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SIMONS TYPE EQUATION IN $\mathbb{S}^2 \times \mathbb{R}$ AND $\mathbb{H}^2 \times \mathbb{R}$ AND APPLICATIONS

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Qual é o segredo da perseverança?
O Amor. - Enamora-te, e não O deixarás.
(Caminho, 999)

Abstract

Equations of Simons type are presented. They are satisfied by a pair of special operators associated to the immersion $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ with constant mean curvature. Some immersions are characterized.

Keywords: Simons Type Equation, Constant Mean Curvature, Immersed Surface.

Resumo

Apresentamos equações tipo Simons. Estas são satisfeitas por um par de operadores especiais associados a imersões $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ com curvatura média constante. Utilizando tais equações, caracterizamos algumas imersões.

Palvras-chave: Equações Tipo Simons, Curvatura Média Constante, Superfícies Imersas.

Contents

Introduction

In 1994, using the traceless second fundamental form $\phi = A - HI$ associated to an immersed hypersurface $M^n \looparrowright \mathbb{S}^{n+1}$, H. Alencar and M. do Carmo, see [AdC], proved that

Theorem. *Let $M^n \looparrowright \mathbb{S}^{n+1}$ be an immersed hypersurface. If M^n is compact and orientable with constant mean curvature H and*

$$|\phi|^2 \leq B_H,$$

where B_H is the square of the positive root of

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + 1).$$

Then:

(a) Either $|\phi|^2 = 0$ (and M^n is totally umbilic) or $|\phi|^2 = B_H$.

(b) The $H(r)$-tori $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r^2 < \dfrac{n-1}{n}$ are the only hypersurfaces with constant mean curvature H and $|\phi|^2 = B_H$.

Motivated by this result we decided to study this problem for surfaces in $M^2(c) \times \mathbb{R}$ with $c = \pm 1$, where $M^2(-1) = \mathbb{H}^2$ and $M^2(1) = \mathbb{S}^2$.

We begin by using the traceless second fundamental form ϕ associated to an immersed surface $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ to characterize some immersed surfaces.

Then we study, in chapter 2, a special tensor S defined by

$$SX = 2HAX - c\langle X, T\rangle T + \frac{c}{2}(1-\nu^2)X - 2H^2X, \tag{1}$$

where $X \in T_p\Sigma$, A is the Weingarten operator associated to the second fundamental form, H is the mean curvature, T is the tangential component of the parallel field ∂_t, tangent to $\mathbb{R}$ in $M^2(c) \times \mathbb{R}$, and $\nu = \langle N, \partial_t\rangle$.

This operator satisfies Codazzi's equation, provided H is constant, with vanishing trace, see Proposition 2.3. We remark that any surface with $|S| = 0$ and constant mean curvature is interesting, because the $(2,0)$-part of the quadratic differential Q,

$$Q(X,Y) = 2H\langle AX, Y\rangle - c\langle X, \partial_t\rangle\langle Y, \partial_t\rangle,$$

of these surfaces vanishes. In [*AR*], section 2, Abresch and Rosenberg described four distinct classes of complete, possibly immersed, constant mean curvature surfaces $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ with vanishing (2,0)-part of quadratic differential Q.

More precisely, the four classes are

(*i*) Σ^2 is an embedded rotationally invariant constant mean curvature sphere S^2_H;

(*ii*) Σ^2 is a convex rotationally invariant constant mean curvature graph D^2_H over the horizontal leaf $M^2(c) \times \{t_0\}$;

(*iii*) Σ^2 is an embedded annulus, rotationally invariant constant mean curvature surface C^2_H with two asymptotically conical ends;

(*iv*) Σ^2 is embedded constant mean curvature surface P^2_H; it is an orbit under some two dimensional solvable subgroup of ambient isometries.

The surface in (*i*) was known to W.Y. Hsiang, in [*Hs*], and to R. Pedrosa and M. Ritoré, in [*PR*]. We shall refer to S^2_H as the embedded rotationally invariant constant mean curvature spheres. In this paper we will call these surfaces described in [*AR*] by Abresch-Rosenberg surfaces.

Remark.1: In $\mathbb{S}^2 \times \mathbb{R}$ only the spheres S^2_H occur.

Next, we obtain an equation of Simons type for S and apply it in some particular cases:

Theorem 0.1 (Simons Type Equation for S)**.** *Let* $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ *be an immersed surface with non zero constant mean curvature* H *and* S *as defined in* (1). *Then,*

$$\langle (\nabla^2 S)x, y\rangle = 2cv^2\langle Sx, y\rangle + 2H\langle Ax, Sy\rangle - \langle A^2x, Sy\rangle +$$

$$+\langle Ay, SAx\rangle - \langle Ax, y\rangle tr(AS)$$

and

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5cv^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST, T\rangle\right) +$$

$$+c|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

Let us consider the polynomial $p_H(t) = -t^2 - \dfrac{1}{H}t + \left(\dfrac{4H^2-1}{2}\right)$. When H is greater than one half, p_H has a positive root denoted by L_H; In fact,

$$L_H = \frac{4H^2 - 1}{\sqrt{8H^4 - 2H^2 + 1} - 1}.$$

One has:

Theorem 0.2. *Let* $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ *be an immersed surface with constant mean curvature* H *greater than one half. If*

$$\Sigma^2 \text{ is complete and } \sup_\Sigma |S| < L_H$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq L_H,$$

then $\Sigma^2 = S^2_H$, *i.e,* Σ^2 *is an embedded rotationally invariant constant mean curvature sphere.*

Let us consider the polynomial $q_H(t) = -t^2 - \dfrac{1}{H}t + \left(\dfrac{8H^4-12H^2-1}{4H^2}\right)$. When H is greater than $\sqrt{\dfrac{12+\sqrt{176}}{16}}$, q_H has a positive root denoted by M_H; In fact,

$$M_H = \frac{8H^4-12H^2-1}{2H(\sqrt{8H^4-12H^2-1}+1)}.$$

One has:

Theorem 0.3. *Let $\Sigma^2 \looparrowright \mathbb{H}^2\times\mathbb{R}$ be an immersed surface with constant mean curvature H greater than $\sqrt{\dfrac{12+\sqrt{176}}{16}} \approx 1.25664$. If*

$$\Sigma^2 \textit{ is complete and } \sup_{\Sigma}|S| < M_H$$

or

$$\Sigma^2 \textit{ is closed and } |S| \leq M_H,$$

then $\Sigma^2 = S^2_H$, i.e, Σ^2 is an embedded rotationally invariant constant mean curvature sphere.

Remark.2: Besides Theorems 0.2 and 0.3, we obtain in chapter 3 further applications of Simons equation of Theorem 0.1.

Remark.3: Besides Simons type equation for S, we obtain the Simons type equation for ϕ.

The organization of this thesis is as follows.

In chapter 1 we comment some notation about covariant derivatives of tensors fields, the curvature tensor and operators associated to the Weingarten operator A that will be used throughout the thesis.

We will recall results on the second covariant derivative of the Weingarten operator, see Theorem 2 in [B], and the result known as the Omori-Yau Maximum Principle to complete manifolds, see Theorem 1 in [Y]. Finally we recall the Gauss' equation for the product space $M^2(c)\times\mathbb{R}$.

In chapter 2, we will obtain an equation of Simons type for the traceless second fundamental form ϕ and for S defined in (1). This chapter involves many curvature computations and it is quite technical.

Finally, in chapter 3, we gives several geometric applications based on the results found in chapter 2 together with the Omori-Yau's Theorem.

CHAPTER 1

Preliminaries

Let $\Sigma^2 \looparrowright M^3$ be an immersed surface. Let $\overline{\nabla}$ denote the Levi-Civitá connection on M^3 and let ∇ denote the Levi-Civitá connection on Σ for the induced metric.

Generally speaking, objects defined on M^3 will be denoted by the same symbols as the corresponding objects defined on Σ plus a bar over the symbol.

