) = \frac{1}{2}$.

To show that $n \mapsto m_n(x)$ is a random walk, its enough to show that T_j are independent. This follows immediately checking that $\mathbb{P}((T_j = t_j) \cap (T_{j'} = t_{j'})) = \frac{1}{4}$. But the last is indeed true, as it was proved in previous subsection, because the branches of $T_{j'}$ are embedded in the branches of T_j and in fact subdivide them since $2q_j \mid q_{j'}\ \forall j' > j$.

By the recurrence property of random walks - sometimes also called *level-crossing phenomenon* - a simple random walk on $\mathbb{Z}$ will cross every point an infinite number of times. In particular if we look for the random walk defined by $n \mapsto m_n(x) - m_n(y)$ it will cross the zero line infinite many times. In other words, and as we will in fact use throughout the text, for almost every pairs of points x and y their random walks defined by $m_n(x)$ and $m_n(y)$ will coincide an infinite number of times.

3 ϕ_n converges to ϕ in $L^P(\mathbb{T})$

The L^P convergence presented in this section depends only on the condition, $\frac{1}{q_j} < \frac{1}{\gamma^j}$ of the irrational number α where γ is a real number greater than 1. Since for all irrational α we have the approximation condition $\|q_j\alpha\|_Z < \frac{1}{q_j}$, in our situation we acquire additionally that $\|q_j\alpha\|_Z < \frac{1}{\gamma^j}$. Note that the divisibility condition $2q_j \mid q_{j+1}$ is enough to guarantee $\frac{1}{q_j} < \frac{1}{\gamma^j}$ and the constant γ will be greater than 2.

3.1 ϕ and ϕ_n belong to $L^P(\mathbb{T})$

As the calculus for ϕ_n is analogous and simpler, we will do that just for ϕ. (For ϕ_n, just put α_{n+1} instead of α and note that $||q_m\alpha_{n+1}||_{\mathbb{Z}} \leq ||q_m\alpha||_{\mathbb{Z}} \forall m \leq n$).

We will show that there exist K such that $\int_{\Lambda_n} |\phi|^P dx < K$. Here, although it depends on P of the L^P, the constant K is independent of n. As $\mu(\Lambda_n)$ tends to 1 as n goes to infinity, this is enough to show that $\phi \in L^P$.

Let $\Lambda_n := \{x \in \mathbb{T} : d(q_jx, C) > ||q_j\alpha||_{\mathbb{Z}} \ \forall j > n\}$, in particular, if $x \in \Lambda_n$, so $\phi(x) = \sum_{j=1}^{n} T(q_jx + q_j\alpha) - T(q_jx)$ and

$$|\phi(x)| \leq \sum_{j=1}^{n} |T(q_jx + q_j\alpha) - T(q_jx)|$$

where $|T(q_jx + q_j\alpha) - T(q_jx)|$ can be 0 or 2.

Consider

$$B_m := \{x \in \mathbb{T} : \sum_{j=1}^{n} |T(q_jx + q_j\alpha) - T(q_jx)| = 2m\}$$

thus $x \in \Lambda_n \cap B_m$ implies that $|\phi(x)| = 2m$.

Whence it follows that

$$\begin{aligned} \int_{\Lambda_n} |\phi|^P &\leq \sum_{m=0}^{n} \int_{B_m} (2m)^P d\,\mathrm{Leb} \\ &= \sum_{m=0}^{n} (2m)^p |B_m| \end{aligned}$$

Now let us estimate $|B_m|$.

We will partition $B_m = \cup_{l=m}^{n} B_{m,l}$, where

$$B_{m,l} := \{x \in \mathbb{T} : \sum_{j=1}^{n} |T(q_j x + q_j\alpha) - T(q_j x)| = 2m \text{ and}$$
$$\text{the last summand non null is in the position } l\}.$$

Since there are m non null elements, $l \geq m$.

We already know that if the j-th summand $T(q_j x + q_j\alpha) - T(q_j x)$ is non null then $C \in Q_j(x) = [q_j x, q_j x + q_j\alpha] \subset \mathbb{T}$, and as the expanding map q_j preserves the Lebesgue measure on the circle, it follows that

$$\text{Leb}(C_j) = \text{Leb}\left(\{x \in \mathbb{T} : C \in [q_j x, q_j x + q_j\alpha]\}\right) = ||q_j\alpha||_{\mathbb{Z}}.$$

In this way, the set B_m is contained in several C_j. All that which the j-th summand is non null. In particular, $B_m \subset C_l$.

Thus

$$|B_m| \leq \sum_{l=m}^{n} |B_{m,l}| \leq \sum_{l=m}^{n} |C_l| = \sum_{l=m}^{n} ||q_l\alpha||_{\mathbb{Z}} \leq 4||q_m\alpha||_{\mathbb{Z}}.$$

So we can calculate

$$\begin{aligned}
\int_{\Lambda_n} |\phi|^P d\,\text{Leb} &\leq \sum_{m=0}^{n} (2m)^P 4||q_m\alpha||_{\mathbb{Z}} \\
&\leq \sum_{m=0}^{\infty} \frac{2^{P+2} m^P}{q_m} \\
&\leq \sum_{m=0}^{\infty} \frac{2^{P+2} m^P}{(\gamma^m)} < K(P).
\end{aligned}$$

As it does not depends on n, it follows that $||\phi||_{L^P}^P < K(P)$.

3.2 ϕ_n is a Cauchy sequence

Given any pair of integers k and n, say $0 < k < n$, then

$$|(\phi_n-\phi_k)(x)| \leq \sum_{j=1}^{k} |T(q_jx+q_j\alpha_{n+1})-T(q_jx+q_j\alpha_{k+1})| + \sum_{j=k+1}^{n} |T(q_jx+q_j\alpha_{n+1})-T(q_jx)|.$$

Following the same reasoning of the previous subsection, let

$$B_m^n := \{x \in \mathbb{T} : \sum_{j=k+1}^{n} |T(q_jx+q_j\alpha_{n+1})-T(q_jx)| = 2m\}$$

$$A_m^{k,n} := \{x \in \mathbb{T} : \sum_{j=1}^{k} |T(q_jx+q_j\alpha_{n+1})-T(q_jx+q_j\alpha_{k+1})| = 2m\}$$

$$\begin{aligned}\int |\phi_n-\phi_k|^P d\,\mathrm{Leb} &\leq \sum_{m=0}^{k} \int_{A_m^{k,n}} (2m)^P d\,\mathrm{Leb} + \sum_{m=0}^{n-k} \int_{B_m^n} (2m)^P d\,\mathrm{Leb} \\ &= \sum_{m=0}^{k} (2m)^P |A_m^{k,n}| + \sum_{m=0}^{n-k} (2m)^P |B_m^n|\end{aligned}$$

