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STOCHASTIC PROCESSES OVER FINITE GRAPHS

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Aos meus pais.

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Abstract

This PhD thesis consists of two parts, both of them related to the study of stochastic processes over discrete structures.

In the first part we study the relation between the performance of the randomized rumor spreading (push model) in a d-regular graph G and the performance of the same algorithm in the percolated graph G_p. We show that if the push model successfully broadcast the rumor within T rounds in the graph G then only $(1+\epsilon)T$ rounds are needed to spread the rumor in the graph G_p when $T = o(pd)$.

In the second part we study the cover time C of a graph G. The expected value of C is well-understood in several families of examples, but much less is known about its fluctuations. In this paper, we give sufficient conditions under which the fluctuations of the cover time converges to the Gumbel extreme value distribution, making progress on a well-known open problem. The distribution of late points for the random walk is also determined under the same conditions. Our methods apply to many "locally transient" families as discrete tori $(\mathbb{Z}/L\mathbb{Z})^d$ with $d \geq 3$ (also proven by Belius) and high girth expanders, as well as to all examples where Gumbel limits were previously known to hold.

Keywords: Push model, cover time, Gumbel law.

Resumo

Esta tese de doutorado é composta por duas partes, ambas relacionadas com o estudo de processos estocásticos em estruturas discretas.

Na primeira parte estudamos a relação entre o desempenho de um algoritmo randomizado (push model) em um grafo d-regular G e o desempenho do mesmo algoritmo no grafo percolado G_p. Nós mostramos que se o algoritmo distribui informação para todos os vértices de G dentro de T rodadas então apenas $(1+\epsilon)T$ rodadas são necessárias para espalhar informação no grafo G_p quando $T = o\,(pd)$.

Na segunda parte estudamos o tempo de cobertura C de um grafo G. O valor esperado de C é bem compreendido em muitas famílias de exemplos, mas muito pouco se sabe sobre suas flutuações. Neste trabalho, damos condições suficientes sob as quais as flutuações do tempo de cobertura convergem para a distribuição de Gumbel, fazendo progresso em um problema em aberto bem conhecido. A distribuição dos pontos cobertos por último pelo passeio aleatório também é determinada sob as mesmas condições. Nossos métodos se aplicam a muitas famílias "localmente transientes" como Toros discretos $(\mathbb{Z}/L\mathbb{Z})^d$ com $d \geq 3$ e expansores com cintura larga, bem como todos exemplos onde a convergência para Gumbel era previamente conhecida.

Palavras-chave: Push model, tempo de cobertura, lei de Gumbel.

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Notation

A concept used throughout this work is stochastic domination. If X and Y are random variables taking values in $\mathbb{R}$, we say that X is *stochastically smaller* than Y and denote this by $X \preceq Y$ if $\mathbb{P}(X \geq t) \leq \mathbb{P}(Y \geq t)$ for any $t \in \mathbb{R}$.

If X has distribution μ and Y has distribution ν, then $X \preceq Y$ if and only if

$$\int f \, d\mu \leq \int f \, d\nu$$

for any continuous and increasing function f. By approximation, $X \preceq Y$ implies the relation above for all increasing upper semicontinuous function f.

In this paper we write $X \sim Y$ when the random variables X and Y have the same distribution. We let $\mathrm{Geo}(p)$, $\mathrm{Be}(p)$, $\mathrm{Bin}(n,p)$, $\mathrm{Po}(\lambda)$ denote, respectively, the *geometric distribution* with parameter p, the *Bernoulli distribution* with parameter p, the *binomial distribution* with parameters (n,p) and the *Poisson distribution* with parameter λ. We also denote $\mu \otimes \nu$ the product measure with marginals distributions μ and ν.

Introduction

The subject of this PhD thesis is the study of stochastic processes over discrete structures. In this introduction we clarify which problems have been addressed and what results have been obtained.

This work consists of two parts that can be read independently. Part I (Chapters 1 to 4) deal with the problem of information spread in networks by a randomized algorithm. In Part II (Chapters 5 to 9) the interest is to study the fluctuations of the cover time and the structure of the last points covered by a random walk in different families of graphs.

In Part I, **Rumor Spreading on Percolation Graphs**, we study the relation between the performance of the randomized rumor spreading (push model) in a d-regular graph G and the performance of the same algorithm in the percolated graph G_p. We show that if the push model successfully broadcast the rumor within T rounds in the graph G then only $(1+\epsilon)T$ rounds are needed to spread the rumor in the graph G_p when $T = o(pd)$. This a joint work with Roberto Oliveira.

Randomized rumor spreading or *randomized broadcasting* is a simple randomized algorithm to spread information in networks. In this work we consider the classical *push model* for rumor spreading, which is described as follows. Initially, one arbitrary vertex knows an information. In the succeding discrete time steps each informed vertex chooses a neighbor independently and uniformly at random and forward the information to it. The fundamental question is: how many time steps are needed until every vertex of the network has been informed?

The push model has been extensively studied. Most of the papers analyze the runtime of this algorithm on different graph classes. On the complete graph, Frieze and Grimmett [22] proved that *with high probability*[1] $(1+o(1))(\log_2 n+\log n)$ rounds are necessary and sufficient to inform all vertices. In [20] Feige *et al.* gave general upper bounds holding for any graph and determined the runtime on random graphs. Also, Chierichetti, Lattanzi and Panconesi [15] proved runtime bounds in terms of the conductance and Sauerwald and Stauffer [34] obtained bounds in terms of the vertex expasion for regular graphs.

[1]With high probability, also denoted w.h.p., refers to an event that holds with probability $1 - o(1)$ as the size of the graph tends to infinity.

The starting point of this part of the thesis is a recent article by Fountoulakis, Huber and Panagiotou [21] which analyze the push protocol on the Erdös-Rényi random graph $G_{n,p}$ where $p \gg \frac{\ln n}{n}$. Among other things, they show that the protocol will inform every vertex in $(1 + o(1))(\log_2 n + \ln n)$ steps w.h.p.. One may restate this result as: the runtime of randomized broadcasting on a complete graph is essentially not affected by random edge deletions, at least up to the connectivity threshold $p = \ln n/n$.

In this part we prove a partial extension of this result to the case of arbitrary percolation graphs. Here one starts with some arbitrary graph G and performs edge percolation on it. Under certain conditions, we show that this will not increase the runtime of the protocol by more than a $1 + o(1)$ factor. This suggests that the push protocol is robust against random edge failures, which is a desirable quality for applications.

We need some definitions in order to state our main Theorem. Given a graph $G = (V, E)$, we let G_p denote the random subgraph of G where each edge is removed independently with probability $1-p$ (the vertex set stays the same). We let $\mathcal{T}_v(G), \mathcal{T}_v(G_p)$ denote the runtimes of the push protocol starting at v over G and G_p (respectively).

Theorem 1. *Let $G_n = (V_n, E_n)$ be a sequence of d_n-regular graph where $|V_n| = n \to \infty$ and $v_n \in V_n$. Suppose that exists $(T_n)_{n\geq 1}$ such that $\mathcal{T}_{v_n}(G_n) \leq T_n$ with high probability. Then, given any $\epsilon > 0$ and $0 < p_n < 1$, such that $p_n \gg \frac{T_n}{d_n}$, we have that $\mathcal{T}_{v_n}(G_{n,p_n}) \leq (1+\epsilon)T_n$ with high probability (here the probability is over the choice of G_{n,p_n} and over the additional randomness of the push protocol). In particular, G_{n,p_n} is connected with high probability.*

One can check in our proofs that the same result holds if G_n has minimum degree d_n. The case where $G_n = K_n$ is complete shows that the condition $p_n \gg \frac{T_n}{d_n}$ cannot be removed in general. We also remark that proving lower bounds for $\mathcal{T}_{v_n}(G_{n,p_n})$ in terms of $\mathcal{T}_{v_n}(G_n)$ is an interesting open problem, but the upper bound we give seems more interesting for applications.

In Part II, **Fluctuations of cover times and the geometry of the set of uncovered points**, we study the cover time C of a graph G. We give sufficient conditions under which the fluctuations of the cover time converges to the Gumbel extreme value distribution, making progress on a well-known open problem. The distribution of the last points covered is also determined under the same conditions. This is a joint work with Roberto Oliveira.

The *cover time* of a connected finite graph $\mathbf{G}$ is the time C it takes for simple random walk to visit all vertices of $\mathbf{G}$. This is one of the most natural random variables associated with simple random walk, and it also has been intensely investigated (see [2, 3, 5, 13, 16, 24, 26, 30]). The expected value of C is well-understood in a variety of examples, and there are also general estimates in terms of hitting times and the Gaussian Free Field [23].

By contrast, relatively little is known about the *fluctuations* of the cover time around its mean and about late points for the random walk. A general concentration result by Aldous [3] gives a necessary and sufficient condition for C to concentrate around its expectation. However, this is quite far from describing the actual limiting distribution of C at finer scales or the law of the set of uncovered points.

Our main motivation is to prove *Gumbel law fluctuations* for cover times. Recall that the Gumbel law is the distribution on $\mathbb{R}$ with cumulative distribution function:

$$F_{\text{Gumbel}}(t) \equiv e^{-e^{-t}} \quad (t \in \mathbb{R}).$$

This law has been known since the early nineties to describe the distributional limit of C in large complete graphs, hypercubes and binomial coefficient graphs [17]. It is a widespread belief that it should also be obtained in much more general families of examples. For instance, the following concrete question has been a well-known open problem for quite some time [5].

Question 1. *Consider the discrete torus $\mathbb{T}_N^d$ with $n = N^d$ vertices and $d \geq 3$ dimensions, and let C_N denote its cover time. Is it true that as $N \to +\infty$,*

$$\frac{\mathsf{C}_N}{\mathcal{G}_d(0)\, n} - \ln n \text{ converges in distribution to the Gumbel law,}$$

where $\mathcal{G}_d(\cdot)$ is Green's function for random walk over $\mathbb{Z}^d$?

Here is a heuristic explanation for why this law might show up. Recall that the hitting time H_a of a vertex a of $\mathbf{G}$ is the time it takes for random walk to visit a for the first time.

1. The cover time is a maximum of hitting times of vertices: $\mathsf{C}_N = \max_{a \in \mathbb{T}_N^d} H_a$;
2. the expected hitting time of a vertex in $\mathbb{T}_N^d$ is $\approx \mathcal{G}_d(0)\, N^d$;
3. rescaled hitting times $H_a/\mathcal{G}_d(0)\, N^d$ are approximately exponentially distributed with mean 1;
4. the Gumbel law describes fluctuations of $\max_{i=1,\ldots,n} E_i - \ln n$, where $E_1, \ldots, E_n$ are iid exponential random variables with mean 1.

Were it not for the *independence* requirement in the last item, this would be very close to a full proof of the Gumbel limit for C_N. This suggests that in order to prove the Gumbel limiting law, one needs to show the approximate independence of hitting times, at least for "good" subsets of vertices. This heuristic also suggests why Gumbel limits should not be specific to discrete tori: except for item 2, which specifies the scaling factor $\mathcal{G}_d(0)\, N^d$, what we described above holds for much more general Markov chains (see [1]).

Very recently, Belius has answered Question 1 in the affirmative [10]. This proof is based on his previous papers [8, 9] and uses several important tools, most notably the theory of random interlacements in $\mathbb{Z}^d$ and very strong couplings of that to the vacant set of random walk in $\mathbb{T}_N^d$. His methods also imply that the (rescaled) set of uncovered points at certain times is approximatelly a Poisson process over the continuous torus. It seems likely that a similar proof could work for other graphs, such as random regular graphs, for which the corresponding infinite interlacements model would work out, but it is not clear if this could be generalized to other examples (eg. graphs with unbounded degree, or oriented graphs).

In this paper we use an abstract version of Belius's argument in [9], which deals with the coverage of level sets in cylinders, to derive stronger and much more general statements. In particular, we obtain a proof of his torus result via a partially independent route. In fact, our proof of the Gumbel limit follows from the fact that the law of late uncovered points has an approximate product law with marginals $\frac{e^{-t}}{n}$. Our results are the first instance of a unified result that proves Gumbel limits (and much more) for a general family of graphs, which includes all of the above examples. We also obtain Gumbel limits in new examples such as high-girth expanders, oriented tori and the Cayley graph of S_n generated by transpositions.

Remark 1. *Our proof employs several ideas present in the paper [9], such as proving that the hitting times of well-separated sets of points behaves as if the individual hitting times were independent. A first version of the results obtained here was obtained shortly after Belius gave a talk at IMPA on [9]. Upon contacting Belius, we learned that he already had a proof of Gumbel law for the torus, which now appears in [10]. His proof also uses many ideas present in [8, 9].*

A key conceptual difference between our work and that of Belius is that we do *not* focus on the specific structure of a given family of graphs. Rather, we obtain our results from a systematic *mean field approximation* of hitting times that uses recent results by Oliveira [33]. There are important conceptual similarities between our approach and that of Miller and Peres in [32, 31] at a high conceptual level, but our conditions do not quite match theirs.

Our results on specific graphs will be derived from the abstract theorem stated below. The natural setting for these is that of continuous-time Markov chains. For simplicity, we will only consider *transitive* chains, which are the chains for which the generator is invariant under a transitive group of permutations of the state space.

Undefined terms will be defined in Chapter 6. We use asymptotic notation, as explained in the same section. Recall also that H_a is the hitting time of a and H_S is the hitting time of set S.

Theorem 2. *Assume that we have a sequence $\mathcal{Q}$ of generators Q of transitive Markov chains over finite state spaces $\mathbf{V}_Q$ with $|\mathbf{V}_Q| \equiv n_Q \to +\infty$ vertices. Let π_Q denote the uniform measure over Q and $\mathrm{t}_{\mathrm{mix}}^{Q}$, $\mathrm{t}_{\mathrm{unif}}^{Q}$ are the standard and uniform mixing times of Q (respectively). Assume that for each $Q \in \mathcal{Q}$ there exists a number $\mathsf{h}(Q)$ such that:*

A0 $\mathbb{E}_{\pi_Q}[H_a] = \left(1 + o\left(\frac{1}{\ln n_Q}\right)\right) \mathsf{h}(Q)$, *where a is any vertex in $\mathbf{V}_Q$;*

A1 $\frac{\mathrm{t}_{\mathrm{mix}}^{Q}}{\mathsf{h}(Q)} = o\left(\frac{1}{\ln^2 n_Q \ln\ln n_Q}\right)$;

A2 *For all $\{a,b\} \in \binom{\mathbf{V}_Q}{2}$, $\mathbb{E}_{\pi_Q}\left[H_{\{a,b\}}\right] \leq \mathsf{h}(Q)/(1+\phi)$, where $\phi > 0$ is constant (ie. does not depend on $Q \in \mathcal{Q}$);*

A3 *There exists $\alpha_3(Q) = o\left(1/\ln n_Q\right)$ such that for all $a \in \mathbf{V}_Q$ the set*

$$B_Q(a) \equiv \{b \in \mathbf{V}_Q \,:\, \mathbb{P}_a\left(H_b \leq \mathrm{t}_{\mathrm{unif}}^{Q}\right) + \mathbb{P}_b\left(H_a \leq \mathrm{t}_{\mathrm{unif}}^{Q}\right) \geq \alpha_3(Q)\}$$

has cardinality $|B_Q(a)| = o\left(n_Q^{\phi}\right)$.

Finally, let $U_{t_0}^Q$ *denote the set of uncovered points of* $\mathbf{V}_Q$ *by* Q *in time* $t_0 = \mathsf{h}(Q)(\ln n_Q + \beta)$*, for some* $\beta \in \mathbb{R}$*. Then:*

$$d_{\mathrm{TV}}(\mathrm{Law}_{\pi_Q}\left(U_{t_0}^Q\right), \bigotimes_{v\in\mathbf{V}_Q} \mathrm{Be}_{\frac{e^{-\beta}}{n_Q}}) = o\left(1\right), \text{ as } n_Q \to \infty. \tag{1}$$

Remark 2. *Since* Q *is transitive, any starting state gives the same Law for* C*.*

Let us comment briefly on the assumptions of this Theorem. In principle, one could remove the transitivity assumption by imposing **A0** to all vertices a. It is not hard to show that we will not obtain the theorem if we do not require nearly equal hitting times for all vertices. **A1** is a “mean-field" assumption that implies that hitting times are approximately exponentially distributed. It is quite possible that it can be weakened to $\mathrm{t}_{\mathrm{mix}}^Q = o\left(\mathsf{h}(Q)\right)$. In applications, **A2** and **A3** will be justified by showing that Q is “locally transient". **A2** will mean that a random walk from a has a chance of drifting off up to time $\mathrm{t}_{\mathrm{unif}}^Q$ without ever hitting b (and vice-versa). Finally, **A3** will be deduced from the fast decay of the associated Green’s function.

As an immediate consequence of Theorem 2 we have the Gumbel law fluctuations for cover times.

Corollary 1. *Assume that the conditions of Theorem 2 are satisfied. If* C_Q *denotes the cover time of* $\mathbf{V}_Q$ *by* Q*, then*

$$\mathrm{Law}_{\pi_Q}\left(\frac{\mathsf{C}_Q}{\mathsf{h}(Q)} - \ln n_Q\right) \text{ converges weakly to the Gumbel law,}$$

that is to say,

$$\forall t \in \mathbb{R},\, \mathbb{P}_{\pi_Q}\left(\frac{\mathsf{C}_Q}{\mathsf{h}(Q)} - \ln n_Q \le t\right) \to e^{-e^{-t}}.$$

Proof: Theorem 2 implies that $|U_{\mathsf{h}(Q)(\ln n_Q + t)}^Q|$ has approximately the same law as a binomial random variable with parameters n_Q and $\frac{e^{-t}}{n_Q}$, therefore

$$\begin{aligned}
\mathbb{P}_{\pi_Q}\left(\frac{\mathsf{C}^Q}{\mathsf{h}(Q)} - \ln n_Q \le t\right) &= \mathbb{P}_{\pi_Q}\left(|U_{\mathsf{h}(Q)(\ln n_Q + t)}^Q| = 0\right) \\
&= \mathbb{P}\left(\mathrm{Bin}\left(n_Q, \frac{e^{-t}}{n_Q}\right) = 0\right) + o\left(1\right) \\
&= \mathbb{P}\left(\mathrm{Po}(e^{-t}) = 0\right) + o\left(1\right) \\
&= e^{-e^{-t}} + o\left(1\right),
\end{aligned}$$

where in the second equality we use equation (1) and in the third one we use the Law of Small Numbers. □

Remark 3. *Besides obtaining the Gumbel law, Belius also showed that the set of uncovered points in the discrete torus, suitably rescaled, approximates a Poisson process over the continuous torus at a fixed time of the form* $h(Q)n + \beta n$*. Theorem 2 is stronger than this.*

We also describe the evolution of the last points covered by Q. We prove that the distribution of the time-rescaled process $\widetilde{U}_c^Q = U^Q_{\mathsf{h}(Q)(\ln n + c)}$ after time β is approximately as the process $\mathcal{A}^Q = \{A_c^Q\}_{c \geq \beta}$ defined as follows:

1. For $c = \beta$, sample A_β^Q from $\bigotimes_{v \in \mathbf{V}_Q} \mathrm{Be}_{\frac{e^{-\beta}}{n_Q}}$;

2. Each vertex of $a \in A_\beta^Q$ survives for time $\mathsf{Exp}(1)$ independently.

The appropriate space for $\widetilde{U}_c^Q$ is the set $\mathbb{D}\left([\beta, \beta'], 2^{\mathbf{V}_Q}\right)$ of set-valued trajectories with right-continuous and left-hands limits. Consider the Prohorov metric d_J on probability measures over $\mathbb{D}\left([\beta, \beta'], 2^{\mathbf{V}_Q}\right)$ induced by Skorohod metric J (both defined in Chapter 6). Then, we have that:

Theorem 3. *Under* **A0 - A3***, for all $\beta, \beta' \in \mathbb{R}$, with $\beta < \beta'$, there exists coupling between the process $\{A_c^Q\}_{c \in [\beta, \beta']}$ and the process $\{\widetilde{U}_c^Q\}_{c \in [\beta, \beta']} = \{U^Q_{\mathsf{h}(Q)(\ln n + c)}\}_{c \in [\beta, \beta']}$ such that*

$$d_J\left(\mathrm{Law}_{\pi_Q}\left(\{\widetilde{U}_c^Q\}_{c \in [\beta, \beta']}\right), \{A_c^Q\}_{c \in [\beta, \beta']}\right) = o\left(1\right), \text{ as } n_Q \to \infty.$$

Remark 4. *Consider again the setting of Belius result. One may use this Theorem to show that, with natural embedding into $(\mathbb{R}/\mathbb{Z})^d$, the process $\{\widetilde{U}_c\}_{c \in [\beta, \beta']}$ converges to an evolving Poisson process where each point disappears independently at rate 1.*

Part I

Rumor Spreading on Percolation Graphs

CHAPTER 1

Key ideas

As mentioned in the Introduction, in this part of the thesis we prove a partial extension of Fountoulakis, Huber and Panagiotou's result [21] to the case of arbitrary percolation graphs. The proof strategy of [21] relies on the geometry of the Erdös-Rényi graph above the connectivity threshold. We have no such information available in our general setting, and instead rely on a very different proof strategy:

Proof strategy: Construct a coupling of:

$$[\text{push over } G_n] \leftrightarrow [\text{random choice of } G_{n,p} + \text{push over } G_{n,p_n}.]$$

Then show that the runtimes over the two graphs are close under the coupling.

