impa Instituto Nacional de Matemática Pura e Aplicada

ABSOLUTE CONTINUITY FOR DIFFEOMORPHISMS WITH NON-COMPACT CENTER LEAVES

José Régis Azevedo Varão Filho

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Advisor: Marcelo Miranda Viana da Silva

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Abstract

We study the measure-theoretical properties of center foliations of volume preserving partially hyperbolic diffeomorphisms with one-dimensional center direction. Recent work of Avila, Viana, Wilkinson [2] dealt with situations where the center leaves are compact or can be compactified in a suitable way. Using different techniques we focus on the non-compact case and obtain very different conclusions.

For one thing, in our context the disintegration of volume may be neither atomic nor Lebesgue. Such examples are found even among Anosov diffeomorphisms. An important tool in this setting is to prove that the disintegration is atomic if and only if the center leaves form a Rokhlin measurable partition. Moreover, even an Anosov may have absolutely continuous center foliation without being C^1-conjugate to its linearization.

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1 Introduction

We study the measure-theoretical properties of the center foliation of partially hyperbolic diffeomorphisms for which the center leaves are non-compact. Two main issues are:

- absolute continuity: when is the center foliation absolutely continuous? What can be said otherwise?
- rigidity: does absolute continuity imply greater regularity?

These issues are fairly well understood in situations when the center leaves are "essentially compact" (it includes perturbations of certain skew-products or of time-one maps of Anosov flows), by recent work of Avila, Viana, Wilkinson [2]:

- *Atomic disintegration*: If the center foliation is non-absolutely continuous, then there exists $k \in \mathbb{N}$ and a full volume subset that intersects each center leaf on exactly k points/orbits.
- *Rigidity*: If the center foliation is absolutely continuous then the diffeomorphism is smoothly conjugate to a rigid model (a rotation extension of an Anosov diffeomorphism or the time-one map of an Anosov flow);

We deal with a different class of partially hyperbolic diffeomorphisms, whose center leaves are non-compact: DA (derived from Anosov) diffeomorphisms, that is, that lie in the isotopy class of some hyperbolic linear automorphism (we refer to these automorphisms as the linearization of the DA diffeomorphisms). The examples exhibited in the results that follow are, actually, Anosov maps, with the weak (stable or unstable) foliation as the center foliation. We also mention that all diffeomorphisms treated on this work are assumed to be at least $C^{1+\alpha}$. This means that volume preserving Anosov on $\mathbb{T}^3$ are all ergodic.

For non-absolutely continuous foliations, our conclusion is different from [2]. Indeed, we show that it is possible to have a disintegration which is non-Lebesgue and non-atomic. This is the first example of this kind:

Theorem A. *There exist volume preserving Anosov diffeomorphisms on $\mathbb{T}^3$ for which the center foliation is non-absolutely continuous and the disintegration of volume on center leaves are not atomic.*

In fact, such diffeomorphisms fill a dense subset of an infinite-dimensional manifold in the neighborhood of any hyperbolic linear automorphisms in the space of volume preserving maps.

The next result shows, in contrast with the compact setting of [2], that absolute continuity has no rigidity implications in our case:

Theorem B. *There exist volume preserving Anosov diffeomorphisms f on $\mathbb{T}^3$ for which the center foliation is absolutely continuous but f is not C^1-conjugate to its linearization.*

In fact, such diffeomorphisms fill a dense subset of an infinite-dimensional manifold in the neighborhood of any hyperbolic linear automorphism in the space of volume preserving maps.

The proofs of these results are given in Sections 3 and 4. In the remaining of this Introduction we outline the structure of the arguments and also present a number of related results. Section 2 contains some background material.

As already mentioned, the partially hyperbolic diffeomorphisms treated by Avila, Viana, Wilkinson [2] have atomic disintegration when the center foliation is non-absolutely continuous. For partially hyperbolic diffeomorphisms with non-compact center leaves there was no information on the nature of these conditional measures when the center foliation is non-absolutely continuous. Some understanding of the behavior of these conditional measures is presented on the following theorems, proved on section 3.

Theorem C. *Let f be a volume preserving Anosov diffeomorphism on $\mathbb{T}^3$, such that the center foliation is non-absolutely continuous. Then*

i) the conditional measures are singular measures with respect to the volume on the center leaf;

ii) if the decomposition is atomic, then there is exactly one atom per leaf. That is, there exists a set of full volume that intersects each center leaf in one point.

We say that a partition $\mathcal{P}$ is measurable with respect to volume if there exist a Borelian family $\{A_i\}_{i\in\mathbb{N}}$ and a set of zero volume F such that for all $P \in \mathcal{P}$, $P \cap F = \bigcap_{i\in\mathbb{N}} B_i \cap F$, where $B_i \in \{A_i, A_i^c\}$.

It turns out that measurability of a foliation is the right condition for atomicity of the conditional measures.

Theorem D. *Let f be a volume preserving Anosov diffeomorphism on $\mathbb{T}^3$. The disintegration of volume is atomic if and only if the partition by center leaves is a measurable partition.*

Hence, to prove Theorem A we just need to construct an Anosov diffeomorphism with non-absolutely continuous foliation for which the partition by center leaves is not a measurable partition.

We begin section 4 studying the implications of the C^1-conjugacy.

Let h be a diffeomorphism commuting with two Anosov diffeomorphisms g and f, that is $f \circ h = h \circ g$, iterating we get $f^n \circ h = h \circ g^n$. Differentiating the former expression on a periodic point p, $g^n(p) = p$, we get $Df^n(h(p))Dh(p) = Dh(g^n(p)) \circ Dg^n(p)$. Hence

$$Dg^n(p) = (Dh(p))^{-1} Df^n(h(p)) Dh(p).$$

The similarity between the matrices $Dg^n(p)$ and $Df^n(h(p))$ on their respective periodic points is a necessary condition for C^1 conjugation. It turns out that on $\mathbb{T}^3$ this is also a sufficient condition by Gogolev, Guysinsky [9]. We improve their result, as follows:

Theorem E. *Let f be a volume preserving Anosov diffeomorphism on $\mathbb{T}^3$. Then, for all $p, q \in Per(f)$, $\lambda^*(p) = \lambda^*(q)$, $* \in \{s, c, u\}$ if and only if f is C^1 conjugate to its linearization.*

The strategy to prove Theorem B is to begin with a linear Anosov, keep one of the exponents (which will give us the absolute continuity by the work of Gogolev [8]) and change another exponent (to break the C^1-conjugacy, by Theorem E).

Although we have shown that absolute continuity of the center foliation does not imply a rigid condition, we obtain a rigidity theorem imposing stronger conditions on the center foliation:

Theorem F. *Let f be a volume preserving DA diffeomorphism on $\mathbb{T}^3$, with the linearization A of the form $E^s \oplus E^{wu} \oplus E^{uu}$. If the center foliation is a C^1 foliation and the center holonomies inside the leaves $\mathcal{F}_f^{wu+uu}$ and $\mathcal{F}_f^{s+wu}$ is uniformly bounded, then f is C^1 conjugate to its linearization and, hence, is an Anosov diffeomorphism.*

Assuming that the center foliation is C^1 and the bounded condition on the center holonomies, we construct conditional measures, $\{m_x\}_{x \in \mathbb{T}^3}$, on the center foliation with some dynamical meaning. That is, the push forward of a conditional measure satisfies $f_* m_x = \lambda^{-1} m_{f(x)}$, where λ is the central exponent of the linearization. By iteration

$$\frac{df_*^n m_x}{dm_{f^n(x)}} = \lambda^{-n}.$$

Since $\lambda_f^c(x) = \lim_{n\to\infty} \frac{df_*^n m_x}{dm_{f^n(x)}}$, the above relation implies that the central Lyapunov exponent equals λ. As the center foliation is C^1, these conditional measures are defined everywhere. Consequently, the center exponent is defined everywhere and equals λ. Therefore, f is an Anosov diffeomorphism. We apply Theorem E to get the C^1 conjugation, thus proving Theorem F.

2 Prerequisites

The reader may consult this section when needed, we mainly state the results that shall be used throughout this work. We make an exception to the subsection on absolute continuity. Since this is a key concept we give a few more words concerning this topic on subsection 2.6.

2.1 Measure theory

For some basic definitions of measure theory the reader may check Rudin's book [16].

Definition 2.1. Let X be a metric space and μ a Borel probability measure. Then μ is a regular measure if for any measurable set B and $\varepsilon > 0$ given, there are compact set $K_\varepsilon \subset B$ and open set $U_\varepsilon \supset B$ such that $\mu(B - K_\varepsilon) < \varepsilon$ and $\mu(U_\varepsilon - B) < \varepsilon$.

