INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA

On Asymptotic Behavior of Economies with Complete Markets: the role of ambiguity aversion

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Abstract

The aim of this work is to analyze some equilibrium consequences of the behavior of heterogeneous ambiguity averse agents in inter-temporal general equilibrium models. As the focus is on the influence of ambiguity aversion, the choice of the models treated here was based on their previous use to attain similar results, allowing their use as a parameter. Because of that, it is considered here a general equilibrium model which fits in frameworks like in Araujo and Sandroni (1999), Sandroni (2000), Blume and Easley (2006) and Condie (2008).

Two different approaches have been used: in the first (Chapter 2) we strive to find out what condition of beliefs is necessary to achieve equilibrium, and in the second (Chapter 3) we want to know what conditions are related to survival. There is also a chapter of preliminaries where the framework is discussed and some auxiliary results are presented.

The first results are within the framework of Araujo and Sandroni (1999) where there is a complete market of contingent claims and bankruptcy is permitted, though incurring a penalty. Agents have the *smooth ambiguity* preferences presented by Klibanoff et al. (2005, 2009). The main result follows those others presented in the literature, but provides quite a different interpretation. It proves that a necessary condition for equilibrium existence is the convergence of *ambiguity perception* reduction.

Other results are placed within the context of Blume and Easley (2006), where behavior is analyzed from the point of view of Pareto Optimal allocations. In this case, agents' behavior is determined by *variational* preferences (axiomatized by Macheroni et al. (2006a,b)). These preferences are more general than the expected utility (used by Sandroni (2000) and Blume and Easley (2006)) and *maxmin* utility (used by Condie

(2008)). The main result contrasts with the result derived by Condie (2008). Under conditions in which a *maxmin* agent does not survive, another ambiguity averse agent can survive.

Contents

Chapter 1

Preliminaries

This chapter provides essential definitions and results for the development of this thesis. The first section presents the basic framework where the main results are attained. The next section contains the needed background on ambiguity averse preferences and explanations about the motivation behind them. Finally, the third section presents the Blackwell-Dubins theorem and related results.

1.1 Dynamic Model of General Equilibrium

This thesis is grounded on general equilibrium models with infinite time horizon. At first, models with infinite time horizon would appear quite unrealistic. However, such models provide theorists an environment in with to test how robust some hypotheses are. Rubinstein (1991)'s defense of the use of infinite horizon models, despite dealing with game models, can be invoked to enrich the discussion.

> *(...) By using infinite horizon games we do not assume that the real world is infinite. Models are not supposed to be isomorphic with reality. (...) Using the terminology of formal logic, we can say that finite horizon models are suitable only for modeling situations in which the last period appears as an "individual constant" (a specified element) in the players' model.* (Rubinstein, 1991, p. 918)

To suppose that decision makers have finite lives is useful when one wants to analyze the effect of taking this particular fact into account; assuming they have infinite lives is useful to analyze the effects of their long-term decisions. Long-term must be understood as something that occurs within a remote period, assuming also that the individual knows that he is going to be alive. The option to use directly a model of infinite time instead of using limits for finite horizons follows the same reasoning.

Consider a dynamic model with discrete time $\mathcal{T} = \{0, 1, \dots\}$. There is a finite set of agents $\mathcal{I} = \{1, \dots, I\}$, which have common information modeled by a filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathcal{T}})$, where $\Omega := \{\omega_0\} \times \prod_{t\geq 1} \mathcal{S}_t$, with ω_0 the sure state occurring for the first time and $\mathcal{S}_t = \{1, \dots, S_t\}$ the set of possible states occurring at each time $t \geq 1$. A representative element of Ω will be denoted by $\omega = (\omega_0, \omega_1, \dots)$ and time-t history $\omega^t = (\omega_0, \dots, \omega_t) \in \Omega^t := \{\omega_0\} \times \prod_{\tau=1}^{t} \mathcal{S}_\tau$. Let $\mathcal{F}_t$ be the σ-algebra generated by $(t+1)$-dimensional cylinders, i.e., $\mathcal{F}_t = \sigma(\{G_t(\omega); \omega \in \Omega\})$, where $G_t(\omega) := \{\omega^t\} \times \prod_{\tau>t} \mathcal{S}_\tau$.

Let $\mathcal{F}^0 = \cup_{t\in\mathcal{T}} \mathcal{F}_t$ be the algebra of finite-time events and $\mathcal{F} = \sigma(\mathcal{F}^0)$ the σ-algebra generated by $\mathcal{F}^0$. The filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathcal{F})$ represents the informational process known by agents. Process ω_t is governed by probability $\mathbb{P}(\cdot|\omega^{t-1})$ on $\mathcal{S}_t$, which can be understood as the conditional probability given ω^{t-1} in the past. These probabilities generate law $\mathbb{P}$ on $(\Omega, \mathcal{F})$ by constructing the partials $\mathbb{P}(\omega^t) = \mathbb{P}(\omega^{t-1})\mathbb{P}(\omega_t|\omega^{t-1})$ on $\mathcal{F}_t$ for each $t \in \mathcal{T}$, and evoking Kolmogorov's extension theorem (see Shiryayev (1984) chapter II, section 3).

The set of all probabilities on a measurable space $(A, \mathcal{A})$ is denoted by $\Delta(A, \mathcal{A})$, or $\Delta(\Omega)$ instead $\Delta(\Omega, \mathcal{F})$ for simplicity. If $P \in \Delta(\Omega)$, P_t denotes its restriction to $\mathcal{F}_t$, and $P_{t+1}(s|\omega^t) := \frac{P_{t+1}(\omega^t, s)}{P_t(\omega^t)}$ denotes the conditional one-step-ahead probability from P. Note that we can consider $P_{t+1}(\cdot|\omega^t) \in \Delta(G_t(\omega), \mathcal{F}_{t+1})$.

For two probabilities, P and Q, we say that Q is absolutely continuous with respect to P if for $A \in \mathcal{F}$, $P(A) = 0$ implies $Q(A) = 0$, and we denote $Q \ll P$. We say that Q is locally absolutely continuous with respect to P if for $A \in \mathcal{F}^0$, $P(A) = 0$ implies $Q(A) = 0$, and we denote $Q \overset{loc}{\ll} P$. If $Q \ll P$ and $P \ll Q$ we say that P and Q are equivalents and denote $Q \sim P$. Again for P and Q we denote the total variation distance[1] between P and Q by $\|P - Q\|$.

The acts considered by the agents must be based on their knowledge of the world, hence the consequences of an act at period t will be contingent to events at $\mathcal{F}_t$. The individual choice space is a subset of

$$X = \left\{ (x_t)_{t \in \mathcal{T}};\ x_t : \Omega \to \mathbb{R} \text{ is } \mathcal{F}_t\text{-adapted and } \sup_{t,\omega} |x_t(\omega)| < \infty \right\},$$

and the space of prices is

$$Y = \left\{ (p_t)_{t \in \mathcal{T}};\ p_t : \Omega \to \mathbb{R} \text{ is } \mathcal{F}_t\text{-adapted and } \sum_{t,\omega} |p_t(\omega)| < \infty \right\},$$

considering the duality pair $\langle x, p \rangle = \sum_{t,\omega} x_t(\omega) p_t(\omega)$ that generates the Mackey topology $\tau(X, Y)$ on X and the weak topology $\sigma(Y, X)$ on Y. It is interesting to note that X could be identified by

$$\left\{ x : \bigcup_{t \in \mathcal{T}} (\{t\} \times \Omega^t) \to \mathbb{R};\ \sup_{t,\omega} |x(t, \omega^t)| < \infty \right\},$$

which in turn is basically ℓ^∞.

1.2 Ambiguity Averse Preferences

In recent years the ambiguity aversion phenomenon, indicated by Ellsberg (1961), has had a lot of attention from economists. While researchers devoted to Decision Theory have created axiomatic models that incorporate this kind of behavior, other economists have used such models to provide new interpretations for various economic phenomena (see Epstein and Schneider (2010) for example).

One of the most successful models that incorporates ambiguity aversion is the *maxmin* model by Gilboa and Schmeidler (1989), where ambiguity aversion is represented by pessimism in face of a set of multiple priors

$$V(c) = \min_{P \in C} \mathbb{E}_P \left[U(c) \right]. \tag{1.1}$$

[1] For two probabilities on a σ-algebra $\mathcal{G}$ the total variation distance is defined by $\| P - Q \| := \sup_{A \in \mathcal{G}} |P(A) - Q(A)|$

Despite the widespread theoretic improvement that has been made based on this model, it does present some setbacks. The most evident is the homogeneous treatment for beliefs in set C. An individual with utility function (1.1) behaves as if his confidence were the same over beliefs in C (see Chateaneuf and Faro (2009) for details). This feature is not natural in some cases. Other models do not present this problem, such as *smooth ambiguity* and *variational* preferences.

1.2.1 Smooth Ambiguity Preferences

This kind of preference, differently from the *maxmin* preferences, allows for smoother behavior face uncertainty. An agent i has such a preference if his utility has the form

$$V^i(c) = \int_{\Delta(\Omega)} \phi_i \left(\int_\Omega U_i(c(\omega))P(d\omega) \right) \mu^i(dP)$$

Here, ambiguity is understood as an uncertainty about what probability governs the system and this is modeled with a subjective probability μ^i with finite support on $\Delta(\Omega)$, whereas ambiguity attitude is captured by ϕ_i. Such a separation between ambiguity and the attitude toward ambiguity has been claimed by Klibanoff et al. (2005, 2009)[2] as an important feature of this model. A probability $P \in \Delta(\Omega)$ is considered by agent i if μ^i assigns positive valuefor P, therefore $supp(\mu^i)$ represents ambiguity perceived by agent i. Probability μ^i is called *second-order belief*, and its reduction $\mathbb{E}_{\mu^i}[P(\cdot)] \in \Delta(\Omega)$ corresponds to subjective belief in expected utility case. Function ϕ_i works with respect to ambiguity aversion as utility index u_i does with respect to risk aversion; concavity of ϕ_i represents ambiguity aversion while convexity means ambiguity propensity, and if it is linear, agent i is ambiguity neutral.