It is not hard to sketch a coupling that solves the analogous problem over oriented graphs. Assume $D = (V, F)$ is a n-vertex *digraph*. Define D_p as the random digraph obtained from D by deleting each oriented edge with probability $1-p$. We consider a variant of push over D and D_p, where each informed vertex v pushes the rumour along outgoing edges chosen uniformly but *without replacement*. We will assume that all vertices have the same out-degree d and that this modified push protocol over D typically takes $T \ll pd$ steps.

We now couple

$$[\text{push over } D \text{ up to time } T] \leftrightarrow [\text{random choice of } D_p + \text{push over } D_p \text{ up to time } T.]$$

For this we need two Bernoulli (indicator) random variables $A_{v\to w}$ and $I_{v\to w}$ for each oriented edge $v \to w$. All of those variables will be assumed independent, and we take:

$$\mathbb{P}\left(A_{v\to w} = 1\right) = p \text{ and } \mathbb{P}\left(I_{v\to w} = 1\right) = \frac{CT}{pd} \text{ for some } C > 0.$$

Notice that $T \geq \log_2 n$ as the number of informed vertices can only double at each time step. Chernoff bounds (see [7]) imply that, if C is sufficiently large, then w.h.p., for all $v \in V$, the set

$$\mathcal{N}(v) \equiv \{w \in V \,:\, v \to w \in F,\, A_{v\to w} I_{v\to w} = 1\}$$

will have at least T elements, as each of the d out-neighbors of v belongs to this set with probability CT/d. This means we can run modified push over D by having each v select the edges $v \to w$ with $w \in \mathcal{N}(v)$ in a random order, up to time T. Notice that this gives the right distribution because, conditionally on $|\mathcal{N}(v)| = k \geq T$, $\mathcal{N}(v)$ is uniform over all k-subsets of out-neighbors of v in D.

We now let D_p be the digraph whose oriented edges are the pairs $v \to w$ with $A_{v\to w} = 1$. To run modified push on D_p, have each v select the edges $v \to w$, $w \in \mathcal{N}(v)$, in the same order as in the protocol over D. The key points are that:

- This gives the right distribution because conditionally on D_p and on $|\mathcal{N}(v)| = k$, $\mathcal{N}(v)$ is uniform over k-subsets of the out-neighbors of v in D_p.
- The set of informed vertices in D and D_p coincide up to time T. In particular, if all vertices are informed in D up to time T w.h.p., the same will hold in D_p.

This shows that the modified push protocol cannot take longer in D_p than its typical runtime in D.

As presented, this proof strategy cannot work for non-oriented graphs. The first problem is that the neighbors to be informed in the push protocol are chosen with replacement. This, however, is not hard to deal with; see Proposition 1 below.

A second and more serious problem is that, if one tries to copy the above coupling, the events $w \in \mathcal{N}(v)$ and $v \in \mathcal{N}(v)$ will be positively correlated given $\{v, w\} \in G_{n,p_n}$. The solution to this will be to introduce a few *extra steps* in the push protocol over G_{n,p_n}. This is a kind of sprinkling idea. The upshot will be that the set of neighbors of v chosen in push-over-G_{n,p_n} will dominate the set chosen in push-over-G, but the difference between the two sets will be so small that this will not matter much. (Incidentally, this is where the ϵ in the Theorem 1 comes from).

CHAPTER 2

Preliminaries

Let $G = (V, E)$ be an unweighted, undirect, simple and connected graph, where V is the set of vertices, E the set of edges and $|V|$ denotes the size of the graph. We consider families of graphs $G_n = (V_n, E_n)$ where $|V_n| = n$. For a vertex $v \in V$, $\deg(v)$ denotes the degree of v and $\Gamma(v)$ denotes the set of neighbors of v.

Given some parameter $p \in [0, 1]$, we consider bond percolation in G by removing each edge of G, independently and with probability $1 - p$. The graph obtained from this process is denoted by G_p. Also, $\deg_p(v)$ denotes the degree of v in G_p and $\Gamma_p(v)$ denotes the set of neighbors of v in G_p.

As mentioned before, we consider the randomized broadcasting algorithm called push algorithm. In fact, the process just explained can be described as follows. Let $I(t)$ be the set of vertices informed at discrete time t. Initially $t = 0$ and $I(0) = \{v\}$, for some choice of $v \in V$. While $I(t) \neq V$, each vertex $u \in I(t)$ chooses a neighbor v_u^t independently and uniformly at random. The new informed set is

$$I(t + 1) = I(t) \cup \{v_u^t; u \in I(t)\}.$$

The process stops when $I(t) = V$.

We are interested in how many time steps are needed for the process to stop. For this, define $\mathcal{T}_v(G) = \min\{t \in \mathbb{N} | I(t) = V\}$ as the first time step in which every vertex of G has been informed.

The rest of this chapter is devoted to the study of the negative hypergeometric distribution. A negative hypergeometric random variable X records the waiting time in trials until the r-th sucess is obtained in repetead random sampling without replacement from a dichotomous population of size N with d successes. In the following we show that a negative hypergeometric random variable is stochastically smaller than a sum of geometric random variables. After that we can bound the probability that the sum of negative hypergeometric distribution deviates above the mean using concentration inequalities for the sum of geometric random variables.

Fix $N \geq 1$, $d \in \{1, \dots, N\}$ and $r \in \{1, \dots, d\}$. The random variable X is *negative hypergeometric*, denoted by $X \sim \mathrm{NH}(N, d, r)$, if

$$\mathbb{P}(X = k) = \frac{\binom{k-1}{r-1}\binom{N-k}{d-r}}{\binom{N}{d}}.$$

The expected value of X is $r\frac{N+1}{d+1}$.

We start studying the relation between geometric and negative hypergeometric random variables.

Lemma 1. *For each $j = 1, \ldots, k_1$, let X_j be the waiting time until the jth sucess in trials without replacement from a population of size $k_1 + k_2$ with k_1 possibilities of sucess (each X_j has distribution $NH(k_1 + k_2, k_1, j)$). Then:*

$$X_1 \preceq Geo\left(\frac{k_1}{k_1+k_2}\right) \preceq Geo\left(\frac{k_1}{k_1+k_2} - \frac{j}{k_1+k_2}\right); \tag{2.1}$$

$$(X_{j+1} - X_j)\,|X_j \preceq Geo\left(\frac{k_1}{k_1+k_2} - \frac{j}{k_1+k_2}\right). \tag{2.2}$$

Proof: Since the number of failures in the population of size $k_1 + k_2$ is k_2, we have that

$$\begin{aligned}
\mathbb{P}(X_1 \geq m) &= \frac{k_2}{k_1+k_2}\,\frac{k_2-1}{k_1+k_2-1}\cdots\frac{k_2-(m-2)}{k_1+k_2-(m-2)} \\
&\leq \left(\frac{k_2}{k_1+k_2}\right)^{m-1} \\
&= \mathbb{P}\left(\mathrm{Geo}\left(\frac{k_1}{k_1+k_2}\right) \geq m\right),
\end{aligned}$$

where the last inequality follows observing that $\frac{k_2-j}{k_1+k_2-j} \leq \frac{k_2}{k_1+k_2}$ for all $j \in \{1, \ldots, \min\{k_1, k_2\}\}$. The second inequality in (2.1) is immediate.

Now to prove (2.2) observe that conditioned in $X_j = k$ the remaining population has size k_1+k_2-k and $k_2 - (k-j)$ failures. Thus,

$$\begin{aligned}
\mathbb{P}(X_{j+1} - X_j \geq m | X_j = k) &= \frac{k_2-(k-j)}{k_1+k_2-k}\,\frac{k_2-(k-j)-1}{k_1+k_2-k-1}\cdots\frac{k_2-(k-j)-(m-2)}{k_1+k_2-k-(m-2)} \\
&\leq \left(\frac{k_2}{k_1+k_2} + \frac{j}{k_1+k_2}\right)^{m-1} \\
&= \mathbb{P}\left(\mathrm{Geo}\left(\frac{k_1}{k_1+k_2} - \frac{j}{k_1+k_2}\right) \geq m\right),
\end{aligned}$$

where the above inequality holds because

$$\begin{aligned}
\frac{k_2-(k-j)}{k_1+k_2-k} &= \frac{\frac{k_2}{k_1+k_2}(k_1+k_2) - k\frac{k_2}{k_1+k_2}}{k_1+k_2-k} + \frac{-k\frac{k_1}{k_1+k_2} + j}{k_1+k_2-k} \\
&\leq \frac{k_2}{k_1+k_2} + \frac{j}{k_1+k_2}
\end{aligned}$$

and

$$\frac{k_2 - (k-j) - l}{k_1 + k_2 - k - l} \leq \frac{k_2 - (k-j)}{k_1 + k_2 - k},$$

for $l = 1, \ldots, m-2$. $\square$

The next lemma enables us to dominate a negative hypergeometric random variable by a sum of geometric random variables. Its proof can be found in the appendix.

Lemma 2. *Let $X_1, \ldots, X_j$ and $Y_1, \ldots, Y_j$ be random variables taking values in $\mathbb{R}$ such that $X_0 \equiv 0$ and $X_{i+1} - X_i | X_i \preceq Y_{i+1}$, for $i = 0, \ldots, j-1$, then*

$$X_j \preceq \widetilde{Y}_1 + \cdots + \widetilde{Y}_j$$

where $\widetilde{Y}_i \sim Y_i$ and $\widetilde{Y}_i$ is independent of $\widetilde{Y}_k$ for $k \neq i$.

We use Lemma 1 combined with Lemma 2 to obtain that if $X_j \sim NH(k_1 + k_2, k_1, j)$ with $j \leq k_1$, then $X_j \preceq G_1 + \cdots + G_j$, where $G_1, \ldots, G_j$ are independent geometric random variables with parameter $\dfrac{k_1}{k_1 + k_2} - \dfrac{j}{k_1 + k_2}$.

Now, consider the following standard fact about stochastic domination, which is a corollary of the Lemma 2.

Corollary 2. *Let $X_1, \ldots, X_r, Y_1, \ldots, Y_r$ be random variables taking values in $\mathbb{R}$ such that $\{X_1, \ldots, X_r\}$ are independent, as well as $\{Y_1, \ldots, Y_r\}$. If $X_i \preceq Y_i$ for $i = 1, \ldots, r$, then*

$$X_1 + \cdots + X_r \preceq Y_1 + \cdots + Y_r.$$

If $X_j \sim \mathrm{NH}(k_1 + k_2, k_1, j)$ and $X_l \sim \mathrm{NH}(k_1 + k_2, k_1, l)$ are independent random variables, we can use Lemma 2 to guarantee that $X_j + X_l \preceq G_1 + \cdots + G_{j+l}$, where $G_1, \ldots, G_{j+l}$ are independent geometric random variables with parameter $\frac{k_1}{k_1+k_2} - \frac{\max\{j,l\}}{k_1+k_2}$.

At this point, the problem of bounding the probability of the sum of negative hypergeometric random variables deviations above the mean is transformed into the problem of bounding the sum of geometric random variables. In the next lemma we give a bound for the sum of geometric random variables with parameter $1 - o(1)$.

Lemma 3. *Given any $\epsilon > 0$ and $C > 1$ there exists $\delta = \delta(\epsilon, C) > 0$ such that if $G_1, \ldots, G_r$ are independent random variables with distribution geometric with parameter $p \geq 1 - \delta$, then*

$$\mathbb{P}\left(G_1 + \cdots + G_r > (1+\epsilon) r\right) \leq \exp\left(-(C-1) r\right).$$

Proof: Begin by taking $\epsilon > 0$ and $C > 1$. If the parameter is $1 - \delta$,

$$\begin{aligned}
\mathbb{P}\left(G_1 + \cdots + G_r \geq (1+\epsilon)r\right) &\leq \exp\left(-\frac{C}{\epsilon} r\epsilon\right) \mathbb{E}\left[\exp\left(\frac{C}{\epsilon}\sum_{i=1}^{r}(G_i - 1)\right)\right] \\
&= \exp(-Cr)\left(\sum_{k=1}^{\infty}(1-\delta)\,\delta^{k-1}\exp\left(\frac{C}{\epsilon}(k-1)\right)\right)^r \\
&\leq \exp(-Cr)\left(\sum_{k=1}^{\infty}\left(\delta\exp\left(\frac{C}{\epsilon}\right)\right)^{k-1}\right)^r.
\end{aligned}$$

For δ sufficiently small $\delta\exp(\frac{C}{\epsilon}) \leq \delta^{\frac{1}{2}} \leq \frac{1}{2}$ and so

$$\begin{aligned}
\exp(-Cr)\left(\sum_{k=1}^{\infty}\left(\delta\exp\left(\frac{C}{\epsilon}\right)\right)^{k-1}\right)^r &= \exp(-Cr)\left(\frac{1}{1-\delta^{\frac{1}{2}}}\right)^r \\
&\leq \exp(-Cr)\left(1 + 2\delta^{\frac{1}{2}}\right)^r \\
&\leq \exp\left(-\left(C - 2\delta^{\frac{1}{2}}\right) r\right) \\
&\leq \exp\left(-\left(C-1\right) r\right).
\end{aligned}$$

This gives the result if the parameter of the geometrics is $1-\delta$. The case $p \geq 1-\delta$ follows via stochastic domination. □

CHAPTER 3

Fast broadcast on G_p

We begin by defining another process, called *push without replacement* and denoted by PWR or PWR(G): at time $t = 0$ an arbitrary vertex knows an information. In the succeeding time steps each informed vertex chooses a neighbor independently and uniformly at random from its neighbors not yet chosen and fowards the information according to this list.

In fact, let $J(t)$ be the set of vertices informed by PWR at time t. Initially $t = 0$ and $J(0) = \{v\}$, for some choice of $v \in V$. While $J(t) \neq V$, each vertex $u \in J(t)$ chooses a neighbor v_u^t independently and uniformly at random in $\Gamma(u)\backslash\{v_u^s; 0 < s < t\}$. The new informed set is

$$J(t+1) = J(t) \cup \{v_u^t; u \in J(t)\}.$$

The process stops when $J(t) = V$.

Define $\mathcal{J}_v(G) = \min\{t \in \mathbb{N} | J(t) = V\}$ as the first time step in which every vertex of G has been informed by the PWR. The next result relates $\mathcal{T}_v(G)$ and $\mathcal{J}_v(G)$. We assume that $I(0) = J(0) = \{v_n\}$ and throughout this chapter we denote $G = G_n, p = p_n, T = T_n, d = d_n$ and $v = v_n$, we also omit v from $\mathcal{T}_v(G)$ and $\mathcal{J}_v(G)$.

Proposition 1. *For any graph G we have that $\mathcal{J}(G) \preceq \mathcal{T}(G)$. Moreover, let $G_n = (V_n, E_n)$ be a sequence of d_n-regular graphs, with $n \to \infty$, and $T_n = o(d_n)$ such that $\mathcal{J}(G_n) \leq T_n$ with high probability. Then given any $\epsilon > 0$, $\mathcal{T}(G_n) \leq (1+\epsilon)T_n$ with high probability.*

Proof: Given any graph G, we start building a coupling between the push and the PWR in G as follows. For each $u \in V$, let $\Theta^u = \{\Theta_i^u\}_{i=1}^{\infty}$ be a sequence of independent random variables with uniform distribution in $\Gamma(u)$. Run the push in G according to the realization of $\{\Theta^u\}_{u\in V}$ in the following sense: the first neighbor to be chosen by a vertex u is Θ_1^u, the second is Θ_2^u and so on.

Now, for $u \in V$, define $X_1^u = 1$ and, for each $k \in \mathbb{N}$,

$$X_k^u = \inf\{m \in \mathbb{N} | \Theta_m^u \notin \{\Theta_{X_1^u}^u, \dots, \Theta_{X_{k-1}^u}^u\}\}.$$

Also, define $U_k^u = \Theta^u_{X_k^u}$. We will use the following fact (proof omitted):

Claim 1. *For an arbitrary $u \in V$, let $\deg(u) = s$. The random variable $(U_j^u)_{j=1}^s$ has uniform distribuition in the permutations of $\Gamma(u)$. Moreover, for each $k \leq s$, $X_k^u - X_{k-1}^u$ has geometric distribution with parameter $1 - \frac{k-1}{s}$ and the vector $(U_j^u)_{j=1}^s$ and the random variables $X_1^u, X_2^u - X_1^u, \ldots, X_s^u - X_{s-1}^u$ are mutually independent.*

Now, run the PWR according to the realization of $\{(U_1^u, \ldots, U_{\deg(u)}^u)\}_{u \in V}$ in the following sense: each vertex u informs its neighbors in the order given by the list $(U_1^u, \ldots, U_{\deg(u)}^u)$.

With the processes constructed in this way, we have $I(t) \subset J(t)$ for all $t \in \mathbb{N}$, which implies $\mathcal{J}(G) \preceq \mathcal{T}(G)$ and we have the first part of the Proposition.

It remains to prove the second assertion. Let $v = I(0) = J(0)$ and G a d-regular graph with $T = o(d)$, then

$$\begin{aligned}
\mathbb{P}\left(\mathcal{T}(G) > (1+\epsilon)T\right) &= \mathbb{P}\left(\mathcal{T}(G) > (1+\epsilon)T, \mathcal{J}(G) \leq T\right) \\
&\quad + \mathbb{P}\left(\mathcal{T}(G) > (1+\epsilon)T, \mathcal{J}(G) > T\right) \\
&= \mathbb{P}\left(\mathcal{T}(G) > (1+\epsilon)T, \mathcal{J}(G) \leq T\right) + o(1) \\
&= \mathbb{P}\left(\exists\, w \in V; \mathcal{T}_{v,w}(G) > (1+\epsilon)T, \mathcal{J}(G) \leq T\right) + o(1) \\
&\leq n \cdot \mathbb{P}\left(\mathcal{T}_{v,w'}(G) > (1+\epsilon)T, \mathcal{J}(G) \leq T\right) + o(1),
\end{aligned}$$

where $\mathcal{T}_{v,w}(G) = \min\{t \in \mathbb{N} | w \in I(t)\}$ is the first time step in which w is informed and w' is defined by the equality $\max_{w \in V} \mathbb{P}\left(\mathcal{T}_{v,w}(G) > (1+\epsilon)T\right) = \mathbb{P}\left(\mathcal{T}_{v,w'}(G) > (1+\epsilon)T\right)$.

Observe that

$$\mathbf{1}\{\mathcal{T}_{v,w'}(G) > (1+\epsilon)T, \mathcal{J}(G) \leq T\} = f((X_k^u - X_{k-1}^u)_{k \leq d, u \in V}, (U_{X_k^u})_{k \leq d, u \in V})$$

where f is a measurable function. Recall that $\{X_k^u - X_{k-1}^u | k \leq d, u \in V\}$ and $\{U_{X_k^u} | k \leq d, u \in V\}$ are mutually independent, then by the substitution principle, we have that

$$\begin{aligned}
\mathbb{P}\left(\mathcal{T}_{v,w'}(G) > (1+\epsilon)T, \mathcal{J}(G) \leq T\right) &= \mathbb{E}\left[\mathbb{E}\left[f(\{X_k^u - X_{k-1}^u\}_{k,u}, \{(U_{X_k^u})_k^d\}_u) | \{(U_{X_k^u})_k^d\}_u\right]\right] \\
&= \mathbb{E}\left[\phi(U_{X_k^u})_k\}_u)\right],
\end{aligned}$$

where $\phi(\{(x_k^u)_k\}_u = \mathbb{E}\left[f(\{X_k^u - X_{k-1}^u\}_{k,u}, \{(x_k^u)_k\}_u)\right]$.