Proposition 2.1. *A Borel probability measure μ on a metric space X is regular.*

Theorem 2.2 (Lusins' theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a Borel probability space and $f : X \to \mathbb{R}$ a measurable function. Then given $\varepsilon > 0$, there exists a set $A_\varepsilon \subset X$ such that $\mu(A) \geq 1 - \varepsilon$ and f restricted to the set A is a continuous function.*

2.2 Decomposition of measure

Let $(M, \mu, \mathcal{B})$ be a probability space, where M is a compact metric space, μ a probability and $\mathcal{B}$ the borelian σ-algebra. Given a partition $\mathcal{P}$ of M by measurable sets, we associate the measurable set

$$(\mathcal{P}, \widetilde{\mu}, \widetilde{\mathcal{B}})$$

by the following way. Let $\pi : M \to \mathcal{P}$ be the canonical projection associate to a point of M the partition element that contains it. Then we define $\widetilde{\mu} := \pi_*\mu$ and $\widetilde{\mathcal{B}} := \pi_*\mathcal{B}$.

Definition 2.3. Given a partition $\mathcal{P}$. A family $\{\mu_P\}_{p\in\mathcal{P}}$ is a *system of conditional measures* for μ (with respect to $\mathcal{P}$) if

i) given $\phi \in C^0(M)$, then $P \mapsto \int \phi\mu_P$ is measurable;

ii) $\mu_P(P) = 1$ $\widehat{\mu}$-a.e.;

iii) if $\phi \in C^0(M)$, then $\displaystyle\int_M \phi d\mu = \int_{\mathcal{P}} (\int_P \phi d\mu_P) d\widetilde{\mu}$.

Observe that the conditions *i*) and *iii*) also hold for bounded ϕ by the Dominated Convergence theorem. When it is clear which partition we are referring to, we say that the family $\{\mu_P\}$ *disintegrates* the measure μ.

Proposition 2.2. *If $\{\mu_P\}$ and $\{\nu_P\}$ are conditional measures that disintegrate μ, then $\mu_P = \nu_P$ $\widetilde{\mu}$-a.e.*

Corollary 2.1. *If $T : M \to M$ preserves a probability μ and the partition $\mathcal{P}$, then $T_*\mu_P = \mu_P$ $\widetilde{\mu}$-a.e.*

Proof. It follows from the fact that $\{T_*\mu_P\}_{P\in\mathcal{P}}$ is also a disintegration of μ. □

Definition 2.4. (Measurable partition) We say that a partition $\mathcal{P}$ is measurable if there exists a borelian family $\{A_i\}_{i\in\mathbb{N}}$ such that

$$\mathcal{P} = \{A_1, A_1^c\} \vee \{A_2, A_2^c\} \vee \ldots \mathrm{mod}\ 0.$$

Theorem 2.5 (Rokhlin' disintegration [18])**.** *Let $\mathcal{P}$ be a measurable partition of a compact metric space M and μ a borelian probability. Then there exists a disintegration by conditional measures for μ.*

2.3 Anosov diffeomorphism

We collect a series of results concerning Anosov diffeomorphism, we restrict to the results needed on forthcoming sections.

Consider $f : M \to M$ a diffeomorphism on a Riemannian manifold M.

A closed set $\Lambda \subset M$ invariant by f is called a **hyperbolic set** if there exist $C > 0$, $\lambda \in (0, 1)$ and for every $x \in \Lambda$ there are $E^s(x)$, $E^u(x) \subset T_xM$ such that

1. $T_xM = E^s(x) \oplus E^u(x)$;

2. $||df_x^n v^s|| \leq C\lambda^n ||v^s||$, $\forall v^s \in E^s(x)$ e $n \geq 0$;

3. $||df_x^{-n} v^u|| \leq C\lambda^n ||v^u||$, $\forall v^u \in E^u(x)$ e $n \geq 0$;

4. $df_x E^s(x) = E^s(f(x))$ e $df_x E^u(x) = E^u(f(x))$.

We say that a set Λ, as above, is an **isolated hyperbolic set** if there exists a neighborhood $\mathcal{U}$ of Λ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\mathcal{U}).$$

We call f an **Anosov diffeomorphism** if $\Lambda = M$, in particular M is an isolated hyperbolic set.

We say that a sequence $\{x_i\}$ on M is a δ-**pseudo orbit** for f if $d(f(x_i), x_{i+1}) \leq \delta$. A point $y \in M$ ε-**shadows** a sequence $\{x_i\}$ if $d(f^i(y), x_i) \leq \varepsilon$. Although we'll be using the next result for Anosov system we state it on its classical version.

Theorem 2.6 (Shadowing lemma)**.** *Let $\Lambda \subset M$ be a hyperbolic set for f. Then, given $\varepsilon > 0$ there exist $\eta, \delta > 0$ such that if $\{x_i\}_{i=j_1}^{j_2}$ is a δ-pseudo orbit for f with $d(x_i, \Lambda) < \eta$, then there exists $y \in M$ with $d(y, \Lambda) < \eta$ such that y ε-shadows $\{x_i\}$. Furthermore,*

- *if* $j_1 = -\infty$ *and* $j_2 = \infty$*, then* y *is unique;*

- *if* $\{x_i\}_{i=j_1}^{j_2}$ *is periodic then* y *is a periodic point;*

- *if* Λ *is isolated, then* $y \in \Lambda$*.*

The above theorem proves the structural stability of Anosov systems, that is if f is an Anosov system and g is close enough to f, then there is a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$.

Since we can describe the stable (and unstable) manifold topologically, that is

$$W_f^s(x) = \{y \in M \mid d(f^n(x), f^n(y)) \to 0,\ n \to \infty\},$$

then the conjugacy h, which is a homeomorphism, satisfies

$$h(W_f^s(x)) = W_g^s(h(x)).$$

Since h is a homeomorphism it might not be able to distinguish velocities inside the unstable foliation. That is, suppose f is an Anosov diffeomorphism with splitting $E^s \oplus E^c \oplus E^{uu}$ and these distributions integrate to foliations, as well for every diffeomorphism g on a neighborhood of f. Although we know that $h(\mathcal{W}_f^u) = \mathcal{W}_g^u$ ($\mathcal{W}^u$ is tangent to $E^{wu} \oplus E^u$), we cannot guarantee that $h(\mathcal{W}_f^c) = \mathcal{W}_g^c$ ($\mathcal{W}^c$ is tangent to E^c). At least in some cases we are able to guarantee that center foliation goes to center foliation.

Proposition 2.3 (Gogolev, Guysinsky [9])**.** *Consider a linear Anosov diffeomorphism* A *on* $\mathbb{T}^3$ *and* $\mathcal{U}$ *a small neighborhood of* A*. If* $f \in \mathcal{U}$*, then the conjugacy* h *satisfies*

$$h(W_f^c) = W_A^c.$$

The shadowing lemma implies the existence of a homeomorphism that conjugates any two closed Anosov diffeomorphisms. In fact, it can be proved that the conjugation h is Hölder. It can't always be C^1, since there is a restricted relation with the periodic points (see below) of both diffeomorphism. But on $\mathbb{T}^3$ we have the following equivalence.

Theorem 2.7 ([9]). *Let f and g be two Anosov diffeomorphism and h a conjugacy between them, $h \circ f = g \circ h$.*

For all $x \in \mathbb{T}^3$ such that $f^p(x) = x$, $Df^p(x)$ and $Dg^p(h(x))$ are conjugated matrices if and only if h is a C^1 diffeomorphism.

We call this relation on the periodic points **periodic data**. Recall that two matrices A and B are conjugated if there is an invertible matrix C such that $A = C^{-1}BC$.

2.3.1 Equilibrium state for Anosov system

We give the classical results concerning equilibrium states for Anosov diffeomorphism. For the results the reader may check Bowen's book [7]. For the definition of metric entropy and topological entropy we refer the reader to the book of Walter [20].

Definition 2.8. Given a Hölder function $\psi : M \to \mathbb{R}$ (we shall call it as the potential ψ) we then define the pressure of f with respect to ψ as

$$P(f, \psi) = \sup_{\mu \in \mathcal{M}(f)} \{h_\mu(f) + \int \phi d\mu\},$$

where $\mathcal{M}(f)$ is the set of invariant probabilities for f and $h_\mu(f)$ is the metric entropy for f with respect to μ.

Observe that the pressure of the zero potential gives the topological entropy of the system: $P(f, 0) = h(f)$.

We say that an invariant measure μ for f is an *equilibrium state* for the potential ϕ if it satisfies the following equality

$$h_\mu(f) + \int \phi d\mu = P_f(\phi).$$

From now on we consider $f : M \to M$ a C^2 transitive Anosov diffeomorphism on a compact manifold M of any dimension.

Theorem 2.9. *If $\phi : M \to \mathbb{R}$ is a Hölder continuous function, then ϕ has a unique equilibrium state, μ_ϕ.*

The next theorem relates the equilibrium state for different potentials.

