INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA

ON NON-TYPE (D) OPERATORS IN NON-REFLEXIVE BANACH SPACES AND CLOSURES OF MONOTONE OPERATORS IN TOPOLOGICAL VECTOR SPACES

OPERADORES QUE NÃO SÃO DO TIPO (D) EM ESPAÇOS DE BANACH NÃO REFLEXIVOS E FECHOS DE OPERADORES MONÓTONOS EM ESPAÇOS VETORIAIS TOPOLÓGICOS

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En memoria de mi padre

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Resumo

Nesta tese, estudamos certas propriedades dos operadores monótonos em dois contextos diferentes. No primeiro contexto, mostramos sob que condições podemos construir operadores lineares que não são de tipo (D) em espaços de Banach não reflexivos. Além disso, damos resposta negativa a duas conjeturas dadas por Marques-Alves e Svaiter, e Borwein, respectivamente. No segundo contexto, estudamos a relação existente entre o fecho monótono polar e o fecho representável de um operador monótono num espaço topológico localmente convexo e Hausdorff. Isto estende um resultado dado por Martínez-Legaz e Svaiter.

Palavras chave: operadores monótonos de tipo (D), fecho polar monótono, fecho representável

Abstract

In this thesis, we study certain properties of monotone operators in two different contexts. In the first context, we show under which conditions we can construct linear monotone operators which are not of type (D). Moreover, we give negative answers to two conjectures due to Marques-Alves and Svaiter, and Borwein, respectively. In the second context, we study the relationship between the monotone polar closure and the representable closure of a monotone operator in a Hausdorff and locally convex topological vector space. This extends a result due to Martínez-Legaz and Svaiter.

Keywords: type (D) monotone operators, monotone polar closure, representable closure

Contents

Notations

X^*	dual space of X
X^{**}	bidual space of X
$\langle x, x^* \rangle$	$x^*(x)$
$\pi, \langle \cdot, \cdot \rangle$	duality product
$\xrightarrow{\|\cdot\|}$	strong convergence
$\xrightarrow{w}$	weak convergence
$\xrightarrow{w*}$	weak$*$ convergence
$\overline{S}^{\mathcal{O}}$	$\mathcal{O}$-closure, where $\mathcal{O}$ is a topology
$\overline{S}$	strong topology closure
$f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$	extended real valued function
$\mathrm{cl}(f)$	largest lsc function majorized by f
$\mathrm{conv}(f)$	largest convex function majorized by f
$\mathrm{cl}\,\mathrm{conv}(f)$	largest lsc convex function majorized by f
$\mathrm{ed}(f)$	efective domain of f
∂f	subdifferential of f
f^*	Fenchel conjugate of f
$T : X \rightrightarrows X^*$	multivalued operator
$\mathrm{dom}(T)$	domain of T
$\mathrm{ran}(T)$	range of T
$\mathrm{gra}(T)$	graph of T
φ_T	Fitzpatrick function of T
$f \leq g$	$f(x)$ is less or equal than $g(x)$, for every $x \in X$
$\{f \leq g\}$	set of points in which f is less or equal than g
$\{f = g\}$	set of points in which f is equal than g
δ_T	indicator function of $\mathrm{gra}(T)$
c_0	real sequences convergent to 0
c	convergent real sequences
ℓ^1	absolutely convergent real sequences
ℓ^∞	bounded sequences

Introduction

Monotone operators were defined and used in the early sixties as a theoretical framework for the study of electrical networks, and, later on, for the study of non-linear partial differential equations. The first works on monotone operators were due to Zarantonello [50], Minty [39], Kato [30], Browder [14], Rockafellar [41], Brézis [11], among others. Since then, monotone operators were object of intense study. See [10] for a survey on the subject.

One of the most important developments in the theory of monotone operators was the use of convex functions to represent them. This breakthrough was due to Fitzpatrick, which proved that any maximal monotone operator is representable by a convex function [22]. This work remained unnoticed until Martínez-Legaz and Therá, and Burachik and Svaiter independently rediscovered this result. Since then, the use of convex representations of maximal monotone operators was subject of intense research.

This thesis has two parts: the first one deals with non-type (D) monotone operators in non-reflexive Banach spaces and the second one is about closures of monotone operators in topological vector spaces. We provide negative answers to two conjectures which appeared in this research area, and extend one previous result to topological vector spaces.

Non-type (D) operators in Banach spaces

In a Banach space X, a maximal monotone operator $T : X \rightrightarrows X^*$ is of *type (D)* if every point $(x^{**}, x^*) \in X^{**} \times X^*$ such that

$$\langle x^{**} - y, x^* - y^* \rangle \geq 0, \qquad \forall\, (y, y^*) \in \operatorname{gra}(T),$$

is the $\sigma(X^{**}, X^*)\times$strong limit of a bounded net in $\text{gra}(T)$. This class, which was originally called "dense type", was introduced by Gossez in [26], to recover some very nice properties maximal monotone operators have in reflexive Banach spaces. Later on, many other classes with nice properties were defined, as for example, type (NI) operators, by Simons [45], and locally maximal monotone operators, by Fitzpatrick and Phelps [23]. Very recently, type (D) and type (NI) classes were proved to be equivalent by Marques-Alves and Svaiter [36], and later on, the complete equivalence of the three aforementioned classes was given by Borwein, et al. [6].

The most important examples of maximal monotone operators of type (D) are: maximal monotone operators in reflexive Banach spaces and subdifferentials of lower semi-continuous proper convex functions. An essential property of type (D) operators in non-reflexive Banach spaces is the uniqueness of its extension to the bidual. This property does not hold in general, as showed by Gossez in [28].

If $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex, proper and lower semi-continuous function, then its subdifferential $\partial f : X \rightrightarrows X^*$ is a type (D) operator whose unique maximal monotone extension to the bidual is $(\partial f^*)^{-1} : X^{**} \rightrightarrows X^*$, where $f^* : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the *Fenchel conjugate* of f. Thus, the inverse of such extension is also a subdifferential, explicitly, $\partial f^* : X^* \rightrightarrows X^{**}$, and hence of type (D). So, in the case of subdifferentials, the property of being of type (D) is "hereditary", that is, the inverse of their extensions to the bidual is also of type (D). This fact led Marques-Alves and Svaiter to formulate the following conjecture [31]:

> The inverse of the unique maximal monotone extension to the bidual of a type (D) maximal monotone operator is also of type (D).

This leads to the following problem:

Problem 1. Given a maximal monotone type (D) operator $T : X \rightrightarrows X^*$ and its unique extension to the bidual $\widetilde{T} : X^{**} \rightrightarrows X^*$: is $\widetilde{T}^{-1} : X^* \rightrightarrows X^{**}$ also of type (D)?

In chapter 2, we answer negatively Problem 1, by giving an explicit example of a type (D) operator for which the inverse of its unique extension to the bidual is not of type (D) (Proposition 2.9 in section 2.3). Moreover, we also give sufficient conditions for such an operator be constructed in general non-reflexive Banach spaces (Theorem 2.7 in section 2.3).

Examples of non-type (D) operators are scarce. The first example is due to Gossez, who defined an example of a non-type (D) operator on ℓ^1 [27]. This operator, which is called the Gossez operator, has a unique extension to the bidual despite being non-type (D). Not long after that, Gossez gave an example, also in ℓ^1, of a non-type (D) operator which has infinite extensions to the bidual [28]. Later, Fitzpatrick and Phelps gave an example of a non-type (D) operator on $L^1[0, 1]$ [24].

In [3], Borwein and Bauschke proved that if a monotone continuous linear operator in a Banach space has a monotone conjugate, then this operator is of type (D), and defined *conjugate monotone spaces* as those Banach spaces X such that the conjugate of any continuous monotone linear operator form X to X^* is monotone as well. Also in [3], it was observed that c_0, c, ℓ^∞ and $L^\infty[0, 1]$ are conjugate monotone spaces while ℓ^1, $L^1[0, 1]$, $(\ell^\infty)^*$ and $(L^\infty[0, 1])^*$ are not conjugate monotone spaces. These facts led Borwein to define *Banach spaces of type (D)* as those Banach spaces where every maximal monotone operator is of type (D), and to formulate the following conjecture [10, §4, question 3]:

- ... I conjecture 'weakly' that if X contains no copy of $\ell^1(\mathbb{N})$ then X is type (D) as would hold in $X = c_0$.

This leads to the following problem:

Problem 2. Can a non-type (D) operator be defined in c_0? What about in spaces containing a isometric copy of c_0?

In chapter 2, we present an example of a linear non-type (D) operator on c_0 (Proposition 2.11 in section 2.4). Moreover, if there exists a non-type (D) operator in a Banach space X, and X can be embedded *norm-isomorphically* into a *bigger* Banach space Ω, then there also exists a non-type (D) operator in Ω (section 2.4.1). In particular, a non-type (D) operator can be found in every space which contains a norm-isomorphic (in particular, isometric) copy of c_0 (Theorem 2.18).

Closures of monotone operators

The use of convex functions to represent monotone operators was due to Fitzpatrick, by defining the *Fitzpatrick function* of a maximal monotone operator. Fitzpatrick's work was not widely recognized until its rediscovery by Martínez-Legaz and Théra [38] and Burachik and Svaiter [20]. Following this, convex representations of maximal monotone operators were subject of intense study, and have many theoretical and practical applications [2, 25, 43, 46, 48, 51].

In 2005, Martínez-Legaz and Svaiter [37], extended the notion of representability to a broader class of monotone operators in Banach spaces, by defining *representable operators*. Moreover, they defined for a monotone operator two kinds of closures: its *representable closure*, which is the smallest (in sense of graph inclusion) representable operator which contains it; and its *monotone polar closure* which is the intersection of every maximal monotone operator containing it.

Theorem 31 in [37] states that these two closures coincide in finite dimensional spaces. Furthermore, an example in Hilbert spaces where these two closures are different was given in [47]. So we state the following problem.

Problem 3. In a topological vector space, under which conditions the two closures are identical (or different)?

In chapter 3, we study the relationship between these two closures, in the context of Hausdorff locally convex topological vector spaces. Our main result is Theorem 3.5, which states that, when the closures do not coincide, the monotone operator has a unique maximal monotone extension, and such extension has a very particular structure. In its proof, we must stand out Lemma 3.9, which guarantees such particular structure, under a condition on the effective domain of the Fitzpatrick function of the operator. We also give a new proof of Theorem 31 in [37] (Theorem 3.13) and corollaries concerning monotone operators with non-monotone affine hull (Corollary 3.14), and studying the case of Banach spaces (Corollary 3.15). Moreover, we give a proposition on the structure of monotone operators with unique maximal monotone extension (Proposition 3.18). Finally, we study the closures of an explicit operator defined in c_0 (section 3.4). For this, we use the operator defined in section 2.5.

Chapter 1

Basic results

Let X be a real Banach space with norm $\|\cdot\|$, and let X^* be its topological dual. For $x \in X$ and $x^* \in X^*$, we will use the notation $\langle x, x^* \rangle = x^*(x)$, and thus we have the *duality product* $\pi : X \times X^* \longrightarrow \mathbb{R}$, $\pi(x, x^*) = \langle x, x^* \rangle$.

Since X^* is also a Banach space, it has its own dual, which is is denoted by $X^{**} = (X^*)^*$ and called *bidual* of X. The *canonical injection* $J : X \longrightarrow X^{**}$ is defined, for $x \in X$, as the (continuous) linear functional

$$J(x) : \begin{array}{ccc} X^* & \longrightarrow & \mathbb{R} \\ x^* & \mapsto & \langle x, x^* \rangle \end{array} . \tag{1.1}$$

It follows from (1.1) that J is everywhere defined, linear, continuous and embeds X *isometrically* into X^{**}. If J is also surjective then X is called *reflexive*. Thus, we can identify X with its canonical image in the bidual space X^{**} and, for the sake of notation, to consider X as included in X^{**}. Clearly, this inclusion will be proper when X is not reflexive.

In X^* is defined the *weak$*$ topology*, $\sigma(X^*, X)$, as the smallest topology which makes the functionals $\{J(x)\}_{x\in X} \subset X^{**}$ continuous. In addition, the topology induced by the norm of X^* is called *strong* topology.

