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ROBUST TRANSITIVITY FOR ENDOMORPHISMS

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Abstract. *The main goal of this work is to give some necessary and some sufficient conditions for endomorphisms on compact manifolds without boundary to be robustly transitive. More concretely, under what conditions a differentiable map, not necessarily invertible, having a dense orbit, verifies that a sufficiently close perturbed map also exhibits a dense orbit.*

In the case of robustly transitive diffeomorphisms is known that a necessary condition is that the tangent bundle admits a dominated splitting. For the case of endomorphisms, that is no longer true. In consequence, conditions that guarantee robustness for transitive endomorphisms cannot depend on the existence of decomposition of the tangent bundle.

For local diffeomorphisms, we show that a necessary condition for robust transitivity is to be volume expanding. Although volume expanding is not a sufficient condition to have endomorphisms robustly transitive. Because of this, we must ask for more hypothesis that guarantee robustness. Indeed the additional hypothesis that we ask is: given any arc in a certain region with a large enough diameter to have a point that its future orbit remains in the expanding region, which implies the existence of a locally maximal expanding invariant set for the original system that intersects every arc big enough.

Fácilmente aceptamos la realidad,
acaso porque intuimos que nada es real.
J.L. Borges

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Contents

Introduction

One goal in dynamics is to look for conditions that guarantee that certain phenomena are robust under perturbations, that is, some main feature of a dynamical system is shared by all nearby systems. In particular, we are interested in the hypotheses under which an endomorphism would be robust transitive.

In the diffeomorphism case, there are many examples of robust transitive systems. The best known is the transitive Anosov diffeomorphism; the example given by Shub in $\mathbb{T}^4$ in 1971, see for instance [PS06b]; another example is the Mañé's Derived from an Anosov in $\mathbb{T}^3$ in [Mañ78]; Bonatti and Díaz gave some geometrical construction that induces a robust transitive system in [BD96]. One of the newest examples is one from Bonatti-Viana that gives a robust transitive diffeomorphism with dominated splitting which is not partially hyperbolic, see [BV00].

On the other hand, any C^1-robust transitive diffeomorphism exhibits a dominated splitting. This is not true anymore for endomorphisms (see endomorphisms version of Bonatti-Viana example, it is explained in example 1 in section § 3.1). Therefore, conditions that imply robust transitivity cannot hinge on the existence of splitting.

The first question that arises is what necessary condition a robust transitive local diffeomorphism has to verify. We show in the first chapter that volume expanding would be a C^1 necessary condition. However, volume expanding is not a sufficient condition that guarantees robust transitivity for a local diffeomorphism, for instance a product of an expanding endomorphism times an irrational rotation is volume expanding and transitive but not robust transitive. Hence, we have to ask for extra conditions that allow us to conclude the robustness, more precisely, we need a property over the initial system that would be robust. As we said before this property cannot depend on the existence of any type of splitting. In fact, the hypothesis that we require is to ask for any arc of large enough diameter to have a point such that its forward iterates remain in the expanding region, implying the existence of an invariant expanding locally maximal set for the initial system that intersects every large arc. Therefore, that topological property persists under perturbations for a certain class of arc.

Instead of transitivity we may ask for the density of the pre-orbit of any point. This hypothesis implies transitivity, but we do not know if the reciprocal holds. But the fact of having just one point which pre-orbit is dense is not enough to conclude transitivity. We must note that if the initial system verifies the density of every pre-orbit, then the perturbed one has "almost" density of the pre-orbit of any point.

This thesis is divided in three chapters, each one with a brief introduction giving the main goals of every chapter. We also define many of the concepts involved throughout the work and pose many questions and remarks related to our results.

In the first chapter, we address the main problem, to find necessary and sufficient conditions in order to have robust transitive endomorphisms. In section § 1.1.1, we present the main result of this work:

Main Theorem *Let $f \in E^1(\mathbb{T}^n)$ be a volume expanding map satisfying the following properties:*

1. *There is an open set U_0 in $\mathbb{T}^n$ such that $f|_{U_0^c}$ is expanding and $diam_{ext}(U_0) < 1$.*

2. *$\{f^{-k}(x)\}_{k\geq 0}$ is dense for every $x \in \mathbb{T}^n$.*

3. *There exists* $0 < \delta_0 < diam_{int}(U_0^c)$ *and there exists an open neighborhood* U_1 *of* $\overline{U}_0$ *such that for every arc* γ *in* U_0^c *with diameter larger than* δ_0*, there is a point* $y \in \gamma$ *such that* $f^k(y)$ *is not in* U_1 *for any* $k \geq 1$.

4. *For every* $z \in U_1^c$*, there exists* $\bar{z} \in U_1^c$ *such that* $f(\bar{z}) = z$.

Then, for every g *close enough to* f*,* $\{g^{-k}(x)\}_{k\geq 0}$ *is dense for every* $x \in \mathbb{T}^n$.

We recommend to the reader before entering into the proof of the Main Theorem, to give a glance to section § 1.1.2 in order to gain some insight about the proof. We want to highlight that this theorem as it is enunciated, it is not assumed the existence of any tangent bundle splitting. In the case that there exists a partially hyperbolic splitting we may get the same conclusion but with weaker hypotheses. This is given in Theorem 2 in section § 1.2. One question that arises from this formulation is: *if a map satisfies the hypotheses of the Main Theorem, is it true that this map is isotopic to an expanding endomorphism?* The Main Theorem can be recasted in terms of the geometrical properties, see Main Theorem Revisited in section § 1.1.8.

In the second chapter, we show some geometrical and topological consequences from the Main Theorem. Besides, we study the existence of Markov Partitions for maps of our type, via semiconjugation with linear expanding endomorphisms, and how it allows us to extract some information about the transitivity of the map isotopic to a linear expanding endomorphism. Also, we pose some related questions.

In the third and last chapter, we construct examples of robust transitive endomorphisms verifying the hypotheses of the Main Theorem, the Main Theorem Revisited and Theorem 2.

Chapter 1

Main Result

In this chapter, our goal is to give sufficient conditions to get robust transitivity for $n-$dimensional torus endomorphisms.

§ 1.1 Volume expanding endomorphisms without invariant splitting

An endomorphism of a differentiable manifold M is a differentiable function $f : M \to M$ of class C^r with $r \geq 1$. Throughout this work we will assume the endomorphism f to be a local diffeomorphism. That means that given any point x, there exists an open set V containing x such that f from V to $f(V)$ is a diffeomorphism. For the main result we do not assume the existence of any invariant splitting for f. Let us denote by $E^1(M)$ the space of C^1-endomorphisms of M endowed with the usual C^1 topology.

Before entering into the Main Theorem, let us recall some definitions that are involved.

Definition 1.1 (**Volume expanding map**)

We say that a map f is *volume expanding* if there exists $\sigma > 1$ such that $|det(Df)| > \sigma$.

Definition 1.2 (**Invariant set**)

We say that a set $\Lambda \subset M$ is a *forward invariant set* for $f \in E^1(M)$ if $f(\Lambda) \subset \Lambda$ and it is *invariant* for f if $f(\Lambda) = \Lambda$.

Let us introduce some notation that we will use throughout this work: if $L : V \to W$ is a linear isomorphism between normed vector spaces, we denote by $m\{L\}$ the *minimum norm* of L, i.e. $m\{L\} = \|L^{-1}\|^{-1}$.

Definition 1.3 (**Expanding map**)

We say that a map f of class C^1 is *expanding* in U a subset of M if there exists $\lambda > 1$ such that $\min_{x \in U}\{m\{D_x f\}\} > \lambda$. It is said that a compact invariant set Λ is an *expanding set* for an endomorphism f if $f\mid_\Lambda$ is an expanding map.

Definition 1.4 (**Locally maximal set**)

Let Λ be an expanding set for $f \in E^1(M)$. If there is an open neighborhood V of Λ such that $\Lambda = \bigcap_{k \geq 0} f^{-k}(\overline{V})$ then Λ is said to be *locally maximal* (or isolated) set. V is called the *isolating block* of Λ.

Definition 1.5 (**Full orbit**)

A sequence $\{x_k\}_{k \in \mathbb{Z}}$ is called a *full orbit* for f if $f(x_k) = x_{k+1}$ for every $k \in \mathbb{Z}$.

Definition 1.6 (**Topologically transitive**)

Let Λ be an invariant set for an endomorphism $f : M \to M$. It is said that Λ is *topologically transitive* if there exists a point $x \in \Lambda$ such that its forward orbit $\{f^k(x)\}_{k \geq 0}$ is dense in Λ. We say that f is *topologically transitive* if $\{f^k(x)\}_{k \geq 0}$ is dense in M for some $x \in M$.

Lemma 1 *Let $f : M \to M$ be a continuous map of a locally compact separable metric space M into itself. The map f is topologically transitive if and only if for any two* *nonempty open sets* $U, V \subset M$*, there exists a positive integer* $N = N(U, V)$ *such that* $f^N(U) \cap V$ *is nonempty.*

Proof. See for instance [KH95, pp.29].

Definition 1.7 (**Topologically mixing**)

A topological dynamical system $f : M \to M$ is called *topologically mixing* if for any two nonempty open sets $U, V \subset M$, there exists a positive integer $N = N(U, V)$ such that for every $k > N$ the intersection $f^k(U) \cap V$ is nonempty.

Definition 1.8 (**Locally eventually onto**)

A map $f \in E^1(M)$ is called *locally eventually onto* if for any nonempty open set $U \subset M$, there exists a positive integer $N = N(U)$ such that $f^N(U) = M$.

Remark 1.1 In general, it holds that if a map is locally eventually onto, then it is topologically mixing. If a map is topologically mixing, then it is topologically transitive. The reciprocals are not true for endomorphisms case.

Remark 1.2 For endomorphisms, if the pre-orbit of every point is dense in the manifold, then the map is transitive. Note that if there exists just one point whose pre-orbit is dense, it is not enough to conclude that the map is transitive.

Definition 1.9 (**Robustly transitive**)

The set $\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$ is C^r*−robustly transitive* if $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is transitive for every endomorphism g C^r−close enough to f. It is said that a map f is C^r*−robustly transitive* if there exists a C^r neighborhood $\mathcal{U}(f)$ such that every $g \in \mathcal{U}(f)$ is transitive.

Definition 1.10 (**Robust non existence of splitting**)

We say that f restricted to an invariant set Λ has *no splitting in a C^r-robust way* if there exists a C^r open neighborhood $\mathcal{U}(f)$ of f such that for every $g \in \mathcal{U}(f)$ the tangent space $T\Lambda$ does not admit invariant subbundles.

Theorem 1 *Let $f \in E^1(M)$ be a local diffeomorphism and U open set in M such that $\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$ is C^1-robustly transitive set and it has no splitting in a C^1-robust way. Then f is volume expanding.*

Proof. The proof of this theorem is similar to the one of Theorem 4 in [BDP03, pp.361], nevertheless we include the main steps of the proof.

Let us consider $f \in E^1(M)$ a local diffeomorphism and denote by $\Lambda_f(U)$ the C^1-robustly transitive (nontrivial) set for f, note that U could be the entire manifold. The idea of the proof is to assume that f is not volume expanding and show that for every C^1-neighborhood of f $\mathcal{U}(f) \subset E^1(M)$, there exists $\psi \in \mathcal{U}(f)$ such that ψ has a sink and therefore ψ cannot be transitive.

Suppose that f is not volume expanding. Since f is onto, it cannot be uniform volume contracting in the entire manifold, so there are points in the manifold such that we have expansion, i.e. $1 \leq |det(Df^k(x))|$ for some $k \geq 0$, but it does not expand too much, i.e. $|det(Df^k(x))| \leq 1+\epsilon$, with ϵ small. Then there are sequences $x_n \in \Lambda_f(U)$, $k_n \in \mathbb{N}$ and $\tau_n > 1$, with $k_n \to \infty$ and $\tau_n \to 1^+$, such that

$$1 \leq |det(Df^{k_n}(x_n))| < \tau_n^{k_n}.$$

This is equivalent to say that

$$\frac{1}{k_n}\sum_{i=0}^{k_n-1} \log(|det(Df(f^i(x_n)))|) < \log(\tau_n).$$

We may take k_n such that $f^i(x_n) \neq f^j(x_n)$ for all $i \neq j$, $i, j \in \{0, \ldots, k_n\}$. Consider for each n the Dirac measure δ_n supported in $\{x_n, f(x_n), \ldots, f^{k_n}(x_n)\}$, i.e. $\delta_n = \frac{1}{k_n}\sum_{i=0}^{k_n-1}\delta_{f^i(x_n)}$. As the space of probabilities is compact with the weak star topology, there exists a subsequence of $\{\delta_n\}_n$ that converges to an invariant probability measure μ such that

$$\int \log|det(Df(x))|d\mu(x) \leq 0.$$

In fact, a classical argument proves that μ is invariant by f, since $f_*(\mu) - \mu$ is the weak star limit of $\frac{1}{k_{n_i}}(\delta_{f^{k_{n_i}}(x_{n_i})} - \delta_{x_{n_i}})$, which converge to zero. Observe that

$$\int \log|det(Df(x))|d\delta_n = \frac{1}{k_n}\sum_{i=0}^{k_n-1}\log(|det(Df^i(x_n))|) = \frac{1}{k_n}\log(|det(Df^{k_n}(x_n))|) \leq \log(\tau_n),$$

then because $\tau_n \to 1^+$ we deduce that

$$\int \log|det(Df(x))|d\mu(x) \leq 0.$$

By the ergodic decomposition theorem, there is an ergodic and $f-$invariant measure ν such that

$$\int \log|det(Df(x))|d\nu(x) \leq 0.$$

Using the ergodic closing lemma for nonsingular endomorphisms, given $\varepsilon > 0$ there is g close to f and a $g-$periodic point y such that

$$\frac{1}{m_\varepsilon}\sum_{i=0}^{m_\varepsilon-1}\log(|det(Dg(g^i(y)))|) < \varepsilon,$$

where m_ε is the period of y. Note that if $\varepsilon \to 0$, then $m_\varepsilon \to \infty$. So, taking $\varepsilon > 0$ arbitrarily small and m_ε big, using Franks' Lemma [Fra71] we get φ close to g such that $\varphi^{m_\varepsilon}(y) = y \in \Lambda_\varphi(U)$ and

$$\frac{1}{m_\varepsilon}\sum_{i=0}^{m_\varepsilon-1}\log(|det(D\varphi(\varphi^i(y)))|) < 0,$$

this means that $|det(D\varphi^{m_\varepsilon}(y))| < \lambda < 1$. Observe that we are assuming the dimension of the manifold greater or equal to 2, so the fact that the modulus of the jacobian of φ be lower than 1 does not imply that all the eigenvalues have modulus smaller than 1.

Since $\Lambda_\varphi(U)$ is C^1-robustly transitive, after a perturbation, we may assume that the relative homoclinic class $H(y, \varphi, U)$ of y is the whole $\Lambda_\varphi(U)$. Now, consider the dense subset $\Sigma \subset \Lambda_\varphi(U)$ consisting of all the hyperbolic periodic points of $\Lambda_\varphi(U)$ homoclinically related to y.

If φ is close enough to f, then the tangent bundle does not admit a splitting as well. Using the idea of the proof of Lemma 6.1 in [BDP03, pp. 407] and, after that, Franks' Lemma, we obtain that there exists ψ a perturbation of φ and a point $p \in \Sigma$ such that all the eigenvalues of $D\psi^{m(p)}(p)$ have modulus strictly lower than 1, where $m(p)$ is the period of p. This means that the maximal invariant set in U of ψ contains a sink, but this is a contradiction since we choose ψ sufficiently close to f such that $\Lambda_\psi(U)$ is still transitive. ∎

Remark 1.3 If $\Lambda_f(U)$ admits a splitting, then the extremal indecomposable subbundle is volume expanding.

Remark 1.4 Theorem 1 implies that volume expanding is a necessary condition for an endomorphism, which is local diffeomorphism, to be a robust transitive map. However, volume expanding is not a sufficient condition that guarantees robust transitivity for a local diffeomorphism. For instance, consider a product of an expanding endomorphism times an irrational rotation: this map is volume expanding and transitive but not robust transitive.

Remark 1.5 It is expected that if f is robustly transitive and has no invariant subbundles in a robust way, then f is a local diffeomorphism. It depends on whether the Ergodic Closing Lemma holds even if there are critical points, since for maps with critical points already exists a version of Connecting Lemma, Closing Lemma and Franks' Lemma, which are the principal results involved in the proof of Theorem 1.