Theorem 2.10. *Let $\phi, \psi : M \to \mathbb{R}$ be two Hölder continuous functions. The following are equivalent*

- $\mu_\phi = \mu_\psi$*;*
- *there exists a constant K such that for all periodic points $x \in M$ we have the equality $\frac{1}{m}\sum_{i=1}^{m} \phi \circ f^n(x) - \frac{1}{m}\sum_{i=1}^{m} \psi \circ f^n(x) = K$.*

Theorem 2.11. *If f leaves invariant a probability measure μ absolutely continuous with respect to volume, then μ is the equilibrium state for the potential $\phi(x) = -log\phi^u(x)$, where ϕ^u is the Jacobian of $Df : E^u \to E^u$.*

Later on we shall consider volume preserving Anosov systems. By volume we mean a smooth measure, that is, let M be a Riemannian manifold and consider ω the volume induced by the metric. A smooth measure will be a measure of the form $\rho\omega$ where the density $\rho : M \to \mathbb{R}_+$ is a positive smooth function. The next theorem provides a condition for a smooth measure to be invariant by a transitive C^2 Anosov diffeomorphism.

Theorem 2.12. *Let f be a C^2 Anosov diffeomorphism, the following are equivalent:*

i) f admits an invariant measure μ absolute continuous to volume (induced by a Riemannian metric);

ii) $Df^n TM_x \to TM_x$ has determinant 1, for $f^n(x) = x$.

This is particularly important for us, because when considering transitive conservative Anosov diffeomorphism with a splitting $TM = E^s \oplus E^c \oplus E^u$, the above theorem gives us property ii), which implies that $\lambda^s(p) + \lambda^c(p) + \lambda^u(p) = 0$, where λ^* is the Lyapunov exponent. It means that determining two of them give us the other one.

2.4 Partially hyperbolic diffeomorphisms

We begin with the definition of partially hyperbolic diffeomorphisms, followed by some classical facts.

Definition 2.13. A diffeomorphism f of a compact Riemannian manifold M is called partially hyperbolic if there are constants $\lambda < \hat{\gamma} < 1 < \gamma < \mu$ and $C > 1$ and a Df -invariant splitting of $TM = E^u(x) \oplus E^c(x) \oplus E^s(x)$ where

$$\begin{aligned}
\frac{1}{C}\mu^n||v|| < \quad & ||Df^n v||, & & v \in E^s_x - \{0\}; \\
\frac{1}{C}\hat{\gamma}^n||v|| < \quad & ||Df^n v|| & < C\gamma^n||v||, \quad & v \in E^c_x - \{0\}; \\
& ||Df^n v|| & < C\lambda^n||v||, \quad & v \in E^u_x - \{0\}.
\end{aligned}$$

Theorem 2.14. *The stable and unstable subbundles E^s and E^u integrate uniquely into foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ which are transversely absolutely continuous.*

The reader may check [11, 12] for the proof of the above theorem. We say that a partially hyperbolic diffeomorphism is **dynamically coherent** if the subbundles $E^s \oplus E^c$ and $E^c \oplus E^u$ integrate into invariant foliations. This implies in particular that there is a center foliation $\mathcal{F}^c$, which is obtained by an intersection of the other two: $\mathcal{F}^c_x = \mathcal{F}^{sc} \cap \mathcal{F}^{cu}$.

It was proved by Brin, Buragov, Ivanov [5] that

Theorem 2.15. *Every partially hyperbolic diffeomorphism on $\mathbb{T}^3$ is dynamically coherent.*

We don't want to get into the discussion of the different types of definitions of partially hyperbolic, but we observe that we are working with the one known as *absolute partially hyperbolic*. Another notion is the *relative partially hyperbolic*. The difference is that in the absolute case we have uniformity on the comparison of contraction and expansions of the subbundles. These two notions are quite similar, on both we have stable and unstable foliations, transversely absolutely continuous. But to point out a difference, in the relative case it was constructed a non-dynamically coherent diffeomorphism [14].

We state Proposition 0.3 from [3], that shall be used later on for the proofs of theorems A and B.

Theorem 2.16. *Let f be a linear conservative Anosov on $\mathbb{T}^3$, seen as partially hyperbolic, and h be a C^r diffeomorphism volume preserving which preserves the direction relative to E^s or E^u. We assume h is C^1 close enough to the identity in order to have $f \circ h$ partially hyperbolic. Then*

$$\int_{\mathbb{T}^3} log J^c_{f \circ h}(x) dVol(x) \neq \int_{\mathbb{T}^3} log J^c_f(x) dVol(x).$$

2.5 Geometric property

By a Derived from Anosov (DA) diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ we mean a partially hyperbolic homotopic to a linear Anosov diffeomorphism A. We call this linear Anosov as the linearization of f. In fact, f is semi-conjugated to its linearization. The semi-conjugacy has good properties. The proof of the next result can be found on Sambarino [17].

Theorem 2.17. *Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear hyperbolic isomorphism. Then, there exists $C > 0$ such that if $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism such that $sup\{||G(x) - Bx|| \mid x \in \mathbb{R}^3\} = K < \infty$ then there exists $H : \mathbb{R}^3 \to \mathbb{R}^3$ continuous and surjective such that:*

- $B \circ H = H \circ G$;
- $||H(x) - x|| \leq CK$ *for all* $x \in \mathbb{R}^3$;
- $H(x)$ *is characterized as the unique point* y *such that*

$$||B^n(y) - G^m(x)|| \leq CK, \ \forall n \in \mathbb{Z};$$

- $H(x) = H(y)$ *if and only if* $||G^n(x) - G(y)|| \leq 2CK \forall n \in \mathbb{Z}$ *and if and only if* $sup_{n\in\mathbb{Z}}\{||G^n(x) - G^n(y)||\} < \infty$;
- *if* $B \in SL(3, \mathbb{Z})$ *and* G *is the lift of* $g : \mathbb{T}^3 \to \mathbb{T}^3$ *then* H *induces* $h : \mathbb{T}^3 \to \mathbb{T}^3$ *continuous and onto such that* $B \circ h = h \circ g$ *and* $dist_{C^0}(h, id) \leq C dist_{C^0}(B, g)$.

The geometrical property we shall need later is given by Hammerlindl [10]:

Proposition 2.4. *Let f be a partially hyperbolic and A be its linearization. Denote by $\tilde{f}$ and $\tilde{A}$ the lift to $\mathbb{R}^n$ of f and A respectively. Then for each $k \in \mathbb{Z}$ and $C > 1$ there is $M > 0$ and a linear map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x, y \in \mathbb{R}^n$*

$$||x - y|| > M \Rightarrow \frac{1}{C} < \frac{||\pi(\tilde{f}^k(x) - \tilde{f}^k(y))||}{||\pi(\tilde{A}^k(x) - \tilde{A}^k(y))||} < C.$$

2.6 Absolute continuity

We show two notions of absolute continuity, namely leafwise and transverse absolute continuity. The tranverse one has appeared in the proof of ergodicity of certain Anosov systems via Hopf's argument. It has been noted that leafwise absolute continuity is as well appropriate. We are going to use the weaker one, leafwise absolute continuity, when referring to just absolute continuity.

Definition 2.18. A foliation $\mathcal{F}$ is called

- *lower leafwise* absolutely continuous if $vol_L << m_L$;
- *upper leafwise* absolutely continuous if $m_L << vol_L$;

where L is a leaf of the foliated box, m_L the conditional measure of the volume measure m and vol_L the Riemannian volume restricted to the leaf L.

Proposition 2.5. *A foliation $\mathcal{F}$ is*

- *lower leafwise absolutely continuous if for every set Y such that $m(Y) = 0$, then for almost every $z \in M$ the leaf L through z meets Y in a set of zero Lebesgue measure;*
- *upper leafwise absolutely continuous if given a measurable set Y such that $vol_L(Y) = 0$ for every leaf L through a full measure subset of points $z \in M$ implies $m(Y) = 0$.*

Therefore a foliation is called **leafwise absolutely continuous** if it is upper and lower leafwise absolutely continuous. Which, of course, is equivalent to have the conditional measures equivalent to the volume on the leaf.

Definition 2.19. A foliation $\mathcal{F}$ is transversely absolutely continuous if given two differentiable transversal sections Σ_1 and Σ_2 to the foliation, the holonomy map $h : \Sigma_1 \to \Sigma_2$ is absolutely continuous.