As before, $B_m^n = \bigcup_{l=k+m}^{n} B_{m,l}^n$, where

$$\begin{aligned}B_{m,l}^n := \{x \in \mathbb{T} : \sum_{j=k+1}^{n} |T(q_jx+q_j\alpha_{n+1})-T(q_jx)| = 2m \text{ and} \\ \text{the last summand non null is in the position } l\}\end{aligned}$$

and defining $C_l^n := \{x \in \mathbb{T} : C \in [q_lx, q_jx+q_l\alpha_{n+1}]\}$, we can estimate

$$\begin{aligned}|B_m^n| &\leq \sum_{l=k+m}^{n} |B_{m,l}^n| \leq \sum_{l=k+m}^{n} |C_l^n| \leq \sum_{l=k+m}^{n} ||q_l\alpha_{n+1}||_{\mathbb{Z}} \leq \sum_{l=k+m}^{n} ||q_l\alpha||_{\mathbb{Z}} \\ &\leq 4||q_{k+m}\alpha||_{\mathbb{Z}}\end{aligned}$$

Analogously, take

$$C_j^{k,n} := \{x \in \mathbb{T} : T(q_jx+q_j\alpha_{n+1})-T(q_jx+q_j\alpha_{k+1}) \neq 0\}$$

and we know that

$$\begin{aligned}
\mathrm{Leb}(C_j^{k,n}) &= q_j|\alpha_{n+1} - \alpha_{k+1}| \\
&\leq q_j|\alpha_{k+1} - \alpha| \\
&\leq q_j \frac{||q_{k+1}\alpha||_{\mathbb{Z}}}{q_{k+1}}
\end{aligned}$$

The set $A_m^{k,n}$ is contained in $C_j^{k,n}$ for all j such that the j-th summand is non null. And this happens for at least one index l greater than m, since there are m non null elements.

In this way,

$$\begin{aligned}
|A_m^{k,n}| &\leq \sum_{l=m}^{k} |C_j^{n,k}| \\
&\leq \sum_{l=m}^{k} q_l \frac{||q_{k+1}\alpha||_{\mathbb{Z}}}{q_{k+1}} \\
&\leq (k-m) q_k \frac{||q_{k+1}\alpha||_{\mathbb{Z}}}{q_{k+1}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int |\phi_n - \phi_k|^P d\,\mathrm{Leb} &\leq \sum_{m=0}^{k} (2m)^P (k-m) \frac{q_k ||q_{k+1}\alpha||_{\mathbb{Z}}}{q_{k+1}} + \sum_{m=0}^{n-k} (2m)^P \;\, 4||q_{k+m}\alpha||_{\mathbb{Z}} \\
&= \frac{q_k ||q_{k+1}\alpha||_{\mathbb{Z}}}{q_{k+1}} \; 2^P \;\, k(\sum_{m=0}^{k} m^P) + 2^{(P+2)} ||q_k\alpha||_{\mathbb{Z}} (\sum_{m=0}^{n-k} m^P \frac{||q_{k+m}\alpha||_{\mathbb{Z}}}{||q_k\alpha||_{\mathbb{Z}}}) \\
&\leq 2^P k \frac{q_k}{q_{k+1}} ||q_{k+1}\alpha||_{\mathbb{Z}} (\sum_{m=0}^{k} m^P) + 2^{(P+2)} ||q_k\alpha||_{\mathbb{Z}} (\sum_{m=0}^{n-k} \frac{m^P}{\gamma^m}) \\
&\leq 2^P k \frac{q_k}{q_{k+1}} \frac{1}{\gamma^{k+1}} (\sum_{m=0}^{k} m^P) + 2^{(P+2)} \frac{1}{\gamma^{k+1}} (\sum_{m=0}^{n-k} \frac{m^P}{\gamma^m}) \\
&\leq 2^P k \frac{q_k}{q_{k+1}} (\sum_{m=0}^{k} \frac{m^P}{\gamma^{k+1}}) + 2^{(P+2)} \frac{1}{\gamma^{k+1}} (\sum_{m=0}^{n-k} \frac{m^P}{\gamma^m}) \\
&\xrightarrow{k\to\infty} 0.
\end{aligned}$$

Where the convergence to zero is due to the fact that on the sums above the numerator has polynomial growth while the denominator increase exponentially fast and $\gamma > 1$.

Concluding that ϕ_n is sequence of Cauchy and because of the completeness of $L^P(\mu)$, it converges.

Let see that the limit is exactly the function $\phi(x) := \sum_{j=1}^{\infty} T(q_j x + q_j \alpha) - T(q_j x)$. In fact, defining

$$\tilde{\phi}_n(x) = \sum_{j=1}^{n} T(q_j x + q_j \alpha) - T(q_j x)$$

By an analogous calculation as before,

$$\begin{aligned}
||\phi_n - \tilde{\phi}_n||_{L^P}^P &\leq \sum_{m=0}^{n} (2m)^P \; 2||q_m|\alpha - \alpha_{n+1}| \;||_{\mathbb{Z}} \\
&\leq 2^{(P+1)} \sum_{m=0}^{n} m^P q_m \frac{||q_{n+1}\alpha||_{\mathbb{Z}}}{q_{n+1}} \\
&\leq 2^{(P+1)} \sum_{m=0}^{n} m^P ||q_{n+1}\alpha||_{\mathbb{Z}} \\
&\leq 2^{(P+1)} \sum_{m=0}^{n} \frac{m^P}{\gamma^{n+1}} \\
&\xrightarrow{n \to \infty} \; 0.
\end{aligned}$$

Where the convergence to zero is due to the fact that on the sums above the numerator has polynomial growth while the denominator increase exponentially fast and $\gamma > 1$.

Concluding that $\phi_n \xrightarrow{L^P} \phi$.

4 Ergodicity and Recurrence

The process developed in this section shows that a set of positive measure when iterated by the skew product F defined in (1.2) fills the hole cylinder $\mathbb{T} \times 2\mathbb{Z}$ almost everywhere. With these ideas we can prove the ergodicity and also the recurrence property of our map F.

As we already said in Section 2, the measure μ we are considering on the cylinder $\mathbb{T} \times 2\mathbb{Z}$ is the measure given by the sum of the Lebesgue measures on the fibers and it is invariant by the skew product F.

4.1 An auxiliary proposition

Definition 4.1. A point z is said to be a *density point* for a set Γ if

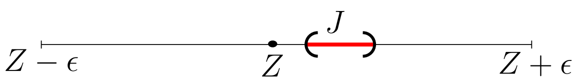
$$\lim_{\epsilon \to 0} \frac{\mu(B_\epsilon(z) \cap \Gamma)}{\mu(B_\epsilon(z))} = 1$$

In other words, given any δ, there exist $\epsilon > 0$ s.t. $\frac{\mu(B_\epsilon(z) \cap \Gamma)}{\mu(B_\epsilon(z))} > 1 - \delta$.

Proposition 4.2. *Let z be a density point for a set $\Gamma \subset \mathbb{R}$, δ and ϵ such that* $\dfrac{\mu(B_\epsilon(z) \cap \Gamma)}{2\epsilon} >$ $1 - \delta$*. Let J be an interval contained in $B_\epsilon(z)$ such that $|J| = \frac{2\epsilon}{k}$, where k is a positive integer such that $k\delta < 1$. Then* $\dfrac{\mu(\Gamma \cap J)}{|J|} > 1 - k\delta$.

Proof. Suppose by contradiction that $\mu(\Gamma \cap J) \leq (1 - k\delta)\frac{2\epsilon}{k}$.

