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Doctoral Thesis

EXISTENCE AND COMPACTNESS THEOREMS FOR THE YAMABE PROBLEM ON MANIFOLDS WITH BOUNDARY

Sérgio de Moura Almaraz

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Adviser: Fernando Codá Marques

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To my parents Jaime and Vanda

Aos meus pais Jaime e Vanda

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Abstract

Let (M, g) be a compact Riemannian manifold with boundary. This work addresses the Yamabe-type problem of finding a conformal scalar-flat metric on M, which has the boundary as a constant mean curvature hypersurface. In the first part we prove an existence theorem, for the umbilic boundary case, that finishes some remaining cases of this problem. In the second part we prove that the whole set of solutions to this problem is compact for dimensions $n \geq 7$ under the generic condition that the boundary trace-free second fundamental form is nonzero everywhere.

Keywords: Manifold with boundary, Yamabe problem, scalar curvature, mean curvature, Weyl tensor, trace-free second fundamental form.

Resumo

Seja (M, g) uma variedade Riemanniana compacta com bordo. Neste trabalho, estamos interessados no problema do tipo Yamabe que se consiste em encontrar uma métrica conforme, com curvatura escalar zero em M, que tenha o bordo como uma hipersuperfície de curvatura média constante. Na primeira parte, é provado um teorema de existência de soluções, no caso de bordo umbílico, que conclui alguns casos remanescentes desse problema. Na segunda parte, é provado que o conjunto de todas as soluções desse problema é compacto para dimensões $n \geq 7$ sob a condição genérica do bordo ser não umbíco em todos os pontos.

Palavras-chave: Variedade com bordo, problema de Yamabe, curvatura escalar, curvatura média, tensor de Weyl, segunda forma fundamental sem traço.

Contents

CHAPTER 1

Introduction and Preliminaries

In 1960, H. Yamabe ([49]) raised the following question:

YAMABE PROBLEM: *Given* (M^n, g)*, a compact Riemannian manifold (without boundary) of dimension* $n \geq 3$*, is there a Riemannian metric, conformal to* g*, with constant scalar curvature?*

In dimension two this result is a consequence of the uniformization theorem for Riemann surfaces. The Yamabe problem can be viewed as a natural uniformization question for higher dimensions.

This question was affirmatively answered after the works of Yamabe himself, N. Trudinger ([48]), T. Aubin ([4]) and R. Schoen ([43]). (See [34] and [46] for nice surveys on the issue.)

In 1992, J. Escobar ([22]) studied the following Yamabe-type problem, for manifolds with boundary:

YAMABE PROBLEM (boundary version): *Given* (M^n, g)*, a compact Riemannian manifold of dimension* $n \geq 3$ *with boundary, is there a Riemannian metric, conformal to* g*, with zero scalar curvature and constant boundary mean curvature?*

In dimension two, the classical Riemann mapping theorem says that any simply connected, proper domain of the plane is conformally diffeomorphic to a disk. This theorem is false in higher dimensions since the only bounded open subsets of $\mathbb{R}^n$, for $n \geq 3$, that are conformally diffeomorphic to Euclidean balls are the Euclidean balls themselves. The Yamabe-type problem proposed by Escobar can be viewed as an extension of the Riemann mapping theorem for higher dimensions.

In analytical terms, this problem corresponds to finding a positive solution to

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + K u^{\frac{n}{n-2}} = 0, & \text{on } \partial M, \end{cases} \tag{1.0.1}$$

for some constant K, where $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ is the conformal Laplacian and $B_g = \frac{\partial}{\partial \eta} - \frac{n-2}{2} h_g$. Here, Δ_g is the Laplace-Beltrami operator, R_g is the scalar curvature, h_g is the mean curvature of ∂M and η is the inward unit normal vector to ∂M.

The solutions of the equations (1.0.1) are the critical points of the functional

$$Q(u) = \frac{\int_M |\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g}{\left(\int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n-2}{n-1}}} ,$$

where dv_g and $d\sigma_g$ denote the volume forms of M and ∂M, respectively. The exponent $\frac{2(n-1)}{n-2}$ is critical for the Sobolev trace embedding $H^1(M) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial M)$. This embedding is not compact and the functional Q does not satisfy the Palais-Smale condition. For this reason, direct variational methods cannot be applied to find critical points of Q. The approach taken by Escobar uses the compactness of the embedding $H^1(M) \hookrightarrow L^{p+1}(\partial M)$, for $1 < p + 1 < \frac{2(n-1)}{n-2}$, to produce solutions to the subcritical equation

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + K u^p = 0, & \text{on } \partial M, \end{cases} \tag{1.0.2}$$

for $p < \frac{n}{n-2}$. In order to study the convergence of the above solutions to a solution of (1.0.1), he introduced the conformally invariant Sobolev quotient

$$Q(M, \partial M) = \inf\{Q(u);\ u \in C^1(\bar{M}), u \not\equiv 0 \text{ on } \partial M\}$$

and proved that it satisfies

$$Q(M, \partial M) \leq Q(B^n, \partial B)\,.$$

Here, B^n denotes the unit ball in $\mathbb{R}^n_+$ endowed with the Euclidean metric.

Under the hypothesis that $Q(M, \partial M)$ is finite (which is the case when $R_g \geq 0$), Escobar also showed that the strict inequality

$$Q(M, \partial M) < Q(B^n, \partial B) \tag{1.0.3}$$

implies the existence of a minimizing solution u to the equations (1.0.1). The method of using the subcritical solutions was inpired by the classical solution of the Yamabe problem for manifolds without boundary.

In the present work we are interested in two problems involving the equations (1.0.1). In Chapter 2, we will discuss the existence of solutions. In Chapter 3, we will be concerned with the compactness of the whole set of solutions to the equations (1.0.2), including both critical and subcritical exponents. We will give the precise statements in the next section.

Notation. In the rest of this work, (M^n, g) will denote a compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M and finite Sobolev quotient $Q(M, \partial M)$.

1.1 Statements

In [22], Escobar proved the following existence result:

Theorem 1.1.1. *(J. Escobar) Assume that one of the following conditions holds:*

(1) $n \geq 6$ *and M has a nonumbilic point on* ∂M*;*
(2) $n \geq 6$*, M is locally conformally flat and* ∂M *is umbilic;*
(3) $n = 4$ *or* 5 *and* ∂M *is umbilic;*
(4) $n = 3$.

Then $Q(M, \partial M) < Q(B^n, \partial B)$ *and there is a minimizing solution to the equations (1.0.1).*

The proof for $n = 6$ under the condition (1) appeared later, in [23].

Further existence results were obtained by F. Marques in [40] and [41]. Together, these results can be stated as follows:

Theorem 1.1.2. *(F. Marques) Assume that one of the following conditions holds:*

(1) $n \geq 8$, $\overline{W}(x) \neq 0$ *for some* $x \in \partial M$ *and* ∂M *is umbilic;*
(2) $n \geq 9$, $W(x) \neq 0$ *for some* $x \in \partial M$ *and* ∂M *is umbilic;*
(3) $n = 4$ *or* 5 *and* ∂M *is not umbilic.*

Then $Q(M, \partial M) < Q(B^n, \partial B)$ *and there is a minimizing solution to the equations (1.0.1).*

Here, W denotes the Weyl tensor of M and $\overline{W}$ the Weyl tensor of ∂M.

Our existence result, to be proved in Chapter 2, deals with the remaining dimensions $n = 6, 7$ and 8 when the boundary is umbilic and $W \neq 0$ at some boundary point:

Theorem 1.1.3. *Suppose that $n = 6, 7$ or 8, ∂M is umbilic and $W(x) \neq 0$ for some $x \in \partial M$. Then $Q(M, \partial M) < Q(B^n, \partial B)$ and there is a minimizing solution to the equations (1.0.1).*

These cases are similar to the case of dimensions 4 and 5 when the boundary is not umbilic, studied in [41].

Now we turn our attention to compactness issues. In the case of manifolds without boundary, the question of compactness of the full set of solutions to the Yamabe equation was first raised by R. Schoen ([44]) in a topics course at Stanford University in 1988. A necessary condition is that the manifold M^n is not conformally equivalent to the sphere S^n. This problem was studied in [17], [18], [35], [36], [38], [39], [45] and [47] and was completely solved in a series of three papers: [10], [11] and [33]. In [10], S. Brendle discovered the first smooth counterexamples for dimensions $n \geq 52$ (see [7] for nonsmooth examples). In [33], Khuri, Marques and Schoen proved compactness for dimensions $3 \leq n \leq 24$. Their proof contains both a local and a global aspect. The local aspect involves the vanishing of the Weyl tensor at any blow-up point and the global aspect involves the Positive Mass Theorem. Finally, in [11], Brendle and Marques extended the counterexamples of [10] to the remaining dimensions $25 \leq n \leq 51$. In [35], [36] and [39] the authors proved compactness for $n \geq 6$ under the condition that the Weyl tensor is nonzero everywhere.

In Chapter 3, we address the question of compactness of the full set of positive solutions to

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + K u^p = 0, & \text{on } \partial M, \end{cases} \tag{1.1.1}$$

where $1 < p \leq \frac{n}{n-2}$. A necessary condition is that M is not conformally equivalent to B^n. As stated by Escobar in [22], $Q(M, \partial M)$ is positive, zero or negative if the first eigenvalue $\lambda_1(B_g)$ of the problem

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + \lambda u = 0, & \text{on } \partial M \end{cases}$$

is positive, zero or negative, respectively. If $\lambda_1(B_g) < 0$, the solution to the equations (1.1.1) is unique. If $\lambda_1(B_g) = 0$, the equations (1.1.1) become linear and the solutions are unique up to a multiplication by a positive constant. Hence, the only interesting case is the one when $\lambda_1(B_g) > 0$.

We expect that, as in the case of manifolds without boundary, there should be counterexamples to compactness of the set of solutions to the equations (1.1.1) in high dimensions. In this work we address the question of whether compactness of these solutions holds generically in any dimension. Our main compactness result, to be proved in Chapter 3, is the following:

Theorem 1.1.4. *Let (M^n, g) be a Riemannian manifold with dimension $n \geq 7$ and boundary ∂M. Assume that $Q(M, \partial M) > 0$. Let $\{u_i\}$ be a sequence of solutions to the equations (1.1.1) with $p = p_i \in [1 + \gamma_0, \frac{n}{n-2}]$ for any small fixed $\gamma_0 > 0$. Suppose there is a sequence $\{x_i\} \subset \partial M$, $x_i \to x_0$, of local maxima points of $u_i|_{\partial M}$ such that $u_i(x_i) \to \infty$. Then the trace-free 2nd fundamental form of ∂M vanishes at x_0.*

By linear elliptic theory, uniform estimates for the solutions of equation (1.1.1) imply $C^{k,\alpha}$-estimates, for some $0 < \alpha < 1$. By the Harnack-type inequality of Lemma C-3 (proved in [29]), uniform estimates on the boundary ∂M imply uniform estimates on M. Hence, an immediate consequence of Theorem 1.1.4 is a compactness theorem for Riemannian manifolds of dimension $n \geq 7$ that satisfy the condition that the boundary trace-free 2nd fundamental form is nonzero everywhere. More precisely:

Theorem 1.1.5. *Let (M^n, g) be a Riemannian manifold with dimension $n \geq 7$ and boundary ∂M. Suppose $Q(M, \partial M) > 0$ and that the trace-free 2nd fundamental form of ∂M is nonzero everywhere. Then, given a small $\gamma_0 > 0$, there is $C > 0$ such that for any $p \in \left[1 + \gamma_0, \frac{n}{n-2}\right]$ and any $u > 0$ solution to the equations (1.1.1), we have*

$$C^{-1} \leq u \leq C \quad \text{and} \quad ||u||_{C^{2,\alpha}(M)} \leq C\,,$$

for some $0 < \alpha < 1$.

In particular, the set

$$\{\tilde{g} \in [g];\ R_{\tilde{g}} = 0 \text{ in } M,\ h_{\tilde{g}} = 1 \text{ on } \partial M\}$$

is compact.

It was pointed out to me by F. Marques that a transversality argument implies that the second fundamental form condition above is generic for

$n \geq 4$. In other words, the set of the Riemannian metrics on M^n such that the trace-free second fundamental form of ∂M is nonzero everywhere is open and dense in the space of all Riemannian metrics on M for $n \geq 4$.

We should mention that Theorem 1.1.5 does not use the Positive Mass Theorem, since the proof of Theorem 1.1.4 contains only a local argument, based in a Pohozaev-type identity.

The problem of compactness of solutions to the equations (1.1.1) was also studied by V. Felli and M. Ould Ahmedou in the conformally flat case with umbilic boundary ([26]) and in the three-dimensional case with umbilic boundary ([27]). Other compactness results for similar equations were obtained by Z. Han and Y. Li in [29] and by Z. Djadli, A. Malchiodi and M. Ould Ahmedou in [15] and [16].

A consequence of Theorem 1.1.5 is the computation of the total Leray-Schauder degree of all solutions to the equations (1.1.1), as in [26], [27] and [29] (see also [33]). When $\lambda_1(B_g) > 0$, we can define a map $F_p : \bar{\Omega}_\Lambda \to C^{2,\alpha}(M)$ by $F_p(u) = u + T(E(u)u^p)$. Here, T is the operator defined by $T(v) = u$, where u is the unique solution to

$$\begin{cases} L_g u = 0, & \text{in } M\,, \\ B_g u = v, & \text{on } \partial M\,, \end{cases}$$

and $\Omega_\Lambda = \{u \in C^{2,\alpha}(M);\ |u|_{C^{2,\alpha}(M)} < \Lambda,\ u > \Lambda^{-1}\}$. From elliptic theory we know that the map $u \mapsto T(E(u)u^p)$ is compact from $\bar{\Omega}_\Lambda$ into $C^{2,\alpha}(M)$, where $E(u) = \int_M |\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 dv_g + \frac{n-2}{2}\int_{\partial M} h_g u^2 d\sigma_g$ is the energy of u. Hence, F_p is of the form I + compact. If $0 \neq F_p(\partial\Omega_\Lambda)$, we may define the Leray-Schauder degree (see [42]) of F_p in the region Ω_Λ with respect to $0 \in C^{2,\alpha}(M)$, denoted by $\deg(F_p, \Omega_\Lambda, 0)$. Observe that $F_p(u) = 0$ if and only if u is a solution to

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + E(u)u^p = 0, & \text{on } \partial M. \end{cases}$$

Observe that these equations imply that $\int_{\partial M} u^{p+1} d\sigma_g = 1$. By the homotopy invariance of the degree, $\deg(F_p, \Omega_\Lambda, 0)$ is constant for all $p \in \left[1, \frac{n}{n-2}\right]$ provided that $0 \neq F_p(\partial\Omega_\Lambda)$ for all $p \in \left[1, \frac{n}{n-2}\right]$. In the linear case, when $p = 1$, we have $\deg(F_1, \Omega_\Lambda, 0) = -1$. This is the content of Lemma 4.2 of [26], which is a modification of the arguments in [29], pp.528-529. Thus, for Λ sufficiently large, Theorem 1.1.5 allow us to calculate the degree for all $p \in \left[1, \frac{n}{n-2}\right]$. Hence, we have:

Theorem 1.1.6. *Let (M^n, g) satisfy the assumptions of Theorem 1.1.5. Then, for Λ sufficiently large and all $p \in \left[1, \frac{n}{n-2}\right]$, we have $deg(F_p, \Omega_\Lambda, 0) = -1$.*

Other works concerning conformal deformation on manifolds with boundary include [1], [3], [2], [8], [21], [24], [25] and [30].

1.2 Notations

Throughout this work we will make use of the index notation for tensors, commas denoting covariant differentiation. We will adopt the summation convention whenever confusion is not possible. When dealing with Fermi coordinates, we will use indices $1 \leq i, j, k, l, m, p, r, s \leq n-1$ and $1 \leq a, b, c, d \leq n$. Lines under or over an object mean the restriction of the metric to the boundary is involved.

We set $\det g = \det g_{ab}$. We will denote by ∇_g or ∇ the covariant derivative and by Δ_g or Δ the Laplacian-Beltrami operator. The full curvature tensor will be denoted by R_{abcd}, the Ricci tensor by R_{ab} and the scalar curvature by R_g or R. The second fundamental form of the boundary will be denoted by h_{ij} and the mean curvature, $\frac{1}{n-1}tr(h_{ij})$, by h_g or h. By π_{kl} we will denote the trace-free second fundamental form, $h_{kl} - h\bar{g}_{kl}$. The Weyl tensor will be denoted by W_{abcd}.

By $\mathbb{R}^n_+$ we will denote the half-space $\{x = (x_1, ..., x_n) \in \mathbb{R}^n;\ x_n \geq 0\}$. If $x \in \mathbb{R}^n_+$ we set $\bar{x} = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1} \cong \partial\mathbb{R}^n$. We will denote by $B^+_\delta(0)$ (or B^+_δ for short) the half-ball $B_\delta(0) \cap \mathbb{R}^n_+$, where $B_\delta(0)$ is the Euclidean open ball of radius $\delta > 0$ centered at the origin of $\mathbb{R}^n$. Given a subset $C \subset \mathbb{R}^n_+$, we set $\partial^+ C = \partial C \cap (\mathbb{R}^n_+ \backslash \partial\mathbb{R}^n_+)$ and $\partial' C = C \cap \partial\mathbb{R}^n_+$.

In various parts of the text, we will identify a point $x_0 \in \partial M$ with the origin of $\mathbb{R}^n_+$, that meaning we are making use of Fermi coordinates $\varphi : B^+_\delta(0) \to M$, centered at x_0. In that case, we will sometimes write $B^+_\delta(x_0)$ or $B^+_\delta(0)$ instead of $\varphi(B^+_\delta(0))$.