For $C \subset X$, denote

$$C^{\perp} = \{x^* \in X^* \mid \langle x, x^* \rangle = 0, \quad \forall\, x \in C\}.$$

Similarly, for $D \subset X^*$,

$$^{\perp}D = \{x \in X \mid \langle x, x^* \rangle = 0, \quad \forall\, x^* \in D\}.$$

From the above definitions, readily follows that $C^{\perp}$ and $^{\perp}D$ are linear subspaces and closed in the weak$*$ and strong topologies, respectively.

1.1 Monotone operators

Let U, V arbitrary sets. A *multivalued* (or point-to-set) operator $T : U \rightrightarrows V$ is a map $T : U \longrightarrow \mathcal{P}(V)$, where $\mathcal{P}(V)$ is the power set of V. Given $T : U \rightrightarrows V$, the *graph* of T is the set

$$\mathrm{gra}(T) := \{(u, v) \in U \times V \mid v \in T(u)\},$$

the *domain* and the *range* of T are, respectively,

$$\mathrm{dom}(T) := \{u \in U \mid T(u) \neq \emptyset\}, \qquad \mathrm{ran}(T) := \{v \in V \mid \exists\, u \in U,\ v \in T(u)\},$$

and the *inverse* of T is the multivalued operator $T^{-1} : V \rightrightarrows U$,

$$T^{-1}(v) = \{u \in U \mid v \in T(u)\}.$$

A multivalued operator $T : U \rightrightarrows V$ is called *single valued*, if $T(u)$ has only one element, for every $u \in \mathrm{dom}(T)$.

Remark 1.1. It is also customary [15, 32, 33, 35, 36, 37] to define the notion of *multivalued operator* not as a function but as a relation, which is equivalent to the former. We will use this modified definition in chapter 3, by identifying multivalued operators with their graph.

Let X be a real Banach space. An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall\, (x, x^*), (y, y^*) \in \mathrm{gra}(T).$$

and it is *maximal monotone* if it is monotone and maximal (with respect to the graph inclusion) in the family of monotone operators of X into X^*. In the same way, an operator $T : X^* \rightrightarrows X$ is *monotone* (resp. *maximal monotone*) if its inverse is monotone (resp. maximal monotone).

Remark 1.2. By using Zorn's Lemma, we can guarantee the existence of a maximal monotone extension for any monotone operator.

Definition 1.3 ([37, Definition 35]). A monotone operator is called *pre-maximal monotone*, if it has a unique maximal monotone extension.

We will say that $(x, x^*), (y, y^*) \in X \times X^*$ are *monotonically related*, if

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

We will also say that a point $(x, x^*) \in X \times X^*$ and an operator $T : X \rightrightarrows X^*$ are *monotonically related*, if (x, x^*) is monotonically related to every point in the graph of T. Furthermore, for an operator $T : X \rightrightarrows X^*$, its *monotone polar* or *μ-polar* [37] is the operator $T^\mu : X \longrightarrow X^*$, such that

$$\mathrm{gra}(T^\mu) := \{(x, x^*) \mid (x, x^*) \text{ is monotonically related to } T\}.$$

Completely analog definitions apply when working in $X^* \times X$.

The following propositions follow immediately from the above definitions.

Proposition 1.4. *Let $T : X \rightrightarrows X^*$ be a multivalued operator. Then the following conditions are equivalent:*

1. *T is monotone;*
2. *every point in $\mathrm{gra}(T)$ is monotonically related to T;*
3. *every two points in $\mathrm{gra}(T)$ are monotonically related;*
4. $\mathrm{gra}(T) \subset \mathrm{gra}(T^\mu)$.

Proposition 1.5. *Let $T : X \rightrightarrows X^*$ be a multivalued operator. Then the following holds:*

1. *T is maximal monotone if, and only if, $T = T^\mu$;*
2. *T^μ is maximal monotone if, and only if, T and T^μ are monotone.*

Proposition 1.6. *Let $T, S : X \rightrightarrows X^*$ be multivalued operators. If* $\mathrm{gra}(T) \subset \mathrm{gra}(S)$ *then* $\mathrm{gra}(S^\mu) \subset \mathrm{gra}(T^\mu)$.

1.2 Linear monotone operators

Definition 1.7. A multivalued operator $T : X \rightrightarrows X^*$ is *linear*, if $\mathrm{gra}(T)$ is a linear subspace of $X \times X^*$.

It follows from the above definition the following facts.

Proposition 1.8 (Cross [21, §I])**.** *Let $T : X \rightrightarrows X^*$ be a linear multivalued operator.*

1. *$T(0)$ is a linear subspace of X^*.*
2. *For every $(x, x^*) \in \mathrm{gra}(T)$, $T(x) = x^* + T(0)$.*
3. *T is single valued if, and only if, $T(0) = \{0\}$.*

Proposition 1.9. *A linear multivalued operator $T : X \rightrightarrows X^*$ is monotone if, and only if,*

$$\langle x, x^* \rangle \geq 0, \quad \forall\, (x, x^*) \in \operatorname{gra}(T).$$

In particular, the linear single valued operators are simply the *linear maps* studied in Linear Algebra.

The following results were presented by Bauschke *et al.* [8] in reflexive Banach spaces and then in general Banach spaces in [4].

Lemma 1.10 ([4, Proposition 5.1, item (i)])**.** *If $T : X \rightrightarrows X^*$ be a monotone linear operator then*

$$T(0) \subseteq \operatorname{dom}(T)^{\perp}. \tag{1.2}$$

Corollary 1.11. *Let $T : X \rightrightarrows X^*$ be a monotone linear operator. If $\operatorname{dom}(T)$ is dense in X then T is single valued.*

Moreover, under the assumption of maximal monotonicity, equality can be obtained in equation (1.2).

Proposition 1.12 ([4, Proposition 5.2, item (i)])**.** *Let $T : X \rightrightarrows X^*$ be a linear maximal monotone operator. Then*

$$T(0) = \operatorname{dom}(T)^{\perp}.$$

We will use the following lemma in section 2.1. For the sake of completeness, we include its proof here.

Lemma 1.13. *Let $T : X \rightrightarrows X^*$ be a monotone linear operator. If $\operatorname{dom}(T) = X$ then T is a maximal monotone single valued operator.*

Proof. Take (x, x^*) monotonically related to T. Since T is everywhere defined, there exists $w^* \in T(x)$. Using the linearity of T,

$$\langle (ty + x) - x, (ty^* + w^*) - x^* \rangle \geq 0, \qquad \forall\, (y, y^*) \in \operatorname{gra}(T),\ t \in \mathbb{R}.$$

This implies, for arbitrary $(y, y^*) \in \operatorname{gra}(T)$, that

$$t^2 \langle y, y^* \rangle + t \langle y, w^* - x^* \rangle \geq 0, \qquad \forall t \in \mathbb{R},$$

which in turn implies

$$\langle y, w^* - x^* \rangle \geq 0, \qquad \forall\, y \in \operatorname{dom}(T) = X.$$

Thus $\langle y, w^* - x^* \rangle = 0$, for all $y \in X$, that is $w^* = x^*$ and $(x, x^*) \in \operatorname{gra}(T)$. $\square$

1.3 The adjoint operator

For X and Y Banach spaces, we consider $X \times Y$ as a Banach space with norm $\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$, dual $(X \times Y)^* = X^* \times Y^*$ and duality product

$$\langle (x, y), (x^*, y^*)\rangle := \langle x, x^*\rangle + \langle y, y^*\rangle.$$

The definition of the adjoint of an operator is defined in several different contexts along the literature [1, 12, 44]. The following definition is a unified version of such definitions and it is based on [8] and [49].

Definition 1.14. Let X and Y be real Banach spaces and let $T : X \rightrightarrows Y$ be a multivalued operator. Define the multivalued operator $T^+ : Y^* \rightrightarrows X^*$ as

$$(y^*, x^*) \in \operatorname{gra}(T^+) \iff (x^*, y^*) \in \operatorname{gra}(T)^\perp. \tag{1.3}$$

Moreover, define the *adjoint operator* $T^* : Y^* \rightrightarrows X^*$ as $T^*(y^*) = -T^+(y^*)$.

Clearly T^+ and T^* are linear multivalued operators. Density of the domain of T gives us an important property.

Proposition 1.15. *Let $T : X \rightrightarrows Y$ be densely defined. Then $T^+, T^* : Y^* \rightrightarrows X^*$ are single valued.*

Proof. Is enough to proof that T^+ is single valued. Take $y^* \in \operatorname{dom}(T^+)$ and take $x^*, w^* \in T^+(y^*)$. Then, by equation (1.3),

$$\langle x, x^*\rangle + \langle y, y^*\rangle = 0,$$
$$\langle x, w^*\rangle + \langle y, y^*\rangle = 0,$$

for all $(x, y) \in \operatorname{gra}(T)$. Substracting these equations we obtain $\langle x, w^* - x^*\rangle = 0$, for all $x \in \operatorname{dom}(T)$. Density of $\operatorname{dom}(T)$ now implies that $w^* = x^*$ and, hence, T^* is single valued. □

Remark 1.16. Definition 1.14 generalizes several concepts, for instance:

1. When $T : X \longrightarrow Y$ is a bounded linear map, then T^* coincides with the *conjugate operator* $T' : Y^* \longrightarrow X^*$ defined in [1, §17.3] as $T'(y^*) = y^* \circ T$, which is also the adjoint operator defined in [44, §4].

2. When X, Y are Hilbert spaces and $T : X \rightrightarrows Y$ is single valued and densely defined, then T^* is the adjoint defined in [1, §20.1].

3. When $Y = X^*$ and $T : X \rightrightarrows X^*$ is a multivalued operator, then $T^+|_X : X \rightrightarrows X^*$ coincides with the $T^\vdash$ operator, defined in [49, Equation (11)].

4. Also when $Y = X^*$ and $T : \mathrm{dom}(T) \subset X \longrightarrow X^*$ is a densely defined linear map, $T^* : \mathrm{dom}(T^*) \subset X^{**} \longrightarrow X^*$ coincides with the adjoint defined in [40, Definition 4.1] or [12, §II.6].

5. If $Y = X^*$ and X is reflexive then, for $T : X \rightrightarrows X^*$, T^* is the adjoint defined in [8, §1].

Definition 1.17 ([49, Definition 2.1])**.** A linear operator $T : X \rightrightarrows X^*$ is called *self-canceling* if

$$\langle x, x^* \rangle = 0, \qquad \forall\, (y, y^*) \in \mathrm{gra}(T).$$

Clearly, every self-canceling operator is monotone. This definition is equivalent to the definition of anti-symmetric operator: a linear operator $T : X \rightrightarrows X^*$ is *anti-symmetric* if $\mathrm{gra}(T) \subset \mathrm{gra}(T^+)$. We will usually use the term "anti-symmetric" for linear single valued operators and "self-canceling" for linear multi-valued operators.

1.4 The Fitzpatrick function and enlargements of maximal monotone operators

Fitzpatrick, in [22], associated to a monotone operator $T : X \rightrightarrows X^*$, the function $\varphi_T : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$, defined as

$$\varphi_T(x, x^*) := \sup_{(y,y^*) \in \mathrm{gra}(T)} \langle y, x^* \rangle + \langle x, y^* \rangle - \langle y, y^* \rangle.$$

Direct manipulation of the last definition yields

$$\begin{aligned} \varphi_T(x, x^*) &= \sup_{(y,y^*) \in \mathrm{gra}(T)} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle, \\ &= \beta_T(x, x^*) + \langle x, x^* \rangle, \end{aligned}$$

where β_T is the *Brezis-Haraux function* [13]. Clearly, φ_T is a convex and lower semi-continuous function, and

$$\mathrm{gra}(T) \subset \{(x, x^*) \mid \varphi_T(x, x^*) = \langle x, x^* \rangle\},$$

so φ_T is proper.