(Fig)



Then,

$$\begin{aligned}\frac{\mu(B_\epsilon(z)\cap\Gamma)}{2\epsilon} &= \frac{\mu((J\cup J^c)\cap\Gamma)}{2\epsilon}\\ &\leq \frac{\mu(J\cap\Gamma)+\mu(J^c\cap\Gamma)}{2\epsilon}\\ &\leq \frac{1-k\delta}{k}+\frac{\mu(J^c\cap\Gamma)}{2\epsilon}\\ &\leq \frac{1-k\delta}{k}+\frac{k-1}{k}\frac{2\epsilon}{2\epsilon}\\ &= 1-\delta\end{aligned}$$

which contradicts the election of ϵ and δ.

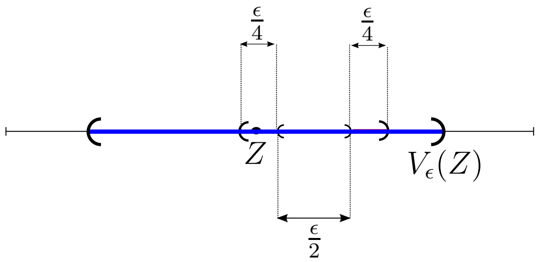
□

As a consequence we have the following

Corollary 4.3. *Adopting the notation and conditions of the previous proposition, if* $\delta < \frac{1}{4}$ *and* $|J| = \epsilon$*, there exists* $\tilde{z} \in J \cap \Gamma$ *s.t.* $d(\tilde{z}, \partial J) > \frac{1}{4}\epsilon$*.*

Proof. Suppose this does not occur. So $\Gamma \cap B_{\frac{\epsilon}{4}}(w) = \emptyset$, where w is the midpoint of J. Therefore $\Gamma \cap J \subset J \backslash B_{\frac{\epsilon}{4}}(w)$.

(Fig)



So we get, using the previous proposition, that

$$\begin{aligned}\frac{\epsilon}{2} > \mu(J\backslash B_{\frac{\epsilon}{4}}(w)) > \mu(\Gamma\cap J) > (1-2\delta)|J|\\ = (1-2\delta)\frac{2\epsilon}{2} = (1-2\delta)\epsilon.\end{aligned}$$

which implies $\delta > \frac{1}{4}$. This contradiction proves the corollary.

□

4.2 Proving the ergodicity

To prove the ergodicity, we will first prove that any invariant set intersecting a fiber with positive measure saturates that entire fiber. The strategy is to observe that fixing a point $(y, 0)$ in the refereed fiber, the orbit until time q_n of any other point x in the basis $\mathbb{T}$ is $\frac{1}{q_n}$-near to y, since the map is the rigid rotation R_α on the basis. Then we can choose k_0 such that the k_0-th iterated of x is $\frac{1}{q_n}$-near to y at the first coordinate. The problem at this point is that the second coordinate could be large and so the iterated $F^{k_0}(x, 0)$ could be really far away from $(y, 0)$. We will show, using the propositions 4.4 and 4.5 below, that x can be chosen in a set of total Lebesgue measure so that $\phi^{k_0}(x) = 0$. This guarantees that $F^{k_0}(x, 0)$ is in fact in the same fiber as $(y, 0)$.

After that we prove by proposition 4.6, that this saturated set jumps fibers with positive measure.

Suppose by contradiction the existence of an invariant set Γ for F, such that $\mu(\Gamma)$ and $\mu(\Gamma^c)$ are both positive.

In these hypothesis there is a fiber in which $\mu(\Gamma\ |_{fiber}) > 0$. Let suppose, by a first case, that the measure of Γ^c is positive when restrict to the same fiber where $\mu(\Gamma\ |_{fiber}) > 0$.

Take points $x \in \Gamma$ and $y \in \Gamma^c$ in a way that they are density points for $\Gamma \cap \Gamma_n$ and Γ^c respectively. Which is possible because $\mu(\Gamma_n) \to 1$, as we showed in section 2.3.

Thus, given any $0 < \delta < 1$, consider n large enough and $\epsilon = \frac{1}{q_{n-1}}$ such that

$$\frac{\mu(B_\epsilon(x) \cap (\Gamma \cap \Gamma_n))}{2\epsilon} > 1 - \delta$$

and

$$\frac{\mu(B_\epsilon(y) \cap \Gamma^c)}{2\epsilon} > 1 - \delta$$

Note that as the embedding $\Gamma_n \subset \Gamma_{n+1}$ occurs, the above happens for all n larger than some n_0, uniformly on δ. Observe also that we do not require that $y \in \Gamma_n$.

Let $I_{n-1}(x)$ the branch of length $\frac{1}{2q_{n-1}}$ containing x and $I_{n-1}(x) \subset B_\epsilon(x)$. Analogously, $I_{n-1}(y) \subset B_\epsilon(y)$.

By the proposition 4.2 and corollary 4.3 above we can suppose the following

$$\frac{\mu(I_{n-1}(x)\cap(\Gamma\cap\Gamma_n))}{\frac{1}{2q_{n-1}}} > 1-2\delta,$$
$$d(x,\partial I_{n-1}(x)) > \frac{1}{6q_{n-1}},$$
$$\frac{\mu(I_{n-1}(y)\cap\Gamma^c)}{\frac{1}{2q_{n-1}}} > 1-2\delta,$$
$$d(y,\partial I_{n-1}(y)) > \frac{1}{6q_{n-1}}.$$

As $||q_n\alpha||_{\mathbb{Z}} < \frac{1}{q_n}$, there exist $k_0 \le q_n$ such that $d(x+k_0\alpha, y) < \frac{1}{q_n}$.
Consider the set

$$B := \{x' \in \mathbb{T}\times\{0\} \cap I_{n-1}(x) \cap (\Gamma\cap\Gamma_n) : x'+k_0\alpha \in I_{n-1}(y)\}$$

Proposition 4.4. *Let k_0 as above, then $F^{k_0}(B)$ is contained in the same fiber as B and $F_1^{k_0}(B) \subset (I_{n-1}(x)\cap(\Gamma\cap\Gamma_n)) + k_0\alpha$, where F_1 is the projection on the first coordinate.*

Proof.

$$\phi^{k_0}(x') = \sum_{j=1}^{n-1} T(q_jx'+q_jk_0\alpha) - T(q_jx') + \sum_{j\ge n} T(q_jx'+q_jk_0\alpha) - T(q_jx')$$

Observe that the second sum is zero because of remark 2.5.

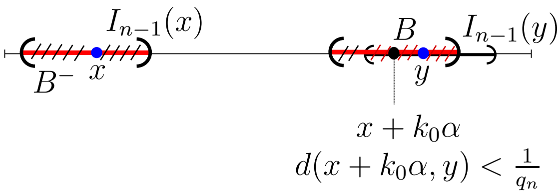
As $x'+k_0\alpha$ belongs to the branch $I_{n-1}(y)$ we know that $T(q_j(x'+k_0\alpha)) = T(q_jy)$ for all $j \le n-1$. By interpreting $x \mapsto \sum_{j=1}^{n-1} T(q_jx) = M_{n-1}(x)$ as a random walk (see Section 2.5), we can suppose without lose of generality that $M_{n-1}(x) = M_{n-1}(y)$. As mentioned also in Section 2.5, almost everywhere two random walks coincide. And since $x' \in I_{n-1}(x)$, we also know that $T(q_jx) = T(q_jx')$ for all $j \le n-1$.