The volume forms of M and ∂M will be denoted by dv_g and $d\sigma_g$, respectively. By η we will denote the inward unit normal vector to ∂M. The n-dimensional sphere of radius r in $\mathbb{R}^{n+1}$ will be denoted by S^n_r. By σ_n we will denote the volume of the n-dimensional unit sphere S^n_1.

For $C \subset M$, we define the energy of a function u in C by

$$E_C(u) = \int_C \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g + \frac{n-2}{2} \int_{\partial' C} h_g u^2 d\sigma_g \,.$$

1.3 Standard solutions in the Euclidean half-space

In this section we will study the Euclidean Yamabe equation in $\mathbb{R}^n_+$ and its linearization.

The simplest example of solution to the Yamabe problem is the ball in $\mathbb{R}^n$ with the canonical Euclidean metric. This ball is conformally equivalent to the half-space $\mathbb{R}^n_+$ by the inversion

$$F : \mathbb{R}^n_+ \to B^n \backslash \{(0, ..., 0, -1)\}$$

with respect to the sphere $S_1^{n-1}(0, ..., 0, -1)$ with center $(0, ..., 0, -1)$ and radius 1. Here, $B^n = B_{1/2}(0, ..., 0, -1/2)$ is the Euclidean ball in $\mathbb{R}^{n+1}$ with center $(0, ..., 0, -1/2)$ and radius $1/2$. The expression for F is

$$F(y_1, ...y_n) = \frac{(y_1, ..., y_{n-1}, y_n + 1)}{y_1^2 + ... + y_{n-1}^2 + (y_n + 1)^2} + (0, ..., 0, -1)$$

and of course its inverse mapping F^{-1} has the same expression. An easy calculation shows that F is a conformal map and $F^* g_{eucl} = U^{\frac{4}{n-2}} g_{eucl}$ in $\mathbb{R}^n_+$, where g_{eucl} is the Euclidean metric and

$$U(y) = (y_1^2 + ... + y_{n-1}^2 + (y_n + 1)^2)^{-\frac{n-2}{2}} .$$

The function U satisfies

$$\begin{cases} \Delta U = 0 \,, & \text{in } \mathbb{R}^n_+ \,, \\ \frac{\partial U}{\partial y_n} + (n-2) U^{\frac{n}{n-2}} = 0 \,, & \text{on } \partial \mathbb{R}^n_+ \,. \end{cases} \tag{1.3.1}$$

Since the equations (1.3.1) are invariant by horizontal translations and scalings with respect to the origin, we obtain the following family of solutions to the equation (1.3.1):

$$U_{\lambda,z}(y) = \left(\frac{\lambda}{\sum_{j=1}^{n-1} (y_j - z_j)^2 + (y_n + \lambda)^2} \right)^{\frac{n-2}{2}} , \tag{1.3.2}$$

where $\lambda \in \mathbb{R}$ and $z = (z_1, ..., z_{n-1}) \in \mathbb{R}^{n-1}$.

The following Liouville-type theorem was proved in [37] (see also [20] and [14]):

Theorem 1.3.1. *(Y. Li, M. Zhu) If $v \geq 0$ is a solution to*

$$\begin{cases} \Delta v = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial v}{\partial y_n} + (n-2) v^{\frac{n}{n-2}} = 0, & \text{on } \partial \mathbb{R}^n_+, \end{cases}$$

then either $v \equiv 0$ or v is of the form (1.3.2).

The following theorem, proved in [32], will be used later:

Theorem 1.3.2. *(B. Hu) If $v \geq 0$ is a solution to*

$$\begin{cases} \Delta v = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial v}{\partial y_n} + (n-2)v^p = 0, & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

with $1 \leq p < \frac{n}{n-2}$ then $v \equiv 0$.

The existence of the family of solutions (1.3.2) has two important consequences. First, we see that the set of solutions to the equations (1.3.1) is non-compact. In particular, the set of solutions to the equations (1.0.1) is not compact when M^n is conformally equivalent to B^n. Secondly, the functions $\frac{\partial U}{\partial y_j}$, for $j = 1, ..., n-1$, and $\frac{n-2}{2}U + y^b\frac{\partial U}{\partial y^b}$, are solutions to the following homogeneous linear problem:

$$\begin{cases} \Delta \psi = 0\,, & \text{in } \mathbb{R}^n_+\,, \\ \frac{\partial \psi}{\partial y_n} + nU^{\frac{2}{n-2}}\psi = 0\,, & \text{on } \partial\mathbb{R}^n_+\,. \end{cases} \tag{1.3.3}$$

Notation. We set $\psi_j = \frac{\partial U}{\partial y_j}$, for $j = 1, ..., n-1$, and $\psi_n = \frac{n-2}{2}U + y^b\frac{\partial U}{\partial y^b}$.

Now, we will show that linear combinations of $\psi_1, ..., \psi_n$ are the only solutions to the equations (1.3.3) under a certain decay hypothesis. This result is similar to the one obtained in [12] for the case of manifols without boundary. More precisely we have:

Lemma 1.3.3. *Suppose ψ is a solution to*

$$\begin{cases} \Delta \psi = 0\,, & \text{in } \mathbb{R}^n_+ \\ \frac{\partial \psi}{\partial y_n} + nU^{\frac{2}{n-2}}\psi = 0\,, & \text{on } \partial\mathbb{R}^n_+\,. \end{cases} \tag{1.3.4}$$

If $\psi(y) = O((1+|y|)^{-\alpha})$ for some $\alpha > 0$, then there exist constants $c_1, ..., c_n$ such that

$$\psi(y) = \sum_{a=1}^{n} c_a\psi_a(y)\,.$$

The following result will be used in the proof of Lemma 1.3.3:

Lemma 1.3.4. *The eigenvalues λ of the problem*

$$\begin{cases} \Delta \bar{\psi} = 0\,, & \text{in } B^n\,, \\ \frac{\partial \bar{\psi}}{\partial \eta} + \lambda\bar{\psi} = 0\,, & \text{on } \partial B^n \end{cases} \tag{1.3.5}$$

are given by $\{\lambda_k = 2k\}_{k=0}^{\infty}$. *The corresponding eigenvectors are the harmonic homogeneous polynomials of degree k restricted to* B^n. *Here, the coefficients of the polynomials are given by the coordinate functions of* $\mathbb{R}^n$ *with center* $(0,...,0,-1/2)$. *In particular, the constant function* 1 *generates the eigenspace associated to the eigenvalue* $\lambda_0 = 0$ *and the coordinate functions* $z_1,...,z_n$ *restricted to* B^n *generate the eigenspace associated to the eigenvalue* λ_1.

Moreover, F takes z_j *to* $\frac{-1}{n-2}U^{-1}\psi_j$, *for* $j = 1,...,n-1$, *and* z_n *to* $\frac{1}{n-2}U^{-1}\psi_n$.

Proof. The first part is an easy consequence of the fact that the spherical harmonics generate $L^2(S^{n-1}_{1/2})$. The last part is a straightforward computation. □

Proof of Lemma 1.3.3. The conformal Laplacian satisfies

$$L_{\phi^{\frac{4}{n-2}}g}(\phi^{-1}u) = \phi^{-\frac{n+2}{n-2}}L_g u,$$

for any smooth functions $\phi > 0$ and u. Similarly, the boundary operator B_g satisfies

$$B_{\phi^{\frac{4}{n-2}}g}(\phi^{-1}u) = \phi^{-\frac{n}{n-2}}B_g u\,.$$

Hence, the equations (1.3.4) are equivalent to

$$\begin{cases} \Delta\bar{\psi} = 0\,, & \text{in } B^n\backslash\{(0,...,0,-1)\}\,,\\ \frac{\partial\bar{\psi}}{\partial\eta} + 2\bar{\psi} = 0\,, & \text{on } \partial B^n\backslash\{(0,...,0,-1)\}\,, \end{cases}$$

where $\bar{\psi} = U^{-1}\psi$. The hypothesis $\psi(y) = O((1+|y|)^{-\alpha})$, $0 < \alpha < n-2$ implies that $\bar{\psi} \in L^p(B^n)$, for any $\frac{n}{n-2} < p < \frac{n}{n-2-\alpha}$. Lemma C-1 ensures that $\bar{\psi}$ is a weak solution to

$$\begin{cases} \Delta\bar{\psi} = 0\,, & \text{in } B^n\,,\\ \frac{\partial\bar{\psi}}{\partial\eta} + 2\bar{\psi} = 0\,, & \text{on } \partial B^n\,. \end{cases}$$

It follows from elliptic theory that $\bar{\psi} \in C^{\infty}(B^n)$. In other words, ψ is a solution to the equations (1.3.4) if and only if $\bar{\psi}$ is an eigenfunction associated to the first nontrivial eigenvalue $\lambda_1 = 2$ of the problem (1.3.5). The result now follows from Lemma 1.3.4. □

1.4 Coordinate expansions for the metric

In this section we will write expansions for the metric g in Fermi coordinates. We will also discuss the concept of conformal Fermi coordinates, introduced by Marques in [40], that will simplify the computations of the next chapters. The conformal Fermi coordinates play the same role that the conformal normal coordinates (see [34]) did in the case of manifolds without boundary. The results of this section are basically proved on pages 1602-1609 and 1618 of [40].

Definition 1.4.1. Let $x_0 \in \partial M$. We choose geodesic normal coordinates $(x_1, ..., x_{n-1})$ on the boundary, centered at x_0. We say that $(x_1, ..., x_n)$, for small $x_n \geq 0$, are the *Fermi coordinates* (centered at x_0) of the point $\exp_x(x_n\eta(x)) \in M$. Here, we denote by $\eta(x)$ the inward unit vector normal to ∂M at $x \in \partial M$.

It is easy to see that in these coordinates $g_{nn} \equiv 1$ and $g_{jn} \equiv 0$, for $j = 1, ..., n-1$.

We fix $x_0 \in \partial M$. Using Fermi coordinates centered at x_0, we work in $B^+_\delta(0) \subset \mathbb{R}^n_+$, for some small $\delta > 0$.

Notation. Set

$$|\partial^r g| = \max_{x\in B^+_\delta(0)} \sum_{|\alpha|=r} \sum_{a,b=1}^{n} |\partial^\alpha g_{ab}|(x)\,,$$

where α denotes a multiindex. We write $|\partial g| = |\partial^1 g|$ for short.

The existence of *conformal Fermi coordinates* and some of its consequences are stated as follows:

Proposition 1.4.2. *For any given integer $N \geq 1$, there is a metric $\tilde{g}$, conformal to g, such that in $\tilde{g}$-Fermi coordinates centered at x_0,*

$$\det \tilde{g} = 1 + O(|x|^N)\,.$$

Moreover, $\tilde{g}$ can be writen as $\tilde{g} = fg$, $f > 0$, with $f(0) = 1$ and $\frac{\partial f}{\partial x_k}(0) = 0$ for $k = 1, ..., n-1$. We also have

$$h(x) = O(|x|^N)\,,$$

where N can be taken arbirarily large.

Proof. The first part is Proposition 3.1 of [40]. The last statement follows from the fact that

$$h_g = \frac{-1}{2(n-1)} g^{ij} g_{ij,n} = \frac{-1}{2(n-1)} (\log\det g)_{,n}\,.$$

□

The next three lemmas will also be used in the computations of the next chapters. The following lemma gives the expansion for the Riemannian metric g in Fermi coordinates:

Lemma 1.4.3. *In Fermi coordinates centered at x_0,*

$$g^{ij}(x) = \delta_{ij} + 2h_{ij}(x_0)x_n + \frac{1}{3}\bar{R}_{ikjl}(x_0)x_kx_l + 2h_{ij;\underline{k}}(x_0)x_nx_k \\ + (R_{ninj} + 3h_{ik}h_{kj})(x_0)x_n^2 + O(|\partial^3 g||x|^3).$$

Suppose further that ∂M is umbilic. Then, in conformal Fermi coordinates centered at x_0, $h_{ij}(x) = O(|x|^N)$, where N can be taken arbirarily large, and

$$g^{ij}(x) = \delta_{ij} + \frac{1}{3}\bar{R}_{ikjl}x_kx_l + R_{ninj}x_n^2 + \frac{1}{6}\bar{R}_{ikjl;m}x_kx_lx_m + R_{ninj;k}x_n^2x_k + \frac{1}{3}R_{ninj;n}x_n^3 \\ + \left(\frac{1}{20}\bar{R}_{ikjl;mp} + \frac{1}{15}\bar{R}_{iksl}\bar{R}_{jmsp}\right)x_kx_lx_mx_p \\ + \left(\frac{1}{2}R_{ninj;kl} + \frac{1}{3}Sym_{ij}(\bar{R}_{iksl}R_{nsnj})\right)x_n^2x_kx_l \\ + \frac{1}{3}R_{ninj;nk}x_n^3x_k + \left(\frac{1}{12}R_{ninj;nn} + \frac{2}{3}R_{nins}R_{nsnj}\right)x_n^4 + O(|\partial^5 g||x|^5)\,.$$

Here, every coefficient is computed at x_0.

Remark 1.4.4. Because of the fact that $h_{ij}(x) = O(|x|^N)$ when the boundary is umbilic, we do not need to use underlined indices in this case.

Lemma 1.4.5. *In conformal Fermi coordinates centered at x_0, we have*

$$\bar{R}_{ij}(x_0) = \bar{R}_{ij;\underline{k}}(x_0) = 0$$

and

$$R_{nn}(x_0) + (h_{ij})^2(x_0) = 0\,.$$

If we suppose that ∂M is umbilic, we have

(i) $\bar{R}_{kl} = Sym_{klm}(\bar{R}_{kl;m}) = 0$;
(ii) $R_{nn} = R_{nn;k} = Sym_{kl}(R_{nn;kl}) = 0$;
(iii) $R_{nn;n} = 0$;
(iv) $Sym_{klmp}(\frac{1}{2}\bar{R}_{kl;mp} + \frac{1}{9}\bar{R}_{ikjl}\bar{R}_{imjp}) = 0$;
(v) $R_{nn;nk} = 0$;
(vi) $R_{nn;nn} + 2(R_{ninj})^2 = 0$;
(vii) $R_{ij} = R_{ninj}$;
(viii) $R_{ijkn} = R_{ijkn;j} = 0$;

(ix) $R = R_{,j} = R_{,n} = 0$*;*
(x) $R_{,ii} = -\frac{1}{6}(\overline{W}_{ijkl})^2$*;*
(xi) $R_{ninj;ij} = -\frac{1}{2}R_{;nn} - (R_{ninj})^2$*;*
where all the quantities are computed at x_0*.*

The idea to prove the first two equations and the items (i),...,(vi) of Lemma 1.4.5 is to express g_{ij} as the exponencial of a matrix A_{ij}. Then we just observe that trace$(A_{ij}) = O(|x|^N)$ for any integer N arbitrarily large. The items (vii)...(xi) are applications of the Gauss and Codazzi equations and the Bianchi identity. We should mention that the item (x) uses the fact that Fermi coordinates are normal on the boundary.

Lemma 1.4.6. *Suppose that* ∂M *is umbilic. Then, in conformal Fermi coordinates centered at* $x_0 \in \partial M$*,* $W_{abcd}(x_0) = 0$ *if and only if*

$$R_{ninj}(x_0) = \overline{W}_{ijkl}(x_0) = 0\,.$$

For the sake of the reader we include the proof of Lemma 1.4.6 here.

Proof of Lemma 1.4.6. Recall that the Weyl tensor is defined by

$$\begin{aligned} W_{abcd} = R_{abcd} - \frac{1}{n-2}\left(R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}\right) \\ + \frac{R}{(n-2)(n-1)}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)\,. \end{aligned} \tag{1.4.1}$$

By the symmetries of the Weyl tensor, $W_{nnnn} = W_{nnni} = W_{nnij} = 0$. By the identity (1.4.1) and Lemma 1.4.5 (viii), $W_{nijk}(x_0) = 0$. From the identity (1.4.1) again and from Lemma 1.4.5 (ii), (vii), (ix),

$$W_{ninj} = \frac{n-3}{n-2}R_{ninj}$$

and

$$W_{ijkl} = \overline{W}_{ijkl} - \frac{1}{n-2}\Big(R_{nink}g_{jl} - R_{ninl}g_{jk} + R_{njnl}g_{ik} - R_{njnk}g_{il}\Big)$$

at x_0. In the last equation we also used the Gauss equation. Now the result follows from the above equations. ∎

CHAPTER 2

The existence theorem

This chapter is devoted to the proof of Theorem 1.1.3. We assume that ∂M is umbilic and choose $x_0 \in \partial M$ such that $W(x_0) \neq 0$. Our proof is explicitly based on constructing a test function ψ such that

$$Q(\psi) < Q(B^n, \partial B)\,. \tag{2.0.1}$$

The function ψ has support in a small half-ball around the point x_0. The usual strategy in this kind of problem (which goes back to Aubin in [4]) consists in defining the function ψ, in the small half-ball, as one of the standard entire solutions to the corresponding Euclidean equations. In our context those are

$$U_\epsilon(x) = U_{\epsilon,(0,\ldots,0)}(x) = \left(\frac{\epsilon}{x_1^2 + \ldots + x_{n-1}^2 + (\epsilon + x_n)^2}\right)^{\frac{n-2}{2}}\,. \tag{2.0.2}$$

where $x = (x_1, \ldots, x_n)$, $x_n \geq 0$.

The next step would be to expand the quotient of ψ in powers of ϵ and, by exploiting the local geometry around x_0, show that the inequality (2.0.1) holds if ϵ is small. In order to simplify the asymptotic analysis, we use conformal Fermi coordinates centered at x_0. This concept, introduced in [40], plays the same role the conformal normal coordinates (see [34]) did in the case of manifolds without boundary.