Within Klibanoff et al. (2005) decision makers have preferences on first and second-order acts, but here it is inconvenient because we consider only assets that are first-order acts, hence markets would be incomplete. Fortunately, Seo (2009) presents an alternative approach based on Anscombe and Aumann (1963) that avoids such a problem.

[2]Epstein (2010) argues that this separation does not occur, but Epstein and Schneider (2010) still classify smooth ambiguity preference as an important model.

1.2.2 Variational Preferences

To make his decision, the agent behaves as if considering, at first, every belief (probability) in $\Delta(\Omega)$. His utility is determined as if he were playing a game against a malevolent Nature that tries to choose a model that minimizes agent's expected utility, but Nature has a kind of cost to realize a probability as effective model. *Variational preferences* were developed by (Macheroni et al. (2006a,b)), and there they explore in detail the behavioral properties of this kind of preference.

Agent i's utility functional is given by

$$V^i(c) = \min_{P\in\Delta(\Omega)} \left\{ \mathbb{E}_P \left[\sum_{t\in\mathcal{T}} \beta^t u_i(c_t) \right] + \Gamma^i(P) \right\}$$

and by its recursive form

$$V_t^i(\omega, c) = u_i(c_t(\omega)) + \min_{P\in\Delta(\Omega,\mathcal{F}_{t+1})} \left\{ \mathbb{E}_P \left[V_{t+1}^i(\omega, c) \right] + \gamma_t^i(\omega, P) \right\}.$$

Where $\beta \in (0,1)$ is the inter-temporal discount factor, common to all agents, $u_i : \mathbb{R}_+ \to \mathbb{R}$ is agent i's utility index, $\Gamma^i : \Delta(\Omega) \to [0,\infty]$ and $\gamma_t^i(\omega,\cdot) : \Delta(\Omega,\mathcal{F}_{t+1}) \to [0,\infty]$ are the ambiguity index and dynamic ambiguity index, respectively.

It is supposed that Γ^i and γ_t^i are convex, lower semi-continuous and with 0 in their image; furthermore, γ_t^i satisfies: fixed P, $\gamma_t^i(\cdot, P)$ is $\mathcal{F}_t$-measurable and fixed ω

$$dom\gamma_t^i(\omega,\cdot) := \{P \in \Delta(\Omega);\ \gamma_t^i(\omega, P) < \infty\} \subset \Delta(G_t(\omega), \mathcal{F}_{t+1}).$$

Conditions on ambiguity indexes ensure that beliefs in $dom\Gamma$ are updated according to Bayes' rule[3]. By recursiveness we need to treat only with one-step-ahead decisions and beliefs, and it simplifies the analysis.

Examples of variational preferences are the *maxmin* preferences where

$$\Gamma(P) = \begin{cases} 0; \text{ if } P \in C \\ \infty; \text{ otherwise} \end{cases}$$

and

$$\gamma_t(\omega, P) = \begin{cases} 0; \text{ if } P = Q_{t+1}(\cdot|\omega^t) \text{ for some } Q \in C \\ \infty; \text{ otherwise} \end{cases}$$

[3]For details see Macheroni et al. (2006b).

where $C \subset \Delta(\Omega)$ is closed, convex and rectangular (for definitions see Epstein and Schneider (2003)), the *expected utility* preferences that are *maxmin* with $C = \{Q\}$, and the *Q-multiplier* preferences where

$$\Gamma(P) = \begin{cases} \theta \mathbb{E}_P \left[\log\left(\frac{dP}{dQ}\right)\right];\ \text{if } P \ll Q \\ \infty;\ \text{otherwise} \end{cases}$$

and

$$\gamma_t(\omega, P) = \begin{cases} \theta\beta^{-t}\mathbb{E}_P \left[\log\left(\frac{dP}{dQ_{t+1}(\cdot|\omega^t)}\right)\right];\ \text{if } P \ll Q_{t+1}(\cdot|\omega^t) \\ \infty;\ \text{otherwise} \end{cases}$$

with $\theta > 0$.

While *maxmin* individuals deal with beliefs in an "all or nothing" way, the *multiplier* individual has a "smoother" method of dealing with beliefs. We can see that *variational* preferences are able to encompass several kinds of behavior.

1.3 Blackwell-Dubins Theorem

All results about asymptotic behavior of agents in a general equilibrium model make use of a theorem which was presented in Blackwell and Dubins (1962). In short, such a theorem tells us that if a belief agrees with another belief, then the posterior beliefs generated by the first will converge to that generated by the second.

Adapted to our notation and purposes, the Blackwell-Dubins Theorem sets the following:

Theorem (Blackwell-Dubins)**.** *Let P and Q probabilities on $(\Omega, \mathcal{F})$, if $Q \ll P$ then there exists $A \in \mathcal{F}$ with $Q(A) = 1$ such that for all $\omega \in A$*

$$\|P(\cdot|\omega^t) - Q(\cdot|\omega^t)\| \xrightarrow{t\to\infty} 0.$$

It is interesting to compare the power of hypothesis and its consequence. It is obvious that convergence of posterior beliefs does not imply absolute continuity between probabilities, because what happens in a specific time t is irrelevant for such a convergence, but it is important for ensure positiveness of probability of an event. Indeed, agreement over finite-time null events[4] is the additional hypothesis that is sufficient for a converse result[5].

Theorem 1. *If $P, Q \in \Delta(\Omega)$ satisfies $Q \overset{loc}{\ll} P$ and*

$$\|P(\cdot|\omega^t) - Q(\cdot|\omega^t)\| \xrightarrow{t\to\infty} 0 \quad Q\text{-}a.s.,$$

then $Q \ll P$.

Let us suppose two probabilities on $\Omega = \{0,1\}^\infty$ generated by i.i.d. trials, being $P_t(1|\omega^{t-1}) = p$ and $Q_t(1|\omega^{t-1}) = q$ with $0 < p, q < 1$. If $p < q$, then posterior beliefs do not converge, although P and Q agree over finite-time null events, we can see that, according to the Law of Large Numbers, the event $\{\lim_T \frac{1}{T}\sum_{t=1}^T \omega_t < \frac{p+q}{2}\} \in \mathcal{F}$ has total P-probability and has Q-probability equals to zero. This gap between events in $\mathcal{F}^0$ and events in $\mathcal{F}$ is related to asymptotic phenomena that are subject of this thesis.

[4] $Q \overset{loc}{\ll} P$ denotes that finite-time null events of P are null events of Q.

[5] Its proof can be found in Kalai and Lehrer (1994).

Chapter 2

Convergence of Expectations

2.1 Introduction

When markets are complete, one necessary condition for existence of a sequential equilibrium is that beliefs must be locally equivalent[1], i.e., all agent's beliefs assign null probability over the same *finite-time* events. Araujo and Sandroni (1999) shown that if bankruptcy is permitted, with a penalty for it, then equivalence of beliefs is a necessary condition for equilibrium existence. In turn, according to Blackwell-Dubins Theorem, equivalence implies convergence of posterior beliefs.

Whereas within the context of expected utility equilibrium existence condition tells us about homogeneity of expectations, when we deal with ambiguity averse preferences, expectations is not a clear concept. Klibanoff et al. (2005) approach gives us a preference that provides a notion of ambiguity perception based on a probability on probabilities. Such a class of preferences allows us to obtain a similar result within the Araujo-Sandroni framework.

This chapter is organized as follows: in Section 2.2 the market structure of and how agents make their decisions is presented; the main result follows in Section 2.3; Section 2.4 brings one equilibrium existence result; the following section concludes; and Appendix presents some auxiliary results.

[1]Without this assumption some agent will believe in an arbitrage opportunity.

2.2 Framework

At period 0 agent i can trade contingent claims for all periods, in other words, each agent i choose an asset allocation $k^i = (k^i_t)_{t>0} \in X$, where $k^i_t(\omega)$ represents the amount that i will receive (deliver in the negative case) at time t if ω^t occurs. Agent i has a positive consumption at period 0

$$c_0 = e^i_0 - \langle q, k \rangle, \tag{2.1}$$

where $q \in Y_+$ is the price of assets. Each agent is endowed with an initial consumption stream $e^i \in X_+$ satisfying, for all $i \in \mathcal{I}$

$$\underline{e} < e^i < \sum_j e^j < \bar{e},$$

for positive constants $\underline{e}$ and $\bar{e}$. For $t > 0$, agent i's consumption stream derived of his choice k^i is $c^i_t = (e^i_t + k^i_t)^+$, and $d^i_t = (e^i_t + k^i_t)^-$ is the amount which he is short of.

Agent i makes his decision by maximizing his penalized utility given by

$$V^i(c) = \mathbb{E}_{\mu^i}\left\{\phi_i\left(\mathbb{E}_P\left[u_i(c_0) + \sum_{t>0} \beta^t v_i(e^i_t + k_t)\right]\right)\right\}, \tag{2.2}$$

where

$$v_i(x) = \begin{cases} u_i(x) & \text{if} \quad x > 0 \\ -M^i x & \text{if} \quad x \leq 0 \end{cases}.$$

We suppose that the penalty for unity of d^i_t is constant $M^i > 0$ and utility index and function ϕ_i satisfy $u_i(0) = 0$, $u'_i, \phi'_i > 0$, $u''_i < 0$, $\phi''_i \leq 0$ and $u_i(x) \xrightarrow{x \to 0} \infty$.