The Riemannian metric extends to natural inner products on space of tensors and the above connections induce natural covariant derivatives of tensor fields. For example, if $\{e_1, e_2\}$ is a geodesic frame in $p \in \Sigma^2$ and ψ is a tensor on Σ^2, we have

$$\nabla^2\psi(p) = \sum_{i=1}^{2} (\nabla_{e_i}\nabla_{e_i}\psi)(p).$$

For more details about covariant derivatives of tensor fields see [S], sections 1 and 2.

We adopt the following convention for the curvature tensor: if $x, y, z \in T_p\Sigma$, we define $R_{x,y}z$ by

$$R_{x,y}z = R(X, Y)Z(p) = (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)(p),$$

for any local vector fields which extend the given vectors x, y, z.

The second fundamental form is defined by $\alpha(X, Y) = (\overline{\nabla}_X Y)^{\perp}$ and the associated Weingarten operator is given by $Av = -(\overline{\nabla}_v N)^T$, where N is a unit normal field on Σ^2. We use the Weingarten operator to define the following operators

$$\langle \overline{R}(A)x, y\rangle := \sum_{i=1}^{2}(-\langle Ax, \overline{R}_{e_i,y}e_i\rangle - \langle Ay, \overline{R}_{e_i,x}e_i\rangle + \langle Ay, x\rangle\langle N, \overline{R}_{e_i,N}e_i\rangle - 2\langle Ae_i, \overline{R}_{e_i,x}y\rangle) \tag{1.1}$$

and

$$\langle \overline{R}' x, y\rangle := \sum_{i=1}^{2}\{\langle(\overline{\nabla}_x\overline{R})_{e_i,y}e_i, N\rangle + \langle(\overline{\nabla}_{e_i}\overline{R})_{e_i,x}y, N\rangle\},$$

where $\{e_1, e_2\}$ is a orthonormal basis of $T_p\Sigma$.

With this notation we have the following result:

Theorem 1.1. *Let $\Sigma^2 \looparrowright M^3$ be an immersed surface with constant mean curvature H. For any $x, y \in T_p\Sigma$ we have*

$$\langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax, y\rangle + \langle \overline{R}(A)x, y\rangle + \langle \overline{R}' x, y\rangle + 2H\langle \overline{R}_{N,x}y, N\rangle + 2H\langle Ax, Ay\rangle. \tag{1.2}$$

Proof. See Theorem 2 in [B] and observe that the codimension here is one.

■

We will also use the result known as the Omori-Yau Maximum Principle whose proof can be found in [Y], Theorem 1.

Theorem 1.2 (Omori-Yau Maximum Principle)**.** *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in C^\infty(M)$ is bounded from above, then there exist a sequence of points $\{p_j\} \in M$ such that*

$$\lim_{j\to\infty} u(p_j) = \sup_M u, \quad |\nabla u|(p_j) < \frac{1}{j}, \text{ and } \Delta u(p_j) < \frac{1}{j}.$$

Let us recall Gauss' equation for Σ^2 in $M^2(c) \times \mathbb{R}$:

$$\begin{aligned} R(Y,X)Z = \langle AX, Z\rangle AY - \langle AY, Z\rangle AX + c(\langle X, Z\rangle Y - \langle Y, Z\rangle X + \\ -\langle Y, T\rangle\langle X, Z\rangle T - \langle X, T\rangle\langle Z, T\rangle Y + \\ +\langle X, T\rangle\langle Y, Z\rangle T + \langle Y, T\rangle\langle Z, T\rangle X), \end{aligned} \tag{1.3}$$

where X, Y, Z in $T_p\Sigma$, N is a unitary normal field on Σ^2 and T is the tangential component of the parallel field ∂_t. For more details see [D].

CHAPTER 2

Simons type equation for ϕ and S in $M^2(c) \times \mathbb{R}$

In this chapter we will obtain an equation of Simons type for the traceless second fundamental form ϕ and for S defined in (1).

Let $M^2(c) \times \mathbb{R}$, where $M^2(-1) = \mathbb{H}^2$ and $M^2(1) = \mathbb{S}^2$. In this case we have that $\overline{R}' = 0$, because $M^2(c) \times \mathbb{R}$ is locally symmetric.

In Lemmas 2.1 and 2.2 we will consider an immersed surface $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ with constant mean curvature H where A is the Weingarten operator associated to the second fundamental form on Σ^2.

Lemma 2.1. *Denoting the identity by I, we have that*

$$\overline{R}(A) = c(5\nu^2 - 1)A - 4cH\nu^2 I.$$

Proof. Consider an orthonormal basis $\{e_1, e_2\}$ in $T_p\Sigma^2$ such that $Ae_i = k_i e_i$, $i = 1, 2$. Consider $x, y \in T_p\Sigma$. We have

$$x = x_1 e_1 + x_2 e_2 \text{ e } y = y_1 e_1 + y_2 e_2.$$

Computing the first sum in (1.1)

$$\sum_{i=1}^{2} \langle \overline{R}_{e_i,y} e_i, Ax \rangle = \langle \overline{R}_{e_1,y} e_1, k_1 x_1 e_1 + k_2 x_2 e_2 \rangle + \langle \overline{R}_{e_2,y} e_2, k_1 x_1 e_1 + k_2 x_2 e_2 \rangle$$

$$
\begin{aligned}
&= k_2x_2\langle \overline{R}_{e_1,y}e_1, e_2\rangle + k_1x_1\langle \overline{R}_{e_2,y}e_2, e_1\rangle \\
&= k_2x_2\langle \overline{R}_{e_1,y_1e_1+y_2e_2}e_1, e_2\rangle + k_1x_1\langle \overline{R}_{e_2,y_1e_1+y_2e_2}e_2, e_1\rangle \\
&= k_2x_2y_2\langle \overline{R}_{e_1,e_2}e_1, e_2\rangle + k_1x_1y_1\langle \overline{R}_{e_2,e_1}e_2, e_1\rangle \\
&= -\overline{K}_\Sigma\,(k_2x_2y_2 + k_1x_1y_1) = -\overline{K}_\Sigma\langle Ax, y\rangle,
\end{aligned}
$$

where $\overline{K}_\Sigma = \langle \overline{R}_{e_1,e_2}e_2, e_1\rangle$.

Hence,

$$\sum_{i=1}^{2}\langle \overline{R}_{e_i,y}e_i, Ax\rangle = -\overline{K}_\Sigma\langle Ax, y\rangle. \tag{2.1}$$

It's simple see that

$$\sum_{i=1}^{2}\langle \overline{R}_{e_i,x}e_i, Ay\rangle = -\overline{K}_\Sigma\langle Ax, y\rangle. \tag{2.2}$$

In the third sum in (1.1) we have

$$
\begin{aligned}
\langle \overline{R}_{e_i,N}e_i, N\rangle &= -c\{(1-\langle e_i,\partial_t\rangle^2)(1-\nu^2) - \nu^2\langle e_i,\partial_t\rangle^2\} \\
&= -c\{1-\langle e_i,\partial_t\rangle^2 - \nu^2 + \langle e_i,\partial_t\rangle^2\nu^2 - \langle e_i,\partial_t\rangle^2\nu^2\} \\
&= -c\{1-\nu^2-\langle e_i,\partial_t\rangle^2\}.
\end{aligned}
$$

Therefore,

$$\sum_{i=1}^{2}\langle \overline{R}_{e_i,N}e_i, N\rangle = -c(1-\nu^2). \tag{2.3}$$

To finish, we computing the fourth sum.