Fixed the realization $\{(U_{X_k^u})_k\}_u = \{(x_k^u)_k\}_u$, the event $\{\mathcal{J}(G) \leq T\}$ implies the existence of a path $\gamma : v = v_0 \sim \cdots \sim v_l = w'$ such that $J_{v_0,v_1} + \cdots + J_{v_{l-1},v_l} \leq T$, where $J_{v_j,v_{j+1}}$ is the first time step in which v_j choose v_{j+1} to transmit the information in the PWR. As $\{\mathcal{T}_{v,w'}(G) > (1+\epsilon)T\}$, defining $T_{v_j,v_{j+1}}$ as the first time step in which v_j choose v_{j+1} to transmit the information in the push in G, we have that $T_{v_0,v_1} + \cdots + T_{v_{l-1},v_l} \geq (1+\epsilon)T$. So,

$$\mathbb{E}\left[f(\{X_k^u - X_{k-1}^u\}_{k,u}, \{(x_k^u)_k\}_u)\right] \leq \mathbb{P}\left(T_{v_0,v_1} + \cdots + T_{v_{l-1},v_l} \geq (1+\epsilon)T\right).$$

To bound the probability above we find the distribution of $T_{v_j,v_{j+1}}$,

$$\begin{aligned} T_{v_j,v_{j+1}} &= X^{v_j}_{J_{v_j,v_{j+1}}} \\ &= X^{v_j}_1 + (X^{v_j}_2 - X^{v_j}_1) + \cdots + (X^{v_j}_{t_{v_j,v_{j+1}}} - X^{v_j}_{t_{v_j,v_{j+1}}-1}) \\ &\sim G^j_1 + \cdots + G^j_{J_{v_j,v_{j+1}}} \end{aligned}$$

where $G^j_1, \ldots, G^j_{J_{v_j,v_{j+1}}}$, by the Claim 7.10, are independent random variables such that G^j_k has geometric distribution with parameter $1 - \frac{k-1}{d}$, for each $k = 1, \ldots, J_{v_j,v_{j+1}}$.

As $\mathrm{Geo}(1 - \frac{k-1}{d}) \preceq \mathrm{Geo}(1 - \frac{T}{d})$ and $J_{v_0,v_1} + \cdots + J_{v_{l-1},v_l} \le T$, using Lemma 2 we have

$$T_{v_0,v_1} + \cdots + T_{v_{l-1},v_l} \preceq G'_1 + \cdots + G'_T,$$

where $G'_1, \ldots, G'_T$ are independent geometric random variables with parameter $1 - \frac{T}{d}$. Then,

$$\mathbb{P}\left(T_{v_0,v_1} + \cdots + T_{v_{l-1},v_l} \ge (1+\epsilon)T\right) \le \mathbb{P}\left(G'_1 + \cdots + G'_T \ge (1+\epsilon)T\right). \tag{3.1}$$

Now, using Lemma 3 with $C > 1 + \ln 2$ and n sufficiently large for $\frac{T}{d} < \delta$,

$$\mathbb{P}\left(G'_1 + \cdots + G'_T \ge (1+\epsilon)T\right) \le \exp\left(-(C-1)T\right),$$

as $T > \log_2 n$ we have that

$$\begin{aligned} \mathbb{P}\left(G'_1 + \cdots + G'_T \ge (1+\epsilon)T\right) &\le \exp\left(-(C-1)\frac{\ln n}{\ln 2}\right) \\ &= o\left(n^{-1}\right). \end{aligned}$$

Thus, $n \cdot \mathbb{P}\left(\mathcal{T}_{v,w'}(G) > (1+\epsilon)T, \mathcal{J}(G) \le T\right) = o(1)$ and the proof is finished. □

The next lemma enables us to build a coupling between the PWR in G and the PWR in G_p.

Lemma 4. *Let G_n be a sequence of d_n-regular graphs, T_n and p_n as in Theorem 1. Take $C = \left(\frac{pd}{T}\right)^{\frac{1}{2}}$, then, for each $u \in V$, there exist random variables N^p_u with distribution $Bin(d, \frac{CT}{d})$, N_u with distribution $Bin(d, \frac{CT}{d}(1 - \frac{CT}{pd}))$, and random elements $\mathcal{N}_u$ in $\Gamma(u)$, $\mathcal{N}^p_u$ in $\Gamma_p(u)$ such that:*

$$\mathbb{P}\left(\mathcal{N}_u = S | N_u = k\right) = \frac{1}{\binom{d}{k}}, \text{ for each } S \subset \Gamma(u) \text{ with } |S| = k, \tag{3.2}$$

and

$$\mathbb{P}\left(\mathcal{N}^p_u = S' | N^p_u = j\right) = \frac{1}{\binom{deg_p(u)}{j}}, \text{ for each } S' \subset \Gamma_p(u) \text{ with } |S'| = j. \tag{3.3}$$

Moreover, $\mathcal{N}_u \subset \mathcal{N}_u^p$ and there exists $\delta = \delta_n \to 0$ as $n \to \infty$ such that

$$\mathbb{P}\left(|\mathcal{N}_u^p \backslash \mathcal{N}_u| \leq \delta \cdot |\mathcal{N}_u|, \forall u \in V\right) = 1 - o\left(n^{-1}\right). \tag{3.4}$$

Proof: Begin by taking independent random variables $\{I'_{u\to v} | (u,v) \in V^2, u \sim v\}$ with distribution $Be(\frac{CT}{pd})$, also take, for each $e \in e(G)$, independent random variables A_e with distribution $Be(p)$ (independent of the I's).

Now defining, for each $(u,v) \in V^2$ with $u \sim v$, $I^p_{u\to v} = A_e I'_{u\to v}$ and, for each $u \in V$, $\mathcal{N}_u^p = \{v \sim u | I^p_{u\to v} = 1\}$ and $N_u^p = |\mathcal{N}_u^p|$, we have that $I^p_{u\to v}$ has distribution $Be(\frac{CT}{d})$ and therefore N_u^p has distribution $Bin(d, \frac{CT}{d})$ and $\mathcal{N}_u^p$ satisfies (3.3).

We will use the following general fact to build N_u and $\mathcal{N}_u$:

Strassen's Lemma *Let μ and ν be distributions on $\mathbb{R}^2$ such that $\mu \preceq \nu$. Then, there exists a coupling (X, Y) of the (μ, ν) such that $X \leq Y$ in the coordinatewise partial order.*

Let μ be the law of $(I^p_{u\to v}, I^p_{v\to u})$ and ν the distribution of (B_1, B_2) where B_1, B_2 are independent random variables $Be(q)$ such that $q = \frac{CT}{d}(1 - \frac{CT}{2pd})$. To show that $\nu \preceq \mu$, by the symmetry of the distributions, it suffices to show that $S_2 \preceq S_1$, where $S_1 = I^p_{u\to v} + I^p_{v\to u}$ and $S_2 = B_1 + B_2$.

Our choice of q implies, by simple calculations, that $\mathbb{P}(S_2 \geq 2) \leq \mathbb{P}(S_1 \geq 2)$ and $\mathbb{P}(S_2 \geq 1) \leq \mathbb{P}(S_1 \geq 1)$ so, $S_2 \preceq S_1$. Now, using Strassen's Lemma, we obtain that there exists $(I_{u\to v}, I_{v\to u})$ with distribution $Be(q) \times Be(q)$ and such that $I_{u\to v} \leq I^p_{u\to v}$ and $I_{v\to u} \leq I^p_{v\to u}$. Moreover, as

$$\mathbb{P}\left(I^p_{u\to v} - I_{u\to v} = 1\right) = \mathbb{E}\left[I^p_{u\to v} - I_{u\to v}\right] = \frac{1}{2p}\left(\frac{CT}{d}\right)^2$$

we have that $(I^p_{u\to v} - I_{u\to v})$ has distribution $Be\left(\frac{1}{2p}\left(\frac{CT}{d}\right)^2\right)$.

Defining, for each $u \in V$, $\mathcal{N}_u = \{v \sim u | I_{u\to v} = 1\}$ and $N_u = |\mathcal{N}_u|$. It immediately follows that N_u has distribution $Bin\left(d, \frac{CT}{d}\left(1 - \frac{CT}{2pd}\right)\right)$, $\mathcal{N}_u^p$ satisfies (3.2) and $\mathcal{N}_u \subset \mathcal{N}_u^p$. It remains to prove (3.4).

Let us prove (3.4). Start by choosing $\delta = \max\left\{C^{-\frac{1}{2}}, \left(\frac{CT}{pd}\right)^{\frac{1}{2}}\right\}$ and defining $N_u^* = |\mathcal{N}_u^p \backslash \mathcal{N}_u|$ that has distribution $Bin\left(d, \frac{1}{2p}\left(\frac{CT}{d}\right)^2\right)$. Let A be the event $\{|N_u - \mathbb{E}[N_u]| \leq \frac{1}{2}\mathbb{E}[N_u]\}$. As $T \geq \log_2 n$ and $C \to \infty$ we have that $\mathbb{E}[N_u] \gg \ln n$. So, we can use Chernoff bounds for the binomial distribution to obtain $\mathbb{P}(A^c) = o(n^{-2})$. Thus,

$$\begin{aligned} \mathbb{P}(N_u^* > \delta N_u) &= \mathbb{P}(N_u^* > \delta N_u, A) + \mathbb{P}(A^c) \\ &\leq \mathbb{P}\left(N_u^* > \delta \frac{3\mathbb{E}[N_u]}{2}\right) + o\left(n^{-2}\right) \\ &= o\left(n^{-2}\right) \end{aligned}$$

where in the last inequality we use Chernooff bounds again and our choice of δ which ensures $\delta \cdot \mathbb{E}[N_u] \gg \max\{\ln n, \mathbb{E}[N_u^*]\}$. This finishes the proof of Lemma. □

The next result studies the relation between $\mathcal{J}(G)$ and $\mathcal{J}(G_p)$.

Proposition 2. *Let $G_n = (V_n, E_n)$ be a sequence of d_n-regular graphs, with $n \to \infty$, that satisfies $\mathcal{J}(G_n) \leq T_n$ with high probability. Then, given $\epsilon > 0$ and choosing $p_n = o(1)$ such that $T_n = o(p_n \cdot d_n)$, we have that $\mathcal{J}(G_{n,p_n}) \leq (1+\epsilon)T_n$ with high probability.*

Proof: We begin building a coupling between the PWR in the graph G and the PWR in the graph G_p using the previous lemma as follows. Take random variables N_u, N_u^p and random elements $\mathcal{N}_u$, $\mathcal{N}_u^p$ as in Lemma 4 and fix an arbitrary order $\{1, \ldots, n\}$ of the vertices of G. For each $u \in V$, order $\Gamma(u)$ in the following way:

- order $\mathcal{N}_u$ uniformly (from 1 to N_u);
- order $\Gamma(u)\backslash\mathcal{N}_u$ uniformly (from $N_u + 1$ to $deg(u)$).

Equation (3.2) ensures that this is a uniform ordering of the neighbors of u. Denote by $Ord_u = Ord_u(w_1^u, \ldots, w_d^u)$, where $w_1^u < \cdots < w_d^u$ are the neighbors of u, the random vector built by the manner above. Run the PWR(G) as follows. If t is the first time that the vertex u receives information then in step $t+1$ the vertex u informs $w_{i_1}^u$ if the i_1-th coordinate of $Ord_u(w_1^u, \ldots, w_d^u)$ is equal to 1, in step $t+2$ the vertex u informs $w_{i_2}^u$ if the i_2-th coordinate of $Ord_u(w_1^u, \ldots, w_d^u)$ is equal to 2 and so on.

Let us run the PWR(G_p) similarly. For each $u \in V$, order $\Gamma_p(u)$ in the following way:

- order $\mathcal{N}_u^p$ uniformly, but conditioned to coincide with the order of $\mathcal{N}_u$ (from 1 to N_u^p);
- order $\Gamma(u)\backslash\mathcal{N}_u^p$ uniformly (from $N_u^p + 1$ to $deg_p(u)$).

Equation (3.3) ensures that this is a uniform ordering of the neighbors of u. Denote by $Ord_u^p = Ord_u^p(\tilde{w}_1^u, \ldots, \tilde{w}_{deg_p(u)}^u)$, where $\tilde{w}_1^u < \cdots < \tilde{w}_{deg_p(u)}^u$ are the neighbors of u in $\Gamma_p(u)$, the random vector built by the manner above. Run the PWR(G_p) using $\{Ord_u^p\}_{u\in V}$.

Now we will prove that if $\mathcal{J}(G) \leq T$ with high probability then $\mathcal{J}(G_p) \leq (1+\epsilon)T$ with high probability. Take $\epsilon > 0$, we have that

$$\mathbb{P}\left(\mathcal{J}(G_p) > (1+\epsilon)T\right) = \mathbb{P}\left(\mathcal{J}(G_p) > (1+\epsilon)T, A\right) + \mathbb{P}\left(\mathcal{J}(G_p) > (1+\epsilon)T, A^c\right), \tag{3.5}$$

where A is the event $\{\mathcal{J}(G) \leq T\} \cap \{N_u > \frac{CT}{2}, |\mathcal{N}_u^p\backslash\mathcal{N}_u| \leq \delta \cdot |\mathcal{N}_u|, N_u^p > \frac{CT}{2}, \forall u \in V\}$ with $C = \left(\frac{pd}{T}\right)^{\frac{1}{2}}$ and $\delta = \delta_n$ as in Lemma 4. First, we analyze the term $\mathbb{P}\left(\mathcal{J}(G_p) > (1+\epsilon)T, A^c\right)$:

$$\begin{aligned}
\mathbb{P}\left(\mathcal{J}(G_p) > (1+\epsilon)T, A^c\right) \quad \leq \quad & \mathbb{P}\left(\mathcal{J}(G) \geq T\right) \\
& + \mathbb{P}\left(\exists u; N_u \leq \frac{CT}{2}\right) \\
& + \mathbb{P}\left(\exists u; N_u^p \leq \frac{CT}{2}\right) \\
& + \mathbb{P}\left(\exists u; |\mathcal{N}_u^p\backslash\mathcal{N}_u| > \delta \cdot |\mathcal{N}_u|\right).
\end{aligned}$$

By hypothesis $\mathbb{P}\left(\mathcal{J}(G) \geq T\right) = o\left(1\right)$. Using Chernoff bounds for the binomial distribution, $T \geq \log_2 n$ and $C \to \infty$ we have that $\mathbb{P}\left(N_u \leq \frac{CT}{2}\right) = o\left(n^{-1}\right)$ and that $\mathbb{P}\left(N^p_u \leq \frac{CT}{2}\right) = o\left(n^{-1}\right)$. So,

$$\mathbb{P}\left(\mathcal{J}(G_p) > (1+\epsilon)T, A^c\right) = o\left(1\right).$$

Next, let us estimate the other term in equation (3.5). For this, let $\mathcal{J}_{v,u}(G) = \min\{t \in \mathbb{N} | u \in J(t)\}$ be the first time step in which u is informed by PWR(G) and let w' be the vertex such that

$$\mathbb{P}\left(\mathcal{J}_{v,w'}(G_p) > (1+\epsilon)T, A\right) = \max_{u \in V} \mathbb{P}\left(\mathcal{J}_{v,u}(G_p) > (1+\epsilon)T, A\right).$$

Then,

$$\begin{aligned}\mathbb{P}\left(\mathcal{J}(G_p) > (1+\epsilon)T, A\right) &\leq \mathbb{P}\left(\exists u \in V; \mathcal{J}_{v,u}(G_p) > (1+\epsilon)T, A\right)\\ &\leq n \cdot \mathbb{P}\left(\mathcal{J}_{v,w'}(G_p) > (1+\epsilon)T, A\right),\end{aligned}$$

and the last expression is equal to

$$n \cdot \mathbb{E}\left[\mathbf{1}_A \mathbb{P}(\mathcal{J}_{v,w'}(G_p) > (1+\epsilon)T | \{Ord_u\}_{u\in V}, \{N_u\}_{u\in V}, \{N^p_u\}_{u\in V})\right], \tag{3.6}$$

because A is $\sigma(\{Ord_u\}_{u\in V}, \{N_u\}_{u\in V}, \{N^p_u\}_{u\in V})$-measurable.

Defining

$$\mathbb{Q}(\cdot) = \mathbb{P}(\cdot | \{Ord_u\}_{u\in V} = \{(m^u_1, \ldots, m^u_d)\}_{u\in V}, \{N_u\}_{u\in V} = \{n_u\}_{u\in V}, \{N^p_u\}_{u\in V} = \{n^p_u\}_{u\in V})$$

and μ as the law of $(\{Ord_u\}_{u\in V}, \{N_u\}_{u\in V}, \{N^p_u\}_{u\in V})$, we obtain that expression (7.20) is equal to

$$n \cdot \int \mathbf{1}_{\widetilde{A}}(\{\widetilde{m}_u\}_u, \{n_u\}_u, \{n^p_u\}_u) \cdot \mathbb{Q}(\mathcal{J}_{v,w'}(G_p) > (1+\epsilon)T)\, d\mu(\{\widetilde{m}_u\}_u, \{n_u\}_u, \{n^p_u\}_u). \tag{3.7}$$

where $\widetilde{m}_u = (m^u_1, \ldots, m^u_d)$ and $\widetilde{A}$ is a Borel measurable set such that $\mathbf{1}_A$ is equal to $\mathbf{1}_{\widetilde{A}}(\{Ord_u\}_u, \{N_u\}_u, \{N^p_u\}_u)$.

For $u \sim w$, let $J_{u,w}(G)$ be the time until vertex u chooses w to transmit the information according to the PWR(G) and $J_{u,w}(G_p)$ the analogue for PWR(G_p). Fixed $\{Ord_u = \widetilde{m}_u\}_{u\in V}$, the event $\{\mathcal{J}(G) \leq T\}$ implies the existence of a path $\gamma : v = v_0 \sim \cdots \sim v_l = w'$ such that $J_{v_0,v_1}(G) + \cdots + J_{v_{l-1},v_l}(G) \leq T$. Also note that $N_{v_j} > T$ and $J_{v_j,v_{j+1}} \leq T$ implies $(v_j, v_{j+1}) \in G_p$ for each $j = 0, \ldots, l-1$. So, expression (3.8) is less than or equal to

$$n \cdot \int \mathbf{1}_{\widetilde{A}}(\{\widetilde{m}_u\}_u, \{n_u\}_u, \{n^p_u\}_u) \cdot \mathbb{Q}(J_{v_0,v_1}(G_p) + \cdots + J_{v_{l-1},v_l}(G_p) > (1+\epsilon)T)\, d\mu. \tag{3.8}$$

Now, we will find the conditional distribution of $J_{v_j,v_{j+1}}(G_p)$. Take $0 \leq r \leq m \leq l$ and $0 \leq k \leq l$, we want to calculate

$$\mathbb{P}\left(J_{v_j,v_{j+1}}(G_p) = k | J_{v_j,v_{j+1}}(G) = r, N_{v_j} = m, N^p_{v_j} = l\right).$$

As $r \leq m$, we have that $v_{j+1} \in \mathcal{N}_{v_j} \subset \mathcal{N}^p_{v_j}$. Moreover, $J_{v_j,v_{j+1}}(G) = r$ implies that v_{j+1} is the

$(j+1)$-th vertex chosen in uniform ordering of $\mathcal{N}_{v_j}$. As the order of $\mathcal{N}^p_{v_j}$ preserves the order of $\mathcal{N}_{v_j}$, we have that $J_{v_j,v_{j+1}}(G_p)$ is the time until the $(j+1)$-th sucess is obtained in repetead random sampling without replacement from a dichotomous population of size $N^p_{v_j}$ with N_{v_j} successes. Then, in $\{J_{v_j,v_{j+1}}(G) \leq N_{v_j}\}$,

$$\mathbb{P}\left(J_{v_j,v_{j+1}}(G_p) = k | J_{v_j,v_{j+1}}(G), N_{v_j}, N^p_{v_j}\right) = \mathbb{P}\left(NH(N^p_{v_j}, N_{v_j}, J_{v_j,v_{j+1}}(G)) = k\right).$$

So, the expression (3.8) can be written as

$$n \cdot \int \mathbf{1}_{\widetilde{A}}(\{\widetilde{m}_u\}_u, \{n_u\}_u, \{n^p_u\}_u) \cdot \mathbb{P}\left(Y_0 + \cdots + Y_l > (1+\epsilon)T\right) \, d\mu \tag{3.9}$$

where, for each $j = 0, \ldots, l-1$, Y_j has distribution $NH(N^p_{v_j}, N_{v_j}, J_{v_j,v_{j+1}}(G))$ and $J_{v_0,v_1} + \cdots + J_{v_{l-1},v_l} \leq T$ in $\widetilde{A}$.