When $n \geq 9$, the strict inequality (2.0.1) was proved in [40]. The difficulty arises because, when $3 \leq n \leq 8$, the first correction term in the expansion does not have the right sign. When $3 \leq n \leq 5$, Escobar proved the strict inequality by applying the Positive Mass Theorem, a global construction originally due to Schoen ([43]). This argument does not work when $6 \leq n \leq 8$ because the metric is not sufficiently flat around the point x_0.

As we have mentioned before, the situation under the hypothesis of Theorem 1.1.3 is much similar to the cases of dimensions 4 and 5 when the boundary is not umbilic, solved by Marques in [41]. As he pointed out, the test functions U_ϵ are not optimal in these cases but the problem is still local. This kind of phenomenon does not appear in the classical solution of the Yamabe problem for manifolds without boundary. However, perturbed test functions have already been used in the works of Hebey and Vaugon ([31]), Brendle ([9]) and Khuri, Marques and Schoen ([33]).

In order to prove the inequality (2.0.1), inspired by the ideas of Marques, we introduce

$$\phi_\epsilon(x) = \epsilon^{\frac{n-2}{2}} R_{ninj}(x_0) x_i x_j x_n^2 \left(x_1^2 + ... + x_{n-1}^2 + (\epsilon + x_n)^2\right)^{-\frac{n}{2}} .$$

Our test function ψ is defined as $\psi = U_\epsilon + \phi_\epsilon$ around $x_0 \in \partial M$.

In Section 2.1 we prove Theorem 1.1.3 by estimating $Q(\psi)$.

2.1 Estimating the Sobolev quotient

In this section, we will prove Theorem 1.1.3 by constructing a function ψ such that

$$Q(\psi) < Q(B^n, \partial B) .$$

We first recall that the positive number $Q(B^n, \partial B)$ also appears as the best constant in the following Sobolev-trace inequality:

$$\left(\int_{\partial \mathbb{R}^n_+} |u|^{\frac{2(n-1)}{n-2}} d\bar{x}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{Q(B^n, \partial B)} \int_{\mathbb{R}^n_+} |\nabla u|^2 dx ,$$

for every $u \in H^1(\mathbb{R}^n_+)$. It was proven by Escobar ([19]) and independently by Beckner ([6]) that the equality is achieved by the functions U_ϵ, defined in (2.0.2). They are solutions to the boundary-value problem

$$\begin{cases} \Delta U_\epsilon = 0 , & \text{in } \mathbb{R}^n_+ , \\ \frac{\partial U_\epsilon}{\partial y_n} + (n-2) U_\epsilon^{\frac{n}{n-2}} = 0 , & \text{on } \partial \mathbb{R}^n_+ . \end{cases} \tag{2.1.1}$$

One can check, using integration by parts, that

$$\int_{\mathbb{R}^n_+} |\nabla U_\epsilon|^2 dx = (n-2) \int_{\partial \mathbb{R}^n_+} U_\epsilon^{\frac{2(n-1)}{n-2}} dx$$

and also that

$$Q(B^n, \partial B) = (n-2) \left(\int_{\partial \mathbb{R}^n_+} U_\epsilon^{\frac{2(n-1)}{n-2}} dx \right)^{\frac{1}{n-1}} . \tag{2.1.2}$$

Since the Sobolev quotient $Q(M, \partial M)$ is a conformal invariant, we can use conformal Fermi coordinates centered at x_0.

Convention. In what follows, all the curvature terms are evaluated at x_0. We fix conformal Fermi coordinates centered at x_0 and work in a half-ball $B^+_{2\delta} = B^+_{2\delta}(0) \subset \mathbb{R}^n_+$.

In particular, for any N arbitrarily large, we can write the volume element dv_g as

$$dv_g = (1 + O(|x|^N))dx . \tag{2.1.3}$$

In many parts of the text we will use the fact that, for any homogeneous polynomial p_k of degree k,

$$\int_{S^{n-2}_r} p_k = \frac{r^2}{k(k+n-3)} \int_{S^{n-2}_r} \Delta p_k . \tag{2.1.4}$$

We will now construct the test function ψ. Set

$$\phi_\epsilon(x) = \epsilon^{\frac{n-2}{2}} A R_{ninj} x_i x_j x_n^2 \left((\epsilon + x_n)^2 + |\bar{x}|^2 \right)^{-\frac{n}{2}} , \tag{2.1.5}$$

for $A \in \mathbb{R}$ to be fixed later, and

$$\phi(y) = A R_{ninj} y_i y_j y_n^2 \left((1 + y_n)^2 + |\bar{y}|^2 \right)^{-\frac{n}{2}} . \tag{2.1.6}$$

Thus, $\phi_\epsilon(x) = \epsilon^{2-\frac{n-2}{2}} \phi(\epsilon^{-1}x)$. Observe that $U = U_1$. Thus, $U_\epsilon(x) = \epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}x)$. Note that $U_\epsilon(x) + \phi_\epsilon(x) = (1 + O(|x|^2))U_\epsilon(x)$. Hence, if δ is sufficiently small,

$$\frac{1}{2} U_\epsilon \leq U_\epsilon + \phi_\epsilon \leq 2U_\epsilon , \quad \text{in } B^+_{2\delta} .$$

Let $r \mapsto \chi(r)$ be a smooth cut-off function satisfying $\chi(r) = 1$ for $0 \leq r \leq \delta$, $\chi(r) = 0$ for $r \geq 2\delta$, $0 \leq \chi \leq 1$ and $|\chi'(r)| \leq C\delta^{-1}$ if $\delta \leq r \leq 2\delta$. Our test function is defined by

$$\psi(x) = \chi(|x|)(U_\epsilon(x) + \phi_\epsilon(x)) .$$

2.1.1 Estimating the energy of ψ

The energy of ψ is given by

$$\begin{aligned} E_M(\psi) &= \int_M \left(|\nabla_g \psi|^2 + \frac{n-2}{4(n-1)} R_g \psi^2 \right) dv_g + \frac{n-2}{2} \int_{\partial M} h_g \psi^2 d\sigma_g \\ &= E_{B^+_\delta}(\psi) + E_{B^+_{2\delta} \backslash B^+_\delta}(\psi) \,. \end{aligned} \tag{2.1.7}$$

Observe that

$$|\nabla_g \psi|^2 \leq C |\nabla \psi|^2 \leq C |\nabla \chi|^2 (U_\epsilon + \phi_\epsilon)^2 + C \chi^2 |\nabla (U_\epsilon + \phi_\epsilon)|^2 \,.$$

Hence,

$$\begin{aligned} E_{B^+_{2\delta} \backslash B^+_\delta}(\psi) \leq C \int_{B^+_{2\delta} \backslash B^+_\delta} |\nabla \chi|^2 U_\epsilon^2 dx + C \int_{B^+_{2\delta} \backslash B^+_\delta} \chi^2 |\nabla U_\epsilon|^2 dx \\ + C \int_{B^+_{2\delta} \backslash B^+_\delta} R_g U_\epsilon^2 dx + C \int_{\partial' B^+_{2\delta} \backslash \partial' B^+_\delta} h_g U_\epsilon^2 d\bar{x} \,, \end{aligned}$$

Thus,

$$E_{B^+_{2\delta} \backslash B^+_\delta}(\psi) \leq C \epsilon^{n-2} \delta^{2-n} \,. \tag{2.1.8}$$

The first term in the right hand side of (2.1.7) is

$$\begin{aligned} E_{B^+_\delta}(\psi) &= E_{B^+_\delta}(U_\epsilon + \phi_\epsilon) \\ &= \int_{B^+_\delta} \left\{ |\nabla_g (U_\epsilon + \phi_\epsilon)|^2 + \frac{n-2}{4(n-1)} R_g (U_\epsilon + \phi_\epsilon)^2 \right\} dv_g \\ &\quad + \frac{n-2}{2} \int_{\partial' B^+_\delta} h_g (U_\epsilon + \phi_\epsilon)^2 d\sigma_g \\ &= \int_{B^+_\delta} |\nabla (U_\epsilon + \phi_\epsilon)|^2 dx + C \epsilon^{n-2} \delta \\ &\quad + \int_{B^+_\delta} (g^{ij} - \delta^{ij}) \partial_i (U_\epsilon + \phi_\epsilon) \partial_j (U_\epsilon + \phi_\epsilon) dx \\ &\quad\quad + \frac{n-2}{4(n-1)} \int_{B^+_\delta} R_g (U_\epsilon + \phi_\epsilon)^2 dx \,. \end{aligned} \tag{2.1.9}$$

Here, we used the identity (2.1.3) for the volume term and Proposition 1.4.2 for the integral envolving h_g.

Now, we will handle each of the three integral terms in the right hand side of (2.1.9) in the next three lemmas.

Lemma 2.1.1. *We have,*

$$\int_{B^+_\delta} |\nabla(U_\epsilon + \phi_\epsilon)|^2 dx \le Q(B^n, \partial B^n)\left(\int_{\partial M} \psi^{\frac{2(n-1)}{n-2}} dx\right)^{\frac{n-2}{n-1}} + C\epsilon^{n-2}\delta^{2-n}$$
$$- \frac{4}{(n+1)(n-1)}\epsilon^4 A^2 (R_{ninj})^2 \int_{B_{\delta\epsilon^{-1}}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n} dy$$
$$+ \frac{8n}{(n+1)(n-1)}\epsilon^4 A^2 (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}} dy$$
$$+ \frac{12n}{(n+1)(n-1)}\epsilon^4 A^2 (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}} dy$$

Proof. Since $R_{nn} = 0$ (see Lemma 1.4.5(ii)), $\int_{S^{n-2}_r} R_{ninj} y_i y_j d\sigma_r(y) = 0$. Thus, we see that

$$\int_{B^+_\delta} |\nabla(U_\epsilon + \phi_\epsilon)|^2 dx = \int_{B^+_\delta} |\nabla U_\epsilon|^2 dx + \int_{B^+_\delta} |\nabla \phi_\epsilon|^2 dx\,. \tag{2.1.10}$$

Integrating by parts equations (2.1.1) and using the identity (2.1.2) we obtain

$$\int_{B^+_\delta} |\nabla U_\epsilon|^2 dx \le Q(B^n, \partial B^n)\left(\int_{\partial' B^+_\delta} U_\epsilon^{\frac{2(n-1)}{n-2}} dx\right)^{\frac{n-2}{n-1}} \le Q(B^n, \partial B^n)\left(\int_{\partial M} \psi^{\frac{2(n-1)}{n-2}} dx\right)^{\frac{n-2}{n-1}}.$$

In the first inequality above we used the fact that $\frac{\partial U_\epsilon}{\partial \eta} > 0$ on $\partial^+ B^+_\delta$, where η denotes the inward unit normal vector . In the second one we used the fact that $\phi_\epsilon = 0$ on ∂M.

For the second term in the right hand side of (2.1.10), an integration by parts plus a change of variables gives

$$\int_{B^+_\delta} |\nabla \phi_\epsilon|^2 dx \le -\epsilon^4 \int_{B^+_{\epsilon^{-1}\delta}} (\Delta\phi)\phi dy + C\epsilon^{n-2}\delta^{2-n}\,,$$

since $\int_{\partial' B^+_\delta} \frac{\partial \phi_\epsilon}{\partial x_n}\phi_\epsilon d\bar{x} = 0$ and the term $\epsilon^{n-2}\delta^{2-n}$ comes from the integral over $\partial^+ B^+_\delta$.

Claim. The function ϕ satisfies

$$\Delta\phi(y) = 2AR_{ninj}y_iy_j((1+y_n)^2+|\bar{y}|^2)^{-\frac{n}{2}} - 4nAR_{ninj}y_iy_jy_n((1+y_n)^2+|\bar{y}|^2)^{-\frac{n+2}{2}}$$
$$- 6nAR_{ninj}y_iy_jy_n^2((1+y_n)^2+|\bar{y}|^2)^{-\frac{n+2}{2}}\,.$$

In order to prove the Claim we set $Z(y) = ((1+y_n)^2+|\bar{y}|^2)$. Since $R_{nn}=0$,

$$\begin{aligned}\Delta(R_{ninj}y_iy_jy_n^2Z^{-\frac{n}{2}}) &= \Delta(R_{ninj}y_iy_jy_n^2)Z^{-\frac{n}{2}} + R_{ninj}y_iy_jy_n^2\Delta(Z^{-\frac{n}{2}})\\ &\quad + 2\partial_k(R_{ninj}y_iy_jy_n^2)\partial_k(Z^{-\frac{n}{2}}) + 2\partial_n(R_{ninj}y_iy_jy_n^2)\partial_n(Z^{-\frac{n}{2}})\\ &= 2R_{ninj}y_iy_jZ^{-\frac{n}{2}} + 2nR_{ninj}y_iy_jy_n^2Z^{-\frac{n+2}{2}}\\ &\quad - 4nR_{ninj}y_iy_jy_n^2Z^{-\frac{n+2}{2}} - 4nR_{ninj}y_iy_jy_n(y_n+1)Z^{-\frac{n+2}{2}}\\ &= 2R_{ninj}y_iy_jZ^{-\frac{n}{2}} - 6nR_{ninj}y_iy_jy_n^2Z^{-\frac{n+2}{2}}\\ &\qquad - 4nR_{ninj}y_iy_jy_nZ^{-\frac{n+2}{2}}\,.\end{aligned}$$

This proves the Claim.

Using the above claim,

$$\begin{aligned}\int_{B^+_{\delta\epsilon^{-1}}}(\Delta\phi)\phi dy &= 2A^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n}R_{ninj}R_{nknl}y_iy_jy_ky_ly_n^2dy\\ &\quad - 4nA^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n-1}R_{ninj}R_{nknl}y_iy_jy_ky_ly_n^3dy\\ &\quad - 6nA^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n-1}R_{ninj}R_{nknl}y_iy_jy_ky_ly_n^4dy\,.\end{aligned}$$

Since $\Delta^2(R_{ninj}R_{nknl}y_iy_jy_ky_l) = 16(R_{ninj})^2$,

$$\int_{S_r^{n-2}} R_{ninj}R_{nknl}y_iy_jy_ky_ld\sigma_r = \frac{2\sigma_{n-2}}{(n+1)(n-1)}r^{n+2}(R_{ninj})^2\,.$$

Thus,

$$\begin{aligned}\int_{B^+_{\delta\epsilon^{-1}}}(\Delta\phi)\phi dy =&\frac{4}{(n+1)(n-1)}A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy\\ &- \frac{8n}{(n+1)(n-1)}A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy\\ &- \frac{12n}{(n+1)(n-1)}A^2(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy\,.\end{aligned}$$

Hence,

$$\int_{B^+_\delta} |\nabla\phi_\epsilon|^2 dx \leq -\frac{4}{(n+1)(n-1)}\epsilon^4 A^2 (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n} dy$$
$$+ \frac{8n}{(n+1)(n-1)}\epsilon^4 A^2 (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}} dy$$
$$+ \frac{12n}{(n+1)(n-1)}\epsilon^4 A^2 (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}} dy$$
$$+ C\epsilon^{n-2}\delta^{2-n}\,.$$

□

Lemma 2.1.2. *We have,*

$$\int_{B^+_\delta} (g^{ij}-\delta^{ij})\partial_i(U_\epsilon+\phi_\epsilon)\partial_j(U_\epsilon+\phi_\epsilon)dx =$$
$$\frac{(n-2)^2}{(n+1)(n-1)}\epsilon^4 R_{ninj;ij} \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n} dy$$
$$+ \frac{(n-2)^2}{2(n-1)}\epsilon^4 (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^n} dy$$
$$- \frac{4n(n-2)}{(n+1)(n-1)}\epsilon^4 A (R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n+1}} dy + E_1\,,$$

where

$$E_1 = \begin{cases} O(\epsilon^4\delta^{-4}) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,. \end{cases}$$

Proof. Observe that

$$\int_{B^+_\delta} (g^{ij}-\delta^{ij})\partial_i(U_\epsilon+\phi_\epsilon)\partial_j(U_\epsilon+\phi_\epsilon)dx = \int_{B^+_\delta} (g^{ij}-\delta^{ij})\partial_i U_\epsilon \partial_j U_\epsilon dx \qquad (2.1.11)$$
$$+ 2\int_{B^+_\delta} (g^{ij}-\delta^{ij})\partial_i U_\epsilon \partial_j \phi_\epsilon dx + \int_{B^+_\delta} (g^{ij}-\delta^{ij})\partial_i \phi_\epsilon \partial_j \phi_\epsilon dx\,.$$

We will handle separately the three terms in the right hand side of (2.1.11). The first term is

$$\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_i U_\epsilon(x)\partial_j U_\epsilon(x)dx = \int_{B^+_{\delta\epsilon^{-1}}}(g^{ij}-\delta^{ij})(\epsilon y)\partial_i U(y)\partial_j U(y)dy$$
$$= (n-2)^2\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n}(g^{ij}-\delta^{ij})(\epsilon y)y_iy_jdy\,.$$

Hence, using Lemma A-1 we obtain

$$\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_i U_\epsilon(x)\partial_j U_\epsilon(x)dx =$$
$$\frac{(n-2)^2}{(n+1)(n-1)}\epsilon^4 R_{ninj;ij}\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy$$
$$+\frac{(n-2)^2}{2(n-1)}\epsilon^4(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^n}dy + E_1'\,,$$

where

$$E_1' = \begin{cases} O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,.\end{cases}$$