Henceforth, we work in the $n-$torus $\mathbb{T}^n$. Since in dimension one volume expanding endomorphism is equivalent to be expanding map, we also assume that the dimension n is at least 2.

Definition 1.11 (**Internal diameter**)

Let U be an open set in $\mathbb{T}^n$. Denote by $\widetilde{U}$ the lift of U restricted to a fundamental domain. Define the *internal diameter* of U^c by

$$diam_{int}(U^c) = \min_{k \in \mathbb{Z}^n \setminus \{0\}} dist(\widetilde{U}, \widetilde{U} + k),$$

where $dist(A, B) := \inf\{\max_{1 \leq i \leq n} |x_i - y_i| : x = (x_1, \ldots, x_n) \in A, y = (y_1, \ldots, y_n) \in B\}$.

Definition 1.12 (**External diameter**)

Let U be an open set in $\mathbb{T}^n$. Denote by $\widetilde{U}$ the lift of U restricted to a fundamental domain. We say that *external diameter* of U is less than 1, denoting by $diam_{ext}(U) < 1$, if the closure of $\widetilde{U}$ is contained in the interior of $[0, 1]^n$.

Remark 1.6 Observe that volume expanding implies that the map is a local diffeomorphism.

§ 1.1.1 The Main Result

Our main result gives sufficient conditions for volume expanding endomorphisms to be robustly transitive, independently of the existence or not of an invariant splitting.

Main Theorem *Let $f \in E^1(\mathbb{T}^n)$ be a volume expanding map, $n \geq 2$, satisfying the following properties:*

1. *There is an open set U_0 in $\mathbb{T}^n$ such that $f\mid_{U_0^c}$ is expanding and $diam_{ext}(U_0) < 1$.*

2. *$\{f^{-k}(x)\}_{k \geq 0}$ is dense for every $x \in \mathbb{T}^n$.*

3. *There exist $0 < \delta_0 < diam_{int}(U_0^c)$ and an open neighborhood U_1 of $\overline{U}_0$ such that for every arc γ in U_0^c with diameter larger than δ_0, there is a point $y \in \gamma$ such that $f^k(y)$ is not in U_1 for any $k \geq 1$.*

4. *For every $z \in U_1^c$ there exists $\bar{z} \in U_1^c$ such that $f(\bar{z}) = z$.*

Then, for every g close enough to f, $\{g^{-k}(x)\}_{k \geq 0}$ is dense for every $x \in \mathbb{T}^n$.

Let us point out the following observations:

Remark 1.7 The hypothesis of external diameter less than 1 and hypothesis (4) are technical. This means that they are necessary conditions for proving our result, but we do not know if there exist weaker conditions that implies the thesis of our theorem.

Remark 1.8 The condition $diam_{ext}(U_0) < 1$ implies that the closure of $\widetilde{U}_0$ is contained in the interior of $[0,1]^n$, where $\widetilde{U}_0$ is the lift of U_0 restricted to $[0,1]^n$. Note that U_0 do not need to be simply connected and could have finitely many connected components. Actually the important fact is that the closure of the convex hull of the lift of U_0 restricted to $[0,1]^n$ is still contained in $(0,1)^n$. Moreover, $diam_{int}(U_0^c) = diam_{int}(\mathfrak{U}_0^c)$, where $\mathfrak{U}_0$ is the convex hull of $\widetilde{U}_0$.

Remark 1.9 If $f \in E^1(\mathbb{T}^n)$ satisfies the hypotheses of the Main Theorem, then f is topologically transitive. Indeed, as a consequence of the Main Theorem, f is C^1 robustly topologically transitive.

Remark 1.10 The Main Theorem is formulated for $n-$dimensional torus but we believe that it can be extended to any manifold that at least supports an expanding endomorphisms, for example nilmanifolds (see [Shu69]).

Remark 1.11 $\Lambda_0 := \bigcap_{n\geq 0} f^{-n}(U_0^c)$ is an expanding set. Moreover, we know that given any arc γ in U_0^c with diameter greater than δ_0, there exists a point $x \in \gamma$ such that $f^k(x)$ is not in U_1 for any $k \geq 1$. Therefore, $\gamma \cap \Lambda_0 \neq \emptyset$ and Λ_0 is not trivial.

Remark 1.12 Using hypothesis (4) of the Main Theorem, given any point $x \in U_1^c$, we can construct a sequence $\{x_k\}_{k\geq 0}$ such that $x_0 = x$, $x_k \in U_1^c$ and $f(x_{k+1}) = x_k$ for every $k \geq 0$. We call this sequence by *inverse path*.

Remark 1.13 Let us denote $\Lambda_1 := \bigcap_{n\geq 0} f^{-n}(U_1^c)$. This set has the following properties:

1. Λ_1 is an expanding set.

2. By hypothesis (3) of the Main Theorem, given any arc γ in U_0^c with diameter greater than δ_0, there exists a point $x \in \gamma$ such that $f(x) \in \Lambda_1$.

3. Since the hypothesis $0 < \delta_0 < diam_{int}(U_0^c)$ is an open condition, we may take U_1 an open neighborhood of $\overline{U_0}$ such that $\delta_0 < diam_{int}(U_1^c) < diam_{int}(U_0^c)$. Then for every arc γ in U_1^c with diameter greater than δ_0 holds that $\gamma \cap \Lambda_1$ is non empty.

4. Λ_1 is invariant, i.e. $f(\Lambda_1) = \Lambda_1$. It is clear that Λ_1 is forward invariant. So let us prove that $\Lambda_1 \subset f(\Lambda_1)$. Pick a point $x \in \Lambda_1$ and consider the sequence $\{x_k\}_{k\geq 0}$ given by remark (1.12). Let us show that $x_k \notin W$ for any $k \geq 0$, where $W = \cup_{n\geq 0} f^{-n}(U_1) = \Lambda_1^c$. If this is not true, there exist $k \geq 0$ and $n_k \geq 0$ such that $f^{n_k}(x_k) \in U_1$. First, observe that remark (1.12) implies that $f^n(x_k) = x_{k-n}$ for $0 \leq n \leq k$. In particular, $f^k(x_k) = x_0$ if $k \geq 0$. And $f^n(x_k) = f^{n-k}(f^k(x_k)) = f^{n-k}(x_0)$

for $n > k \geq 0$. Therefore, if $-k \leq -n_k \leq 0$, then $f^{n_k}(x_k) = x_{k-n_k}$. Since every x_k belongs to U_1^c, we obtain that $f^{n_k}(x_k)$ belongs to U_1^c which is a contradiction because it was supposed that $f^{n_k}(x_k) \in U_1$. If $-n_k < -k < 0$, then $f^{n_k}(x_k) = f^{n_k-k}(x_0)$. Since $x_0 \in \Lambda_1$, every positive iterate of x_0 by f belongs to U_1^c, thus $f^{n_k}(x_k) \in U_1^c$, which contradicts the fact that $f^{n_k}(x_k) \in U_1$. Thus, $x_k \in \Lambda_1$ for every $k \geq 0$.

5. In section § 1.1.3, we prove that this set is locally maximal or it is contained in an expanding locally maximal set.

Question 1.1 If f satisfies the hypotheses of the Main Theorem, does this implies that f is isotopic to an expanding endomorphism?

§ 1.1.2 Sketch of the Proof of Main Theorem

We want to prove that any small perturbation g of the initial system f has the property that the pre-orbit of any point is dense in the manifold. The mechanism to prove that is the following: *given any open set V, there exist $x \in V$ and $k \in \mathbb{N}$ such that $g^k(V)$ contains a ball of a fixed radius R_0 centered in $g^k(x)$*. If we have the latter property, we may conclude our claim, since given $0 < \varepsilon < R_0$ for g $\varepsilon/2-$close to f the pre-orbit of any point by g are $\varepsilon-$dense, hence $g^k(V)$ intersects $\{g^{-n}(z)\}$ for any z. Therefore, V intersects $\{g^{-n}(z)\}$ for any z.

Note that because f is expanding outside U_0, if x is a point in U_0^c such that its forward orbit stays outside U_0, then $f^k(\mathbb{B}_r(x)) \supset \mathbb{B}_{\lambda_0^k r}(f^k(x))$, where $\lambda_0 > 1$ is the expanding constant of f. The goal is to show that for any open set V, there exists a point $x \in V$ such that the forward orbit of some iterate $f^m(x)$ stay outside U_0.

Now, working in the covering space, since U_0 has external diameter less than 1 and the volume increases, follows that the diameter of the forward iterates of the lift of V grows. Hence there is some iterate with diameter big enough such that we may use hypothesis

(3) to obtain such a point x. The aim is to show that this mechanism is robust.

In order to prove the statement we use a geometrical approach:

1. Hypothesis (3) implies that there is an expanding subset that "separates", meaning that a nice class of arcs in U_0^c intersects this set. (See Lemma 3 in section § 1.1.5)

2. Properly chosen, this set is locally maximal. (See Lemma 2 in section § 1.1.3)

3. Hence, it has a continuation conjugated to the initial one. (See Claim 1.1 in section § 1.1.4)

4. Therefore, that topological property of separation persists. (See Lemma 4 in section § 1.1.5)

Finally, since the initial system has the pre-orbit of any point dense, the perturbed has the pre-orbit of any point "almost" dense. Then using the geometrical approach as above we conclude the density of the pre-orbit of any point. ∎

§ 1.1.3 Existence of a Locally Maximal Set for f

Lemma 2 *Either Λ_1 is a locally maximal set or there exists Λ^* an expanding locally maximal set for f such that $\Lambda_1 \subset \Lambda^*$ and Λ^* verifies that every arc γ in U_0^c with diameter greater than δ_0 has a point such that the image by f belongs to Λ^*. Moreover, every arc γ in U_1^c with diameter greater than δ_0 intersects Λ^*.*

Proof. We may divide the proof in two cases:

Case I. $\Lambda_1 \cap \partial U_1 = \emptyset$.

Let us observe that $\Lambda_1 \cap \partial U_1 = \emptyset$ implies that Λ_1 is contained in the open neighborhood $V = int(U_1^c)$. Then V is an isolating block for Λ_1, therefore Λ_1 is locally maximal.

Case II. $\Lambda_1 \cap \partial U_1 \neq \emptyset$.

Choose $\varepsilon > 0$ sufficiently small such that the open ball $\mathbb{B}_\varepsilon(x)$ is contained in U_0^c for all $x \in \Lambda_1$ and for every $x \in \Lambda_1$, since f is a local diffeomorphism, there exists an open set U_x such that $f\mid_{U_x}: U_x \to \mathbb{B}_\varepsilon(x)$ is a diffeomorphism. Note that the collection $\{\mathbb{B}_\varepsilon(x)\}_{x\in\Lambda_1}$ is an open cover of Λ_1. Since Λ_1 is compact, there is a finite subcover, let us say $\{\mathbb{B}_\varepsilon(x_i)\}_{i=1}^N$.

Fix $\lambda_0^{-1} < \lambda' < 1$, where λ_0 is the expansion constant of f and pick N' greater or equal to N, the cardinal of the finite subcover of Λ_1, such that for every $y \in \Lambda_1$, there is $i = i(y) \in \{1, \dots, N'\}$ such that $\mathbb{B}_{\lambda'\varepsilon}(y) \Subset \mathbb{B}_\varepsilon(x_i)$, i.e. $\overline{\mathbb{B}_{\lambda'\varepsilon}(y)} \subset \mathbb{B}_\varepsilon(x_i)$.

Let us define $W = \bigcup_{i=1}^{N'} \mathbb{B}_\varepsilon(x_i)$ and $\widehat{W} = \bigcup_{i=1}^{N'} \overline{\mathbb{B}_\varepsilon(x_i)}$.

By remark (1.13) Λ_1 is invariant, then we have that for every x_i, there exists at least one $x_i^j \in \Lambda_1$ such that $f(x_i^j) = x_i$. Let us consider for every $1 \le i \le N'$ all the possible pre-images by f of x_i that belongs to Λ_1, i.e. recall that f is a local diffeomorphism, hence for every point $x \in M$, the cardinal $\sharp\{f^{-1}(x)\} = N_f$ is constant, then for every $i \in \{1, \dots, N'\}$, there exist $K_i \subset \{1, \dots, N_f\}$ such that if $j \in K_i$ then $x_i^j \in \Lambda_1$ and $f(x_i^j) = x_i$. Therefore for every $i \in \{1, \dots, N'\}$ and for every $j \in K_i$, there exist open sets U_i^j such that $x_i^j \in U_i^j$ and $f\mid_{U_i^j}: U_i^j \to \mathbb{B}_\varepsilon(x_i)$ is a diffeomorphism. Given $i \in \{1, \dots, N'\}$, for every $j \in K_i$ consider the inverse branches, $\varphi_{i,j} : \mathbb{B}_\varepsilon(x_i) \to U_i^j$ such that

$$\begin{aligned} \varphi_{i,j}(x_i) &= \ x_i^j, \\ f\varphi_{i,j}(x) &= \ x, \quad \forall\, x \in \mathbb{B}_\varepsilon(x_i). \end{aligned}$$

Now, consider $\Lambda^* = \bigcap_{n\geq 0} f^{-n}(\widehat{W})$. Clearly, $\Lambda_1 \subset \Lambda^* \subset U_0^c$ and Λ^* is an expanding set. In order to show that Λ^* is locally maximal, it is enough to show that $\Lambda^* \cap \partial\widehat{W} = \emptyset$, which is equivalent showing that $f^{-1}(\widehat{W})$ is contained in W. Just to make more clear what follows, let us rewrite $f^{-1}(\widehat{W})$ in terms of the inverse branches,

$$f^{-1}(\widehat{W}) = f^{-1}(\bigcup_{i=1}^{N'} \overline{\mathbb{B}_\varepsilon(x_i)}) = \bigcup_{i=1}^{N'} \bigcup_{j\in K_i} \overline{\varphi_{i,j}(\mathbb{B}_\varepsilon(x_i))}.$$

So, it is enough to show that $\overline{\varphi_{i,j}(\mathbb{B}_\varepsilon(x_i))} \subset \mathbb{B}_\varepsilon(x_{m_{i,j}})$, for some $x_{m_{i,j}} \in \{x_1, \ldots, x_{N'}\}$.

In fact,

$$\varphi_{i,j}(\mathbb{B}_\varepsilon(x_i)) = U_i^j \subset \mathbb{B}_{\lambda_0^{-1}\varepsilon}(\varphi_{i,j}(x_i)) \subset \mathbb{B}_{\lambda'\varepsilon}(\varphi_{i,j}(x_i)) = \mathbb{B}_{\lambda'\varepsilon}(x_i^j)$$

then, there exists $m_{i,j} \in \{1, \ldots, N'\}$ such that $\mathbb{B}_{\lambda'\varepsilon}(x_i^j) \Subset \mathbb{B}_\varepsilon(x_{m_{i,j}})$, and the assertion holds. Easily follows that Λ^* has the property that every arc γ in U_0^c with diameter larger than δ_0 has a point such that its image by f belongs to it. ∎

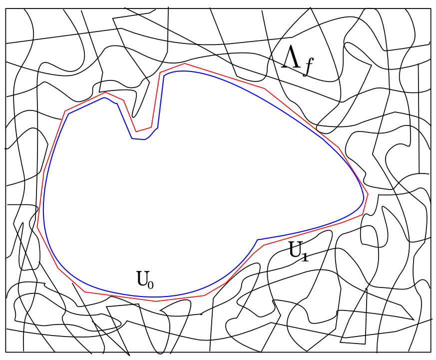


Figure 1.1: Λ_f *looks like a net which is an expanding set that "separates"*

Remark 1.14 We want to highlight that for diffeomorphisms there exist examples of hyperbolic sets that are not contained in any locally maximal hyperbolic set, see for instance [Cro02] and [Fis06]. A similar construction seems feasible for endomorphisms. The hypothesis (4) guarantees that Λ_1 is an invariant set. Moreover, we can consider a finite covering $\{\mathbb{B}_\varepsilon(x_i)\}_{i=1}^{N'}$ for Λ_1, with $x_i \in \Lambda_1$, in such a way that for every point $y \in \Lambda_1$, there is x_i such that $\overline{\mathbb{B}_{\lambda'\varepsilon}(y)} \subset \mathbb{B}_\varepsilon(x_i)$. Thus we conclude that Λ^* is contained in the interior of $\widehat{W}$ and therefore the expanding set Λ_1 is either locally maximal or is contained in a locally maximal expanding set.