Definition 1. *An equilibrium with penalties is allocations and price* $((\bar{c}^i, \bar{k}^i)_{i \in \mathcal{I}}, \bar{q})$ *such that each agent* i *optimizes and markets clear, i.e.*

$$\sum_i c^i = \sum_i e^i,$$

$$\sum_i k^i = 0.$$

Lemma 1. *In every equilibrium with penalties, agent* i*'s first order conditions*

$$\beta^t u'_i(c^i_t(\omega^t)) \mathbb{E}_{\mu^i}\{\phi'_i(\mathbb{E}_P[U_i(c^i)]) P(\omega^t)\} = \lambda(i) q_t(\omega^t) \tag{2.3}$$

are satisfied, where $U_i(c^i) := \sum_t \beta^t u_i(c^i_t)$, $\omega^t \in \Omega^t$ *and* $\lambda(i) > 0$ *is agent* i*'s Lagrange multiplier.*

Proof: Under equilibrium with penalties there is no default, in fact

$$c_t^i - d_t^i = e_t^i + k_t^i$$

by summing over i we get

$$\sum_{i \in \mathcal{I}} (c_t^i - d_t^i) = \sum_{i \in \mathcal{I}} (e_t^i + k_t^i)$$

and by equilibrium conditions

$$\sum_{i \in \mathcal{I}} d_t^i = 0.$$

By assumption of Inada conditions equilibrium allocations are positive. So, because they are interior solutions of maximization problems we get first order conditions (see Luenberger (1969) section 9.3, theorem 1).

□

2.3 Convergence of Expectations

Second-order belief of an agent i with utility form (2.2) induces a probability

$$\mathbb{P}^i(\cdot) = \mathbb{E}_{\mu^i}[P(\cdot)],$$

called its reduction. Such a reduction coincides with agent's (first-order) belief in the ambiguity neutral case. Therefore we can think that a phenomenon which depends on $\mathbb{P}^i$ rather than μ^i does not consider ambiguity aversion.

The next proposition asserts that a necessary condition for existence of equilibrium is equivalence between reductions of every individual. Araujo and Sandroni (1999) assumed that individuals are risk averse, and further features on attitude toward risk plays no role. Similarly, here is assumed that agents are either ambiguity averse or ambiguity neutral, that is $\phi_i'' \leq 0$, the level of attitude toward ambiguity is not mentioned.

Proposition 1. *Let $((\vec{c}^i, \bar{k}^i)_{i \in \mathcal{I}}, \bar{q})$ be an equilibrium with penalties, then*

$$\mathbb{P}^i \ll \mathbb{P}^j \ \forall i, j \in \mathcal{I}.$$

Proof: Suppose $\mathbb{P}^i$ is not absolutely continuous with respect to $\mathbb{P}^j$ for some $i, j \in \mathcal{I}$. Thus, by Lemma 2 in the Appendix, there is a sequence of events $A_t \in \mathcal{F}_t$ such that $\mathbb{P}^j(A_t) \to 0$ but $\mathbb{P}^i(A_t) \geq \delta > 0$.

By the first order conditions 2.3

$$\begin{aligned}\left\langle q_t, \frac{1}{\beta^t}\chi_{A_t}\right\rangle &= \frac{1}{\lambda_i}\mathbb{E}_{\mu^i}\left\{\phi_i'\left(\mathbb{E}_P\left[U_i(\bar{c}^i)\right]\right)\mathbb{E}_P\left[u_i'(\bar{c}^i)\chi_{A_t}\right]\right\}\\ &\geq \frac{1}{\lambda_i}\phi_i'(U_i(\bar{e}))u_i'(\bar{e})\mathbb{P}^i(A_t)\\ &\geq \frac{1}{\lambda_i}\phi_i'(U_i(\bar{e}))u_i'(\bar{e})\delta =: \eta > 0.\end{aligned}$$

As $supp(\mu^j)$ is finite, $\mathbb{P}^j(A_t) \to 0$ implies that for any $n \in \mathbb{N}$ there is $t \in \mathcal{T}$ such that $P(A_t) < \frac{1}{n}$ for all $P \in supp(\mu^j)$. So, let $\bar{t} \in \mathcal{T}$ such that for $P \in supp(\mu^j)$ we get

$$P(A_{\bar{t}}) < \frac{K}{M_j + u_j(\bar{e})}$$

where $K := u_j(\bar{c}_0^j + \eta) - u_j(\bar{c}_0^j) > 0$. Define $(\hat{c}^j, \hat{k}^j)$ by $\hat{c}_0^j := \bar{c}_0^j + \eta$ and $\hat{k}_t^j := \bar{k}_t^j$ if $t \neq \bar{t}$ and $\hat{k}_{\bar{t}}^j := \bar{k}_{\bar{t}}^j - \frac{\chi_{A_{\bar{t}}}}{\beta^{\bar{t}}}$, that is in restrictions because

$$\hat{c}_0^j - e_0^j + \left\langle q, \hat{k}^j\right\rangle = \bar{c}_0^j + \eta - e_0^j + \left\langle q, \bar{k}^j\right\rangle - \left\langle q_t, \frac{\chi_{A_{\bar{t}}}}{\beta^{\bar{t}}}\right\rangle \leq 0$$

increasing $\hat{c}_0^j$ if we need to.

Now, we can verify that $(\hat{c}^j, \hat{k}^j)$ is better than $(\bar{c}^i, \bar{k}^i)$ for j, and this gives us one contradiction.

$$\begin{aligned}V^j(\hat{c}^j) &= \mathbb{E}_{\mu^j}\left\{\phi_i\left(\mathbb{E}_P\left[u_j(\hat{c}_0) + \sum \beta^t v_j(\hat{c}_t^j)\right]\right)\right\}\\ &= \mathbb{E}_{\mu^j}\left\{\phi_i\left(\mathbb{E}_P\left[u_j(\bar{c}_0) + \sum \beta^t v_j(\bar{c}_t^j) + (u_j(\hat{c}_0^j) - u_j(\bar{c}_0^j)) + \beta^{\bar{t}}(v_j(\hat{c}_{\bar{t}}^j) - v_j(\bar{c}_{\bar{t}}^j))\right]\right)\right\}\\ &\geq \mathbb{E}_{\mu^j}\left\{\phi_i\left(\mathbb{E}_P\left[u_j(\bar{c}_0) + \sum \beta^t v_j(\bar{c}_t^j) + K - \chi_{A_{\bar{t}}}\beta^{\bar{t}}(\beta^{-\bar{t}}M_j + u_j(\bar{e}))\right]\right)\right\}\\ &> \mathbb{E}_{\mu^j}\left\{\phi_i\left(\mathbb{E}_P\left[u_j(\bar{c}_0) + \sum \beta^t v_j(\bar{c}_t^j)\right] + K - P(A_{\bar{t}})(M_j + u_j(\bar{e}))\right)\right\}\\ &> V^j(\bar{c}^j).\end{aligned}$$

□

Corollary below is similar to convergence of expectations result found in the literature, but it provides a different interpretation. While posterior beliefs homogeneity within context of expected utilities gives to us an idea that knowledge must be the same between individuals to ensure equilibrium, when agents are smooth ambiguity their perception about how model governs the events could ever differ. For example, let us suppose that there are two agents and consider two non-equivalent probabilities $P, Q \in \Delta$. If second-order beliefs are given by $\mu^1(P) = \mu^1(Q) = 1/2$ and $\mu^2(1/2P + 1/2Q) = 1$, then $\mathbb{P}^1 \sim \mathbb{P}^2$ and equilibrium is possible even if agent 1 is ambiguity averse, but perception of ambiguity is distinct for agents. But if $supp(\mu^2) = \{R\}$, R is not equivalent to P or Q, and $R \notin co(\{P, Q\})^2$, then $\mathbb{P}^1$ and $\mathbb{P}^2$ are not equivalent and there is no equilibrium with penalties.

Corollary 1. *Under equilibrium with penalties* $\|\mathbb{P}^i(\cdot|\omega^t) - \mathbb{P}^j(\cdot|\omega^t)\| \xrightarrow{t\to\infty} 0$ $\mathbb{P}^i$*-a.s., for all* $i, j \in \mathcal{I}$.

Proof: It is an immediate consequence of the Blackwell-Dubins Theorem.

□

2.4 Existence of Equilibrium

Definition 2. *We say that a set of probabilities* $\mathcal{P}$ *displays strong compatibility condition if there is a constant* $K > 0$ *such that*

$$P(A) \leq KQ(A), \ \forall A \in \mathcal{F}$$

for any $P, Q \in \mathcal{P}$.

The following result presents sufficient conditions for existence of an equilibrium with penalties. It is supposed that agents' reductions are strongly compatible, condition that implies equivalence.

Proposition 2. *Suppose that* $\{\mathbb{P}^i\}_{i\in\mathcal{I}}$ *displays strong compatibility condition, then there exists* $(M_i)_{i\in\mathcal{I}}$ *such that one equilibrium with penalties exists.*

[2] $co(A)$ denotes the convex hull of set A.

Proof: If we consider the utilities

$$\tilde{V}_i(c) = \mathbb{E}_{\mu^i}\left\{\phi_i\left(\mathbb{E}_P\left[U_i(c)\right]\right)\right\},$$

by Bewley (1972)[3] there exists $((\bar{c}^i)_{i\in\mathcal{I}}, \bar{p}) \in X_+^I \times X_+^*$ such that

i) $\sum_i (\bar{c}^i - e^i) = 0;$

ii) $\bar{c}^i = \arg\max \left\{\tilde{V}(c^i); \bar{p}(c^i - e^i) \leq 0,\ c^i \geq 0\right\}.$

Define $\bar{k}^i$ by $\bar{k}_0^i := 0$, $\bar{k}_t^i := \bar{c}_t^i - \bar{e}_t^i$, $t \geq 1$ and $\bar{q}$ by $\bar{q}_0 := 0$, $\bar{q}_t := \frac{1}{p_0}\bar{p}_t$ $t \geq 1$. We claim that $((\bar{c}^i, \bar{k}^i)_{i\in\mathcal{I}}, \bar{q})$ is equilibrium with penalty for suitable $(M_i)_{i\in\mathcal{I}}$.