When T is maximal monotone, then $\varphi_T(x, x^*) \geq \langle x, x^* \rangle$, for all $(x, x^*) \in X \times X^*$, and the last inclusion holds as an equality, that is

$$\mathrm{gra}(T) = \{(x, x^*) \mid \varphi_T(x, x^*) = \langle x, x^* \rangle\}.$$

Moreover, the Fitzpatrick function of a monotone operator T is related with its monotone polar. Indeed,

$$\operatorname{gra}(T^\mu) = \{(x, x^*) \mid \varphi_T(x, x^*) \leq \langle x, x^* \rangle\}.$$

Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. By Proposition 1.5, item *1*, $T = T^\mu$, that is,

$$\operatorname{gra}(T) = \{(x, x^*) \mid \langle x - y, x^* - y^* \rangle \geq 0, \forall\, (y, y^*) \in \operatorname{gra}(T)\}.$$

The ε-enlargement of T, defined by Burachik, Iusem and Svaiter in [19], is the operator $T^\varepsilon : X \rightrightarrows X^*$, defined via its graph as

$$\operatorname{gra}(T^\varepsilon) = \{(x, x^*) \mid \langle x - y, x^* - y^* \rangle \geq -\varepsilon, \forall\, (y, y^*) \in \operatorname{gra}(T)\},$$

From this definition, readily follows

$$\begin{gathered} \operatorname{gra}(T^\varepsilon) = \{(x, x^*) \mid \varphi_T(x, x^*) \leq \langle x, x^* \rangle + \varepsilon\}, \\ \operatorname{gra}(T) \subset \operatorname{gra}(T^\varepsilon), \qquad \forall\, \varepsilon \geq 0, \end{gathered}$$

and $T = T^0$.

We say that T is *non-enlargeable* if $T = T^\varepsilon$, for every $\varepsilon > 0$. Follows from this that T is non-enlargeable if, and only if,

$$T = \operatorname{ed}(\varphi_T),$$

where ed stands for *effective domain*. Since T is maximal monotone and φ_T is a convex function, using Lemma 1.2 in [34] (or Theorem 4.2 in [8]), T is an affine linear subspace in $X \times X^*$. This is an extension of a result given by Burachik and Iusem, for linear operators in finite dimensional spaces with certain conditions [18, Theorem 2.15].

Chapter 2

On non-type (D) operators

This chapter is concerned with the study of type (D) operators in non-reflexive Banach spaces.

In section 2.1 we give the preliminary definitions we will use along the chapter, in particular, the definition of a maximal monotone operator of type (D). In section 2.2, we study the *Gossez operator*, which is one of the most important examples of non-type (D) operators, and it is central in our study. Section 2.3 is devoted to the study of Problem 1. We first give conditions to construct a type (D) operator with non-type (D) extension and then we construct an explicit example of operator with this behavior. In section 2.4 we give a counterexample for problem 2. In addition, in section 2.4.1, we extend a monotone operator defined in a Banach space to a *larger* Banach space, and study the relationship between the original operator and the *embedded* operator, in particular, what happens with the maximal monotonicity and type (D) properties of the latter. We later use these results to proof that we can define non-type (D) operators in any Banach space which contains a norm-isomorphic copy of c_0. Finally, in section 2.5 we define an example of non-maximal monotone linear operator which will be further studied in section 3.4.

The results of this chapter were published in [16, 17].

2.1 Classes of monotone operators

Let X be a Banach space. Since $X \subseteq X^{**}$, any multivalued operator $T : X \rightrightarrows X^*$ induces an operator $\widehat{T} : X^{**} \rightrightarrows X^*$,

$$\widehat{T}(x^{**}) = \begin{cases} T(x^{**}), & \text{if } x^{**} \in X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If T is monotone then $\widehat{T}$ is also monotone. Moreover, in reflexive Banach spaces, $\widehat{T}$ is maximal monotone when T is, but, in non-reflexive Banach spaces, $\widehat{T}$ can fail to be maximal monotone even if T is.

Definition 2.1. Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then we say T is

1. *of dense type or type (D)* ([26]), if for every $(x^{**}, x^*) \in X^{**} \times X^*$ such that

$$\langle x^* - y^*, x^{**} - y \rangle \geq 0, \forall (y, y^*) \in \text{gra}(T),$$

   there exists a bounded net $(x_\alpha, x^*_\alpha)_\alpha$ in $\text{gra}(T)$ such that $x_\alpha \xrightarrow{w^*} x^{**}$ and $x^*_\alpha \xrightarrow{\|\cdot\|} x^*$;

2. *of negative infimum type or type (NI)* ([45]), if

$$\inf_{(y,y^*)\in T} \langle x^* - y^*, x^{**} - y \rangle \leq 0,$$

   for all $(x^*, x^{**}) \in X \times X^{**}$.

Note that in *reflexive* Banach spaces, all maximal monotone operators are of type (D). Gossez proved that a maximal monotone operator $T : X \rightrightarrows X^*$ of type (D) has a unique maximal monotone extension to the bidual [29], namely, $(\widehat{T})^\mu : X^{**} \rightrightarrows X^*$, the monotone polar of $\widehat{T}$. Furthermore, maximal monotone operators of type (D) share many properties with maximal monotone operators defined in reflexive Banach spaces, as for example, convexity of the closure of the domain and convexity of the closure of the range [26].

Beside maximal monotone operators in reflexive Banach spaces, the most important examples of type (D) operators are the subdifferentials of lower semicontinuous convex functions [42].

The notion of type (NI) operator, given by Simons in 1996, was recently proved to be equivalent to the notion of type (D) operator by Marques Alves and Svaiter. Along this chapter, we will use many times this characterization of type (D) operator.

Theorem 2.2 ([36, eq. (5) and Theorem 4.4, item 2])**.** *An operator $T : X \rightrightarrows X^*$ is of type (D) if, and only if, is of type (NI), that is*

$$\inf_{(y,y^*)\in T} \langle x^* - y^*, x^{**} - y \rangle \leq 0,$$

for all $(x^*, x^{**}) \in X^* \times X^{**}$.

Moreover, when working with linear operators, the next proposition gives us an alternative characterization of type (D) operator, which we will use in section 2.2.

Proposition 2.3 ([3, Theorem 4.1]). *Let $T : X \longrightarrow X^*$ be a continuous monotone linear map. Then T is of type (D) if, and only if, T^* is monotone.*

This result was recently extended to general monotone linear operators by Bauschke *et al.*, [7].

To study the behavior of $\widehat{T}$ in $X^{**} \times X^*$, in [3, Definition 2.1] were defined the operators $\overline{T}, \widetilde{T} : X^{**} \rightrightarrows X^*$, as follows:

1. $(x^{**}, x^*) \in \operatorname{gra}(\overline{T})$ if, and only if, there exists a bounded net $(x_\alpha, x_\alpha^*)_\alpha$ in $\operatorname{gra}(T)$ such that $x_\alpha \xrightarrow{w*} x^{**}$ and $x_\alpha^* \xrightarrow{\|\cdot\|} x^*$;

2. $(x^{**}, x^*) \operatorname{gra}(\widetilde{T})$ if, and only if,
$$\inf_{(y,y^*) \in \operatorname{gra}(T)} \langle x^* - y^*, x^{**} - y \rangle \geq 0;$$

Thus, if T is monotone then

$$\operatorname{gra}(\widehat{T}) \subseteq \operatorname{gra}(\overline{T}) \subseteq \operatorname{gra}(\widetilde{T}) = \operatorname{gra}(\widehat{T}^\mu). \tag{2.1}$$

And from this, a nicer characterization of Definition 2.1 is easily obtained.

Proposition 2.4 ([26, §2]). *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then,*

1. *T has a unique maximal monotone extension to the bidual if, and only if, $\widetilde{T}$ is monotone.*

2. *T is of type (D) if, and only if, $\overline{T} = \widetilde{T}$.*

Using the previous proposition and Lemma 1.13, we obtain the following lemma, which is a generalization of Theorem 6.7 in [40].

Lemma 2.5. *Let $T : X \rightrightarrows X^*$ be a monotone linear operator. If T is surjective then $\widehat{T}$ is maximal monotone and $\widehat{T}^{-1}$ is single valued. Moreover, T is of type (D).*

Proof. Since $\operatorname{ran}(T) = X^*$, $\widehat{T}^{-1} : X^* \rightrightarrows X^{**}$ is an everywhere defined monotone linear operator. Then, by Lemma 1.13, $\widehat{T}^{-1}$ is a maximal monotone single valued operator, thus $\widehat{T}$ is maximal monotone. Moreover, this implies that all the inclusions in equation 2.1 become equalities and, by Proposition 2.4, item *2*, T is of type (D). □

2.2 The Gossez operator

One of the most important examples of non-type (D) operators was given by Gossez. Gossez defined in [27] the following operator

$$G : \ell^1 \longrightarrow \ell^\infty, \qquad (Gx)_n := \sum_{k>n} x_k - \sum_{k<n} x_k, \tag{2.2}$$

which is everywhere defined, linear, continuous, anti-symmetric and maximal monotone, but not of type (D). Also, as proved by Gossez, G^* is not monotone.

Recall that c_0, ℓ^1 and ℓ^∞ are real Banach spaces, and

$$\ell^1 = (c_0)^*, \qquad \ell^\infty = (\ell^1)^* = (c_0)^{**}.$$

From now on, let

$$e = (1, 1, \ldots) \in \ell^\infty \setminus c_0. \tag{2.3}$$

Additional properties of the Gossez operator are the following.

Proposition 2.6. *Let G be the Gossez operator defined as above. Then:*

1. *If $x = (x_n)_n \in \ell^1$ and $y = (y_n)_n = G(x) \in \ell^\infty$ then*

$$y_n - y_{n+1} = x_n + x_{n+1}. \tag{2.4}$$

2. *G is injective.*

3. *For any $x \in \ell^1$, $G(x) + \langle x, e \rangle e \in c_0$.*

4. *$G(x) \in c_0$ if, and only if, $\langle x, e \rangle = 0$.*

Proof.

1. Let $x = (x_n)_n \in \ell^1$ and $y = (y_n)_n = G(x)$. Then

$$\begin{aligned} y_n - x_n - x_{n+1} &= \sum_{k>n} x_k - \sum_{k<n} x_k - x_n - x_{n+1} \\ &= \sum_{k>n+1} x_k - \sum_{k<n+1} x_k \\ &= y_{n+1}, \end{aligned}$$

which proofs (2.4).

2. Since G is linear, it suffices to prove that its kernel is trivial. Suppose that $x = (x_n)_n \in \ell^1$ and $G(x) = 0$. Using (2.4),

$$0 = x_n + x_{n+1}, \quad \forall\, n \in \mathbb{N}.$$

Hence $x_n = (-1)^{n+1}x_1$ and, since $x \in \ell^1$, we conclude $x_n = 0$, for all $n \in \mathbb{N}$.

3. Let $x \in \ell^1$. Taking limits when $n \longrightarrow \infty$ in (2.2), we obtain

$$\lim_{n\longrightarrow\infty} G(x)_n = -\sum_{k=1}^{\infty} x_k. \tag{2.5}$$

The result follows by observing that $\langle x, e\rangle = \sum_{k=1}^{\infty} x_k$.

4. Follows immediately from (2.5). □

2.3 Non-type (D) extensions of type (D) linear operators

The main results of this section are: to give conditions for a type (D) operator to have a non-type (D) extension; and to give an explicit example of a type (D) operator with non-type (D) extension, that is, a negative answer to problem 1.

A somewhat different version of this theorem was given by the authors in [17].

Theorem 2.7 ([17, Theorem 2.1])**.** *Suppose that $A : X^* \longrightarrow X^{**}$ is a linear (single valued) operator, that $\operatorname{ran}(A) \subseteq X$ and that there exists $\zeta \in X^{**} \setminus X$ such that*

$$\langle A(x^*), x^*\rangle = \langle x^*, \zeta\rangle^2, \quad \forall\, x^* \in X^*. \tag{2.6}$$

Define $T : X \rightrightarrows X^*$ as

$$\operatorname{gra}(T) = \{(A(x^*), x^*) \mid x^* \in X^*\}.$$

Then,

1. *T is maximal monotone of type (D);*
2. *$\widetilde{T} = \widehat{T} = A^{-1}$;*
3. *$\widetilde{T}^{-1}$ is not of type (D) on $X^* \times X^{**}$.*

Proof. Condition (2.6) trivially implies that A is monotone. Since $\operatorname{ran}(A) \subset X$, $\widehat{T} = A^{-1}$ and T is surjective. Therefore, T is a surjective monotone linear operator and, using Lemma 2.5, we conclude that $\widehat{T}$ is maximal monotone, $\widehat{T} = \widetilde{T}$ and T is of type (D). This proves items *1* and *2*.