$$
\begin{aligned}
\sum_{i=1}^{2}\langle \bar{R}_{e_i,x}y, Ae_i\rangle &= \langle \overline{R}_{e_1,x}y, k_1e_1\rangle + \langle \overline{R}_{e_2,x}y, k_2e_2\rangle \\
&= \langle \overline{R}_{e_1,x_1e_1+x_2e_2}y, k_1e_1\rangle + \langle \overline{R}_{e_2,x_1e_1+x_2e_2}y, k_2e_2\rangle \\
&= k_1x_2\langle \overline{R}_{e_1,e_2}y, e_1\rangle + k_2x_1\langle \overline{R}_{e_2,e_1}y, e_2\rangle \\
&= k_1x_2\langle \overline{R}_{e_1,e_2}(y_1e_1+y_2e_2), e_1\rangle + k_2x_1\langle \overline{R}_{e_2,e_1}(y_1e_1+y_2e_2), e_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= k_1x_2y_2\langle\overline{R}_{e_1,e_2}e_2,e_1\rangle + k_2x_1y_1\langle\overline{R}_{e_2,e_1}e_1,e_2\rangle\\
&= \overline{K}_\Sigma\left(k_1x_2y_2 + k_2x_1y_1\right)\\
&= \overline{K}_\Sigma\left([2H-k_2]x_2y_2 + [2H-k_1]x_1y_1\right)\\
&= \overline{K}_\Sigma\left(2Hx_2y_2 - k_2x_2y_2 + 2Hx_1y_1 - k_1x_1y_1\right)\\
&= \overline{K}_\Sigma\left(2H[x_1y_1 + x_2y_2] - [k_1x_1y_1 + k_2x_2y_2]\right)\\
&= \overline{K}_\Sigma\left(2H\langle x,y\rangle - \langle Ax,y\rangle\right),
\end{aligned}
$$

where we used that $2H = k_1 + k_2$.

Thus,

$$\sum_{i=1}^{2}\langle\overline{R}_{e_i,x}y, Ae_i\rangle = \overline{K}_\Sigma\left(2H\langle x,y\rangle - \langle Ax,y\rangle\right). \tag{2.4}$$

Now, we need computing $\overline{K}_\Sigma$. Using the tensor of curvature in $M^2(c)\times\mathbb{R}$ we have:

$$\overline{K}_\Sigma = \langle\overline{R}_{e_1,e_2}e_2,e_1\rangle = c\left(1 - \langle e_1,T\rangle^2 - \langle e_2,T\rangle^2\right) = c(1-|T|^2)$$

Therefore,

$$\overline{K}_\Sigma = cv^2. \tag{2.5}$$

Substituting (2.1), (2.2),(2.3) and (2.4) into (1.1), obtain

$$\langle\overline{R}(A)x,y\rangle = 2\overline{K}_\Sigma\langle Ax,y\rangle - c(1-v^2)\langle Ax,y\rangle - 2\overline{K}_\Sigma\left(2H\langle x,y\rangle - \langle Ax,y\rangle\right).$$

Using (2.5) we obtain

$$\langle\overline{R}(A)x,y\rangle = 5cv^2\langle Ax,y\rangle - c\langle Ax,y\rangle - 4cv^2H\langle x,y\rangle.$$

Thus,

$$\overline{R}(A) = c(5v^2-1)A - 4cHv^2I.$$

■

Lemma 2.2. $\langle \overline{R}_{N,x}y, N\rangle = -c\{\langle x,T\rangle\langle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\}$.

Proof. We observe that

$$\langle x^*, y^*\rangle = \langle x,y\rangle - \langle x,T\rangle\langle y,T\rangle,$$

$$\langle x^*, N^*\rangle = \nu\langle x,T\rangle$$

and

$$\langle N^*, N^*\rangle = 1-\nu^2,$$

where we have used $v^* = v - \langle v,\partial_t\rangle\partial_t$ for any $v\in T_p(M^2(c)\times\mathbb{R})$.

It follows that

$$\langle \overline{R}_{N,x}y, N\rangle = -c\{\langle N^*, x^*\rangle\langle N^*, y^*\rangle - \langle N^*,N^*\rangle\langle x^*,y^*\rangle\}$$

$$= -c\{(\nu\langle x,T\rangle)(\nu\langle y,T\rangle) - (\langle x,y\rangle - \langle x,T\rangle\langle y,T\rangle)\langle T,T\rangle\}$$

$$= -c\{\nu^2\langle x,T\rangle\langle y,T\rangle - \langle x,y\rangle\langle T,T\rangle + \langle x,T\rangle\langle y,T\rangle$$

$$-\nu^2\langle x,T\rangle\langle y,T\rangle\} = -c\{\langle x,T\rangle\langle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\}.$$

This concludes the proof.

∎

Proposition 2.1. *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with constant mean curvature H and let A be the Weingarten operator associated to the second fundamental form on Σ^2. Then,*

$$\langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax,y\rangle + c(5\nu^2-1)\langle Ax,y\rangle - 4cH\nu^2\langle x,y\rangle$$

$$-2cH\{\langle x,T\rangle\langle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\} + 2H\langle Ax,Ay\rangle.$$

Proof. Consider equation (1.2)

$$\langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax,y\rangle + \langle\overline{R}(A)x,y\rangle$$

$$+\langle\overline{R}'x,y\rangle + 2H\langle\overline{R}_{N,x}y,N\rangle + 2H\langle Ax,Ay\rangle.$$

Now, we use Lemmas 2.1 and 2.2 and the fact $\bar{R}'=0$ to obtain

$$\langle(\nabla^2 A)x, y\rangle = -|A|^2\langle Ax,y\rangle + c(5\nu^2-1)\langle Ax,y\rangle - 4cH\nu^2\langle x,y\rangle$$

$$-2Hc\{\langle x,T\rangle\rangle y,T\rangle - \langle x,y\rangle\langle T,T\rangle\} + 2H\langle Ax,Ay\rangle.$$

This concludes the proof.

∎

Consider two tensors Z, W on Σ^2. We define the inner product $\langle Z, W\rangle$ in $p \in \Sigma^2$ as

$$\langle Z, W\rangle = \sum_{i=1}^{2} \langle Ze_i, We_i\rangle,$$

where $\{e_1, e_2\}$ is an orthonormal basis for $T_p\Sigma$.

Lemma 2.3. *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with constant mean curvature and let A be the Weingarten operator associated to the second fundamental form on Σ^2. Then,*

(*a*) $\langle\nabla^2 A, I\rangle = 0$.

(*b*) $\langle\nabla^2 A, A\rangle = -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 - 2cH\langle AT, T\rangle + 4cH^2|T|^2 + 2Htr(A^3)$.

Proof. Consider $\{e_1, e_2\}$ an orthonormal basis of $T_p\Sigma$. We use the definition of the inner product between tensors and the expression in Proposition 2.1 to obtain

$$\langle\nabla^2 A, A\rangle = \sum_{i=1}^{2}\langle(\nabla^2 A)e_i, Ae_i\rangle = -|A|^2\sum_{i=1}^{2}\langle Ae_i, Ae_i\rangle +$$

$$c(5\nu^2-1)\sum_{i=1}^{2}\langle Ae_i, Ae_i\rangle - 4cH\nu^2\sum_{i=1}^{2}\langle Ae_i, e_i\rangle - 2cH\left\{\sum_{i=1}^{2}\langle AT, e_i\rangle\langle e_i, T\rangle\right.$$

$$\left. -\langle T, T\rangle\sum_{i=1}^{2}\langle Ae_i, e_i\rangle\right\} + 2H\sum_{i=1}^{2}\langle A^2 e_i, Ae_i\rangle.$$

Therefore,

$$\langle\nabla^2 A, A\rangle = -|A|^4 + c(5\nu^2-1)|A|^2 - 8cH^2\nu^2 - 2cH\langle AT, T\rangle + 4cH^2|T|^2 + 2Htr(A^3).$$

Using the definition of the inner product and Proposition 2.1 we obtain

$$\langle\nabla^2 A, I\rangle = \sum_{i=1}^{2}\langle(\nabla^2 A)e_i, e_i\rangle = -|A|^2\sum_{i=1}^{2}\langle Ae_i, e_i\rangle + \ldots$$

$$\ldots + c(5\nu^2-1)\sum_{i=1}^{2}\langle Ae_i,e_i\rangle - 8cH\nu^2 - 2cH\left\{\sum_{i=1}^{2}\langle T,e_i\rangle\langle e_i,T\rangle\right.$$
$$\left.-2\langle T,T\rangle\right\} + 2H\sum_{i=1}^{2}\langle A^2e_i,e_i\rangle.$$

Therefore,

$$\begin{aligned}\langle\nabla^2 A, I\rangle &= -2H|A|^2 + c(5\nu^2-1)2H - 8cH\nu^2 + 2cH\langle T,T\rangle + 2H|A|^2 = 0\\ &= -2H|A|^2 + c(5\nu^2-1)2H - 8cH\nu^2 + 2cH(1-\nu^2) + 2H|A|^2\\ &= 10cH\nu^2 - 2cH - 8cH\nu^2 + 2cH - 2cH\nu^2 = 0,\end{aligned}$$

where we have used that $\nu^2 + |T|^2 = 1$.