Inside $\{\forall u \in V; |\mathcal{N}^p_u \backslash \mathcal{N}_u| \leq \delta \cdot |\mathcal{N}_u|, N^p_u > \frac{CT}{2}\}$ we have that

$$\begin{aligned} \frac{N_u}{N^p_u} - \frac{J_{v_j,v_{j+1}}(G)}{N^p_u} &\geq \frac{N_u}{N_u + |\mathcal{N}^p_u \backslash \mathcal{N}_u|} - \frac{T}{N^p_u} \\ &\geq \frac{1}{1+\delta} - \frac{2T}{CT} \\ &= 1 - o(1). \end{aligned}$$

Then, using previous chapter

$$\mathbb{P}\left(Y_0 + \cdots + Y_l > (1+\epsilon)T\right) \leq \mathbb{P}\left(G_0 + \cdots + G_T > (1+\epsilon)T\right)$$

where $G_0, \ldots, G_T$ are independent geometric random variables with parameter $1 - o(1)$. Thus, for n sufficiently large, we can use Lemma 3 and obtain that

$$\mathbb{P}\left(G_0 + \cdots + G_T > (1+\epsilon)T\right) = o\left(n^{-1}\right),$$

because $T \geq \log_2 n$. So, expression (3.9) is bounded by $o(1)$ and this finishes the proof of the proposition. □

Proof of Theorem 1: By hypothesis, $\mathcal{T}(G_n) \leq T_n$ with high probability. Then, using Proposition 1, we have that $\mathcal{J}(G_n) \leq T_n$ w.h.p.. As $T_n = o(p_n.d_n)$ and $p_n = o(1)$ we can use Proposition 2 and obtain that $\mathcal{J}(G_{n,p_n}) \leq (1+\epsilon')T_n$ w.h.p.. Finally, we use the second part of Proposition 1 to conclude that $\mathcal{T}(G_{n,p_n}) \leq (1+\epsilon')^2 T_n$ w.h.p.. Choosing ϵ' appropriately we have the result. □

CHAPTER 4

Appendix

In this appendix we prove Lemma 2. First, consider the following theorem (see [29]):

Proposition 3 (Disintegration)**.** *Let* ξ *and* η *be two random variables in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Consider a measurable function* f *on* $\mathbb{R} \times \mathbb{R}$ *with* $\mathbb{E}\left[|f(\xi, \eta)|\right] < \infty$*. Then*

$$\mathbb{E}\left[f(\xi, \eta)|\eta\right] = \int f(s, \eta)\, \mu(\eta, ds),\ \mathbb{P} - a.s.,$$

where $\mu(\eta, \cdot)$ *is the regular conditional probability of* ξ *given* η*.*

Proof of Lemma 2: We proceed by induction in r. For $r = 1$ the result follows because $X_1 - X_0|X_0 \sim X_1$ and $X_1 \preceq Y_1$.

Now, define $f(\xi, \eta) = 1\{\xi + \eta \geq t\}$. Hence

$$\begin{aligned} \mathbb{P}\left(X_r \geq t\right) &= \mathbb{P}\left(X_{r-1} + X_r - X_{r-1} \geq t\right) \\ &= \mathbb{E}\left[f(X_{r-1}, X_r - X_{r-1})\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f(X_{r-1}, X_r - X_{r-1})|X_{r-1}\right]\right] \end{aligned}$$

Let $\mu(X_{r-1}, \cdot)$ be the regular conditional probability of $X_r - X_{r-1}$ given X_{r-1} and ν the distribution of Y_r. By Proposition 3, the expression above is equal to

$$\mathbb{E}\left[\int f(X_{r-1}, s)\, \mu(X_{r-1}, ds)\right].$$

By assumption, we have $X_r - X_{r-1}|X_{r-1} \preceq Y_r$ and this means that $\mu(X_{r-1}, \cdot) \preceq \nu$, $\mathbb{P}$-almost surely. As $f(\xi, \cdot)$ is increasing upper semicontinuos we have that

$$\mathbb{E}\left[\int f(X_{r-1}, s)\, \mu(X_{r-1}, ds)\right] \leq \mathbb{E}\left[\int f(X_{r-1}, s)\, \nu(ds)\right].$$

Now, take $\widetilde{Y}_r \sim Y_r$ independent of X_{r-1} and let ϑ be the distribution of $\widetilde{Y}_r$. The last expression is equal to

$$\mathbb{E}\left[\int f(X_{r-1}, s)\,\vartheta(ds)\right] = \mathbb{P}\left(X_{r-1} + \widetilde{Y}_r \geq t\right).$$

Since $\widetilde{Y}_r$ is independent of X_{r-1}, we can use the substitution principle to obtain

$$\begin{aligned}
\mathbb{P}\left(X_{r-1} + \widetilde{Y}_r \geq t\right) &= \mathbb{E}\left[\mathbb{E}\left[f(X_{r-1}, \widetilde{Y}_r)|\widetilde{Y}_r\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[f(X_{r-1}, \tilde{y}_r)\right](\tilde{y}_r = \widetilde{Y}_r)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(X_{r-1} \geq t - \tilde{y}_r\right)(\tilde{y}_r = \widetilde{Y}_r)\right].
\end{aligned}$$

By the induction hypothesis, there exist $\widetilde{Y}_i \sim Y_i$, for $i = 1, \ldots, r-1$, independent random variables such that $X_{r-1} \preceq \widetilde{Y}_1 + \cdots + \widetilde{Y}_{r-1}$ (we can take $\widetilde{Y}_1, \ldots, \widetilde{Y}_{r-1}$ independent of $\widetilde{Y}_r$). Therefore,

$$\begin{aligned}
\mathbb{P}\left(X_r \geq t\right) &\leq \mathbb{P}\left(X_{r-1} + \widetilde{Y}_r \geq t\right) \\
&\leq \mathbb{P}\left(\widetilde{Y}_1 + \cdots + \widetilde{Y}_{r-1} + \widetilde{Y}_r \geq t\right)
\end{aligned}$$

and the result follows. □

Part II

Fluctuations of cover times and the geometry of the set of uncovered points

CHAPTER 5

Key ideas

The purpose of this part is to study the distribution of the cover time and the geometry of the last points covered by the random walk on graphs.

Our starting point is the fact that hitting times of moderately large sets are approximately exponentially distributed. That is to say, we show in Chapter 7 that, for moderately large $|S|$, if x is "far" from S (ie. $B(x) \cap S = \emptyset$),

$$\mathbb{P}_x (H_S > t) \approx e^{-\frac{t}{\mathbb{E}_\pi[H_S]}}. \tag{5.1}$$

In particular, if $S = \{s\}$ is a singleton,

$$\mathbb{P}_x (H_s > t) \approx e^{-\frac{t}{\mathfrak{h}(Q)}}.$$

We will need strong, quantititative forms of these inequalities to get the kind of approximation we need. In particular, it is important that the initial point of the trajectory of X_t is *not stationary* and that we get tail bounds that decrease fast with t. This is not achieved by the classical papers [1, 4], and we resort to recent inequalities by Oliveira [33].

By itself, formula (5.1) does not say much about $\mathbb{E}_\pi [H_S]$. An upshot of assumption **A3** is that "most" S of moderate size are "spread out" in the sense that that $\forall (a, b) \in (S)_2 \; b \notin B(a)$. We will use further ideas from [33] to show (see Proposition 4) that in this case:

$$\text{If } S \cup \{x\} \text{ is spread out, } \mathbb{P}_x (H_S > t) \approx e^{-\frac{t\,|S|}{\mathfrak{h}(Q)}}. \tag{5.2}$$

In particular, if S is spread out, the hitting time of S is about $1/|S|$ times the hitting time of a single vertex.

Furthermore, we add a result about the point of S that is reached at time H_S, which may have other applications. We show (see Theorem 5) that, under the restrictions described above,

$$\mathrm{Law}_x\left(\frac{H_S}{\mathsf{h}(Q)}, X_{H_S}\right) \approx \mathsf{Exp}\left(\frac{1}{|S|}\right) \otimes \mathrm{Unif}(S). \tag{5.3}$$

That is, the location of X_{H_S} is approximately uniform over S, and $\frac{H_S}{\mathsf{h}(Q)}$ and X_{H_S} are approximately independent. This will determine the evolution of the set of uncovered points. In fact, it implies that each vertex of S survives for time $\mathsf{Exp}(\mathsf{h}(Q))$ independently.

However, we have the restrictions that S is of moderate size and $S \cup \{x\}$ is spread-out. Our solution to this is the same as Belius's. We divide the process into two phases:

- **Phase** 1 lasts up to time $t_0 = \mathsf{h}(Q)(\ln n + \beta)$. We will see that at the end of this phase the set U_{t_0} of uncovered vertices has approximate product law. This result uses the exponential approximations just described as well as a variant of Brun's Sieve described in Theorem 6 below. The latter Theorem shows that in a "Poissonian" setting, it suffices to show that probabilities of the type:

  $$\mathbb{P}\left(\forall a \in S, a \text{ is uncovered}\right)$$

  behave as they should for "most" S of constant size, and that they are not too large for other S. We note that in principle one could apply sharper bounds, such as those obtained by the Chen-Stein method [6]. However, it seems difficult to check the assumptions of these results via our techniques. Our Theorem might also prove useful in other circumstances.

  As an immediate consequence of the approximate product law we have the Gumbel law fluctuations for cover times as mentioned in the introduction. We also obtain as corollary of the approximate product law that:

  - $|U_{t_0}| = (1 + o(1))\, e^{-\beta}$, if $e^{-\beta} \gg 1$;
  - $\{X_{t_0}\} \cup U_{t_0}$ is spread-out, in the sense described above.

- Assuming this typical event, **phase** 2 lasts from t_0 up to the cover time. As described above, each vertex of U_{t_0} survives for time $\mathsf{Exp}(\mathsf{h}(Q))$ independently.

Therefore we have a complete description of the set of uncovered points after time $t_0 = \mathsf{h}(Q)(\ln n + \beta)$.

Remark 5. *A direct proof of all of our results is quite trivial for complete graphs. Beliu's main conceptual contribution in [8, 9, 10] was to notice that the facts that we just described, which also hold true for complete graphs, suffice to obtain Gumbel limits for tori. We add to his insight by showing that a list of weaker and more abstract conditions actually gives stronger results.*

CHAPTER 6

Preliminaries

6.1 Basic notation

Given a set S, $(S)_k$ denotes the set of all ordered k-tuples of distinct elements of A whereas $\binom{S}{k}$ denotes the set of all subsets of size k.

Given a function $f : \mathbb{R} \to \mathbb{R}$, we let

$$\|f\|_{\mathrm{Lip}} = \sup_{(x,y)\in(\mathbb{R})_2} \left| \frac{f(x) - f(y)}{x - y} \right|.$$

We allow suprema to be infinite.

$M_1(S)$ denotes the set of probability measures over a set S, with the "natural" σ-field (which we will never specify explicitly). The total variation metric on $M_1(S)$ is given by:

$$d_{\mathrm{TV}}(\mu, \lambda) = \max\{\lambda(A) - \mu(A) \,:\, A \subset S \text{ measurable}\}.$$

If X is a A-valued random variable, $\mathrm{Law}\,(X)$ denotes the law or distribution of X.

We now define two distances over $M_1(\mathbb{R})$. The L_1 Wasserstein distance on $M_1(\mathbb{R})$ is given by:

$$d_W(\eta_1, \eta_2) \equiv \int_{\mathbb{R}} |\eta_1((-\infty, t]) - \eta_2((-\infty, t])|\, dt \ \ (\eta_1, \eta_2 \in M_1(\mathbb{R})).$$

This expression defines a metric (with potentially infinite values) which is always finite when restricted to measures with finite first moments. In that case we also have a dual formula [35]:

$$d_W(\eta_1, \eta_2) \equiv \sup\left\{ \int_{\mathbb{R}} f\, d(\eta_1 - \eta_2) \,:\, f : \mathbb{R} \to \mathbb{R},\ \|f\|_{\mathrm{Lip}} \le 1 \right\}.$$

We will sometimes abuse notation and write $d_W(X, Y)$ instead of $d_W(\mathrm{Law}\,(X)\,, \mathrm{Law}\,(Y))$, and note that $|\mathbb{E}\,[X] - \mathbb{E}\,[Y]\,| \le d_W(X, Y)$.

6.2 Markov chains

In this paper Q always denotes (the generator of) a continuous-time Markov chain over a finite state space $\mathbf{V}$, with transition rates $q(x,y)$, $(x,y) \in (\mathbf{V})_2$. Most concrete examples of Q's we discuss come from finite graphs. Indeed, a graph $\mathbf{G}$ over the set $\mathbf{V}$, where each $x \in \mathbf{V}$ has degree d_x, naturally corresponds to a specific Q given by:

$$q(x,y) \equiv \begin{cases} \frac{1}{d_x}, & \text{if } x \text{ and } y \text{ are adjacent in } \mathbf{G}; \\ 0, & \text{otherwise.} \end{cases}$$

Trajectories of Q will be denoted by $\{X_t\}_{t\geq 0}$, and the space of all trajectories is denoted by $\mathbb{D} \equiv \mathbb{D}([0,+\infty),\mathbf{V})$. The law of $\{X_t\}_{t\geq 0}$ when started from $x \in \mathbf{V}$ or $\lambda \in M_1(\mathbf{V})$ will be denoted by $\mathbb{P}_x$ or $\mathbb{P}_\lambda$. Moreover, if $Y : \mathbb{D} \to S$ is a random variable, $\mathrm{Law}_x(Y)$ and $\mathrm{Law}_\lambda(Y)$ denote the law (or distribution) of Y under $\mathbb{P}_x$ or $\mathbb{P}_y$. All chains we consider will be assumed to be *irreducible*, meaning that the digraph with vertex set $\mathbf{V}$ and arcs corresponding to $(a,b) \in (\mathbf{V})_2$ with $q(a,b) > 0$ is strongly connected. It is well-known that an irreducible Q has a unique stationary measure $\pi \in M_1(\mathbf{V})$. Moreover, for any $\epsilon > 0$ the ϵ-mixing-time of Q (defined below) is well-defined and finite.

$$\mathrm{t}^Q_{\mathrm{mix}}(\epsilon) \equiv \inf\{t \geq 0 \,:\, \max_{x\in\mathbf{V}} d_{\mathrm{TV}}(\mathbb{P}_x(X_t \in \cdot), \pi) \leq \epsilon\}.$$

Omitting ϵ correponds to taking $\epsilon = 1/4$. We also define the uniform mixing time:

$$\mathrm{t}^Q_{\mathrm{unif}} \equiv \inf\{t_0 \geq 0 \,:\, \forall t \geq t_0,\, \forall x,y \in \mathbf{V},\, \mathbb{P}_x(X_t = y) \leq 2\pi(y)\}.$$

The hitting time of a set $\emptyset \not\subset S \subset \mathbf{V}$ is:

$$H_S \equiv \inf\{t \geq 0 \,:\, X_t \in S\}.$$

We write H_a or $H_{a,b}$ when $A = \{a\}$ for some $a \in \mathbf{V}$ or $A = \{a,b\} \in \binom{\mathbf{V}}{2}$ (resp.). Finally,

$$\mathsf{C}(S) \equiv \max_{a\in S} H_a$$

and $\mathsf{C} \equiv \mathsf{C}(\mathbf{V}) = \max_{a\in\mathbf{V}} H_a$.

Now, fix $T, T' \in \mathbb{R}$, with $T < T'$, and let $\mathbb{D}([T,T'], 2^{\mathbf{V}(G)})$ be the space of functions w from $[T,T']$ to $2^{\mathbf{V}(G)}$ that are right-continuous and have left-hands limits. Where $2^{\mathbf{V}(G)}$ denotes the space of subsets of $\mathbf{V}(G)$.

Let Λ be the class of strictly increasing, continuous mapping of $[T,T']$ onto itself with $\lambda(T) = T$ and $\lambda(T') = T'$. If we consider the metric defined by

$$J(w_1,w_2) = \inf_{\lambda\in\Lambda} \left\{ \ln||\lambda||_{\mathrm{Lip}} + \ln||\lambda^{-1}||_{\mathrm{Lip}} + \mathbf{1}\{\exists t \in [-T,T] : w_1(t) \neq w_2(\lambda(t))\} \right\},$$

where $||\lambda||_{\mathrm{Lip}} = \sup_{x,y\in[T,T'];x\neq y} \left|\frac{\lambda(x)-\lambda(y)}{x-y}\right|$. It is well known that $\mathbb{D}\left([T,T'], 2^{\mathbf{V}(G)}\right)$ is a Polish space with the metric $J(\cdot,\cdot)$, see [12].

Also denote the Prohorov metric d_J on probability measures over $\mathbb{D}\left([\beta,\beta'],2^{\mathbf{V}(G)}\right)$ by

$$d_P(\mu,\nu)=\inf\left\{\epsilon>0:\forall F\subset\mathbb{D}\left([T,T'],2^{\mathbf{V}(G)}\right)\text{ closed, }\mu(F)\le\nu(F^\epsilon)+\epsilon\text{ and }\nu(F)\le\mu(F^\epsilon)+\epsilon\right\},$$

where μ,ν are probability measures over $\mathbb{D}\left([\beta,\beta'],2^{\mathbf{V}(G)}\right)$ and

$$F^\epsilon=\{w\in\mathbb{D}\left([\beta,\beta'],2^{\mathbf{V}(G)}\right):J(w,w')\le\epsilon\text{ for some }w'\in F\}.$$

CHAPTER 7

Exponential approximation

The main goal of this chapter is to collect the results on exponential approximation of hitting times that we need in the proof of the Main Theorem. We will need the following definitions.

$$\epsilon_0(Q) = C\sqrt{\frac{\mathrm{t}_{\mathrm{mix}}^Q}{\mathbb{E}_\pi\left[H_S\right]}\ \ln\left(\frac{\mathbb{E}_\pi\left[H_S\right]}{\mathrm{t}_{\mathrm{mix}}^Q}\right)} \tag{7.1}$$

$$\epsilon_1(Q) = C\,\epsilon_0(Q)\vee\alpha_3(Q),\ \text{where } \alpha_3(Q) \text{ comes from } \mathbf{A3} \tag{7.2}$$

$$\epsilon_2(Q) = C\,\ln n\sqrt{\frac{\mathrm{t}_{\mathrm{mix}}^Q}{\mathbb{E}_\pi\left[H_S\right]}\ \ln\left(\frac{\mathbb{E}_\pi\left[H_S\right]}{\mathrm{t}_{\mathrm{mix}}^Q}\right)} \tag{7.3}$$

$$\tag{7.4}$$

The constant $C > 0$ is universal, but we will not define it explicitly. We just need it to be big enough so that the estimates presented below hold. There will also be a fourth universal constant $\eta > 0$ which we will require to be small enough, and we will assume $\epsilon_0(Q), \epsilon_1(Q), \epsilon_2(Q) \leq \eta$. Finally, we will allow ourselves to increase C or decrease η in the middle of a proof, so that eg. we could write:

$$e^{\epsilon_0(Q)} \leq 1 + \epsilon_0(Q).$$

Lemma 5. *If C is large enough and η is small enough, we have the inequality*

$$\mathbb{P}_\pi\left(H_S \leq \mathrm{t}_{\mathrm{mix}}^Q(\epsilon_0(Q)^2)\right) \leq \epsilon_0(Q)^2.$$

Proof: This was proved for the hitting time of diagonal sets (ie. meeting times) in [33, Proposition 4.1]. The same argument applies to the present case. □

Henceforth the Markov chain Q will be implicit in some notations. The following definition and Theorem come from [33].

Definition 1. *Given $\alpha > 0$, $\beta \in (0,1)$ and $m > 0$, we say that a distribution μ over $[0,+\infty)$ is* $\mathsf{Exp}(m,\alpha,\beta)$ *if for all $t \geq 0$:*

$$\mu((t,+\infty)) = (1\pm\alpha)_+ \, e^{-\frac{t}{(1\pm\beta)\,m}}.$$

We write $Z =_d \mathsf{Exp}(m,\alpha,\beta)$ if $\mathrm{Law}\,(Z)$ is $\mathsf{Exp}(m,\alpha,\beta)$. We also define $\mathsf{Exp}(m) = \mathsf{Exp}(m,0,0)$.