Now we prove *3*. Since X is a closed subspace of X^{**} and $\zeta \in X^{**} \setminus X$, there exists $L \in X^{***}$, such that

$$L(x) = 0, \quad \forall\, x \in X; \qquad \langle \zeta, L\rangle > \frac{1}{4}.$$

Therefore, using also the assumption $\operatorname{ran}(A) \subset X$ and (2.6), we have

$$\begin{aligned}
\inf_{(y^*,y^{**})\in \operatorname{gra}(A)} \langle \zeta - y^{**}, L - y^*\rangle &= \inf_{y^*\in X^*} \langle \zeta - A(y^*), L - y^*\rangle \\
&= \inf_{y^*\in X^*} \langle \zeta, L\rangle - \langle \zeta, y^*\rangle - \langle A(y^*), L\rangle + \langle A(y^*), y^*\rangle \\
&= \inf_{y^*\in X^*} \langle \zeta, L\rangle - \langle \zeta, y^*\rangle + \langle \zeta, y^*\rangle^2 \\
&= \langle \zeta, L\rangle + \inf_{t\in\mathbb{R}} t^2 - t \\
&= \langle \zeta, L\rangle - \frac{1}{4} \\
&> 0,
\end{aligned}$$

that is, A is not of type (NI). Using Theorem 2.2, we conclude that $\widetilde{T}^{-1} = A$ is not of type (D). □

Remark 2.8. In [17], instead of condition (2.6) the authors used the slightly weaker condition

$$\sup_{x^*\in X^*} \langle x^*, \zeta\rangle - \langle A(x^*), x^*\rangle < \infty.$$

However, condition (2.6) gives additional information about the structure of the operator A. Under such condition, A can be decomposed as

$$A = G + E,$$

where $G : X^* \longrightarrow X^{**}$ is an anti-symmetric operator and $E : X^* \longrightarrow X^{**}$ is such that $E(x^*) = \langle x^*, \zeta\rangle\zeta$.

The following proposition uses the Gossez operator to construct an operator which meets Theorem 2.7 conditions.

Proposition 2.9 ([17, Proposition 3.2])**.** *Define* $A : \ell^1 \longrightarrow \ell^\infty$*,*

$$A(x^*) = G(x^*) + \langle x^*, e\rangle e,$$

where G is the Gossez operator and e is defined as in equation (2.3). *Then* $\operatorname{ran}(A) \subset c_0$ *and the operator* $T : c_0 \rightrightarrows \ell^1$ *defined as*

$$\operatorname{gra}(T) = \{(A(x^*), x^*) \mid x^* \in \ell^1\}, \tag{2.7}$$

is maximal monotone of type (D), but the inverse of its unique maximal monotone extension to the bidual, $\widetilde{T}^{-1} : \ell^1 \rightrightarrows \ell^\infty$, is not of type (D).

Proof. The inclusion $\operatorname{ran}(A) \subset c_0$ is proved in Proposition 2.6, item *3*.

To prove the second part of this proposition, is enough to proof that A satisfies the assumptions of Theorem 2.7 with $X = c_0$ and $\zeta = e$. Clearly A is linear, continuous and everywhere defined, since G and $\langle \cdot, e\rangle e$ are. Moreover, for any $x^* \in \ell^1$,

$$\begin{aligned}\langle A(x^*), x^*\rangle &= \langle G(x^*), x^*\rangle + \langle \langle x^*, e\rangle e, x^*\rangle \\ &= \langle x^*, e\rangle \langle e, x^*\rangle \\ &= \langle x^*, e\rangle^2,\end{aligned}$$

where the second equality holds since G is anti-symmetric. □

2.4 A non-type (D) operator in c_0

In this section we give a negative answer to problem 2. In order to do this, also using the Gossez operator, we define an operator with domain in c_0 which is not of type (D). Furthermore, in section 2.4.1, we extend this result proving that if a Banach space contains a closed subspace which is norm-isomorphic to c_0, then there also exist non-type (D) maximal monotone operators in such space.

The following results are taken from [16].

Lemma 2.10 ([16, Lemma 2.1])**.** *The operator $S : c_0 \rightrightarrows \ell^1$ defined as*

$$\operatorname{gra}(S) = \{(-G(x^*), x^*) \mid G(x^*) \in c_0,\ x^* \in \ell^1\},$$

is single valued in its domain, linear, anti-symmetric, maximal monotone and

$$\operatorname{ran}(S) = \{x^* \in \ell^1 \mid \langle x^*, e\rangle = 0\}.$$

Proof. Observe that $S = -G^{-1}|_{c_0}$. Therefore, since the Gossez operator is injective, linear and anti-symmetric, so are G^{-1} and S.

By Proposition 2.6, item *4*, and the definition of S, we can write

$$\operatorname{gra}(S) = \{(-G(x^*), x^*) \mid \langle x^*, e\rangle = 0\}.$$

Therefore, $x^* \in \operatorname{ran}(S)$ if, and only if, $\langle x^*, e\rangle = 0$.

To prove maximal monotonicity of S, take $(x, x^*) \in c_0 \times \ell^1$ monotonically related to S, that is,

$$\langle x - y, x^* - y^*\rangle \geq 0, \qquad \forall\, (y, y^*) \in \operatorname{gra}(S).$$

Fix $i \in \mathbb{N}$ and define, for $m > i$, $u^m = (u_k^m)_k \in \ell^1$ as

$$u_k^m = \begin{cases} -1, & k = i, \\ 1, & k = m, \\ 0, & \text{otherwise}, \end{cases} \tag{2.8}$$

and let $v^m = (v_k^m)_k = G(u^m)$. Then, using equations (2.2) and (2.8), we have

$$v_k^m = \begin{cases} 1, & k = i \text{ or } k = m, \\ 2, & i < k < m, \\ 0, & \text{otherwise}. \end{cases} \tag{2.9}$$

Since $v^m \in c_0$ and S is linear, for any $t \in \mathbb{R}$,

$$(-tv^m, tu^m) \in \operatorname{gra}(S).$$

Therefore, for any $t \in \mathbb{R}$,

$$\langle x + tv^m, x^* - tu^m\rangle \geq 0$$

which is equivalent to

$$\langle x, x^*\rangle + t[\langle v^m, x^*\rangle - \langle x, u^m\rangle] \geq 0.$$

Since the above inequality holds for any $t \in \mathbb{R}$,

$$\langle x, u^m\rangle = \langle v^m, x^*\rangle, \qquad \forall\, m > i, \tag{2.10}$$

$$\langle x, x^*\rangle \geq 0. \tag{2.11}$$

Equation (2.10), together with (2.8) and (2.9), imply

$$x_m - x_i = x_i^* + 2\sum_{k=i+1}^{m-1} x_k^* + x_m^*.$$

Using the assumptions $x \in c_0$, $x^* \in \ell^1$ and taking the limit $m \longrightarrow \infty$ in the above equation, we conclude that

$$x_i = -\left[x_i^* + 2\sum_{k=i+1}^{\infty} x_k^*\right] = -\left[G(x^*)_i + \sum_{k=1}^{\infty} x_k^*\right].$$

This implies that $x = -G(x^*) - \langle x^*, e\rangle e$. Using equation (2.11) and the fact that G is anti-symmetric,

$$0 \leq \langle x, x^*\rangle = -\langle G(x^*), x^*\rangle - \langle x, e\rangle^2 = -\langle x, e\rangle^2.$$

Follows that $\langle x^*, e\rangle = 0$ and $x = -G(x^*)$. Thus $(x, x^*) \in \operatorname{gra}(S)$ which proves the maximal monotonicity of S. $\square$

As a consequence of Lemma 2.10, we can write

$$\begin{aligned}\operatorname{gra}(S) &= \{(-G(x^*), x^*) \mid x^* \in \ell^1\ \langle x^*, e\rangle = 0\},\\ &= \{(-A(x^*), x^*) \mid x^* \in \ell^1\ \langle x^*, e\rangle = 0\}.\end{aligned}$$

where A is the operator defined in Proposition 2.9.

We now give a negative answer to Problem 2.

Proposition 2.11 ([16, Proposition 2.2])**.** *The maximal monotone operator* $S : c_0 \rightrightarrows \ell^1$ *(defined in Lemma 2.10),*

$$\operatorname{gra}(S) = \{(-G(x^*), x^*) \mid G(x^*) \in c_0,\ x^* \in \ell^1\},$$

has infinitely many maximal monotone extensions to $\ell^\infty \times \ell^1$*. In particular,* S *is a maximal monotone non-type (D) operator in* c_0*.*

Proof. Given $y^* \in \ell^1$ and $\alpha \in \mathbb{R}$, we claim that

$$\langle x - (-G(y^*) + \alpha e), x^* - y^*\rangle = \alpha\langle y^*, e\rangle, \qquad \forall\,(x, x^*) \in \operatorname{gra}(S). \tag{2.12}$$

Take $(x, x^*) \in \operatorname{gra}(S)$. Then, by Lemma 2.10, $x = -G(x^*)$ and $\langle x^*, e\rangle = 0$. Moreover

$$\langle x + G(y^*), x^* - y^*\rangle = -\langle G(x^* - y^*), x^* - y^*\rangle = 0,$$

which readily implies (2.12).

Take $\tilde{y} \in \ell^1$ such that $\langle \tilde{y}, e\rangle > 0$ and define

$$x^\tau = -G(\tau\tilde{y}) + \frac{1}{\tau}e, \qquad 0 < \tau < \infty.$$

In view of (2.12),

$$(x^\tau, \tau\tilde{y}) \in \mathrm{gra}(\widetilde{T}), \qquad 0 < \tau < \infty.$$

Therefore, for each $\tau \in (0, \infty)$, there exists a maximal monotone extension $T_\tau : \ell^\infty \rightrightarrows \ell^1$ of T such that

$$(x^\tau, \tau\tilde{y}) \in G(T_\tau).$$

However, these extensions are distinct because if $\tau, \tau' \in (0, \infty)$ and $\tau \neq \tau'$ then

$$\langle x^\tau - x^{\tau'}, \tau\tilde{y} - \tau'\tilde{y}\rangle = (\tau - \tau')(1/\tau - 1/\tau')\langle \tilde{y}, e\rangle < 0. \qquad \square$$

From the proof of the above proposition, we can state the following.

Corollary 2.12. *Let $S : c_0 \rightrightarrows \ell^1$ be defined as in Lemma 2.10. Then*

$$\mathrm{gra}(\widehat{S}^\mu) = \{(-G(x^*) + \alpha e, x^*) \in \ell^\infty \times \ell^1 \mid \alpha\langle x^*, e\rangle \geq 0\}.$$

2.4.1 Immersions of monotone operators

In this section we will show that if a Banach space contains a (closed) subspace norm-isomorphic to c_0, then there exist non-type (D) maximal monotone operators in such space. In order to do this we extend a monotone operator defined in a Banach space to a *larger* Banach space and study the relationship between the original operator and the *extended* operator.

Let X and Ω be real Banach spaces. A linear map $A : X \longrightarrow \Omega$ is a *norm-isomorphism* from X into a subspace of Ω, if A is injective, continuous and $A^{-1} : \mathrm{ran}(A) \subset \Omega \longrightarrow X$ is continuous.

Recall the definition of adjoint for bounded linear maps, given in Remark 1.16, item *1*: for a bounded linear map $A : X \longrightarrow \Omega$, its adjoint $A^* : \Omega^* \longrightarrow X^*$ is given by $A^*(w^*) = w^* \circ A$. We will use the following well known result.

Lemma 2.13. *If $A : X \longrightarrow \Omega$ is a norm-isomorphism then $A^* : \Omega^* \longrightarrow X^*$ is surjective.*

Proof. Take any $x^* \in X^*$. Since A^{-1} is continuous, $\xi^* = x^* \circ A^{-1}$ is a continuous linear functional defined in $\mathrm{ran}(A)$. Using the Hahn-Banach Theorem, we conclude that there exists $w^* \in \Omega^*$ which extends ξ^*. Therefore, as w^* and ξ^* coincides in $\mathrm{ran}(A)$, $w^* \circ A = \xi^* \circ A = x^*$ and so, $A^*(w^*) = x^*$. $\square$

Definition 2.14 ([16, Equation (11)])**.** Let X, Ω be real Banach spaces, and let $A : X \longrightarrow \Omega$ be a norm-isomorphism from X onto a closed subspace of Ω. Consider the application $\Psi_A : \mathrm{ran}(A) \times \Omega^* \longrightarrow X \times X^*$, defined as

$$\Psi_A(w, w^*) = (A^{-1}(w), A^*(w^*)). \tag{2.13}$$

For $T : X \rightrightarrows X^*$ define $T_A : \Omega \rightrightarrows \Omega^*$ as

$$\begin{aligned}\operatorname{gra}(T_A) &= \left\{ (w,w^*) \in \Omega \times \Omega^* \;\middle|\; \exists\, (x,x^*) \in \operatorname{gra}(T), \quad \begin{array}{c} w = A(x), \\ x^* = A^*(w^*) \end{array} \right\}, \\ &= \Psi_A^{-1}(\operatorname{gra}(T)). \end{aligned} \tag{2.14}$$

A resumed version of the following lemma was given by the authors in [16].