Theorem 2.20. *The stable and unstable foliations for a partially hyperbolic are transversely absolutely continuous with bounded Jacobians.*

Theorem 2.21. *Transverse absolute continuity implies leafwise absolutely continuity.*

Proof (Brin, Stuck [6]): Let $\mathcal{F}$ be the transverse absolutely continuous foliation. Let T be a transversal to the foliation. Let us look on a foliated box such that the foliations are horizontal and the transversal is vertical, $T = h(x_0, I)$. Let $\mathcal{T}$ be a smooth foliation having $T = \mathcal{T}(x_0)$ as a leaf. Since $\mathcal{T}$ is smooth we can integrate the volume as

$$m(A) = \int_{\mathcal{F}(x_0)} \int_{\mathcal{T}(x)} \chi_A(x,y)\Gamma(\mathcal{T}(x), y)\ dm_{\mathcal{T}(x)}(y)\ dm_{\mathcal{F}(x_0)}(x)$$

where $\Gamma(\mathcal{T}(x), y)$ are the densities for the conditional measure on $\mathcal{T}$, since the foliation is smooth these are continuous densities. Let $h_x : \mathcal{T}(x_0) \to \mathcal{T}(x)$ be the holonomy through the foliation $\mathcal{F}$. Then

$$\int_{\mathcal{T}(x)} \chi_A(x,y)\Gamma(\mathcal{T}(x), y)\ dm_{\mathcal{T}(x)}(y) =$$

$$= \int_{\mathcal{T}(x_0)} \chi_A \circ h_x(x,\xi) Jac\ h_x(\xi) \Gamma \circ h_x(\mathcal{T}(x), \xi)\ dm_{\mathcal{T}(x_0)}(\xi).$$

We substitute this equality in the former equality and change, by Fubini, the order of the integral. Recalling that we can do this since $\mathcal{T}$ is smooth. Since $\mathcal{T}(x_0) = T$, $m(A) =$

$$= \int_T \int_{\mathcal{F}(x_0)} \chi_A \circ h_x(x,\xi) Jac\ h_x(\xi) \Gamma \circ h_x(\mathcal{T}(x), \xi)\ \ dm_{\mathcal{F}(x_0)}(x) dm_{\mathcal{T}(x_0)}(\xi)$$

we calculate $\int_{\mathcal{F}(x_0)} \chi_A \circ h_x(x,\xi) Jac\ h_x(\xi)\Gamma \circ h_x(\mathcal{T}(x),\xi)\ \ dm_{\mathcal{F}(x_0)}(x)$ in terms of $\mathcal{F}(\xi)$ by repeating the process analogously. That is, we now have to look for the holonomy from $\mathcal{F}(\xi) \to \mathcal{F}(x_0)$. $\square$

Proposition 2.6. *If $\mathcal{F}$ is a leafwise absolutely continuous foliation, then for every transversely absolute continuous foliation $\mathcal{T}$ which is also a transversal local foliation to $\mathcal{F}$, the local $\mathcal{F}$-holonomy map $h_{\mathcal{F}}$ between m-almost every pair of $\mathcal{T}$-leaves is transverse absolutely continuous.*

Proof (Pugh, Viana, Wilkinson [13]): Fix a leaf F_0 of $\mathcal{F}$, identify this leaf with $\mathbb{R}^n$. Consider the group $G = \mathbb{R}^n$ acting on F_0 by translations. Let $\mathcal{T}$ be as in the hypothesis, then each element of G gives rise to a homeomorphism h_g in the ambient space $\mathcal{U}$ (of the foliation box) by $\mathcal{F}$-holonomies such that the restriction of h_g to $\mathcal{T}_x$ is the $\mathcal{F}$-holonomy to $\mathcal{T}_{gx}$.

Since $\mathcal{F}$ is leafwise absolutely continuous

$$m(A) = \int_{\mathcal{T}_0} \int_{\mathcal{F}_y} 1_A(x,y)\rho(x,y)\ dm_{\mathcal{F}_y}(x) d\hat{m}(y),$$

then

$$\begin{aligned} m(h_g(A)) &= \int_{\mathcal{T}_0} \int_{\mathcal{F}_y} 1_A \circ h_g^{-1}(x,y)\rho(x,y)\ dm_{\mathcal{F}_y}(x) d\hat{m}(y) \\ &= \int_{\mathcal{T}_0} \int_{\mathcal{F}_y} 1_A(x,y)\rho \circ h_g^{-1}(x,y)\ d(h_g^{-1})_* m_{\mathcal{F}_y}(x) d\hat{m}(y). \end{aligned}$$

Since $(h_g^{-1})_* m_{\mathcal{F}_y} = m_{\mathcal{F}_y}$, using the formulas above, if $m(A) = 0$ then $m(h_g(A)) = 0$. That is, h_g is absolutely continuous.

Lemma 2.1. *For every $g \in G$ there exists a set $X_g \subset \mathcal{F}_0$ of full measure, such that for all $x \in X_g$ the restriction of h_g to $\mathcal{T}_x$ is absolutely continuous.*

Proof. By contradiction, there exists a $m_{\mathcal{F}_0}$-measure set $B \subset \mathcal{F}_0$ such that for all $x \in B$ there is a set $Z \subset \mathcal{U}$ such that $m(Z) = 0$ and $m(h_g(Z)) \neq 0$.

But we observed before that h_g is absolutely continuous, therefore $m(h_g(Z))$ should be zero. A contradiction. □

Let us finish the proof of the proposition.

In the space $\mathcal{F}_0 \times \mathcal{F}_0$ we have a foliation $\mathcal{W}$ which leaf through a point $(x_0, g(x_0))$ is an affine space given by

$$\mathcal{W}_g = \{(x, g(x)) \mid x \in \mathcal{F}_0\}.$$

By the above lemma, each leaf of $\mathcal{W}$ has a full measure set such that any (x, x') in this set has the property that the $\mathcal{F}$-holonomy between $\mathcal{T}_x$ and $\mathcal{T}_{x'}$ is absolutely continuous. Fubini implies that the union of these sets have full measure. That is, $m \times m$-a.e. (x, y) the $\mathcal{F}$-holonomy between $\mathcal{T}_x$ and $\mathcal{T}_y$ is absolutely continuous.

□

Example 2.22. Leafwise absolute continuity, but not transverse absolutely continuous. We present the Example 6.1 from Viana, Yang [19], which is based on an example of Kan (see [4], Chapter 11). For more details check the references therein.

Consider a map $f_0 : S^1 \times [0,1] \to S^1 \times [0,1]$, $f_0(x,t) = (2x, g(t))$ preserving the boundaries (i.e. $g(0) = 0$ and $g(1) = 1$), $g(t) < t$ for $t \in (0,1)$, and $0 < g'(t) < 2$ for $t \in [0,1]$. Therefore f_0 is a partially hyperbolic with the vertical leaves as center foliations. We perturb f_0 in such a way that

$$\frac{\partial f}{\partial x}(0,0) \neq \frac{\partial f}{\partial x}(0,1).$$

This perturbed map f still has a center foliation by [11]. The property above implies that the conjugacy (which is given by the holonomy of the center foliations through the boundaries) of the boundaries is not absolutely continuous. Therefore this center foliation is not absolutely continuous. On the other hand g is taken in such a way that the lower boundary is an attractor and the perturbation f still has $S^1 \times [0,1)$ as its basin of attraction. By Pesin's theory it implies that on $S^1 \times [0,1)$ the center foliations is transverse absolutely continuous. Which implies the center foliation to be leafwise absolutely continuous by Proposition 2.6.

Finally we state a result due to Gogolev [8] which shall be our starting point to understand absolute continuity for partially hyperbolic diffeomorphism with non-compact center leaves.

Theorem 2.23. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov diffeomorphism with splitting of the form $E^s \oplus E^{wu} \oplus E^{uu}$, then $\mathcal{F}^c_f$ is absolutely continuous if and only $\lambda^{uu}(p) = \lambda^{uu}(q)$ for all periodic points p and q.*

3 Non-Absolute Continuity of $\mathcal{F}^c$

On this section we are interested on the behavior of the center foliation, for when it is non-absolutely continuous. We begin by proving Theorem C.

Proof of Theorem C. By ergodicity we know that the Birkhoff set

$$B = \{x \in \mathbb{T}^3 \mid 1/n \sum_{i=0}^{n-1} \delta_{f^i(x)} \to Vol \text{ as } n \to \infty\}$$

has full measure.

Lemma 3.1. *If there is a center leaf such that $\mathcal{F}^c \cap B$ has positive Lebesgue measure, then the center foliation is absolutely continuous.*

Proof. Let D be any disc on the central foliation and consider the following construction

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(\frac{m_D}{m_D(D)}),$$

where m_D means the Lebesgue measure on the central leaf. It turns out that these measures converge to a measure μ such that the disintegration of μ on the center leaves are absolutely continuous with respect to the Lebesgue measure. This is a well known construction of such measures, studied by Pesin, Sinai in the eighties. For more references see [4] Chapter 11 and the references therein. Although Pesin, Sinai studied these measures for the case of the disc D in the unstable foliation, for the center foliation, in our case, this construction is the same. Gogolev, Guysinsky [9] have worked explicitly on this case and the reader may check at [9] the construction.

We consider a slightly different construction, we consider instead of the disc D, as above, we consider the disc $D \cap B$ for which it has positive Lebesgue measure on the center leaf, by hypothesis there exists such a disc. It turns out that these measures still converge to a measure with conditional measures absolutely continuous to the Lebesgue measure on the center leaf (Lemma 11.12 [4]). Since the points on B have the property $1/n \sum_{i=0}^{n-1} \delta_{f^i(x)} \to Vol$, it turns out that the sequence μ_n converges to volume. Hence, volume has Lebesgue disintegration on the center leaves. $\square$

By the above lemma, since we are in the case where the center foliation is non-absolutely continuous, we must have that the center foliation intersects B on a set of zero Lebesgue measure. But the conditional measures give full measure to B, since B has full measure. Therefore the conditional measures are singular with respect to the Lebesgue measure. And item i) is proved.