In this way,

$$\sum_{j=1}^{n-1} T(q_j x' + q_j k_0 \alpha) - T(q_j x') = \sum_{j=1}^{n-1} T(q_j y) - T(q_j x')$$
$$= \sum_{j=1}^{n-1} T(q_j x) - \sum_{j=1}^{n-1} T(q_j x')$$
$$= 0$$

Concluding that if x' belongs to B then $\phi^{k_0}(x') = 0$. Thus $F_1^{k_0}(B) \subset (I_{n-1}(x) \cap (\Gamma \cap \Gamma_n)) + k_0\alpha$. $\square$

Let us denote $B^- = I_{n-1}(x) \cap (\Gamma \cap \Gamma_n)$, by the proposition 4.2, $\mu(B^-) > (1 - 2\delta)|I_{n-1}(x)| = (1 - 2\delta)\frac{1}{2q_{n-1}}$.
(Fig)



Proposition 4.5. *Let k_0 as above, then $F^{k_0}(B) \cap \Gamma^c \neq \emptyset$.*

Proof. In fact, $F^{k_0}(B) \subset I_{n-1}(y)$ and

$$\mu(F^{k_0}(B)) \geq \mu(B^-) - \frac{1}{q_n} > \frac{1 - 2\delta}{2q_{n-1}} - \frac{1}{q_n}$$

and

$$\mu(\Gamma^c \cap I_{n-1}(y)) > \frac{1 - 2\delta}{2q_{n-1}}$$

As much as both of the sets $F^{k_0}(B)$ and $\Gamma^c \cap I_{n-1}(y)$ are contained in $I_{n-1}(y)$, if their intersection is empty is because

$$\mu(I_{n-1}(y)) \geq \mu(F^{k_0}(B)) + \mu(\Gamma^c \cap I_{n-1}(y))$$

which is equivalent to

$$\frac{1}{2q_{n-1}} \geq \frac{1-2\delta}{2q_{n-1}} - \frac{1}{q_n} + \frac{1-2\delta}{2q_{n-1}}$$
$$\Updownarrow$$
$$\frac{1}{q_n} \geq \frac{1-2\delta}{q_{n-1}} - \frac{1}{2q_{n-1}}$$
$$\Updownarrow$$
$$\frac{1}{q_n} \geq \frac{1-4\delta}{2q_{n-1}}$$
$$\Updownarrow$$
$$\frac{2}{1-4\delta} \geq \frac{q_n}{q_{n-1}}$$

and since δ can be taken small enough, the above contradicts the hypothesis of divisibility os the continuants q_n, which implies $\frac{q_n}{q_{n-1}} > \lambda > 2$.

□

Now remember that Γ is an invariant set and B is a subset of Γ, so $F^{k_0}(B) \subset \Gamma$. This together with the above proposition implies that $\Gamma \cap \Gamma^c \neq \emptyset$, which is an absurd.

The absurd cames from suppose that $\mu(\Gamma^c) > 0$, therefore $\mu(\Gamma^c) = 0$. In other words, any invariant set intersecting a fiber with positive measure saturates the entire fiber.

Proposition 4.6. *Given Γ contained in a fiber, let say $\Gamma \subset \mathbb{T} \times \{0\}$, such that $\mu(\Gamma) = 1$, then there is a $\tilde{\Gamma} \subset \Gamma$, still of positive measure and such that $|\phi(x)| = 2$ for all x in $\tilde{\Gamma}$*

Proof. In fact, in order that $|\phi(x)| = 2$ is enough that

1. $q_j(x + \alpha)$ and $q_j x$ belong to the same plateau for all $j \geq 2$ and

2. $q_1(x + \alpha)$ and $q_1 x$ belong to distinct plateaus

The measure of the set of points of the circle that satisfy (1) is grater than $1 - \sum_{j\geq 2} ||q_j\alpha||_{\mathbb{Z}}$, while the set in (2) has measure $||q_1\alpha||_{\mathbb{Z}}$.

By the choice of the sequence q_n, $||q_1\alpha||_{\mathbb{Z}} > \sum_{j\geq 2} ||q_j\alpha||\mathbb{Z}$. So we have that the sum of the measures of the two sets above is grater than 1, concluding that they must intersect in set of positive measure.

Take $\hat{\Gamma}$ the set of points of Γ that satisfies (1) and (2). And as much as $\mu(\Gamma) = 1$, this set also has positive measure.

□

The proposition above implies that an invariant set of hole measure in a fiber, jump over fibers with positive measure. And once an invariant set is of positive measure in a fiber, we can saturate such fiber as explained before. Concluding that the measure of the complement of any invariant set is null in any fiber. Recalling that the measure μ in $\mathbb{T} \times 2\mathbb{Z}$ is defined by the sum of the Lebesgue measure in each fiber, we get $\mu(\Gamma^c) = 0$.

Observe that the way we prove ergodicity in particular implies recurrence since taking $x = y$ the orbit will return near the initial point.

This concludes the proof of Theorem A.

5 Counting Procedure

Our aim in this section is counting the return times

$$S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,z) = \sum_{k=0}^{q_{n+1}} \mathbb{1}_A \circ F_n^k(x,z) \tag{5.1}$$

to the skew product map F_n defined in (2.4) and A being the fiber $\mathbb{T} \times \{0\}$.

An integer k is a return time for a point (x,z) to the set A when $F_n^k(x,z) \in A$ which holds if and only if $\phi_n^k(x) = 0$ (bring up definitions in (2.5)). Therefore, to estimate this number of returns, we should determine the number of integers $k < q_{n+1}$ such that $\phi_n^k(x) = 0$. But observe that

$$\begin{aligned}\phi_n^k(x) = 0 &\Leftrightarrow \sum_{j=1}^{n} T(q_j x + k q_j \alpha_{n+1}) - T(q_j x) = 0 \\ &\Leftrightarrow \sum_{j=1}^{n} T(q_j x + k q_j \alpha_{n+1}) = m_n(x) = m.\end{aligned}$$

So, fixed a point x and consequently an $m = m_n(x)$ (see (2.6)), we are going to count the number of integers $k < q_{n+1}$ such that $\sum_{j=1}^n T(q_j x + k q_j \alpha_{n+1}) = m$.

Now consider sequences $\overrightarrow{t} = (t_1, \ldots, t_n) \in \{-1,1\}^n$, such that $\sum_{j=1}^n t_j = m$ and for each $\overrightarrow{t}$ take the set of integers

$$K_{\overrightarrow{t}} := \{k < q_{n+1} : T(q_j x + k q_j \alpha_{n+1}) = t_j \ \forall j = 1, \ldots, n\}$$

Consequently, the number (5.1) we are looking for is given by $\sum_{\overrightarrow{t}} \# K_{\overrightarrow{t}}$. But if for some reason, $\# K_{\overrightarrow{t}}$ does not depend on the sequence $\overrightarrow{t}$, then

$$S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,z) = \# K_{\overrightarrow{t}_0} \cdot \binom{n}{\frac{n+m}{2}}$$

that is, the cardinal of the sets $K_{\overrightarrow{t}}$ times the number of sequences of n elements whose sum of the terms is m, which is represented here by the binomial $\binom{n}{\frac{n+m}{2}}$.