Lemma 2.15 ([16, Lemma 3.2])**.** *Consider Ψ_A as in equation* (2.13) *and, for $T : X \rightrightarrows X^*$, T_A as in equation* (2.14)*. Then*

1. *Ψ_A is surjective.*
2. *Ψ_A maps $\operatorname{gra}(T_A)$ onto $\operatorname{gra}(T)$.*
3. *For every $(w, w^*) \in T_A$, $w^* + \ker(A^*) \subset T_A(w)$.*
4. *$\operatorname{dom}(T_A^\mu) \subset \operatorname{ran}(A)$.*
5. *If (x, x^*) and (y, y^*) are monotonically related, $(w, w^*) \in \Psi_A^{-1}(x, x^*)$ and $(z, z^*) \in \Psi_A^{-1}(y, y^*)$, then (w, w^*) and (z, z^*) also are.*
6. *Ψ_A maps $\operatorname{gra}(T_A^\mu)$ onto $\operatorname{gra}(T^\mu)$.*
7. *If T is monotone then T_A is monotone.*
8. *If T is maximal monotone then T_A is maximal monotone.*

Proof.

1. This follows from the fact that A is an norm-isomorphism from X onto $\operatorname{ran}(A)$ and A^* is surjective, by Lemma 2.13.
2. Note that $\operatorname{dom}(T_A) \subset \operatorname{ran}(A)$. Hence Ψ_A is well defined. To prove that this application takes $\operatorname{gra}(T_A)$ onto $\operatorname{gra}(T)$, take $(x, x^*) \in \operatorname{gra}(T)$ and, by item *1*, let (w, w^*) be such that $\Psi_A(x, x^*) = (w, w^*)$. Therefore, $w = A(x)$, $x^* = A(w^*)$ and, by Definition 2.14, $(w, w^*) \in \operatorname{gra}(T_A)$ and Ψ_A maps this point into (x, x^*).
3. This follows from item *2* and the fact that $\Psi_A(w, w^*) = \Psi_A(w, w^* + u^*)$, for all $u^* \in \ker(A^*)$.
4. Note that the assumptions on A imply that $\operatorname{ran}(A)$ is a closed subspace. Let $w_0 \in \operatorname{dom}(T_A^\mu)$ and suppose that $w_0 \notin \operatorname{ran}(A)$. Take $w_0^* \in T_A^\mu(w_0)$ and

$(w, w^*) \in \text{gra}(T_A)$. Since $w \in \text{ran}(A)$, $w_0 - w \notin \text{ran}(A)$ and, using the Hahn-Banach theorem, we conclude that there exists $u^* \in \Omega^*$ such that

$$\langle z, u^* \rangle = 0, \ \ \forall\, z \in \text{ran}(A), \tag{2.15}$$
$$\langle w_0 - w, u^* \rangle > \langle w_0 - w, w_0^* - w^* \rangle. \tag{2.16}$$

Equation (2.15) implies that $u^* \in \ker(A^*)$ and, by item *3*, $w^* + u^* \in T_A(w)$. Thus

$$\begin{aligned} 0 &\leq \langle w_0 - w, w_0^* - (w^* + u^*) \rangle \\ &= \langle w_0 - w, w_0^* - w^* \rangle - \langle w_0 - w, u^* \rangle < 0, \end{aligned}$$

where the first inequality comes from the fact that (w_0, w_0^*) is monotonically related to $(w, w^* + u^*)$ and the last comes from (2.16). This contradiction proves the result.

5. Since $(x, x^*) = \Psi_A(w, w^*) = (A^{-1}(w), A^*(w^*))$ and $(y, y^*) = \Psi_A(z, z^*) = (A^{-1}(z), A^*(z^*))$,

$$\begin{aligned} \langle w - z, w^* - z^* \rangle &= \langle A(x) - A(y), w^* - z^* \rangle \\ &= \langle A(x - y), w^* - z^* \rangle \\ &= \langle x - y, A^*(w^* - z^*) \rangle \\ &= \langle x - y, x^* - y^* \rangle. \end{aligned}$$

This easily implies the result.

6. Take $(x, x^*) \in \text{gra}(T^\mu)$ and let (w, w^*) such that $(x, x^*) = (A^{-1}(w), A^*(w^*))$. Now, take $(z, z^*) \in \text{gra}(T_A)$ and let $(y, y^*) = (A^{-1}(z), A^*(z^*)) \in \text{gra}(T)$. Since $(x, x^*) \in \text{gra}(T^\mu)$ and $(y, y^*) \in \text{gra}(T)$, (x, x^*) and (y, y^*) are monotonically related and, by item *5*, (w, w^*) and (z, z^*) also are. Therefore, as $(z, z^*) \in \text{gra}(T_A)$ was taken arbitrarily, $(w, w^*) \in \text{gra}(T_A^\mu)$.

7. Assume T is monotone. Take $(w, w^*), (z, z^*) \in \text{gra}(T_A)$ and let $(x, x^*) = \Psi_A(w, w^*)$ and $(y, y^*) = \Psi_A(z, z^*)$. By item *2*, $(x, x^*), (y, y^*) \in \text{gra}(T)$ and, since T is monotone, (x, x^*) and (y, y^*) are monotonically related. Item *5* now implies that $(w, w^*), (z, z^*) \in \text{gra}(T_A)$ also are monotonically related, therefore T_A is monotone.

8. Assume T is maximal monotone. By item *7*, T_A is monotone, so remains to prove that T_A is maximal. Since T is maximal monotone, $T = T^\mu$ and, by

items 2 and 6,

$$\text{gra}(T_A^\mu) = \Psi_A^{-1}(\text{gra}(T^\mu)) = \Psi_A^{-1}(\text{gra}(T)) = \text{gra}(T_A).$$

Therefore $T_A = T_A^\mu$ and T_A is maximal monotone. □

Lemma 2.16 ([16, Lemma 3.2]). *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator and consider $T_A : \Omega \rightrightarrows \Omega^*$ as in Definition 2.14. If T_A is of type (D) on $\Omega \times \Omega^*$ then T is of type (D) on $X \times X^*$.*

Proof. Suppose that $(\widehat{x}^*, \widehat{x}^{**}) \in X^* \times X^{**}$. Using Lemma 2.13, we can find $\widehat{w}^* \in (A^*)^{-1}(\widehat{x}^*)$. Using Lemma 2.15, item 2, we have

$$\begin{aligned}
\inf_{(x,x^*)\in\text{gra}(T)} \langle \widehat{x}^* - x^*, \widehat{x}^{**} - x\rangle &= \inf_{(w,w^*)\in\text{gra}(T_A)} \langle \widehat{x}^* - A^*(w^*), \widehat{x}^{**} - A^{-1}(w)\rangle \\
&= \inf_{(w,w^*)\in\text{gra}(T_A)} \langle A^*(\widehat{w}^*) - A^*(w^*), \widehat{x}^{**} - A^{-1}(w)\rangle \\
&= \inf_{(w,w^*)\in\text{gra}(T_A)} \langle \widehat{w}^* - w^*, A^{**}(\widehat{x}^{**}) - A(A^{-1}(w))\rangle \\
&= \inf_{(w,w^*)\in\text{gra}(T_A)} \langle \widehat{w}^* - w^*, A^{**}(\widehat{x}^{**}) - w\rangle
\end{aligned}$$

Since $(\widehat{x}^*, \widehat{x}^{**})$ is a generic element of $X^* \times X^{**}$ and T_A is type (D), in view of the above result and Theorem 2.2, we conclude that T is also of type (D). □

Using Lemmas 2.15 and 2.16, we obtain the following Theorem.

Theorem 2.17 ([16, Theorem 3.3]). *Let X and Ω be Banach spaces for which there exists a linear map $A : X \longrightarrow \Omega$ such that A is a norm-isomorphism from X onto a closed subspace of Ω. If there exists a non-type (D) maximal monotone operator on $X \times X^*$, then there also exists a non-type (D) maximal monotone operator on $\Omega \times \Omega^*$.*

Proof. Let $T : X \times X^*$ be a non-type (D) operator on $X \times X^*$ and define $T_A : \Omega \rightrightarrows \Omega^*$ as in Definition 2.14. Using Lemma 2.15 we conclude that T_A is maximal monotone. If T_A is of type (D) then, by Lemma 2.16, T is also of type (D), in contradiction with the assumptions of the theorem. Therefore T_A is a maximal monotone non-type (D) operator. □

Combining Proposition 2.11 and Theorem 2.17, we have

Theorem 2.18 ([16, Corollary 3.4]). *Any real Banach space Ω which contains a norm-isomorphic copy of c_0 has a non-type (D) maximal monotone operator.*

Remark 2.19. Not long after the submission of [16], Borwein *et al.* [5] extended our results by giving conditions similar to Theorem 2.7 to construct non-type (D) operators in general Banach spaces. Combining Theorem 2.7 and Theorem 3.6 in [5] we can state the following result.

Theorem 2.20. *Let $A : X^* \longrightarrow X^{**}$ be a linear single valued operator. Assume that* $\operatorname{ran}(A) \subseteq X$ *and that there exists $\zeta \in X^{**} \setminus X$ such*

$$\langle A(x^*), x^* \rangle = \langle x^*, \zeta \rangle^2, \quad \forall\, x^* \in X^*.$$

Consider the operators $T, S : X \rightrightarrows X^*$ be defined as

$$\begin{aligned} \operatorname{gra}(T) &= \{(A(x^*), x^*) \mid x^* \in X^*\}, \\ \operatorname{gra}(S) &= \{(-A(x^*), x^*) \mid x^* \in X^*, \langle x^*, \zeta \rangle = 0\}. \end{aligned}$$

Then,

1. *T is maximal monotone of type (D).*
2. *$\widetilde{T} = \widehat{T} = A^{-1}$.*
3. *$\widetilde{T}^{-1}$ is not of type (D) on $X^* \times X^{**}$.*
4. *$-S$ is not maximal monotone.*
5. *S is anti-symmetric and maximal monotone but not of type (D).*

2.5 Further examples

We now present an operator in $c_0 \times \ell^1$, which will be used in section 3.4. Recall the definition of $T : c_0 \rightrightarrows \ell^1$, defined as in equation (2.7),

$$\begin{aligned} \operatorname{gra}(T) &= \{(A(x^*), x^*) \mid x^* \in \ell^1\}, \\ &= \{(G(x^*) + \langle x^*, e \rangle e, x^*) \mid x^* \in \ell^1\}. \end{aligned}$$

Lemma 2.21. *The operator $N : c_0 \rightrightarrows \ell^1$ defined as*

$$\operatorname{gra}(N) = \{(G(x^*), x^*) \mid G(x^*) \in c_0,\ x^* \in \ell^1\}, \tag{2.17}$$

is single valued in its domain, anti-symmetric and monotone, but not maximal monotone. Moreover, $\operatorname{ran}(N) = \{x^* \in \ell^1 \mid \langle x^*, e \rangle = 0\}$ *and $N^{\mu} = T$.*

Proof. Clearly $N = -S$ hence, by Lemma 2.10, N is single valued in its domain, anti-symmetric, monotone and

$$\operatorname{ran}(N) = \operatorname{ran}(S) = \{x^* \in \ell^1 \mid \langle x^*, e\rangle = 0\}.$$

Moreover, by Proposition 2.6, item *4*, we can write

$$\operatorname{gra}(N) = \{(G(x^*), x^*) \mid x^* \in \ell^1,\ \langle x^*, e\rangle = 0\}.$$

Now we prove that $N^\mu = T$. Take $(G(x^*), x^*) \in \operatorname{gra}(N)$, then $\langle x^*, e\rangle = 0$ and

$$(G(x^*), x^*) = (G(x^*) + \langle x^*, e\rangle e, x^*) = (A(x^*), x^*) \in \operatorname{gra}(T).$$

Thus, $\operatorname{gra}(N) \subset \operatorname{gra}(T)$. This implies,

$$\operatorname{gra}(T) = \operatorname{gra}(T^\mu) \subset \operatorname{gra}(N^\mu),$$

where the equality follows from Proposition 1.5, item *1*, and the fact that T is maximal monotone, and the inclusion follows from Proposition 1.6.