For item ii), suppose that the disintegration of volume on center leaves is atomic. Since f is Anosov, consider $\{R_1, \ldots, R_k\}$ a Markov partition of $\mathbb{T}^3$. Let us suppose that $\mathcal{F}^c$ is expanding, the analogous argument works for the contracting case. Note that since center leaf goes to center leaf, a Markov property implies that $f(\mathcal{F}^c_{R(x)}(x)) \supset \mathcal{F}^c_{R(f(x))}(f(x))$.

The following lemmas conclude the proof of item ii).

Lemma 3.2. *All the atoms have the same weight when considering the disintegration of volume on the center leaves of R_i.*

Proof. On each Markov rectangle we may apply Rokhlin's disintegration theorem on center leaves. Therefore, when writing m_x we mean the conditional measure for the disintegration on Markov rectangle that contains x. Consider the set $A_\delta = \{x \in A \mid m_x(x) \leq \delta\}$. Since $f(\mathcal{F}^c_{R(x)}(x)) \supset \mathcal{F}^c_{R(f(x))}(f(x))$, we have that $f_* m_x(I) \leq m_{f(x)}(I)$ where I is inside the connected component of $\mathcal{F}^c_{f(x)} \cap R(f(x))$ that contains $f^n(x)$. If $f(x) \in A_\delta$, then

$$m_x(x) = f_* m_x(f(x)) \leq m_x(f(x)) \leq \delta.$$

Hence, $f^{-1}(A_\delta) \subset A_\delta$.

By ergodicity, since our Anosov is volume preserving on $\mathbb{T}^3$, A_δ has full measure or zero measure. Let δ_0 be the discontinuity point of the function $\delta \in [0,1] \mapsto Vol(A_\delta)$. This implies that almost every atom has weight δ_0. $\square$

Lemma 3.3. *On every Markov partition R_i the conditional measures have the same number of atoms, with the same weight.*

Proof. This is a direct consequence from the above lemma. Since all the atoms have the same weight δ_0 the conditional measures must have $1/\delta_0$ number of atoms. $\square$

Lemma 3.4. *There is a set of full volume B_1, of atoms, such that if $x \in B_1$, then $B_1 \cap \mathcal{F}^c_x$ is contained in the connected component of $R_{i(x)} \cap \mathcal{F}^c_x$ that contains x.*

Proof. Let A be the set of atoms and T be the set of transitive points. Both sets have full volume measure by ergodicity. Suppose, by contradiction, that there is a subset $A_1 \subset A$ of positive volume measure such that $\forall x \in A_1$ we get $A \cap R^c_{i(x)} \neq \emptyset$, where $R^c_{i(x)}$ is the complement of the Markov partition that contains x, note that $Vol(A_1 \cap T) > 0$. Define the following map

$$\begin{aligned} h : A_1 \cap T &\to \mathbb{R} \\ x &\mapsto h(x) = d_{\mathcal{F}^c_x}(R_{i(x)}, R'_{i(x)}), \end{aligned}$$

where $d_{\mathcal{F}^c_x}(R_{i(x)}, R'_{i(x)})$ means the distance inside the center leaf of the Markov rectangle $R_{i(x)}$ to the closest Markov rectangle that has an atom which we call $R'_{i(x)}$.

Since h is a measurable map, there exists $K_1 \subset A_1 \cap T$, with $Vol(K_1) > 0$ for which h is a continuous map when restricted to K_1. And since volume is a regular measure, there is compact set $K_2 \subset K_1$, also with positive volume measure.

Let $\alpha = Max_{x \in K_2} h(x)$. Fix $z_0 \in R_{i(z_0)}$, and consider a ball small enough such that $B(z_0, r) \subset int R_{i(z_0)}$. Hence, $\forall y \in K_2$, let $n_y \in \mathbb{N}$ be an integer big enough so that, since f is uniformly expanding in the center direction, $f^{-n_y}(\mathcal{F}^c(y, \alpha)) \subset B(z_0, r) \subset int R_{i(z_0)}$.

It means that we have at least doubled the number of atoms inside $R_{i(z_0)}$, which is an absurd since we have already shown that the number of atoms are constant on each Markov partition. □

Lemma 3.5. *There is a set of full volume $B_2 \subset B_1$ such that the center foliation intersects B_2 at most on one point.*

Proof. By contradiction suppose that the number of atoms on all Markov partition are greater than one. Let A_2 be a set with full volume measure inside the union of the Markov rectangle such that if $x \in A_2$, then $A_2 \cap \mathcal{F}^c_{x,loc}$ has the same number of points, in this case greater than one. Where $F^c_{x,loc}$ is the connected set of the center foliation restricted to the Markov rectangle that intersects x. We define the map

$$\begin{array}{rcl} h : A_2 & \to & \mathbb{R} \\ x & \mapsto & h(x) \end{array}$$

where $h(x)$ is the smallest distance between the atoms of $\mathcal{F}^c_{x,loc}$. By Lusin's theorem there is a set $K_1 \subset A_2$ of positive measure for which h is continuous. Since volume is regular, there is a compact subset K_2 of K_1 with positive measure. Let $\alpha = \min_{x \in K_2} h(x)$.

Let $\beta > 0$ be an inferior bound for the length of $\mathcal{F}^c_{loc}$. Let $n_0 \in \mathbb{N}$ big enough so that any segment of a center leaf with length greater than or equal to α has the length of its n_0th iterate greater than β. This means that $f^{n_0}(K_2)$, which has positive measure, have all the atoms separated from each other with respect to the Markov partition. Since we have a finite number of Markov partition, one of them must have a set with positive measure such that its leaves have only one atom. Hence all Markov partition must have one atom, absurd. □

□

Still on the same context, of volume preserving Anosov on $\mathbb{T}^3$, we have:

Theorem 3.1. *The disintegration of volume is atomic if and only if the partition by center leaves is a measurable partition.*

Proof. Suppose $\{\mathcal{F}^c_x\}_{x\in M}$ is a measurable partition, then we can apply Rokhlin's theorem and we decompose volume on probabilities m_x on center leaves. Let

$$A_L = \{x \in M \mid m_x(\mathcal{F}^c_L(x)) \geq 0.6\},$$

where $\mathcal{F}^c_L(x)$ is the segment of $\mathcal{F}^c(x)$ of length L on the induced metric and centered at x.

Note that there is $L \in \mathbb{R}$ such that $vol(A_L) > 0$. Let us suppose that f contracts the center leaf, then $f^{-1}(\mathcal{F}^c_L(f(x))) \supset \mathcal{F}^c_L(x)$. Since $f_* m_x = m_{f(x)}$, for $x \in A_L$,

$$m_{f(x)}(\mathcal{F}^c_L(f(x))) = m_x(f^{-1}(\mathcal{F}^c_L(f(x)))) \geq m_x(\mathcal{F}^c_L(x)) \geq 0.6.$$

So $f(x) \in A_L$, by ergodicity $f(A_L) \subset A_L$ implies $Vol(A_L) = 1$.

Claim: $diam^c A_L \cap \mathcal{F}^c_x \leq 2L$, where $diam^c$ means the diameter of the set inside the center leaf.

Suppose there exist $y_1, y_2 \in A_L \cap \mathcal{F}^c_x$ with $d^c(y_1, y_2) > 2L$. Then

$$\mathcal{F}^c_L(y_1) \cap \mathcal{F}^c_L(y_2) = \emptyset \text{ and } m_x(\mathcal{F}^c_L(y_i)) \geq 0.6, \ i = 1, 2.$$

Then

$$1 \geq m_x(\mathcal{F}^c_L(y_1) \cup \mathcal{F}^c_L(y_2)) = m_x(\mathcal{F}^c_L(y_1)) + m_x(\mathcal{F}^c_L(y_2)) \geq 0.6 + 0.6 = 1.2.$$

This absurd concludes the proof of the claim.

Claim: The decomposition has atom.

Define

$$L_0 = inf\{L \in [0, \infty) \mid Vol(A_L) = 1\}.$$

Note that $Vol(A_{L_0}) = 1$, to see that take a sequence $L_n \to L_0$ and observe that $A_{L_0} = \cap_i A_{L_n}$. Let $\lambda = inf||Df^{-1}|E^c||$, let $\varepsilon < 1$ such that $\varepsilon\lambda > 1$. For $x \in A_{\lambda L_0}$

$$m_{f(x)}(\mathcal{F}^c_{\varepsilon L_0}(f(x))) = m_x(f^{-1}(\mathcal{F}^c_{\varepsilon L_0}(f(x)))) \geq m_x(\mathcal{F}^c_{L_0}(x)) \geq 0.6.$$

Therefore $f(x) \in A_{\varepsilon L_0}$. By ergodicity we may suppose A_{L_0} f-invariant, hence $Vol(A_{\varepsilon L_0}) = 1$. Absurd since $\varepsilon L_0 < L_0$. This means that $L_0 = 0$, which implies atom.

Let us prove the converse. Suppose we have atomic decomposition, we want to see that the partition through center leaves is a measurable partition.