The second term is

$$2\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_i U_\epsilon(x)\partial_j\phi_\epsilon(x)dx \tag{2.1.12}$$
$$= -2\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_i\partial_j U_\epsilon(x)\phi_\epsilon(x)dx - 2\int_{B^+_\delta}(\partial_i g^{ij})(x)\partial_j U_\epsilon(x)\phi_\epsilon(x)dx$$
$$+ O(\epsilon^{n-2}\delta^{2-n})$$
$$= -2\epsilon^2\int_{B^+_{\delta\epsilon^{-1}}}(g^{ij}-\delta^{ij})(\epsilon y)\partial_i\partial_j U(y)\phi(y)dy - 2\epsilon^3\int_{B^+_{\delta\epsilon^{-1}}}(\partial_i g^{ij})(\epsilon y)\partial_j U(y)\phi(y)dy$$
$$+ O(\epsilon^{n-2}\delta^{2-n})\,.$$

But,

$$-2\epsilon^2\int_{B^+_{\delta\epsilon^{-1}}}(g^{ij}-\delta^{ij})(\epsilon y)\partial_i\partial_j U(y)\phi(y)dy \tag{2.1.13}$$

$$= -2(n-2)\epsilon^2 A\int_{B^+_{\delta\epsilon^{-1}}}((1+y_n)^2+|\bar{y}|^2)^{-n-1}(g^{ij}-\delta^{ij})(\epsilon y)$$

$$\cdot\left\{ny_iy_j-((1+y_n)^2+|\bar{y}|^2)\delta_{ij}\right\}R_{nknl}y_ky_ly_n^2dy$$

$$= -\frac{4n(n-2)}{(n+1)(n-1)}\epsilon^4A(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy+E_2',$$

where

$$E_2'=\begin{cases}O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,.\end{cases}$$

In the last equality of 2.1.13, we used Lemma A-2 and the fact that Lemma 1.4.3, together with Lemma 1.4.5(i),(ii),(iii), implies

$$\int_{S_r^{n-2}}(g^{ij}-\delta^{ij})(\epsilon y)\delta_{ij}R_{nknl}y_ky_ld\sigma_r(y)=\int_{S_r^{n-2}}O(\epsilon^4|y|^4)R_{nknl}y_ky_ld\sigma_r(y)\,.$$

We also have, by Lemma 1.4.3 and Lemma 1.4.5(i),

$$-2\epsilon^3\int_{B^+_{\delta\epsilon^{-1}}}(\partial_ig^{ij})(\epsilon y)\partial_jU(y)\phi(y)dy=E_3'=\begin{cases}O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,.\end{cases}$$

Hence,

$$2\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_iU_\epsilon(x)\partial_j\phi_\epsilon(x)dx=E_2'+E_3'$$

$$-\frac{4n(n-2)}{(n+1)(n-1)}\epsilon^4A(R_{ninj})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy\,.$$

Finally, the third term in the right hand side of (2.1.11) is written as

$$\int_{B^+_\delta}(g^{ij}-\delta^{ij})(x)\partial_i\phi_\epsilon(x)\partial_j\phi_\epsilon(x)dx=\epsilon^4\int_{B^+_{\delta\epsilon^{-1}}}(g^{ij}-\delta^{ij})(\epsilon y)\partial_i\phi(y)\partial_j\phi(y)dy$$

$$=\begin{cases}O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,.\end{cases}$$

The result now follows if we choose ϵ small such that $\log(\delta\epsilon^{-1}) > \delta^{2-n}$. □

Lemma 2.1.3. *We have,*

$$\frac{n-2}{4(n-1)}\int_{B^+_\delta} R_g(U_\epsilon+\phi_\epsilon)^2dx = \frac{n-2}{8(n-1)}\epsilon^4 R_{;nn}\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy$$
$$-\frac{n-2}{24(n-1)^2}\epsilon^4(\overline{W}_{ijkl})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy + E_2\,,$$

where

$$E_2 = \begin{cases} O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,.\end{cases}$$

Proof. We first observe that

$$\int_{B^+_\delta} R_g(U_\epsilon+\phi_\epsilon)^2dx = \int_{B^+_\delta} R_gU_\epsilon^2dx + 2\int_{B^+_\delta} R_gU_\epsilon\phi_\epsilon dx + \int_{B^+_\delta} R_g\phi_\epsilon^2dx\,. \quad (2.1.14)$$

We will handle each term in the right hand side of (2.1.14) separately. Using Lemma A-3, we see that the first term is

$$\int_{B^+_\delta} R_g(x)U_\epsilon(x)^2dx = \epsilon^2\int_{B^+_{\delta\epsilon^{-1}}} R_g(\epsilon y)U^2(y)dy \quad (2.1.15)$$
$$= \frac{1}{2}\epsilon^4R_{;nn}\int_{B^+_{\delta\epsilon^{-1}}}\frac{y_n^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy + E_4'$$
$$-\frac{1}{12(n-1)}\epsilon^4(\overline{W}_{ijkl})^2\int_{B^+_{\delta\epsilon^{-1}}}\frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy\,,$$

where

$$E_4' = \begin{cases} O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,.\end{cases}$$

By Lemma 1.4.5(ix), the second term is

$$2\int_{B_\delta^+} R_g(x)U_\epsilon(x)\phi_\epsilon(x)dx = 2\epsilon^4\int_{B^+_{\delta\epsilon^{-1}}} R_g(\epsilon y)U(y)\phi(y)dy$$
$$= \begin{cases} O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8 \end{cases}$$

and the last term is

$$\int_{B_\delta^+} R_g\phi_\epsilon^2 dx = \begin{cases} O(\epsilon^4\delta) & \text{if } n=6\,,\\ O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n\geq 8\,. \end{cases}$$

□

2.1.2 Proof of Theorem 1.1.3

Now, we proceed to the proof of Theorem 1.1.3.

Proof of Theorem 1.1.3. It folows from Lemmas 2.1.1, 2.1.2 and 2.1.3 and the identities (2.1.7), (2.1.8) and (2.1.9) that

$$
\begin{aligned}
E_M(\psi) \leq Q(B^n, \partial B^n)\left(\int_{\partial M} \psi^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} &+ E \\
&- \epsilon^4 \frac{4A^2}{(n+1)(n-1)}(R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy \\
&+ \epsilon^4 \frac{(n-2)^2}{(n+1)(n-1)} R_{ninj;ij} \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^n}dy \\
&+ \epsilon^4 \frac{8nA^2}{(n+1)(n-1)}(R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^3|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy \\
&+ \epsilon^4 \frac{12nA^2}{(n+1)(n-1)}(R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy \\
&- \epsilon^4 \frac{4n(n-2)A}{(n+1)(n-1)}(R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^4}{((1+y_n)^2+|\bar{y}|^2)^{n+1}}dy \\
&+ \epsilon^4 \frac{(n-2)^2}{2(n-1)}(R_{ninj})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^n}dy \\
&+ \epsilon^4 \frac{n-2}{8(n-1)} R_{;nn} \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy \\
&- \epsilon^4 \frac{n-2}{48(n-1)^2}(\overline{W}_{ijkl})^2 \int_{B^+_{\delta\epsilon^{-1}}} \frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy\,. \qquad (2.1.16)
\end{aligned}
$$

where

$$
E = \begin{cases} O(\epsilon^4\delta^{-4}) & \text{if } n = 6\,, \\ O(\epsilon^5 \log(\delta\epsilon^{-1})) & \text{if } n = 7\,, \\ O(\epsilon^5) & \text{if } n \geq 8\,. \end{cases}
$$

We divide the rest of the proof in two cases.

<u>The case $n = 7, 8$.</u>

Set $I = \int_0^\infty \frac{r^n}{(r^2+1)^n}dr$. We will apply the change of variables $\bar{z} = (1+y_n)^{-1}\bar{y}$ and Lemmas B-1 and B-2 in order to compare the different integrals in the expansion (2.1.16).

These integrals are

$$I_1 = \int_{\mathbb{R}^n_+} \frac{y_n^2|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^n} dy_n d\bar{y} = \int_0^\infty y_n^2(1+y_n)^{3-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^n} d\bar{z}$$
$$= \frac{2(n+1)\,\sigma_{n-2}\,I}{(n-3)(n-4)(n-5)(n-6)}\,,$$

$$I_2 = \int_{\mathbb{R}^n_+} \frac{y_n^3|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^{n+1}} dy_n d\bar{y} = \int_0^\infty y_n^3(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^{n+1}} d\bar{z}$$
$$= \frac{3(n+1)\,\sigma_{n-2}\,I}{n(n-2)(n-3)(n-4)(n-5)}\,,$$

$$I_3 = \int_{\mathbb{R}^n_+} \frac{y_n^4|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^{n+1}} dy_n d\bar{y} = \int_0^\infty y_n^4(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^{n+1}} d\bar{z}$$
$$= \frac{12(n+1)\,\sigma_{n-2}\,I}{n(n-2)(n-3)(n-4)(n-5)(n-6)}\,,$$

$$I_4 = \int_{\mathbb{R}^n_+} \frac{y_n^4|\bar{y}|^2}{((1+t)^2+|\bar{y}|^2)^n} dy_n d\bar{y} = \int_0^\infty y_n^4(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^2}{(1+|\bar{z}|^2)^n} d\bar{z}$$
$$= \frac{24\,\sigma_{n-2}\,I}{(n-2)(n-3)(n-4)(n-5)(n-6)}$$

and

$$I_5 = \int_{\mathbb{R}^n_+} \frac{y_n^2}{((1+t)^2+|\bar{y}|^2)^{n-2}} dy_n d\bar{y} = \int_0^\infty y_n^2(1+y_n)^{3-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{1}{(1+|\bar{z}|^2)^{n-2}} d\bar{z}$$
$$= \frac{8(n-2)\,\sigma_{n-2}\,I}{(n-3)(n-4)(n-5)(n-6)}\,.$$

Thus,

$$
\begin{aligned}
E_M(\psi) \le{}& Q(B^n,\partial B^n)\left(\int_{\partial M}\psi^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} + E' \\
&+ \epsilon^4\left\{-\frac{4A^2}{(n+1)(n-1)}I_1 + \frac{8nA^2}{(n+1)(n-1)}I_2 + \frac{(n-2)^2}{2(n-1)}I_4\right\}(R_{ninj})^2 \\
&+ \epsilon^4\left\{\frac{12nA^2}{(n+1)(n-1)} - \frac{4n(n-2)A}{(n+1)(n-1)}\right\} I_3\cdot(R_{ninj})^2 \\
&+ \epsilon^4\frac{(n-2)^2}{(n+1)(n-1)}I_1\cdot R_{ninj;ij} + \epsilon^4\frac{n-2}{8(n-1)}I_5\cdot R_{;nn} \\
&- \epsilon^4\frac{n-2}{48(n-1)^2}(\overline{W}_{ijkl})^2\int_{\mathbb{R}^n_+}\frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy\,. \qquad (2.1.17)
\end{aligned}
$$

where

$$
E' = \begin{cases} O(\epsilon^5\log(\delta\epsilon^{-1})) & \text{if } n=7\,,\\ O(\epsilon^5) & \text{if } n=8\,.\end{cases}
$$

Using Lemma 1.4.5(xi) and substituting the expressions obtained for I_1,...,I_5 in the expansion (2.1.17), the coefficients of $R_{ninj;ij}$ and $R_{;nn}$ cancel out and we obtain

$$
\begin{aligned}
E_M(\psi) \le{}& Q(B^n,\partial B^n)\left(\int_{\partial M}\psi^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} + E' \\
&+ \epsilon^4\sigma_{n-2}I\cdot\gamma\left\{16(n+1)A^2 - 48(n-2)A + 2(8-n)(n-2)^2\right\}(R_{ninj})^2 \\
&- \epsilon^4\frac{n-2}{48(n-1)^2}(\overline{W}_{ijkl})^2\int_{\mathbb{R}^n_+}\frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}}dy\,, \qquad (2.1.18)
\end{aligned}
$$

where

$$
\gamma = \frac{1}{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}\,.
$$

Choosing $A=1$, the term $16(n+1)A^2 - 48(n-2)A + 2(8-n)(n-2)^2$ in the expansion (2.1.18) is -62 for $n=7$ and -144 for $n=8$. Thus, for small ϵ, since $W_{abcd}(x_0)\neq 0$, the expansion (2.1.18) together with Lemma 1.4.6 implies that

$$
E_M(\psi) < Q(B^n,\partial B^n)\left(\int_{\partial M}\psi^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}}
$$

for dimensions 7 and 8.

The case $n = 6$.

We will again apply the change of variables $\bar{z} = (1+y_n)^{-1}\bar{y}$ and Lemma B-1 in order to compare the different integrals in the expansion (2.1.16). In the next estimates we are always assuming $n = 6$.

In this case, the first integral is

$$\begin{aligned} I_{1,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^n} dy_n d\bar{y} \\ &= \int_{B^+_{\delta\epsilon^{-1}} \cap \{y_n \leq \delta/2\epsilon\}} \frac{y_n^2|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^n} dy_n d\bar{y} + O(1) \\ &= \int_{\mathbb{R}^n_+ \cap \{y_n \leq \delta/2\epsilon\}} \frac{y_n^2|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^n} dy_n d\bar{y} + O(1)\,. \end{aligned}$$

Hence,

$$\begin{aligned} I_{1,\delta/\epsilon} &= \int_0^{\delta/2\epsilon} y_n^2(1+y_n)^{3-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^n} d\bar{z} + O(1) \\ &= \log(\delta\epsilon^{-1}) \frac{n+1}{n-3} \sigma_{n-2}\, I + O(1)\,. \end{aligned}$$

The second integral is

$$I_{2,\delta/\epsilon} = \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^3|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^{n+1}} dy_n d\bar{y} = O(1)\,.$$

Similarly to $I_{1,\delta/\epsilon}$, the others integrals are

$$\begin{aligned} I_{3,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^4}{((1+t)^2+|\bar{y}|^2)^{n+1}} dy_n d\bar{y} \\ &= \int_0^{\delta/2\epsilon} y_n^4(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^4}{(1+|\bar{z}|^2)^{n+1}} d\bar{z} + O(1) \\ &= \log(\delta\epsilon^{-1}) \frac{n+1}{2n} \sigma_{n-2}\, I + O(1)\,, \end{aligned}$$

$$\begin{aligned} I_{4,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^4|\bar{y}|^2}{((1+t)^2+|\bar{y}|^2)^{n}} dy_n d\bar{y} \\ &= \int_0^{\delta/2\epsilon} y_n^4(1+y_n)^{1-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^2}{(1+|\bar{z}|^2)^{n}} d\bar{z} \\ &= \log(\delta\epsilon^{-1}) \sigma_{n-2}\, I + O(1)\,, \end{aligned}$$

$$\begin{aligned}
I_{5,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{y_n^2}{((1+t)^2+|\bar{y}|^2)^{n-2}} dy_n d\bar{y} \\
&= \int_0^{\delta/2\epsilon} y_n^2(1+y_n)^{3-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{1}{(1+|\bar{z}|^2)^{n-2}} d\bar{z} + O(1) \\
&= \log(\delta\epsilon^{-1}) \frac{4(n-2)}{n-3} \sigma_{n-2}\, I + O(1)
\end{aligned}$$

and

$$\begin{aligned}
I_{6,\delta/\epsilon} &= \int_{B^+_{\delta\epsilon^{-1}}} \frac{|\bar{y}|^2}{((1+y_n)^2+|\bar{y}|^2)^{n-2}} dy_n d\bar{y} \\
&= \int_0^{\delta/2\epsilon} (1+y_n)^{5-n} dy_n \int_{\mathbb{R}^{n-1}_+} \frac{|\bar{z}|^2}{(1+|\bar{z}|^2)^{n-2}} d\bar{z} + O(1) \\
&= \log(\delta\epsilon^{-1}) \frac{4(n-1)(n-2)}{(n-3)(n-5)} \sigma_{n-2}\, I + O(1)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
E_M(\psi) \leq\; & Q(B^n, \partial B^n) \left(\int_{\partial M} \psi^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} + O(\epsilon^4\delta^{-4}) \\
&+ \epsilon^4 \left\{ -\frac{4A^2}{(n+1)(n-1)} I_{1,\delta/\epsilon} + \frac{(n-2)^2}{2(n-1)} I_{4,\delta/\epsilon} \right\} (R_{ninj})^2 \\
&+ \epsilon^4 \left\{ \frac{12nA^2}{(n+1)(n-1)} - \frac{4n(n-2)A}{(n+1)(n-1)} \right\} I_{3,\delta/\epsilon} \cdot (R_{ninj})^2 \\
&+ \epsilon^4 \frac{(n-2)^2}{(n+1)(n-1)} I_{1,\delta/\epsilon} \cdot R_{ninj;ij} + \epsilon^4 \frac{n-2}{8(n-1)} I_{5,\delta/\epsilon} \cdot R_{;nn} \\
&- \epsilon^4 \frac{n-2}{48(n-1)^2} I_{6,\delta/\epsilon} \cdot (\overline{W}_{ijkl})^2\,. \qquad (2.1.19)
\end{aligned}$$

Using Lemma 1.4.5(xi) and substituting the expressions obtained for $I_{1,\delta/\epsilon}$,...,$I_{6,\delta/\epsilon}$ in expansion (2.1.19), the coefficients of $R_{ninj;ij}$ and $R_{;nn}$ cancel out and we obtain

$$
\begin{aligned}
E_M(\psi) \le\; & Q(B^n, \partial B^n)\left(\int_{\partial M} \psi^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} + O(\epsilon^4\delta^{-4}) \\
& + \epsilon^4 \log(\delta\epsilon^{-1})\sigma_{n-2} I \cdot \\
& \quad \left\{\frac{6(n-3)-4}{(n-1)(n-3)}A^2 - \frac{2(n-2)}{n-1}A + \frac{(n-2)^2(n-5)}{2(n-1)(n-3)}\right\}(R_{ninj})^2 \\
& - \epsilon^4 \log(\delta\epsilon^{-1})\sigma_{n-2} I \frac{(n-2)^2}{12(n-1)(n-3)(n-5)}(\overline{W}_{ijkl})^2 . \qquad (2.1.20)
\end{aligned}
$$

Choosing $A = 1$, the term $\frac{6(n-3)-4}{(n-1)(n-3)}A^2 - \frac{2(n-2)}{n-1}A + \frac{(n-2)^2(n-5)}{2(n-1)(n-3)}$ in the expansion (2.1.20) is $-\frac{2}{15}$ for $n = 6$. Thus, for small ϵ, since $W_{abcd}(x_0) \neq 0$, the expansion (2.1.20) together with Lemma 1.4.6 implies that

$$
E_M(\psi) < Q(B^n, \partial B^n)\left(\int_{\partial M} \psi^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}}
$$

for dimension $n = 6$. □

CHAPTER 3

The compactness theorem

This chapter is devoted to the proof of our compactness result, Theorem 1.1.4. We will now outline the proof of this theorem. The strategy of the proof is similar to the one proposed by Schoen in the case of manifolds without boundary. It is based on finding local obstructions to blow-up by means of a Pohozaev-type identity. We suppose that there is a simple blow-up point for a sequence $\{u_i\}$. We then approximate the sequence $\{u_i\}$ by the standard Euclidean solution plus a correction term ϕ_i. The function ϕ_i is defined as a solution to a non-homogeneous linear equation. We then use the Pohozaev identity to prove that the boundary trace-free 2nd fundamental form vanishes at the blow-up point. Finally we apply the Pohozaev identity to establish, after rescaling arguments, a sign condition that allows the reduction to the simple blow-up case.