§ 1.1.4 Continuation of the Locally Maximal Set

Definition 1.13 (**$\delta-$pseudo orbit**)

The sequence $\{x_n\}_{n\in\mathbb{Z}}$ is said to be a $\delta-$*pseudo orbit* for f if $d(f(x_n), x_{n+1}) \leq \delta$ for every $n \in \mathbb{Z}$.

Definition 1.14 (**$\varepsilon-$shadowed**)

We say that a $\delta-$pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ for f is $\varepsilon-$*shadowed* by a full orbit $\{y_n\}_{n\in\mathbb{Z}}$ for f if $d(y_n, x_n) \leq \varepsilon$ for every $n \in \mathbb{Z}$.

Definition 1.15 (**Topological conjugacy**)

$f : M \to M$ is topologically conjugate to $g : N \to N$ if there exists a homeomorphism $h : M \to N$ such that $h \circ f = g \circ h$.

In order to fix some notation for what follows, we will denote by Λ_f the expanding locally maximal set for f, it means that Λ_f is either Λ_1, in the case it is locally maximal, or it is Λ^* given in Lemma 2; and denote by U the isolating block of Λ_f.

Claim 1.1 There exists $\mathcal{V}_1(f)$ an open neighborhood of f in $E^1(M)$ such that if $g \in \mathcal{V}_1(f)$, then g is expanding on $\Lambda_g = \bigcap_{n\geq 0} g^{-n}(U)$ and there exists an homeomorphism $h_g : \Lambda_g \to \Lambda_f$ that gives the topological conjugacy and h_g is closed to the identity.

Proof. In order to get the conjugacy we use the Shadowing Lemma for expanding endomorphisms, see for instance [Liu91].

Since Λ_f is an expanding locally maximal set for f, there exists $\beta > 0$ such that f is expansive with constant β in Λ_f.

Fix $0 < \eta < \beta$. By the endomorphism version of the Shadowing Lemma, there exists $\varepsilon > 0$ such that any $\varepsilon-$pseudo orbit for f within ε of Λ_f is uniquely $\eta-$shadowed by a full orbit in Λ_f.

Take N such that

$$\bigcap_{j=0}^{N} f^{-j}(U) \subset \{q : \, d(q, \Lambda_f) < \varepsilon/2\}.$$

There exists a C^0 neighborhood $\mathcal{V}(f)$ of f such that for g in $\mathcal{V}(f)$

$$\bigcap_{j=0}^{N} g^{-j}(U) \subset \{q : \, d(q, \Lambda_f) < \varepsilon/2\}$$

and for any $x \in \bigcap_{j=0}^{N} g^{-j}(U)$, we may consider $\{x_n\}_{n\in\mathbb{Z}}$ a full orbit for g, where $x_0 = x$, getting that $\{x_n\}_n$ is an $\varepsilon-$pseudo orbit for f.

Let $\Lambda_g = \bigcap_{n\geq 0} g^{-n}(U)$. Taking an open subset $\mathcal{V}_1(f)$ of $\mathcal{V}(f)$ small enough in the C^1 topology, then for $g \in \mathcal{V}_1(f)$, Λ_g is an expanding locally maximal set for g. If g is close enough to f, then g is also expansive with constant β. Moreover, the Shadowing Lemma also holds for g.

Take $g \in \mathcal{V}_1(f)$. Given $x \in \Lambda_g$, consider $\{x_n\}_{n\in\mathbb{Z}}$ a full orbit for g, where $x_0 = x$. As $\{x_n\}_n$ is an $\varepsilon-$pseudo orbit for f, there exists a unique full orbit $\{y_n\}_{n\in\mathbb{Z}}$ for f with $y_0 = y \in \Lambda_f$ that $\eta-$shadows $\{x_n\}_{n\in\mathbb{Z}}$.

Let us define $h_g : \Lambda_g \to \Lambda_f$ by $h_g(x) = y$, where y is given by the Shadowing Lemma. By the uniqueness of the shadowing point, this map is well defined. The continuity of h_g follows from the shadowing lemma.

Moreover, $h_g \circ g = f \circ h_g$. In fact, consider the sequence $\{z_n\}_{n\in\mathbb{Z}}$ where $z_n = g(x_n) = x_{n+1}$. This $\varepsilon-$pseudo-orbit is $\eta-$shadowed by a unique full orbit $\{w_n\}_{n\in\mathbb{Z}}$ for f, with $w_0 = w \in \Lambda_f$. Then, for every $n \in \mathbb{Z}$,

$$\begin{aligned} d(w_n, z_n) &= d(f^n(w_0), x_{n+1}) = d(f^n(h_g(z_0)), g^n(g(x_0))) \\ &= d(f^n(h_g \circ g(x_0)), g^n(g(x_0))) = d(f^{n+1} \circ f^{-1} \circ h_g \circ g(x_0), g^{n+1}(x_0)) < \eta \end{aligned}$$

Observe that $f^{-1} \circ h_g \circ g(x_0) = w_{-1}$ is $\eta-$shadowing x_0. So, by uniqueness, we have that $f^{-1} \circ h_g \circ g(x_0) = y_0$; i.e. $h_g \circ g(x) = f \circ h_g(x)$.

Since we can apply the Shadowing Lemma for Λ_g using the same constants as in the construction of h_g, we define a map $h_f : \Lambda_f \to \Lambda_g$ such that $h_f \circ f = g \circ h_f$. In fact, if $\{y_n\}_{n\in\mathbb{Z}}$ is a full orbit for f with $y_0 \in \Lambda_f$, then it is an $\varepsilon-$pseudo orbit for g. Hence, this pseudo orbit is uniquely shadowed by a full orbit $\{x_n\}_{n\in\mathbb{Z}}$ for g, with $x_0 \in \Lambda_g$. Thus, $h_f(y_0) = x_0$ and $d(y_n, x_n) < \eta$ for every $n \in \mathbb{Z}$; moreover, h_f is continuous and satisfies $h_f \circ f = g \circ h_f$ just as h_g.

Next, let us verify that h_g is one to one. Let $p_1,\ p_2 \in \Lambda_g$ be two points such that $h_g(p_1) = h_g(p_2)$. Note that $d(f^n(h_g(p_1)), g^n(p_1)) < \eta$ and $d(f^n(h_g(p_2)), g^n(p_2)) < \eta$ by construction. Then $h_g(p_1)$ is $\eta-$shadowed by p_1 and p_2, which by uniqueness gives that $p_1 = p_2$.

Finally, for $y \in \Lambda_f$, consider a full orbit of $h_f(y)$ by g. Since $d(g^n(h_f(y)), f^n(y))$ is small for all n and some f full orbit of y shadows the g full orbit of $h_f(y)$, we have that $h_g(h_f(y)) = y$. Hence, h_g is onto and therefore is a homeomorphism. ∎

The next claim is a version for expanding endomorphisms that was already provided for the case of hyperbolic diffeomorphisms in [Rob76, Theorem 4.1]. The goal is to show that we can extend the conjugation between $f|_{\Lambda_f}$ and $g|_{\Lambda_g}$ to an open neighborhood U of Λ_f in such a way that still is an homeomorphism that conjugate $f|_U$ and $g|_U$, noting that the conjugation is unique just in Λ_f. We are going to use this extension in next section for proving the robustness of the property of Λ_f disconnects a "nice" class of sets.

Claim 1.2 The homeomorphism $h_f : \Lambda_f \to \Lambda_g$ in claim (1.1) can be extended as an homeomorphism H to an open neighborhood of Λ_f such that $H \circ f = g \circ H$.

Proof. This geometrical proof is inspired in the proof given by Palis in [Pal68] and also used to prove the Grobman-Hartman Theorem in [Shu87, pp.96].

Other alternative proof consist in using inverse limit space, by this an expanding endomorphism becomes a hyperbolic diffeomorphism and so Theorem 4.1 in [Rob76] could be applied.

The goal is to choose U an isolating neighborhood of Λ_f and to construct an homeomorphism from U onto itself, using the inverse branches of f and g, and a fundamental domain D_f for f, i.e. for every $x \in U \setminus \Lambda_f$, there exists $n \in \mathbb{N}$ such that $f^n(x) \in D_f$.

Observe that the isolating block of Λ_f is also an isolating block of Λ_g. Now we can take in the same way a fundamental domain for g, D_g. After it is taken an homeomorphism H between both fundamental domains D_f and D_g. Then this homeomorphism is saturated to $U \setminus \Lambda_f$ by backward iteration, i.e. if $x \in U \setminus \Lambda_f$, let n be such that $f^n(x) \in D_f$, take $H \circ f^n(x)$ and then $g^{-n} \circ H \circ f^n(x)$ where g^{-n} is taken carefully using the corresponding inverse branches.

Denote by N_f the cardinal of $\{f^{-1}(x)\}$, since f is a local diffeomorphism, N_f is constant. Let $K \subset \{1, \dots, N_f\}$ be such that for every $i \in K$, there exist $U_i^f \subset U$ and $\varphi_i^f : U \to U_i^f$ inverse branch of f such that $\varphi_i^f(U) = U_i^f$ and $f(U_i^f) = f \circ \varphi_i^f(U) = U$. Also, for g as in claim (1.1), for every $i \in K$, there exist $U_i^g \subset U$ and $\varphi_i^g : U \to U_i^g$ the inverse branch of g such that $\varphi_i^g(U) = U_i^g$ and $g(U_i^g) = g \circ \varphi_i^g(U) = U$.

We wish to construct an homeomorphism H on U satisfying $H \circ f = g \circ H$ and $H \mid_{\Lambda_f} = h_f$. We can begin as follows. Suppose that the restriction $H : \partial U \to \partial U$ is any well-defined orientation preserving diffeomorphism. The restriction of H to ∂U_i^f is then defined as follows $H(x) = \varphi_i^g \circ H \circ f(x)$ if $x \in \partial U_i^f$ because H conjugate f and g. Extend H to a diffeomorphism which send $U \setminus \cup_{i \in K} U_i^f$ bounded by ∂U and ∂U_i^f onto $U \setminus \cup_{i \in K} U_i^g$ bounded by ∂U and ∂U_i^g. We may assume that the Hausdorff distance between U and Λ_f is small, see Lemma 2, then the initial H is close to the identity. Let us say that $d(H(x), x) < \eta$, where $\eta > 0$ is given arbitrarily.

Given $i, j \in K$, denote $U_{j,i}^f = \varphi_j^f \circ \varphi_i^f(U)$ and $U_{2\,i}^f = U_i^f \setminus \bigcup_{j \in K} U_{j,i}^f$. If $x \in \partial U_{j,i}^f$ then $H(x) = \varphi_j^g \circ \varphi_i^g \circ H \circ f^2(x) \in \partial U_{j,i}^g$. If $x \in U_{2\,i}^f \setminus \Lambda_f$ then $H(x) = \varphi_i^g \circ H \circ f(x) \in U_{2\,i}^g$.

Doing this process inductively we have that: Given $i_1, \dots, i_n \in K$, denote $U^f_{i_n,\dots,i_1} = \varphi^f_{i_n} \circ \cdots \circ \varphi^f_{i_1}(U)$ and $U^f_{n\ (i_{n-1},\dots,i_1)} = U^f_{i_{n-1},\dots,i_1} \setminus \bigcup_{i_n \in K} U^f_{i_n,\dots,i_1}$. If $x \in \partial U^f_{i_n,\dots,i_1}$ then $H(x) = \varphi^g_{i_n} \circ \cdots \circ \varphi^g_{i_1} \circ H \circ f^n(x)$. If $x \in U^f_{n\ (i_{n-1},\dots,i_1)} \setminus \Lambda_f$ then $H(x) = \varphi^g_{i_{n-1}} \circ \cdots \circ \varphi^g_{i_1} \circ H \circ f^{n-1}(x)$. And $H(x) = h_f(x)$ if $x \in \Lambda_f$.

Let us prove that H is continuous.

Given $x \in \Lambda_f$, let $(x_n)_n$ be a sequence in $U \setminus \Lambda_f$ such that $x_n \to x$, when $n \to \infty$. Let us prove that $H(x_n) \to H(x)$, when $n \to \infty$.

First, consider $\{z_k\}_{k \in \mathbb{Z}}$ an $f-$full orbit in Λ_f such that $z_0 = x$ and for every $n \in \mathbb{N}$, consider $\{z^n_k\}_{k \in \mathbb{Z}}$ a full orbit by f associated to each x_n using the corresponding inverse branches (for the backward iterates) given by the full orbit of x, where $z^n_0 = x_n$. Since f is continuous, for every $k \in \mathbb{Z}$, we have that $z^n_k \to z_k$ when $n \to \infty$.

Note that for every $n \in \mathbb{N}$, there exists $k_n > 0$ such that $z^n_{k_n} \in U \setminus \cup_{i \in K} U^f_i$. Furthermore, $z^n_k \in U$ for every $k \in [-k_n, k_n]$. Since $H \circ f = g \circ H$, we get that $H(x_n) \in \bigcap_{k=-k_n}^{k_n} g^k(U)$.

Hence, for η and ε as in claim (1.1) and for every $n \in \mathbb{N}$, we have that $\{z^n_k\}_{k=-k_n}^{k_n}$ is a finite $\varepsilon-$pseudo orbit for g and it is $\eta-$shadowed by a $g-$orbit of $H(x_n)$ until k_n for forward iterates and $-k_n$ for backward iterates.

Observe that as m goes to infinity, the finite pseudo orbit $y^m_n = \{z^n_k\}_{k=-m}^m$ becomes longer. Consider now the sequence $\{y^m_n\}_n$. Then $y^m_n \to \{z_k\}_{k=-m}^m$ when $n \to \infty$. Hence, the sets of shadowing points of the finite pseudo orbits $y^{k_n}_n$ converge to the shadowing point of the infinite pseudo orbit $\{z_k\}_k$, then $H(x_n) \to h_f(x) = H(x)$ when $n \to \infty$. ∎

§ 1.1.5 The Locally Maximal Set "Separates"

The main goal of this section is to show that the locally maximal set for f has a topological property that persist under perturbation, roughly speaking means that Λ_f and Λ_g

disconnect small open sets. We prove that Λ_f intersects "some nice" class of arcs in U_1^c and which Λ_g also intersects for all g nearby f. The first question that arise is: which arcs belong to this "nice" class?, the second questions in the context of proving the Main Theorem is: why is this property enough? and the third question is: why do the "nice class" exist? All these questions are answered along the section, but to give some brief insight about the main ideas observe that:

1. These "nice" arcs have the property that we can build a "nice cylinder" (see definition 1.18) containing the initial arc and Λ_f "separates" (see definition 1.20) this cylinder in a "robust way".

2. It is enough to consider these "nice" class of arcs to finish the proof of the Main Theorem. Suppose we have the existence of this class of arcs and suppose that given any open set there is an iterate by g that has a "nice" arc. Then there is a point in this iterate which its forward orbits stay in the expanding region, hence the internal radius growth until a fixed radius in finitely many iterates. It allows us to conclude the density of the pre-orbit of any point by the perturbation, just noting that the density of every pre-orbit by the initial map implies $\varepsilon-$density of the pre-orbit by the perturbed map.

3. We show in claim (1.3) that every large arc admits a "nice" arc.

Let us define the concepts involved in this section.

Definition 1.16 (**Cylinder**)

Given γ a differentiable arc and $r > 0$, it is said that $C(\gamma, r)$ is a *cylinder* centered at γ with radius r if

$$C(\gamma, r) := \bigcup_{x \in \gamma} ([T_x\gamma]^{\perp})_r,$$

where $([T_x\gamma]^\perp)_r$ denotes the ball $\mathbb{B}_r(x)$ centered at x with radius r intersected with $[T_x\gamma]^\perp$ the orthogonal to the tangent to γ in x.

Definition 1.17 (**Simply connected cylinder**)

Given γ a differentiable arc and $r > 0$, it is said that a cylinder $C(\gamma, r)$ is *simply connected* if it is retractile to a point.

Remark 1.15 Fixed the radius, the cylinder could be not retractile to a point. In this case, working in the universal covering space, consider the convex hull of its lift and then project it on the manifold. We call the resulting set as *simply connected cylinder* as well and denote in the same way as above.

Definition 1.18 (**Nice cylinder**)

Given γ an arc and $r > 0$, it is said that a cylinder $C(\gamma, r)$ is a *nice cylinder* if it is simply connected cylinder and if x_A and x_B are the extremal points of γ then $A := ([T_{x_A}\gamma]^\perp)_r \subset \partial C(\gamma, r)$ and $B := ([T_{x_B}\gamma]^\perp)_r \subset \partial C(\gamma, r)$. In this case, we say that A and B are the *top and bottom sides* of the cylinder.