■

Proposition 2.2. *Let $\Sigma \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with constant mean curvature H and ϕ the traceless second fundamental form, then*

(*a*) $|\phi|^2 = |A|^2 - 2H^2$.

(*b*) $\nabla\phi = \nabla A$.

(*c*) $trA^3 = 3H|\phi|^2 + 2H^3$.

Proof. The proof of item (a) is:

$$\begin{aligned}|\phi|^2 = \langle\phi,\phi\rangle = \langle A-HI, A-HI\rangle &= \langle A,A\rangle - 2H\langle A,I\rangle + H^2\langle I,I\rangle\\ &= |A|^2 - 4H^2 + 2H^2 = |A|^2 - 2H^2,\end{aligned}$$

where $\langle A,I\rangle = 2H$ and $\langle I,I\rangle = 2$.
To prove item (b), we consider tangent fields X, Y. Then,

$$\begin{aligned}(\nabla_X\phi)Y = (\nabla_X A)Y - (\nabla_X(HI))Y &= (\nabla_X A)Y - \nabla_X HI(Y) + H\nabla_X Y\\ &= (\nabla_X A)Y - H\nabla_X Y - X(H)Y + H\nabla_X Y = (\nabla_X A)Y,\end{aligned}$$

because H is constant.
Finally, the proof of item(c) is:

$$tr(A^3) = \sum_{i=1}^{2}\langle A^3e_i,e_i\rangle = \sum_{i=1}^{2}\langle(\phi+HI)^3e_i,e_i\rangle$$

$$= \sum_{i=1}^{2} \langle (\phi^3 + 3H\phi^2 + 3H^2\phi + H^3 I)e_i, e_i \rangle = 3H|\phi|^2 + 2H^3,$$

because $tr\phi = tr\phi^3 = 0$.

∎

Next we shall derive an equation of Simons type for the traceless second fundamental form ϕ:

Theorem 2.1. *Let $\Sigma \hookrightarrow M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature H and ϕ the traceless second fundamental form. Then*

$$\langle \nabla^2\phi, \phi \rangle = -|\phi|^4 + (2H^2 + 5cv^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle$$

and

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - |\phi|^4 + (2H^2 + 5cv^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle.$$

Proof. We use Proposition 2.2 to show that

$$\langle \nabla^2\phi, \phi \rangle = \langle \nabla^2 A, A - HI \rangle = \langle \nabla^2 A, A \rangle - H\langle \nabla^2 A, I \rangle.$$

Now, we use Lemma 2.3 to obtain

$$\langle \nabla^2\phi, \phi \rangle = -|A|^4 + c(5v^2 - 1)|A|^2 - 8cH^2v^2 + 2cH\langle AT, T \rangle$$
$$+4cH^2|T|^2 + 2Htr(A^3).$$

Therefore,

$$\langle \nabla^2\phi, \phi \rangle = -(|\phi|^2 + 2H^2)^2 + c(5v^2 - 1)(|\phi|^2 + 2H^2) - 8cH^2v^2 +$$
$$-2cH\langle (\phi + HI)T, T \rangle + 4cH^2|T|^2 + 2H(3H|\phi|^2 + 2H^3),$$

which brings us to

$$\langle \nabla^2\phi, \phi \rangle = -|\phi|^4 - 4H^2|\phi|^2 - 4H^4 + c(5v^2 - 1)|\phi|^2 + 2c(5v^2 - 1)H^2 - 8cH^2v^2$$
$$-2cH\langle \phi T, T \rangle - 2cH^2|T|^2 + 4cH^2|T|^2 + 6H^2|\phi|^2 + 4H^4.$$

Hence,

$$\langle \nabla^2\phi, \phi \rangle = -|\phi|^4 + 2H^2|\phi|^2 + c(5v^2 - 1)|\phi|^2 - 2cH\langle \phi T, T \rangle.$$

To finish, we use that $\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \langle\nabla^2\phi, \phi\rangle$.

∎

Now we evaluate the Laplacian of $|S|^2$ where S is defined by (1), i.e,

$$S = 2HA - c\langle T, \cdot\rangle T + \frac{c}{2}(1 - \nu^2)I - 2H^2 I.$$

Proposition 2.3 (Codazzi's Equation)**.** *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature and S defined in* (1). *Then*

$$(\nabla_X S)Y = (\nabla_Y S)X,$$

for all tangent fields X, Y on Σ^2 and $tr(S) = 0$.

Proof. We consider (u, v) an isothermals parameters of the surface Σ^2. Now, we consider the complex parametric, $z = u + iv$. Let us set

$$T_S(X, Y) := (\nabla_X S)Y - (\nabla_Y S)X = \nabla_X(SY) - \nabla_Y(SX) - S[X, Y].$$

We will prove that T_S is null. For this, consider the derivative

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

We will compute T_S in the basis $\{\partial_z, \partial_{\bar{z}}\}$. First note that,

$$\begin{aligned} \langle T_S(\partial_z, \partial_{\bar{z}}), \partial_z\rangle &= \partial_z\langle S\partial_{\bar{z}}, \partial_z\rangle - \langle S\partial_{\bar{z}}, \nabla_{\partial_z}\partial_z\rangle + \\ &\quad -\partial_{\bar{z}}\langle S\partial_z, \partial_z\rangle + \langle S\partial_z, \nabla_{\partial_{\bar{z}}}\partial_z\rangle \\ &= -Q^{(2,0)}_{\bar{z}} = 0, \end{aligned}$$

because $Q^{(2,0)}$ is holomorphic and using the fact that $\nabla_{\partial_z}\partial_{\bar{z}} = 0$, $\nabla_{\partial_z}\partial_z = \frac{\lambda_{\bar{z}}}{\lambda}\partial_z$, $\langle S\partial_z, \partial_z\rangle = Q^{(2,0)}$ and $\langle S\partial_z, \partial_{\bar{z}}\rangle = 0$, where $\lambda = \langle\partial_z, \partial_{\bar{z}}\rangle$.

Next,

$$\begin{aligned} \langle T_S(\partial_z, \partial_{\bar{z}}), \partial_{\bar{z}}\rangle &= \partial_{\bar{z}}\langle \partial_{\bar{z}}, S\partial_z\rangle - \langle S\partial_z, \nabla_{\partial_{\bar{z}}}\partial_{\bar{z}}\rangle + \\ &\quad -\partial_z\langle S\partial_{\bar{z}}, \partial_{\bar{z}}\rangle + \langle S\partial_{\bar{z}}, \nabla_{\partial_z}\partial_{\bar{z}}\rangle \\ &= -\overline{Q^{(2,0)}_{\bar{z}}} = 0, \end{aligned}$$

where we have used that $\nabla_{\partial_{\bar{z}}}\partial_{\bar{z}} = \dfrac{\lambda_{\bar{z}}}{\lambda}\partial_{\bar{z}}$ and $\overline{\overline{Q^{(2,0)}}_z} = Q^{(2,0)}_{\bar{z}}$. It follows that $T_S = 0$.