Our purpose now is show the following

Lemma 5.1. *$\# K_{\overrightarrow{t}}$ is independent on the sequence $\overrightarrow{t}$, moreover $\# K_{\overrightarrow{t}} = \frac{q_{n+1}}{2^n}$.*

The rest of this section is devoted to prove the lemma above.

To deal with the formal counting procedure, to each k, let us associate the sequence $(s_n, \cdots, s_1) \in \{-1, 1\}^n$ in the following way

$$s_j = T(q_j x + k q_j \alpha_{n+1}); \quad j = n, \cdots, 1.$$

Thai allow us to consider the following partition of the interval of integers $[q_{n+1}] := \{0, 1, \cdots, q_{n+1} - 1\}$, which can be write as the disjoint union

$$[q_{n+1}] = \biguplus_{s_i \in \{-1,1\}} A_{(s_n, \cdots, s_j)}$$

where $A_{(s_n,\cdots,s_j)} := \{k < q_{n+1} : T_n(x + k\alpha_{n+1}) = s_n, \ldots, T_j(x + k\alpha_{n+1}) = s_j\}$. Note that by definition we have the properties

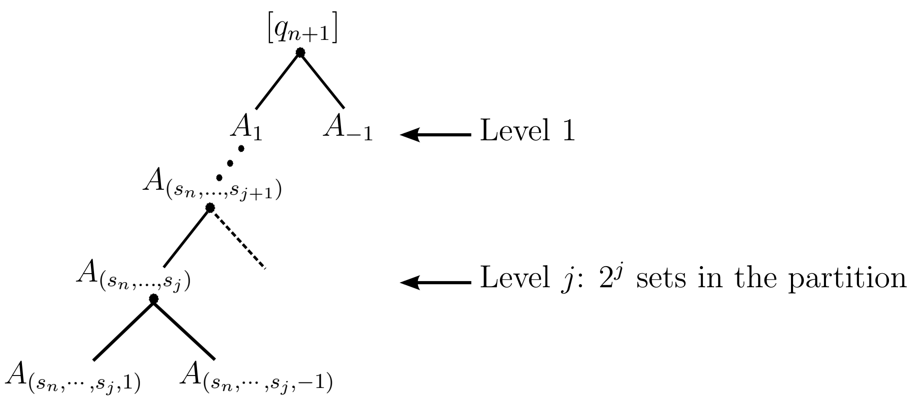
$$A_{(s_n,\cdots,s_2,s_1)} \subset A_{(s_n,\cdots,s_2)} \subset \cdots \subset A_{(s_n,s_{n-1})} \subset A_{(s_n)}$$

and

$$A_{(s_n,\cdots,s_j)} = A_{(s_n,\cdots,s_j,1)} \biguplus A_{(s_n,\cdots,s_j,-1)}.$$

This last property induces a tree associated to the sets $A_{(s_n,\cdots,s_j)}$ as represented in the picture below

(Fig)



Our objective now is show that it is a binary tree, i.e., in each level the quantity of elements in each set $A_{(s_n,\cdots,s_j)}$ is independent of the sequence $(s_n, \cdots, s_j)$. In other words, we want to see that $\#A_{(s_n,\cdots,s_{j+1},s_j)} = \frac{\#A_{(s_n,\cdots,s_{j+1})}}{2}$, obtaining

$$\#A_{(s_n,\cdots,s_j)} = \frac{\#[q_{n+1}]}{2^{n-j}}$$

(note: 2^{n-j} is the number of distinct sequences in the level j).

But the above is equivalent to show

$$\begin{aligned}&\#\{k < q_{n+1} : T(q_{j+1}x + kq_{j+1}\alpha_{n+1}) = s_{j+1} \text{ and } T(q_jx + kq_j\alpha_{n+1}) = s_j\} = \\ &= \frac{1}{2}\#\{k < q_{n+1} : T(q_{j+1}x + kq_{j+1}\alpha_{n+1}) = s_{j+1}\}\end{aligned}$$

Rewriting $q_jx + kq_j\alpha_{n+1}$ as $x_j + k\frac{p_{n+1}}{\frac{q_{n+1}}{q_j}}$, where $x_j = q_jx$, the above yet is equivalent to show that

$$\begin{aligned}&\#\{k < q_{n+1} : x_{j+1} + k\frac{p_{n+1}}{\frac{q_{n+1}}{q_{j+1}}} \in L_{s_{j+1}} \text{ and } x_j + k\frac{p_{n+1}}{\frac{q_{n+1}}{q_j}} \in L_{s_j}\} = \\ &= \frac{1}{2}\{k < q_{n+1} : x_{j+1} + k\frac{p_{n+1}}{\frac{q_{n+1}}{q_{j+1}}} \in L_{s_{j+1}}\}.\end{aligned}$$

Proposition 5.2. *If r is the rest of the quotient from kp_{n+1} by $\frac{q_{n+1}}{q_{j+1}}$, then*

$$T(x_j + k\frac{p_{n+1}}{\frac{q_{n+1}}{q_j}}) = T(x_j + rx_{j+1} + k\frac{1}{\frac{q_{n+1}}{q_{j+1}}}).$$

Proof. It is enough to write

$$\begin{aligned}x_j + k\frac{p_{n+1}}{\frac{q_{n+1}}{q_j}} &= x_j + (L\frac{q_{n+1}}{q_j} + r)\frac{1}{\frac{q_{n+1}}{q_j}} \\ &= x_j + L + r\frac{1}{\frac{q_{n+1}}{q_j}}\end{aligned}$$

where L belongs to $\mathbb{Z}$ and since T is $\mathbb{Z}$ periodic, we get the equality above. $\square$

Using that, our goal becomes to demonstrate that

$$\begin{aligned}&\#\{r < q_{n+1} : x_{j+1} + r\frac{1}{\frac{q_{n+1}}{q_{j+1}}} \in L_{sj+1} \text{ and } x_j + r\frac{1}{\frac{q_{n+1}}{q_j}} \in L_{s_j}\} = \\ &= \frac{1}{2}\#\{r < q_{n+1} : x_{j+1} + r\frac{1}{\frac{q_{n+1}}{q_{j+1}}} \in L_{s_{j+1}}\}\end{aligned} \tag{5.2}$$

Writing $kp_{n+1} = L\frac{q_{n+1}}{q_j} + r$ with $0 < k < q_{n+1}$, the function $k \mapsto kp_{n+1} \pmod{\frac{q_{n+1}}{q_j}}$ has q_j pre-images for each rest r in its codomain. Thus the above counting is the same in each interval os integers of length $\frac{q_{n+1}}{q_j}$.

Which reduce our purpose to show

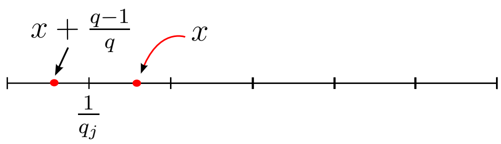
$$\#\{r < \frac{q_{n+1}}{q_j} : x_{j+1} + r\frac{1}{\frac{q_{n+1}}{q_{j+1}}} \in L_{s_{j+1}} \text{and } x_j + r\frac{1}{\frac{q_{n+1}}{q_j}} \in L_{s_j}\} =$$
$$= \frac{1}{2}\#\{r < q_{n+1} : x_{j+1} + r\frac{1}{\frac{q_{n+1}}{q_{j+1}}} \in L_{s_{j+1}}\}$$

Note that in the above equation, if we multiply the both sides by q_j, we get back the equality (5.2).