Remark 1.16 Note that in general a cylinder does not have top and bottom sides and does not be simply connected.

Hereafter, fix U_2 an open set such that $\overline{U_1} \subset U_2$ and $\delta_0 < diam_{int}(U_2^c) < diam_{int}(U_0^c)$. Let $d_1 = d_H(U_2, U_1) > 0$, where d_H denotes the Hausdorff metric, and let $k \in \mathbb{N}$ such that $\delta_0' = \delta_0 + \frac{d_1}{3k} < diam_{int}(U_2^c)$.

Let us denote by $\widetilde{U}$ the lift of U_0, π the projection of $\mathbb{R}^n$ onto M and $\mathfrak{U}_0$ the convex hull of $\widetilde{U} \cap [0,1]^n$. Consider $\mathrm{P}_i(\mathfrak{U}_0)$ the projection of $\mathfrak{U}_0$ in the $i-$th coordinate in the $n-$dimensional cube $[0,1]^n$. Since $diam_{ext}(U_0) < 1$ and remark (1.8), for every $1 \le i \le n$, there exist $0 < k_i^- < k_i^+ < 1$ such that $k_i^- < \mathrm{P}_i(\mathfrak{U}_0) < k_i^+$. Note that $1 + k_i^- - k_i^+ > \delta_0'$ for every i, because $1 + k_i^- - k_i^+ > diam_{int}(U_0^c)$ by construction.

Let $R_i^m = \{x \in \mathbb{R}^n : k_i^- + m < x_i < k_i^+ + m\}$ with $m \in \mathbb{Z}$, $1 \leq i \leq n$ and x_i is the $i-$th coordinate of x. Thus, $\mathfrak{U}_0 \subset \bigcap_{m \in \mathbb{Z}, 1 \leq i \leq n} R_i^m$. Denote by $L_i^+ = \{x \in \mathbb{R}^n : x_i = k_i^+\}$ and $L_i^- = \{x \in \mathbb{R}^n : x_i = k_i^-\}$. Let $\tilde{f}$ be the lift of f.

The next claim answer the third question stated at the beginning of the section.

Claim 1.3 Let $m > 2\sqrt{n}$ be fixed. Given any arc γ in $\mathbb{R}^n$ with $diam(\gamma) > m$, there exist an arc $\gamma' \subset \gamma$, $1 \leq i \leq n$ and $j \in \mathbb{Z}$ such that $\partial\gamma' \cap (L_i^+ + j)$, $\partial\gamma' \cap (L_i^- + j + 1)$ and $P_i^j(\gamma') \subset [k_i^+ + j, k_i^- + j + 1]$. Moreover, γ' admits a nice cylinder, $\gamma^* = \pi(\gamma')$ is in U_2^c, diameter of γ^* is larger than δ_0 and γ^* also admits a nice cylinder contained in U_1^c.

Proof. Take γ an arc with diameter larger than m, then the projection of γ in the $i-$th coordinate contains an interval of the kind formed by k_i^+ and $1 + k_i^-$ for some i (or formed by $k_i^+ + j$ and $k_i^- + j + 1$ for some $j \in \mathbb{Z}$). If it is not true, γ would be in a $n-$dimensional cube with sides smaller than $k_i^+ - k_i^- < 1$ and this cube has diameter smaller than $\sqrt{n}$, but this contradict the fact that $diam(\gamma) > m$. Hence, we may pick an arc γ' in γ such that $\partial\gamma' \cap L_i^+ + j$, $\partial\gamma' \cap L_i^- + j + 1$ and $P_i^j(\gamma') \subset [k_i^+ + j, k_i^- + j + 1]$ for some $1 \leq i \leq n$ and some $j \in \mathbb{Z}$. Therefore, diameter of γ' is greater than δ_0 and in consequence its projection in M also has diameter greater than δ_0.

Moreover, since the property of the arc γ' to have a projection in between $k_i^+ + j$ and $k_i^- + 1 + j$, we may construct a cylinder centered at γ' and radius $\frac{d_1}{2}$ such that this cylinder is "far" away from $\widetilde{U}$, so this cylinder could be simply connected or, if it is not simply connected cylinder, it has holes that are different from $\widetilde{U}$. In the case that the cylinder is not simply connected, we consider the convex hull of the cylinder, since the original cylinder is bounded by $L_i^+ + j$ and $L_i^- + j + 1$, then the convex hull stay in between these two hyperplanes and therefore it does not intersect $\widetilde{U}$. By abuse of notation, let us denote this set by $C(\gamma', \frac{d_1}{2})$, it is a simply connected cylinder. Observe that by construction, this cylinder will have top and bottom sides, thus $C(\gamma', \frac{d_1}{2})$ is a nice cylinder.

Take $\gamma^* = \pi(\gamma')$, note that γ' can be choose such that γ^* is contained in U_2^c and the diameter of γ^* is larger than δ_0, then projecting the nice cylinder for γ' in M we obtain a nice cylinder for γ^* which is denoted by $C(\gamma^*, \frac{d_1}{2})$. This nice cylinder has the property that every arc that goes from bottom to top side has diameter at least δ_0 and all this process can be made in such a way that the nice cylinder is in U_1^c. ∎

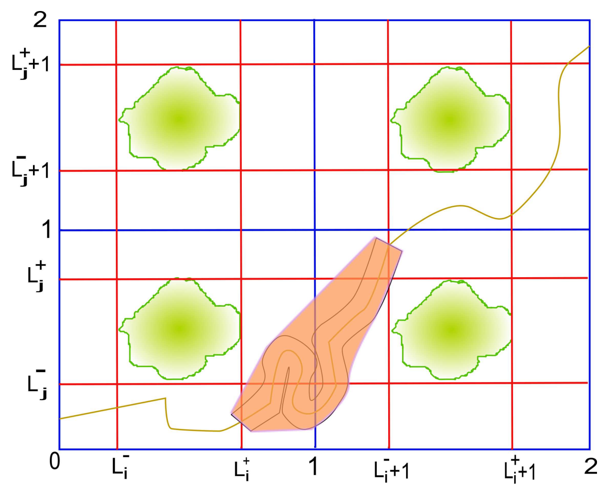


Figure 1.2: *Nice cylinders*

Definition 1.19 (**Lateral border**)

Given γ a differentiable arc and $r > 0$. The *lateral border* S of the cylinder $C(\gamma, r)$ is $\partial C(\gamma, r)$ minus the top and bottom sides of the cylinder if they exist.

Definition 1.20 (**Separated horizontally**)

We say that a nice cylinder $C(\gamma, r)$ is *separated horizontally* by a set Λ if there exists a connected component of Λ such that intersects the nice cylinder across the lateral border and $C(\gamma, r)$ minus that connected component of Λ has at least two connected component.

Now, we are going to prove that the locally maximal set for f, found in section § 1.1.3, has the geometrical property of separating horizontally these nice cylinders of a certain radius.

Lemma 3 *Given any arc γ in U_2^c with diameter greater than δ_0 that admits a nice cylinder as in claim (1.3) holds that Λ_f separates horizontally this nice cylinder.*

Proof. Let us denote by T the nice cylinder associated to γ as in the statement and let A and B denote the top and bottom sides of T respectively. Let $\varepsilon > 0$ be arbitrarily small.

Let T' be a bigger cylinder containing T joint together with two security regions, denote by S_A and S_B, and such that the distance between the lateral border of T and the lateral border of T' is small, for instance $d_H(T, T') = \frac{d_1}{6k}$, see figure (1.3). For security regions S_A and S_B we means two strips of $\frac{d_1}{6k}$ thickness glued to the sides A and B of T, or in other words, S_A (respectively S_B) is the set of points in T^c such that the distance from these points to A (respectively B) is less or equal to $\frac{d_1}{6k}$. This set T' was constructed in such a way that its diameter is greater than δ_0'.

Since γ is in U_2^c and diameter is greater than δ_0, we can assure that $T \cap \Lambda_f$ is non empty. Consider all the connected components of $T \cap \Lambda_f$. For every $x \in T \cap \Lambda_f$, we assign K_x the connected component of $T \cap \Lambda_f$ that contains x. Observe that we may define an equivalence relation: $x \sim x'$ if and only if $K_x = K_{x'}$. Then we pick one component from each class, or in other words we pick just the connected components that are two by two disjoints.

We claim that Λ_f separates T horizontally. If there exists one component K_x that separates T horizontally in more than one connected component, the assertion holds.

Suppose it does not happen, i.e. none of the K_x separates T horizontally. Take U_x open set in T' such that $K_x \subset U_x$, $\partial U_x \cap \Lambda_f = \emptyset$, ∂U_x is connected and ∂U_x does not divides T horizontally. If there are many K_y accumulating in one K_x, then we could have a same open set U_x containing more than one connected component K_y.

Observe that the collection $\{U_x\}$ is an open cover of $T \cap \Lambda_f$. Since it is compact, there is a finite subcover $\{U_i\}_{i=1}^N$, i.e. $T \cap \Lambda_f \subset \mathcal{U} = \cup_{i=1}^N U_i$.

If the connected components of $\mathcal{U}$ does not separates horizontally T, it is easy to construct a curve going from A to B with diameter greater than δ_0 and empty intersection with the U_i's; hence, this curve does not intersects the set Λ_f. But this contradicts the fact that every curve in U_1^c with diameter larger than δ_0 intersects Λ_f. Then the connected components of $\mathcal{U}$ separate T horizontally, denote by C_j the connected components of T minus these connected components of $\mathcal{U}$ that separates T horizontally.

Observe that every C_j is path connected, since they are the complement of a finite union of open sets in a simply connected set T. There exist a finite quantity of C_j, let us say m. We can reorder these sets enumerating from the top side. If we denote by V_j each of the connected components of $T \cap \mathcal{U}$ that separate T horizontally, we have two cases, either C_j is in between two consecutive V_j and V_{j+1} (or V_{j-1} and V_j) or C_j just intersects one V_j on the border.

The idea is to build a curve from top to bottom of T connecting C_j with C_{j+1} in such a way that the diameter of the arc is greater than δ_0 but without intersecting Λ_f, which is an absurd because it is again in U_1^c and has diameter greater than δ_0, then this curve must intersects Λ_f.

It is enough to show that we can pass from C_j to C_{j+1} without touching Λ_f. For this, we must observe that every V_j is a union of finitely many U_i, let us say $U_{i_1}, \ldots, U_{i_j}$. Pick a curve γ_j in C_j going from top to bottom, i.e. γ_j goes from ∂V_j to ∂V_{j+1} (or ∂V_{j-1} and ∂V_j) and γ_j does not intersects the interior of V_j and V_{j+1} (or V_{j-1} and V_j), then there exists $i_s \in \{i_1, \ldots, i_j\}$ such that $\gamma_j \cap \partial U_{i_s} \neq \emptyset$. After that continue this arc picking a curve following by the border of U_{i_s} until C_{j+1}, which has empty intersection with Λ_f by construction, if it is not possible to do in one step, pick another U_{i_k} and repeat the process. Note that this process finish in finitely many times. The resulting arc from joint together all this segment has diameter greater than δ_0 and with empty intersection with Λ_f as we wanted. ∎

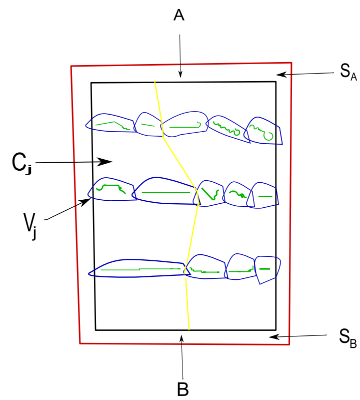


Figure 1.3: Λ_f *splits "horizontally" every nice cylinder in at least two connected component*

Remark 1.17 In claim (1.1), remembering that $d(h, id) < \eta$, we may fix $\eta < \min\{\frac{d_1}{6k}, \delta_0, \beta\}$. So for this η, there exists $\varepsilon_0 > 0$ given by the shadowing lemma and this determine $\mathcal{V}_1(f)$ given in claim (1.1).

Lemma 4 *Given $g \in \mathcal{V}_1(f)$ and given γ an arc in U_2^c with diameter greater than δ_0' such that it admits a nice cylinder $C(\gamma, \frac{d_1}{2})$, then $\gamma \cap \Lambda_g$ is not empty.*

Proof. Let $g \in \mathcal{V}_1(f)$, it means that g is an endomorphism within distance of f less than ε_0 in the C^1 topology. Take γ an arc in U_2^c with diameter greater than δ_0' such that $C(\gamma, \frac{d_1}{2})$ is a nice cylinder.

By construction, we may assume that every arc taken in the nice cylinder that goes from top to bottom has diameter greater or equal to the diameter of γ. Take two security regions inside the cylinder, in the top and bottom sides of the cylinder respectively, with

$\frac{d_1}{6k}$ of thickness each one, i.e. two strips glued to the top and bottom sides of the cylinder such that each one is the set of points in the cylinder within distance to top (respectively bottom) side less or equal to $\frac{d_1}{6k}$, see figure (1.4). Let us denote by C' the cylinder resulting of taking out these two security strips from the original cylinder $C(\gamma, \frac{d_1}{2})$, then the diameter of C' is still greater than δ_0.

Hence, the diameter of $\gamma' = \gamma \cap C'$ is greater than δ_0 and it is in U_1^c. Lemma 3 implies that Λ_f separates horizontally C', hence γ' intersects Λ_f, let us denote by x_f the point in the intersection.

Since $x_f \in \Lambda_f$, claim (1.1) and remark (1.17), there exists $x_g \in \Lambda_g \cap \mathbb{B}_\eta(x_f)$. Note that Λ_f separates $\mathbb{B}_\eta(x_f)$ in at least two connected component. Hence, Λ_g separates $\mathbb{B}_\eta(x_f)$ in at least two connected component as well, because $f\mid_U$ and $g\mid_U$ are conjugated. Therefore, Λ_g must intersects γ. ∎

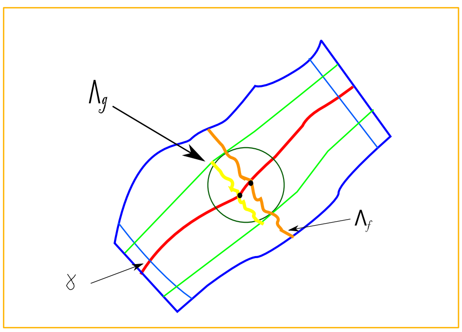


Figure 1.4: Λ_g *intersects* γ

§ 1.1.6 Getting Sets of Large Diameter

Lemma 5 *There exist* $\mathcal{V}_2(f)$ *and* $R > 0$ *such that for every* $g \in \mathcal{V}_2(f)$*, if there is* $x \in M$ *such that* $g^n(x) \notin U_0$ *for every* $n \geq 0$*, then there is* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$*, there exists* $N = N(\varepsilon) \in \mathbb{N}$ *such that* $\mathbb{B}_R(g^N(x)) \subset g^N(\mathbb{B}_\varepsilon(x))$*.*

Proof. We may pick U_3 an open subset contained in U_0 such that $m\{Df\mid_{U_3^c}\} > \lambda'$, with $1 < \lambda' < \lambda_0$. Take $\mathcal{V}_2(f)$ an open subset perhaps smaller than $\mathcal{V}_1(f)$ such that $m\{Dg\mid_{U_3^c}\} > \lambda'$ holds for every $g \in \mathcal{V}_2(f)$. Let us fix $R = d_H(U_0, U_3) > 0$.

Given $0 < \varepsilon < R$, take $N \in \mathbb{N}$ such that $(\lambda')^{-N}R < \varepsilon/2$. Then $\mathbb{B}_{(\lambda')^{-N}R}(x) \subset \mathbb{B}_\varepsilon(x)$.

Observe that $\mathbb{B}_R(g^n(x)) \cap U_3 = \emptyset$, for every $n \geq 0$. Also,

$$g^k(\mathbb{B}_{(\lambda')^{-N}R}(x)) = \mathbb{B}_{(\lambda')^{-N+k}R}(g^k(x)) \subset \mathbb{B}_R(g^k(x)),$$

for every $0 \leq k \leq N$. In particular, $g^k(\mathbb{B}_{(\lambda')^{-N}R}(x)) \cap U_3 = \emptyset$, for every $0 \leq k \leq N$. Then

$$g^N(\mathbb{B}_{(\lambda')^{-N}R}(x)) = \mathbb{B}_R(g^N(x)) \subset g^N(\mathbb{B}_\varepsilon(x)).$$

∎

Remark 1.18 Let us note that Lemma 5 holds for every point in Λ_g.