Theorem 4 ([33])**.** *There exists a universal constant $C > 0$ such that the following holds. In the above Markov chain setting, assume that $0 < \epsilon \leq \delta < 1/10C$ are such that:*

$$\mathbb{P}_\pi \left(H_S \leq \mathrm{t}_{\mathrm{mix}}(\delta\epsilon)\right) \leq \delta\epsilon.$$

Let $t_\epsilon(S)$ be the ϵ-quantile of $\mathrm{Law}_\pi\,(H_S)$, ie. the unique number $t_\epsilon(S) \in [0,+\infty)$ with $\mathbb{P}_\pi\,(H_S \leq t_\epsilon(S)) = \epsilon$ (this is well-defined since $\mathbb{P}_\pi\,(H_S \leq t)$ is a continuous and strictly increasing function of t in our setting).Given $\lambda \in M_1(\mathbf{V})$, write:

$$r_\lambda(S) \equiv \mathbb{P}_\lambda \left(H_S \leq \mathrm{t}_{\mathrm{mix}}(\delta\epsilon)\right).$$

Then:

$$\mathrm{Law}_\lambda\,(H_S) = \mathsf{Exp}\left(\frac{t_\epsilon(S)}{\epsilon}, C\,\epsilon + 2r_\lambda(S), C\,\delta\right).$$

Moreover,

$$\left|\frac{\epsilon\mathbb{E}_\pi\,[H_S]}{t_\epsilon(S)} - 1\right| = O\,(\delta + r_\lambda)$$

and:

$$\mathrm{Law}_\lambda\,(H_S) =_d \mathsf{Exp}\left(\mathbb{E}_\pi\,[H_S]\,, C\,\epsilon + 2r_\lambda, C\,\delta\right).$$

We will not prove this, but the main ideas in this proof are in Proposition 4 below. Let us begin to prove the results we will need later on.

Lemma 6. *If C_0, C_1 are large enough, for any $t \geq 0$:*

$$\mathbb{P}_\pi\,(H_a > t) = \{1 \pm \epsilon_0(Q)\}\,\exp\left(-\frac{t}{(1\pm\epsilon_0(Q))\,\mathsf{h}(Q)}\right).$$

Moreover, there exists some universal $\eta > 0$ such that if $S \subset \mathbf{V}$ and $0 < |S| < \eta/\epsilon_0(Q)$,

$$\mathbb{P}_\pi\,(H_S > t) = \{1 \pm |S|\,\epsilon_0(Q)\}\,\exp\left(-\frac{t}{(1\pm|S|\,\epsilon_0(Q))\,\mathbb{E}_\pi\,[H_S]}\right).$$

and

$$\mathbb{E}_\pi\,[H_S] \geq (1 - |S|\,\epsilon_0(Q))\,\frac{\mathsf{h}(Q)}{|S|}.$$

Proof: This proof will contain a few statements that are not necessary for the end result, which will be useful for later lemmas.

If $S = \{a\}$ is a singleton, one may apply Theorem 4 with

$$\epsilon \in [\epsilon_0(Q), 2\epsilon_0(Q)],\ \delta = 2\epsilon_0(Q)$$

to deduce:

$$\forall \epsilon \in [\epsilon_0(Q), 2\epsilon_0(Q)] \,:\, \mathrm{Law}_\lambda\left(H_a\right) = \mathsf{Exp}\left(\mathsf{h}(Q), C\,\epsilon_0(Q) + 2r_\lambda(\{a\}), C\,\epsilon_0(Q)\right) \quad (7.5)$$

$$\text{and } t_\epsilon(\{a\}) = \left(1 + O\left(\epsilon_0(Q)\right)\right)\epsilon\,\mathsf{h}(Q). \quad (7.6)$$

Notice that this implies the Lemma in the case $|S| = 1$ since (increasing C_0 if necessary) for $|A| = 1$, by definition of $\mathsf{Exp}(m, \alpha, \beta)$, since $\mathbb{P}_\pi\left(H_a \le \mathrm{t}^Q_{\mathrm{mix}}(2\epsilon_0(Q)^2)\right) \le \epsilon_0(Q)^2$.

Now consider some general set S with $|S| \le \eta/\epsilon_0(Q)$. Choose:

$$\delta = 2|S|\,\epsilon_0(Q) \text{ and } \epsilon \in [\epsilon_0(Q), 2\epsilon_0(Q)]$$

and again apply Theorem 4 (assuming η is small enough). We have

$$\mathbb{P}_\pi\left(H_S \le \mathrm{t}^Q_{\mathrm{mix}}(\delta\epsilon)\right) \le \sum_{a\in S}\mathbb{P}_\pi\left(H_a \le \mathrm{t}^Q_{\mathrm{mix}}(\epsilon_0(Q)^2)\right) = \frac{|S|\,\epsilon_0(Q)^2}{4} \le \delta\epsilon.$$

Thus the conditions of the Theorem are satisfied, and we obtain:

$$\forall \epsilon \in [\epsilon_0(Q), 2\epsilon_0(Q)],\, \mathrm{Law}_\lambda\left(H_S\right) = \mathsf{Exp}\left(\mathbb{E}_\pi\left[H_S\right], C\,|S|\epsilon_0(Q) + 2r_\lambda(S), C\,|S|\epsilon_0(Q)\right) \quad (7.7)$$

$$\text{and } t_{|S|\epsilon}(S) = \left(1 + O\left(\epsilon_0(Q)\,|S|\right)\right)\epsilon|S|\mathbb{E}_\pi\left[H_S\right]. \quad (7.8)$$

This implies the two equations in the Theorem (by increasing C_0 and decreasing δ, if necessary), as:

$$\mathbb{P}_\pi\left(H_S \le \mathrm{t}^Q_{\mathrm{mix}}(\epsilon_0(Q)^2)\right) \le \sum_{a\in S}\mathbb{P}_\pi\left(H_S \le \mathrm{t}^Q_{\mathrm{mix}}(\epsilon_0(Q)^2)\right) \le |S|\epsilon_0(Q)^2.$$

The last step is the inequality relating $\mathbb{E}_\pi\left[H_S\right]$ to $\mathsf{h}(Q)$. By transitivity, $t_\epsilon(\{a\})$ is independent of a, therefore:

$$\mathbb{P}_\pi\left(H_S \le t_\epsilon(\{a\})\right) \le \sum_{b\in S}\mathbb{P}_\pi\left(H_b \le t_\epsilon(\{b\})\right) = |S|\,\epsilon.$$

Therefore, $t_\epsilon(\{a\}) \le t_{|S|\,\epsilon}(S)$ and (7.8), (7.6) imply:

$$\mathbb{E}_\pi\left[H_S\right] \ge \left(1 - O\left(\epsilon_0(Q)|S|\right)\right)\frac{\mathsf{h}(Q)}{|S|}. \quad (7.9)$$

□

We will also need the following definition.

Definition 2. *We define the collection of well-separated sets:*

$$\mathcal{S} \equiv \{S \subset V \,:\, S \neq \emptyset,\, \forall (a,b) \in (S)_2,\, b \notin B(a).\}.$$

Define the graph $\mathcal{G} = (V, \{ab \,:\, a \in B_Q(b) \text{ or } b \in B_Q(a)\})$. Also define $\mathcal{S}^k_l = \{S \in \binom{V_n}{k} \,:$ $\mathcal{G}[S]$ has l connected components$\}$, this way $\binom{V_n}{k} = \cup_{l=1}^k \mathcal{S}^k_l$, moreover, $S \in \mathcal{S}^k_l$ is independent set in the graph $\mathcal{G}_n$ if and only if $l = k$. For $S \in \mathcal{S}^k_l$, we can write $S = \cup_{i=1}^l S_i$, where the S_i's are the connected components of S in $\mathcal{G}$. We also have $H_S = \min_{1\leq i\leq l} H_{S_i}$.

The following result show that if $S \in \mathcal{S}_l^k$, then $\mathbb{P}_\pi(H_S > t) \approx \prod_{i=1}^l \exp\left(-\frac{t}{\mathbb{E}_\pi\left[H_{S_i}\right]}\right)$.

Proposition 4. *For $S \in \mathcal{S}_l^k$ and $0 \leq t \leq 2\mathsf{h}(Q)\ln n$, we have that*

$$\mathbb{P}_\pi(H_S > t) = (1 + o(1))\exp\left(-\frac{t}{\mu}\right), \tag{7.10}$$

where $\mu = \frac{1}{\sum_{i=1}^l \frac{1}{\mathbb{E}_\pi\left[H_{S_i}\right]}}$ and the term $o(1)$ is uniform for $k = |S|$ fixed.

Proof: Initially, take $S \in \mathcal{S}_l^k$, so $S = \cup_{i=1}^l S_i$. Define $\mu_i = \mathbb{E}_\pi[H_{S_i}]$ and $\mu = \frac{1}{\sum_{i=1}^l \frac{1}{\mu_i}}$. First, we show that $\mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) \approx \frac{\epsilon_0(Q)\mu}{\mu_i}$, that is, for $1 \leq i \leq l$, we have that

$$\mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) = (1 + O(\delta))\frac{\epsilon_0(Q)\mu}{\mu_i}, \tag{7.11}$$

where $\delta = 2\epsilon_0(Q)|S|$.

Note that $\mu_i \leq \mathbb{E}_\pi[H_a]$, for all $a \in S_i$, and by **A0** we have $\mathbb{E}_\pi[H_a] \leq (1 + o\left(\frac{1}{\ln n}\right))\mathsf{h}(Q)$. For other side, by Lemma 6, $\mathbb{E}_\pi[H_{S_i}] \geq (1 - |S_i|\epsilon_0(Q))\frac{\mathsf{h}(Q)}{|S_i|}$. Then, for n large, $\frac{\mu}{\mu_i} \geq \frac{1}{2|S|}$. Taking $\epsilon' \in [\frac{\epsilon_0(Q)\mu}{2|S|\mu_i}, \frac{\epsilon_0(Q)\mu}{\mu_i}]$ follows that $\delta\epsilon' \geq \epsilon_0(Q)^2$ and so $\mathrm{t}_{\mathrm{mix}}^Q(\delta\epsilon') \leq \mathrm{t}_{\mathrm{mix}}^Q(\epsilon_0(Q)^2)$. Therefore, by Lemma 5

$$\mathbb{P}_\pi\left(H_{S_i} \leq \mathrm{t}_{\mathrm{mix}}^Q(\delta\epsilon')\right) \leq \sum_{s\in S_i}\mathbb{P}_\pi\left(H_s \leq \mathrm{t}_{\mathrm{mix}}^Q(\epsilon_0(Q)^2)\right) \leq |S_i|\epsilon_0(Q)^2 \leq \delta\epsilon'.$$

So, we can use Theorem 4 and obtain that

$$\left|\frac{\epsilon'\mu_i}{t_{\epsilon'(S_i)}} - 1\right| \leq O(\delta)$$

and then

$$(1 - O(\delta))\frac{\epsilon_0(Q)\mu}{\mu_i} \leq \mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) \leq (1 + O(\delta))\frac{\epsilon_0(Q)\mu}{\mu_i},$$

and we have (7.11).

Now, we will show that

$$\mathbb{P}_\pi(H_S \leq \epsilon_0(Q)\mu) = (1 + O(\delta))\epsilon_0(Q). \tag{7.12}$$

To get (7.12), first observe that, using (7.11), we have

$$\begin{aligned}\mathbb{P}_\pi(H_S \leq \epsilon_0(Q)\mu) &\leq \sum_{i=1}^l \mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) \\ &\leq (1 + O(\delta))\epsilon_0(Q)\sum_{i=1}^l \frac{\mu}{\mu_i} \\ &= (1 + O(\delta))\epsilon_0(Q).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{P}_\pi\left(H_S \le \epsilon_0(Q)\mu\right) &\ge \sum_{i=1}^{l} \mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu\right) \\ &\quad - \sum_{1\le i<j\le l} \mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu, H_{S_j} \le \epsilon_0(Q)\mu\right).\end{aligned}$$

The last term can be written as

$$\begin{aligned}\mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu, H_{S_j} \le \epsilon_0(Q)\mu\right) &= \mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu, H_{S_j} \le \epsilon_0(Q)\mu, H_{S_i} < H_{S_j}\right) \\ &\quad + \mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu, H_{S_j} \le \epsilon_0(Q)\mu, H_{S_j} < H_{S_i}\right).\end{aligned}$$

Using Markov property we arrive at

$$\mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu, H_{S_j} \le \epsilon_0(Q)\mu, H_{S_i} < H_{S_j}\right) \le \mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu\right) \max_{s\in S_i} \mathbb{P}_s\left(H_{S_j} \le \epsilon_0(Q)\mu\right).$$

Remember that $s_j \notin B(s)$ for all $s_j \in S_j$. So, for $s \in S_i$, we can use Markov property and we get

$$\begin{aligned}\mathbb{P}_s\left(H_{S_j} \le \epsilon_0(Q)\mu\right) &\le \sum_{s_j\in S_j} \mathbb{P}_s\left(H_{s_j} \le \epsilon_0(Q)\mu\right) \\ &\le \sum_{s_j\in S_j} \left(\alpha_3(Q) + 2\mathbb{P}_\pi\left(H_{s_j} \le \epsilon_0(Q)\mu\right)\right) \\ &\le |S_j|\left(\alpha_3(Q) + (1+O(\delta))\frac{\epsilon_0(Q)\mu}{\mu_i}\right)\end{aligned}$$

Using Lemma 6, the last inequality reduces to

$$\mathbb{P}_s\left(H_{S_j} \le \epsilon_0(Q)\mu\right) \le \epsilon_1(Q)(1+O(\delta))^2|S|.$$

So,

$$\mathbb{P}_\pi\left(H_{S_i} \le \epsilon_0(Q)\mu, H_{S_j} \le \epsilon_0(Q)\mu\right) \le (1+O(\delta))^3\epsilon_1(Q)|S|\frac{\epsilon_0(Q)\mu}{\mu_i}$$

and we can write

$$\mathbb{P}_s\left(H_S \le \epsilon_0(Q)\mu\right) \ge (1-O(\delta))\epsilon_0(Q),$$

proving (7.12).

Finally, let us finish the proof of the claim. By (7.12) there exists a number $\rho = (1+O(\delta))\epsilon_0(Q)$ such that $\epsilon_0(Q)\mu = t_\rho(S)$. As $\delta\rho \ge \epsilon_0(Q)^2$ we have that

$$\mathbb{P}_\pi\left(H_S \le \mathrm{t_{mix}}(\delta\rho)\right) \le \sum_{i=1}^{l} \mathbb{P}_\pi\left(H_{S_i} \le \mathrm{t_{mix}}(\delta\rho)\right) \le \epsilon_0(Q)^2|S| = \delta\rho.$$

Then we can apply Theorem 4 and obtain that

$$\mathrm{Law}_{\pi}(H_S) = \mathsf{Exp}\left(\frac{t_\rho(S)}{\rho}, O(\delta), O(\delta)\right),$$

with

$$\frac{t_\rho(S)}{\rho} = \frac{\epsilon_0(Q)\mu}{(1+O(\delta))\epsilon_0(Q)} = (1+O(\delta))\mu.$$

Therefore,

$$\mathbb{P}_\pi(H_S > t) = (1 \pm O(\delta))_+ e^{-\frac{t}{(1\pm O(\delta))\mu}}.$$

As $\mu \geq (1-O(\delta))\frac{\mathsf{h}(Q)}{|S|}$ and $t \leq 2\mathsf{h}(Q)\ln n$, we have

$$\begin{aligned}\mathbb{P}_\pi(H_S > t) &= (1 \pm O(\delta))_+ e^{-\frac{t}{\mu}} e^{\pm\frac{O(\delta)t}{\mu}} \\ &= (1 \pm O(\delta))_+ e^{-\frac{t}{\mu}} e^{\pm\frac{O(\delta)}{|S|}}.\end{aligned}$$

Remember that $\delta = \epsilon_0(Q)|S|$ and $\epsilon_0(Q) = o\left(\frac{1}{\ln n}\right)$, so

$$\mathbb{P}_\pi(H_S > t) = (1+o(1))e^{-\frac{t}{\mu}},$$

and it finishes the proof of the Proposition. □

The next result shows that if $\{x\} \cup S$ is well separated, then

$$\mathrm{Law}_x\left(\frac{H_S}{\mathsf{h}(Q)}, X_{H_S}\right) \approx \mathsf{Exp}\left(\frac{1}{|S|}\right) \otimes \mathrm{Unif}(S).$$

The marginal distribution of $\frac{H_S}{\mathsf{h}(Q)}$ comes from [33] and we add here a new result about the point of S reached.

Theorem 5. *Let $S = S_1 \cup S_2 \cup \cdots \cup S_l$ be a subset of $\mathbf{V}$ with $|S| \cdot l = O(\ln n)$. Assume that for all distinct $1 \leq i, j \leq l$ and all $x \in S_i$ we have that $\mathbb{P}_x\left(H_{S_j} \leq \mathrm{t}_{\mathrm{unif}}^Q\right) \leq \epsilon$. Also let $v \in \mathbf{V}$ be such that $\mathbb{P}_v\left(H_S \leq \mathrm{t}_{\mathrm{unif}}^Q\right) \leq \epsilon$. If $I = \min\{i \in [l]; X_{H_S} \in S_i\}$ and $p_i = \frac{\mu}{\mu_i}$, then there exists a coupling of the pair (H_S, I) and $(\widetilde{H}_S, \widetilde{I})$, where $(\widetilde{H}_S, \widetilde{I})$ is a random vector with distribution $\mathsf{Exp}(\mu) \otimes \{p_i\}_{i=1}^l$, such that*

$$\mathbb{P}\left(I \neq \widetilde{I}\right) = O\left(|S|^2 \epsilon_0(Q)\right) \tag{7.13}$$

and

$$\mathbb{E}\left[\frac{|H_S - \widetilde{H}_S|}{\mu}\right] = O\left(|S|^2 \epsilon_0(Q)\right). \tag{7.14}$$

Proof: We start by showing (7.13). It is well known that we can couple I and $\widetilde{I}$ so that

$$\mathbb{P}\left(I \neq \widetilde{I}\right) = d_{TV}(I, \widetilde{I}) = \sum_{i=1}^{l} (\mathbb{P}_x(I = i) - p_i)_+$$

Let us find an upper bound for $\mathbb{P}_x(I = i)$. Note that,

$$\mathbb{P}_x(I = i) = \sum_{k=0}^{\infty} \mathbb{P}_x(I = i, H_S \in (k\epsilon_0(Q)\mu, (k+1)\epsilon_0(Q)\mu]). \tag{7.15}$$

Observe that, for n sufficiently large, we have

$$\mathbb{P}_\pi\left(H_S \leq \mathrm{t}_{\mathrm{mix}}^{Q}(\delta\epsilon_0(Q))\right) \leq \delta\epsilon_0(Q) < (1 + O(\delta))\epsilon_0(Q) = \mathbb{P}_\pi(H_S < \epsilon_0(Q)\mu) \tag{7.16}$$

then $T := \mathrm{t}_{\mathrm{mix}}^{Q}(\delta\epsilon_0(Q)) < \epsilon_0(Q)\mu$. Using this, we can bound $\mathbb{P}_x(I = i, H_S \in (k\epsilon_0(Q)\mu, (k+1)\epsilon_0(Q)\mu])$ by

$$\mathbb{P}_x\left(H_S > k\epsilon_0(Q)\mu, H_{S_i} \circ \Theta_{k\epsilon_0(Q)\mu} \leq T\right) + \mathbb{P}_x\left(H_S > k\epsilon_0(Q)\mu, H_{S_i} \circ \Theta_{k\epsilon_0(Q)\mu+T} \leq \epsilon_0(Q)\mu\right). \tag{7.17}$$

As $T = \mathrm{t}_{\mathrm{mix}}^{Q}(\delta\epsilon_0(Q))$, the conditional law of $X_{k\epsilon_0(Q)\mu+T}$ given $\mathcal{F}_{k\epsilon_0(Q)\mu}$ and the conditional law of $X_{(k-1)\epsilon_0(Q)\mu+T}$ given $\mathcal{F}_{(k-1)\epsilon_0(Q)\mu}$ are $\delta\epsilon_0(Q)$ close to π. Then, using Markov property, the expression (7.17) is less than or equal to

$$\mathbb{P}_x(H_S > (k-1)\epsilon_0(Q)\mu)(\mathbb{P}_\pi(H_{S_i} \leq T) + \delta\epsilon_0(Q)) + \mathbb{P}_x(H_S > k\epsilon_0(Q)\mu)(\mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) + \delta\epsilon_0(Q)) \tag{7.18}$$

Now, we will proceed as in Lemma 3.2 of [33]. Using an idea as above, we can prove that, for all $k \in \mathbb{N}\backslash\{1\}$,

$$\mathbb{P}_x(H_S > k\epsilon_0(Q)\mu) \leq (1 - \epsilon_0(Q) + O(\delta\epsilon_0(Q)))^k. \tag{7.19}$$

The case $k = 1$ follows by Markov property and (7.16).