Proposition 5.3. *If $2 \mid q$ then the sequence $\{x + r\frac{1}{q}\}_{r<q}$ is such that*

$$\#\{r < q : x + r\frac{1}{q} \in I_1\} = \#\{r < q : x + r\frac{1}{q} \in L_{-1}\} = \frac{q}{2}$$

Proof. First of all note that's enough to demonstrate it for $0 \leq x < \frac{1}{q}$, because (Fig)



If $x \in [\frac{1}{q}, \frac{2}{q})$, take $\tilde{x} = ||x + \frac{q-1}{q}||_{\mathbb{Z}} < \frac{1}{q}$, then $\#\{r < q : \tilde{x} + r\frac{1}{q} \in I_i\} = \#\{r < q : x + r\frac{1}{q} \in I_i\}$, $\quad i = 1, -1$ and analogous for an $x \in [l\frac{1}{q}, \frac{l+1}{q})$. This means that "does not matter the displacement".

Since it is already true for $x = 0$, take an x such that $0 < x < \frac{1}{q}$. Then

$$\#\{r < q : x + r\frac{1}{q} \in I_1\} = \frac{q}{2}$$

holds because for $r \in \{0, 1, \cdots, \frac{q}{2} - 1\}$, we have

$$\begin{aligned} x + r\frac{1}{q} &< x + (\frac{q}{2} - 1)\frac{1}{q} \\ &= x + \frac{1}{2} - \frac{1}{q} \\ &< \frac{1}{2} \end{aligned}$$

then $x + r\frac{1}{q} \in I_1$.

And for $r \in \{\frac{q}{2}, \frac{q}{2} + 1, \cdots, q - 1\}$, we have

$$\begin{aligned} x + r\frac{1}{q} &> x + \frac{q}{2}\frac{1}{q} \\ &= x + \frac{1}{2} \\ &> \frac{1}{2} \end{aligned}$$

then $x + r\frac{1}{q} \in L_{-1}$.

$\square$

And again we have reduced our purpose to show that

Proposition 5.4.

$$\begin{aligned} \#\{r < \frac{q_{n+1}}{q_j} : r\frac{1}{\frac{q_{n+1}}{q_{j+1}}} \in L_{s_{j+1}} \ and\ r\frac{1}{\frac{q_{n+1}}{q_j}} \in L_{s_j}\} = \\ = \frac{1}{2}\#\{r < \frac{q_{n+1}}{q_j} : r\frac{1}{\frac{q_{n+1}}{q_{j+1}}} \in L_{s_{j+1}}\} \end{aligned}$$

Proof. To simplify the notation, let us denote $q_1 := \frac{q_{n+1}}{q_{j+1}}$ and $q_2 := \frac{q_{n+1}}{q_j}$. So $q_2 > q_1$ and by the choice of the sequence q_j, $2q_1 \mid q_2$.

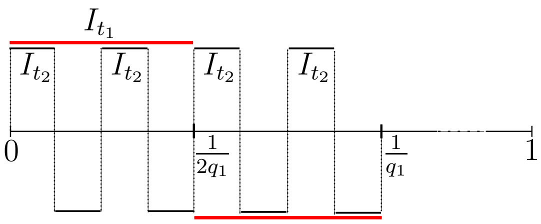
Thus we would like to prove that

$$\begin{aligned} \#\{r < q_2 : r\frac{1}{q_1} \in L_{t_1} \text{ and } r\frac{1}{q_2} \in L_{t_2}\} \\ \frac{1}{2}\#\{r < q_2 : r\frac{1}{q_1} \in L_{t_1}\} \end{aligned}$$

Since we know that $2q_1 \mid q_2$, let's say $q_2 = 2Lq_1$, then

$$\#\{r < q_2 : r\frac{1}{q_1} \in L_{t_1} \text{ and } r\frac{1}{q_2} \in L_{t_2}\} = q_1 L = \frac{q_2}{2} \tag{5.3}$$

(Fig)



This is true because after all the reductions that we made, the intervals $[\frac{r}{q_2}, \frac{r+1}{q_2})$ are embedded in $[0, \frac{1}{2q_1})$ or $[\frac{1}{2q_1}, 1)$ and $\#\{r < q_2 : \frac{r}{q_2} \in L_{t_2}\} = L$ for each L_{t_1}. But since there are q_1 intervals of the type L_{t_1} to complete the circle, the equality in (5.3) holds.

Now let's count $\#\{r < q_2 : r\frac{1}{q_1} \in L_{t_1}\}$. We have an r for each interval L_{t_1} and there are q_1 intervals of the type L_{t_1} in the circle. Then, if r goes until q_2 we are going through $\frac{q_2}{q_1}$ rounds in the circle. Accordingly, $\#\{r < q_2 : r\frac{1}{q_1} \in L_{t_1}\} = q_1 \frac{q_2}{q_1} = q_2$

Therefore,

$$\#\{r < q_2 : r\frac{1}{q_1} \in L_{t_1} \text{ and } r\frac{1}{q_2} \in L_{t_2}\}$$
$$\frac{1}{2}\#\{r < q_2 : r\frac{1}{q_1} \in L_{t_1}\}$$

$\square$

This concludes the binary character of the tree presented before.

In particular, for an arbitrary fixed sequence $(s_n, \cdots, s_1)$, we have got

$$\#A_{(s_n,\ldots,s_1)} = \#K_{\vec{s}} = \frac{\#[q_{n+1}]}{2^n}$$

and the counting of the Birkhoff sum then is

$$\begin{aligned}
S^{F_n}_{q_{n+1}}(x,z) &= \#\{k < q_{n+1} : \sum_{j=1}^{n} T(q_j x + k q_j \alpha_{n+1}) = m_n(x)\} \\
&= \#K_{\vec{s}} \cdot \#\{(s_n, \cdots, s_1) \in \{-1,1\}^n : \sum_{j=1}^{n} s_j = m = m_n(x)\} \\
&= \frac{q_{n+1}}{2^n} \binom{n}{\frac{n+m}{2}}
\end{aligned}$$

Remark 5.5. The counting process done before holds for almost everywhere in $\mathbb{T}$, since we are excluding the points x such that $q_j x + k q_j \alpha_{n+1} = \{0, \frac{1}{2}\} (\text{mod } 1)$ for $j = 1, \cdots, n$ and for $k = 0, \cdots, q_{n+1} - 1$. But outside this set of zero measure, we can calculate $S^{F_n}_{q_n+1}(x, z)$.

5.1 Pushing counting process to F

Let

$$\phi_n(x) := \sum_{j=1}^{n} T(q_j x + q_j \alpha_{n+1}) - T(q_j x)$$

$$\tilde{\phi}_n(x) := \sum_{j=1}^{n} T(q_j x + q_j \alpha) - T(q_j x)$$

$$\phi(x) := \sum_{j=1}^{\infty} T(q_j x + q_j \alpha_{n+1}) - T(q_j x)$$

The purpose is to compare the Birkhoff sums in each case.