Note that $(\bar{c}^i, \bar{k}^i)$ is in the constraint (2.1) since by $\langle \bar{p}, \bar{c}^i - e^i\rangle = 0$ we get $\bar{c}_0^i - e_0^i + \langle \bar{q}, \bar{k}^i\rangle = 0$, furthermore $\sum_i \bar{k}^i = \sum_i (\bar{c}^i - e^i)$.

Now, consider an arbitrary (c^i, k^i) satisfying the constraint (2.1). As $c_t^i - d_t^i = e_t^i + k_t^i$, multiplying both sides by $\bar{p}_t$ and summing over $t > 0$ we get

$$\sum_{t>0} \bar{p}_t c_t^i - \sum_{t>0} \bar{p}_t d_t^i = \sum_{t>0} \bar{p}_t e_t^i + \bar{p}_0 \sum_{t>0} \bar{q}_t k_t^i,$$

by summing over all nodes and rearranging we get

$$\langle \bar{p}, c^i\rangle = \langle \bar{p}, e^i\rangle + \langle \bar{p}, d^i\rangle.$$

Denote by $\bar{r} = \langle \bar{p}, e^i\rangle$ and

$$\psi_i(r) = \max\left\{\tilde{V}_i(c^i); c^i \in X_+ \text{ and } \langle \bar{p}, c^i\rangle \leq r\right\}.$$

Since ψ_i is concave (see Lemma 3) if $r < r'$

$$\psi_i(r) - \psi_i(r') \geq -D_+\psi_i(r)(r' - r)$$

where $D_+\psi_i(r)$ denotes the derivative from the right of ψ_i at r.

By the first order conditions for (ii)

$$\beta^t u_i'(c_t^i(\omega^t))\mathbb{E}_{\mu^i}\left\{\phi_i'\left(\mathbb{E}_P\left[U_i(c^i)\right]\right)P(\omega^t)\right\} = \lambda_i \bar{p}_t(\omega^t),\ \forall \omega^t \in \Omega^t,$$

[3] Since μ^i has finite support (see Section 1.2.1) $\tilde{V}_i$ is a finite sum of Mackey continuous functions, hence is also Mackey continuous.

so, by Lemma 5 we get quota $q_t(\omega^t) \leq L_i \beta^t \mathbb{P}^i(\omega^t)$, where the constant incorporates p_0. Therefore, by this bound and the concavity of ϕ_i and ψ_i

$$
\begin{aligned}
V_i(\bar{c}^i) - V_i(c^i) =& \mathbb{E}_{\mu^i}\left\{\phi_i\left(\mathbb{E}_P\left[\sum \beta^t u_i(\bar{c}^i_t)\right]\right)\right\} - \mathbb{E}_{\mu^i}\left\{\phi_i\left(\mathbb{E}_P\left[\sum \beta^t (u_i(c^i_t) - M_i d^i_t)\right]\right)\right\} \\
\geq& \mathbb{E}_{\mu^i}\left\{\phi_i\left(\mathbb{E}_P\left[\sum \beta^t u_i(\bar{c}^i_t)\right]\right)\right\} - \mathbb{E}_{\mu^i}\left\{\phi_i\left(\mathbb{E}_P\left[\sum \beta^t u_i(c^i_t)\right]\right)\right\} \\
&+ \mathbb{E}_{\mu^i}\left\{\phi'_i\left(\mathbb{E}_P\left[\sum \beta^t u_i(c^i_t)\right]\right)\mathbb{E}_P\left[\sum \beta^t M_i d^i_t\right]\right\} \\
\geq& \psi_i(\bar{r}) - \psi_i(\bar{r} + \langle \bar{p} + d^i \rangle) + \phi'_i(U_i(\bar{e})) M_i \mathbb{E}_{\mathbb{P}^i}\left[\sum \beta^t d^i_t\right] \\
\geq& -D_+\psi_i(\bar{r})\langle \bar{p}, d^i \rangle + \phi'_i(U_i(\bar{e})) M_i \mathbb{E}_{\mathbb{P}^i}\left[\sum \beta^t d^i_t\right] \\
\geq& \left(\phi'_i(U_i(\bar{e})) M_i - D_+\psi_i(\bar{r}) L_i\right) \mathbb{E}_{\mathbb{P}^i}\left[\sum \beta^t d^i_t\right],
\end{aligned}
$$

thus, if $M_i \geq \frac{D_+\psi_i(\bar{r})L_i}{\phi'_i(U_i(\bar{e}))}$, $((\bar{c}^i, \bar{k}^i)_{i\in\mathcal{I}}, \bar{q})$ is an equilibrium with penalties.

□

2.5 Conclusion

The main result of this chapter extends conclusions of Araujo and Sandroni (1999) within a wider context where ambiguity aversion is included. While compatibility over beliefs is a necessary condition for ensuring equilibrium when agents have expected utilities and attitudes toward risk play no role in such a case, if agents have smooth ambiguity preferences there exits equilibrium only if their second-order beliefs' reduction are equivalents and ambiguity attitudes have no importance.

On the other hand, for attain an equilibrium existence result it is found constants M^i that depends on risk and ambiguity attitude. The importance of Proposition 2 resides in the guarantee of non-vacuity of Proposition 1.

2.6 Appendix

Lemma 2. *Let $P, Q \in \Delta$, if P is not absolutely continuous with respect to Q then there exists a sequence $A_t \in \mathcal{F}_t$ such that $Q(A_t) \to 0$ and $P(A_t) > \delta > 0$ $\forall t \in \mathcal{T}$.*

Proof: By hypothesis there exists $A \in \mathcal{F}$ such that $Q(A) = 0$ but $P(A) > \delta$ for some $\delta \in (0,1)$. Since $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$ and $\cup_t \mathcal{F}_t$ is an algebra, by Carathéodory extension (see Shiryayev (1984)) $Q(A) = \inf\{Q(B); A \subset B \in \cup_t \mathcal{F}_t\}$. So, for each $n \in \mathbb{N}$, $\exists A_{t_n} \in \mathcal{F}_{t_n}$ such that $A \subset A_{t_n}$ and $Q(A_{t_n}) < \frac{1}{n}$, with $t_n < t_{n+1}$ because $\mathcal{F}_t \subset \mathcal{F}_{t+1}$. On the other hand $P(A_{t_n}) \geq P(A) > \delta$. For $t \in (t_n, t_{n+1})$ put $A_t = A_{t_n}$.

$\square$

Lemma 3. *ψ_i is an increasing and concave function.*

Proof: If $r < r'$, then $\left\{\tilde{V}_i(c); \langle \bar{p}, c\rangle \leq r, c \geq 0\right\} \subset \left\{\tilde{V}_i(c); \langle \bar{p}, c\rangle \leq r', c \geq 0\right\}$, and so $\psi_i(r) \leq \psi_i(r')$.

If $r, r' > 0$ and $\alpha \in [0,1]$, by concavity of V_i

$$\tilde{V}_i(\alpha c + (1-\alpha)c') \geq \alpha \tilde{V}_i(c) + (1-\alpha)\tilde{V}_i(c'),$$

therefore

$$\begin{aligned}
& \max\left\{\tilde{V}_i(\alpha c + (1-\alpha)c'); \langle \bar{p}, c\rangle \leq r, \langle \bar{p}, c'\rangle \leq r' \text{ and } c, c' \geq 0\right\} \\
\geq\ & \max\left\{\alpha\tilde{V}_i(c) + (1-\alpha)\tilde{V}_i(c'); \langle \bar{p}, c\rangle \leq r, \langle \bar{p}, c'\rangle \leq r' \text{ and } c, c' \geq 0\right\} \\
=\ & \max\left\{\alpha\tilde{V}_i(c); \langle \bar{p}, c\rangle \leq r, c \geq 0\right\} + \max\left\{(1-\alpha)\tilde{V}_i(c'); \langle \bar{p}, c'\rangle \leq r', c' \geq 0\right\} \\
=\ & \alpha\psi_i(r) + (1-\alpha)\psi_i(r').
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \psi_i(\alpha r + (1-\alpha)r') \\
=\ & \max\left\{\tilde{V}_i(\alpha c + (1-\alpha)c'); \langle \bar{p}, \alpha c + (1-\alpha)c'\rangle \leq \alpha r + (1-\alpha)r' \text{ and } c, c' \geq 0\right\} \\
\geq\ & \max\left\{\tilde{V}_i(\alpha c + (1-\alpha)c'); \langle \bar{p}, c\rangle \leq r, \langle \bar{p}, c'\rangle \leq r' \text{ and } c, c' \geq 0\right\}.
\end{aligned}$$

$\square$

Lemma 4. *If $\{\mathbb{P}^i\}_{i\in\mathcal{I}}$ displays strong compatibility condition and $(c^i)_{i\in\mathcal{I}}$ is an equilibrium, then there exists $l_i > 0$ (depending on the equilibrium) such that $c^i_t \geq l_i$ for each $i \in \mathcal{I}$ and all $t \in \mathcal{T}$.*

Proof: Fix $i \in \mathcal{I}$ and $t \in \mathcal{T}$, since $\underline{e} < \sum_i e^i = \sum_i c^i$, fixed ω^t either $c_t^i(\omega^t) \geq \underline{e}$ or for some $j \in \mathcal{I}$ $c_t^j(\omega^t) \geq \underline{e}$. Assume the last case, from the first order conditions (2.3)

$$\frac{u_j'(c_t^j(\omega^t))\mathbb{E}_{\mu^j}\{\phi_j'(\mathbb{E}_P[U_j(c^j)]P(\omega^t))\}}{u_i'(c_t^i(\omega^t))\mathbb{E}_{\mu^i}\{\phi_i'(\mathbb{E}_P[U_j(c^i)]P(\omega^t))\}} = \frac{\lambda_j}{\lambda_i},$$

so

$$\frac{u_j'(\underline{e})}{u_i'(c_t^i(\omega^t))}\frac{\phi_j'(u_j(c_0^j))K\mathbb{P}^i(\omega^t)}{\phi_i'(U_i(\bar{e}))\mathbb{P}^i(\omega^t)} \geq \frac{\lambda_j}{\lambda_i},$$

therefore $u_i'(c_t^i(\omega^t)) \leq \frac{\lambda_i u_j'(\underline{e})\phi_j'(u_j(c_0^j))K}{\lambda_j \phi_i'(U_i(\bar{e}))}$ and $c_t^i(\omega^t) \geq u_i'^{-1}\left(\frac{\lambda_i u_j'(\underline{e})\phi_j'(u_j(c_0^j))K}{\lambda_j \phi_i'(U_i(\bar{e}))}\right) > 0$.