Lift f to $\mathbb{R}^3$, by Hammerlindl [10] we may find a disk $\tilde{D}^2$ transverse to the center foliation, by quasi-isometry of the center foliation we may take this disk as big as we want. So take a disk such that its projection $D^2 = \pi(\tilde{D}^2)$ has the property:

$$\mathcal{F}^c_x \cap D^2 \neq \emptyset, \ \forall x \in \mathbb{T}^3.$$

Since the decomposition is atomic, we already know that it has one atom per leaf. Let us define the following set of full measure:

$$\hat{M} = \bigcup_{p \in A} \mathcal{F}^c_{loc}(p),$$

where A is the set of atoms, $\mathcal{F}^c_{loc}(p)$ is the segment of center leaf such that the right extreme point is p and the left extreme point is on D^2 and $\#\mathcal{F}^c_{loc}(p) \cap D^2 = 1$.

Since D^2 is a separable metric space, $\{\mathcal{F}^c_{loc}(p)\}_{p \in A}$ is a measurable partition for $\hat{M}$. Therefore we have a family of subsets $\{A_i\}_{i \in \mathbb{N}}$ of $\hat{M}$ such for all $p \in A$

$$\mathcal{F}^c_{loc}(p) = \bigcap_{i \in \mathbb{N}} B_i, \text{ where } B_i \in \{A_i, A_i^c\}.$$

□

3.1 Non-atomic singular measures

We are now ready to show the existence of a DA (in fact an Anosov) for which the disintegration of volume on the center leaves is neither Lebesgue nor atomic.

Proof of Theorem A. Consider a linear Anosov with the following split $TM = E^{ss} \oplus E^{ws} \oplus E^u$. Let ϕ be a volume preserving diffeomorphism which preserves the E^u direction. By Baraviera, Bonnatti [3] $\int \lambda^{ws}_A \neq \int \lambda^{ws}_{A\circ\phi}$. Let h be the conjugacy between A and f, $f \circ h = h \circ A$. Let us see that

Claim: $h_* Vol = Vol$.

Note that h is a conjugacy of A and f, then they have the same topological entropy λ^u_A. Hence, $h_* Vol$ is a measure of maximal entropy. Observe that the perturbation $A \circ \psi$ of A is such that it preserves the E^u direction, which means that the potentials 0 and $-log\lambda^u_f$ are cohomological and therefore gives the same equilibrium states. That is, $h_* Vol = Vol$.

We have already seen that on this case $h(\mathcal{F}^c) = \mathcal{F}^c$. This means that

$$\{\mathcal{F}^c_A(x)\}_{x \in \mathbb{T}^3} \text{ measurable partition} \Leftrightarrow \{\mathcal{F}^c_{A\circ\psi}(x)\}_{x \in \mathbb{T}^3} \text{ measurable partition.}$$

But we know that $\{\mathcal{F}^c_A(x)\}_{x\in\mathbb{T}^3}$ is not a measurable partition, since its decomposition is Lebesgue, and therefore not atomic.

Claim: $\mathcal{F}^c_{A\circ\psi}$ is not absolutely continuous.

Suppose it is absolutely continuous, then $\lambda^{ss}_f(p) = cte$ for all periodic point p, by construction $\lambda^u_f(p) = \lambda^u_A$. Since we are on the volume preserving case, $\lambda^{ws}(p)$ is also constant. Hence f is C^1-conjugate to A, but this would imply $\int \lambda^{ws}_f dVol = \int \lambda^{ws}_A dVol$. Absurd.

The above claims imply that $\mathcal{F}^c_f$ is non-absolutely continuous and form a non-measurable partition, hence it is not atomic and the conditional measures are singular with respect to the Lebesgue measure on the center leaves. □

4 Conjugacy and Foliation: A C^1 point of view

We begin by understanding how Lyapunov exponents vary with respect to their linearization.

Proposition 4.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic, not necessarily ergodic nor volume preserving, and let A be its linearization. Then $\int \lambda^u(f) dVol \leq \lambda^u(A)$.*

Proof. Suppose that $\int \lambda^u_f(x) dVol(x) > \lambda^u_A$, then there exists a set B of positive volume and a constant α such that $\lambda^u_f(x) > \alpha > \lambda^u_{f^*}$ $\forall x \in B$. Define

$$B_N = \{x \in B \mid ||Df^n|E^u_x|| \geq e^{n\alpha};\ \forall n \geq N\}.$$

Note that

$$B = \bigcup_{N=1}^{\infty} B_N,$$

this means that there is N_0 such that $Vol(B_{N_0}) > 0$. Since $\mathcal{F}^u_f$ is absolutely continuous then there is $x \in B$ such that $\mathcal{F}^u_f(x) \cap B_{N_0}$ has positive volume on the unstable leaf.

Let $I \subset \mathcal{F}^u_f(x)$ be a compact segment with $Vol^c(I \cap B_{N_0}) > 0$ and $length(I) =: l(I) > M$. Then

$$\begin{aligned} l(f^n(I)) &= \int_{f^n(I)} dVol^u = \int_I (f^n)^* dVol^u \geq \int_{I \cap A_{N_0}} (f^n)^* dVol^u \\ &\geq \int_{I \cap B_{N_0}} ||Df^n|E^u_x|| dVol^u(x) \geq e^{n\alpha} Vol^c(I \cap A_{N_0}). \end{aligned}$$

Consider x, y the extremes of $I = [x, y]$. Then $d^u(f^n(x), f^n(y)) = l(f^n(I))$. Using quasi-isometry on the first inequality below we get

$$
\begin{aligned}
\frac{d(f^n(x), f^n(y))}{d(A^n(x), A^n(y))} &\geq cte \frac{d^u(f^n(x), f^n(y))}{d(A^n(x), A^n(y))} \\
&\geq cte \frac{e^{n\alpha}}{e^{n\lambda_A^u}} \frac{Vol(I \cap B_{N_0})}{d(x, y)} \\
&\longrightarrow \infty \text{ as } n \to \infty.
\end{aligned}
$$

By Proposition 2.4 this ratio should be bounded. Absurd.

□

The same type of argument above give us:

Corollary 4.1.

$$\int \lambda^s(f) \geq \lambda^s(A).$$

We consider the following for the case of Anosov systems, for it will be used later.

Corollary 4.2. *Let f be an Anosov diffeomorphism with the following split on the tangent space $TM = E^{ss} \oplus E^{ws} \oplus E^u$ and $\mathcal{F}^{ws}$ absolutely continuous. Then $\lambda_f^{ws} \geq \lambda_A^{ws}$.*

Proof. The prove goes as before, with a minor change. We proceed, as previously, applying Proposition 2.4 with the following linear map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ which is the projection onto a center foliation of the linearization. The projection is with respect to the system of coordinate given by the foliations of the linearization $(x_{ss}, x_{ws}, x_u) \in \mathbb{R}^n$. □

Proof of Theorem E: We only have to prove the implication, as the converse is a direct consequence of the C^1-conjugacy.

Let us suppose that f is partially hyperbolic with the following split of the tangent space: $TM = E^{ss} \oplus E^{ws} \oplus E^u$. The next three lemmas concern this case, the other case is reduced to this one by applying the inverse.

Lemma 4.1.

$$\lambda^u_f(m) = \lambda^u_f(p),\ \forall p \in Per(f).$$

Proof. By ergodicity the set of transitive points $\mathcal{T}$ has total volume. We may assume that all points of $\mathcal{T}$ have well defined Lyapunov exponents. For $x \in \mathcal{T}$; given $\varepsilon > 0$ let $\delta > 0$ be such that by uniform continuity

$$|\ log||Df|E^u_{y_1}|| - log||Df|E^u_{y_2}||\ | < \varepsilon,\ \text{if } d(y_1, y_2) < \delta.$$

From the Shadowing lemma there is α such that for every α-pseudo orbit is δ shadowed by a real orbit. Given $N_0 \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ and $n_0 > N_0$ such that $\{\ldots, f^{n_0-1}(x), x, f(x), \ldots, f^{n_0-1}(x), \ldots\}$ is an α-pseudo orbit. Since it is a pseudo-periodic orbit it is δ shadowed by a periodic point with period n_0, call this point q. Using that E^u is one dimensional, then

$$\left| \frac{1}{n_0} log||Df^{n_0}|E^u_{y_1}|| - \frac{1}{n_0} log||Df^{n_0}|E^u_{y_2}|| \right| < \varepsilon$$

Since we already know that $\lambda^u_f(x)$ exists, this implies that $\lambda^u_f(x) = \lambda^u_f(q)$, hence $\lambda^u_f(m) = \lambda^u_f(p)$ as we wanted. □

Lemma 4.2.

$$\lambda^u_f(m) = \lambda^u_A.$$

Proof. We know that the topological entropy of A is λ^u_A, the conjugacy gives $h_{top}(f) = h_{top}(A)$. From the theory of equilibrium states (section 2.3.1) the measure of maximal entropy is given by the potential $\psi = 0$ and the equilibrium state for the potential $\psi = -log\lambda^u$ gives the SRB measure, which is m in our case. And to see that both equilibrium states are the same we just need to see that both potential are cohomologous (section 2.3.1). It means that both measures coincide if, and only if,

$$\frac{1}{n}\sum_{i=1}^{n}(-log||Df_{f^i(x)}|E^u||) = cte,\ \forall x \text{ such that } f^n(x) = x.$$

Which is true by hypothesis.