An important part in our proof is the use of the correction term ϕ_i to obtain refined pointwise blow-up estimates. The idea of using a correction term first appeared in [31] and was significantly improved in [9]. This type of blow-up estimate was derived in [33] where the authors studied compactness in the case of manifolds without boundary. Although we do not have the kind of explicit control of the terms ϕ_i the authors had in [33], a key observation is that some orthogonality conditions are sufficient to obtain the vanishing of the boundary trace-free 2nd fundamental form.

In Section 3.1 we discuss the equations we will work with. In Section 3.2 we prove the Pohozaev identity we will work with. In Section 3.3 we discuss the concepts of isolated and isolated simple blow-up points and state some basic properties. In Section 3.4 we find the correction term ϕ_i and prove its properties. In Section 3.5 we obtain the pointwise estimates for u_i. In Section 3.6 we prove the vanishing of the trace-free 2nd fundamental form at any isolated simple blow-up point and prove the Pohozaev sign condition. In Section 3.7 we reduce our analysis to the case of isolated simple blow-up points and prove Theorem 1.1.4.

3.1 Conformal scalar and mean curvature equations

In this section we will introduce the partial differential equation we will work with in the next sections. We will also discuss some of its properties related to conformal deformation of metrics.

Let u be a positive smooth solution to

$$\begin{cases} L_g u = 0, & \text{in } M, , \\ B_g u + (n-2)f^{-\tau}u^p = 0, & \text{on } \partial M\,, \end{cases} \tag{3.1.1}$$

where $\tau = \frac{n}{n-2} - p$, $1 + \gamma_0 \leq p \leq \frac{n}{n-2}$ for some fixed $\gamma_0 > 0$ and f is a positive function. The equations (3.1.1) have an important scaling invariance property. Fix $x_0 \in \partial M$. Let $\delta > 0$ be small. Given $s > 0$ define the renormalized function

$$v(y) = s^{\frac{1}{p-1}}u(sy)\,, \quad \text{for } y \in B^+_{\delta s^{-1}}(0)\,.$$

Here, we work with Fermi coordinates centered at x_0. Then

$$\begin{cases} L_{\hat{g}} v = 0, & \text{in } B^+_{\delta s^{-1}}(0)\,, \\ B_{\hat{g}} v + (n-2)\hat{f}^{-\tau}v^p = 0, & \text{on } \partial' B^+_{\delta s^{-1}}(0)\,, \end{cases}$$

where $\hat{f}(y) = f(sy)$ and the coefficients of the metric $\hat{g}$ in Fermi coordinates are given by $\hat{g}_{kl}(y) = g_{kl}(sy)$.

Notation. We say that $u \in \mathcal{M}_p$ if u is a positive smooth solution to the equations (3.1.1).

The reason to work with the equations (3.1.1) instead of the equations (1.1.1) is that the first one has an important conformal invariance property.

Suppose $\tilde{g} = \phi^{\frac{4}{n-2}} g$ is a metric conformal to g. Recall that the conformal Laplacian satisfies

$$L_{\phi^{\frac{4}{n-2}}g}(\phi^{-1}u) = \phi^{-\frac{n+2}{n-2}} L_g u, \tag{3.1.2}$$

for any smooth functions $\phi > 0$ and u. Similarly, the boundary operator B_g satisfies

$$B_{\phi^{\frac{4}{n-2}}g}(\phi^{-1}u) = \phi^{-\frac{n}{n-2}} B_g u\,. \tag{3.1.3}$$

Hence, if u is a solution to the equations (3.1.1), then $\phi^{-1}u$ satisfies

$$\begin{cases} L_{\tilde{g}}(\phi^{-1}u) = 0, & \text{in } M\,,\\ B_{\tilde{g}}(\phi^{-1}u) + (n-2)(\phi f)^{-\tau}(\phi^{-1}u)^p = 0, & \text{on } \partial M\,, \end{cases}$$

which is again equations of the same type.

Notation. Let $\Omega \subset M$ be a domain in a Riemannian manifold (M, g). Let $\{g_i\}$ be a sequence of metrics on M. We say that $u_i \in \mathcal{M}_i$ if u_i satisfies

$$\begin{cases} L_{g_i}u_i = 0, & \text{in } \Omega\,,\\ B_{g_i}u_i + (n-2)f_i^{-\tau_i}u_i^{p_i} = 0, & \text{on } \partial'\Omega\,, \end{cases} \tag{3.1.4}$$

where $\tau_i = \frac{n}{n-2} - p_i$ and $1 + \gamma_0 \leq p_i \leq \frac{n}{n-2}$ for some fixed $\gamma_0 > 0$.

In this chapter we will work with sequences $\{u_i \in \mathcal{M}_i\}$ and assume $f_i \to f > 0$ uniformly, and $g_i \to g_0$ in $C^2(M)$ for some metric g_0.

By the conformal invariance stated above, we are allowed to replace the metric g_i by $\phi_i^{\frac{4}{n-2}} g_i$ as long as we have control of the conformal factors ϕ_i. In this case we replace the sequence $\{u_i\}$ by $\{\phi_i^{-1}u_i\}$ which we also denote by $\{u_i\}$. In particular, we can use conformal Fermi coordinates centered at some point $x_i \in \partial M$.

3.2 A Pohozaev-type identity

In this section we will prove the Pohozaev-type identity we will use in the subsequent blow-up analysis.

Proposition 3.2.1. *Let u be a solution to*

$$\begin{cases} \Delta_g u - \frac{n-2}{4(n-1)} R_g u = 0\,, & \text{in } B^+_\delta\,,\\ \frac{\partial u}{\partial y_n} - \frac{n-2}{2} h_g u + K f^{-\tau} u^p = 0\,, & \text{on } \partial' B^+_\delta\,, \end{cases}$$

where K is a constant. Let $0 < r < \delta$. Set

$$P(u,r) = \int_{\partial^+ B_r^+}\left(\frac{n-2}{2}u\frac{\partial u}{\partial r} - \frac{r}{2}|\nabla u|^2 + r\left|\frac{\partial u}{\partial r}\right|^2\right)d\sigma_r + \frac{r}{p+1}\int_{\partial(\partial' B_r^+)} Kf^{-\tau}u^{p+1}d\bar{\sigma}_r\,.$$

Then

$$P(u,r) = -\int_{B_r^+}\left(x^a\partial_a u + \frac{n-2}{2}u\right)A_g(u)dx + \frac{n-2}{2}\int_{\partial' B_r^+}\left(\bar{x}^k\partial_k u + \frac{n-2}{2}u\right)h_g u d\bar{x}$$
$$-\frac{\tau}{p+1}\int_{\partial' B_r^+} K(\bar{x}^k\partial_k f)f^{-\tau-1}u^{p+1}d\bar{x} + \left(\frac{n-1}{p+1} - \frac{n-2}{2}\right)\int_{\partial' B_r^+} Kf^{-\tau}u^{p+1}d\bar{x}\,,$$

where $A_g = \Delta_g - \Delta - \frac{n-2}{4(n-1)}R_g$.

Proof. Observe that

$$\int_{B_r^+}(x^b\partial_b u)\partial_{aa}u dx + \int_{B_r^+}\delta^{ab}(\partial_b u)(\partial_a u)dx + \frac{1}{2}\int_{B_r^+}x^b\partial_b(\partial_a u)^2 dx$$
$$= \frac{1}{r}\int_{\partial^+ B_r^+}(x^b\partial_b u)(x^a\partial_a u)d\sigma_r - \int_{\partial' B_r^+}(\bar{x}^k\partial_k u)(\partial_a u)\delta_n^a d\bar{x}\,.$$

Summing in $a = 1, ..., n$ we obtain

$$\int_{B_r^+}(x^b\partial_b u)\Delta u dx + \int_{B_r^+}|\nabla u|^2 dx + \frac{1}{2}\sum_a\int_{B_r^+}x^b\partial_b(\partial_a u)^2 dx$$
$$= r\int_{\partial^+ B_r^+}\left|\frac{\partial u}{\partial r}\right|^2 d\sigma_r - \int_{\partial' B_r^+}(\bar{x}^k\partial_k u)(\partial_n u)d\bar{x}\,. \tag{3.2.1}$$

But, integrating by parts, we have

$$\frac{1}{2}\sum_a\int_{B_r^+}x^b\partial_b(\partial_a u)^2 dx = -\frac{n}{2}\sum_a\int_{B_r^+}(\partial_a u)^2 dx + \frac{r}{2}\sum_a\int_{\partial^+ B_r^+}(\partial_a u)^2 d\sigma_r$$
$$-\frac{1}{2}\sum_a\int_{\partial' B_r^+}x^b\delta_b^n(\partial_a u)^2 d\bar{x}$$
$$= -\frac{n}{2}\int_{B_r^+}|\nabla u|^2 dx + \frac{r}{2}\int_{\partial^+ B_r^+}|\nabla u|^2 d\sigma_r\,, \tag{3.2.2}$$

$$
\begin{aligned}
\int_{\partial' B_r^+}(\bar{x}^k\partial_k u)(\partial_n u)d\bar{x} &= -\int_{\partial' B_r^+}(\bar{x}^k\partial_k u)(Kf^{-\tau}u^p - \frac{n-2}{2}h_g u)d\bar{x}\\
&= -\frac{1}{p+1}\int_{\partial' B_r^+}K\bar{x}^k\partial_k(u^{p+1})f^{-\tau}d\bar{x}\\
&\quad + \frac{n-2}{2}\int_{\partial' B_r^+}(\bar{x}^k\partial_k u)h_g u d\bar{x}\\
&= \frac{n-1}{p+1}\int_{\partial' B_r^+}Kf^{-\tau}u^{p+1}d\bar{x} + \frac{1}{p+1}\int_{\partial' B_r^+}K(\bar{x}^k\partial_k f^{-\tau})u^{p+1}d\bar{x}\\
&\quad - \frac{r}{p+1}\int_{\partial(\partial' B_r^+)}Kf^{-\tau}u^{p+1}d\bar{\sigma}_r\\
&\quad\quad + \frac{n-2}{2}\int_{\partial' B_r^+}(\bar{x}^k\partial_k u)h_g u d\bar{x}\,.
\end{aligned}
\tag{3.2.3}
$$

Substituting equalities (3.2.2) and (3.2.3) in (3.2.1) we obtain

$$
\begin{aligned}
\int_{B_r^+}(x^b\partial_b u)\Delta u dx &- \frac{n-2}{2}\int_{B_r^+}|\nabla u|^2 dx + \frac{r}{2}\int_{\partial^+ B_r^+}|\nabla u|^2 d\sigma_r\\
&= r\int_{\partial^+ B_r^+}\left|\frac{\partial u}{\partial r}\right|^2 d\sigma_r - \frac{n-1}{p+1}\int_{\partial' B_r^+}Kf^{-\tau}u^{p+1}d\bar{x}\\
&\quad - \frac{1}{p+1}\int_{\partial' B_r^+}K(\bar{x}^k\partial_k f^{-\tau})u^{p+1}d\bar{x} + \frac{r}{p+1}\int_{\partial(\partial' B_r^+)}Kf^{-\tau}u^{p+1}d\bar{\sigma}_r\\
&\quad - \frac{n-2}{2}\int_{\partial' B_r^+}(\bar{x}^k\partial_k u)h_g u d\bar{x}\,.
\end{aligned}
\tag{3.2.4}
$$

Using

$$
\int_{B_r^+}|\nabla u|^2 dx = -\int_{B_r^+}u\Delta u dx + \int_{\partial^+ B_r^+}u\frac{\partial u}{\partial r}d\sigma_r + \int_{\partial' B_r^+}(Kf^{-\tau}u^{p+1} - \frac{n-2}{2}h_g u^2)d\bar{x}
$$

and $\Delta u = -A_g(u)$ in equality (3.2.4) we get the result. □

3.3 Isolated and isolated simple blow-up points

In this section we will discuss the notions of isolated and isolated simple blow-up points and prove some of their properties. These notions are slight modifications of the ones used by Felli and Ould Ahmedou in [26] and [27] and are inspired by similar definitions in the case of manifolds without boundary.

Definition 3.3.1. Let $\Omega \subset M$ be a domain in a Riemannian manifold (M, g). We say that $x_0 \in \partial'\Omega$ is a *blow-up point* for the sequence $\{u_i \in \mathcal{M}_i\}_{i=1}^{\infty}$, if there is a sequence $\{x_i\} \subset \partial'\Omega$ such that

(1) $x_i \to x_0$;

(2) $u_i(x_i) \to \infty$;

(3) x_i is a local maximum of $u_i|_{\partial M}$.

Briefly we say that $x_i \to x_0$ is a blow-up point for $\{u_i\}$. The sequence $\{u_i\}$ is called a blow-up sequence.

Convention. If $x_i \to x_0$ is a blow-up point, we work in $B^+_\delta(0) \subset \mathbb{R}^n_+$, for some small $\delta > 0$, using g_i-Fermi coordinates centered at x_i.

Notation. If $x_i \to x_0$ is a blow-up point we set $M_i = u_i(x_i)$ and $\epsilon_i = M_i^{-(p_i-1)}$.

3.3.1 Isolated blow-up points

We define the notion of an isolated blow-up point as follows:

Definition 3.3.2. We say that $x_i \to x_0$ is an *isolated* blow-up point if it is a blow-up point and there exist $\delta, C > 0$ such that

$$u_i(x) \leq C|x|^{-\frac{1}{p_i-1}}\,, \quad \text{for all } x \in \partial' B^+_\delta(0)\backslash\{0\}\,. \tag{3.3.1}$$

Remark 3.3.3. Note that the definition of isolated blow-up point is invariant under renormalization, which was descrided in Section 3.1. This follows from the fact that if $v_i(y) = s^{\frac{1}{p_i-1}} u_i(sy)$, then

$$u_i(x) \leq C|x|^{-\frac{1}{p_i-1}} \Longleftrightarrow v_i(y) \leq C|y|^{-\frac{1}{p_i-1}}\,,$$

where $x = sy$.

The first result concerning isolated blow-up points states that the inequality (3.3.1) also holds for points $x \in B^+_\delta(0)\backslash\{0\}$.

Lemma 3.3.4. *Let $x_i \to x_0$ be an isolated blow-up point. Then $\{u_i\}$ satisfies*

$$u_i(x) \leq C|x|^{-\frac{1}{p_i-1}}\,, \quad \textit{for all } x \in B^+_\delta(0)\backslash\{0\}\,.$$

Proof. Let $0 < s < \frac{\delta}{3}$ and set $v_i(y) = s^{\frac{1}{p_i-1}} u_i(sy)$ for $|y| < 3$. Then v_i satisfies

$$\begin{cases} L_{\tilde{g}_i} v_i = 0, & \text{in } B^+_3(0), \\ (B_{\tilde{g}_i} + (n-2)\tilde{f}_i^{-\tau_i} v_i^{p_i-1}) v_i = 0, & \text{on } \partial' B^+_3(0)\,, \end{cases}$$

where $(\tilde{g}_i)_{kl}(y) = (g_i)_{kl}(sy)$ and $\tilde{f}(y) = f(sy)$. Hence, Lemma C-3 gives

$$\max_{B_2^+(0)\backslash B_{1/2}^+(0)} v_i \leq C(n, \max_{\partial' B_3^+(0)} v_i) \min_{B_2^+(0)\backslash B_{1/2}^+(0)} v_i \,. \tag{3.3.2}$$

By the scaling invariance (Remark 3.3.3) v_i is uniformly bounded in compact subsets of $\partial' B_3^+(0)\backslash\{0\}$. Hence, the result follows from inequality (3.3.2). □

A corollary of the proof of Lemma 3.3.4 is the following Harnack-type inequality:

Lemma 3.3.5. *Let $x_i \to x_0$ be an isolated blow-up point and δ as in Definition 3.3.2. Then $\exists C > 0$ such that $\forall 0 < s < \frac{\delta}{3}$,*

$$\max_{B_{2s}^+(0)\backslash B_{s/2}^+(0)} u_i \leq C \min_{B_{2s}^+(0)\backslash B_{s/2}^+(0)} u_i \,.$$

The next proposition says that, in the case of an isolated blow-up point, the sequence $\{u_i\}$, when renormalized, converges to the standard Euclidean solution U.