On the other hand, take $(x, x^*) \in c_0 \times \ell^1$ monotonically related to N, that is,

$$\langle x - G(y^*), x^* - y^*\rangle \geq 0, \qquad \forall\, y^* \in \ell^1,\ \langle y^*, e\rangle = 0.$$

Fix $i \in \mathbb{N}$ and, for every $m > i$, consider u^m and v^m defined as in equations (2.8) and (2.9). Now we proceed as in the proof of Lemma 2.10. Since, for any $t \in \mathbb{R}$ and $m > i$, $(tv^m, tu^m) \in \operatorname{gra}(N)$, we conclude that

$$\begin{aligned} \langle x, u^m\rangle &= -\langle v^m, x^*\rangle, \qquad \forall\, m > i \\ \langle x, x^*\rangle &\geq 0. \end{aligned}$$

which implies,

$$x_i - x_m = x_i^* + 2\sum_{k=i+1}^{m-1} x_k^* + x_m^*.$$

Taking the limit $m \longrightarrow \infty$,

$$x_i = x_i^* + 2\sum_{k=i+1}^{\infty} x_k^* = G(x^*)_i + \sum_{k=1}^{\infty} x_k^*.$$

This implies that $x = G(x^*) + \langle x^*, e\rangle e = A(x^*)$. Therefore, $(x, x^*) \in \operatorname{gra}(T)$ and $T = N^\mu$. This also implies that N is not maximal monotone, since $\operatorname{ran}(N) \neq \operatorname{ran}(T) = \ell^1$. □

Chapter 3

Representable and monotone polar closures of monotone operators

This chapter is concerned to Problem 3, that is, the study of the relationship between the representable and the polar monotone closures of monotone operators in topological vector spaces. In sections 3.1 and 3.2 we give the definitions and properties of the representable and polar monotone closures, respectively. In section 3.3, we give an extension of Theorem 31 in [37], along with several corollaries. Finally, in section 3.4 we study the closures of an explicit operator defined in c_0

Most of the results in this chapter are in [15].

In this chapter, X is a real topological vector space, Hausdorff and locally convex, with topology τ, X^* is its topological dual and $w^* = \sigma(X^*, X)$ is the weak$*$ topology of X^*.

For the sake of notation, from now on, we will identify a multivalued operator $T : X \rightrightarrows X^*$ with its graph $\operatorname{gra}(T)$, that is, we will write $(x, x^*) \in T$ instead of $(x, x^*) \in \operatorname{gra}(T)$ and $T \subset S$ will mean $\operatorname{gra}(T) \subset \operatorname{gra}(S)$.

As in the case of Banach spaces, the *duality product* $\pi : X \times X^* \longrightarrow \mathbb{R}$ is defined as

$$\pi(x, x^*) = \langle x, x^* \rangle = x^*(x), \quad \forall\, x \in X,\ x^* \in X^*.$$

Furthermore, the definitions of *monotone* and *maximal monotone* operator (page 2), linear operator (Definition 1.7), Fitzpatrick function and non-enlargeable operators (section 1.4) are the same as in chapter 1.

3.1 The representable closure

If $h : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and proper function, such that

$$h(x, x^*) \geq \pi(x, x^*), \quad \forall\, (x, x^*) \in X \times X^*,$$

then the set

$$\{(x, x^*) \in X \times X^* \mid h(x, x^*) = \pi(x, x^*)\}$$

is a monotone operator [37, Theorem 5]. In this case, it is said that such operator is *represented* by h. We are interested in monotone operators which are represented by lower semi-continuous functions.

Definition 3.1. Let $\mathcal{O}$ be a topology in $X \times X^*$. A multivalued operator $A : X \rightrightarrows X^*$ is *representable in the $\mathcal{O}$ topology* if there exists $h : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ convex, proper and $\mathcal{O}$-lower semi-continuous such that

$$h \geq \pi, \qquad A = \{h = \pi\}.$$

For instance, maximal monotone operators in a Hausdorff locally convex real topological vector space are $\tau \times w^*$-representable, as proved by Fitzpatrick [22]. Indeed, if $A : X \rightrightarrows X^*$ is a maximal monotone operator, then it is represented — in the $\tau \times w^*$ topology— by its Fitzpatrick function, $\varphi_A : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$,

$$\begin{aligned} \varphi_A(x, x^*) &= \sup_{(y,y^*)\in A} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \\ &= \sup_{(y,y^*)\in A} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle, \end{aligned} \tag{3.1}$$

which was defined in section 1.4. On the other hand, in general not every monotone operator is $\tau \times w^*$-representable. For instance, take $T : \mathbb{R} \rightrightarrows \mathbb{R}$, $T = \{(0, 0), (0, 1)\}$.

Let $\mathcal{R}$ be the family of the $\tau \times w^*$-representable operators in $X \times X^*$, that is,

$$\mathcal{R} = \{R : X \rightrightarrows X^* \mid R \text{ is } \tau \times w^*\text{-representable}\}.$$

From now on, unless otherwise stated, we will always use the $\tau \times w^*$ topology. Given a monotone operator $A : X \rightrightarrows X^*$, a *representable extension* of A is any operator $R \in \mathcal{R}$ such that $A \subset R$. Thus, the *family of representable extensions* of A is

$$\mathcal{R}(A) = \{R \in \mathcal{R} \mid A \subset R\}.$$

The following proposition was proved by Martinez-Legaz and Svaiter [37] in the context of Banach spaces, but its extension to topological vector spaces is trivial.

Proposition 3.2 ([37, Corollary 10]). *Let $A : X \rightrightarrows X^*$ be a multivalued operator. Then A is monotone if, and only if, $\mathcal{R}(A) \neq \emptyset$. Moreover, for any family $\{R_i\}_{i\in I} \subset \mathcal{R}(A)$,*

$$\bigcap_{i\in I} R_i \in \mathcal{R}(A).$$

The *representable closure* of a monotone operator $A : X \rightrightarrows X^*$, is defined as

$$\mathrm{cl}_{\mathcal{R}}(A) = \bigcap_{R\in\mathcal{R}(A)} R. \tag{3.2}$$

By Proposition 3.2, $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{R}(A)$, so it is the smallest (in the sense of graph inclusion) representable extension of A.

For $A : X \rightrightarrows X^*$, consider $\delta_A : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$\delta_A(x, x^*) = \begin{cases} 0, & \text{if } (x, x^*) \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

This function is known as the *indicator function* of A. Moreover, the *$\mathcal{S}$-function* of A is $\mathcal{S}_A = \mathrm{cl}\,\mathrm{conv}_{\tau\times w^*}(\pi + \delta_A)$, which is the largest convex and $\tau \times w^*$-lower semi-continuous function majorized by $\pi + \delta_A$. The next proposition, also given in [37] for Banach spaces, relates the $\mathcal{S}$-function and the representable closure of a monotone operator.

Proposition 3.3 ([37, §3]).

1. *A is representable if, and only if, $\mathcal{S}_A = \mathrm{cl}\,\mathrm{conv}_{\tau\times w^*}(\pi + \delta_A)$ represents A.*
2. *If A is monotone then $\mathrm{cl}_{\mathcal{R}}(A) = \{\mathcal{S}_A \leq \pi\}$.*
3. *Let A be monotone, $(x_0, x_0^*) \in X \times X^*$ and define*

   $$\tau_{(x_0,x_0^*)}(A) = A - \{(x_0, x_0^*)\} = \{(x - x_0, x^* - x_0^*) \mid (x, x^*) \in A\}. \tag{3.3}$$

   Then $\mathrm{cl}_{\mathcal{R}}\big(\tau_{(x_0,x_0^*)}(A)\big) = \tau_{(x_0,x_0^*)}\big(\mathrm{cl}_{\mathcal{R}}(A)\big)$.

3.2 The monotone polar closure

Monotonicity can also be analyzed using the classical notion of polarity [9], as follows. Recall the definition of *monotone polar* of an operator $A : X \rightrightarrows X^*$, given in section 1.1,

$$A^\mu = \{(x, x^*) \in X \times X^* \mid \langle x - y, x^* - y^*\rangle \geq 0, \forall\, (y, y^*) \in A\}.$$

Thus, the *monotone polar closure* or *μ-polar closure* of A [37, §4] is

$$A^{\mu\mu} = (A^{\mu})^{\mu}.$$

Let $\mathcal{M}$ be the family of maximal monotone operators in $X \times X^*$, that is

$$\mathcal{M} = \{T : X \rightrightarrows X^* \mid M \text{ is maximal monotone}\},$$

and for a monotone operator $A : X \rightrightarrows X^*$, consider the *family of maximal monotone extensions* of A,

$$\mathcal{M}(A) = \{T \in \mathcal{M} \mid A \subset T\}.$$

The following proposition subsumes some properties of the monotone polar, which were also proved for Banach spaces [37], but also trivially holds for topological vector spaces. Some of these properties were already stated in sections 1.1 and 1.4.

Proposition 3.4 ([37, §4])**.**

1. *If $A \subset B \subset X \times X^*$ then $B^{\mu} \subset A^{\mu}$.*
2. *A is monotone if, and only if, $A \subset A^{\mu}$.*
3. *A is maximal monotone if, and only if, $A = A^{\mu}$.*
4. *A^{μ} is maximal monotone if, and only if, A and A^{μ} are monotone.*
5. *$A^{\mu} = \{\varphi_A \leq \pi\}$.*
6. *If A is monotone then*
$$A^{\mu} = \bigcup_{M \in \mathcal{M}(A)} M$$
and
$$A^{\mu\mu} = \bigcap_{M \in \mathcal{M}(A)} M$$
7. *If A is monotone then $A \subset A^{\mu\mu} \subset A^{\mu} \subset \operatorname{ed}(\varphi_A)$.*
8. *Let A be monotone, $(x_0, x_0^*) \in X \times X^*$ and define $\tau_{(x_0,x_0^*)}(A)$ as in* (3.3)*. Then*
$$\big(\tau_{(x_0,x_0^*)}(A)\big)^{\mu} = \tau_{(x_0,x_0^*)}\big(A^{\mu}\big).$$

3.3 Relationship between the closures

Let $A : X \rightrightarrows X^*$ be a monotone operator. Since any maximal monotone operator is representable, by Proposition 3.2, equation (3.2) and Proposition 3.4, item *6*, $A^{\mu\mu}$ is representable and

$$\mathrm{cl}_{\mathcal{R}}(A) \subseteq A^{\mu\mu}.$$

Our aim is to prove the following theorem.

Theorem 3.5 ([15, Theorem 2.2])**.** *Assume that $X \times X^*$ is endowed with the $\tau \times w^*$ topology. Let $A : X \rightrightarrows X^*$ be a monotone operator. If $\mathrm{cl}_{\mathcal{R}}(A) \neq A^{\mu\mu}$ then A^μ is affine linear, maximal monotone, non-enlargeable and there exists (x_0, x_0^*) such that $\varphi_A(x_0, x_0^*) < \langle x_0, x_0^* \rangle$. In particular, A is pre-maximal monotone.*

Observe that for any $A : X \rightrightarrows X^*$,

$$(0,0) \notin A^{\mu\mu} \qquad \iff \qquad A^\mu \cap \{\pi < 0\} \neq \emptyset.$$

The following proposition gives sufficient conditions for $(0,0) \notin A^{\mu\mu}$, and it was proved in the context of Banach spaces in [37]. For the sake of completeness, we provide its proof in the context of topological vector spaces.

Proposition 3.6 ([37, Propositions 26, 28 and 30])**.** *Let A be monotone. Then any of the following conditions are sufficient for $(0,0) \notin A^{\mu\mu}$:*

1. $\varphi_A(0,0) < 0$ *and* $\mathrm{ed}(\varphi_A) \cap \{\pi < 0\} \neq \emptyset$*;*
2. $\varphi_A(0,0) = 0$ *and* $\{\varphi_A < 0\} \cap \{\pi < 0\} \neq \emptyset$*;*
3. $\varphi_A(0,0) > 0$.