Finally Pesin's formula gives that $h_f(m) = \int \lambda_f^u dm = \lambda_f^u$. Let us put all this equalities below.

$$\lambda_A^u = h_{top}(A) = h_{top}(f) = h_f(m) = \int \lambda_f^u dm = \lambda_f^u(p).$$

The lemma is then proved. □

Lemma 4.3.

$$\lambda_f^{ws}(p) = \lambda_A^{ws}$$

Proof. By the above lemma we already know that $\lambda_f^u(p) = \lambda_A^u$; and $\lambda_f^{ss}(p) \geq \lambda_A^{ss}$ by Corollary 4.1. Hence, since we are on the volume preserving case $\lambda_f^{ss}+\lambda_f^{ws}+\lambda_f^u = \lambda_A^{ss}+\lambda_A^{ws}+\lambda_A^u$, therefore we just need to see that $\lambda_f^{ws}(p) \geq \lambda_A^{ws}$ which is the Corollary 4.2. □

The above lemmas imply,

$$\lambda_f^*(p) = \lambda_A^*(h(p)),\ \forall p \in Per(f).$$

The above equality gives periodic data. Theorem 2.7 implies that f is C^1 conjugate to the linear one. □

4.1 Failure of the rigidity of center foliation

We are now ready to prove Theorem B.

Proof of Theorem B. We start from a linear Anosov with splitting $TM = E^{ss} \oplus E^{ws} \oplus E^u$. Let ϕ be a volume preserving diffeomorphism which preserves the E^{ss} direction. This means it is absolutely continuous by Gogolev [8] and by Theorem E it is not C^1 conjugate to the linear since we changed the integral of the center foliation. From Theorem F we indeed got an absolutely continuous foliation that is not C^1. □

4.2 Conditional measures with dynamical meaning

The goal of this subsection is to prove Theorem F. We shall associate to each center leaf a class of measures differing from each other by a multiplication of a positive real number in such a way that on each foliated box the normalized element of this class will give the Rokhlin disintegration of the measure. When the foliation satisfies the hypothesis on Theorem F we shall be able to pick measurably on each leaf a representative with some dynamical meaning, it will then help us to obtain some information on the center Lyapunov exponent of f.

Lemma 4.4 (Avila, Viana, Wilkinson [2])**.** *For any foliation boxes $\mathcal{B}$, $\mathcal{B}'$ and m-almost every $x \in \mathcal{B} \cap \mathcal{B}'$ the restriction of $m_x^{\mathcal{B}}$ and $m_x^{\mathcal{B}'}$ to $\mathcal{B} \cap \mathcal{B}'$ coincide up to a constant factor.*

Proof. Let $\mu_{\mathcal{B}}$ be the measure on Σ obtained as the projection of $m|\mathcal{B}$ along local leaves. Consider any $\mathcal{C} \subset \mathcal{B}$ and let $\mu_{\mathcal{C}}$ be the projection of $m|\mathcal{C}$ on Σ,

$$\frac{d\mu_{\mathcal{C}}}{d\mu_{\mathcal{B}}} \in (0,1],\ \nu_{\mathcal{C}} \text{ almost every point.}$$

For any measurable set $E \subset \mathcal{C}$

$$m(E) = \int_{\Sigma} m_\xi^{\mathcal{B}}(E)\ d\mu_{\mathcal{B}}(\xi) = \int_{\Sigma} m_\xi^{\mathcal{B}}(E) \frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}}(\xi)\ d\mu_{\mathcal{C}}(\xi).$$

By essential uniqueness, this proves that the disintegration of $m|\mathcal{C}$ is given by

$$m_\xi^{\mathcal{C}} = \frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}}(\xi)\ m_\xi^{\mathcal{B}};\ \mu_{\mathcal{C}}(\xi) \text{ almost every point.}$$

Take $\mathcal{C} = \mathcal{B} \cap \mathcal{B}'$. Therefore $\frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}}(\xi) m_\xi^{\mathcal{B}}|\mathcal{C} = m_\xi^{\mathcal{C}} = \frac{d\mu_{\mathcal{B}'}}{d\mu'_{\mathcal{C}}}(\xi) m_\xi^{\mathcal{B}'}|\mathcal{C}$. Where $\mu'_{\mathcal{C}}$ is the projection of measure μ on the transversal Σ' relative to the $\mathcal{B}'$ box. Hence

$$m_\xi^{\mathcal{B}}|\mathcal{C} = a(\xi)m_\xi^{\mathcal{B}'}|\mathcal{C},$$

where $a(\xi) = \frac{d\mu_{\mathcal{B}'}}{d\mu'_{\mathcal{C}}}(\xi)(\frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}}(\xi))^{-1}$. $\square$

The above lemma implies the existence of a family $\{[m_x] \mid x \in M\}$ of measures defined up to scaling and satisfying $m_x(M\backslash\mathcal{F}_x) = 0$. The map $x \mapsto [m_x]$ is constant on leaves of $\mathcal{F}$ and the conditional probabilities $m_x^{\mathcal{B}}$ coincide almost everywhere with the normalized restrictions of $[m_x]$.

We observe that disintegration of a measure is an almost everywhere concept, but in our case, since we shall be considering a C^1 center foliation, we look to the conditional measures, of volume, defined everywhere. And, more important, the number $a(\xi) = \frac{d\mu_{\mathcal{B}'}}{d\mu'_{\mathcal{C}}}(\xi)(\frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}}(\xi))^{-1}$ is indeed defined everywhere.

From now on we work on the lift. Let $B := \mathcal{W}^{su}(0)$ which is the saturation by unstable leaves of the stable manifold of $0 \in \mathbb{R}^3$. By the semi-conjugacy we know that every segment of center leaf which has size large enough keep increasing by forward iteration. Let γ_0 be a length with this property. Let B_0 be the two-dimensional topological surface such that each center leaf intersects B and B_0 on two points, that are on the same center leaf and at a distance γ_0 inside the center leaf. Let $B_k := f^k(B_0)$ therefore, for each point $\xi \in B$ there is a unique point $q_k(\xi) \in B_k$ that is on the same center leaf as ξ. Since it will be clear to which point ξ $q_k(\xi)$ is associate, we use q_k instead to simplify notation.

Define the measure $m_{\xi,k}$ by

$$m_{\xi,k}([0, q_k]) = \lambda^k,$$

where λ is the center eigenvalue of the linearization, $[0, q_k]$ means the segment $[\xi, q_k(\xi)]$ inside the center leaf of ξ.

Lemma 4.5.

$$f_* m_{x,k} = \lambda^{-1} m_{f(x),k+1}.$$

Proof. Just see that

$$f_* m_{x,k}([0, q_{k+1}]) = \lambda^{-1} m_{f(x),k+1}([0, q_{k+1}]).$$

□

Therefore if the sequence $m_{x,k}$ converges we would get

$$f_* m_x = \lambda^{-1} m_{f(x)}.$$

In general, by Lemma 4.4, for two foliated boxes $\mathcal{B}$ and $\mathcal{B}'$ we have

$$m_x^{\mathcal{B}} \frac{d\nu_{\mathcal{B}}}{d\nu_{\mathcal{C}}} = m_x^{\mathcal{B}'} \frac{d\nu_{\mathcal{B}'}}{d\nu'_{\mathcal{C}}}.$$

We apply this formula to the following boxes: $\mathcal{B}$ and $\mathcal{B}_k$, where $\mathcal{B}$ comprehend the segment of center leaves between B and B_0, similarly $\mathcal{B}_k$ is formed by the segment of center leaves bounded by B and B_k. Then

$$m_x^{\mathcal{B}}.1 = \frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}} m_x^{\mathcal{B}_k} = \frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}} \lambda^{-k} m_{x,k}.$$

Note that $\lambda^k m_{x,k} = m_x^{\mathcal{B}_k}$ by the definition of the disintegration. The above proves

Lemma 4.6. *On $\mathcal{B}$:*

$$m_{x,k} = \left(\frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}}\right)^{-1} \lambda^k m_x^{\mathcal{B}}.$$

To establish the convergence of the measures we shall need

Lemma 4.7. *If $\mathcal{F}^c$ satisfies the hypothesis of Theorem F then, there is a uniform constant α such that*

$$\frac{1}{\alpha} \frac{l(\mathcal{F}_x^c \cap \mathcal{B}_k)}{l(\mathcal{F}_x^c \cap \mathcal{B})} \leq \frac{d\mu_{B_k}}{d\mu_{\mathcal{B}}}(x) \leq \alpha \frac{l(\mathcal{F}_x^c \cap \mathcal{B}_k)}{l(\mathcal{F}_x^c \cap \mathcal{B})}.$$

Proof. To calculate $\frac{l(\mathcal{F}_x^c \cap \mathcal{B}_k)}{l(\mathcal{F}_x^c \cap \mathcal{B})}$ we need to estimate the volume of a rectangular box. The center holonomy on the center unstable and center stable foliation are bounded by hypothesis. Therefore the volume can be calculated (estimated) by height times base.