Finally, put $l_i = \min\left\{\underline{e}, u_i'^{-1}\left(\frac{\lambda_i u_j'(\underline{e})\phi_j'(u_j(c_0^j))K}{\lambda_j \phi_i'(U_i(\bar{e}))}\right); j \in \mathcal{I} \setminus \{i\}\right\}$, which is independent of ω^t.

□

Lemma 5. *Under the same assumptions of the previous lemma, we get* $q_t(\omega^t) \leq L_i \beta^t \mathbb{P}^i(\omega^t)$, *where* L_i *is a positive constant.*

Proof: This is achieved only by using the Lemma 4 bound and first order conditions (2.3).

□

Chapter 3

Survival

3.1 Introduction

The *market selection hypothesis* has long been invoked by economists to justify the assumption that economic agents have rational expectations, i.e., that their beliefs are identical to the probabilistic model that governs the events. The rationale is the following: in an economy populated by heterogeneous agents, those who have such a feature will obtain advantages over others and in the long-term will accumulate more wealth; their decisions will be more important to the economy; asymptotically, such individuals will be the ones to influence prices and dominate the market. However, for this reasoning to work, it is necessary to assume certain hypotheses. To achieve positive results, i.e., those where selection for who makes accurate predictions happen, Sandroni (2000) and Blume and Easley (2006) suppose that agents have expected utilities and markets are complete. Without market completeness there are negative results as in Blume and Easley (2006), Beker and Chattopadhyay (2010) and Coury and Sciubba (2010).

Our focus is on the exclusion of the first assumption, since we want to study the effects of ambiguity aversion. An important work in such a direction is that of Condie (2008), whose main result indicates that the influence of *maxmin* agents in complete markets becomes irrelevant when compared with rational expectations individuals. Such a result could make it seem that ambiguity averse preferences are economically unimportant, but this is not true. This kind of preference has been used to improve economic theory in many areas, providing new approaches and solving problems with a realistic appeal (see Epstein and Schneider (2010) for a brief survey in finance).

The survival analysis by Sandroni (2000) and Blume and Easley (2006) shows that if agents behave according to expected utilities, then what matters in determining survival are the inter-temporal discount factors and beliefs. Condie (2008) analyzes survival of *maxmin* agents, who are ambiguity averse, and concludes that survival for this type of agent is difficult to happen if a rational expectation agent is present, due to the aggregate risk. By considering a general type of ambiguity averse preference, we can reconcile survival of ambiguity averse agents with the presence of aggregate risk.

Considerations with respect to Pareto optimality are in Section 3.2. The Section 3.3 brings examples of asymptotic behavior of consumption decisions in different situations where optimality conditions are met. The mean results are presented in Section 3.4 and the Section 3.5 concludes.

3.2 Pareto Optimality

Following Blume and Easley (2006) the analysis will be made from the Pareto optimal allocations, hence, the consequences will be valid for complete markets.

We suppose that each agent is endowed with an initial consumption stream $e^i \in X_+$.

Definition 3. *An allocation $(c^i)_{i\in\mathcal{I}}$ is called **Pareto optimal** if it is feasible, that is, $\sum_i c^i = \sum_i e^i$, and there is no feasible allocation $(\dot{c}^i)_{i\in\mathcal{I}}$ such that $V^i(\dot{c}^i) \geq V^i(c^i)$ $\forall i$ and $V^{i_0}(\dot{c}^{i_0}) > V^{i_0}(c^{i_0})$ for some $i_0 \in \mathcal{I}$.*

We consider only consumptions in X_{++}, and if $c^* = (c^{1*}, \ldots, c^{I*}) \in X_{++}^{\mathcal{I}}$ is Pareto optimal, there is $(\lambda_1, \ldots, \lambda_I) \gg \mathbf{0}$ such that c^* is the solution for problem

$$\begin{cases} \max\limits_{(c^1,\ldots,c^I)} \sum\limits_i \lambda_i V^i(c^i) \\ s.t. \ \sum\limits_i (c^i - e^i) \leq \mathbf{0}. \end{cases} \tag{3.1}$$

By the first order conditions (Ekeland and Turnbull, 1983, 124) for that problem there are constants $\eta_t(\omega) > 0$ such that

$$\lambda_i p_t^i(\omega) = \eta_t(\omega) \tag{3.2}$$

for some $p^i = (p^i_t) \in \partial V^i(c^{i*})$, for any $i \in \mathcal{I}$.

Next lemma is part of Theorem 18 of Macheroni et al. (2006a) and characterizes the superdifferential of a variational utility.

Lemma 6. *The superdifferential of variational utility V has the form*

$$\partial V(c) = \left\{ (\beta^t u'(c_t) dP_t); P \in \arg\min_{Q \in \Delta(\Omega)} \left\{ E_Q \left[\sum_{t \in \mathcal{T}} \beta^t u(c_t) \right] + \Gamma(Q) \right\} \right\},$$

for any $c \in X_{++}$.

Proof: This follows by proposition 6 of section 4.3.3 in Aubin (1982).

□

If $(c^1, \ldots, c^I)$ is Pareto optimal, by equation (3.2) and by previous lemma we get for each $i \in \mathcal{I}$ a probability $\mathbb{P}^i \in \arg\min_{Q \in \Delta(\Omega)} \left\{ E_Q \left[\sum_{t \in \mathcal{T}} \beta^t u(c_t) \right] + \Gamma(Q) \right\}$ that is the effective belief of agent i. Such probabilities are related with the fixed allocation and carry all information needed to determine survival[1].

We can derive from (3.2) some useful relations: for all $t \in \mathcal{T}$, $\omega \in \Omega$ and $i, j \in \mathcal{I}$

$$\lambda_i \beta^t u'_i(c^i_t(\omega)) \mathbb{P}^i_t(\omega) = \lambda_j \beta^t u'_j(c^j_t(\omega))\ \mathbb{P}^j_t(\omega), \tag{3.3}$$

moreover, we get for every $s \in \mathcal{S}_t$

$$\frac{u'_i(c^i_t(\omega^{t-1}, s))}{u'_i(c^i_{t-1}(\omega^{t-1}))} \mathbb{P}^i_t(s|\omega^{t-1}) = \frac{u'_j(c^j_t(\omega^{t-1}, s))}{u'_j(c^j_{t-1}(\omega^{t-1}))} \mathbb{P}^j_t(s|\omega^{t-1}) \tag{3.4}$$

and, $\forall r, s \in \mathcal{S}_t$

$$\frac{u'_i(c^i_t(\omega^{t-1}, s))}{u'_j(c^j_t(\omega^{t-1}, s))} \frac{\mathbb{P}^i_t(s|\omega^{t-1})}{\mathbb{P}^j_t(s|\omega^{t-1})} = \frac{u'_i(c^i_t(\omega^{t-1}, r))}{u'_j(c^j_t(\omega^{t-1}, r))} \frac{\mathbb{P}^i_t(r|\omega^{t-1})}{\mathbb{P}^j_t(r|\omega^{t-1})}. \tag{3.5}$$

By Lemma 6 and recursive form of utilities we get

$$\mathbb{P}^i_t(\cdot|\omega^{t-1}) \in \arg\min_{P \in \Delta(G_t(\omega), \mathcal{F}_{t+1})} \left\{ E_P \left[u(c^i_t) \right] + \gamma^i_t(\omega, P) \right\}.$$

[1]Remember that we assume the same inter-temporal discount factor for every agent.

3.3 Examples

This section presents some representative situations for general results about the survival problem. The context, in terms of uncertainty and endowments, is the same in all cases. There are two states of nature and two agents, $\mathcal{S} = \{1,2\} = \mathcal{I}$, $\mathbb{P}$ is generated by i.i.d. trials uniformly on $\mathcal{S}$, i.e., $\mathbb{P}_t(1|\omega^{t-1}) = 1/2$, $\forall\ t$. Agent 1 always has expected utility with correct belief and his utility is given by

$$V_1(c^1) = \mathbb{E}_{\mathbb{P}}\left[\sum_{t=0}^{\infty}\left(\frac{1}{2}\right)^t \log c_t^1\right].$$

Agent 2 is different in each case, allowing for a comparative analysis. The endowments depend only on the current nature state, $e_t^1(1) = e_t^2(1) = 1/2$, $e_t^1(2) = e_t^2(2) = 1/2 + \delta/2$, with $\delta > 0$.