Proposition 3.3.6. *Let $x_i \to x_0$ be an isolated blow-up point. Set*

$$v_i(y) = M_i^{-1} u_i(M_i^{-(p_i-1)} y) \,, \quad \textit{for } y \in B^+_{\delta M_i^{p_i-1}}(0) \,.$$

Then given $R_i \to \infty$ and $\beta_i \to 0$, after choosing subsequences, we have

(a) $|v_i - U|_{C^2(B^+_{R_i}(0))} < \beta_i$;
(b) $\lim_{i\to\infty} \frac{R_i}{\log M_i} = 0$;
(c) $\lim_{i\to\infty} p_i = \frac{n}{n-2}$.

The proof of Proposition 3.3.6 is analogous to Lemma 2.6 of [26] or Proposition 4.3 of [39]. It uses Theorems 1.3.1 and 1.3.2.

Remark 3.3.7. Once we have proved Proposition 3.3.6 it is not difficult to see that, if we change the metric by an uniformly bounded conformal factor $f_i > 0$, with $f_i(0) = 1$ and $\frac{\partial f_i}{\partial x_k}(0) = 0$ for $k = 1, ..., n-1$, then isolated blow-up points are preserved. This is the case of conformal Fermi coordinates, for example (see Proposition 1.4.2).

The following lemma will be used later when we consider the set of blow-up points.

Lemma 3.3.8. *Given $R, \beta > 0$, there exists $C_0 > 0$ such that if $u \in \mathcal{M}_p$ and $S \subset \partial M$ is a compact set, we have the following:*

If $\max_{x \in \partial M \backslash S} \left(u(x) d_{\bar{g}}(x, S)^{\frac{1}{p-1}} \right) \geq C_0$, then $\frac{n}{n-2} - p < \beta$ and there exists $x_0 \in \partial M \backslash S$, local maximum of u, such that

$$\left| u(x_0)^{-1} u(x) - U(u(x_0)^{p-1} x) \right|_{C^2(B^+_{2r_0}(x_0))} < \beta \,, \qquad (3.3.3)$$

where $r_0 = R u(x_0)^{-(p-1)}$. If $\emptyset$ is the empty set, we define $d_{\bar{g}}(x, \emptyset) = 1$.

Proof. Suppose by contradiction that there exist $R, \beta > 0$ such that, for all $C_0 > 0$, there exist $u \in \mathcal{M}_p$ and $S \subset \partial M$ compact such that

$$\max_{x \in \partial M \backslash S} \left(u(x) d_{\bar{g}}(x, S)^{\frac{1}{p_i-1}} \right) \geq C_0$$

and there is no such point x_0. Hence, we can suppose that there are sequences

$$w_i(x_i') = \max_{x \in \partial M \backslash \, S_i} w_i(x) \to \infty,$$

where $w_i(x) = u_i(x) d_{\bar{g}}(x, S_i)^{\frac{1}{p_i-1}}$ and $x_i' \in \partial M$. Here, $S_i \subset \partial M$ is compact. We assume that $p_i \to p_0$, for some $p_0 \in \left(1, \frac{n}{n-2}\right]$, and $x_i' \to x_0'$ for some $x_0' \in \partial M$. Set $N_i = u_i(x_i')$. Observe that $N_i \to \infty$.

We use Fermi coordinates centered at x_i'. Set $v_i(y) = N_i^{-1} u_i(N_i^{-(p_i-1)} y)$ for $y \in B^+_{\delta N_i^{p_i-1}}(0)$. It follows from the discussion in Section 3.1 that v_i satisfies

$$\begin{cases} L_{\hat{g}_i} v_i = 0 \,, & \text{in } B^+_{\delta N_i^{p_i-1}}(0) \,, \\ B_{\hat{g}_i} v_i + (n-2) \hat{f}_i^{-\tau_i} v_i^{p_i} = 0 \,, & \text{on } \partial' B^+_{\delta N_i^{p_i-1}}(0) \,, \end{cases}$$

where $\hat{f}_i(y) = f(N_i^{-(p_i-1)} y)$ and $\hat{g}_i$ stands for the metric with coefficients $(\hat{g}_i)_{kl}(y) = g_{kl}(N_i^{-(p_i-1)} y)$.

Claim. $v_i \leq C$ in compacts of $\mathbb{R}^n_+$.

Let $x \in \partial' B^+_\delta(0)$. Since $w_i(x) \leq w_i(x_i')$, we have

$$\frac{d_{\bar{g}}(S_i, x_i') - d_{\bar{g}}(x_i', x)}{d_{\bar{g}}(S_i, x_i')} \leq \frac{d_{\bar{g}}(S_i, x)}{d_{\bar{g}}(S_i, x_i')} \leq \left(N_i u_i(x)^{-1} \right)^{p_i - 1} \,.$$

On the other hand,

$$\frac{d_{\bar{g}}(S_i,x_i')-d_{\bar{g}}(x_i',x)}{d_{\bar{g}}(S_i,x_i')}=1-\frac{N_i^{-(p_i-1)}|y|}{d_{\bar{g}}(S_i,x_i')}=1-w_i(x_i')^{-(p_i-1)}|y|=1-o_i(1)|y|,$$

where we have set $y=N_i^{p_i-1}x$. This proves that $v_i\le C$ in compacts of $\partial\mathbb{R}^n_+$. Now the Claim follows from Lemma C-3.

Hence, we can suppose that $v_i\to v$ in $C^2_{loc}(\mathbb{R}^n_+)$, for $v>0$ satisfying

$$\begin{cases}\Delta v=0, & \text{in } \mathbb{R}^n_+,\\ \frac{\partial v}{\partial y_n}+(n-2)v^{p_0}=0, & \text{on } \partial\mathbb{R}^n_+\end{cases}$$

and $v(0)=1$. Then, by Theorems 1.3.1 and 1.3.2, $p_0=\frac{n}{n-2}$ and v is of the form (1.3.2). Hence, we can find $y_{(i)}\in\partial' B^+_{\delta N_i^{p_i-1}}(0)$ local maxima of v_i, such that $y_{(i)}\to(z_1,...,z_{n-1},0)\in\mathbb{R}^n_+$. Then, after a renormalization such that $v_i(y_{(i)})=1$, v_i satisfies the estimate (3.3.3), for i large, with $x_0=N_i^{-(p_i-1)}y_{(i)}$. This is a contradiction. □

Once we have proved Lemma 3.3.8, the proof of the following proposition is analogous to Proposition 5.1 of [38] (see also Lemma 3.1 of [47] or Proposition 1.1 of [29]):

Proposition 3.3.9. *Given small $\beta>0$ and large $R>0$ there are constants $C_0,C_1>0$, depending only on β, R and (M^n,g), such that if $u\in\mathcal{M}_p$ and $\max_{\partial M}u\ge C_0$, then $\frac{n}{n-2}-p<\beta$ and there are $x_1,...,x_N\in\partial M$ local maxima of u, such that:*

(1) If $r_j=Ru(x_j)^{-(p-1)}$ for $j=1,...,N$, then $\{\partial' B^+_{r_j}(x_j)\subset\partial M\}_{j=1}^N$ is a disjoint collection;

(2) For $j=1,...,N$, $\left|u(x_j)^{-1}u(x)-U(u(x_j)^{p-1}x)\right|_{C^2(B^+_{2r_j}(x_j))}<\beta;$

(3) We have

$$u(x)\,d_{\bar{g}}(x,\{x_1,...,x_N\})^{\frac{1}{p-1}}\le C_1\,,\quad \text{for all } x\in\partial M\,,$$

$$u(x_j)\,d_{\bar{g}}(x_j,x_k)^{\frac{1}{p-1}}\ge C_0\,,\quad \text{for any } j\neq k\,.$$

3.3.2 Isolated simple blow-up points

Let us introduce the notion of an isolated simple blow-up point. Let $x_i \to x_0$ be an isolated blow-up point. Set

$$\bar{u}_i(r) = \frac{2}{\sigma_{n-1} r^{n-1}} \int_{\partial^+ B_r^+(0)} u_i d\sigma_r$$

and $w_i(r) = r^{\frac{1}{p_i-1}} \bar{u}_i(r)$.

Note that the definition of w_i is invariant under renormalization, which was descrided in Section 3.1. More precisely, if $v_i(y) = s^{\frac{1}{p_i-1}} u_i(sy)$, then

$$r^{\frac{1}{p_i-1}} \bar{v}_i(r) = (sr)^{\frac{1}{p_i-1}} \bar{u}_i(sr)\,.$$

Definition 3.3.10. An isolated blow-up point $x_i \to x_0$ is *simple* if there is $\delta > 0$ such that w_i has exactly one critical point in $(0, \delta)$.

Remark 3.3.11. Let us handle the case $U(y) = ((1+y_n)^2 + \sum_{j=1}^{n-1} y_j^2)^{\frac{2-n}{2}}$. Observe that

$$\bar{U}(r) = area\,(\partial^+ B_r^+)^{-1} \int_{\partial^+ B_r^+} U(y) d\sigma_r(y) = 2(\sigma_{n-1})^{-1} \int_{\partial^+ B_1^+} U(rw) d\sigma_1(w)\,.$$

Hence,

$$\frac{d}{dr}(r^{\frac{n-2}{2}} \bar{U}(r)) = 2r^{\frac{n-4}{2}} (\sigma_{n-1})^{-1} \int_{\partial^+ B_1^+} \left(\frac{n-2}{2} U(rw) + r\frac{\partial}{\partial r}(U(rw)) \right) d\sigma_1.$$

Since $\frac{n-2}{2} U + y^b \frac{\partial U}{\partial y^b} = \frac{n-2}{2}((1+y_n)^2 + \sum_{j=1}^{n-1} y_j^2)^{-\frac{n}{2}}(1-|y|^2)$, we conclude that $\frac{d}{dr}(r^{\frac{n-2}{2}} \bar{U}(r)) = 0$, for $r > 0$, if and only if $r = 1$.

Now, let $x_i \to x_0$ be an isolated blow-up point and $R_i \to \infty$. Using Proposition 3.3.6 we see that, choosing a subsequence, $r \mapsto r^{\frac{1}{p_i-1}} \bar{u}_i(r)$ has exactly one critical point in $(0, r_i)$. Moreover, its derivative is negative right after the critical point. Hence, if $x_i \to x_0$ is isolated simple then there is $\delta > 0$ such that $w_i'(r) < 0$ for all $r \in [r_i, \delta)$.

The next proposition is an important property of isolated simple blow-up points.

Proposition 3.3.12. *Let $x_i \to x_0$ be an isolated simple blow-up point. Then there exists $C, \delta > 0$ such that* *(a)* $M_i u_i(x) \leq C|x|^{2-n}$ *for all* $x \in B^+_\delta(0)\backslash\{0\}$*;*

(b) $M_i u_i(x) \geq C^{-1}G_i(x)$ *for all* $x \in B^+_\delta(0)\backslash B^+_{r_i}(0)$*, where* G_i *is the Green's function so that:*

$$\begin{cases} L_{g_i}G_i = 0, & \text{in } B^+_\delta(0)\backslash\{0\},\\ G_i = 0, & \text{on } \partial^+ B^+_\delta(0),\\ B_{g_i}G_i = 0, & \text{on } \partial' B^+_\delta(0)\end{cases}$$

and $|x|^{n-2}G_i(x) \to 1$*, as* $|x| \to 0$*.*

The remaining part of this section will be dedicated to the proof of Proposition 3.3.12. We will use the following lemma:

Lemma 3.3.13. *Let* $x_i \to x_0$ *be an isolated simple blow-up point and let* $\rho > 0$ *be small. Then there exist* $C, \delta > 0$ *such that*

$$M_i^{\lambda_i}|\nabla^r u_i|(x) \leq C|x|^{2-r-n+\rho},$$

for $x \in B^+_\delta(0)$ *and* $r = 0, 1, 2$*. Here,* $\lambda_i = (p_i - 1)(n - 2 - \rho) - 1$*.*

The proof of Lemma 3.3.13 is analogous to Lemma 2.7 of [26]. For the sake of the reader we include this proof below. It uses the following maximum principle, which is Lemma A.2 of [29]:

Lemma 3.3.14. *Let* (N, g) *be a Riemannian manifold and* $\Omega \subset N$ *be a connected open set with piecewise smooth boundary* $\partial\Omega = \Gamma \cup \Sigma$*. Let* $h \in L^\infty(\Omega)$ *and* $\sigma \in L^\infty(\Sigma)$*. Suppose that* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *,* $u > 0$ *in* $\bar{\Omega}$*, satisfies*

$$\begin{cases} \Delta_g u + hu \leq 0\,, & \text{in } \Omega\,,\\ \frac{\partial u}{\partial \nu} + \sigma u \leq 0\,, & \text{on } \Sigma\end{cases}$$

and $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies*

$$\begin{cases} \Delta_g v + hv \leq 0\,, & \text{in } \Omega\,,\\ \frac{\partial v}{\partial \nu} + \sigma v \leq 0\,, & \text{on } \Sigma\,,\\ v \geq 0\,, & \text{on } \Gamma\,,\end{cases}$$

where ν *denotes the unit normal of* Σ *pointing inwards. Then* $v \geq 0$ *in* $\bar{\Omega}$*.*

Proof. Let $w = v/u$. Then

$$\begin{cases} \Delta_g w + 2u^{-1}\nabla u \cdot \nabla w + (\Delta_g u + hu)u^{-1}w \leq 0\,, & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} + u^{-1}(\frac{\partial u}{\partial \nu} + \sigma u)w \leq 0\,, & \text{on } \Sigma\,, \\ w \geq 0\,, & \text{on } \Gamma\,. \end{cases}$$

Thus, the usual maximum principle implies that $w \geq 0$ in $\bar{\Omega}$. Therefore, $v \geq 0$ in $\bar{\Omega}$. □

Proof of Lemma 3.3.13. Let $R_i \to \infty$. We define $r_i = M_i^{-(p_i-1)}R_i$ and choose a subsequence according to Proposition 3.3.6. We will first prove the following two claims:

Claim 1. There exists $C > 0$ such that

$$M_i^{(p_i-1)(n-2)-1}u_i(x) \leq C|x|^{2-n},$$

for $x \in B^+_{r_i}$. (Observe that $(p_i - 1)(n-2) - 1 = 1 - (n-2)\tau_i$, for $\tau_i = \frac{n}{n-2} - p_i$.)

Claim 2. There exists $C > 0$ such that

$$M_i^{\lambda_i}u_i(x) \leq C|x|^{2-n+\rho},$$

for $x \in B^+_{r_i}$.

For $|y| \leq R_i$, by Remark 3.3.18,

$$M_i^{-1}u_i(M_i^{-(p_i-1)}y) = v_i(y) \leq CU(y) = C\left(\frac{1}{(1+y_n)^2 + \sum_{j=1}^{n-1} y_j^2}\right)^{\frac{n-2}{2}}.$$

If $y = M_i^{p_i-1}x$, then

$$M_i^{-1}u_i(x) \leq C\left(\frac{M_i^{-2(p_i-1)}}{(M_i^{-(p_i-1)} + x_n)^2 + \sum_{j=1}^{n-1} x_j^2}\right)^{\frac{n-2}{2}}. \tag{3.3.4}$$

Claims 1 and 2 follow from the inequality (3.3.4).

Now, we need to extend the estimate of Claim 2 to $B^+_\delta \backslash B^+_{r_i}$. We set $L_i = L_{g_i}$ and $K_i = B_{g_i} + (n-2)f_i^{-\tau_i}u_i^{p_i-1}$. Then u_i satisfies

$$\begin{cases} L_i u_i = 0, & \text{in } B^+_\delta\,, \\ K_i u_i = 0, & \text{on } \partial' B^+_\delta \end{cases}$$

and, by Lemma 3.3.14, (L_i, K_i) satisfies the following maximum principle:

Lemma 3.3.15. *If v_i satisfies*

$$\begin{cases} L_i v_i \le 0, & \text{in } B^+_\delta \backslash B^+_{r_i}, \\ K_i v_i \le 0, & \text{on } \partial' B^+_\delta \backslash \partial' B^+_{r_i}, \\ v_i \ge 0, & \text{on } \partial^+ B^+_\delta \cup \partial^+ B^+_{r_i}, \end{cases}$$

then $v_i \ge 0$ in $B^+_\delta \cup B^+_{r_i}$.

We set

$$\phi_{i,\nu}(x) = |x|^{-\nu} - \epsilon_0 |x|^{-\nu-1} x_n,$$

where $\epsilon_0 > 0$ is small to be fixed later, and

$$\psi_i(x) = C\left(M_i^{-\lambda_i}\phi_{i,n-2-\rho}(x) + \eta_i \phi_{i,\rho}(x)\right),$$

where $\eta_i = \max_{\partial^+ B^+_\delta} u_i$ and $C > 0$ is chosen such that

$$u_i \le \psi_i,$$

on $\partial^+ B^+_\delta \backslash \partial^+ B^+_{r_i}$. This is possible by Claim 2.