To finish,

$$tr(S) = 2Htr(A) - c|T|^2 + c(1-\nu^2) - 4H^2 = 0,$$

where we used that $|T|^2 + \nu^2 = 1$ and $tr(A) = 2H$.

■

The result below is known, see [S], p. 81, adapted for codimension 1, but, for completeness we will give its proof.

Lemma 2.4. *Let Z be a symmetric operator satisfying Codazzi's equation and $tr(Z) = 0$, then*

$$\langle(\nabla^2 Z)x, y\rangle = \sum_{i=1}^{2}\{-\langle Zy, R_{e_i,x}e_i\rangle - \langle Ze_i, R_{e_i,x}y\rangle\}, \tag{2.6}$$

where $\{e_1, e_2\}$ is an orthonormal basis of $T_p\Sigma$.

Proof. Let us consider a geodesic frame $\{E_1, E_2\}$ in $p \in \Sigma^2$ which extends the basis $\{e_1, e_2\}$ and X, Y local vector parallel fields which extend the given vectors x, y. Compute

$$\begin{aligned}
(\nabla^2 Z)X &= \sum_{i=1}^{2}(\nabla_{E_i}\nabla_{E_i}Z)X = \sum_{i=1}^{2}\nabla_{E_i}((\nabla_{E_i}Z)X) \\
&= \sum_{i=1}^{2}\nabla_{E_i}((\nabla_X Z)E_i = \sum_{i=1}^{2}(\nabla_{E_i}\nabla_X Z)E_i \\
&= \sum_{i=1}^{2}(\nabla_X\nabla_{E_i}Z)E_i + \sum_{i=1}^{2}(R(E_i, X)Z)E_i \\
&= \sum_{i=1}^{2}\nabla_X((\nabla_{E_i}Z)E_i) + \sum_{i=1}^{2}(R(E_i, X)Z)E_i \\
&= \nabla_X\left(\sum_{i=1}^{2}(\nabla_{E_i}Z)E_i\right) + \sum_{i=1}^{2}(R(E_i, X)Z)E_i,
\end{aligned}$$

taking into account that Z satisfies Codazzi's equation.

Then,

$$\begin{aligned}\langle(\nabla^2 Z)x,y\rangle &= \langle\nabla_X\left(\sum_{i=1}^{2}(\nabla_{E_i}Z)E_i\right),Y\rangle+\sum_{i=1}^{2}\langle(R(E_i,X)Z)E_i,Y\rangle\\ &= X\left(\sum_{i=1}^{2}\langle(\nabla_{E_i}Z)E_i,Y\rangle\right)+\sum_{i=1}^{2}\langle(R(E_i,X)Z)E_i,Y\rangle\\ &= X(tr(\nabla_Y Z))+\sum_{i=1}^{2}\langle(R(E_i,X)Z)E_i,Y\rangle\\ &= XY(tr(Z))+\sum_{i=1}^{2}(\langle(R(E_i,X)(ZE_i),Y\rangle-\langle Z(R(E_i,X)E_i),Y\rangle).\end{aligned}$$

By noting that $trZ=0$, and computing the above expression at the point p, we obtain

$$\langle(\nabla^2 Z)x,y\rangle=\sum_{i=1}^{2}\{-\langle(R(e_i,x)y,Ze_i\rangle-\langle R(e_i,x)e_i,Zy\rangle\}.$$

■

Let us evaluate each summand in expression (2.6).

Lemma 2.5. *Let Z be an operator as in Lemma* 2.4. *Then,*

$$i)\ \sum_{i=1}^{2}\langle Zy,R_{e_i,x}e_i\rangle=-cv^2\langle Zx,y\rangle-2H\langle Ax,Zy\rangle+\langle A^2x,Zy\rangle.$$

and

$$ii)\ \sum_{i=1}^{2}\langle Ze_i,R_{e_i,x}y\rangle=-cv^2\langle Zx,y\rangle-\langle Ay,ZAx\rangle+\langle Ax,y\rangle tr(AZ).$$

Proof. Consider $\{e_1,e_2\}$ an orthonormal basis of $T_p\Sigma$. Using Gauss' equation (1.3) we find

$$\langle Zy,R_{e_i,x}e_i\rangle=-c\{\langle x,Zy\rangle-\langle x,e_i\rangle\langle Zy,e_i\rangle-\langle x,T\rangle\langle Zy,T\rangle+...$$

$$\begin{gathered}
... - \langle e_i, T\rangle^2\langle x, Zy\rangle + \langle e_i, T\rangle\langle x, e_i\rangle\langle Zy, T\rangle + \\
+\langle x, T\rangle\langle e_i, T\rangle\langle e_i, Zy\rangle\} - \langle Ae_i, e_i\rangle\langle Ax, Zy\rangle + \\
+\langle Ax, e_i\rangle\langle Ae_i, Zy\rangle.
\end{gathered}$$

Therefore,

$$\begin{gathered}
\sum_{i=1}^{2}\langle Zy, R_{e_i,x}e_i\rangle = -c\{2\langle x, Zy\rangle - \sum_{i=1}^{2}\langle x, e_i\rangle\langle Zy, e_i\rangle + \\
-2\langle x, T\rangle\langle Zy, T\rangle - \langle x, Zy\rangle\sum_{i=1}^{2}\langle e_i, T\rangle^2 + \\
+\langle Zy, T\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle x, e_i\rangle + \langle x, T\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle e_i, Zy\rangle\} + \\
-\langle Ax, Zy\rangle\sum_{i=1}^{2}\langle Ae_i, e_i\rangle + \sum_{i=1}^{2}\langle Ax, e_i\rangle\langle Ae_i, Zy\rangle,
\end{gathered}$$

which implies that

$$\begin{gathered}
\sum_{i=1}^{2}\langle Zy, R_{e_i,x}e_i\rangle = -c\{2\langle x, Zy\rangle - \langle Zx, y\rangle - 2\langle x, T\rangle\langle Zy, T\rangle + \\
-\langle x, Zy\rangle|T|^2 + \langle Zy, T\rangle\langle x, T\rangle + \langle x, T\rangle\langle T, Zy\rangle\} + \\
-\langle Ax, Zy\rangle 2H + \langle Ax, AZy\rangle.
\end{gathered}$$

Hence,

$$\sum_{i=1}^{2}\langle Zy, R_{e_i,x}e_i\rangle = -c(1 - |T|^2)\langle Zx, y\rangle - 2H\langle Ax, Zy\rangle + \langle A^2x, Zy\rangle,$$

which shows the validity of (i). Now, one may verify that

$$\begin{gathered}
\langle Ze_i, R_{e_i,x}y\rangle = -c\{\langle e_i, y\rangle\langle Ze_i, x\rangle - \langle x, y\rangle\langle Ze_i, e_i\rangle + \\
-\langle x, T\rangle\langle Ze_i, T\rangle\langle e_i, y\rangle - \langle e_i, T\rangle\langle y, T\rangle\langle x, Ze_i\rangle + \\
+\langle e_i, T\rangle\langle x, y\rangle\langle Ze_i, T\rangle + \langle x, T\rangle\langle y, T\rangle\langle e_i, Ze_i\rangle\} + ...
\end{gathered}$$

$$... - \langle Ae_i, y\rangle\langle Ax, Ze_i\rangle + \langle Ax, y\rangle\langle Ae_i, Ze_i\rangle.$$

Therefore

$$\sum_{i=1}^{2}\langle Ze_i, R_{e_i,x}y\rangle = -c\left\{\sum_{i=1}^{2}\langle e_i, y\rangle\langle Ze_i, x\rangle - \langle x, y\rangle\sum_{i=1}^{2}\langle Ze_i, e_i\rangle + \right.$$

$$-\langle x, T\rangle\sum_{i=1}^{2}\langle Ze_i, T\rangle\langle e_i, y\rangle - \langle y, T\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle x, Ze_i\rangle +$$

$$+\langle x, y\rangle\sum_{i=1}^{2}\langle e_i, T\rangle\langle Ze_i, T\rangle + \langle x, T\rangle\langle y, T\rangle\sum_{i=1}^{2}\langle e_i, Ze_i\rangle\} +$$

$$-\sum_{i=1}^{2}\langle Ae_i, y\rangle\langle Ax, Ze_i\rangle + \langle Ax, y\rangle\sum_{i=1}^{2}\langle Ae_i, Ze_i\rangle.$$

Therefore

$$\sum_{i=1}^{2}\langle Ze_i, R_{e_i,x}y\rangle = -c\{\langle Zx, y\rangle - \langle x, T\rangle\langle Zy, T\rangle - \langle y, T\rangle\langle Zx, T\rangle +$$

$$+\langle ZT, T\rangle\langle x, y\rangle\} - \langle Ay, ZAx\rangle + \langle Ax, y\rangle tr(AZ),$$

noting that $trZ = 0$.