3.3.1 Expected Utility Example

Beginning with a well known example based on Sandroni (2000) where there are two agents with expected utilities, one of whom has a wrong belief, being driven out of the market by the other one with correct belief. The key to achieve this result is the *law of large numbers*.

Here, agent 2 also has expected utility, but with wrong belief, his utility is given by

$$V_2(c^2) = \mathbb{E}_{\bar{\mathbb{P}}}\left[\sum_{t=0}^{\infty}\left(\frac{1}{2}\right)^t \log c_t^2\right],$$

with $\bar{\mathbb{P}}_t(1|\omega^t) = 1/2 - \varepsilon$ and $0 < \varepsilon < 1/2$.

By (3.3) we get

$$\frac{(\frac{1}{2})^t c_t^2(\omega^t)}{(\frac{1}{2}-\varepsilon)^n(\frac{1}{2}+\varepsilon)^{t-n} c_t^1(\omega^t)} = \frac{\lambda_2}{\lambda_1},\ \forall t \in \mathbb{N},$$

where n is the number of times that state 1 occurs.

The law of large numbers gives us $n \approx t/2$, then

$$\frac{(\frac{1}{2})^t}{(\frac{1}{2}-\varepsilon)^n(\frac{1}{2}+\varepsilon)^{t-n}} \approx \frac{(\frac{1}{2})^t}{(\frac{1}{2}-\varepsilon)^{t/2}(\frac{1}{2}+\varepsilon)^{t/2}} = \left(\frac{\frac{1}{4}}{\frac{1}{4}-\varepsilon^2}\right)^{t/2} \xrightarrow{t\to\infty} \infty.$$

Whereas $\frac{\lambda_2}{\lambda_1}$ is a positive constant, $\frac{c_t^2(\omega^t)}{c_t^1(\omega^t)} \to 0$ with probability 1, and by $c_t^1(\omega^t) \leq 1 + \delta$ we get $c_t^2(\omega^t) \to 0$ $\mathbb{P}$ a.s.

This example is related to Proposition 2 (1) of Sandroni (2000) and Theorem 3 (ii) of Blume and Easley (2006). Note that the only important fact about endowments is their limitation. Below we will show that for survival of an averse ambiguity agent, other features matter.

3.3.2 Maxmin Utility Example

The next example is based on Condie (2008) where an agent with maxmin utility cannot survive in the presence of an expected utility with correct belief. In turn, this conclusion strongly depends on the aggregate risk. The ambiguity averse customer acts as if he were an expected utility with wrong belief, and he cannot survive as well as in the previous case. But there is a particularity of maxmin utility that does not occur in the more general model of variational utilities. A maxmin agent deals with his possible beliefs in a homogeneous way, so aggregate risk forces him to take a precautionary attitude that moves away from the one that ensures survival.

In this example the utility of agent 2 is given by

$$V_2(c^2) = \min_{P\in\Delta(\Omega)} \left[\mathbb{E}_P \left(\sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^t \log c_t^2 \right) + \Gamma(P) \right],$$

where $\gamma_t(P_t) = \begin{cases} 0; \text{ if } P_t(1|\omega) \in [1/3, 2/3] \\ \infty; \text{ otherwise.} \end{cases}$

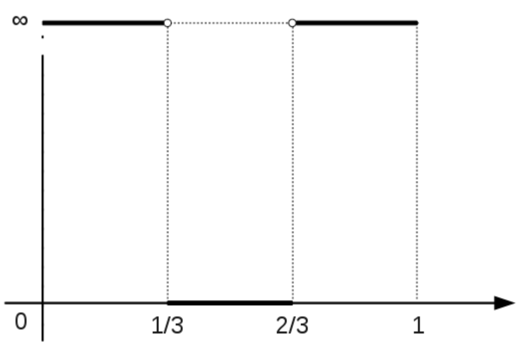

Figure 3.1: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

By (3.5) we get

$$\frac{c_t^2(\omega^{t-1},1)}{c_t^1(\omega^{t-1},1)}\frac{\mathbb{P}_t(1|\omega^{t-1})}{\mathbb{P}_t^2(1|\omega^{t-1})}=\frac{c_t^2(\omega^{t-1},2)}{c_t^1(\omega^{t-1},2)}\frac{\mathbb{P}_t(2|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})}.$$

and by market clearing

$$\frac{1+\delta-c_t^2(\omega^{t-1},2)}{1-c_t^2(\omega^{t-1},1)}\frac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)}=\frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})}. \tag{3.6}$$

If $c_t^2(\omega^{t-1},1) > c_t^2(\omega^{t-1},2)$ then $\mathbb{P}_t(1|\omega^{t-1}) = 1/3$, because $\mathbb{P}_t^2(\cdot|\omega^{t-1})$ minimizes $\mathbb{E}_P\left[\left(\frac{1}{2}\right)^t(\log c_t^2(\omega^{t-1},\cdot))\right]$ subject to $P(1)\in[1/3,2/3]$, and by (3.6) $\frac{1+\delta-c_t^2(\omega^{t-1},2)}{1-c_t^2(\omega^{t-1},1)} < 1/2$ whence we get

$$1+\delta-c_t^2(\omega^{t-1},2) < 1/2-1/2c_t^2(\omega^{t-1},1) < 1-c_t^2(\omega^{t-1},1)$$

so $c_t^2(\omega^{t-1},2) > c_t^2(\omega^{t-1},1)$, a contradiction.

If $c_t^2(\omega^{t-1},1) = c_t^2(\omega^{t-1},2)$, since consumption is positive, from equation (3.6) we get[2] $\frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})} > 1+\delta$. If $c_t^2(\omega^{t-1},2) > c_t^2(\omega^{t-1},1)$, then agent 2 acts like an expected utility assigning probability 2/3 for state 1. In both cases we get $\frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})} \geq \min\{1+\delta,2\}$, so agent 2 does not survive as in the previous example because he always makes inaccurate predictions. Such an example fits Theorem 1 of Condie (2008).

3.3.3 Motivating Example

The last example gives an idea of how a variational agent can survive even in a presence of an expected utility with correct belief agent, and with aggregate risk. An individual could be ambiguity averse and survive as long as his ambiguity index is not so small. Such a constraint depends on how big the aggregate risk is.

While agent 1 has expected utility with correct belief, agent 2's utility is given by

$$V_2(c^2)=\min_{P\in\Delta(\Omega)}\left[\mathbb{E}_P\left(\sum_{t=0}^{\infty}\left(\frac{1}{2}\right)^t\log c_t^2\right)+\Gamma(P)\right],$$

where

$$\gamma_t(P_t)=\begin{cases}(\frac{1}{2}-P_t(1|\omega^{t-1}))\varepsilon; & \text{if } P_t(1|\omega^{t-1})\leq\frac{1}{2}\\(P_t(1|\omega^{t-1})-\frac{1}{2})\varepsilon; & \text{if } P_t(1|\omega^{t-1})\geq\frac{1}{2}\end{cases}$$

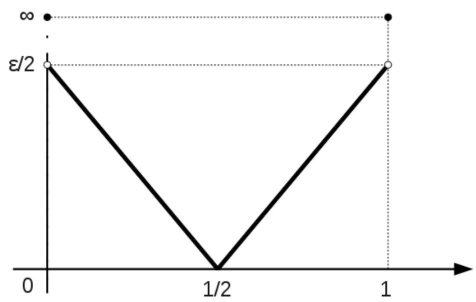

Figure 3.2: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

Again, as in (3.6)

$$\frac{1+\delta-c_t^2(\omega^{t-1},2)}{1-c_t^2(\omega^{t-1},1)}\frac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)}=\frac{\mathbb{P}_t^2(1|\omega^{t-1})}{\mathbb{P}_t^2(2|\omega^{t-1})}$$

by rearranging this expression

$$\mathbb{P}_t^2(1|\omega^{t-1})\left(\frac{1}{c_t^2(\omega^{t-1},1)}-1\right)=\mathbb{P}_t^2(2|\omega^{t-1})\left(\frac{1+\delta}{c_t^2(\omega^{t-1},2)}-1\right)$$

Consider the possibilities for $\mathbb{P}_t^2(\cdot|\omega^{t-1})$.

If $\mathbb{P}_t^2(1|\omega^{t-1})<\mathbb{P}_t^2(2|\omega^{t-1})$ we get $\dfrac{c_t^2(\omega^{t-1},2)}{c_t^2(\omega^{t-1},1)}>1+\delta>1$, then

$$\begin{aligned}&\mathbb{P}_t^2(1|\omega^{t-1})\log c_t^2(\omega^{t-1},1)+\mathbb{P}_t^2(2|\omega^{t-1})\log c_t^2(\omega^{t-1},2)+\gamma_t(P_t)\\&>1/2\log c_t^2(\omega^{t-1},1)+1/2\log c_t^2(\omega^{t-1},2)\end{aligned}$$

and $\mathbb{P}_t^2$ is not a minimizer.

If $\mathbb{P}_t^2(1|\omega^{t-1})>\mathbb{P}_t^2(2|\omega^{t-1})$ we get $\dfrac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)}>\dfrac{1}{1+\delta}$. Therefore

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_t^2(\cdot|\omega^{t-1})}[\log c_t^2(\omega^{t-1},\cdot)]+\gamma_t(\omega,P_t)-\mathbb{E}_{\mathbb{P}_t(\cdot|\omega^{t-1})}[\log c_t^2(\omega^{t-1},\cdot)]\\&=(\mathbb{P}_t^2(1|\omega^{t-1})-1/2)\log c_t^2(\omega^{t-1},1)+\\&+(\mathbb{P}_t^2(2|\omega^{t-1})-1/2)\log c_t^2(\omega^{t-1},2)+(\mathbb{P}_t^2(1|\omega^{t-1})-1/2)\varepsilon\\&=(\mathbb{P}_t^2(1|\omega^{t-1})-1/2)[\log\left(\frac{c_t^2(\omega^{t-1},1)}{c_t^2(\omega^{t-1},2)}\right)+\varepsilon]\\&>(\mathbb{P}_t^2(1|\omega^{t-1})-1/2)[\log\left(\frac{1}{1+\delta}\right)+\varepsilon]\end{aligned}$$

So if $\varepsilon-\log(1+\delta)>0$, $P_t=\mathbb{P}_t$ is the only minimizer.