Proof. Suppose that item *1* holds. Take $(y, y^*) \in \mathrm{ed}(\varphi_A)$ such that $\langle y, y^* \rangle < 0$. Using the convexity of φ_A and the assumptions $\varphi_A(0,0) < 0$ and $\varphi_A(y, y^*) < \infty$, we conclude that there exists $t \in (0,1)$, small enough, such that

$$\begin{aligned}\varphi_A((1-t)(0,0) + t(y,y^*)) &\leq (1-t)\varphi_A(0,0) + t\varphi_A(y,y^*)\\ &< t^2\langle y, y^*\rangle\\ &< 0,\end{aligned}$$

where the first inequality follows from the convexity of φ_A. Define

$$(y_t, y_t^*) = (1-t)(0,0) + t(y,y^*) = t(y,y^*).$$

Since $t^2\langle y, y^* \rangle = \langle y_t, y_t^* \rangle$, we conclude that

$$\varphi_A(y_t, y_t^*) < \langle y_t, y_t^* \rangle < 0.$$

Using item *5* of Proposition 3.4 we conclude that $(y_t, y_t^*) \in A^\mu$, and the last inequality in the above equation imply that $(0,0) \notin A^{\mu\mu}$.

Item *2* is proved by the same reasoning. Indeed, if this item holds, take (y, y^*) such that $\varphi_A(y, y^*) < 0$ and $\langle y, y^*\rangle < 0$. Since φ_A is convex and $\varphi_A(0,0) = 0$, for any $t \in [0,1]$,

$$\begin{aligned}\varphi_A((1-t)(0,0) + t(y,y^*)) &\leq (1-t)\varphi_A(0,0) + t\varphi_A(y,y^*)\\ &= t\varphi_A(y,y^*)\end{aligned} \tag{3.4}$$

Since $\varphi_A(y, y^*) < 0$, there exists $t \in (0,1)$ small enough, such that

$$t\varphi_A(y,y^*) < t^2\langle y, y^*\rangle < 0. \tag{3.5}$$

Define again $(y_t, y_t^*) = t(y, y^*)$. Since $t^2\langle y, y^*\rangle = \langle y_t, y_t^*\rangle$, and combining equations (3.4) and (3.5),

$$\varphi_A(y_t, y_t^*) < \langle y_t, y_t^*\rangle < 0,$$

which implies $(y_t, y_t^*) \in A^\mu \cap \{\pi < 0\}$ and $(0,0) \notin A^{\mu\mu}$.

If item *3* holds, direct use of Definition 3.1 shows that $0 \notin A^\mu$. Since A is monotone, $A \subset A^\mu$, $A^{\mu\mu} \subset A^\mu$ and the conclusion follows. □

The next lemma generalizes to topological linear spaces a result proved in reflexive Banach spaces.

Lemma 3.7 ([37, Proposition 29])**.** *Let A be monotone. If $\varphi_A(0,0) = 0$ and $\mathcal{S}_A(0,0) > 0$ then $(0,0) \notin A^{\mu\mu}$.*

Proof. Since $\varphi_A(x, x^*) = (\pi + \delta_A)^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$, by Fenchel-Moreau Theorem for locally convex spaces,

$$\begin{aligned}\mathcal{S}_A(x,x^*) &= (\pi+\delta_A)^{**}(x,x^*) = (\varphi_A)^*(x^*,x),\\ &= \sup_{(y,y^*)} \langle y, x^*\rangle + \langle x, y^*\rangle - \varphi_A(y,y^*).\end{aligned} \tag{3.6}$$

In particular, $\sup_{(y,y^*)} -\varphi_A(y^*, y) = \mathcal{S}_A(0,0) > 0$, so there exists $(y, y^*) \in X \times X^*$ such that

$$\varphi_A(y,y^*) < 0. \tag{3.7}$$

The above inequality trivially implies that there exists $t \in (0,1)$, small enough, such that

$$t\big(t\langle y,y^*\rangle + (1-t)\varphi_A(y,y^*)\big) = t^2\langle y,y^*\rangle + t(1-t)\varphi_A(y,y^*) < 0,$$

Since $\sup_{(x,x^*)\in A} -\langle x, x^*\rangle = \varphi_A(0,0) = 0$ and $t, 1-t \in (0,1)$, there exists

$(z, z^*) \in A$ such that

$$t^2\langle y, y^*\rangle + t(1-t)\varphi_A(y, y^*) + (1-t)\langle z, z^*\rangle < 0, \tag{3.8}$$
$$t\varphi_A(y, y^*) + (1-t)\langle z, z^*\rangle < 0 \tag{3.9}$$

where we used again (3.7) for the second inequality. Define

$$(z_0, z_0^*) = t(y, y^*) + (1-t)(z, z^*).$$

Then

$$\begin{aligned}\langle z_0, z_0^*\rangle &= t^2\langle y, y^*\rangle + t(1-t)(\langle z, y^*\rangle + \langle y, z^*\rangle) + (1-t)^2\langle z, z^*\rangle \\ &= t^2\langle y, y^*\rangle + t(1-t)\big(\langle z, y^*\rangle + \langle y, z^*\rangle - \langle z, z^*\rangle\big) + (1-t)\langle z, z^*\rangle \\ &\le t^2\langle y, y^*\rangle + (1-t)t\varphi_A(y, y^*) + (1-t)\langle z, z^*\rangle, \end{aligned} \tag{3.10}$$

where the last inequality follows from the inclusion $(z, z^*) \in A$ and (3.1). On the other hand, using the convexity of φ_A and again the inclusion $(z, z^*) \in A$, we have

$$\begin{aligned}\varphi_A(z_0, z_0^*) &\le t\varphi_A(y, y^*) + (1-t)\varphi_A(z, z^*) \\ &= t\varphi_A(y, y^*) + (1-t)\langle z, z^*\rangle. \end{aligned} \tag{3.11}$$

From equations (3.8) and (3.10) we conclude that $(z_0, z_0^*) \in \{\pi < 0\}$. In addition, combining equations (3.9) and (3.11) we obtain that $\varphi_A(z_0, z_0^*) < 0$, which, combined with Proposition 3.6, item *2*, ends the proof. □

Remark 3.8. Notice that the last equality in (3.6) uses the fact that X^* is endowed with w^* topology. This is the reason why in Theorem 3.5 and Corollary 3.15 the space $X \times X^*$ is endowed with the $\tau \times w^*$ and the strong×weak∗ topologies, respectively.

The following lemma is central in our study. Imposing a condition on the Fitzpatrick function of a monotone operator, we can obtain a rather strong structure of such operator.

Lemma 3.9 ([15, Lemma 4.4])**.** *Let A be monotone. If* $\mathrm{ed}(\varphi_A) \subset \{\pi \ge 0\}$ *then $A^\mu = \mathrm{ed}(\varphi_A)$ and A^μ is maximal monotone and affine linear. Moreover,* $\mathrm{ed}(\varphi_{A^\mu}) = A^\mu$*, so A^μ is non-enlargeable.*

Proof. First, we prove that $\mathrm{ed}(\varphi_A)$ is monotone. Since $\mathrm{ed}(\varphi_A)$ is convex,

$$\left(\frac{x+y}{2}, \frac{x^*+y^*}{2}\right) \in \mathrm{ed}(\varphi_A), \quad \forall\,(x, x^*), (y, y^*) \in \mathrm{ed}(\varphi_A).$$

Hence

$$0 \leq 4 \left\langle \frac{x+y}{2}, \frac{x^*+y^*}{2} \right\rangle = \langle x+y, x^*+y^* \rangle, \quad \forall\,(x,x^*),(y,y^*) \in \operatorname{ed}(\varphi_A). \tag{3.12}$$

Fix $(x,x^*) \in \operatorname{ed}(\varphi_A)$, then, by (3.12),

$$\langle y-(-x), y^*-(-x^*) \rangle = \langle x+y, x^*+y^* \rangle \geq 0, \quad \forall\,(y,y^*) \in A \subset \operatorname{ed}(\varphi_A).$$

Thus, $(-x,-x^*) \in A^\mu \subset \operatorname{ed}(\varphi_A)$ and, again by (3.12),

$$\langle x-y, x^*-y^* \rangle = \langle (-x)+y, (-x^*)+y^* \rangle \geq 0, \quad \forall\,(y,y^*) \in \operatorname{ed}(\varphi_A),$$

which implies that $(x,x^*) \in \operatorname{ed}(\varphi_A)^\mu$. Therefore,

$$\operatorname{ed}(\varphi_A) \subset \operatorname{ed}(\varphi_A)^\mu,$$

proving that $\operatorname{ed}(\varphi_A)$ is monotone.

Since A is monotone, $A \subset \operatorname{ed}(\varphi_A)$. Therefore, $\operatorname{ed}(\varphi_A)^\mu \subset A^\mu \subset \operatorname{ed}(\varphi_A)$, which, combined with the above inclusion, shows that $A^\mu = \operatorname{ed}(\varphi_A)$. Proposition 3.4, item *4*, now implies that A^μ is maximal monotone. Furthermore, since $\operatorname{ed}(\varphi_A)$ is convex and maximal monotone, it is affine linear by [34, Lemma 1.2].

Now we prove that A^μ is non-enlargeable. From $A \subset A^\mu$, we have $\varphi_A \leq \varphi_{A^\mu}$. This means that $\operatorname{ed}(\varphi_{A^\mu}) \subset \operatorname{ed}(\varphi_A) = A^\mu$. The other inclusion follows trivially from the monotonicity of A^μ. Hence $A^\mu = \operatorname{ed}(\varphi_{A^\mu})$ and A^μ is non-enlargeable. □

Lemma 3.10 ([15, Lemma 4.5])**.** *Let A be monotone. If $\varphi_A(0,0) < 0$ and $(0,0) \in A^{\mu\mu}$ then $A^\mu = \operatorname{ed}(\varphi_A)$ and A^μ is maximal monotone, affine linear and non-enlargeable.*

Proof. First use Proposition 3.6, item *1*, to conclude that under these assumptions, $\{\pi < 0\} \cap \operatorname{ed}(\varphi_A) = \emptyset$. But this is equivalent to $\operatorname{ed}(\varphi_A) \subset \{\pi \geq 0\}$, which is precisely the condition needed by Lemma 3.9. □

We now are in conditions to present the proof of our main result, Theorem 3.5.

Proof of Theorem 3.5. Suppose $\operatorname{cl}_{\mathcal{R}}(A) \neq A^{\mu\mu}$, that is, $A^{\mu\mu} \setminus \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$. Since $\operatorname{cl}_{\mathcal{R}}(A)$ and $A^{\mu\mu}$ are preserved by translations, without loss of generality, we can assume that $(0,0) \in A^{\mu\mu} \setminus \operatorname{cl}_{\mathcal{R}}(A)$, that is,

$$(0,0) \in A^{\mu\mu}, \tag{3.13}$$

$$(0,0) \notin \operatorname{cl}_{\mathcal{R}}(A). \tag{3.14}$$

By Proposition 3.6, item *3*, equation (3.13) implies $\varphi_A(0,0) \leq 0$. On the other hand, by Proposition 3.3, item *2*, equation (3.13) is equivalent to $\mathcal{S}_A(0,0) > 0$, which combined with (3.14) and Lemma 3.7 allow us to conclude that

$$\varphi_A(0,0) < 0.$$

This inequality, equation (3.13) and Lemma 3.10 now imply that $A^{\mu} = \mathrm{ed}(\varphi_A)$ and A^{μ} is maximal monotone, affine linear and non-enlargeable. Moreover, A is pre-maximal monotone, A^{μ} is the unique maximal monotone extension of A and $A^{\mu\mu} = A^{\mu} = \mathrm{ed}(\varphi_A)$.

Finally, if $(x_0, x_0^*) \in A^{\mu\mu} \backslash \mathrm{cl}_{\mathcal{R}}(A)$, then $(0,0) \in (\tau_{(x_0,x_0^*)}(A))^{\mu\mu} \backslash \mathrm{cl}_{\mathcal{R}}(\tau_{(x_0,x_0^*)}(A))$. Therefore $\varphi_{\tau_{(x_0,x_0^*)}(A)}(0,0) < 0$, which implies

$$\varphi_A(x_0, x_0^*) = \sup_{(x,x^*)\in A} \langle x - x_0, x_0^* - x^* \rangle + \langle x_0, x_0^* \rangle < \langle x_0, x_0^* \rangle. \qquad \square$$

Remark 3.11. Follows from the proof of Theorem 3.5 that

$$A^{\mu\mu} \setminus \mathrm{cl}_{\mathcal{R}}(A) \subset \{\varphi_A < \pi\}.$$

The general setting in which we state Theorem 3.5 allow us to have many important cases as a particular case. We begin giving a new proof of Theorem 31 in [37]. For this, we need first to state the following lemma, which is due to Svaiter [49]. Recall that for a linear operator $T : X \rightrightarrows X^*$, $T^{\vdash} = T^{+}|_X$ (Remark 1.16, item *3*) or, explicitly,

$$T^{\vdash} = \{(x, x^*) \in X \times X^* \mid \langle x, y^* \rangle + \langle y, x^* \rangle = 0, \forall\, (y, y^*) \in T\}.$$

Lemma 3.12 ([49, Lemma 2.1])**.** *Let $T : X \rightrightarrows X^*$ be a maximal monotone linear operator. Then*

1. $T^{\vdash} \subset \{(x, x^*) \mid \varphi_T(x, x^*) = 0\}$.
2. $T \cap T^{\vdash} = T \cap \{(x, x^*) \mid \langle x, x^* \rangle = 0\}$.