Considering that

$$-\langle x, T\rangle\langle Zy, T\rangle - \langle y, T\rangle\langle Zx, T\rangle + \langle ZT, T\rangle\langle x, y\rangle$$

$$= -(1 - \nu^2)\langle Zx, y\rangle,$$

we find

$$\sum_{i=1}^{2}\langle Ze_i, R_{e_i,x}y\rangle = -c\nu^2\langle Zx, y\rangle - \langle Ay, ZAx\rangle + \langle Ax, y\rangle tr(AZ),$$

which demonstrates (*ii*).

■

Theorem 2.2. *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with non zero constant mean curvature H and let Z be an operator on Σ^2 satisfying Codazzi's equation with $tr(Z) = 0$. Then,*

$$\langle(\nabla^2 Z)x, y\rangle = 2cv^2\langle Zx, y\rangle + 2H\langle Ax, Zy\rangle - \langle A^2x, Zy\rangle +$$

$$+\langle Ay, ZAx\rangle - \langle Ax, y\rangle tr(AZ).$$

Proof. We use the expressions of Lemma 2.5 in equation (2.6) obtained in Lemma 2.4.

■

Next we derive an equation of Simons type for the operator S as defined in (1).

Theorem 2.3 (Thm 0.1 in Introduction)**.** *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be an immersed surface with non zero constant mean curvature H and S as defined in* (1)*. Then,*

$$\langle(\nabla^2 S)x, y\rangle = 2cv^2\langle Sx, y\rangle + 2H\langle Ax, Sy\rangle - \langle A^2x, Sy\rangle +$$

$$+\langle Ay, SAx\rangle - \langle Ax, y\rangle tr(AS)$$

and

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5cv^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST, T\rangle\right) +$$

$$+c|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

Proof. First, since S satisfies Proposition 2.3, we can use the Theorem 2.2 with $Z = S$.

Now, we know that $\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + \langle\nabla^2 S, S\rangle$. Furthermore, we finds that

$$\langle\nabla^2 S, S\rangle = 2cv^2|S|^2 + 2Htr(AS^2) - [tr(AS)]^2.$$

Now, we need to compute $tr(AS^2)$ and $tr(AS)$, as follows:

$$tr(AS^2) = tr\{S^2(S + \frac{c}{2H}\langle T, \cdot\rangle T - \frac{c}{4H}(1 - v^2)I + HI)\}$$

$$= trS^3 + \frac{c}{2H}tr(\langle T, S^2\cdot\rangle T) - \left(\frac{c}{4H}(1 - v^2) - H\right)trS^2$$

$$= 0 + \frac{c}{2H}|ST|^2 - \left(\frac{c}{4H}(1-\nu^2) - H\right)|S|^2$$

and

$$tr(AS) = tr\{S(S + \frac{c}{2H}\langle T,\cdot\rangle T - \frac{c}{4H}(1-\nu^2)I - HI)\}$$
$$= trS^2 + \frac{c}{2H}tr(\langle T, S\cdot\rangle T) - (\frac{c}{4H}(1-\nu^2) - H)trS$$
$$= |S|^2 + \frac{c}{2H}\langle ST, T\rangle - 0,$$

noting that $tr\ S = tr\ S^3 = 0$, also that

$$tr(\langle T, S\cdot\rangle T) = \sum_{i=1}^{2}\langle T, Se_i\rangle\langle T, e_i\rangle = \langle ST, T\rangle$$

and that

$$tr(\langle T, S^2\cdot\rangle T) = \sum_{i=1}^{2}\langle T, S^2e_i\rangle\langle T, e_i\rangle = \langle S^2T, T\rangle.$$

Therefore,

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2c\nu^2|S|^2 + 2H\left(\frac{c}{2H}|ST|^2 - \left(\frac{c}{4H}(1-\nu^2) - H\right)|S|^2\right)$$
$$-\left(|S|^2 + \frac{c}{2H}\langle ST, T\rangle\right)^2,$$

in this way,

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2c\nu^2|S|^2 + c|ST|^2 - \left(\frac{c}{2}(1-\nu^2) - 2H^2\right)|S|^2 +$$
$$-|S|^4 - \frac{c}{H}\langle ST, T\rangle|S|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

Rearranging terms, we obtain finally

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST, T\rangle\right) +$$
$$+c|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

■

CHAPTER 3

Applications

In this chapter, we will apply the results found in chapter 2 together with the Omori-Yau's Theorem to classify some surfaces in $M^2(c) \times \mathbb{R}$.

Theorem 3.1. *Let $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be an oriented complete immersed minimal surface. Assume that*

$$\sup_{\Sigma}(|A|^2 + 5\nu^2) < 1.$$

Then Σ^2 is a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$.

Proof. Using Theorem 2.1 with $H = 0$ and $c = -1$, one finds

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4 + (1 - 5\nu^2)|A|^2 \geq |A|^2 \left(-|A|^2 + 1 - 5\nu^2\right).$$

Let $\frac{d}{2} := -\sup_{\Sigma}(|A|^2 + 5\nu^2) + 1 > 0$. Therefore,

$$\Delta|A|^2 \geq d \cdot |A|^2. \tag{3.1}$$

Using Gauss' equation (1.3) in $\mathbb{H}^2 \times \mathbb{R}$ we have

$$K_\Sigma = K_{ext} - \nu^2 = -\frac{|A|^2 + 5\nu^2}{2} + \frac{3\nu^2}{2} \geq -\frac{1}{2}.$$

Now we can use Theorem 1.2 with $u = |A|^2$, i.e, there exist $\{p_j\}$ in Σ^2 such that

$$\lim_{j\to\infty} |A|^2(p_j) = \sup_{\Sigma} |A|^2 \ \textit{and} \ \lim_{j\to\infty} \Delta|A|^2(p_j) \leq 0.$$

Next, we use inequality (3.1) to conclude that $\sup_\Sigma |A|^2 = 0$, i.e, Σ^2 is totally geodesic with $|\nu| < \sqrt{0.2}$.

Since Σ^2 is totally geodesic and $|\nu| < \sqrt{0.2}$ it cannot be a slice, it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$.

■

Theorem 3.2. *Let $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be a complete immersed surface with constant mean curvature H. Assume that*

$$\sup_{\Sigma}(|\phi|^2 + 5\nu^2) < 2H^2 + 1 \ \textit{ and } \ \langle \phi T, T\rangle \geq 0.$$

Then Σ^2 is a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$.