We can apply (7.17), (7.18) and (7.19) in (7.15), concluding

$$\begin{aligned} \mathbb{P}_x(I = i) &\leq (\mathbb{P}_\pi(H_{S_i} \leq T) + \delta\epsilon_0(Q)) \sum_{i=0}^{\infty} (1 - \epsilon_0(Q) + O(\delta\epsilon_0(Q)))^{k-1} \\ &\quad + (\mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) + \delta\epsilon_0(Q)) \sum_{i=0}^{\infty} (1 - \epsilon_0(Q) + O(\delta\epsilon_0(Q)))^{k} \\ &\leq (\mathbb{P}_\pi(H_{S_i} \leq T) + \delta\epsilon_0(Q)) \frac{1}{\epsilon_0(Q) - O(\delta\epsilon_0(Q))} \\ &\quad + (\mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) + \delta\epsilon_0(Q)) \frac{1}{\epsilon_0(Q) - O(\delta\epsilon_0(Q))}. \end{aligned}$$

In Proposition 4 we have shown that $\mathbb{P}_\pi(H_{S_i} \leq \epsilon_0(Q)\mu) = (1 + O(\delta))\epsilon_0(Q)p_i$. As $\mathbb{P}_\pi(H_{S_i} \leq T) \leq$

$O\left(\delta\epsilon_0(Q)\right)$, it follows that

$$\mathbb{P}_x\left(I=i\right) \leq p_i + O\left(\delta\right)$$

and we have (7.13).

Now, we will study (7.14). Start building a coupling between H_S and $\widetilde{H}_S$ so that

$$\mathbb{E}\left[|H_S-\widetilde{H}_S||I=i,\widetilde{I}=k\right] = d_W(\mathrm{Law}_x\left(H_S|I=i\right),\mathsf{Exp}(\mu)).$$

The existence of this coupling is discussed in [35].

We can write

$$\mathbb{E}\left[|H_S-\widetilde{H}_S||I,\widetilde{I}\right] = \sum_{i,k=1}^{l}\mathbb{E}\left[|H_S-\widetilde{H}_S||I=i,\widetilde{I}=k\right]\mathbf{1}\{I=i,\widetilde{I}=k\}. \tag{7.20}$$

For $\{I=i,\widetilde{I}=i\}$, we have

$$\mathbb{E}\left[\mathbb{E}\left[|H_S-\widetilde{H}_S||I=i,\widetilde{I}=i\right]\mathbf{1}\{I=i,\widetilde{I}=i\}\right] \leq \mathbb{E}\left[\int_0^\infty |\mathbb{P}_x\left(H_S>t|I=i\right)-e^{-\frac{t}{\mu}}|dt\mathbf{1}\{I=i\}\right],$$

using Fubini's theorem we have that the last term is equal to

$$\int_0^\infty |\mathbb{P}_x\left(H_S>t,I=i\right)-e^{-\frac{t}{\mu}}\mathbb{P}_x\left(I=i\right)|dt. \tag{7.21}$$

Further,

$$|\mathbb{P}_x\left(H_S>t,I=i\right)-\mathbb{P}_x\left(H_S>t,I\circ\Theta_{t+T}=i\right)| \leq \mathbb{P}_x\left(H_S\in(t,t+T],I=i\right).$$

Therefore, equation (7.21) is less than or equal to

$$\int_0^\infty |\mathbb{P}_x\left(H_S>t,I\circ\Theta_{t+T}=i\right)-e^{-\frac{t}{\mu}}\mathbb{P}_x\left(I=i\right)| + \int_0^\infty \mathbb{P}_x\left(H_S\in(t,t+T],I=i\right)dt. \tag{7.22}$$

Summing the second term in equation (7.22), we get

$$\begin{aligned}\int_0^\infty \mathbb{P}_x\left(H_S\in(t,t+T]\right)dt &= \mathbb{E}_x\left[\int_{H_S-T}^{H_S} dt\right]\\ &= T\\ &= o\left(\mu\right),\end{aligned}$$

where we use Fubini's theorem in first equality.

Use Markov property to obtain

$$\mathbb{P}_x\left(H_S>t,I\circ\Theta_{t+T}=i\right) = \mathbb{P}_x\left(H_S>t\right)\left(\mathbb{P}_\pi\left(I=i\right)\pm\delta\epsilon_0(Q)\right).$$

Now, note that (7.10) still valid with δ_x instead π. So, we can bound the first term in equation (7.22)

$$\int_0^\infty |(1+o(1))e^{-\frac{t}{\mu}}\mathbb{P}_\pi(I=i) - e^{-\frac{t}{\mu}}\mathbb{P}_\pi(I=i)|dt = o(1)\mathbb{P}_\pi(I=i)\mu.$$

Finally,

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{l}\mathbb{E}\left[\frac{|H_S-\widetilde{H}_S|}{\mu}|I=i,\widetilde{I}=i\right]\mathbf{1}\{I=i,\widetilde{I}=i\}\right] &= \frac{1}{\mu}o(1)\mathbb{P}_\pi(I=i)\mu+\frac{T}{\mu}\\ &= o(1).\end{aligned}$$

The way which H_S and $\widetilde{H}_S$ were coupled implies that there exists a constant C such that

$$\mathbb{E}\left[\frac{|H_S-\widetilde{H}_S|}{\mu}|I=i,\widetilde{I}=k\right]\mathbf{1}\{I=i,\widetilde{I}=k\}\leq C.$$

As $\mathbb{P}\left(I\neq\widetilde{I}\right)\leq l\cdot O(\delta)$, we have that equation (7.14) follows using the statements above in (7.20). □

CHAPTER 8

The approximate product law

In this chapter we prove Theorem 2. We show that the uncovered set at time $t_0 = \mathsf{h}(Q)(\ln n + \beta)$ is approximately distributed as $\bigotimes_{v\in\mathbf{V}} \mathrm{Be}_{\frac{e^{-\beta}}{n}}$. We will deduce Theorem 2 from the following result, which we prove at the end of this chapter.

Theorem 6. *Suppose $\mathcal{G}_n = (\mathbf{V}_n, \mathcal{E}_n)$ is a sequence of graphs where each $\mathcal{G}_n$ has n vertices and maximal degree $\Delta_n = n^{o(1)}$. Assume that $(X_n(v))_{v\in\mathbf{V}_n}$ is a collection of indicator random variables indexed by $\mathbf{V}_n$ with $p_n(v) \equiv \mathbb{P}(X_n(v) = 1) = (1 + o(1))\frac{e^{-\beta}}{n}$, where $\beta \in \mathbb{R}$ is a constant. Assume further that, as $n \to +\infty$,*

$$\forall k \in \mathbb{N} : \sup_{S\in\binom{\mathbf{V}_n}{k} \text{ independent}} \left| \frac{\mathbb{E}\left[\prod_{v\in S} X_n(v)\right]}{e^{-\beta k}/n^k} - 1 \right| = o(1), \tag{8.1}$$

where the $o(1)$ term may depend on k. Also assume that there exists a constant $c > 0$ such that:

$$\forall k \in \mathbb{N}, \forall U \in \binom{\mathbf{V}_n}{k} \text{ not independent} : \mathbb{E}\left[\prod_{v\in U} X_n(v)\right] \le n^{-c-r(U)} \tag{8.2}$$

where $r(U)$ is the number of connected components in $\mathcal{G}_n[U]$. Then:

$$d_{\mathrm{TV}}(\mathrm{Law}\,(X_n(v) : v \in \mathbf{V}_n), \bigotimes_{v\in\mathbf{V}_n} \mathrm{Be}_{\frac{e^{-\beta}}{n}}) = o(1).$$

Let us apply the above theorem to the current case. Consider the graph:

$$\mathcal{G}_n = (\mathbf{V}_n, \{ab : a \in B_Q(b) \text{ or } b \in B_Q(a)\}).$$

If $\Delta_n = \max \mathrm{d}_{\mathcal{G}_n}(a)$, by **A3**, we have that $\Delta_n = n^{o(1)}$. If we take $S \in \binom{\mathbf{V}_n}{k}$ with S independent set, we can use **A0** and Proposition 4 to conclude (8.1). For $S = \cup_{i=1}^{l} \in \mathcal{S}_l^k$, with $l < k$, by **A2** there exists some S_i with at least two elements a, b, and then $\mathbb{E}_\pi[H_{S_i}] \leq \mathbb{E}_\pi[H_{a,b}] \leq \frac{\mathsf{h}(Q)}{1+\phi}$. Again using **A0**, it follows that:

$$\begin{aligned}\mathbb{P}_\pi(H_S > t) &\leq (1+o(1))\exp\left(-\frac{t(l+\phi)}{\mathsf{h}(Q)}\right)\\ &= (1+o(1))n^{-(l+\phi)}\end{aligned}$$

and we have (8.2). Therefore, we can apply Theorem 6 and we get

$$d_{\mathrm{TV}}(\mathrm{Law}_\pi(U_{t_0}), \bigotimes_{v\in\mathbf{V}_n}\mathrm{Be}_{e^{-\beta}/n}) = o(1), \tag{8.3}$$

proving Theorem 2.

The Bernoulli approximation immediately implies the concentration of $|U_{t_0}|$. Moreover, consider the event:

$$E \equiv \{\{X_{t_0}\}\cup U_{t_0}\in\mathcal{S}\}\cap\{||U_{t_0}|-e^{-\beta}|\leq\delta\, e^{-\beta}\}$$

where $\delta \geq 1/e^{-\beta}$. Theorem 2 implies the next result that will be needed after.

Lemma 7. *Given $\epsilon > 0$, there exists $\beta = \beta(\epsilon)\in\mathbb{R}$ such that*

$$\mathbb{P}_\pi(E) \geq 1-\epsilon,$$

for n large enough.

Proof of the Lemma 7: For a given $t \geq 0$, define the set of *bad pairs.*

$$B_{t_0} \equiv \{(a,b)\in(\mathbf{V})_2 : b\in B(a),\, H_{a,b} > t_0\}.$$

Clearly,

$$\{\{X_{t_0}\}\cup U_{t_0}\in\mathcal{S}\} = \{B_{t_0}=\emptyset\}\cap\left(\bigcap_{a\in\mathbf{V}}\{H_a > t_0 \Rightarrow X_{t_0}\notin B(a)\}\right).$$

Hence:

$$\mathbb{P}_\pi(E^c) \leq \mathbb{P}_\pi(B_{t_0}\neq\emptyset) + \sum_{a\in\mathbf{V}}\mathbb{P}_\pi(H_a > t_0 \text{ and } X_{t_0}\in B(a)) + \mathbb{P}_\pi\left(||U_{t_0}|-e^{-\beta}| > \delta\, e^{-\beta}\right). \tag{8.4}$$

We bound the three terms separately. We first compute:

$$\begin{aligned}\mathbb{P}_\pi(B_{t_0}\neq\emptyset) &= \mathbb{P}_\pi\left(\mathrm{Po}(e^{-\beta})>0\right)+o(1)\\ &= e^{-e^{-\beta}}+o(1)\\ &< \frac{\epsilon}{2},\end{aligned}$$

choosing β properly.

Define $s_0 = t_0 - \mathrm{t}_{\mathrm{unif}}^{Q}$. We now bound:

$$\begin{aligned}
\sum_{a\in\mathbf{V}} \mathbb{P}_\pi\left(H_a > t_0 \text{ and } X_{t_0}\in B(a)\right) &\le \sum_{a\in\mathbf{V}} \mathbb{P}_\pi\left(H_a > s_0 \text{ and } X_{t_0}\in B(a)\right)\\
\text{(Lemma 6 + “}t_0 \le 2\mathsf{h}(Q)\ln n\text{")} &\le \sum_{a\in\mathbf{V}} (1+\epsilon_1(Q))\, e^{-\frac{s_0}{\mathsf{h}(Q)}}\, \mathbb{P}_\pi\left(X_{\mathrm{t}_{\mathrm{unif}}^{Q}}\circ\Theta_{s_0}\in B(a)\mid H_a > s_0\right)\\
\text{(defn. of } \mathrm{t}_{\mathrm{unif}}^{Q}) &\le \sum_{a\in\mathbf{V}} (1+\epsilon_1(Q))\, e^{-\frac{s_0}{\mathsf{h}(Q)}}\, \pi(B(a))\\
(\pi \text{ uniform + } \mathbf{A3}) &\le \sum_{a\in\mathbf{V}} (1+\epsilon_1(Q))\, e^{-\frac{s_0}{\mathsf{h}(Q)}}\, \frac{o\left(n^{\phi}\right)}{n}\\
\text{(defn. of } s_0) &\le (1+\epsilon_1(Q))\, e^{\frac{\mathrm{t}_{\mathrm{unif}}^{Q}}{\mathsf{h}(Q)}}\, e^{-\beta}\, \frac{o\left(n^{\phi}\right)}{n}\\
(\mathrm{t}_{\mathrm{unif}}^{Q} \le \mathsf{h}(Q)) &\le (1+\epsilon_1(Q))\, e\, e^{-\beta}\, \frac{o\left(n^{\phi}\right)}{n}\\
&= o(1)\,.
\end{aligned}$$

Indeed, $\mathrm{t}_{\mathrm{unif}}^{Q} \le \mathsf{h}(Q)$ follows from $\mathrm{t}_{\mathrm{mix}}^{Q} \le \eta\,\mathsf{h}(Q)/\ln^2 n \ln\ln n$ and $\mathrm{t}_{\mathrm{unif}}^{Q} \le C\ln n \mathrm{t}_{\mathrm{mix}}^{Q}$ (which hold if C is large enough).

Finally, by Theorem 6, we have $|U_{t_0}| \approx \mathrm{Binom}(n, \frac{e^{-\beta}}{n})$. As $\mathrm{Binom}(n, \frac{e^{-\beta}}{n})$ is concentrated around $e^{-\beta}$, the same applies to $|U_{t_0}|$. It finishes the proof of the Lemma. □

Proof of Theorem 6: We start by denoting $\mathrm{Law}\,(X_n(v) : v\in\mathbf{V}_n)$ by μ and $\bigotimes_{v\in\mathbf{V}} \mathrm{Be}_{e^{-\beta}/n}$ by ν. Also denote $\mathcal{S} = \{S\subset\mathbf{V}_n : S \text{ is independent set in } \mathcal{G}_n\}$ and take $\epsilon > 0$. The total variation distance between μ and ν can be calculated as

$$\begin{aligned}
d_{\mathrm{TV}}(\mu,\nu) &= \sum_{S\subset\mathbf{V}_n} (\mu(S)-\nu(S))_+\\
&\le \sum_{S\notin\mathcal{S}} \mu(S) + \sum_{S\in\mathcal{S},\,|S|>R} \mu(S) + \sum_{S\in\mathcal{S}:\,|S|\le R} (\mu(S)-\nu(S))_+\\
&= (I)+(II)+(III).
\end{aligned}$$

Let us take care of the first term (I):

$$\begin{aligned}
\sum_{S\notin\mathcal{S}} \mu(S) &= \mathbb{P}\left(\exists v,w\in\mathbf{V}_n : v\sim w, X_n(v)=1, X_n(w)=1\right)\\
&\le \sum_{v\in V_n}\sum_{w\sim v} \mathbb{E}\left[X_n(v)X_n(w)\right]\\
&= n^{o(1)-c},
\end{aligned}$$

where we have the last equality because $\Delta_n = n^{o(1)}$ and by (8.2). So, $\sum_{S\notin\mathcal{S}}\mu(S) < \epsilon$ for n large enough.

Now, we bound the second term (II). Choose R such that $\frac{e^{-\beta(R+1)}}{(R+1)!} \leq \epsilon$, then

$$\begin{aligned}
\sum_{S\in\mathcal{S}:|S|>R} \mu(S) &= \mathbb{P}\left(\exists W \subset \{v \in V_n : X_n(v) = 1\} : |W| = R+1, W \in \mathcal{S}\right) \\
&\leq \sum_{W\in\mathcal{S},|W|=R+1} \mathbb{E}\left[\prod_{v\in W} X_n(v)\right] \\
&\leq (1+o(1))\binom{n}{R+1}\left(\frac{e^{-\beta}}{n}\right)^{R+1} \\
&\leq (1+o(1))\frac{e^{-\beta(R+1)}}{(R+1)!} \\
&\leq (1+o(1))\epsilon,
\end{aligned}$$

the last inequality follows by our choice of R.

At last, we study (III). By the inclusion-exclusion principle it can be shown that

$$\begin{aligned}
\nu(S) &= \mathbb{E}\left[\prod_{v\in S} X_n(v) \prod_{w\notin S}(1 - X_n(w))\right] \\
&= \sum_{l=0}^{n-|S|}(-1)^l \sum_{W\in\binom{V_n\setminus S}{l}} \mathbb{E}\left[\prod_{v\in S\cup W} X_n(v)\right].
\end{aligned}$$

Using Bonferroni inequalities, we have that for all $K = 1, \ldots, \frac{n-|S|}{2}$

$$\sum_{l=0}^{2K-1}(-1)^l \sum_{W\in\binom{V_n\setminus S}{l}} \mathbb{E}\left[\prod_{v\in S\cup W} X_n(v)\right] \leq \nu(S) \leq \sum_{l=0}^{2K}(-1)^l \sum_{W\in\binom{V_n\setminus S}{l}} \mathbb{E}\left[\prod_{v\in S\cup W} X_n(v)\right].$$

Choose K such that $\frac{e^{-\beta(2K)}}{2K!} < \epsilon$. We can bound $\sum_{S\in\mathcal{S}:|S|\leq K}(\mu(S) - \nu(S))_+$ by

$$\sum_{S\in\mathcal{S}:|S|\leq R}\left[\sum_{l=0}^{2K}(-1)^l \sum_{W\in\binom{V_n\setminus S}{l}}\left(\mathbb{E}\left[\prod_{v\in S\cup W} X_n(v)\right] - \left(\frac{e^{-\beta}}{n}\right)^{|S|+l}\right)\right]_+ + \sum_{S\in\mathcal{S}:|S|\leq R}\sum_{W\in\binom{V_n\setminus S}{2K}}\left(\frac{e^{-\beta}}{n}\right)^{|S|+2K} \tag{8.5}$$

To bound the second term in (8.5), note that

$$\begin{aligned}
\sum_{S\in\mathcal{S}:|S|\leq R}\sum_{W\in\binom{V_n\setminus S}{2K}}\left(\frac{e^{-\beta}}{n}\right)^{|S|+2K} &\leq \sum_{s=0}^{R}\binom{n}{s}\binom{n-s}{2K}\left(\frac{e^{-\beta}}{n}\right)^{|S|+2K} \\
&\leq \sum_{s=0}^{R}\frac{n^s}{s!}\cdot\frac{n^{2K}}{2K!}\cdot\left(\frac{e^{-\beta}}{n}\right)^{s+2K} \\
&\leq \frac{e^{-\beta 2K}e^{e^{-\beta}}}{2K!} \\
&< \epsilon e^{e^{-\beta}},
\end{aligned}$$

where the last inequality follows by our choice of K.

The other term in (8.5) is less than or equal to

$$\sum_{S\in\mathcal{S}:|S|\le R}\sum_{l=0}^{2K}\sum_{W\in\binom{V_n\setminus S}{l}:S\cup W\in\mathcal{S}}\left|\mathbb{E}\left[\prod_{v\in S\cup W}X_n(v)\right]-\left(\frac{e^{-\beta}}{n}\right)^{|S|+l}\right|+ \tag{8.6}$$

$$\sum_{S\in\mathcal{S}:|S|\le R}\sum_{l=0}^{2K}\sum_{W\in\binom{V_n\setminus S}{l}:S\cup W\notin\mathcal{S}}\left|\mathbb{E}\left[\prod_{v\in S\cup W}X_n(v)\right]-\left(\frac{e^{-\beta}}{n}\right)^{|S|+l}\right|. \tag{8.7}$$

We want to show that

$$\sum_{S\in\mathcal{S}:|S|\le R}\sum_{l=0}^{2K}\sum_{W\in\binom{V_n\setminus S}{l}:S\cup W\in\mathcal{S}}\left|\mathbb{E}\left[\prod_{v\in S\cup W}X_n(v)\right]-\left(\frac{e^{-\beta}}{n}\right)^{|S|+l}\right|<\epsilon, \tag{8.8}$$

for n sufficiently large. Denoting $\sup_{S\in\binom{\mathrm{V}_n}{l}:S\in\mathcal{S}}\left|\frac{\mathbb{E}[\prod_{v\in S}X_n(v)]}{e^{-\beta l}/n^l}-1\right|=f_l(n)$, by (8.1), we have that $\sup_{0\le l\le 2K}f_l(n)=o(1)$. Then, the left hand side of (8.8) is less than or equal to

$$\begin{aligned}\sum_{S\in\mathcal{S}:|S|\le R}\sum_{l=0}^{2K}\sum_{W\in\binom{V_n\setminus S}{l}:S\cup W\in\mathcal{S}}f_l(n)\left(\frac{e^{-\beta}}{n}\right)^{|S|+l} &\le \sum_{S\in\mathcal{S}:|S|\le R}\sum_{l=0}^{2K}f_l(n)\cdot\frac{n^l}{l!}\cdot\left(\frac{e^{-\beta}}{n}\right)^{|S|+l}\\ &\le \sum_{S\in\mathcal{S}:|S|\le R}o(1)\cdot\frac{e^{-\beta|S|}}{n^{|S|}}\cdot e^{e^{-\beta}}\\ &\le \sum_{s=0}^{R}o(1)\cdot\frac{n^s}{s!}\cdot\frac{e^{-e^{-\beta}}s}{n^s}\cdot e^{e^{-\beta}}\\ &\le o(1)\,e^{2e^{-\beta}}\end{aligned}$$

and we have (8.8) for n large enough.