Theorem 3.13 ([15, Theorem 31])**.** *Let X be finite dimensional and $A \subset X \times X^*$ be monotone. Then $\mathrm{cl}_{\mathcal{R}}(A) = A^{\mu\mu}$.*

Proof. By Theorem 3.5, if $\mathrm{cl}_{\mathcal{R}}(A) \neq A^{\mu\mu}$ then A^{μ} is affine linear, maximal monotone, non-enlargeable and there exists (x_0, x_0^*) such that $\varphi_A(x_0, x_0^*) < \langle x_0, x_0^* \rangle$. Without loss of generality, since $\mathrm{cl}_{\mathcal{R}}(A)$ and $A^{\mu\mu}$ are preserved by translations, we can assume that $(x_0, x_0^*) = (0,0) \in A^{\mu}$. Thus, A^{μ} is linear and $\varphi_A(0,0) < 0$.

Let $T = A^{\mu}$. By Lemma 3.12, item *1*,

$$T^{\vdash} \subset \{(x, x^*) \mid \varphi_T(x, x^*) = 0\} \subset \mathrm{ed}(\varphi_T) = T.$$

We now prove the opposite inclusion. Since X is finite dimensional, $\dim T^{\vdash} = 2n - \dim T$. Furthermore, since T is maximal monotone, $\dim T = n$ by Minty's Theorem. Hence $\dim T^{\vdash} = 2n - n = n = \dim T$ and

$$T^{\vdash} = T.$$

But this implies $T \subset \{(x, x^*) \mid \langle x, x^* \rangle = 0\}$, by Lemma 3.12, item 2; thus, since $\varphi_A(0,0) < 0$ and $A \subset A^{\mu} = T \subset \{\pi = 0\}$,

$$0 = \langle a, 0 \rangle + \langle 0, a^* \rangle - \langle a, a \rangle \leq \varphi_A(0,0) < 0,$$

for any $(a, a^*) \in A$. This is clearly a contradiction. □

For those monotone operators that do not admit an affine linear maximal monotone extensions, the representable closure and the monotone polar closure are identical.

Corollary 3.14 ([15, Corollary 2.3])**.** *If $A : X \rightrightarrows X^*$ is monotone, $X \times X^*$ is endowed with the $\tau \times w^*$ topology and the convex hull of A is not monotone then*

$$\mathrm{cl}_{\mathcal{R}}(A) = A^{\mu\mu}.$$

Proof. Suppose $\mathrm{cl}_{\mathcal{R}}(A) \neq A^{\mu\mu}$. Then A^{μ} is affine linear and maximal monotone. In particular A^{μ} is convex and so,

$$A \subset \mathrm{conv}(A) \subset A^{\mu}.$$

This contradicts the fact of $\mathrm{conv}(A)$ not being monotone. □

In the particular case where X is a Banach space and τ is the norm-topology, we have the following corollary.

Corollary 3.15 ([15, Corollary 2.4])**.** *Let X be a real Banach space and let $A : X \rightrightarrows X^*$ be a monotone operator. Assume that $X \times X^*$ is endowed with the strong×weak∗ topology. If $\mathrm{cl}_{\mathcal{R}}(A) \neq A^{\mu\mu}$ then A^{μ} is affine linear, maximal monotone, non-enlargeable and there exists (x_0, x_0^*) such that $\varphi_A(x_0, x_0^*) < \langle x_0, x_0^* \rangle$. In particular, A is pre-maximal monotone.*

Remark 3.16. If X is a reflexive Banach space, then $X \times X^*$ is also reflexive and, for proper convex functions, lower semi-continuity in the strong, weak, weak∗ and hence strong×weak∗ topologies are equivalent. Therefore, in this case, Corollary 3.15 still holds if we replace the strong×weak∗ topology by the strong topology of $X \times X^*$.

On the other hand, when X is a non-reflexive Banach space, the strong topology of $X \times X^*$ is finer than the strong×weak∗ topology. So, for a monotone operator $A : X \rightrightarrows X^*$,

$$A \subset \mathrm{cl}_{s\times s-\mathcal{R}}(A) \subset \mathrm{cl}_{\mathcal{R}}(A) \subset A^{\mu\mu}.$$

To finalize this section, we present two results on the structure of pre-maximal monotone operators.

Lemma 3.17. *Let $A : X \rightrightarrows X^*$ be a monotone operator. Then 1 $\Longrightarrow$ 2 $\Longrightarrow$ 3.*

1. *A is pre-maximal monotone and $\varphi_A(0, 0) < 0$;*
2. *$(0, 0) \in A^{\mu\mu}$ and $\varphi_A(0, 0) < 0$;*
3. *$\mathrm{ed}(\varphi_A) \subset \{\pi \geq 0\}$.*

Proof. Assume item *1* holds. Since $\varphi_A(0, 0) < 0$, by Proposition 3.4, item *5*, $(0, 0) \in A^\mu$. This trivially implies item *2*, since $A^\mu = A^{\mu\mu}$.

On the other hand, using Proposition 3.6, item *2*, we see that $(0, 0) \in A^{\mu\mu}$ and $\varphi_A(0, 0) < 0$ implies that $\mathrm{ed}(\varphi_A) \cap \{\pi < 0\}$ is empty. □

Proposition 3.18. *Let $A : X \rightrightarrows X^*$ be a pre-maximal monotone operator. Then, either $\varphi_A \geq \pi$, or $A^\mu = \mathrm{ed}(\varphi_A)$, A^μ is affine linear and non-enlargeable.*

Proof. Suppose that doesn't hold the condition $\varphi_A \geq \pi$. Without loss of generality, we can assume that $\varphi_A(0, 0) < \langle 0, 0\rangle = 0$. This, together with the hypothesis of pre-maximality of A implies, by Lemma 3.17, that $\mathrm{ed}(\varphi_A) \subset \{\pi \geq 0\}$. The result now follows by Lemma 3.9. □

Remark 3.19. In a non-reflexive Banach space X, a maximal monotone operator $T : X \rightrightarrows X^*$ has a unique extension to the bidual if, and only if, $\widehat{T}^{-1} : X^* \rightrightarrows X^{**}$ is pre-maximal monotone. Thus, by Proposition 3.18, if $T : X \rightrightarrows X^*$ has a unique extension to the bidual then, either $\varphi_{\widehat{T}} \geq \pi$, or T is affine linear and non-enlargeable. This is precisely Theorem 1.3 in [34].

3.4 An example in Banach spaces

In this section we apply the above stated results to a particular operator defined in the Banach space c_0, and study the relationship between the representable closures in the two different topologies, strong×strong and weak×weak∗.

Let $N : c_0 \rightrightarrows \ell^1$ be the operator defined in (2.17), that is,

$$\begin{aligned}\operatorname{gra}(N) &= \{(G(x^*), x^*) \mid G(x^*) \in c_0,\ x^* \in \ell^1\}\\ &= \{(G(x^*), x^*) \mid x^* \in \ell^1,\ \langle x^*, e\rangle = 0\},\end{aligned} \tag{3.15}$$

which, by Lemma 2.21, is single valued, anti-symmetric, monotone but not maximal monotone, $\operatorname{ran}(N) = \{x^* \in \ell^1 \mid \langle x^*, e\rangle = 0\}$ and

$$N^\mu = \{(G(x^*) + \langle x^*, e\rangle e, x^*) \mid x^* \in \ell^1\}. \tag{3.16}$$

By Proposition 2.9 and equation (2.7), N^μ is maximal monotone. Therefore, N is pre-maximal monotone and $N^{\mu\mu} = N^\mu$.

On the other hand, equation (3.15) implies that $N^{-1} = G|_{\operatorname{ran}(N)}$ and, since G is continuous and $\operatorname{ran}(N)$ is closed in ℓ^1, $\operatorname{gra}(N)$ is (strongly-)closed in $c_0 \times \ell^1$. Therefore, the function $\delta_N + \pi : c_0 \times \ell^1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ is proper, convex and (strongly-)lower semi-continuous, and it represents N. Thus N is representable in the strong topology of $c_0 \times \ell^1$. In conclusion,

$$\operatorname{cl}_{s\times s-\mathcal{R}}(N) = N \subsetneq N^{\mu\mu}.$$

Now we deal with the weak×weak∗ topology. For every $m \in \mathbb{N}$, define $e^m = (e^m_i)_i$, as

$$e^m_i = \begin{cases} 1 & i = m,\\ 0 & \text{otherwise}.\end{cases}$$

Clearly $e^m \in \ell^1 \subset c_0 \subset \ell^\infty$, for all $m \in \mathbb{N}$. Is also clear, from (2.2) and (3.16), that $(e_1, e_1) \in N^\mu$.

Define, for $\varepsilon > 0$, the neighborhoods of e^1 in the weak and weak∗ topologies of c_0 and ℓ^1, respectively,

$$\begin{aligned} V &= \{x \in c_0 \mid |\langle x - e^1, e^i\rangle| < \varepsilon,\ i = 1,2,3\},\\ W &= \{x^* \in \ell^1 \mid |\langle e^i, x^* - e^1\rangle| < \varepsilon,\ i = 1,2,3\},\end{aligned}$$

so $V \times W$ is a weak×weak∗ neighborhood of (e_1, e_1). Note that a generic element of W has the form

$$x^* = (1 + \varepsilon_1, \varepsilon_2, \varepsilon_3, x^*_4, x^*_5 \ldots),$$

where $|\varepsilon_i| < \varepsilon$, for $i = 1, 2, 3$.

We will prove that, for $\varepsilon \in (0, 1)$, $V \times W$ does not intersect N. Suppose, on the contrary, that there exists $(x, x^*) \in (V \times W) \cap N$. Therefore, $x = G(x^*)$,

$x^* \in W \cap \operatorname{ran}(N)$ and we can write $x^* = (1+\varepsilon_1, \varepsilon_2, \varepsilon_3, x_4^*, x_5^*, \ldots)$, where

$$\langle x^*, e\rangle = 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \sum_{k\geq 4} x_k^* = 0.$$

Using the fact that $x = G(x^*)$, and writing $x = (x_1, x_2, x_3, \ldots)$, we obtain

$$\begin{aligned} x_1 &= -1 - \varepsilon_1, \\ x_2 &= -2 - 2\varepsilon_1 - \varepsilon_2, \\ x_3 &= -2 - 2\varepsilon_1 - 2\varepsilon_2 - \varepsilon_3. \end{aligned}$$

Since $x \in V$,

$$|\langle x - e^1, e^1\rangle| = |2 + \varepsilon_1| < \varepsilon,$$

which implies

$$2 - \varepsilon < 2 - |\varepsilon_1| \leq |2 + \varepsilon_1| < \varepsilon.$$

This is a contradiction to the fact that $\varepsilon < 1$. Therefore $(V \times W) \cap N = \emptyset$ and, thus, $N^{\mu\mu} \neq \overline{N}^{w\times w^*}$. Using the same argument as in the strong case, $\delta_{\overline{N}^{w\times w^*}} + \pi$ is a proper, convex and weak$\times$weak$*$-lower semi-continuous function, so $\overline{N}^{w\times w^*}$ is weak$\times$weak$*$ representable. Therefore

$$N \subset \operatorname{cl}_{w\times w^* - \mathcal{R}}(N) \subset \overline{N}^{w\times w^*} \subsetneq N^{\mu\mu}.$$

This implies that the representable closure of N in any topology finer than the weak$\times$weak$*$ topology is properly contained in its monotone polar closure, in particular, the representable closure in the strong$\times$weak$*$ topology. In addition, Theorem 3.5 implies that N^μ is non-enlargeable.

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