In order to apply Lemma 3.3.15 to prove that $\psi_i - u_i \ge 0$ in $B^+_\delta \backslash B^+_{r_i}$, we need to show that $\psi_i - u_i$ satisfies

$$\begin{cases} L_i(\psi_i - u_i) \le 0, & \text{in } B^+_\delta \backslash B^+_{r_i}, \\ K_i(\psi_i - u_i) \le 0, & \text{on } \partial' B^+_\delta \backslash \partial' B^+_{r_i}. \end{cases} \tag{3.3.5}$$

Let us prove the first inequality of (3.3.5). Set $r = r(x) = |x|$. Then

$$\Delta_{g_i}(r^{-\nu}) = -\nu(-\nu + n - 2) r^{-\nu-2} + O(r^{-\nu-1})$$

and

$$\begin{aligned} \Delta_{g_i}(r^{-\nu-1} x_n) &= \Delta_{g_i}(r^{-\nu-1}) x_n - 2(\nu+1) r^{-\nu-3} x_n + O(r^{-\nu-1}) \\ &= -(\nu+1)(n-1-\nu) r^{-\nu-3} x_n + O(r^{-\nu-1}). \end{aligned}$$

Hence,

$$\Delta_{g_i}(\phi_{i,\nu}) = r^{-\nu-2}\left(-\nu(n-2-\nu) + \epsilon_0(\nu+1)(n-1-\nu)\frac{x_n}{r} + O(r)\right)$$

and

$$\begin{aligned} L_i \psi_i = {} & C M_i^{-\lambda_i}\left\{\Delta_{g_i}(\phi_{i,n-2-\rho}) - \frac{n-2}{4(n-1)} R_{g_i} \phi_{i,n-2-\rho}\right\} \\ & + C\eta_i \left\{\Delta_{g_i}(\phi_{i,\rho}) - \frac{n-2}{4(n-1)} R_{g_i} \phi_{i,\rho}\right\} \\ \le {} & C\left(M_i^{-\lambda_i}|x|^{-n+\rho} + \eta_i |x|^{-2-\rho}\right)\left(-\rho(n-2-\rho) + \epsilon_0 c(n,\rho)\frac{x_n}{r} + O(r)\right), \end{aligned}$$

where $c(n,\rho)=(\rho+1)(n-1-\rho)$. Here, we choose ϵ_0 small such that $L_i\psi_i\leq 0$ for $r_i\leq|x|\leq\delta$, since δ is also chosen small.

Now we will prove the second inequality of (3.3.5). Observe that

$$\begin{aligned}K_i\psi_i &= CM_i^{-\lambda_i}K_i(\phi_{i,n-2-\rho})+C\eta_iK_i(\phi_{i,\rho})\\ &= C\left(M_i^{-\lambda_i}|x|^{1-n+\rho}+\eta_i|x|^{-1-\rho}\right)\left(-\epsilon_0-\frac{n-2}{2}h_{g_i}|x|+(n-2)f_i^{-\tau_i}u_i(x)^{p_i-1}|x|\right).\end{aligned}$$

The fact that $K_i\psi_i(x)\leq 0$ for $r_i\leq|x|\leq\delta$ follows from the following claim:

Claim 3. Given $\gamma>0$, we have $u_i(x)^{p_i-1}|x|<\gamma$ for i large and $r_i\leq|x|<\delta$.

By the second paragraph of Remark 3.3.11, $r\mapsto r^{\frac{1}{p_i-1}}\bar{u}_i(r)$ is decreasing in (r_i,δ). This, together with the Harnack inequality of Lemma 3.3.5, implies that

$$u_i(x)|x|^{\frac{1}{p_i-1}}\leq C\bar{u}_i(|x|)|x|^{\frac{1}{p_i-1}}\leq C\bar{u}_i(r_i)r_i^{\frac{1}{p_i-1}}$$

for all $r_i\leq|x|<\delta$. Applying Claim 1 to the right hand side of this inequality, we obtain

$$u_i(x)|x|^{\frac{1}{p_i-1}}\leq CM_i^{-1+(n-2)\tau_i}r_i^{2-n+\frac{1}{p_i-1}}$$

for all $r_i\leq|x|<\delta$. On the other hand,

$$M_i^{-1+(n-2)\tau_i}r_i^{2-n+\frac{1}{p_i-1}}=R_i^{2-n+\frac{1}{p_i-1}}\to 0$$

as $i\to\infty$, since $R_i\to\infty$. This proves Claim 3.

Hence, we have proved that

$$u_i(x)\leq C\left(M_i^{-\lambda_i}|x|^{2-n+\rho}+\eta_i|x|^{-\rho}\right)\tag{3.3.6}$$

in B^+_δ. It follows from Lemma 3.3.5 that

$$\delta^{\frac{1}{p_i-1}}\eta_i\leq C\delta^{\frac{1}{p_i-1}}\bar{u}_i(\delta)\leq Cr^{\frac{1}{p_i-1}}\bar{u}_i(r)$$

for $r\leq\delta$, where in the last inequality we used again the 2nd paragraph of Remark 3.3.11. Thus, by the estimate (3.3.6),

$$\begin{aligned}\delta^{\frac{1}{p_i-1}}\eta_i &\leq Cr^{\frac{1}{p_i-1}}\left(M_i^{-\lambda_i}r^{2-n+\rho}+\eta_ir^{-\rho}\right)\\ &\leq CM_i^{-\lambda_i}r^{2-n+2\rho}+C\eta_ir^{\frac{1}{p_i-1}-\rho},\end{aligned}$$

for $\rho\leq\frac{n-2}{4}=\frac{1}{2}\lim_{i\to\infty}\frac{1}{p_i-1}$. Hence,

$$r^{-\rho}\left(\delta^{\frac{1}{p_i-1}}-Cr^{\frac{1}{p_i-1}-\rho}\right)\eta_i\leq CM_i^{-\lambda_i}r^{2-n+\rho}.\tag{3.3.7}$$

Choosing $r \le \delta'$ for some $\delta' > 0$ small fixed, we use the estimate (3.3.7) in the estimate (3.3.6) to conclude the proof of Lemma 3.3.13. □

Remark 3.3.16. Set $v_i(y) = M_i^{-1} u_i(M_i^{-(p_i-1)} y)$ and suppose that $x_i \to x_0$ is isolated simple. Then, as a consequence of Lemma 3.3.13 and Proposition 3.3.6, we see that there is $C > 0$ such that

$$|\nabla^r v_i|(y) \le C M_i^{\rho(p_i-1)} (1+|y|)^{2-r-n}$$

for any $y \in B^+_{\delta M_i^{p_i-1}}(0)$ and $r = 0, 1, 2$.

Now we are going to estimate τ_i.

Proposition 3.3.17. *Let $x_i \to x_0$ be an isolated simple blow-up point and let $\rho > 0$ be small. Then there is $C > 0$ such that*

$$\tau_i \le \begin{cases} C\epsilon_i^{1-2\rho+o_i(1)}, & \text{for } n \ge 5, \\ C\epsilon_i^{1-2\rho+o_i(1)} \log(\epsilon_i), & \text{for } n = 4. \end{cases} \tag{3.3.8}$$

Proof. We write the Pohozaev identity of Proposition 3.2.1 as

$$P(u_i, r) = F_i(u_i, r) + \bar{F}_i(u_i, r) + \frac{\tau_i}{p_i + 1} Q_i(u_i, r), \tag{3.3.9}$$

for $r \le \delta$, where
$F_i(u, r) = -\int_{B_r^+} (x^b \partial_b u + \frac{n-2}{2} u)(L_{g_i} - \Delta) u \, dx$,
$\bar{F}_i(u, r) = \frac{n-2}{2} \int_{\partial' B_r^+} (\bar{x}^b \partial_b u + \frac{n-2}{2} u) h_{g_i} u \, d\bar{x}$,
$Q_i(u, r) = \frac{(n-2)^2}{2} \int_{\partial' B_r^+} f_i^{-\tau_i} u^{p_i+1} d\bar{x} - (n-2) \int_{\partial' B_r^+} (\bar{x}^k \partial_k f) f_i^{-\tau_i - 1} u^{p_i+1} d\bar{x}$.

It follows from Proposition 3.3.6 that we can choose a subsequence such that

$$\int_{\partial' B_{r_i}^+} u_i^{p_i-1} \ge c > 0,$$

where $r_i = R_i \epsilon_i \to 0$. Hence, for $r > 0$ small, $Q_i(u_i, r) \ge c > 0$.

Using Lemma 3.3.13 we obtain

$$P_i(u_i, r) \le C\epsilon_i^{\frac{2\lambda_i}{p_i-1}} = C\epsilon_i^{n-2-2\rho+o_i(1)}. \tag{3.3.10}$$

Changing variables,

$$\bar{F}_i(u_i, r) = -\epsilon_i^{-\frac{2}{p_i-1}+n-2} \int_{\partial' B^+_{r\epsilon_i^{-1}}} \left(\bar{y}^b \partial_b v_i + \frac{n-2}{2} v_i \right) h_{g_i}(\epsilon_i \bar{y}) v_i(\bar{y}) d\bar{y}.$$

Observe that $-\frac{2}{p_i-1}+n-2=-(n-2)\frac{\tau_i}{p_i-1}=o_i(1)$. By Remark 3.3.16 and the fact that we can suppose that $h(0)=0$ (see Proposition 1.4.2),

$$\bar{F}_i(u_i,r)=\epsilon_i^{-2\rho+o_i(1)}\int_{\partial' B^+_{r\epsilon_i^{-1}}} O((1+|\bar{y}|)^{2-n})O(\epsilon_i|\bar{y}|)O((1+|\bar{y}|)^{2-n})d\bar{y}$$
$$\geq -C\epsilon_i^{1-2\rho+o_i(1)}\cdot\begin{cases}1, & \text{for } n\geq 5,\\ \log\epsilon_i, & \text{for } n=4.\end{cases} \tag{3.3.11}$$

Similarly,

$$F_i(u_i,r)=-\epsilon_i^{-\frac{2}{p_i-1}+n-2}\int_{B^+_{r\epsilon_i^{-1}}}(y^b\partial_b v_i+\frac{n-2}{2}v_i)(L_{\hat{g}_i}-\Delta)v_i dy$$
$$=\epsilon_i^{-2\rho+o_i(1)}\int_{B^+_{r\epsilon_i^{-1}}}O((1+|y|)^{2-n})O(\epsilon_i|y|)O((1+|y|)^{-n})dy$$

Hence, $F_i(u_i,r)\geq -C\epsilon_i^{1-2\rho+o_i(1)}$, for $n\geq 4$. This, together with the identities (3.3.9), (3.3.10), (3.3.11) and the fact that $Q_i(u_i,r)\geq c>0$, gives the result. □

Now, we are able to prove Proposition 3.3.12.

Proof of Proposition 3.3.12. We will first need the following two claims.

Claim 1. Given a small $\sigma>0$, there is $C>0$ such that

$$\int_{\partial' B^+_\sigma} u_i^{p_i}d\bar{x}\leq CM_i^{-1}\,.$$

If follows from Proposition 3.3.6 that we can choose a subsequence such that

$$\int_{\partial' B^+_{r_i}} u_i^{p_i}(\bar{x})d\bar{x}=M_i^{-(p_i-1)(n-1)+p_i}\int_{\partial' B^+_{R_i}} v_i(\bar{y})^{p_i}d\bar{y}\leq CM_i^{-1}\,.$$

Here, $r_i=R_iM_i^{-(p_i-1)}$ and $R_i\to\infty$. On the other hand, by Lemma 3.3.13,

$$\int_{\partial' B^+_\sigma\backslash\partial' B^+_{r_i}} u_i^{p_i}(\bar{x})d\bar{x}\leq CM_i^{-\lambda_i p_i}\int_{\partial' B^+_\sigma\backslash\partial' B^+_{r_i}}|\bar{x}|^{(2-n+\rho)p_i}d\bar{x}\leq o_i(1)M_i^{-1}\,.$$

This proves Claim 1.

Claim 2. There is $\sigma_1 > 0$ such that for all $0 < \sigma < \sigma_1$ there is $C = C(\sigma)$ such that

$$u_i(x)u_i(x_i) \leq C$$

for any $x \in \partial^+ B^+_\sigma(0)$.

It is not difficult to see that if $\sigma_1 > 0$ is small we can find a conformal metric, still denoted by g_i, such that $R_{g_i} \equiv 0$ in $B^+_{\sigma_1}(0)$ and $h_{g_i} \equiv 0$ on $\partial' B^+_{\sigma_1}(0)$.

We fix $\sigma \in (0, \sigma_1)$ and choose $x_\sigma \in \partial^+ B^+_\sigma(0)$.

If we set $w_i = u_i(x_\sigma)^{-1} u_i$, then w_i satisfies

$$\begin{cases} \Delta_{g_i} w_i = 0, & \text{in } B^+_\sigma(0)\,, \\ \frac{\partial w_i}{\partial \eta} + (n-2)u_i(x_\sigma)^{p_i - 1} w_i^{p_i} = 0, & \text{on } \partial' B^+_\sigma(0)\,. \end{cases} \tag{3.3.12}$$

By the Harnack inequality of Lemma C-3, for each $\beta > 0$ there is $C_\beta > 0$ such that

$$C_\beta^{-1} \leq w_i(x) \leq C_\beta$$

if $|x| > \beta$. Observe that Lemma 3.3.13 implies that $u_i(x_\sigma)^{p_i - 1} \to 0$ as $i \to \infty$. Hence, we can suppose that $w_i \to w > 0$ in $C^2_{loc}(B^+_\sigma(0) \backslash \{0\})$ and w satisfies

$$\begin{cases} \Delta_{g_0} w = 0, & \text{in } B^+_\sigma(0) \backslash \{0\}, \\ \frac{\partial w}{\partial \eta} = 0, & \text{on } \partial' B^+_\sigma(0) \backslash \{0\}. \end{cases} \tag{3.3.13}$$

Here, g_0 is the C^2-limit of g_i. It follows from elliptic linear theory that

$$w = aG(x) + b(x) \quad \text{for } x \in B^+_\sigma(0) \backslash \{0\}\,,$$

where $a \geq 0$. Here, G is the Green's function so that

$$\begin{cases} \Delta_{g_0} G = 0, & \text{in } B^+_\sigma(0) \backslash \{0\}, \\ G = 0, & \text{on } \partial^+ B^+_\sigma(0), \\ \frac{\partial G}{\partial \eta} = 0, & \text{on } \partial' B^+_\sigma(0) \backslash \{0\}, \\ \lim_{|x| \to 0} |x|^{2-n} G(x) = 1\,, \end{cases}$$

and b satisfies

$$\begin{cases} \Delta_{g_0} b = 0, & \text{in } B^+_\sigma(0), \\ \frac{\partial b}{\partial \eta} = 0, & \text{on } \partial' B^+_\sigma(0)\,. \end{cases}$$

We will prove that $a > 0$. Set $r = |x|$. Since the blow-up is isolated simple, $r \mapsto r^{\frac{1}{p_i - 1}} \bar{u}_i(r)$ is decreasing in (r_i, σ) (see the 2nd paragraph of Remark 3.3.11).

Taking the limit as $i \to \infty$, we conclude that $r \mapsto r^{\frac{n-2}{2}}\overline{W}(r)$ is decreasing in $(0,\sigma)$. Hence, w has a non-removable singularity at the origin. Therefore $a > 0$.

Fix $\delta > 0$ small. Then there is $c_1 > 0$ such that

$$-\int_{\partial^+ B_\delta^+} \frac{\partial w}{\partial r} d\sigma_r > c_1. \tag{3.3.14}$$

Integrating by parts the first equation of (3.3.12) we obtain

$$\begin{aligned} 0 = \int_{B_\delta^+} \Delta_{g_0} w_i dx &= \int_{\partial^+ B_\delta^+} \frac{\partial w_i}{\partial r} d\sigma_\delta - \int_{\partial' B_\delta^+} \frac{\partial w_i}{\partial \eta} d\bar{x} \\ &= \int_{\partial^+ B_\delta^+} \left(\frac{\partial w}{\partial r} + o_i(1) \right) d\sigma_\delta + (n-2) u_i(x_\sigma)^{-1} \int_{\partial' B_\delta^+} u_i^{p_i} d\bar{x} \\ &\leq -c_1 + C u_i(x_\sigma)^{-1} u_i(x_i)^{-1}, \end{aligned} \tag{3.3.15}$$

where we used the estimate (3.3.14) and Claim 1 in the last inequality. This proves Claim 2.

Now we are going to prove the item (a). Suppose by contradiction it does not hold. Then passing to a subsequence we can choose $\{x_i'\} \subset M$ such that $|x_i'| \to 0$ and

$$u_i(x_i) u_i(x_i') |x_i'|^{n-2} \to \infty. \tag{3.3.16}$$

By Proposition 3.3.6 we can assume that $R_i u_i(x_i)^{-(p_i-1)} \leq |x_i'| \leq \delta$ where $R_i \to \infty$. Set $v_i(y) = |x_i'|^{\frac{1}{p_i-1}} u_i(|x_i'|\, y)$ for $y \in B^+_{\delta |x_i'|^{-1}}(0)$. Hence, the origin is an isolated simple blow-up point for $\{v_i\}$. Thus, by Claim 2, there is $C > 0$ such that

$$|x_i'|^{\frac{2}{p_i-1}} u_i(x_i) u_i(x_i') = v_i(0) v_i(y_i') \leq C$$

where $y_i' = |x_i'|^{-1} x_i'$. This contradicts the hypothesis (3.3.16).

Item (b) is just an application of Lemma 3.3.14. □

Remark 3.3.18. Set $v_i(y) = M_i^{-1} u_i(M_i^{-(p_i-1)} y)$ and suppose that $x_i \to x_0$ is isolated simple. Then, as a consequence of Propositions 3.3.6 and 3.3.12, we see that $v_i \leq CU$ in $B^+_{\delta M_i^{p_1-1}}(0)$.