Lemma 6 *For every $g \in \mathcal{V}_2(f)$ and given V an open path connected set in M, there exists $m_0 = m_0(V, g) \in \mathbb{N}$ such that $diam(\tilde{g}^{m_0}(\widetilde{V})) > m$, where $\tilde{g}$ and $\widetilde{V}$ are the lift of g and V, respectively. In particular, it contains an arc with diameter greater than m.*

Proof. Let $g \in \mathcal{V}_2(f)$ and V be an open path connected set in M. Since g is volume expanding, let us say with expanding constant $\lambda > 1$, we have that $vol(\tilde{g}^k(\widetilde{V})) > \lambda^k vol(\widetilde{V})$, for $k \geq 1$. Iterating by $\tilde{g}$, the volume increase and furthermore the diameter of its iterates growth also in the covering space. Hence, there exists $m_0 \in \mathbb{N}$ such that $diam(\tilde{g}^{m_0}(\widetilde{V})) > m$. ∎

Remark 1.19 For the case that V is an open connected set, observe that given a point in V there exists an open ball centered in this point and contained in V such that it is path connected. Then we may apply Lemma 6 to this ball and obtain a similar statement for V.

§ 1.1.7 Proof of The Main Theorem

Let $f \in E^1(\mathbb{T}^n)$ be such as in the statement. Lemma 2 implies that we may assume the existence of Λ_f an expanding locally maximal set for f.

Fix $0 < \alpha < R$, arbitrarily small. Given $x \in \mathbb{T}^n$, since $\bigcup_{i\in\mathbb{N}}\{f^{-i}(x)\}$ is dense, there exists $n_0 \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{n_0}\{f^{-i}(x)\} \text{ is } \alpha/2\text{-dense.}$$

Take a neighborhood $\mathcal{U}(f) \subset \mathcal{V}_2(f)$, where $\mathcal{V}_2(f)$ was given in Lemma 5, such that for every $g \in \mathcal{U}(f)$ follows that

$$\bigcup_{i=0}^{n_0}\{g^{-i}(x)\} \text{ is } \alpha/2\text{-close to } \bigcup_{i=0}^{n_0}\{f^{-i}(x)\}.$$

Hence, $\bigcup_{i=0}^{n_0}\{g^{-i}(x)\}$ is α-dense.

Let V be an open connected set in $\mathbb{T}^n$. By Lemma 6, there exists $m_0 \in \mathbb{N}$ such that $diam(\tilde{g}^{m_0}(\widetilde{V})) > m$. Then we may pick an arc γ in $\tilde{g}^{m_0}(\widetilde{V})$ with diameter larger than m and applying claim (1.3) follows that there exists a connected piece γ' of γ such that $\gamma^* = \pi(\gamma')$ is in U_2^c, diameter of γ^* is larger than δ_0' and it admits a nice cylinder $C(\gamma^*, \frac{d_1}{2})$. By Lemma 4 follows that $\gamma^* \cap \Lambda_g$ is not empty, let y be a point in the intersection.

Hence, for this point y, there exists $\varepsilon_0 = \varepsilon_0(y) > 0$ such that $\mathbb{B}_{\varepsilon_0}(y) \subset g^{m_0}(V)$, by Lemma 5 taking $0 < \varepsilon < \varepsilon_0$, we get that there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\mathbb{B}_R(g^N(y)) \subset g^N(\mathbb{B}_\varepsilon(y)) \subset g^{m_0+N}(V).$$

Hence, $\mathbb{B}_\alpha(g^N(y)) \subset g^{m_0+N}(V)$. Since the $\alpha-$density, we have that

$$\bigcup_{i=0}^{n_0}\{g^{-i}(x)\} \cap \mathbb{B}_\alpha(g^N(y)) \neq \emptyset.$$

Therefore, denoting by $p = m_0 + N$,

$$\bigcup_{i=0}^{n_0}\{g^{-i}(x)\} \cap g^p(V) \neq \emptyset.$$

Taking the $p-$th pre-image by g, we obtain that there is $i_0 \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{i_0}\{g^{-i}(x)\} \cap V \neq \emptyset.$$

Thus, for every $g \in \mathcal{U}(f)$ follows that $\bigcup_{i\in\mathbb{N}}\{g^{-i}(x)\}$ is dense in $\mathbb{T}^n$ for every $x \in \mathbb{T}^n$. ∎

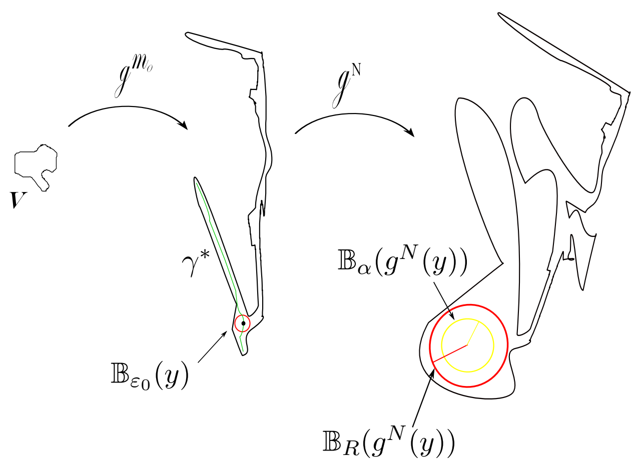


Figure 1.5: *Iterations by the perturbed map*

§ 1.1.8 The Main Theorem Revisited

In this section, we enunciate a weaker version of the Main Theorem. Observe that using hypothesis (3) and (4) of the Main Theorem, we showed in sections § 1.1.3 and § 1.1.5 the existence of a locally maximal expanding set for f which separates large nice cylinders, and by section § 1.1.4 follows that this geometrical property persist under perturbation, i.e. there is a set Λ_f locally maximal which intersects a nice class of arcs in U_0^c and this property also holds for the perturbed. The hypothesis of f being volume expanding guarantees that given any open set in the covering space, we are able to choose some iterates such that it contains an arc with diameter big enough to apply claim (1.3) and Lemma 3. Hence, the Main Theorem may be enunciated in a weaker version as follows:

Main Theorem Revisited *Let $f \in E^1(\mathbb{T}^n)$ be volume expanding such that the pre-orbit of every point are dense. Suppose that there exist an open set U_0 with $diam_{ext}(U_0) < 1$ and Λ_f a locally maximal expanding set for f in U_0^c such that every arc γ in U_0^c with diameter large enough intersect Λ_f. Then, the pre-orbit of every point are C^1 robustly dense.*

Remark 1.20 Observe that the Main Theorem implies the Main Theorem Revisited, the reciprocal could be false. The hypotheses of the Main Theorem assures the existence of a nice Λ_f with certain geometrical properties, but there might exist weaker conditions that guarantee the existence of such a set.

§ 1.2 Volume expanding endomorphisms with invariant splitting

Definition 1.21 (**Unstable cone family**)

Given $f : M \to M$ a local diffeomorphism, let V be an open subset of M such that $f|_V$ is a diffeomorphism onto its image. Denote by φ the inverse branches of f restricted to V; more precisely, $\varphi : f(V) \to V$ such that $f \circ \varphi(x) = x$ if $x \in f(V)$. A continuous cone field $\mathcal{C}^u = \{\mathcal{C}^u_x\}_x$ defined on V is called *unstable* if it is forward invariant:

$$Df(x')\,\mathcal{C}^u_{x'} \subset \mathcal{C}^u_{f(x')}$$

for every $x' \in V \cap \varphi(V)$.

Remark 1.21 Given a point x, there is not necessarily a unique unstable subbundle, i.e. for each inverse path $\{x_k\}_{k\geq 0}$, it means $x_0 = x$ and $f(x_{k+1}) = x_k$ for $k \geq 0$, there exists an unstable direction belonging to $\mathcal{C}^u$.

Definition 1.22 (**Complementary splitting**)

We say that a splitting $\mathbb{E}^c_x + \mathcal{C}^u_x$ is *complementary* if the unstable cone $\mathcal{C}^u_x$ contains an invariant subspace whose dimension is equal to the dimension of the manifold minus the dimension of the central subbundle.

Definition 1.23 (**Partially hyperbolic endomorphism with expanding extremal direction**)

It is said that an endomorphism f is *partially hyperbolic with expanding extremal direction* provided the tangent bundle splits into two non-trivial subbundles $TM = \mathbb{E}^c \oplus \mathbb{E}^u$ which are invariant under the tangent map Df, i.e. for every $x \in M$, there exists a complementary splitting $\mathbb{E}^c_x + \mathcal{C}^u_x$, where $\{\mathcal{C}^u_x\}_x$ is a family of unstable cones, and there exists $0 < \lambda < 1$ such that for every inverse branches φ of f follows that

1. $\|D\varphi(x)\, v\| < \lambda$, for all $v \in \mathcal{C}^u_x$.

2. $\|Df(x') \mid_{\mathbb{E}^c(x')}\| \|D\varphi(x)v\| < \lambda$, for all $v \in \mathcal{C}^u_x$, where $\varphi(x) = x',\ f(x') = x$.

§ 1.2.1 Theorem 2: Splitting Version

In section § 1.1.1, we gives sufficient conditions for volume expanding local diffeomorphisms without invariant subbundles be robustly transitive. Now, we state a version for the case when tangent bundle splits into two non-trivial subbundles, one with an expanding behavior and the other one with nonuniform behavior but dominated by the expanding one.

Theorem 2 *Let $f \in E^1(\mathbb{T}^n)$ be a locally diffeomorphism partially hyperbolic with expanding extremal direction satisfying the following properties:*

1. $\{f^{-k}(x)\}_{k\geq 0}$ *is dense for every* $x \in \mathbb{T}^n$.

2. *There exist $\delta_0 > 0$, $\lambda_0 > 1$ and $k_0 \in \mathbb{N}$ such that for every $x \in \mathbb{T}^n$, if γ is a disc tangent to the unstable cone $\mathcal{C}^u_x$ with internal diameter larger than δ_0, there exists a point $y \in \gamma$ such that $m\{Df^i |_{\mathbb{E}^c(f^k(y))}\} > \lambda_0^i$, for all $i > 0$, for all $k > k_0$.*

Then, for every g close enough to f, $\{g^{-k}(x)\}_{k\geq 0}$ is dense for every $x \in \mathbb{T}^n$.

§ 1.2.2 Proof of Theorem 2

The proof of Theorem 2 is pretty similar to the proof given in [PS06b], where it is proved that any partially hyperbolic diffeomorphism satisfying a hypothesis like the one stated in Theorem 2 and such that the strong stable foliation is minimal, then the strong stable foliation is robustly minimal. That property says that in any compact piece of the unstable foliation, there exists a point such that the central bundle has uniform expanding behavior along the forward orbit, and this is exactly what we have. The key is to prove that this property is robust under perturbation.

Given a local diffeomorphism f as in the statement of Theorem 2, we want to show that any small perturbation g preserve the property of density of the pre-orbit of any point. Our strategy is to prove that given any disc tangent to the unstable cones for g with large enough internal diameter has a point such that the central direction along the forward orbit by g is uniformly expanding.

Observe that given any open set, since we have a direction that is indeed expanding, the diameter along the unstable direction of the iterates growth. Then we are able to pick a disc inside this iterate such that the disc is tangent to the unstable cones with diameter large enough to apply the last property. Hence, there exists a point which its forward orbit is expanding in all direction, then there is some iterate such that it contains a ball of a fix radius ε.

Since g is close enough to f, we have that the pre-orbit by g are $\varepsilon-$dense. Therefore, given any open set, by the property of the unstable discs, there exists an iterate such that it intersects the pre-orbit by g of any point. Thus, we conclude the density of the pre-orbit of any point by the perturbation.

Moreover, the proof of Theorem 2 can also be performed in the spirit of Main Theorem. In fact, it is possible to show that

$$\bigcap_{l\geq 0} f^{-l}(\{x : m\{Df^n \mid_{\mathbb{E}^c(f^k(x))}\} > \lambda_0^n,\ n > 0,\ k > k_0\})$$

is an invariant expanding set such that separates unstable discs. This provides a geometrical interpretation. ∎

§ 1.3 Remarks About the Main Theorem and Theorem 2

On regard the similarities between Main Theorem and Theorem 2, we must note that in both theorems we assume that f is volume expanding, since we know by Theorem 1 that volume expanding is a necessary condition in order to have robust transitivity. Also, we assume that the pre-orbit by f of every point is dense, actually this hypothesis is stronger than transitivity. Moreover, this hypothesis does not depend on the existence of splitting.

Besides, in the Main Theorem we asked for large arcs to contain points such that its forward iterations remain in the expanding region. The same is required in Theorem 2 but just for unstable discs, and the equivalent for the splitting version to say that the forward orbit is “in an expanding region” is that the central bundle along the forward orbit of such points has uniform expanding behavior.

The main difference in their proof arise from the fact that in the version with splitting, since we know that we have uniform expansion in one direction, any disc with internal diameter larger than δ_0 and tangent to this direction, growth until length $\delta_1 > \delta_0$ in a bounded uniform time, independently of the disc. Note that we cannot guarantee that this happens just having volume expansion.

Observe that in the Main Theorem is not assumed that f does not have any splitting. In fact, it could also be partially hyperbolic. However, knowing in advance that the endomorphism is partially hyperbolic then it is possible to get sufficient conditions for robust transitivity weaker than the one requires by the Main Theorem.

Chapter 2

Existence of a semiconjugation to a linear expanding endomorphism

In this chapter, we show some consequences from the Main Theorem. The main goals are to study the properties of the semiconjugation between a linear expanding endomorphism and an isotopic endomorphism to the initial one, and the relation between the Markov Partition and Transitivity of these maps.

§ 2.1 Dynamical Consequences: Geometrical and Topological

Definition 2.1 (**Totally disconnected set**)

A set Λ is said to be totally disconnected if every point is a connected component.

By Lemma 3 we have that

Corollary 1 *Λ_f is not totally disconnected. Furthermore, Λ_f intersects every large arc in U_0^c. If the connected components of U_0 are simply connected, then Λ_f is a basic set.*

Remark 2.1 Since hypothesis (3) of the Main Theorem, the connected components of $\cup_{n\geq 1} f^{-n}(U_0)$ have diameter less than δ_0.

Remark 2.2 The fact of Λ_f be expanding locally maximal set implies that there exists a Markov Partition for this set.

§ 2.2 The Case f is Isotopic to a Linear Expanding Endomorphism $\mathcal{E}$

In this section, given an endomorphism f isotopic to a linear expanding endomorphism $\mathcal{E}$, we construct the semiconjugation between these two maps and study the properties of the semiconjugation and the relation with the existence of Markov Partition for f and how it can help us to deduce if f is transitive. Moreover, we are interested in the case that we know that there exists an expanding locally maximal set Λ_f with the properties given in the Main Theorem and how this gives some information about the Markov partition. Finally, we pose some related questions.

§ 2.2.1 Existence of the Semiconjugation

Let $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ be a linear expanding endomorphism and let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an endomorphism isotopic to $\mathcal{E}$.

Let us remember that the lift of f is given by $\tilde{f} = \widetilde{E} + p$, where $\widetilde{E} \in SL(n, \mathbb{Z})$ (called the linear part) is the lift of $\mathcal{E}$ and p is $\mathbb{Z}^n-$periodic isotopic to a constant.

Lemma 7 *There exists a semiconjugation between $\tilde{f}$ and $\widetilde{E}$; i.e. there exists $H : \mathbb{R}^n \to \mathbb{R}^n$ continuous, onto and $H \circ \tilde{f} = \widetilde{E} \circ H$. Moreover, there exists a constant $K_1 > 0$ such that $\|H - I\| < K_1$. Also, for any $m \in \mathbb{Z}^n$ we have that $H(x + m) = H(x) + m$.*

Proof.