Proposition 5.6. *Out of a set of measure $2q_{n+1}||q_{n+1}\alpha||_{\mathbb{Z}}$, we have the equality of the following Birkhoff sums*

$$S^{R_\alpha}_k(\phi)(x) = S^{R_\alpha}_k(\tilde{\phi}_n)(x) \quad \forall k = 1, \cdots, q_{n+1}$$

Proof. In fact,

$$S_k(\phi, R_\alpha)(x) = S_k(\tilde{\phi}_n, R_\alpha)(x) \Leftrightarrow \sum_{j=n+1}^{\infty} T(q_j x + k q_j \alpha) - T(q_j x) = 0$$

and for this is enough that $q_jx + kq_j\alpha$ and q_jx belong to the same plateau for all $k = 1, \cdots, q_{n+1}$, i.e, $d(q_jx, C) > ||kq_j\alpha||_{\mathbb{Z}} \forall j \geq n+1$.

But attending that the sets where this does not occur are embedded, it's enough to take out just the bigger of them., i.e, $k = q_{n+1}$.

$$\begin{gathered} d(q_jx, C) > ||q_{n+1}q_j\alpha||_{\mathbb{Z}} \forall j \geq n+1 \\ \Downarrow \\ d(q_jx, C) > ||kq_j\alpha||_{\mathbb{Z}} \forall j \geq n+1 \text{ and } \forall k = 1, \cdots, q_{n+1} \end{gathered}$$

since for $j \geq n+1$, $||q_{n+1}q_j\alpha||_{\mathbb{Z}} = q_{n+1}||q_j\alpha||_{\mathbb{Z}}$

and for $k < q_j$, $q_{n+1}||q_j\alpha||_{\mathbb{Z}} > k||q_j\alpha||_{\mathbb{Z}} = ||kq_j\alpha||_{\mathbb{Z}}$.

Thus $\sum_{j=n+1}^{\infty} T(q_jx + kq_j\alpha) - T(q_jx) = 0$ out of a set of measure $\sum_{j=n+1}^{\infty} q_{n+1}||q_j\alpha||_{\mathbb{Z}}$ which is smaller than $2q_{n+1}||q_{n+1}\alpha||_{\mathbb{Z}}$. □

Proposition 5.7. *Out of a set of measure $(q_{n+1})^{1+\epsilon}||q_{n+1}\alpha||_{\mathbb{Z}}$, we have the equality of the following Birkhoff sums*

$$S_k^{R_\alpha}(\tilde{\phi}_n)(x) = S_k^{R_{\alpha_{n+1}}}(\phi_n)(x) = \quad \forall k = 1, \cdots, q_{n+1}$$

Proof. In fact,

$$S_k^{R_\alpha}(\tilde{\phi}_n)(x) - S_k^{R_{\alpha_{n+1}}}(\phi_n)(x) = \sum_{j=1}^{n} T(q_jx + kq_j\alpha) - T(q_jx + kq_j\alpha_{n+1})$$

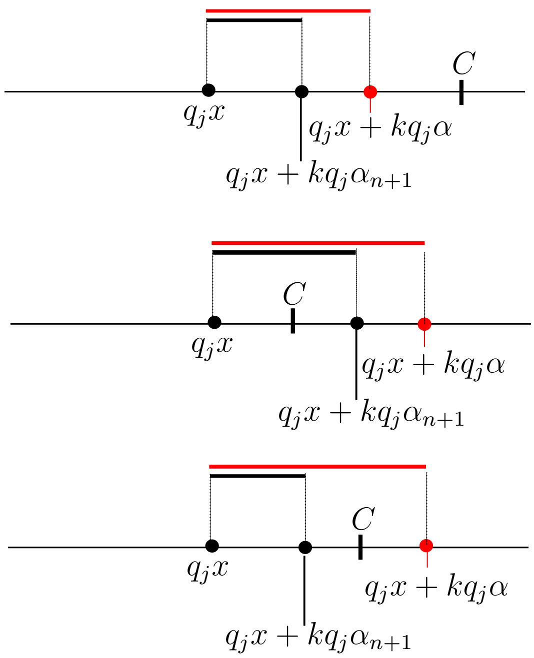
but this sum is null if $q_jx + kq_j\alpha$ and $q_jx + kq_j\alpha_{n+1}$ are in the same plateau for all $j = 1, \cdots, n$. And for each j the measure of the set such that they are not in the same plateau is $|kq_j\alpha - kq_j\alpha_{n+1}| = kq_j|\alpha - \alpha_{n+1}| = kq_j\frac{||q_{n+1}\alpha||_{\mathbb{Z}}}{q_{n+1}}$.

(Fig)

Therefore the equality of that Birkhoff sums holds for all $k = 1, cdots, q_{n+1}$ out of a set of measure

$$\begin{aligned} \sum_{j=1}^{n}\sum_{k=1}^{q_{n+1}} kq_j\frac{||q_{n+1}\alpha||_{\mathbb{Z}}}{q_{n+1}} &= \left(\sum_{j=1}^{n} q_j\right)\left(\sum_{k=1}^{q_{n+1}} k\right)\frac{||q_{n+1}\alpha||_{\mathbb{Z}}}{q_{n+1}} \quad \approx \left(\sum_{j=1}^{n} q_j\right) q_{n+1}||q_{n+1}\alpha||_{\mathbb{Z}} \\ &\leq q_{n+1}^{1+\epsilon}||q_{n+1}\alpha||_{\mathbb{Z}} \end{aligned}$$

□



Putting together the two propositions above, we get the following relation among the Birkhoff sums

$$S_k^{R_\alpha}(\phi)(x) = S_k^{R_{\alpha_{n+1}}}(\phi_n)(x) \quad \forall k = 1, \cdots, q_{n+1}$$

out of a set of measure

$$\begin{aligned} 2q_{n+1}||q_{n+1}\alpha||_{\mathbb{Z}} + q_{n+1}^{1+\epsilon}||q_{n+1}\alpha||_{\mathbb{Z}} &\leq \frac{2}{q_{n+1}^{2+2\epsilon}} + \frac{1}{q_{n+1}^{2+\epsilon}} \\ &\leq \frac{2}{q_{n+1}^{2+\epsilon}} \quad \text{for a large enough } n. \end{aligned}$$

6 Renyi inequality

In this section we will establish the Renyi inequality

$$\int_A \left(S^{F_n}_{q_{n+1}}\right)^2 d\mu \leq M \left(\int_A S^{F_n}_{q_{n+1}} d\mu\right)^2 \tag{6.1}$$

for the map

$$\begin{array}{rccc} F_n: & \mathbb{T}\times\mathbb{Z} & \longrightarrow & \mathbb{T}\times\mathbb{Z} \\ & (x,z) & \mapsto & (x+\alpha_{n+1}, z+\phi_n(x)) \end{array}$$

in the sequence of times q_{n+1}. Where M is a real number that does not depends on n.

For us the set A will be take as a fiber $\mathbb{T}\times\{z\}$, for simplicity, take the fiber $\mathbb{T}\times\{0\}$.