Proof. We consider the expression in Theorem 2.1 for the particular case $c = -1$:

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - |\phi|^4 + (2H^2 + 1 - 5\nu^2)|\phi|^2 + 2H\langle \phi T, T\rangle.$$

As $\langle \phi T, T\rangle \geq 0$, we find

$$\frac{1}{2}\Delta|\phi|^2 \geq -|\phi|^4 + (2H^2 + 1 - 5\nu^2)|\phi|^2.$$

Consider $\dfrac{d}{2} := 2H^2 + 1 - \sup\limits_{\Sigma}(|\phi|^2 + 5\nu^2) > 0$. Then

$$\Delta|\phi|^2 \geq 2|\phi|^2(2H^2 + 1 - 5\nu^2 - |\phi|^2) \geq d|\phi|^2,$$

which implies,

$$\Delta|\phi|^2 \geq d|\phi|^2. \tag{3.2}$$

Using Gauss' equation (1.3) in $\mathbb{H}^2 \times \mathbb{R}$ we have

$$K_\Sigma = K_{ext} - \nu^2 = -\frac{|\phi|^2 + 5\nu^2 - 2H^2}{2} + \frac{3\nu^2}{2} \geq -\frac{1}{2}.$$

Now we can use Theorem 1.2 with $u = |\phi|^2$, i.e, there exist $\{p_j\}$ in Σ^2 such that

$$\lim_{j\to\infty}|\phi|^2(p_j) = \sup_{\Sigma}|\phi|^2 \ \textit{and} \ \lim_{j\to\infty}\Delta|\phi|^2(p_j) \le 0.$$

Furthermore, we use inequality (3.2) to conclude that $\sup_{\Sigma}|\phi|^2 = 0$, i.e, Σ^2 is totally umbilical. Next, we use that if Σ^2 is totally umbilical with constant mean curvature in $\mathbb{H}^2\times\mathbb{R}$ then Σ^2 is totally geodesic, which follows from [*ST*] section 4. Since Σ^2 is totally geodesic and $|\nu| < \sqrt{0.2}$ it must be a vertical plane $\gamma\times\mathbb{R}$ for some geodesic γ in $\mathbb{H}^2$. This concludes the proof.

■

We need the following result:

Lemma 3.1. *Let $\Sigma^2 \looparrowright M^2(c)\times\mathbb{R}$ be a complete immersed surface with non zero constant mean curvature H. Then $|S| = 0$ if and only if Σ^2 is an Abresch-Rosenberg surface.*

Proof. We consider (u, v) an isothermals parameters to a surface Σ^2. Now, we consider the complex parameter, $z = u + iv$ and (2,0)-part of the Abresch-Rosenberg differential

$$Q(x, y) = 2H\langle Ax, y\rangle - c\langle x, T\rangle\langle y, T\rangle.$$

We can rewrite Q as

$$Q(x, y) = \langle Sx, y\rangle - \frac{c}{2}(1-\nu^2)\langle x, y\rangle + 2H^2\langle x, y\rangle.$$

Next we evaluate $Q(\partial_z, \partial_z)$ noting that $\langle\partial_z, \partial_z\rangle = 0$:

$$Q(\partial_z, \partial_z) = \langle S\partial_z, \partial_z\rangle = \left(\frac{\tilde{e}-\tilde{g}}{4}\right) - i\frac{\tilde{f}}{2},$$

where $\tilde{e} = \langle S\partial_u, \partial_u\rangle = -\langle S\partial_v, \partial_v\rangle = -\tilde{g}$ and $\tilde{f} = \langle S\partial_u, \partial_v\rangle$. Therefore

$$|Q^{(2,0)}| = \sqrt{\left(\frac{\tilde{e}-\tilde{g}}{4}\right)^2 + \frac{\tilde{f}^2}{4}} = \sqrt{\frac{\tilde{e}^2}{4} + \frac{\tilde{f}^2}{4}} = \frac{\lambda^2}{2\sqrt{2}}|S|,$$

where $\lambda = |\partial_u| > 0$. This concluses the proof.

■

Let us consider the polynomial $p_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{4H^2-1}{2}\right)$. When H is greater than one half there is a positive root for p_H. Let L_H be the positive root. One has:

Theorem 3.3 (Thm 0.2 in Introduction)**.** *Let $\Sigma^2 \looparrowright \mathbb{S}^2\times\mathbb{R}$ be an immersed surface with constant mean curvature H greater than one half. If*

$$\Sigma^2 \textit{ is complete and } \sup_{\Sigma}|S| < L_H;$$

or

$$\Sigma^2 \textit{ is closed and } |S| \leq L_H.$$

Then $\Sigma^2 = S^2_H$, i.e, Σ^2 is an embedded rotationally invariant constant mean curvature sphere.

Proof. Let consider two cases. First, Σ is complete and second, Σ is closed.
First Case. Consider the expression in Theorem 2.3 with $c = 1$:

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5\nu^2}{2} - \frac{1}{2} + 2H^2 - \frac{1}{H}\langle ST, T\rangle\right)$$

$$+|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

As $|\langle ST, T\rangle| \leq |ST| \leq |S|$, we have

$$\frac{1}{2}\Delta|S|^2 \geq -|S|^4 + |S|^2\left(\frac{5\nu^2}{2} + \frac{4H^2-1}{2} - \frac{1}{H}|S|\right) + \left(\frac{4H^2-1}{4H^2}\right)\langle ST, T\rangle^2,$$

hence,

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2\left(\frac{4H^2-1}{2} - \frac{1}{H}|S| - |S|^2\right) + \frac{5}{2}\nu^2|S|^2, \tag{3.3}$$

because $H > \frac{1}{2}$.
Observe that

$$\frac{4H^2-1}{2} - \frac{1}{H}|S| - |S|^2 \geq p_H(\sup_{\Sigma}|S|) =: \frac{d}{2} > 0$$

and $\nu^2|S|^2 \geq 0$. Therefore
$$\Delta|S|^2 \geq d|S|^2. \tag{3.4}$$
Now we estimate $|S|$.
$$|S| \geq 2H|A| - |\langle T, \cdot\rangle T| - (1-\nu^2) - 4H^2 \geq 2H|A| - 2(1-\nu^2) - 4H^2,$$
that is,
$$L_H \geq |S| \geq 2H|A| - 2 - 4H^2.$$
Using Gauss' equation (1.3) in $\mathbb{S}^2 \times \mathbb{R}$ we find
$$K_\Sigma = K_{ext} + \nu^2 = -\frac{|A|^2}{2} + 2H^2 + \nu^2 \geq -\frac{1}{2}\left(\frac{L_H + 2 + 4H^2}{2H}\right)^2.$$
Now we can use Theorem 1.2 with $u = |S|^2$, i.e, there exists a $\{p_j\}$ in Σ^2 such that
$$\lim_{j\to\infty} |S|^2(p_j) = \sup_\Sigma |S|^2 \ \textit{and} \ \lim_{j\to\infty} \Delta|S|^2(p_j) \leq 0.$$
By means of inequality (3.4) we conclude that $\sup_\Sigma |S|^2 = 0$, i.e, $|S| = 0$ in Σ^2. Using Lemma 3.1 and Remark 1 of the Introduction we conclude the proof.

Second case. Let us consider expression (3.3)
$$\frac{1}{2}\Delta|S|^2 \geq |S|^2\left(\frac{4H^2-1}{2} - \frac{1}{H}|S| - |S|^2\right) + \frac{5}{2}\nu^2|S|^2.$$
As $|S| \leq L_H$, we have $\dfrac{4H^2-1}{2} - \dfrac{1}{H}|S| - |S|^2 \geq 0$. Hence,
$$\frac{1}{2}\Delta|S|^2 \geq \frac{5}{2}\nu^2|S|^2.$$
Integrating and using Stokes' Theorem we find
$$0 \geq \frac{5}{2}\int_\Sigma \nu^2|S|^2 d\Sigma \geq 0.$$
It follows that
$$|S| \cdot \nu = 0. \tag{3.5}$$

Let $\Theta = \{p \in \Sigma^2 : \nu(p) = 0\} = \nu^{-1}(0)$ be the nodal lines of ν. We knows that

$$\Delta\nu + (|A|^2 + Ric(N, N)\nu = 0.$$

Hence, we could apply Theorem 2.5 in [C], p. 49, to conclude that Θ has empty interior. Thus, using (3.5), $|S|$ vanishes in an open and dense set. By continuity, $|S| = 0$ in Σ.