Since $\tilde{f} = \widetilde{E} + p$ and p is bounded, fixing $K > \|p\|$, every full orbit $\{x_k\}_{k\in\mathbb{Z}}$ for $\tilde{f}$ is a $K-$pseudo-orbit for $\widetilde{E}$. In fact, given x in $\mathbb{R}^n$, we may associate a full orbit $\{x_k\}_{k\in\mathbb{Z}}$ for $\tilde{f}$, this means that $\tilde{f}(x_k) = x_{k+1}$ and $x_0 = x$, then for every $k \in \mathbb{Z}$ follows that

$$\begin{aligned} \|\widetilde{E}(x_k) - x_{k+1}\| &= \|\widetilde{E}(\tilde{f}^k(x_0)) - \tilde{f}^{k+1}(x_0)\| = \|(\widetilde{E} - \tilde{f})(\tilde{f}^k(x_0))\| \\ &\leq \|\widetilde{E} - \tilde{f}\| = \|p\|. \end{aligned}$$

Note that $\widetilde{E}$ is an expanding map and linear, therefore it has global product structure and in consequence the (endomorphism version) shadowing lemma holds for any pseudo orbit of $\widetilde{E}$; that is, for any $K > 0$, there is $K_1 = K_1(K) > 0$ such that for any pseudo orbit, $\|\widetilde{E}(x_k) - x_{k+1}\| < K$ for every $k \in \mathbb{Z}$, there exists a unique $y \in \mathbb{R}^n$ such that y K_1-shadows $\{x_k\}_{k\in\mathbb{Z}}$. Then for every x there is a unique point $y = H(x)$ such that

$$\|\widetilde{E}^k(H(x)) - \tilde{f}^k(x)\| < K_1, \quad \forall\, k \in \mathbb{Z}. \tag{2.1}$$

As a consequence of the shadowing lemma, the map $H : \mathbb{R}^n \to \mathbb{R}^n$ is well defined and continuous. Furthermore, $H \circ \tilde{f} = \widetilde{E} \circ H$ and $\|H - I\| < K_1$. Hence H is onto.

Now, given $m \in \mathbb{Z}^n$ we have that

$$\begin{aligned} \|\widetilde{E}^k(H(x) + m) - \tilde{f}^k(x + m)\| &= \|\widetilde{E}^k(H(x)) + \widetilde{E}^k(m) - [\widetilde{E}^k(x + m) + p_k(x + m)]\| \\ &= \|\widetilde{E}^k(H(x)) - [\widetilde{E}^k(x) + p_k(x)]\| \\ &= \|\widetilde{E}^k(H(x)) - \tilde{f}^k(x)\| < K_1, \quad \forall\, k \in \mathbb{Z}. \end{aligned}$$

Then by uniqueness follows that $H(x + m) = H(x) + m$. ∎

Consider $\pi : \mathbb{R}^n \to \mathbb{T}^n$ the canonical projection. Let us define $h : \mathbb{T}^n \to \mathbb{T}^n$ by $h(\pi(x)) = \pi(H(x))$. Hence, we have the following

Corollary 2 *There exists a semiconjugation between f and $\mathcal{E}$, i.e. there exists a map $h : \mathbb{T}^n \to \mathbb{T}^n$ continuous, onto and $h \circ f = \mathcal{E} \circ h$.*

Remark 2.3 The semiconjugation h has the following properties:

1. f transitive implies $int(h^{-1}(x)) = \emptyset$ for every $x \in \mathbb{T}^n$. In fact, let us suppose that there exists a point x such that the interior of $h^{-1}(x)$ is not empty, call $U = int(h^{-1}(x))$, so $h(U) = x$. Since f is transitive, there is $n > 0$ such that $f^n(U) \cap U$ is nonempty. Then $h(f^n(U)) = \mathcal{E}^n(h(U)) = \mathcal{E}^n(x)$, hence $h(f^n(U) \cap U) = x = \mathcal{E}^n(x)$. Therefore, x is periodic and U is an open periodic set for f, but this is not possible since f is transitive.

2. $h^{-1}(x)$ is connected. Let $\pi(\tilde{x}) = x$, by the construction of h, $h^{-1}(x)$ is connected if and only if $H^{-1}(\tilde{x})$ is connected. So, let us suppose that $H^{-1}(\tilde{x})$ is not connected, then there exist $\tilde{z}, \tilde{y} \in H^{-1}(\tilde{x})$ in two different connected components, i.e. there exists L hyperplane dividing $\mathbb{R}^n$ in two component such that $\tilde{z}$ and $\tilde{y}$ are in different components. Since $H(\tilde{z}) = H(\tilde{y}) = \tilde{x}$, we have that $\widetilde{E}^n(H(\tilde{z})) = \widetilde{E}^n(H(\widetilde{y})) = \widetilde{E}^n(\widetilde{x})$, for every $n \geq 0$, then $H(\tilde{f}^n(\tilde{z})) = H(\widetilde{f}^n(\tilde{y}))$. Hence, by (2.1), $\|\tilde{f}^n(\tilde{z}) - \tilde{f}^n(\tilde{y})\| \leq \|\tilde{f}^n(\tilde{z}) - H(\tilde{f}^n(\tilde{z}))\| + \|H(\widetilde{f}^n(\tilde{y})) - \tilde{f}^n(\tilde{y})\| < 2K_1$. On the other hand, we have that for any $\tilde{w} \in L$, $H(\tilde{w}) \neq \tilde{x}$, then $\|H(\tilde{f}^n(\tilde{w})) - H(\tilde{f}^n(\tilde{z}))\| = \|\widetilde{E}^n(H(\tilde{w})) - \widetilde{E}^n(H(\tilde{z}))\| \geq \lambda^n \|H(\tilde{w}) - H(\tilde{z})\|$, with $\lambda > 1$. Since $\|\tilde{f}^n(\tilde{w}) - \tilde{f}^n(\tilde{z})\| \geq \|H(\tilde{f}^n(\tilde{w})) - H(\tilde{f}^n(\tilde{z}))\| - \|H(\widetilde{f}^n(\tilde{w})) - \tilde{f}^n(\tilde{w})\| - \|H(\widetilde{f}^n(\tilde{z})) - \tilde{f}^n(\tilde{z})\| \geq \lambda^n \|H(\tilde{w}) - H(\tilde{z})\| - 2K_1$, we have that the distance between $\tilde{f}^n(\tilde{z})$ and L goes to infinity and the same happen for $\tilde{f}^n(\tilde{y})$. So, the distance between $\tilde{f}^n(\tilde{z})$ and $\tilde{f}^n(\tilde{y})$ cannot be bounded contradicting that $\|\tilde{f}^n(\tilde{z}) - \tilde{f}^n(\tilde{y})\| < 2K_1$. Thus, $H^{-1}(\tilde{x})$ is connected and therefore $h^{-1}(x)$ is connected.

§ 2.2.2 Markov Partition and Transitivity

Let us define Markov Partition for an endomorphism

Definition 2.2 (**Markov partition for endomorphism**)

A *Markov Partition* for an endomorphism $\mathcal{E} : M \to M$ is a family $\mathcal{P} = \{R_1, \ldots, R_N\}$

of compact sets covering M such that:

1. For every $1 \leq i < j \leq N$, $R_i \cap R_j = \partial R_i \cap \partial R_j$.
2. For all $1 \leq i \leq N$, $\mathcal{E} \mid_{int(R_i)}$ is injective.
3. For all $1 \leq i \leq N$, $\overline{int(R_i)} = R_i$.
4. If $\mathcal{E}(R_j) \cap int(R_i) \neq \emptyset$, then $R_i \subset \mathcal{E}(R_j)$.

Remark 2.4 All expanding endomorphism has a Markov Partition.

Let $\mathcal{P}^{\mathcal{E}}$ be a Markov Partition of the linear map $\mathcal{E}$, denote by R_i every element of the partition. Then $\mathbb{T}^n = \bigcup_{i=1}^{N} R_i$. Consider the following family

$$\mathcal{P}^f := \{h^{-1}(R_i) : i = 1, \ldots, N\}.$$

Claim 2.1 $\mathcal{P}^f$ is a Markov Partition for f.

Proof. Let us denote by $\tilde{R}_i = h^{-1}(R_i)$.

• First, observe that $int(R_i) \cap int(R_j) = \emptyset$ if $i \neq j$, so $int(\tilde{R}_i) \cap int(\tilde{R}_j) = h^{-1}(int(R_i)) \cap h^{-1}(int(R_j)) = \emptyset$. Then $\tilde{R}_i \cap \tilde{R}_j = \partial \tilde{R}_i \cap \partial \tilde{R}_j$.

• $f \mid_{int(\tilde{R}_i)}$ is a diffeomorphism. Let us suppose that $f \mid_{int(\tilde{R}_i)}$ is not an homeomorphism. So, take $x, y \in int(\tilde{R}_i)$ such that $f(x) = f(y)$. Note that $h(x)$ and $h(y)$ belong to $int(R_i)$, where $\mathcal{E}$ restricted to it is injective, then $\mathcal{E} \circ h(x) = h \circ f(x) = h \circ f(y) = \mathcal{E} \circ h(y)$ imply that $h(x) = h(y)$. Pick a curve γ in $int(\tilde{R}_i)$ from x to y, then $f(\gamma)$ is a curve with homology, and therefore $h(f(\gamma))$ has homology also. On the other hand, $h(\gamma)$ does not have homology and $\mathcal{E} \mid_{int(R_i)}$ is an homeomorphism, then $\mathcal{E}(h(\gamma)) = h(f(\gamma))$ does not have homology as well.

• $\overline{int(\tilde{R}_i)} = \tilde{R}_i$ holds for all $1 \leq i \leq N$.

• If $f(\tilde{R}_j) \cap int(\tilde{R}_i) \neq \emptyset$, then $\tilde{R}_i \subset f(\tilde{R}_j)$. Suppose that $y \in f(\tilde{R}_j) \cap int(\tilde{R}_i)$, therefore $h(y) = h \circ f(\bar{y}) = \mathcal{E} \circ h(\bar{y})$ belongs to $\mathcal{E}(R_j) \cap int(R_i)$, where $\bar{y} \in \tilde{R}_j$ is such that $f(\bar{y}) = y$. Then holds that $R_i \subset \mathcal{E}(R_j)$. Given $\bar{z} \in \tilde{R}_i$, there exist $z \in R_i$ and $\hat{z} \in R_j$ such that $\bar{z} \in h^{-1}(z)$ and $\mathcal{E}(\hat{z}) = z$. Then $\bar{z} \in h^{-1} \circ \mathcal{E}(\hat{z}) = f \circ h^{-1}(\hat{z})$, i.e. $\bar{z} \in f(\tilde{R}_j)$. ∎

For what follows, we study the refinement of the partition of f induced by the semiconjugation and see what information can be deduced from it, such as under which condition on the refinement can be concluded that h is an homeomorphism. Moreover, we give necessary and sufficient conditions on the refinement that characterize the transitivity of f.

Given a point $x \in \mathbb{T}^n$, denote by $\mathcal{P}_0(x)$ the element of the partition $\mathcal{P}^f$ that contains x. Let us consider a refinement of the partition,

$$\mathcal{P}_n^f(x) = \{y : \; f^k(y) \in \mathcal{P}_0(f^k(x)), \; k = 0, \ldots, n\}.$$

Claim 2.2 $\mathcal{P}_n^f(x) = h^{-1}(\mathcal{P}_n^{\mathcal{E}}(h(x)))$ for every $n \in \mathbb{N}$.

Proof. First observe that $\mathcal{P}_n^{\mathcal{E}}(h(x)) = \{z : \mathcal{E}^k(z) \in \mathcal{R}_0(\mathcal{E}^k(h(x))), \; k = 0, \ldots, n\}$, where $\mathcal{R}_0(w)$ denote the element of the partition for $\mathcal{E}$ that contains w.

Let us show first that $h^{-1}(\mathcal{P}_n^{\mathcal{E}}(h(x))) \subset \mathcal{P}_n^f(x)$. Given $z \in \mathcal{P}_n^{\mathcal{E}}(h(x))$, we have that $f^k \circ h^{-1}(z) = h^{-1} \circ \mathcal{E}^k(z) \in h^{-1}(\mathcal{R}_0(\mathcal{E}^k(h(x))))$. Since $\mathcal{E}^k(h(x)) = h(f^k(x))$, $f^k(h^{-1}(z)) \in \mathcal{P}_0(f^k(x))$.

Now, let us prove the equality. Given $y \in \mathcal{P}_n^f(x)$ follows that $\mathcal{E}^k(h(y)) = h(f^k(y)) \in h(\mathcal{P}_0(f^k(x)))$. Therefore, $\mathcal{E}^k(h(y)) \in \mathcal{R}_0(h(f^k(x))) = \mathcal{R}_0(\mathcal{E}^k(h(x)))$. Thus, the equality holds. ∎

Let us define

$$\mathcal{P}^f_\infty(x) := \bigcap_{n\geq 0} \mathcal{P}^f_n(x).$$

Claim 2.3 *$\mathcal{P}^f_n(x)$ is a compact connected set for every n. Moreover, $\mathcal{P}^f_\infty(x)$ is a compact connected set.*

Proof. First observe that

$$\mathcal{P}^f_n(x) = \mathcal{P}_0(x) \cap f^{-1}(\mathcal{P}_0(f(x))) \cap \ldots \cap f^{-n}(\mathcal{P}_0(f^n(x)))$$

and $f^{-1}(\tilde{R}_j) \cap \tilde{R}_i$ is a compact connected set contained in $\tilde{R}_i$ for every $1 \leq i, j \leq N$. Then, we get that $\mathcal{P}^f_1(x)$ is a compact connected set. Let us assume that $\mathcal{P}^f_{n-1}(x)$ is compact and connected, then $\mathcal{P}^f_n(x) = \mathcal{P}^f_{n-1}(x) \cap f^{-n}(\mathcal{P}_0(f^n(x)))$ is a compact connected set, because $f^{-n}(\mathcal{P}_0(f^n(x)))$ has one connected component in $\mathcal{P}^f_{n-1}(x)$ and it is compact as well.

Now, since $\{\mathcal{P}^f_n(x)\}_n$ is a nested sequence of compact and connected sets, we have that $\mathcal{P}^f_\infty(x)$ is a compact connected set. ∎

Claim 2.4 *h is a conjugacy if and only if for every $x \in \mathbb{T}^n$, $\mathcal{P}^f_\infty(x) = \{x\}$.*

Proof. Take $z \in \mathcal{P}^f_\infty(x)$ and suppose that z and x are not equals. Since h is a conjugacy, $h(x) \neq h(z)$. Because $\mathcal{E}$ is an expanding endomorphism, let us call $\lambda_0 > 1$ the expanding constant, $d(\mathcal{E}^k(h(x)), \mathcal{E}^k(h(z))) > \lambda_0^k d(h(x), h(z))$. Then, there exists $k_0 > 0$ such that $\mathcal{E}^k(h(z)) \notin \mathcal{R}_0(\mathcal{E}^k(h(x)))$ for all $k \geq k_0$. On the other hand, $z \in \mathcal{P}^f_\infty(x)$ implies that $h(z) \in \mathcal{P}^{\mathcal{E}}_n(h(x))$, consequently $\mathcal{E}^k(h(z)) \in \mathcal{R}_0(\mathcal{E}^k(h(x)))$ for all $k \geq 0$. Thus, $x = z$.

Now let us assume that $\mathcal{P}^f_\infty(x) = \{x\}$. We have to prove that h is one-to-one. Consider x and y such that $h(x) = h(y)$. Then $h(y) \in \mathcal{P}^{\mathcal{E}}_\infty(h(x))$. Thus, $y \in h^{-1}(\mathcal{P}^{\mathcal{E}}_\infty(h(x))) = \mathcal{P}^f_\infty(x)$. ∎

Question 2.1 How can transitivity be recovered from the partition?

Lemma 8 *If the set of points $\{x : \mathcal{P}^f_\infty(x) = \{x\}\}$ is dense in $\mathbb{T}^n$, then f is transitive.*

Proof. Let us suppose that $\{x : \mathcal{P}^f_\infty(x) = \{x\}\}$ is a dense set. So, it is enough to show that given x in that set, $\mathcal{P}_0(x)$ the element of the partition that contain it for finitely many iterates by f cover $\mathbb{T}^n$. Observe that $f^n(\mathcal{P}^f_n(x)) = \mathcal{P}_0(f^n(x))$. Then $f^{n+1}(\mathcal{P}^f_n(x)) = f(\mathcal{P}_0(f^n(x))) = \mathbb{T}^n$. ∎

Remark 2.5 Actually, if the set of points $\{x : \mathcal{P}^f_\infty(x) = \{x\}\}$ is dense in $\mathbb{T}^n$, then f is locally eventually onto. In fact, given V an open set, there exists $z \in \{x : \mathcal{P}^f_\infty(x) = \{x\}\} \cap V$. So, $\mathcal{P}_0(z) \cap V \neq \emptyset$. Taking n large enough we get that $\mathcal{P}^f_n(z) \subset V$. Therefore, $f^{n+1}(V) = \mathbb{T}^n$.