First of all, note that in a fixed fiber the set of points x that has the same $m_n(x) = \sum_{j=1}^n T(q_j x)$ has measure

$$Leb_{\mathbb{T}}(\{x\in\mathbb{T} : m_n(x) = m\} = \frac{1}{2}\binom{n}{\frac{n+m}{2}}$$

since $x \mapsto \sum_{j=1}^n T(q_j x) = m_n(x)$ is a random walk with $\mathbb{P}(T(q_j x) = 1) = \mathbb{P}(T(q_j x) = -1) = \frac{1}{2}$, m is in the range $[-n, n]$ and $m \equiv n(\bmod 2)$. Thus it determines a Gaussian distribution, and in a set of measure $\frac{1}{2^n}\binom{n}{\frac{n+m}{2}}$, with $|m| \leq n$ and $m \equiv n(\bmod 2)$, the Birkhoff sum is given by

$$S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,z) = \frac{q_{n+1}}{2^n}\binom{n}{\frac{n+m}{2}}$$

6.1 The return sequences

Once the calculus of the return times $S^{F_n}_{q_{n+1}}(A)(x,y)$ has been done in the previous section, we can obtain the return sequences $a_{q_{n+1}}$ for the skew product F_n.

$$
\begin{aligned}
a_{q_{n+1}} = a_{q_{n+1}}(A) = &= \int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,y)d\mu \\
&= \sum_{m=-n, m\equiv n(2)}^{n} \frac{1}{2^n}\binom{n}{\frac{n+m}{2}}\frac{q_{n+1}}{2^n}\binom{n}{\frac{n+m}{2}} \\
&= \frac{q_{n+1}}{2^{2n}}\sum_{i=0}^{n}\binom{n}{i}^2 \\
&= \frac{q_{n+1}}{2^{2n}}\binom{2n}{n} \\
&\simeq \frac{q_{n+1}}{2^{2n}}\cdot\frac{2^{2n}}{\sqrt{\pi n}} = \frac{q_{n+1}}{\sqrt{\pi n}}
\end{aligned}
$$

6.2 Renyi inequality for F_n

$$
\left(\int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,y)d\mu\right)^2 = \frac{q_{n+1}^2}{2^{4n}}\binom{2n}{n}^2
$$

$$
\begin{aligned}
\int_A \left(S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,y)\right)^2 d\mu &= \sum_{m=-n, m\equiv n(2)}^{n} \frac{1}{2^n}\binom{n}{\frac{n+m}{2}}\left(\frac{q_{n+1}}{2^n}\binom{n}{\frac{n+m}{2}}\right)^2 \\
&= \frac{q_{n+1}^2}{2^{3n}}\sum_{i=0}^{n}\binom{n}{i}^3
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\int_A \left(S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,y)\right)^2 d\mu}{\left(\int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,y)d\mu\right)^2} &= \frac{\frac{q_{n+1}^2}{2^{3n}}\sum_{i=0}^{n}\binom{n}{i}^3}{\frac{q_{n+1}^2}{2^{4n}}\binom{2n}{n}^2} \\
&= \frac{2^n\sum_{i=0}^{n}\binom{n}{i}^3}{\binom{2n}{n}^2}
\end{aligned}
$$

Now remembering that the estimate of the central binomial term is $\binom{n}{\frac{n}{2}} \approx \frac{2^n}{\sqrt{\pi\frac{n}{2}}}$ and rewriting

$$
\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i}^3 &= \sum_{i=0}^{n} \binom{n}{i}^2 \binom{n}{i} \\
&\leq \sum_{i=0}^{n} \binom{n}{i}^2 \binom{n}{\frac{n}{2}} \\
&= \binom{2n}{n} \binom{n}{\frac{n}{2}} \\
&\approx \frac{2^{2n}}{\sqrt{\pi n}} \frac{2^n}{\sqrt{\pi \frac{n}{2}}} \\
&= \frac{2^{3n}\sqrt{2}}{\pi n}
\end{aligned}
$$

Therefore

$$
\frac{\int_A \left(S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,y)\right)^2 d\mu}{\left(\int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A)(x,y) d\mu\right)^2} \lesssim \frac{2^n \frac{2^{3n}\sqrt{2}}{\pi n}}{\frac{2^{4n}}{\pi n}} = \sqrt{2}
$$

The above let us to conclude that the Renyi inequality is valid with constant $M = \sqrt{2}$ for each one of the functions F_n.

6.3 Pushing the Renyi Inequality to F

In this section we will use the Renyi inequality for F_n obtained above and the relations found in propositions 5.6 and 5.7 to compare the integrals of the Birkhoff sums of F_n and F. Then we will use that to get the Renyi inequality to the skew product F.

$$
\begin{aligned}
\left| \int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A) d\mu - \int_A S^{F}_{q_{n+1}}(\mathbb{1}_A) d\mu \right| &\leq q_{n+1} \cdot (2q_{n+1} + q_{n+1}^{1+\epsilon})||q_{n+1}\alpha||_{\mathbb{Z}} \\
&= (2q_{n+1}^2 + q_{n+1}^{2+\epsilon})||q_{n+1}\alpha||_{\mathbb{Z}} \\
&\xrightarrow{n\to\infty} 0
\end{aligned}
$$

And as each one of the integrals above diverge to infinity when n goes to infinity, for a large n is true that

$$\frac{1}{2}\int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A)d\mu \leq \int_A S^{F}_{q_{n+1}}(\mathbb{1}_A)d\mu \leq 2\int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A)d\mu$$

and analogously

$$\frac{1}{2}\int_A \left(S^{F_n}_{q_{n+1}}(\mathbb{1}_A)\right)^2 d\mu \leq \int_A \left(S^{F}_{q_{n+1}}(\mathbb{1}_A)\right)^2 d\mu \leq 2\int_A \left(S^{F_n}_{q_{n+1}}(\mathbb{1}_A)\right)^2 d\mu$$

since we also have that

$$\begin{aligned}\left|\int_A \left(S^{F}_{q_{n+1}}(\mathbb{1}_A)\right)^2 d\mu - \int_A \left(S^{F}_{q_{n+1}}(\mathbb{1}_A)\right)^2 d\mu\right| &\leq q_{n+1}^2 \cdot (2q_{n+1} + q_{n+1}^{1+\epsilon})||q_{n+1}\alpha||_{\mathbb{Z}} \\ &= (2q_{n+1}^3 + q_{n+1}^{3+\epsilon})||q_{n+1}\alpha||_{\mathbb{Z}} \\ &\xrightarrow{n\to\infty} 0\end{aligned}$$

Now as we already know that the Renyi inequality is true for F_n, it follows that

$$\begin{aligned}\int_A \left(S^{F}_{q_{n+1}}(\mathbb{1}_A)\right)^2 d\mu &\leq 2\int_A \left(S^{F_n}_{q_{n+1}}(\mathbb{1}_A)\right)^2 d\mu \\ &\leq 2\sqrt{2}\left(\int_A S^{F_n}_{q_{n+1}}(\mathbb{1}_A)d\mu\right)^2 \\ &\leq 2\sqrt{2}\left(2\int_A S^{F}_{q_{n+1}}(\mathbb{1}_A)d\mu\right)^2 \\ &\leq 8\sqrt{2}\left(\int_A S^{F}_{q_{n+1}}(\mathbb{1}_A)d\mu\right)^2\end{aligned}$$

Which is the Renyi inequality for F, proving the rationally ergodicity of the skew products F and completing the remainder of the theorem B.

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