Now we will show that the other sum in (8.6) is small, that is,

$$\sum_{S\in\mathcal{S}:|S|\le R}\sum_{l=0}^{2K}\sum_{W\in\binom{V_n\setminus S}{l}:S\cup W\notin\mathcal{S}}\left|\mathbb{E}\left[\prod_{v\in S\cup W}X_n(v)\right]-\left(\frac{e^{-\beta}}{n}\right)^{|S|+l}\right|<\epsilon. \tag{8.9}$$

Let $\mathcal{S}_t^m=\{U\subset\binom{V_n}{m}:U$ have t connected components$\}$. Let us find an upper bound for the number of elements in $\mathcal{S}_t^m$. If $S\in\mathcal{S}_t^m$, there exist $S_1,\dots,S_t$ connected components of S. First we choose $s_1,\dots,s_t$ vertices in V to be the root of each connected component of S. Thereafter, we choose an element s_{t+1} adjacent to one of the s_i: we have at most $t\Delta$ possibilities. We may continue to choose $s_{t+2},s_{t+3},\dots,s_k$ so that s_i is adjacent to s_j for some $1\le j<i$: we have at most $(k-1)\Delta$ possibilities for each choice. It is easy to see that all choices of $S\in\mathcal{S}_t^m$ can be obtained in this way. So:

$$|\mathcal{S}_t^m|\le n^t\Delta^{k-t}(k-1)^k=n^{t+o(1)}. \tag{8.10}$$

Then we have that there exists $h_t^m(n) \to 0$ as $n \to \infty$ such that $|S_t^m| \le n^{t+h_t^m(n)}$.

Therefore, the left hand side of (8.9) is bounded by

$$\begin{aligned}
\sum_{U \notin \mathcal{S}:|U| \le R+2K} \left(\mathbb{E}\left[\prod_{v \in U} X_n(v) \right] + \left(\frac{e^{-\beta}}{n} \right)^{|U|} \right) &= \sum_{m=1}^{R+2K} \sum_{U \notin \mathcal{S}:|U|=m} \left(\mathbb{E}\left[\prod_{v \in U} X_n(v) \right] + \left(\frac{e^{-\beta}}{n} \right)^{|U|} \right) \\
&= \sum_{m=1}^{R+2K} \sum_{t=1}^{R+2K} \sum_{U \in \mathcal{S}_t^m} \left(\mathbb{E}\left[\prod_{v \in U} X_n(v) \right] + \left(\frac{e^{-\beta}}{n} \right)^{|U|} \right) \\
&\le \sum_{m=1}^{R+2K} \sum_{t=1}^{R+2K} n^{t+h_t^m(n)} \left(n^{-c-t} + \left(\frac{e^{-\beta}}{n} \right)^{m} \right) \\
&< \epsilon,
\end{aligned}$$

and (8.9) follows for n large enough. Combining (8.8) and (8.9) we have that (III) is bounded by $(2 + e^{e^{-\beta}})\epsilon$.

Finally, using our bounds to (I), (II) and (III) we arrive at

$$d_{\mathrm{TV}}(\mathrm{Law}\,(X_n(v)\,:\,v \in \mathbf{V}_n)\,, \bigotimes_{v \in \mathbf{V}_n} \mathrm{Be}_{\frac{e^{-\beta}}{n}}) = d_{\mathrm{TV}}(\mu, \nu) \le (4 + e^{e^{-\beta}})\epsilon,$$

for n large enough. Since $\epsilon > 0$ is arbitrary, this finishes the proof of the theorem. □

CHAPTER 9

The evolution of the last points covered

In this chapter we analyze the evolution of the uncovered set after time $t_0 = \mathsf{h}(Q)(\ln n + \beta)$. For this, define $\tau_0 = t_0$, $U_0 = U_{t_0}$ and, for each $i = 1, \dots, |U_{t_0}|$,

$$\tau_i = \inf\{t \geq 0 : X_{\tau_{i-1}+t} \in U_{i-1}\},$$

$$I_i = X_{\tau_{i-1}+\tau_i},$$

$$U_i = U_{i-1} \backslash \{I_i\}.$$

Now, define the process $\mathcal{A} = \{A_c\}_{c \geq \beta}$ as follows:

1. For $c = \beta$, sample A_β from $\bigotimes_{v \in \mathbf{V}_n} \mathrm{Be}_{\frac{e^{-\beta}}{n}}$;
2. Each vertex of $a \in A_\beta$ survives for time $\mathsf{Exp}(1)$ independently.

The lack of memory property of exponentials implies that the process defined above is consistent. For each $a \in \mathbf{V}_n$, let E_a denote the exponential random variable with parameter 1 as described above. For the process $\mathcal{A}$ denote $\overline{\tau}_0 = \beta$ and, for each $i = 1, \dots, |A_0|$, define

$$\overline{\tau}_i = \min\{E_a : a \in A_{\overline{\tau}_{i-1}}\},$$

$$\overline{I}_i = \{a \in \mathbf{V}_n : \overline{\tau}_i = E_a\},$$

$$A_i = A_{i-1} \backslash \{\overline{I}_i\}.$$

Note that, for each i, $\overline{\tau}_i$ has distribution $\mathsf{Exp}\left(\frac{1}{A_{i-1}}\right)$. Moreover, $\overline{\tau}_1, \dots, \overline{\tau}_{|A_0|}$ are independent random variables.

Theorem 7. *For all $\beta, \beta' \in \mathbb{R}$, with $\beta < \beta'$, there exists a coupling between $\{A_c\}_{c \in [\beta, \beta']}$ and the process*

$\{\widetilde{U}_c\}_{c\in[\beta,\beta']} = \{U_{\mathsf{h}(Q)(\ln n+c)}\}_{c\in[\beta,\beta']}$ *such that*

$$\mathbb{P}\left(\widetilde{U}_\beta \neq A_\beta\right) = o\left(1\right), \tag{9.1}$$

$$\mathbb{P}\left(\bigcup_{i=1}^{|\widetilde{U}_\beta|}\{I_i \neq \overline{I}_i\}\right) = o\left(1\right), \tag{9.2}$$

and

$$\mathbb{E}\left[\min\left(\sum_{i=1}^{|\widetilde{U}_\beta|}\frac{|\frac{\tau_i}{\mathsf{h}(Q)} - \overline{\tau}_i|}{|\widetilde{U}_\beta|}, 1\right)\right] = o\left(1\right). \tag{9.3}$$

Proof: The proof follows from Theorem 6 and Theorem 8 below. □

Theorem 7 directly implies:

Theorem 3. *For all $\beta,\beta'\in\mathbb{R}$, with $\beta<\beta'$, there exists coupling between the process $\{A_c\}_{c\in[\beta,\beta']}$ and the process $\{\widetilde{U}_c\}_{c\in[\beta,\beta']} = \{U_{\mathsf{h}(Q)(\ln n+c)}\}_{c\in[\beta,\beta']}$ such that*

$$d_J\left(\mathrm{Law}_{\pi_Q}\left(\{\widetilde{U}_c\}_{c\in[\beta,\beta']}\right), \{A_c\}_{c\in[\beta,\beta']}\right) \to 0, \tag{9.4}$$

as $n\to\infty$, for all $\epsilon>0$.

Proof [sketch]: Define a function $\lambda = \lambda(c)$ by $\lambda(\beta)=\beta$, $\lambda(\beta')=\beta'$, and

$$\lambda\left(\frac{\sum_{i=1}^{i}\tau_i}{\mathsf{h}(Q)} - \beta\right) = \sum_{i=1}^{i}\overline{\tau}_i - \beta,$$

when $\sum_{j\leq i}\tau_j \leq \beta'$, for other c define $\lambda(c)$ via linear interpolation.. By the properties of the coupling, $U_{\lambda(c)} = A_{\lambda(c)}$ for all c. Checking that $||\lambda||_{\mathrm{Lip}}, ||\lambda^{-1}||_{\mathrm{Lip}} \approx 1$ since $\left|\frac{\tau_i}{\mathsf{h}(Q)} - \widetilde{\tau}_i\right| \ll \widetilde{\tau}_i$, we will have

$$\mathbb{P}\left(J\left(\mathrm{Law}_\pi\left(\{\widetilde{U}_c\}_{c\in[\beta,\beta']}\right), \{A_c\}_{c\in[\beta,\beta']} > \epsilon\right)\right) \to 0. \tag{9.5}$$

Moreover, equation (9.5) implies (9.4) and it finishes the proof of the Theorem. □

We know, by Theorem 6, that $\mathrm{Law}\left(U_{t_0}\right) \approx \bigotimes_{v\in\mathbf{V}}\mathrm{Be}_{\frac{e^{-\beta}}{n}}$. To prove Theorem 7, we will study

$$\mathrm{Law}\left(\tau_1,\ldots,\tau_{U_{t_0}}, I_1,\ldots,I_{|U_{t_0}|}|X_{t_0},U_{t_0}\right).$$

The next result completes the purpose of this chapter.

Theorem 8. *There exists a coupling between $\left(\tau_1,\ldots,\tau_{|U_{t_0}|}, I_1,\ldots,I_{|U_{t_0}|}\right)$ and $\left(\widetilde{\tau}_1,\ldots,\widetilde{\tau}_{|U_{t_0}|}, \widetilde{I}_1,\ldots,\widetilde{I}_{|U_{t_0}|}\right)$,* *such that* $\mathrm{Law}\left(\widetilde{\tau}_1,\ldots,\widetilde{\tau}_{|U_{t_0}|},\widetilde{I}_1,\ldots,\widetilde{I}_{|U_{t_0}|}|X_{t_0},U_{t_0}\right)$ *has distribution*

$$\bigotimes_{i=0,\ldots,|U_{t_0}|-1}\mathsf{Exp}\left(\frac{\mathsf{h}(Q)}{|U_{t_0}|-i|}\right)\bigotimes_{i=0,\ldots,|U_{t_0}|-1}\mathsf{Unif}(U_{t_0}\backslash\{\widetilde{I}_0,\ldots,\widetilde{I}_i\}),\tag{9.6}$$

where $\widetilde{I}_0=\emptyset$. *Moreover,*

$$\mathbb{P}\left(\bigcup_{i=1}^{|U_{t_0}|}\{I_i\neq\widetilde{I}_i\}\right)=o\left(1\right)\tag{9.7}$$

and

$$\mathbb{E}\left[\min\left(\sum_{i=1}^{|U_{t_0}|}\frac{|\tau_i-\widetilde{\tau}_i|}{\mathsf{h}(Q)\backslash|U_{t_0}|},1\right)\right]=o\left(1\right).\tag{9.8}$$

Proof: Begin by taking

$$F=\{\{X_{t_0}\}\cup U_{t_0}\in\mathcal{S}\}\cap\{|U_{t_0}|\leq K\}$$

where K is chosen so that

$$\mathbb{P}_\pi\left(|U_{t_0}|>K\right)\leq\frac{\mathbb{E}_\pi\left[|U_{t_0}]\,|\right.}{K}\leq\left(1+o\left(1\right)\right)\frac{e^{-\beta}}{K}=o\left(1\right)$$

and $K^2\epsilon_0(Q)=o\left(1\right)$. Note that Lemma 7 and our choice of K imply that

$$\mathbb{P}_\pi\left(F^C\right)\leq\mathbb{P}_\pi\left(\{X_{t_0}\}\cup U_{t_0}\notin\mathcal{S}\right)+\mathbb{P}_\pi\left(|U_{t_0}|>K\right)=o\left(1\right).$$

We prove Theorem 8 by induction. First, define $I_S=X_{H_S}$ and observe that, by the Markov property,

$$\mathrm{Law}_\pi\left(\tau_1,I_1|X_{t_0},U_{t_0}\right)=\mathrm{Law}_{X_{t_0}}\left(H_{U_{t_0}},I_{U_{t_0}}\right).$$

So, we can use Theorem 5 and obtain a vector $(\widetilde{\tau}_1,\widetilde{I}_1)$ coupled to (τ_1,I_1) such that

$$\mathrm{Law}_\pi\left(\widetilde{\tau}_1,\widetilde{I}_1|X_{t_0},U_{t_0}\right)=\mathsf{Exp}\left(\frac{\mathsf{h}(Q)}{|U_{t_0}|}\right)\times\mathsf{Unif}(U_{t_0}),$$

$$\mathbb{P}_\pi\left(I_1\neq\widetilde{I}_1|X_{t_0},U_{t_0}\right)=O\left(|U_{t_0}|^2\epsilon_0(Q)\right)\text{ in }F$$

and

$$\mathbb{E}_\pi\left[\frac{|\tau_1-\widetilde{\tau}_1|}{\mathsf{h}(Q)\backslash|U_{t_0}|}|X_{t_0},U_{t_0}\right]=O\left(|U_{t_0}|^2\epsilon_0(Q)\right)\text{ in }F.$$

The way in which K was chosen implies that $O\left(|U_{t_0}|^2\epsilon_0(Q)\right)=o\left(1\right)$. Moreover, as $\mathbb{P}_\pi\left(F\right)=1-o\left(1\right)$, we have that $(\widetilde{\tau}_1,\widetilde{I}_1)$ satisfies (9.6), (9.7) and (9.8) for the case $j=1$.

Now, suppose that there exists $\left(\widetilde{\tau}_1,\ldots,\widetilde{\tau}_{|U_{t_0}|},\widetilde{I}_1,\ldots,\widetilde{I}_j\right)$ coupled to $\left(\tau_1,\ldots,\tau_{|U_{t_0}|},I_1,\ldots,I_j\right)$ satisfying (9.6), (9.7) and (9.8). By Markov property,

$$\mathrm{Law}_\pi\left(\tau_{i+1}, I_{i+1}|X_{t_0}, U_{t_0}, \tau_1, \dots, \tau_{|U_{t_0}|}, I_1, \dots, I_j\right)$$

is the same as

$$\mathrm{Law}_{X_{t_0+\tau_1+\cdots+\tau_j}}\left(H_{U_{t_0}\backslash\{I_1,\dots,I_j\}}, I_{U_{t_0}\backslash\{I_1,\dots,I_j\}}\right).$$

In $\bigcap_{i=1}^j\{I_i = \widetilde{I}_i\}$, event that occurs with probability $1 - o(1)$ by induction hypothesis, the above distribution is the same as

$$\mathrm{Law}_{X_{t_0+\tau_1+\cdots+\tau_j}}\left(H_{U_{t_0}\backslash\{\widetilde{I}_1,\dots,\widetilde{I}_j\}}, I_{U_{t_0}\backslash\{\widetilde{I}_1,\dots,\widetilde{I}_j\}}\right).$$

In F, we have that $\{X_{t_0}\}\cup U_{t_0} \in \mathcal{S}$, in particular $\{X_{t_0+\tau_1+\cdots+\tau_j}\}\cup U_{t_0}\backslash\{\widetilde{I}_1,\dots,\widetilde{I}_j\} \in \mathcal{S}$. So, we can use Theorem 5 to obtain $(\widetilde{\tau}_{j+1}, \widetilde{I}_{j+1})$ coupled to (τ_{j+1}, I_{j+1}) such that $\mathrm{Law}_\pi\left(\widetilde{\tau}_{j+1}, \widetilde{I}_{j+1}|X_{t_0}, U_{t_0}\backslash\{\widetilde{I}_1,\dots,\widetilde{I}_j\}\right)$ has distribution

$$\mathsf{Exp}\left(\frac{\mathsf{h}(Q)}{|U_{t_0}| - j}\right) \times \mathsf{Unif}(U_{t_0}\backslash\{\widetilde{I}_1,\dots,\widetilde{I}_j\}),$$

moreover

$$\mathbb{P}_\pi\left(I_{j+1} \neq \widetilde{I}_{j+1}|X_{t_0}, U_{t_0}, \tau_1, \dots, \tau_j, I_1, \dots, I_j\right) = O\left(|U_{t_0}|^2\epsilon_0(Q)\right) \text{ in } F$$

and

$$\mathbb{E}_\pi\left[\frac{|\tau_j - \widetilde{\tau}_j|}{\mathsf{h}(Q)\backslash(|U_{t_0}| - j)}|X_{t_0}, U_{t_0}, \tau_1, \dots, \tau_j, I_1, \dots, I_j\right] = O\left(|U_{t_0}|^2\epsilon_0(Q)\right) \text{ in } F.$$

This way the vector $(\tau_1, \dots, \tau_j, I_1, \dots, I_j)$ has the desired distribution. Further, using that $\mathbb{P}_\pi(F) = 1 - o(1)$, more $O\left(|U_{t_0}|^2\epsilon_0(Q)\right) = o(1)$ in F and the induction hypothesis we get

$$\mathbb{P}_\pi\left(\bigcup_{i=1}^{j+1}\{I_i \neq \widetilde{I}_i\}\right) = o(1)$$

and

$$\mathbb{E}_\pi\left[\min\left(\sum_{i=1}^{j+1}\frac{|\tau_i - \widetilde{\tau}_i|}{\mathsf{h}(Q)\backslash(|U_{t_0}| - i)}, 1\right)\right] = o(1).$$

This finishes the proof of the theorem. □

CHAPTER 10

Examples

In this chapter we show that Theorem 2 is applicable when the graph studied is a torus or a high-girth expander. We show that for each case there exists a number $\mathsf{h}(Q)$ satisfying the hypotheses of Theorem 2. Theorem 2 also applies to hypercubes, oriented-tori and transposition random walks over S_n".

10.1 Preliminaries

In this section we prove a preliminary result that will be useful in the examples. Through this section, Q denotes (the generator of) a continuous-time Markov chain over a finite state space $\mathbf{V}$ and uniform stationary measure. Also denote $n = |\mathbf{V}|$.

Define the probability of escape of the set $A \subset \mathbf{V}$ starting form x as

$$p_{esc}(x, A) = \sum_{y \in \mathbf{V}} q(x, y) \mathbb{P}_y \left(H_A > \mathrm{t}_{\mathrm{unif}}^{Q} \right) \tag{10.1}$$

and denote $p_{esc}(x) = p_{esc}(x, \{x\})$.

The next statement employs ideas present in [18], it relates $\mathbb{E}_\pi [H_A]$ and $p_{esc}(x, A)$.

Lemma 8. *If $A \subset \mathbf{V}$ is such that $\frac{\mathrm{t}_{\mathrm{mix}}^{Q}}{\mathbb{E}_\pi[H_A]} = o\left(\frac{1}{\ln^2 n \ln \ln n}\right)$ and $\sum_{x \in A} p_{esc}(x, A) = \Omega(1)$, then*

$$n = \left(1 + o\left(\frac{1}{\ln n}\right)\right) \left(\sum_{x \in A} p_{esc}(x, A)\right) \mathbb{E}_\pi [H_A] .$$

Remark 6. *In this setting is always the case that $\mathbb{E}_\pi [H_A] = \Omega\left(\frac{n}{|A|}\right)$, so it suffices that $\frac{\mathrm{t}_{\mathrm{mix}}^{Q} |A|}{n} = o\left(\frac{1}{\ln^2 n \ln \ln n}\right)$ for the Claim to be applicable.*

Proof: Firstly, note that we can use Theorem 4 (this is possible by Lemma 5) to obtain

$$\sum_{u\in\mathbf{V}} \pi(u) d_W(\mathrm{Law}_u(H_A), \mathsf{Exp}(\mathbb{E}_\pi[H_A])) \le \epsilon_0(Q)|A|\mathbb{E}_\pi[H_A], \tag{10.2}$$

remember the definiton of $\epsilon_0(Q)$ in Chapter 7.