Question 2.2 In the case that f is isotopic to a linear expanding endomorphism $\mathcal{E}$. If f is transitive, does it imply that the pre-orbit of any point by f are dense? does it imply that the set $\{x : \mathcal{P}^f_\infty(x) = \{x\}\}$ is dense in $\mathbb{T}^n$? Do they have total (Lebesgue) measure?

Question 2.3 Knowing that there exists an expanding locally maximal set Λ with certain geometrical properties such as the ones given by the Main Theorem. What can it be said about the Markov Partition?

Definition 2.3 (**Physical measure**)

An invariant Borel probability measure μ for a dynamical system $f : M \to M$ is said to be a *physical or Sinai-Ruelle-Bowen(SRB) measure* if its basin of attraction,

$$B(\mu) = B(\mu; f) := \{z \in M : \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(z)} \to \mu \text{ weakly* as } n \to \infty\},$$

has positive Lebesgue measure in M.

Question 2.4 Does there exist physical measures? If there exists, is it unique?

Question 2.5 Since f is semiconjugated to $\mathcal{E}$, by the Bowen's formula, see [Bow71], we get that the topological entropy of f is bounded by below by the topological entropy of the linear map. There are examples, such as Derived from an expanding linear endomorphism where the central direction is one dimensional, the topological entropy of the original map and the semiconjugated are equal. Are there examples such as the topological entropy of the semiconjugated map is greater than the initial map's entropy?

Remark 2.6 The linear expanding maps preserve volume, they are Lebesgue invariant.

So, this make us ask the following:

Question 2.6 Is it possible to construct a map f satisfying the hypotheses of the Main Theorem and being volume preserving?

Remark 2.7 If f satisfy the hypotheses of the Main Theorem and assuming that f is C^2 and volume preserving, then f is ergodic respect to Lebesgue measure.

Chapter 3

Existence of Robust Transitive Endomorphisms

In this chapter we show that there exist examples of robust transitive endomorphisms verifying the hypotheses of our main results.

§ 3.1 Example 1: Applying Main Theorem

Consider $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ an expanding endomorphism, with $n \geq 2$. Note that taking a large $m > 1$, $\mathcal{E}^m$ has the topological degree to the power $m-th$ elements in the Markov Partition, so without loss of generality we may assume that the initial map has many elements in the partition. More precisely, if $N = deg(\mathcal{E})$, we may assume that N is large and therefore the Markov partition has N elements. Denote by R_i the elements of the partition, with $1 \leq i \leq N$; R_i is closed, $int(R_i)$ is nonempty and $int(R_i) \cap int(R_j) = \emptyset$ if $i \neq j$.

Now, consider $\psi : \mathbb{T}^n \to \mathbb{T}^n$ a map isotopic to the identity and denote by $\widehat{R}_i = \psi(R_i)$ for every i. The idea of using this map is to deform the elements of the Markov partition and get a new partition which elements are not all of the same size, there could be some very small, others very big.

Set U_0 an open set in $\mathbb{T}^n$ such that if $\widetilde{U}$ is the convex hull of the lift of U_0, then $\widetilde{U} \cap [0,1]^n$ is contained in the interior of $[0,1]^n$, i.e. $diam_{ext}(U_0) < 1$. Note that there exist $\widehat{R}_i$ such that $\widehat{R}_i \cap U_0$ is nonempty. We ask one more condition for U_0, there are many $\widehat{R}_i$ contained in U_0^c, this condition is feasible since we asked for the initial map to have many elements in the partition.

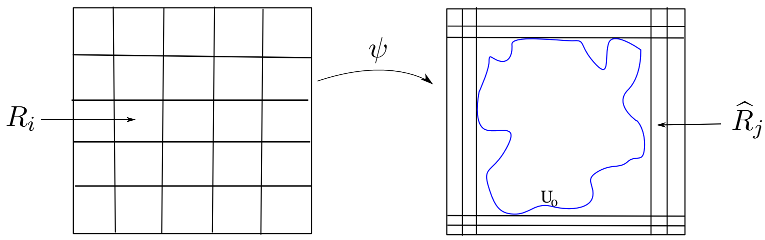


Figure 3.1: *Deforming the initial Markov Partition*

Define $f_0 : \mathbb{T}^n \to \mathbb{T}^n$ by $f_0 = \psi \circ \mathcal{E}$. We assume that there exist $p \in U_0$ and $q_i \in U_0^c$ fixed points for f_0, with $1 \leq i \leq n-1$, this is possible because we may start with an expanding map which has as many fixed points as we need.

Let us suppose that p and q_i are expanding by f_0 in all directions, it means that all the eigenvalues associated to these points are in modulus greater than 1. Pick $\varepsilon > 0$ small enough such that $\mathbb{B}_\varepsilon(q_i) \cap U_0 = \emptyset$ and $\mathbb{B}_\varepsilon(q_i) \cap \mathbb{B}_\varepsilon(q_j) = \emptyset$ if $i \neq j$.

Let us denote the decomposition of the tangent space as follows

$$T_x(\mathbb{T}^n) = \mathbb{E}_1^u \prec \mathbb{E}_2^u \prec \cdots \prec \mathbb{E}_{n-1}^u \prec \mathbb{E}_n^u,$$

where $\prec$ denotes that $\mathbb{E}_i^u$ dominates the expanding behavior of $\mathbb{E}_{i-1}^u$.

Next we deform f_0 by a smooth isotopy supported in $U_0 \cup (\bigcup \mathbb{B}_\varepsilon(q_i))$ in such a way that:

1. The continuation of p goes through a pitchfork bifurcation, appearing two periodic points r_1, r_2, such that both are repeller and p becomes a saddle point. But the new map f still expand volume in U_0.

2. Two expanding eigenvalues of q_i become complex expanding eigenvalues. More precisely, we mix the two expanding subbundles of $T_{q_i}(\mathbb{T}^n)$ corresponding to $\mathbb{E}^u_i(q_i)$ and $\mathbb{E}^u_{i+1}(q_i)$, obtaining $T_{q_i}(\mathbb{T}^n) = \mathbb{E}^u_1 \prec \mathbb{E}^u_2 \prec \cdots \prec \mathbb{F}^u_i \prec \mathbb{E}^u_{n-1} \prec \mathbb{E}^u_n$, where $\mathbb{F}_i$ is two dimensional and correspond to the complex eigenvalues.

3. Outside $U_0 \cup (\bigcup \mathbb{B}_\varepsilon(q_i))$, f coincides with f_0.

4. f is expanding in U_0^c.

5. There exists $\sigma > 1$ such that $|det(Df(x))| > \sigma$ for every $x \in \mathbb{T}^n$.

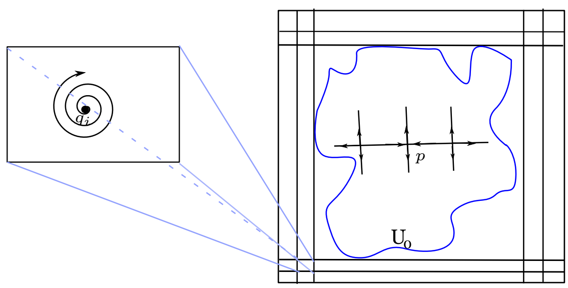


Figure 3.2: f *isotopic to* f_0

§ 3.1.1 Property of Large Arcs

Claim 3.1 Every large arc in U_0^c has a point such that its forward orbits remain in U_0^c.

Proof. Take d the maximum of the diameter of the elements of the partition contained in U_0^c. Note that every arc in U_0^c with diameter larger than d cannot be contained in the interior of any element of the partition, more precisely has to intersect at least two elements of the partition. Hence, the image by f of this arc γ has diameter 1. So there exists a piece of $f(\gamma)$ in U_0^c intersecting at least one element of the partition across two parallel sides, let us call γ^1. Choose a pre-image of γ^1 in γ and call it γ_1.

Repeating the process for γ^1, we have that there is γ^2 a piece of $f(\gamma^1)$ verifying the same condition as γ^1. Then, choose γ_2 a pre-image of γ_2 by f^2 in γ.

Thus, we construct a sequence of nested arcs in γ. The intersection is non empty, a point in this intersection satisfy our claim. ∎

§ 3.1.2 Remarks About Example 1

1. q_i's are fixed points for f with complex expanding eigenvalues. Note that the existence of these points ensures that the tangent bundle does not admit any invariant subbundle. We could also start with an expanding map having, besides p, periodic points q_i with complex eigenvalues. In such a case, it is enough to make p goes through a pitchfork bifurcation.

2. This example shows that U_0 can be as big as we desired while it verifies the hypothesis of having external diameter less than 1.

3. It can be constructed in any dimension.

§ 3.2 Example 2: Applying the Main Theorem Revisited

Let us consider $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ an expanding endomorphism, with $n \geq 2$. Assume that the initial map has many elements in the Markov partition, let us say N elements.

Denote by R_i the elements of the partition, with $1 \leq i \leq N$. Since $\mathcal{E}$ is expanding, R_i

are closed, $int(R_i)$ are nonempty and $int(R_i) \cap int(R_j) = \emptyset$ if $i \neq j$. Choose finitely many of these elements, $\{R_{i_j}\}_{j=1}^{k}$, such that $R_{i_j} \cap R_{i_s} = \emptyset$ if $i_j \neq i_s$, i.e. they are two by two disjoints. Consider the pre-images of every R_{i_j}, let us say $\mathcal{E}^{-1}(R_{i_j}) = \{P_{i_j}^{l}\}_{l=1}^{N}$. Denote by $P_{i_j}^{0} = R_{i_j}$. Next, we keep $P_{i_j}^{r}$ such that $P_{i_j}^{r} \cap P_{i_s}^{l} = \emptyset$ whenever $0 \leq r \neq l \leq N$ and $i_j \neq i_s$. Finally, let us denote by $\{P_i\}_i$ the collection of these latter subsets, so they are two by two disjoints.

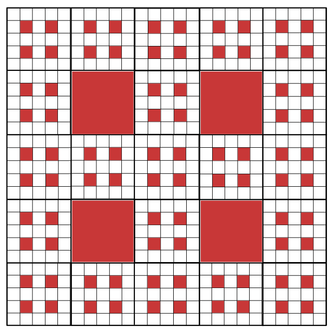
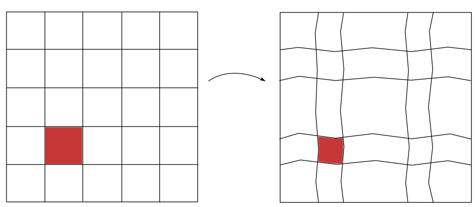


Figure 3.3: $\{P_i\}_i$ *collection*

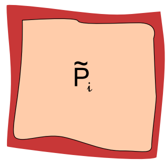
Now, consider $\psi : \mathbb{T}^n \to \mathbb{T}^n$ a map isotopic to the identity and denote by $\widehat{P}_i = \psi(P_i)$ for every i.



Choose $\widetilde{P}_i$ an open connected subset such that its closure is contained in the interior of $\widehat{P}_i$. Let $\phi_i : \mathbb{T}^n \to \mathbb{T}^n$ be a map isotopic to the identity such that

- $\phi_i \mid_{\widetilde{P}_i}$ is not expanding.

- $\phi_i \mid_{\widehat{P}_i^c}$ is the identity.



Define $\phi : \mathbb{T}^n \to \mathbb{T}^n$ by

$$\phi(x) = \begin{cases} \phi_i(x), & \text{if } x \in \widehat{P}_i \\ x, & \text{if } x \notin \bigcup_i \widehat{P}_i \end{cases}$$

Hence, ϕ is equal to the identity in $[\bigcup_i \widehat{P}_i]^c$, expands volume but is not expanding in $\bigcup_i \widehat{P}_i$.

Once we have defined all these maps, we consider the map $f = \phi \circ \psi \circ \mathcal{E}$ from $\mathbb{T}^n$ onto itself and denote by $U_0 = int(\bigcup_i \widehat{P}_i)$. Observe that f verifies that:

(i) f is a volume expanding endomorphism.

(ii) f is an expanding map in U_0^c.

(iii) $\Lambda_f = \bigcap_{n\geq 0} f^{-n}(U_0^c)$ is an expanding locally maximal set for f which has the property that separate large nice cylinders.

Since (i) and (ii) are immediate from the construction of f, we concentrate our interest in to prove (iii).

§ 3.2.1 Λ_f Separates Large Nice Cylinders

Note that by the construction of U_0, we have that the elements of the pre-orbit of U_0 are two by two disjoints. Let us consider $d_0 = \max\{diam_{ext}(c.c. \bigcup_{n\geq 0} f^{-n}(U_0))\}$. Since the definition of U_0, $0 < d_0 < 1$.

Claim 3.2 If γ is an arc in U_0^c with diameter 1, then γ intersects Λ_f.

Proof. Let γ be an arc in U_0^c such that $diam(\gamma) = 1$. Suppose that γ does not intersect Λ_f.

Remember that $\Lambda_f = \mathbb{T}^n \setminus \bigcup_{n\geq 0} f^{-n}(U_0)$, it means that if $x \in \Lambda_f$, then $f^n(x) \notin U_0$. Therefore, γ is contained in one pre-image of U_0 or in a union of pre-images of U_0.

Observe that γ cannot be contained in just one pre-image of U_0, because if it is contained in $f^{-n}(U_0)$ for some $n \geq 0$, then $diam(\gamma) < diam(f^{-n}(U_0)) < d_0$, which is absurd because $d_0 < 1$.

Hence, γ should be contained in a union of pre-images of U_0, since γ is compact we can cover with a finite union of pre-images of U_0. But we know that the pre-images of U_0 are two by two disjoints, hence there exist points in γ that cannot be covers by the pre-images of U_0. In particular, γ intersects Λ_f. ∎

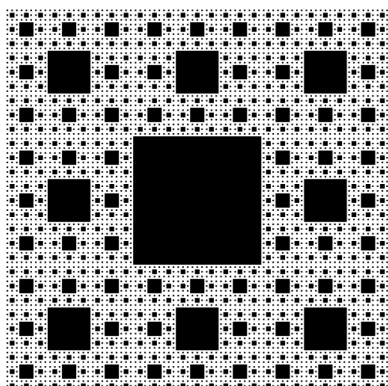


Figure 3.4: Λ_f *looks like a Sierpinski set*

Remark 3.1 We have already the existence of the invariant expanding locally maximal set Λ_f. Moreover, by claim (3.2) we get that this invariant set intersects every arc with large diameter. Then by the Main Theorem Revisited follows that this map is robust transitive.

§ 3.2.2 Remarks About Example 2

1. We can apply our Main Theorem Revisited to this example, obtaining in particular that f is robustly transitive.

2. The $\widehat{P}_i$'s can be as many and as big as we want.

3. We can construct many examples starting with this initial map. In particular, we can construct examples without invariant subbundles, such as putting a fix point in the complement of the U_0 with complex eigenvalues and doing a derived from an expanding endomorphisms inside of some $\widehat{P}_i$.

§ 3.3 Example 3: Applying Theorem 2

The idea of next example is to build an endomorphism in the 2-Torus which is a skew-product and in the dynamic there is a blender. This example is more or less a standard adaptation for endomorphisms of examples obtained in [BD96] for diffeomorphisms.

First, let us establish some notation before defining the map. Pick $0 < a < 1/2 < b < 1$ and denote $I_{12} = [a, 1]$ and $I_{34} = [0, b]$. Note that $I_{12} \cap I_{34} = [a, b]$. This decomposition is associated to the horizontal fibers.

Next, fix $N > 3$ and pick $0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < a_4 < b_4 < 1$ such that $b_i - a_i = 1/N$. Let us denote by $I_i = [a_i, b_i]$ with $1 \leq i \leq 4$. Note that they are two by two disjoint and do not contain 0 or 1. We associate this decomposition to the vertical fibers.

Let us call $R_i = I_{12} \times I_i$ with $i = 1, 2$, and $R_i = I_{34} \times I_i$ with $i = 3, 4$.