□

Hence,

$$\frac{d\mu_{B_k}}{d\mu_{\mathcal{B}}}(x) = \alpha_{x,k} \frac{l(\mathcal{F}_x^c \cap \mathcal{B}_k)}{l(\mathcal{F}_x^c \cap \mathcal{B})},$$

where $\alpha_{x,k} \in [1/\alpha, \alpha]$, for all $x \in \mathbb{R}^3$ and $k \in \mathbb{N}$.

Therefore using Lemma 4.6 we get on $\mathcal{B}$

$$m_{x,k} = \left(\alpha_{x,k} \frac{l(\mathcal{F}_x^c \cap \mathcal{B}_k)}{l(\mathcal{F}_x^c \cap \mathcal{B})} \right)^{-1} \lambda^k m_x^{\mathcal{B}}.$$

For each x there is a subsequence $\alpha_{x,k_{i(x)}}$ that converges to some $\tilde{\alpha}_x$ as $i(x) \to \infty$.

Lemma 4.8. *There is $\beta > 0$ such that $\lambda^k / l(\mathcal{F}_x^c \cap \mathcal{B}_k) \in [1/\beta, \beta]$ for all x.*

Proof. We need to estimate the fraction

$$\frac{||f^n(H(x)) - f^n(H(y))||}{||A^n(x) - A^n(y)||} = \frac{||H \circ A^n(x) - H \circ A^n(x)||}{||A^n(x) - A^n(y)||}.$$

By the triangular inequality:

$$\begin{aligned} \frac{||H \circ A^n(x) - H \circ A^n(y)||}{||A^n(x) - A^n(y)||} &\leq \frac{||H(A^n(x)) - A^n(x)||}{||A^n(x) - A^n(y)||} + \frac{||A^n(x) - A^n(y)||}{||A^n(x) - A^n(y)||} \\ &+ \frac{||H(A^n(y)) - A^n(y)||}{||A^n(x) - A^n(y)||}, \end{aligned}$$

and

$$\begin{aligned} \frac{||H \circ A^n(x) - H \circ A^n(y)||}{||A^n(x) - A^n(y)||} &\geq -\frac{||H(A^n(x)) - A^n(x)||}{||A^n(x) - A^n(y)||} + \frac{||A^n(x) - A^n(y)||}{||A^n(x) - A^n(y)||} \\ &- \frac{||H(A^n(y)) - A^n(y)||}{||A^n(x) - A^n(y)||}. \end{aligned}$$

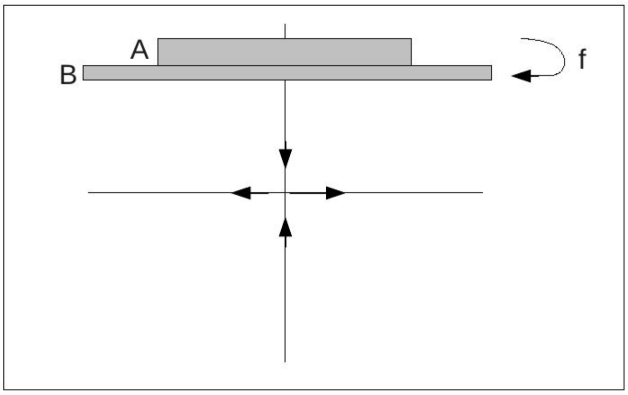
We know that H is at a bounded distance of the identity and $||A^n(x) - A^n(y)||$ is big. □

By the above lemma we may assume that $\lambda^k/l(\mathcal{F}^c_x \cap \mathcal{B}_k)$ goes to one as k increases, otherwise incorporate it to the constant $\alpha_{x,k}$. Then sending $k_{i(x)}$ to infinity

$$m_x := \lim_{k_{i(x)}\to\infty} m_{x,k_{i(x)}} = (l(\mathcal{F}^c_x \cap \mathcal{B})/\tilde{\alpha}_x)m^{\mathcal{B}}_x. \tag{1}$$

By going to a subsequence we obtained a convergent measure, but we want it to have a specific property. Therefore we have to be more careful on how to define them. We've seen above that $f_* m_{x,k} = \lambda^{-1} m_{f(x),k+1}$, hence for fixed x there is $k_{i(x)}$ defined as above, but if we define $k_{i(f(x))} = k_{i(x)} + 1$ we obtain the convergence satisfying $f_* m_x = \lambda^{-1} m_{f(x)}$. This means that for fixed x we can define on the orbit of x measures satisfying the mentioned dynamical property.

The measures are in fact indexed on a two dimensional plane manifold W^{su}. Hence, to define properly on the whole space, consider the rectangle A as in the figure below representing W^{su}, the intersection of A to the stable manifold of the origin is a fundamental domain. The two biggest sides are unstable leaves. Hence defining the measures as we mentioned above on A and on its iterates we get measures with dynamical properties.



From the above we conclude that we did get measures on each center leaf with the property that $f_* m_x = \lambda^{-1} m_{f(x)}$. The construction of such measures will help us to get information of the center Lyapunov exponent, since we may recover λ by the equality

$$\frac{df_* m_x}{dm_{f(x)}} = \lambda^{-1}.$$

Let us explore more deeply the above relation.

Lemma 4.9. *By the above notation, the center Lyapunov exponent of f exists everywhere and it is equal to λ.*

Proof. Note that

$$\frac{df_*^n m_x}{dm_{f^n(x)}}(f^n(x)) = \lambda^{-n}.$$

Let us calculate the Radon-Nikodym derivative by another way. Let $I_\delta^n \subset \mathcal{F}^c_{f^n(x)}$ be a segment of length δ around $f^n(x)$. Then

$$\frac{df_*^n m_x}{dm_{f^n(x)}}(f^n(x)) = \lim_{\delta \to 0} \frac{f_*^n m_x(I_\delta^n)}{m_{f^n(x)}(I_\delta^n)}.$$

And

$$\begin{aligned}
\frac{df_*^n m_x}{dm_{f^n(x)}}(f^n(x)) &= \lim_{\delta \to 0} \frac{m_x(f^{-n}(I_\delta^n))}{m_{f^n(x)}(I_\delta^n)} = \lim_{\delta \to 0} \frac{\int_{f^{-n}(I_\delta^n)} \rho_x d\lambda_x}{\int_{I_\delta^n} \rho_{f^n(x)} d\lambda_{f(x)}} \\
&\approx \frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))} \lim_{\delta \to 0} \frac{\int_{f^{-n}(I_\delta^n)} d\lambda_x}{\int_{I_\delta^n} d\lambda_{f(x)}} \approx \lim_{\delta \to 0} \frac{\rho_x(x)}{\rho_{f^n(x)}} \frac{\int_{I_\delta^n} ||Df^{-n}|| d\lambda_x}{\int_{I_\delta^n} d\lambda_{f(x)}} \\
&\approx \frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))} ||Df^{-n}(x)||.
\end{aligned}$$

We then have

$$\lim_{\delta \to 0} \frac{df_*^n m_x}{dm_{f^n(x)}}(I_\delta^n) = \frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))} ||Df^{-n}(x)||.$$

From the other equalities we have

$$\frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))} ||Df^{-n}(x)|| = \lambda^{-n}.$$

By applying "$lim_{n\to\infty} 1/n\ log$" to the above equality we get

$$\lambda^c(x) = \lambda,$$

since the densities of m_x are uniformly limited. □

We are now ready for the

Proof of Theorem F: First, let us prove that f is an Anosov diffeomorphism. We just need to analyze the behavior of Df on the center direction. Let $\varepsilon > 0$ be such that $\lambda_\varepsilon := \lambda - \varepsilon > 0$. Since the center exponent exists for every x then, given $x \in \mathbb{T}^3$, there are $n_x \in \mathbb{N}$ and a neighborhood $\mathcal{U}_x$ of x such that $\forall x \in \mathcal{U}_x\ |Df^{n_x}|E^c| \geq e^{n_x \lambda_\varepsilon}$. Since $\mathbb{T}^3$ is a compact manifold take a finite cover $\mathcal{U}_{x_1} \ldots \mathcal{U}_{x_l}$. Let $C_i < 1$ small enough so that for $x \in \mathcal{U}_{x_i}$ then $|Df^n(x)|E^c| \geq C_{x_i} e^{n\lambda_\varepsilon}$ for all $n \in \{0, 1, \ldots, n_{x_i}\}$. Let $C := \min_i\ C_{x_i}$, we then have that $|Df^n(x)|E^c| \geq Ce^{n\lambda_\varepsilon}$ for all $x \in \mathbb{T}^3$ and $n \in \mathbb{N}$.

Since, in particular, the center foliation is absolutely continuous, from Gogolev [8], one of the extremal exponents is constant on periodic points. On the other hand the above theorem gives that in particular on the periodic points the central exponent is also constant. Since we are on the conservative case all Lyapunov exponents are constant on periodic points. Then Theorem E gives that f is C^1-conjugate to its linearization. □

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