[2]Note that $1+\delta<\frac{1+\delta-x}{1-x}<\infty,\ \forall\ 0<x<1$.

Therefore, agent 2 acts as an expected utility with correct belief if, for example, $\varepsilon = \delta = 1$, which fits into the context of previous examples. The message given to us by these examples is that the relation between survival of an ambiguity averse agent and the presence of aggregate risk could be made in a more precise way than that found in Condie (2008). Proposition 4 is an effort in that direction.

3.4 Survival

A Pareto optimal allocation $(c_i)_{i\in\mathcal{I}}$ and beliefs $\mathbb{P}^i$ given in (3.3), for each $i \in \mathcal{I}$, are fixed.

Definition 4. *Agent i survives on the path ω if $\overline{\lim}\, c_t^i(\omega) > 0$. We say that i survives if there is $A \in \mathcal{F}$ with $\mathbb{P}(A) = 1$ such that i survives on all $\omega \in A$.*

Some assumptions are needed to achieve the results.

Assumption 1. *Let $e := \sum_i e^i$. For every $i \in \mathcal{I}$ endowments satisfy $\underline{e} < e^i < e < \bar{e}$, for positive constants $\underline{e}$ and $\bar{e}$.*

Assumption 2. *$u_i' > 0$, $u_i'' < 0$ and $u_i'(x) \xrightarrow{x\to 0} \infty$ for all $i \in \mathcal{I}$.*

Assumption 3. *For all path ω, suppose that $\mathbb{P}_t(\cdot|\omega^{t-1}) > 0$ and*

$$dom\ \gamma_t^i(\omega,\cdot) \subset \Delta^+(G_t(\omega), \mathcal{F}_{t+1}) := \{r \in \Delta(G_t(\omega), \mathcal{F}_{t+1}); r(A) > 0\ \forall A \in \mathcal{F}_{t+1}\backslash\{\emptyset\}\}.$$

Assumption 4. *Agent 1 has expected utility with correct belief.*

Assumptions 1 and 2 guarantee that the solutions to (3.1) are in X_{++}. Assumption 3 says that every state has a positive chance of occurring any time and after any history; furthermore, relevant beliefs have this same property. Assumption 4 is supposed to test other agents in an unfavorable environment, since they are competing with a well informed agent.

The next lemmata are known results and can be found in Blume and Easley (2006).

Lemma 7. *Consider $i \neq j$. Agent i does not survive on the event $\left\{\frac{u_i'(c_t^i(\omega))}{u_j'(c_t^j(\omega))} \to \infty\right\}$. If agent i does not survive on ω, then for some $j \in \mathcal{I}$, $\overline{\lim}\, \frac{u_i'(c_t^i(\omega))}{u_j'(c_t^j(\omega))} = \infty$.*

Proof: If $\frac{u_i'(c_t^i(\omega))}{u_j'(c_t^j(\omega))} \to \infty$, then $c_t^i(\omega) \to 0$, by assumptions 1 and 2.

On the other hand, if $c_t^i(\omega) \to 0$, by assumption 1 there is $j \in \mathcal{I}$ such that $c_t^j > \underline{e}/I$ for infinite indexes t. Hence, the denominator of $\frac{u_i'(c_t^i(\omega))}{u_j'(c_t^j(\omega))}$ is upper bounded, and the result follows by assumption 2.

$\square$

Lemma 8. *Agent i survives $\mathbb{P}^i$ almost surely.*

Proof: Let $j \neq i$ in $\mathcal{I}$. Define the following random variables on $(\Omega, \mathcal{F})$,

$$L_t(\omega) = \frac{\mathbb{P}_t^j(\omega)}{\mathbb{P}_t^i(\omega)}.$$

Will be proven that $\{L_t\}$ is martingale with respect to $(\mathcal{F}_t)$ and $\mathbb{P}^i$. Indeed,

$$\mathbb{E}_{\mathbb{P}^i}[L_{t+1}|\mathcal{F}_t](\omega) = \sum_{s\in\mathcal{S}} \frac{\mathbb{P}_{t+1}^j(\omega^t, s)}{\mathbb{P}_{t+1}^i(\omega^t, s)} \frac{\mathbb{P}^i(\{(\omega^t, s)\} \times \Omega)}{\mathbb{P}^i(\{\omega^t\} \times \Omega)} = \sum_{s\in\mathcal{S}} \mathbb{P}_{t+1}^j(\omega^t, s)/\mathbb{P}_t^i(\omega^t) = L_t(\omega).$$

It is also easy to see that $\mathbb{E}[L_t] = 1$, $\forall t$. Therefore, by martingale convergence (see Shiryayev (1984)) (L_t) converges and its limit is finite $\mathbb{P}^i$ almost surely. Finally, by equation (3.3) and by Lemma 7 agent i survives $\mathbb{P}^i$ almost surely.

$\square$

By the previous lemma, a criterion for survival of an agent i is that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}^i$, Lemma 9 in the Appendix shows that this condition is also necessary.

The main results of this chapter are presented below. Proofs are left to the Appendix. First, though, let us look at a definition for aggregate risk.

Definition 5. *Define the functional $\delta : X \times \Omega \to \mathbb{R}$ by*

$$\delta(x, \omega) := \varliminf_t \left(\sup\{|x_t(\omega^{t-1}, r) - x_t(\omega^{t-1}, s)|; r, s \in \mathcal{S}_t\}\right).$$

There is ***aggregate risk on the path*** *ω if $\delta(e, \omega) > 0$, if there is aggregate risk $\mathbb{P}$ almost surely we simply say that there is aggregate risk.*

The next definition is analogous to the *strict minimum consensus property* of Condie (2008), and the following proposition is a generalization of his Theorem 1 for variational preferences.

Definition 6. *We say that agent i satisfies* ***property P*** *if $\exists\, T \in \mathcal{T}$ and $\varepsilon > 0$ such that $\forall t > T$, if $P \in \Delta(G_t(\omega), \mathcal{F}_{t+1})$ satisfies $\|P(s) - \mathbb{P}_t(s|\omega^{t-1})\| \leq \varepsilon$, then $\gamma_t^i(\omega, P) = 0$.*

Proposition 3. *Assume that there is aggregate risk. If agent i satisfies property P, then i does not survive.*

The subsequent result can be understood as limiting the scope of maxmin utilities in survival analysis, because a variational agent can survive even believing in "distributions which differ from the truth in all feasible directions"[3].

Proposition 4. *Suppose that $u_i(0) > -\infty$ and $\mathcal{S}_t = \mathcal{S}$ for all $t > 0$. If there is $T \in \mathcal{T}$ such that for every $t > T$, $\gamma_{t-1}^i(\omega, \mathbb{P}_t(\cdot|\omega^{t-1})) = 0$ and*

$$\gamma_{t-1}^i(\omega, P) \geq S \max\{|u_i(0)|, |u_i(\bar{e})|\} \|P(s) - \mathbb{P}_t(s|\omega^{t-1})\|,$$

then i survives on ω.

Lemma 8 and Theorem 1 together compose the main tool to attain survival results. Lemma 8 tell us that an individual always acts to guarantee his survival based on his effective belief, and if its posteriors converge to the truth posteriors then, according to Theorem 1, such an agent survives.

According to the proof of Proposition 4, we can see that relevant one-step-ahead beliefs at time t belongs to set

$$A_t^i(\omega) = \left\{P \in \Delta(G_t(\omega), \mathcal{F}_{t+1}); \gamma_t^i(\omega, P) \leq S(|u_i(0)| \vee |u_i(\bar{e})|)\|P - \mathbb{P}_{t+1}(\cdot|\omega^t)\|\right\},$$

for each $t \in \mathcal{T}$. So, if $B_t^i = \{P \in \Delta(\Omega); P_{\tau+1}(\cdot|\omega^\tau) \in A_\tau^i(\omega)\ \forall \tau \leq t\}$ the set of relevant beliefs[4] is $B^i = \cap_{t \in \mathcal{T}} B_t$.

[3]This quotation from Condie (2008) is part of his explanation about property P that ensures the non survival of maxmin agents.

[4]We refer to belief as relevant when it is a candidate to minimize $E_Q\left[\sum_{t \in \mathcal{T}} \beta^t u_i(c_t)\right] + \Gamma^i(Q)$.

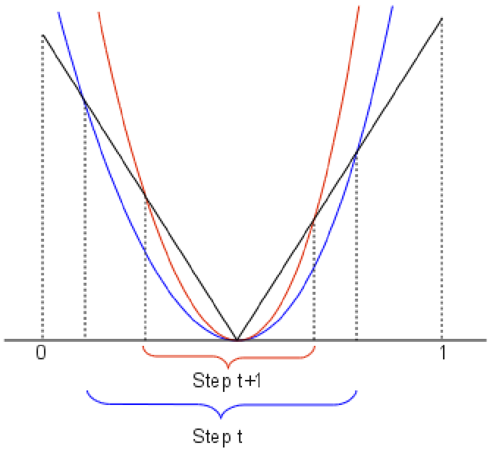

Figure 3.3: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

In many situations it is natural to suppose that ambiguity aversion vanishes over time. In such a case, dynamic ambiguity indexes will increase with t and sets A_t^i will decrease, as sketched in Figure 3.3. If sets A_t^i collapse in a point, by the same hypotheses made in Proposition 4, any probability in B^i will be equivalent to $\mathbb{P}$[5]. Therefore, we have conditions on ambiguity indexes that ensure survival.