Now, define $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$\Phi(x, y) = (\varphi_y(x), \mathcal{E}(y)),$$

where $\varphi_y, \mathcal{E} : S^1 \to S^1$ are defined as follows:

1. $\mathcal{E}$ is an expanding endomorphism such that:
   - $\mathcal{E}(I_i) = [0, 1]$ for every i.
   - There exist $a_i < c_i < b_i$ such that $\mathcal{E}(c_i) = c_i$.
2. φ_y is defined by $\varphi_y(x) = f_i(x)$, if $y \in I_i$, where $f_i : S^1 \to S^1$ are differentiable maps defined as follows:
   - f_1 and f_2 satisfy the following properties:
     - (i) f_1 has two fixed points, 0 and a. 0 is a repeller and a is an attractor for f_1.
     - (ii) f_2 has two fixed points, 0 and a', where $0 < a' < a$. 0 is an attractor and a' is a repeller for f_2.
     - (iii) $f_1(I_{12})$ overlaps $f_2(I_{12})$ and $f_1(I_{12}) \cup f_2(I_{12}) = I_{12}$.
     - (iv) $|f_1' \,|_{I_{12}}| < 1$ and $|f_2' \,|_{I_{12}}| < 1$.
   - f_3 and f_4 satisfy the following properties:
     - (i') f_3 has two fixed points, 0 and b', where $b < b' < 1$. 0 is a repeller and b' is an attractor for f_3.
     - (ii') f_4 has two fixed points, b and 1. b is an attractor and 1 is a repeller for f_4.
     - (iii') $f_3(I_{34})$ overlaps $f_4(I_{34})$ and $f_3(I_{34}) \cup f_4(I_{34}) = I_{34}$.
     - (iv') $|f_3' \,|_{I_{34}}| < 1$ and $|f_4' \,|_{I_{34}}| < 1$.
3. $|det(Df)| = |\frac{\partial \varphi_y}{\partial x} \mathcal{E}'| > 1$.
4. $\mathcal{E}' \gg \frac{\partial \varphi_y}{\partial y}$.

Hence, the horizontal fibers $F_i = S^1 \times c_i$ are invariant by f. Moreover, by condition (4), the image of every vertical fiber is almost a vertical fiber, in the sense that the tangent vector is close to a vertical one; more precisely, the unstable cones family are almost vertical.

Next, we consider $\Lambda_1^+ = \bigcap_{n\geq 0} \Phi^{-n}(R_1 \cup R_2)$ and $\Lambda_2^+ = \bigcap_{n\geq 0} \Phi^{-n}(R_3 \cup R_4)$. Let $\Lambda_1 = \bigcap_{n\in\mathbb{Z}} \Phi^{-n}(R_1 \cup R_2)$ and $\Lambda_2 = \bigcap_{n\in\mathbb{Z}} \Phi^{-n}(R_3 \cup R_4)$, note that both sets are expanding invariant locally maximal sets.

§ 3.3.1 Λ_1 and Λ_2 Separate Large Vertical Segments

Let us denote by $\ell_1^u(p)$ the vertical segment passing through p and length 1.

Claim 3.3 For every $p \in R_1 \cup R_2$, follows that $\ell_1^u(p) \cap \Lambda_1^+ \neq \emptyset$.

Proof. Let us suppose that $p \in R_1 \cup R_2$. Note that for $i = 1, 2$, the image of $L_i = \ell_1^u(p) \cap R_i$ by Φ has length 1 and by property (4) of Φ follows that $\Phi(L_i)$ is almost vertical. Moreover, $L_i \cap F_i \neq \emptyset$ and $\Phi(L_i \cap F_i) \in F_i \subset R_i$ with $i = 1, 2$. Then $\Phi(\ell_1^u(p)) \cap (R_1 \cup R_2) \neq \emptyset$. Take K_1^i one connected component of $\Phi(\ell_1^u(p)) \cap R_i$, for $i = 1, 2$, such that $P_2(K_1^i) = I_i$, where P_2 is the projection in the second coordinate. Consider the pre-image of K_1^i by Φ in L_i and call it S_1^i.

Now, iterate K_1^i by Φ, doing a similar process we obtain K_2^i a connected component of $\Phi(K_1^i) \cap R_i$ such that $P_2(K_2^i) = I_i$. Again take a pre-image of K_2^i by Φ^2, giving a compact segment $S_2^i \subset S_1^i$. Repeating this process, we may construct a nested sequence of compact segment $\{S_k^i\}_k$ in each R_i. Thus, $\bigcap_k S_k^i$ is not empty and belong to $\ell_1^u(p) \cap \Lambda_1^+$. ∎

Claim 3.4 For every $p \in R_1 \cup R_2$, follows that $\ell_1^u(p) \cap \Lambda_1 \neq \emptyset$.

Proof. By claim (3.3), we know that there exist a point $z \in \ell_1^u(p) \cap \Lambda_1^+$, this means that $\Phi^n(z) \in R_1 \cup R_2$ for every $n \geq 0$.

Then, just remain to show that there exist a sequence $\{z_k\}_{k\geq 0} \subset R_1 \cup R_2$ such that $z_0 = z$ and $\Phi(z_k) = z_{k-1}$. The idea of the construction of such a sequence is to use now the property (2-*iii*) of overlapping in the horizontal dynamics.

Knowing that $\Phi(R_1) = f_1(I_{12}) \times [0,1]$ and $\Phi(R_2) = f_2(I_{12}) \times [0,1]$, since property (2-*iii*) we get that $\Phi(R_1) \cap \Phi(R_2) = f_1(I_{12}) \times [0,1]$. Hence, $z_0 \in (R_1 \cup R_2) \cap \Phi(R_1)$ or $z_0 \in (R_1 \cup R_2) \cap \Phi(R_2)$, then there exists $z_1 \in R_1 \cup R_2$ such that $\Phi(z_1) = z_0$. Repeating this process we construct the requires sequence. ∎

Claim 3.5 For every $p \in R_3 \cup R_4$, follows that $\ell_1^u(p) \cap \Lambda_2 \neq \emptyset$.

Proof. The proof is similar to claim (3.4) just making the necessary arrangement. ∎

Claim 3.6 For every $q \in \mathbb{T}^2$, we have that either $\ell_1^u(q) \cap \Lambda_1 \neq \emptyset$ or $\ell_1^u(q) \cap \Lambda_2 \neq \emptyset$.

Proof. Given any point $q \in \mathbb{T}^2$, note that $\ell_1^u(q) \cap R_i \neq \emptyset$ for every $1 \leq i \leq 4$. Hence, taking $p_i \in \ell_1^u(q) \cap R_i$ and noting that $\ell_1^u(p_i) = \ell_1^u(q)$, we may use claim (3.4) or (3.5) to conclude that either $\ell_1^u(q) \cap \Lambda_1 \neq \emptyset$ or $\ell_1^u(q) \cap \Lambda_2 \neq \emptyset$. ∎

§ 3.3.2 Remarks About Example 3

This example was constructed in the 2-Torus with one dimensional central bundle, but we can construct it in any $\mathbb{T}^n$ and the dimension of the central bundle not need to be 1. Also, we can use more than 4 dynamics in the horizontal, more precisely we put 2 blenders in the dynamic but we can consider as many blenders as we want.

§ 3.4 Example 4: Applying Theorem 2

Let $\mathbb{B}_0$ be an open ball in $\mathbb{T}^m$ centered at 0 with radius $\alpha < 1$ and $\varphi_0 : \mathbb{T}^m \to \mathbb{T}^m$ be a differentiable map isotopic to the identity such that:

- $\varphi_0(0) = 0$
- There exist $0 < \lambda_0 < \lambda_1 < 1$ such that $\lambda_0 < m\{D\varphi_0\} < |D\varphi_0 \,|_{\mathbb{B}_0}\,| < \lambda_1$, i.e. φ_0 is a contraction in a disk.

Let us consider $\mathbb{D}_0$ the lift of $\mathbb{B}_0$ to $\mathbb{R}^m$ and $\widetilde{\varphi}_0$ the lift of φ_0. Note that $\widetilde{\varphi}_0(0) = 0$ and $\lambda_0 < m\{D\widetilde{\varphi}_0\} < |D\widetilde{\varphi}_0 \,|_{\mathbb{D}_0}\,| < \lambda_1$. By Proposition 2.3 of Nassiri's PhD Thesis [Nas06], there exists $k \in \mathbb{N}$ such that for every small $\varepsilon > 0$, there exist $c_1, \dots, c_k \in \mathbb{B}_\varepsilon(0)$ and $\delta > 0$ such that $\mathbb{B}_\delta(0) \subset \overline{Orbit^+_{\mathcal{G}}(0)}$, where $\mathcal{G} = \mathcal{G}(\widetilde{\varphi}_0, \widetilde{\varphi}_0 + c_1, \dots, \widetilde{\varphi}_0 + c_k)$ and $Orbit^+_{\mathcal{G}}(0)$ is the set of points lying on some orbit of 0 under the iterated function system (IFS) $\mathcal{G}$; more precisely, if we denote by $\widetilde{\phi}_0 = \widetilde{\varphi}_0$ and $\widetilde{\phi}_i = \widetilde{\varphi}_0 + c_i$ for $i = 1, \dots, k$, then $Orbit^+_{\mathcal{G}}(0)$ is the set of sequence $\{\widetilde{\phi}_{\Sigma_l}(0)\}_{l=1}^{\infty}$ where $\Sigma_l = (\sigma_1, \dots, \sigma_l)$, $\widetilde{\phi}_{\Sigma_l} = \widetilde{\phi}_{\sigma_l} \circ \cdots \circ \widetilde{\phi}_{\sigma_1}$ and $\{\sigma_i\}_{i\in\mathbb{N}} \in \{0, \dots, k\}^{\mathbb{N}}$. (For more details about IFS see Nassiri's PhD Thesis)

Now choose $p_1, \dots, p_r \in \mathbb{T}^m$ such that $\mathbb{T}^m \subset \bigcup_j \mathbb{B}_\delta(p_j)$.

If ϕ_i is the projection of $\widetilde{\phi}_i$ on $\mathbb{T}^m$, define for every j the IFS $\mathcal{G}_j = \mathcal{G}_j(\phi_0 + p_j, \phi_1 + p_j, \dots, \phi_k + p_j)$. Then $\mathbb{B}_\delta(p_j) \subset \overline{Orbit^+_{\mathcal{G}_j}(0)}$. Therefore, there exists an open set $\mathbb{D}_0 \subset \mathbb{B}_0$ such that $\bigcup \phi_i(\mathbb{D}_0) \supset \mathbb{D}_0$, i.e. the IFS has the covering property. Hence, $\bigcup_i \phi_i(\mathbb{B}_{\delta'}(p_j)) \supset \mathbb{B}_{\delta'}(p_j)$, with $0 < \delta' \leq \delta$. Moreover, $\mathcal{G}_j$ has also the overlapping property as in Example 3, in the previous section.

Define the skew-product $F : \mathbb{T}^m \times \mathbb{T}^n \to \mathbb{T}^m \times \mathbb{T}^n$ by

$$F(x, y) = (\psi_y(x), \mathcal{E}(y)),$$

where:

- $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ is an expanding map with $(k+1)r$ fixed points, let us denote the fixed points by $e_1^i, \dots, e_r^i$ with $0 \le i \le k$.
- For every $y \in \mathbb{T}^n$, $\psi_y : \mathbb{T}^m \to \mathbb{T}^m$ is a differentiable map isotopic to the identity such that $\psi_{e_j^i} = \phi_i + p_j$, with $0 \le i \le k$ and $1 \le j \le r$.

Hence, every fiber $\mathbb{T}^m \times \{e_j^i\}$ is invariant by F. Set $R_j^i = \mathbb{B}_{\delta'}(p_j) \times Q_j^i$, where Q_j^i is a small neighborhood of e_j^i in $\mathbb{T}^n$ such that $\mathcal{E}(Q_j^i) = \mathbb{T}^n$ and they are all disjoints for every i, j. Note that R_j^i are the analogous of R_i in the previous example.

Let $\Lambda_F := \bigcap\limits_{n\in\mathbb{Z}} F^n(\bigcup\limits_{i,j} R_j^i)$.

§ 3.4.1 Λ_F Separate Large Unstable Discs

Claim 3.7 Λ_F verifies that for every $z \in \bigcup_{i,j} R_j^i$ follows that $\ell_1^u(z) \cap \Lambda_F \neq \emptyset$, where $\ell_1^u(z)$ is an unstable disc of internal diameter 1 passing through z.

Proof. We may prove that there exists a point $z \in \bigcup_{i,j} R_j^i$ such that $F^n(z) \in \bigcup_{i,j} R_j^i$ for every $n \ge 0$ in a similar way as we proved claim (3.3) in previous example.

Moreover, for this z there exist $z_1 \in \bigcup_{i,j} R_j^i$ such that $F(z_1) = z$. In fact, the idea is more or less the same as in previous example, we must note that $F(R_j^i) = \psi_{e_j^i}(\mathbb{B}_{\delta'}(p_j)) \times \mathcal{E}(Q_j^i) = \phi_i(\mathbb{B}_{\delta'}(p_j)) \times \mathbb{T}^n$.

On the other hand, using the property of covering and overlapping follows that

$$\bigcup_{i,j} R_j^i = \bigcup_{i,j} \mathbb{B}_{\delta'}(p_j) \times Q_j^i \subset \bigcup_{i,j} \phi_i(\mathbb{B}_{\delta'}(p_j)) \times \mathcal{E}(Q_j^i) = F(\bigcup_{i,j} R_j^i).$$

Therefore, since $z \in \bigcup_{i,j} R_j^i$, there exist $z_1 \bigcup_{i,j} R_j^i$ such that $F(z_1) = z$. Inductively we can construct a sequence $\{z_k\}_{k\ge 0} \subset \bigcup_{i,j} R_j^i$ such that $z_0 = z$ and $F(z_k) = z_{k-1}$.

Thus, $z \in \ell_1^u(z) \cap \Lambda_F$. ∎

§ 3.4.2 Remarks About Example 4

This example is a generalization of Example 3. The intention here is to show that we may apply Theorem 2 without taking into account the dimension of the central bundle and this could be as large as we want. Another observation is that the existence of blenders guarantee that our examples are robust transitive and this example verifies the property over the unstable discs with sufficiently large internal diameter intersecting the invariant expanding locally maximal set for the skew-product.

Bibliography

[BD96] Christian Bonatti and Lorenzo J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math. (2)*, 143(2):357–396, 1996.

[BDP03] C. Bonatti, L. J. Díaz, and E. R. Pujals. A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math. (2)*, 158(2):355–418, 2003.

[Bow71] Rufus Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, 153:401–414, 1971.

[BV00] Christian Bonatti and Marcelo Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[Cro02] Sylvain Crovisier. Une remarque sur les ensembles hyperboliques localement maximaux. *C. R. Math. Acad. Sci. Paris*, 334(5):401–404, 2002.

[Fis06] Todd Fisher. Hyperbolic sets that are not locally maximal. *Ergodic Theory Dynam. Systems*, 26(5):1491–1509, 2006.

[Fra71] John Franks. Necessary conditions for stability of diffeomorphisms. *Trans. Amer. Math. Soc.*, 158:301–308, 1971.

[KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[Liu91] Pei Dong Liu. The R-stability of C^1 self-mapping orbit spaces. *Chinese Ann. Math. Ser. A*, 12(4):415–421, 1991.

[Mañ78] Ricardo Mañé. Contributions to the stability conjecture. *Topology*, 17(4):383–396, 1978.

[Nas06] Meysam Nassiri. *Robustly Transitive Sets in Nearly Integrable Hamiltonian Systems*. PhD thesis, IMPA, 2006.

[Pal68] J. Palis. On Morse-Smale dynamical systems. *Topology*, 8:385–404, 1968.

[PS06a] Enrique R. Pujals and Martin Sambarino. Homoclinic bifurcations, dominated splitting, and robust transitivity. In *Handbook of dynamical systems. Vol. 1B*, pages 327–378. Elsevier B. V., Amsterdam, 2006.

[PS06b] Enrique R. Pujals and Martin Sambarino. A sufficient condition for robustly minimal foliations. *Ergodic Theory and Dynamical Systems*, 26(01):281–289, 2006.

[Rob76] Clark Robinson. Structural stability of C^1 diffeomorphisms. *J. Differential Equations*, 22(1):28–73, 1976.

[Shu69] Michael Shub. Endomorphisms of compact differentiable manifolds. *American Journal of Mathematics*, 91(1):175–199, 1969.

[Shu87] Michael Shub. *Global stability of dynamical systems*. Springer-Verlag, New York, 1987. With the collaboration of Albert Fathi and Rémi Langevin, Translated from the French by Joseph Christy.