Using Lemma 3.1 and Remark.1 of the Introduction we conclude the proof.

■

Theorem 3.4. *There exists no $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ complete immersed surface with constant mean curvature greater than one half such that $|S| = L_H$.*

Proof. Suppose that there exist $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ satisfying the condition of the theorem. Using expression (3.3)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2\left(\frac{4H^2-1}{2} - \frac{1}{H}|S| - |S|^2\right) + \frac{5}{2}\nu^2|S|^2,$$

with $|S| = L_H$ one find that

$$0 \geq 0 + \frac{5}{2}\nu^2 L_H^2 \geq 0.$$

Hence $\nu = 0$, i.e, $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ is a cylinder $\gamma \times \mathbb{R}$ for some $\gamma \in \mathbb{S}^2$ with constant curvature $2H$.

On the other hand, for a cylinder $\gamma \times \mathbb{R}$, where $\gamma \in \mathbb{S}^2$ with constant curvature $2H$, we may write

$$S = \begin{pmatrix} 2H^2 + \frac{1}{2} & 0 \\ 0 & -2H^2 - \frac{1}{2} \end{pmatrix}.$$

As $|S| = \frac{\sqrt{2}}{2}(4H^2 + 1) > L_H$ we have a contradiction.

■

In next theorem we need the following result:

Lemma 3.2. *Any Abresch-Rosenberg surface* $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ *with* $H > \dfrac{1}{2}$ *is an embedded rotationally invariant constant mean curvature sphere.*

Proof. See Proposition 4.3 in [*AR*], p. 19.

■

Let us consider the polynomial $q_H(t) = -t^2 - \dfrac{1}{H}t + \left(\dfrac{8H^4 - 12H^2 - 1}{4H^2}\right)$. When H is greater than a positive root of the polynomial $r(x) = 8x^4 - 12x^2 - 1$, i.e, H is greater than $\sqrt{\dfrac{12+\sqrt{176}}{16}}$, there is a positive root for q_H. Let M_H be the positive root.

Theorem 3.5 (Thm 0.3 in Introduction)**.** *Let* $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ *be an immersed surface with constant mean curvature H greater than* $\sqrt{\dfrac{12+\sqrt{176}}{16}} \approx 1.25664$. *If*

$$\Sigma^2 \textit{ is complete and } \sup_{\Sigma} |S| < M_H$$

or

$$\Sigma^2 \textit{ is closed and } |S| \leq M_H,$$

then $\Sigma^2 = S^2_H$, *i.e,* Σ^2 *is an embedded rotationally invariant constant mean curvature sphere.*

Proof. Let us consider two cases. First, Σ is complete and second, Σ is closed.

First case. Consider the expression in Theorem 2.3 with $c = -1$

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2\left(-\frac{5v^2}{2} + \frac{1}{2} + 2H^2 + \frac{1}{H}\langle ST, T\rangle\right)$$

$$-|ST|^2 - \frac{1}{4H^2}\langle ST, T\rangle^2.$$

As $|\langle ST, T\rangle| \leq |ST| \leq |S|$, we may write

$$\frac{1}{2}\Delta|S|^2 \geq -|S|^4 + |S|^2\left(\frac{4H^2 + 1 - 5v^2}{2} - \frac{1}{H}|S|\right) - \left(\frac{4H^2+1}{4H^2}\right)|S|^2,$$

i.e,
$$\frac{1}{2}\Delta|S|^2 \geq |S|^2\left(\frac{4H^2-4+5-5\nu^2}{2}-\frac{1}{H}|S|-\frac{4H^2+1}{4H^2}-|S|^2\right).$$
This may be rewritten as,
$$\frac{1}{2}\Delta|S|^2 \geq |S|^2\left(\frac{8H^4-12H^2-1}{4H^2}-\frac{1}{H}|S|-|S|^2\right)+\frac{5}{2}(1-\nu^2)|S|^2. \tag{3.6}$$

Observe that
$$\frac{8H^4-12H^2-1}{4H^2}-\frac{1}{H}|S|-|S|^2 \geq q_H(\sup_\Sigma |S|) =: \frac{d}{2} > 0$$

and $(1-\nu^2)|S|^2 \geq 0$. Therefore,
$$\Delta|S|^2 \geq d|S|^2. \tag{3.7}$$

Next we estimate $|S|$.
$$|S| \geq 2H|A| - |\langle T,\cdot\rangle T| - (1-\nu^2) - 4H^2 \geq 2H|A| - 2(1-\nu^2) - 4H^2,$$

i.e,
$$M_H \geq |S| \geq 2H|A| - 2 - 4H^2.$$
Using Gauss' equation (1.3) in $\mathbb{H}^2\times\mathbb{R}$ we find
$$K_\Sigma = K_{ext} - \nu^2 = -\frac{|A|^2}{2} + 2H^2 - \nu^2 \geq -\frac{1}{2}\left(\frac{M_H+2+4H^2}{2H}\right)^2.$$

Now we can use Theorem 1.2 with $u = |S|^2$, i.e, there exists a $\{p_j\}$ in Σ^2 such that
$$\lim_{j\to\infty}|S|^2(p_j) = \sup_\Sigma |S|^2 \textit{ and } \lim_{j\to\infty}\Delta|S|^2(p_j) \leq 0.$$
Inequality (3.7) allows us conclude that $\sup_\Sigma |S|^2 = 0$, i.e, $|S| = 0$ in Σ^2. Then, by using Lemmas 3.1 and 3.2, we conclude the proof.

Second case. Let us consider expression (3.6)
$$\frac{1}{2}\Delta|S|^2 \geq |S|^2\left(\frac{8H^4-12H^2-1}{4H^2}-\frac{1}{H}|S|-|S|^2\right)+\frac{5}{2}(1-\nu^2)|S|^2.$$

As $|S| \leq M_H$, we have that $\dfrac{8H^4 - 12H^2 - 1}{4H^2} - \dfrac{1}{H}|S| - |S|^2 \geq 0$. Hence,

$$\frac{1}{2}\Delta|S|^2 \geq \frac{5}{2}(1 - \nu^2)|S|^2.$$

Integrating and using Stokes' Theorem we write

$$0 \geq \frac{5}{2}\int_\Sigma (1 - \nu^2)|S|^2 d\Sigma \geq 0.$$

Moreover

$$(1 - \nu^2) \cdot |S|^2 = 0. \tag{3.8}$$

Consider $\Theta = \{p \in \Sigma^2; \nu^2(p) = 1\} \subset \mathbb{H}^2 \times \{t_0\}$, for any t_0. Since H is positive we have that Θ has empty interior. Thus, using (3.8), we conclude that $|S|$ vanishes in an open and dense set. By continuity, $|S| = 0$ in Σ. Using Lemma 3.1 and the fact that the only closed surface is S_H^2 we conclude the proof.

■

Theorem 3.6. *There exists no $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ a complete immersed surface with constant mean curvature greater than $\sqrt{\dfrac{12 + \sqrt{176}}{16}} \approx 1.25664$ such that $|S| = M_H$.*

Proof. Suppose that there exists $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ satisfying the condition of the theorem. Using expression (3.6)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2\left(\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{H}|S| - |S|^2\right) + \frac{5}{2}(1 - \nu^2)|S|^2$$

with $|S| = M_H$ we obtain:

$$0 \geq 0 + \frac{5}{2}(1 - \nu^2)M_H^2 \geq 0.$$

Hence $\nu^2 = 1$, i.e, $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ is a slice $\mathbb{H}^2 \times \{t_0\}$. But $\mathbb{H}^2 \times \{t_0\}$ has zero mean curvature, and this is impossible because H is positive.

■

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