Kac's theorem shows that

$$\frac{1}{\pi(A)} = 1 + \sum_{x\in A}\frac{\pi(x)}{\pi(A)}\sum_{y\in\mathbf{V}} q(x,y)\mathbb{E}_y[H_A].$$

Then, using Markov property, we get

$$\frac{1}{\pi(A)} \le 1 + \sum_{x\in A}\frac{1}{|A|}\sum_{y\in\mathbf{V}} q(x,y)\left(\mathrm{t}^Q_{\mathrm{unif}} + \sum_{u\in\mathbf{V}}\mathbb{P}_y\left(X_{\mathrm{t}^Q_{\mathrm{unif}}} = u, H_A > \mathrm{t}^Q_{\mathrm{unif}}\right)\mathbb{E}_u[H_A]\right).$$

As

$$|\mathbb{E}_u[H_A] - \mathbb{E}_\pi[H_A]| \le d_W(\mathrm{Law}_u(H_A), \mathsf{Exp}(\mathbb{E}_\pi[H_A])),$$

we have that

$$\begin{aligned} n-1 \le & \sum_{x\in A}\sum_{y\in\mathbf{V}} q(x,y)\mathrm{t}^Q_{\mathrm{unif}} \\ & + \sum_{x\in A}\sum_{y\in\mathbf{V}} q(x,y)\mathbb{P}_y\left(H_A > \mathrm{t}^Q_{\mathrm{unif}}\right)\mathbb{E}_\pi[H_A] \\ & + \sum_{x\in A}\sum_{y\in\mathbf{V}} q(x,y)\sum_{u\in\mathbf{V}}\mathbb{P}_y\left(X_{\mathrm{t}^Q_{\mathrm{unif}}} = u\right) d_W(\mathrm{Law}_u(H_A), \mathsf{Exp}(\mathbb{E}_\pi[H_A])). \end{aligned}$$

Using the definition of $\mathrm{t}^Q_{\mathrm{unif}}$, equation (10.2) and that $\sum_{x\in A} p_{esc}(x,A) = \Omega(1)$, we obtain

$$\begin{aligned} n-1 \le & \sum_{x\in A}\sum_{y\in\mathbf{V}} q(x,y)\mathrm{t}^Q_{\mathrm{unif}} \\ & + \sum_{x\in A} p_{esc}(x,A)\mathbb{E}_\pi[H_A] \\ & + \sum_{x\in A} p_{esc}(x,A)C\epsilon_0(Q)\mathbb{E}_\pi[H_A]. \end{aligned}$$

Thus, using that $\frac{\mathrm{t}^Q_{\mathrm{mix}}}{\mathbb{E}_\pi[H_A]} = o\left(\frac{1}{\ln^2 n \ln\ln n}\right)$ and $\epsilon_0(Q) = o\left(\frac{1}{\ln n}\right)$, we have

$$n \le \left(1 + o\left(\frac{1}{\ln n}\right)\right)\left(\sum_{x\in A} p_{esc}(x,A)\right)\mathbb{E}_\pi[H_A].$$

Analogously, we get

$$n \ge \left(1 + o\left(\frac{1}{\ln n}\right)\right)\left(\sum_{x\in A} p_{esc}(x,A)\right)\mathbb{E}_\pi[H_A]$$

and the Lemma 8 follows. □

10.2 Discrete Tori

The $d-$dimensional torus $\mathbb{T}_L^d$ is a graph whose vertex set is the Cartesian product $\otimes_{i=1}^d\{-L,\dots,L\}$ and edges j and k whenever $j+L\equiv k+1$ (mod 2L). Vertices $x=(x_1,\dots,x_d)$ and $y=(y_1,\dots,y_d)$ are neighbors in $\mathbb{T}_L^d$ if for some $j\in\{-L,\dots,L\}$, we have $x_i=y_i$ for all $i\neq j$ and $x_j+L\equiv y_j\pm 1$ (mod 2L).

Define the Green's function $\mathcal{G}(x,y)=\mathcal{G}(y,x)=\mathcal{G}(y-x)$ by

$$\mathcal{G}(x)=\mathbb{E}_x\left[\sum_{t=0}^{\infty}\mathbf{1}\{\overline{S}_t=0\}\right],$$

where $\{\overline{S}_t\}_{t\in\mathbb{N}}$ is the simple random walk (SRW) in discrete time on $\mathbb{Z}^d$. We will show that choosing $\mathsf{h}(Q)=L^d\cdot\mathcal{G}(0)$ the hypotheses of Theorem 2 will be satisfied for the torus case.

Remember that, for the torus with $d\geq 3$ (see [26]), $\mathrm{t}_{\mathrm{unif}}^Q=O\left(L^2\right)$ and by our choice of $\mathsf{h}(Q)$ we have that **A1** follows immediately, that is,

$$\frac{\mathrm{t}_{\mathrm{mix}}^Q}{\mathsf{h}(Q)}=o\left(\frac{1}{\ln^2 L^d\ln\ln L^d}\right).$$

For $A\subset\mathbb{Z}_L^d$ and $A'\subset\mathbb{Z}^d$, we define

$$K_A^L=\inf\{t\in\mathbb{N}:S_t^L\in A\}$$

and

$$\overline{K}_A=\inf\{t\in\mathbb{N}:\overline{S}_t\in A'\},$$

where $\{S_t^L\}_{t\in\mathbb{N}}$ is the SRW on $\mathbb{Z}_L^d$ and $\{\overline{S}_t\}_{t\in\mathbb{N}}$ is the SRW on $\mathbb{Z}^d$.

The next statement shows that $p_{esc}(x,A)\approx\mathbb{P}_x\left(\overline{K}_A=\infty\right)$.

Claim 2. *Take* $0\not\subset A\subset\mathbb{Z}_L^d$, *we have that*

$$|p_{esc}(x,A)-\mathbb{P}_x\left(\overline{K}_A=\infty\right)|=O\left(L^{-\gamma}\right),$$

for some $\gamma>0$.

Proof: The constant $C>0$ written below may change from one equation to another. As $\mathrm{t}_{\mathrm{unif}}^Q\leq CL^2$, we have that (10.1) implies that

$$p_{esc}(x,A)\leq\sum_{y\in V}q(x,y)\mathbb{P}_y\left(H_A>CL^2\right).$$

Using the link between the continuous time and discrete walk, we have

$$p_{esc}(x,A)\leq\mathbb{P}_x\left(K_A^{CL}>CL^2\right)+O\left(L^{-\gamma}\right),$$

for some $\gamma > 0$.

Assume without loss of generality that $x = 0$ and put $K_A = K_A^{CL}$. We want to bound

$$\mathbb{P}_x\left(K_A \leq CL^2\right) \leq \mathbb{P}_x\left(K_A \leq \frac{CL^2}{\ln^2 L}\right) + \mathbb{P}_x\left(\frac{CL^2}{\ln^2 L} \leq K_A \leq CL^2\right).$$

Note that,

$$\begin{aligned}\mathbb{P}_x\left(K_A \leq CL^2\right) &\leq \mathbb{P}_x\left(K_A \leq \frac{CL^2}{\ln^2 L}, \max_{0\leq t\leq \frac{CL^2}{\ln^2 L}} |S_t^{(i)}| \leq CL\right) \\ &+ \sum_{i=1}^{d} \mathbb{P}_x\left(\max_{0\leq t\leq \frac{CL^2}{\ln^2 L}} |S_t^{(i)}| > CL\right) \\ &+ \mathbb{P}_x\left(\frac{CL^2}{\ln^2 L} \leq K_A \leq CL^2\right),\end{aligned}$$

where $S_t^{(i)}$ is the i-th coordinates of S_t.

The first term can be bounded by

$$\mathbb{P}_x\left(K_A \leq \frac{CL^2}{\ln^2 L}, \max_{0\leq t\leq \frac{CL^2}{\ln^2 L}} |S_t^{(i)}| \leq L\right) \leq \mathbb{P}_x\left(\overline{K}_A < \infty\right).$$

Using martingale maximal inequality and Chernoff's bounds (see [7]), we write the second one as

$$\begin{aligned}\sum_{i=1}^{d} \mathbb{P}_x\left(\max_{0\leq t\leq \frac{CL^2}{\ln^2 L}} |S_t^{(i)}| > CL\right) &\leq 2d\exp\left(\frac{-C^2L^2}{\frac{2C^2L^2}{\ln^2 L}}\right) \\ &= O\left(L^{-C\ln L}\right).\end{aligned}$$

Finally, the last one

$$\begin{aligned}\mathbb{P}_x\left(\frac{CL^2}{\ln^2 L} \leq K_A \leq CL^2\right) &\leq \mathbb{P}_x\left(\max_{0\leq t\leq \frac{CL^2}{\ln^2 L}} |S_t^{(i)}| > CL\right) \\ &+ \sum_{y\in\{-CL,\ldots,CL\}^d} \mathbb{P}_x\left(\overline{S}_{\frac{CL^2}{\ln^2 L}} = y\right)\mathbb{P}_y\left(K_A < CL^2\right).\end{aligned}$$

Lawler (see [25]) show that $\mathbb{P}_x(S_t = y) \leq Ct^{-\frac{d}{2}}$, for all $x, y \in \mathbb{Z}^d$. Teixeira (see [14]) show that if $A' = \{-(CL)^{\frac{1}{2}}, \ldots, (CL)^{\frac{1}{2}}\}^d$, so $A \in A'$ for L large enough and there exists $C > 0$ such that

$$\sup_{x\in\mathbb{Z}^d_{CL}\setminus A'} \mathbb{P}_x\left(K_A \leq CL^2\right) = O\left(L^{-\delta}\right),$$

for some $\delta > 0$. Then, splitting the sum over $y \in \mathbb{Z}^d_{CL}$ in $y \in \mathbb{Z}^d_{CL} \backslash A'$ and A' we obtain

$$\begin{aligned}\mathbb{P}_x\left(\frac{CL^2}{\ln^2 L} \leq K_A \leq CL^2\right) &\leq O\left(L^{-CL\ln L}\right) + CL^d \cdot \frac{\ln^d L}{L^d} \cdot L^{-\delta} + CL^{\frac{d}{2}} \cdot \frac{\ln^d L}{L^d} \cdot 1 \\ &= O\left(L^{-\gamma}\right),\end{aligned}$$

for some $\gamma > 0$.

This shows that,

$$\left|\mathbb{P}_x\left(K_A \leq CL^2\right) - \mathbb{P}_x\left(\overline{K}_A < \infty\right)\right| = O\left(L^{-\gamma}\right)$$

and it finishes the proof of Claim 2. □

Now, we will show that the torus satisfies the others hypotheses of Theorem 2. For the torus (see [26]) there exists constant $C > 0$ such that $CL^d \leq \mathbb{E}_\pi[H_x]$, so the hypotheses of Lemma 8 are satisfied and using Claim 2 also, we have

$$\begin{aligned}\mathbb{E}_\pi[H_x] &= \left(1 + o\left(\frac{1}{\ln n}\right)\right)\frac{n}{p_{esc}(x)} \\ &= \left(1 + o\left(\frac{1}{\ln n}\right)\right)\frac{n}{\mathbb{P}_x\left(\overline{K}_x = \infty\right)} \\ &= \left(1 + o\left(\frac{1}{\ln n}\right)\right) n\mathcal{G}(0).\end{aligned}$$

Thus we have **A0**.

Also, we can use Lemma 8 to $A = \{x, y\}$. Then,

$$\begin{aligned}\mathbb{E}_\pi\left[H_{\{x,y\}}\right] &= \left(1 + o\left(\frac{1}{\ln n}\right)\right)\frac{n}{p_{esc}(x, \{x,y\}) + p_{esc}(y, \{x,y\})} \\ &= \left(1 + o\left(\frac{1}{\ln n}\right)\right)\frac{n}{\mathbb{P}_x\left(\overline{K}_{\{x,y\}} = \infty\right) + \mathbb{P}_y\left(\overline{K}_{\{x,y\}} = \infty\right)} \\ &= \left(1 + o\left(\frac{1}{\ln n}\right)\right)\frac{n}{2}(\mathcal{G}(0) + \mathcal{G}(x-y)) \\ &\leq \frac{n}{1+\phi}\mathcal{G}(0),\end{aligned}$$

for some $\phi > 0$ and n large enough. The last inequality follows using $\sup_{z \neq 0} \mathcal{G}(z) < \mathcal{G}(0)$. Therefore we have **A2**.

To finish let us prove **A3**. Consider the box $B = \{-l^{\frac{1}{d-2}}, \ldots, l^{\frac{1}{d-2}}\}$, where $l = \ln^2 L^d$. So, $|B| = l^{\frac{d}{d-2}} = (L^d)^{o(1)}$. Moreover, for all $y \in V \backslash B$ we have

$$\begin{aligned}\mathbb{P}_0\left(H_y < \mathrm{t}^Q_{\mathrm{unif}}\right) &\leq \mathbb{P}_0\left(\overline{K}_y < \infty\right) + O\left(L^{-\gamma}\right) \\ &\leq \mathcal{G}(y) + O\left(L^{-\gamma}\right) \\ &\leq O\left(||y||^{2-d}\right) + O\left(L^{-\gamma}\right) \\ &= o\left(\frac{1}{\ln L^d}\right),\end{aligned}$$

where we use Claim 2 in the first inequality, in the third one we use the classical result $\mathcal{G}(y) = O\left(||y||^{2-d}\right)$ for $d \geq 3$ (see [25]) and in the last one we use our choice of l. This shows that **A3** is satisfied.

10.3 High-Girth Expanders

For a graph $G = (\mathbf{V}, E)$ and $S, T \subset \mathbf{V}$, denote the set of edges from S to T by $E(S,T) = \{(x,y)|x \in S, y \in T, (x,y) \in E\}$. Also define the (edge) expasion rate of G by

$$h(G) = \min_{S \subset \mathbf{V}: |S| \leq \frac{n}{2}} \frac{|E(S,S^C)|}{|S|}.$$

A family of expander graphs is a sequence of d-regular graphs $\{G_n\}_{n\in\mathbb{N}}$ of size increasing with n for which there exists $\epsilon > 0$ such tht $h(G_n) \geq \epsilon$ for all $n \in \mathbb{N}$. The girth $g(G)$ of a graph G is the lenght of the shortest cycle in G. A graph G with size n is said to have high girth if $g(G) = \omega(\ln \ln n)$. In this example we consider a family of vertex-transitive high girth expanders for instance Ramanujan graphs (see [28]). A proof that these graphs have high girth is available from [11].

For $A \subset \mathbb{Z}^d$ and $A' \subset \mathcal{T}_d$, where $\mathcal{T}_d$ is the infinite $d-$regular tree, we define

$$K_A = \inf\{t \in \mathbb{N} : S_t \in A\}$$

and

$$\overline{K}_{A'} = \inf\{t \in \mathbb{N} : \overline{S}_t \in A'\},$$

where $\{S_t\}_{t\in\mathbb{N}}$ and $\{\overline{S}_t\}_{t\in\mathbb{N}}$ are, respectively, the SRW in discrete time on G and $\mathcal{T}_d$.

Define the Green's function on $\mathcal{T}_d$, denoted by $\mathcal{G}(x,y) = \mathcal{G}(y,x) = \mathcal{G}(y-x)$ as

$$\mathcal{G}(x) = \mathbb{E}_x\left[\sum_{t=0}^{\infty} \mathbf{1}\{\overline{S}_t = 0\}\right].$$

We will show that choosing $\mathsf{h}(Q) = n \cdot \mathcal{G}(0)$ the hypotheses of Theorem 2 will be satisfied for the high girth expanders case. Note that for expanders praphs $\mathrm{t}_{\mathrm{mix}}^{Q} = O\left(\ln n\right)$ (see [19]), then **A1** follows immediately by our choice of $\mathsf{h}(Q)$.

For each $x \in V$, let $\phi_x : G|_{B_G(x, \frac{g(G)}{2})} \mapsto \mathcal{T}_d|_{B_{\mathcal{T}_d}(0, \frac{g(G)}{2})}$ be the natural isomorphism which takes the $\frac{g(G)}{2}-$neighborhood of x in G on the $\frac{g(G)}{2}-$neighborhood of 0 in $\mathcal{T}_d$, where 0 is the root of $\mathcal{T}_d$. We define $B_G(x,r) = \{y \in G : d(x,y) \leq r\}$, where $d(\cdot,\cdot)$ is the distance on graph G.

Claim 3. *Take $x \in G$ and $A \subset \mathbf{V}$. If $d(x,A) \leq \frac{g(G)}{2}$, we have that*

$$|p_{esc}(x,A) - \mathbb{P}_0\left(\overline{K}_{\phi_x(A)} = \infty\right)| \leq C(\ln n)^{-\omega(1)}.$$

If $d(x,A) > \frac{g(G)}{2}$, *then*

$$|p_{esc}(x,A)| \leq C(\ln n)^{-\omega(1)}.$$

Proof: Using the link between continuous time and discrete time walk it is enough show that

$$|\mathbb{P}_x\left(K_A < \mathrm{t}_{\mathrm{unif}}\right) - \mathbb{P}_0\left(\overline{K}_{\phi_x(A)} < \infty\right)| \leq C(\ln n)^{-\omega(1)}.$$

Take $r = \frac{g(G)}{2}$, and denote

$$\tau_r = \inf t \in \mathbb{N} : d(x, S_t) = r$$

and

$$\overline{\tau}_r = \inf t \in \mathbb{N} : d(x, \overline{S}_t) = r.$$

As $G|_{B(x,r)}$ is isomorphic to $\mathcal{T}_d|_{B(0,r)}$, for $y \in \mathbf{V}$, we have that

$$\mathbb{P}_x\left(S_{\tau_r} = y\right) = \begin{cases} 0 & , \quad d(x,y) \neq r; \\ \frac{1}{d(d-1)^{r-1}} & , \quad d(x,y) = r. \end{cases}$$

Suppose that $d(x,A) \leq \frac{g(G)}{2}$ and denote $A' = \phi\left(A|_{B\left(x,\frac{g(G)}{2}\right)}\right)$. Therefore

$$\begin{aligned} \mathbb{P}_x\left(K_A \leq \mathrm{t}^Q_{\mathrm{unif}}\right) &= \mathbb{P}_x\left(K_A \leq \tau_r\right) + \mathbb{P}_x\left(\tau_r < K_A \leq \mathrm{t}^Q_{\mathrm{unif}}\right) \\ &\leq \mathbb{P}_0\left(\overline{K}_{A'} \leq \overline{\tau}_r\right) + \sum_{0 \leq t \leq \mathrm{t}^Q_{\mathrm{unif}} - r} \sum_{a \in A} \mathbb{P}_x\left(S_{\tau_r + t} = a\right) \\ &\leq \mathbb{P}_0\left(\overline{K}_{A'} \leq \overline{\tau}_r\right) + O\left(|A| \mathrm{t}^Q_{\mathrm{unif}} \frac{1}{d(d-1)^r}\right) \\ &\leq \mathbb{P}_0\left(\overline{K}_{A'} \leq \overline{\tau}_r\right) + C(\ln n)^{-\omega(1)}, \end{aligned}$$

where in the second inequality we use that for a simple random walk in a regular graph we have $\max_{v \in V} \mathbb{P}_x\left(S_{\tau_r + t} = v\right) \leq \max_{v \in V} \mathbb{P}_x\left(S_{\tau_r} = v\right)$. In the last one we use that $\mathrm{t}^Q_{\mathrm{mix}} = O\left(\ln n\right)$ and that $r = \omega(\ln \ln n)$. The case $d(x,A) > \frac{g(G)}{2}$ follows immediately.

For other side,

$$\mathbb{P}_0\left(\overline{K}_{A'} < \infty\right) = \mathbb{P}_0\left(\overline{K}_{A'} \leq \overline{\tau}_r\right) + \mathbb{P}_0\left(\overline{\tau}_r < \overline{K}_{A'} < \infty\right),$$

and note that

$$\begin{aligned} \mathbb{P}_0\left(\overline{\tau}_r < \overline{K}_{A'} < \infty\right) &\leq \mathbb{P}_0\left(r < \overline{K}_{A'} < \infty\right) \\ &\leq \sum_{l > r} |A| \frac{1}{d(d-1)^l} \\ &= C(\ln n)^{-\omega(1)}. \end{aligned}$$

It finishes the proof of the Claim. $\square$

For a expander graph (see [27]) we have that $\mathbb{E}_x[H_y] = \Theta(n)$ for all x and y in $\mathbf{V}$, so the hypotheses of Lemma 8 are satisfied for $A = \{y\}$ and $A = \{x, y\}$. Therefore **A0** and **A2** follow as in the torus case.

To finish let us show that **A3** is satisfied. For this, note that for all $y \in V \backslash B\left(x, \frac{g(G)}{2}\right)$ we have that

$$\mathbb{P}_x\left(H_y < \mathrm{t}_{\mathrm{unif}}^{Q}\right) \leq C(\ln n)^{-\omega(1)},$$

moreover $|B\left(x, \frac{g(G)}{2}\right)| = o\left(n^{\phi}\right)$ for all $\phi > 0$.

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