In the next proposition we assume that there are only two agents. While agent 1 has an expected utility, agent 2 has a more general variational utility. For agent 2 we consider that two distinct types are possible, a and b. Type a is less ambiguity averse than b, so their utility index are the same and the ambiguity index of a is greater than the ambiguity index of b. Note that if (c^1, c^2) is a Pareto optimal allocation when agent 2 is of type b, then, assuming that $\Gamma^a(\mathbb{P}^b) = \Gamma^b(\mathbb{P}^b)$, the same allocation is Pareto optimal even when agent 2 is of type a. Proposition 5 gives an inverse relationship between the level of ambiguity aversion and survival.

Proposition 5. *Suppose that a is less ambiguity averse than b and $\Gamma^a(\mathbb{P}^b) = \Gamma^b(\mathbb{P}^b)$. If type b survives, then type a also survives.*

[5]If $\gamma_t^i(\omega, \mathbb{P}_{t+1}(\cdot|\omega^t)) = 0$, then $\mathbb{P}_{t+1}(\cdot|\omega^t) \in A_t^i(\omega)$. We know that $\mathbb{P}_{t+1}^i(\cdot|\omega^t) \in A_t^i(\omega)$, so if the sequence of sets $A_t^i(\omega)$ collapses into a single point we get $\|\mathbb{P}_{t+1}(\cdot|\omega^t) - \mathbb{P}_{t+1}^i(\cdot|\omega^t)\| \to 0$.

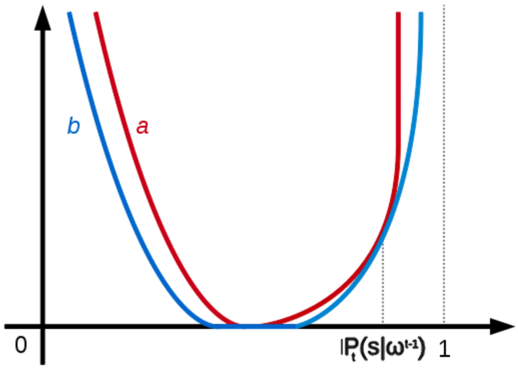

Figure 3.4: $\gamma_t^2(P_t)$ *versus* $P_t(1|\omega^{t-1})$

3.5 Conclusion

Survival of individuals behaving according to expected utility depends on inter-temporal discount factors and compatibility between beliefs and the truth as shown in Sandroni (2000) and Blume and Easley (2006). To study the influence of ambiguity aversion, the step taken by Condie (2008) was to introduce agents with *maxmin* utilities.

Considering $\beta^i = \beta^j \; \forall i, j$ to isolate aversion ambiguity effects, he finds that ambiguity averse agents survive under aggregate risk only in special cases. By introducing *variational* preferences that are more general than *maxmin*, we find that ambiguity averse individuals, with analogous characteristics to those in Condie (2008)'s case, can survive under aggregate risk. Moreover, in particular cases it is possible to make finer relations between the level of ambiguity aversion and the magnitude of aggregate risk that lead to survival.

3.6 Appendix

Lemma 9. *Agent i survives if, and only if,* $\mathbb{P} \ll \mathbb{P}^i$.

Proof: If $\mathbb{P} \ll \mathbb{P}^i$ then, by Lemma 8, agent i survives $\mathbb{P}$ almost surely.

Note that, by Assumption 3, $\mathbb{P}_t$ and $\mathbb{P}_t^i$ are equivalents. If i survives then, according to Lemma 7,

$$\mathbb{P}\left(\frac{u_i'(c_t^i(\omega))}{u_1'(c_t^1(\omega))} \nrightarrow \infty\right) = \mathbb{P}\left(L_t(\omega) \nrightarrow \infty\right) = 1,$$

where $L_t(\omega) = \frac{\mathbb{P}_t(\omega)}{\mathbb{P}^i_t(\omega)}$. By the proof of Theorem 1 p. 493 of Shiryayev (1984) we get $\mathbb{P}(\exists \lim L_t(\omega)) = 1$, therefore $\mathbb{P}(\lim L_t(\omega) < \infty) = 1$. Finally, by Theorem 2 p. 495 of Shiryayev (1984), $\mathbb{P} \ll \mathbb{P}^i$.

$\square$

Proof of Proposition 3: Suppose that i survives.

If $c^i_t(\omega^{t-1}, \cdot)$ is constant for a large enough t, by (3.4) and Theorem 1 for any j that survives we get

$$\frac{u_i'(c^i_t(\omega^{t-1}, s))}{u_i'(c^i_{t-1}(\omega^{t-1}))} \approx \frac{u_j'(c^j_t(\omega^{t-1}, s))}{u_j'(c^j_{t-1}(\omega^{t-1}))}.$$

Since $c^i_t(\omega^{t-1}, \cdot)$ is constant, $c^j_t(\omega^{t-1}, \cdot)$ is asymptotically constant, i.e., $\delta(c^j)=0$. Then $\delta(e) = \delta(\sum_{j \ survives} c^j) = 0$, a contradiction.

So, for any $\tau \in \mathcal{T}$ there is $t > \tau$ such that $c^i_t(\omega^{t-1}, \cdot)$ is not constant. Then

$$\|\mathbb{P}^i_t(\cdot|\omega^{t-1}) - \mathbb{P}_t(\cdot|\omega^{t-1})\| \geq \varepsilon$$

for a sequence $t \nearrow \infty$, and by Theorem 1 again, i does not survive.

$\square$

Proof of Proposition 4: Let $c \in X$ and $\omega \in \Omega$. For any $P \in dom\ \gamma^i_{t-1}(\omega, \cdot)$ we get

$$\begin{aligned}
& \left\{\mathbb{E}_P\left[u_i(c_t(\omega^{t-1}, \cdot))\right] + \gamma^i_{t-1}(\omega, P)\right\} - \left\{\mathbb{E}_{\mathbb{P}_t(\cdot|\omega^{t-1})}\left[u_i(c_t(\omega^{t-1}, \cdot))\right] + \gamma^i_{t-1}(\omega, \mathbb{P}_t(\cdot|\omega^{t-1})\right\} \\
= & \sum_{s \in \mathcal{S}} u_i(c_t(\omega^{t-1}, s))(P(s) - \mathbb{P}_t(s|\omega^{t-1})) + \gamma^i_{t-1}(\omega, P) \\
\geq & \sum_{s \in \mathcal{S}} u_i(c_t(\omega^{t-1}, s))(P(s) - \mathbb{P}_t(s|\omega^{t-1})) \\
+ & S \max\{|u_i(0)|, |u_i(\bar{e})|\} \|P(s) - \mathbb{P}_t(s|\omega^{t-1})\| \\
\geq & 0.
\end{aligned}$$

So $\{\mathbb{P}_t(\cdot|\omega^{t-1})\} = \arg \min_{P \in dom\ \gamma^2_{t-1}(\omega, \cdot)} \left\{\mathbb{E}_P\left[u_2(c_t(\omega^{t-1}, \cdot))\right] + \gamma^2_{t-1}(\omega, P)\right\}$.

$\square$

Proof of Proposition 5: An agent a with utility V^a is less ambiguity averse than another with utility V^b if $u^a = u^b$ and $\Gamma^a \geq \Gamma^b$. If $\mathbb{P}^b$ minimizes $\mathbb{E}_P[u^b(c^2)] + \Gamma^b(P)$ and $\Gamma^a(\mathbb{P}^b) = \Gamma^b(\mathbb{P}^b)$, then, $\forall P \in \Delta(\Omega)$

$$\mathbb{E}_{\mathbb{P}^b}[u^a(c^2)] + \Gamma^a(\mathbb{P}^b) = \mathbb{E}_{\mathbb{P}}[u^b(c^2)] + \Gamma^b(\mathbb{P}^b) \leq \mathbb{E}_P[u^b(c^2)] + \Gamma^b(P) \leq \mathbb{E}_P[u^a(c^2)] + \Gamma^a(P).$$

So $\mathbb{P}^b$ minimizes $\mathbb{E}_P[u^a(c^2)] + \Gamma^a(P)$ too, and if b survives then a survives.

$\square$

Conclusion

Ambiguity aversion plays an important role in dynamic models, but it cannot be seen in a simplistic way like "ambiguity aversion implies that...". There are many levels of ambiguity aversion and many ways to model it. This thesis attempts to show through distinct problems that such two directions matter in the analysis.

We deal with two models of decision under uncertainty that are more general than those used in literature about related problems. Despite the criticism involving it, the smooth ambiguity model provides a tractable way of tackling problems in which there is a need to compare beliefs taken a priori. Specifically, within the problem approached in Chapter 2, second-order beliefs allow for a similar analysis to that of the subjective first-order beliefs case, without leaving its ambiguity modeling. On the other hand, in Chapter 3 we explore the potential of variational preference that has the advantage of being a generalization of the maxmin model, which is the most successful model for ambiguity, and encompasses many different ambiguity averse behaviors.

Problems approached in this thesis are distinct in their formulation but share the same interest in knowing whether ambiguity aversion is robust within a context of dynamic general equilibrium. In both cases we get positive answers, but restricted. In Chapter 2 we have a result which generalizes the one found in the literature about expectation convergence. The necessity of equivalence between reductions for second-order beliefs enables one to embody an ambiguity averse individual within an equilibrium model, but restricts the discrepancy between ambiguity perceptions. In Chapter 3 we generalize the existing results as those considered expected utility and maxmin preference, but our analysis has the virtue of relating intensities of ambiguity aversion and aggregate risk in a more precisely way with the survival problem.

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