3.4 The linearized equation

In this section we will be interested in solutions of a certain type of linear problem. These solutions will be used in the blow-up estimates of Section 3.5.

Convention. In this section, we will always use the conformal equivalence between $\mathbb{R}^n_+ \cup \{\infty\}$ and B^n realized by the inversion F (see Section 1.3).

Let $r \mapsto \chi(r)$ be a smooth cut-off function such that $\chi(r) \equiv 1$ for $0 \le r \le \delta$ and $\chi(r) \equiv 0$ for $r > 2\delta$. Set $\chi_{\epsilon_i}(r) = \chi(\epsilon_i r)$. Thus, $\chi_{\epsilon_i}(r) \equiv 1$ for $0 \le r \le \delta\epsilon_i^{-1}$ and $\chi_{\epsilon_i}(r) \equiv 0$ for $r > 2\delta\epsilon_i^{-1}$.

Proposition 3.4.1. *Let $x_i \in \partial M$ and $0 < \epsilon_i \to 0$ be sequences and choose Fermi coordinates centered at each x_i. Then there is a solution ϕ_i to*

$$\begin{cases} \Delta\phi_i(y) = -\chi_{\epsilon_i}(|y|)\epsilon_i h_{kl}(0) y_n (\partial_k\partial_l U)(y)\,, & \text{for } y \in \mathbb{R}^n_+\,, \\ \frac{\partial\phi_i}{\partial y_n}(\bar{y}) + nU^{\frac{2}{n-2}}\phi_i(\bar{y}) = 0\,, & \text{for } \bar{y} \in \partial\mathbb{R}^n_+\,, \end{cases} \tag{3.4.1}$$

satisfying

$$|\nabla^r \phi_i|(y) \le C(r)\epsilon_i |h_{kl}(0)|(1+|y|)^{3-r-n}\,, \quad \text{for } y \in \mathbb{R}^n_+\,,\ r = 0,1,2\,, \tag{3.4.2}$$

$$\phi_i(0) = \frac{\partial\phi_i}{\partial y_1}(0) = \dots = \frac{\partial\phi_i}{\partial y_{n-1}}(0) = 0\,, \tag{3.4.3}$$

$$\int_{\partial\mathbb{R}^n_+} U^{\frac{n}{n-2}}(\bar{y})\phi_i(\bar{y})\, d\bar{y} = 0\,. \tag{3.4.4}$$

Proof. Set

$$f_i(F(y)) = -\chi_{\epsilon_i}(y)\epsilon_i h_{kl}(0) y_n (\partial_k\partial_l U)(y) U^{-\frac{n+2}{n-2}}(y) \quad \text{for } y \in \mathbb{R}^n_+\,.$$

Observe that f_i can be extended as a smooth function to B^n and is $L^2(B^n)$-ortogonal to the coordinate functions $z_1, \dots, z_n$, taken with center $(0, \dots, 0, -1/2)$. To see this orthogonality, we use the conformal equivalence between B^n and $\mathbb{R}^n_+ \cup \{\infty\}$ and the fact that, for every homogeneous polynomial p_k of degree k, we have

$$\int_{S^{n-2}_r} p_k = \frac{r^2}{k(k+n-3)} \int_{S^{n-2}_r} \Delta p_k\,. \tag{3.4.5}$$

By Lemma 1.3.4 and elliptic linear theory, it is possible to find a smooth solution $\bar{\phi}_{\epsilon_i}$ to

$$\begin{cases} \Delta\bar{\phi}_{\epsilon_i} = f_i\,, & \text{in } B^n\,, \\ \frac{\partial\bar{\phi}_{\epsilon_i}}{\partial\eta} + 2\bar{\phi}_{\epsilon_i} = 0\,, & \text{on } \partial B^n\,, \end{cases} \tag{3.4.6}$$

also $L^2(B^n)$-ortogonal to the coordinate functions $z_1, \dots, z_n$.

Set $D=\{(z,w)\in B^n\times B^n;\ z=w\}$. Let G be the Green's function so that

$$\begin{cases}\Delta G(z,w)=\sum_{a=1}^n q_a(w)z_a\,, & \text{in } (B^n\times B^n)\backslash D\,,\\ \left(\frac{\partial}{\partial\eta}+2\right)G(z,w)=0\,, & \text{on } (\partial B^n\times\partial B^n)\backslash D\cap(\partial B^n\times\partial B^n)\,,\end{cases}$$

where Δ and $\frac{\partial}{\partial\eta}$ are taken with respect to z, and $|z-w|^{n-2}G(z,w)\to 1$ as $|z-w|\to 0$.

Then $\bar{\phi}_{\epsilon_i}$ satisfies

$$\bar{\phi}_{\epsilon_i}(z)=-\int_{B^n}G(z,w)f_i(w)dw\,.$$

Therefore,

$$|\bar{\phi}_{\epsilon_i}(z)|\leq C\epsilon_i|h_{kl}(0)|\int_{B^n}|z-w|^{2-n}|w+(0,...,0,1)|^{-3}dw\,.$$

It follows from the result in [28], p.150 (see also [5], p.108) that

$$\bar{\phi}_{\epsilon_i}(z)\leq C\epsilon_i|h_{kl}(0)||z+(0,...,0,1)|^{-1}\leq C\epsilon_i|h_{kl}(0)|(|F(z)|+1)\,.$$

Hence, $\phi_{\epsilon_i}=U\bar{\phi}_{\epsilon_i}$ satisfies the estimate (3.4.2). By the properties (3.1.2) and (3.1.3) of the operators L_g and B_g, ϕ_{ϵ_i} is a solution to the equations (3.4.1).

Now, we choose coefficients $c_{j,i}=\frac{1}{n-2}\frac{\partial\phi_{\epsilon_i}}{\partial y_j}(0)$, $j\in\{1,...,n-1\}$, and $c_{n,i}=-\frac{2}{n-2}\phi_{\epsilon_i}(0)$ and define

$$\phi_i=\phi_{\epsilon_i}+\sum_{a=1}^n c_{a,i}\psi_a\,.$$

Then ϕ_i is also a solution to the equations (3.4.1) and satisfies the identity (3.4.3). Since ϕ_{ϵ_i} satisfies the estimate (3.4.2), we see that $|c_{a,i}|\leq C|h_{kl}(0)|\epsilon_i$ for $a=1,...,n$. Hence, ϕ_i also satisfies the estimate (3.4.2).

Let us prove the identity (3.4.4). Observe that $\bar{\phi}_i=U^{-1}\phi_i$ also satisfies the equations (3.4.6) and f_i is $L^2(B^n)$-ortogonal to the constant function 1. Hence, integrating by parts the first equation of (3.4.6) we see that $\bar{\phi}_i$ is $L^2(\partial B^n)$-ortogonal to the funcion 1. This is the identity (3.4.4). □

The following result is an important estimate that will be used in the subsequent local blow-up analysis.

Proposition 3.4.2. *Let ϕ_i and ϵ_i be as in Proposition 3.4.1 and suppose that $n \geq 5$. Then ϕ_i satisfies*

$$-\int_{B^+_{\delta\epsilon_i^{-1}}}\left(y^b\partial_b\phi_i+\frac{n-2}{2}\phi_i\right)\epsilon_i h_{kl}(0)y_n\partial_k\partial_l U\,dy$$
$$-\int_{B^+_{\delta\epsilon_i^{-1}}}\left(y^b\partial_b U+\frac{n-2}{2}U\right)\epsilon_i h_{kl}(0)y_n\partial_k\partial_l\phi_i\,dy \geq -C(n)|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}\,.$$

Proof. Integrating by parts,

$$\begin{aligned}
&-\int_{B^+_{\delta\epsilon_i^{-1}}}\left(y^b\partial_b\phi_i+\frac{n-2}{2}\phi_i\right)(\epsilon_i h_{kl}(0)y_n\partial_k\partial_l U)dy\\
&\geq \int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k\phi_i\partial_l U dy+\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n y_b\partial_b\partial_k\phi_i\partial_l U dy\\
&\quad+\frac{n-2}{2}\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k\phi_i\partial_l U dy-C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}
\end{aligned}\tag{3.4.7}$$

and

$$\begin{aligned}
&-\int_{B^+_{\delta\epsilon_i^{-1}}}\left(y^b\partial_b U+\frac{n-2}{2}U\right)(\epsilon_i h_{kl}(0)y_n\partial_k\partial_l\phi_i)dy\\
&\geq \int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k U\partial_l\phi_i dy+\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n y_b\partial_b\partial_k U\partial_l\phi_i dy\\
&\quad+\frac{n-2}{2}\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k U\partial_l\phi_i dy-C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}\,.
\end{aligned}\tag{3.4.8}$$

Here, the term $C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}$ comes from the integrals over $\partial^+B^+_{\delta\epsilon_i^{-1}}$ using the estimate (3.4.2). Another integration by parts gives

$$\begin{aligned}
&\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n y_b(\partial_b\partial_k\phi_i)\partial_l U dy+\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n y_b(\partial_b\partial_k U)\partial_l\phi_i dy\\
&\qquad\geq -(n+1)\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k\phi_i\partial_l U dy-C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}
\end{aligned}$$

This, together with the inequalities (3.4.7) and (3.4.8), gives

$$
\begin{aligned}
&-\int_{B^+_{\delta\epsilon_i^{-1}}}\left(y^b\partial_b\phi_i+\frac{n-2}{2}\phi_i\right)(\epsilon_i h_{kl}(0)y_n\partial_k\partial_l U)dy\\
&\quad-\int_{B^+_{\delta\epsilon_i^{-1}}}\left(y^b\partial_b U+\frac{n-2}{2}U\right)(\epsilon_i h_{kl}(0)y_n\partial_k\partial_l\phi_i)dy\\
&\qquad\geq-\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k\phi_i\partial_l U dy-C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}\,.
\end{aligned}
$$

The result now follows from the following Claim:

Claim. $-\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k\phi_i\partial_l U dy\geq -C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}$

Integrating by parts,

$$
\begin{aligned}
-\int_{B^+_{\delta\epsilon_i^{-1}}}\epsilon_i h_{kl}(0)y_n\partial_k\phi_i\partial_l U dy&\geq\int_{B^+_{\delta\epsilon_i^{-1}}}\phi_i\epsilon_i h_{kl}(0)y_n\partial_k\partial_l U-C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}\\
&=-\int_{B^+_{\delta\epsilon_i^{-1}}}(\Delta\phi_i)\phi_i dy-C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}\,.
\end{aligned}
$$

It follows from the estimate (3.4.2) and the assumption over the dimension that

$$
-\int_{B^+_{\delta\epsilon_i^{-1}}}(\Delta\phi_i)\phi_i dy\geq-\int_{\mathbb{R}^n_+}(\Delta\phi_i)\phi_i dy-C|h_{kl}(0)|^2\epsilon_i^{n-2}\delta^{2-n}\,.
$$

Hence, in order to prove the Claim, we will show that

$$
-\int_{\mathbb{R}^n_+}(\Delta\phi_i)\phi_i dy\geq 0 \tag{3.4.9}
$$

Set $\bar{\phi}_i=U^{-1}\phi_i$. Then

$$
-\int_{\mathbb{R}^n_+}(\Delta\phi_i)\phi_i dy=-\int_{B^n}(\Delta_{B^n}\bar{\phi}_i)\bar{\phi}_i dz\,, \tag{3.4.10}
$$

where we have used the property (3.1.2) of the conformal Laplacian. Now, integrating by parts in B^n, we obtain

$$
-\int_{B^n}(\Delta_{B^n}\bar{\phi}_i)\bar{\phi}_i dz=\int_{B^n}|\nabla\bar{\phi}_i|^2_{B^n}dz-2\int_{\partial B^n}\bar{\phi}_i^2 d\sigma\,, \tag{3.4.11}
$$

where η points inwards on ∂B^n. The last equality is due to the equations (3.4.1) and the property (3.1.3) of the boundary operator B_g.

By Lemma 1.3.4,

$$\inf_{\bar{\phi}\in C_1} \frac{\int_{B^n} |\nabla\bar{\phi}|^2 dz}{\int_{\partial B^n} \bar{\phi}^2 d\sigma} = 2\,,$$

where $C_1 = \{\bar{\phi} \in H^1(B^n); \int_{\partial B^n} \bar{\phi} d\sigma = 0\}$. Hence, by the identity (3.4.4),

$$\int_{B^n} |\nabla\bar{\phi}_i|^2_{B^n} dz - 2\int_{\partial B^n} \bar{\phi}_i^2 d\sigma \geq 0\,. \tag{3.4.12}$$

Now inequality (3.4.9) follows from equalities (3.4.10) and (3.4.11) and inequality (3.4.12). This proves the Claim. □

3.5 Blow-up estimates

In this section we will give a pointwise estimate for a blow-up sequence $\{u_i\}$ in a neighborhood of an isolated simple blow-up point. The arguments given here are modifications of the ones given in [33] and [39] for the case of manifolds without boundary.

Assumption. In this section we assume $n \geq 5$.

Let $x_i \to x_0$ be an isolated simple blow-up point for the sequence $\{u_i \in \mathcal{M}_i\}$. Set $v_i(y) = \epsilon_i^{\frac{1}{p_i-1}} u_i(\epsilon_i y)$ for $y \in B^+_{\delta\epsilon_i^{-1}} = B^+_{\delta\epsilon_i^{-1}}(0)$. We know that v_i satisfies

$$\begin{cases} L_{\hat{g}_i} v_i = 0, & \text{in } B^+_{\delta\epsilon_i^{-1}},\\ B_{\hat{g}_i} v_i + (n-2)\hat{f}_i^{-\tau_i} v_i^{p_i} = 0, & \text{on } \partial' B^+_{\delta\epsilon_i^{-1}}, \end{cases} \tag{3.5.1}$$

where $\hat{f}(y) = f(\epsilon_i y)$ and $\hat{g}_i$ is the metric with coefficients $(\hat{g}_i)_{kl}(y) = (g_i)_{kl}(\epsilon_i y)$. Let ϕ_i be the solution to the linearized equation obtained in Proposition 3.4.1.

The main result of this section is

Proposition 3.5.1. *There exist $C, \delta > 0$ such that, after passing to conformal Fermi coordinates,*

$$\begin{aligned} |v_i - (U+\phi_i)|(y) &\leq C(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2(1+|y|)^{4-n} + C\epsilon_i^{n-3}(1+|y|)^{-1}\,,\\ |\nabla v_i - \nabla(U+\phi_i)|(y) &\leq C(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2(1+|y|)^{3-n} + C\epsilon_i^{n-3}(1+|y|)^{-2}\,,\\ |\nabla^2 v_i - \nabla^2(U+\phi_i)|(y) &\leq C(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2(1+|y|)^{2-n} + C\epsilon_i^{n-3}(1+|y|)^{-3}\,, \end{aligned}$$

for $y \in B^+_{\delta\epsilon_i^{-1}}$.

In order to prove Proposition 3.5.1 we will first prove some auxiliary results.

Lemma 3.5.2. *There exist* $\delta, C > 0$ *such that*

$$|v_i - U - \phi_i|(y) \leq C \max\{(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2, \epsilon_i^{n-3}, \tau_i\}\,,$$

for $y \in B^+_{\delta\epsilon_i^{-1}}$.

Proof. Set

$$\Lambda_i = \max_{y \in B^+_{\delta\epsilon_i^{-1}}} |v_i - U - \phi_i|(y) = |v_i - U - \phi_i|(y_i)\,,$$

for some $y \in B^+_{\delta\epsilon_i^{-1}}$. From Remark 3.3.18 we know that $v_i \leq CU$ in $B^+_{\delta\epsilon_i^{-1}}$. Hence, if there is $c > 0$ such that $|y_i| \geq c\epsilon_i^{-1}$, then

$$\Lambda_i = |v_i - U - \phi_i|(y_i) \leq C\,|y_i|^{2-n} \leq C\,\epsilon_i^{n-2}$$

where we used the estimate (3.4.2) in the first inequality. This implies the inequality $|v_i - U - \phi_i|(y) \leq C\,\epsilon_i^{n-2}$, for $|y| \leq \delta\epsilon_i^{-1}$. Hence, we can suppose that $|y_i| \leq \delta\epsilon_i^{-1}/2$.

Suppose, by contradiction, the result is false. Then, choosing a subsequence if necessary, we can suppose

$$\Lambda_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2 \to 0,\;\; \Lambda_i^{-1}\epsilon_i^{n-3},\;\; \Lambda_i^{-1}\tau_i \to 0\,. \tag{3.5.2}$$

Define

$$w_i(y) = \Lambda_i^{-1}(v_i - U - \phi_i)(y)\,, \quad \text{for } y \in B^+_{\delta\epsilon_i^{-1}}\,.$$

By the equations (1.3.1) and (3.5.1), w_i satisfies

$$\begin{cases} L_{\hat{g}_i} w_i = Q_i\,, & \text{in } B^+_{\delta\epsilon_i^{-1}}\,,\\ B_{\hat{g}_i} w_i + b_i w_i = \overline{Q}_i\,, & \text{on } \partial' B^+_{\delta\epsilon_i^{-1}}\,, \end{cases} \tag{3.5.3}$$

where

$$b_i = (n-2)\hat{f}_i^{-\tau_i}\frac{v_i^{p_i} - (U+\phi_i)^{p_i}}{v_i - (U+\phi_i)}\,,$$
$$Q_i = -\Lambda_i^{-1}\left\{(L_{\hat{g}_i} - \Delta)(U + \phi_i) + \Delta\phi_i\right\},$$
$$\overline{Q}_i = -\Lambda_i^{-1}\left\{(n-2)\hat{f}_i^{-\tau_i}(U+\phi_i)^{p_i} - (n-2)U^{\frac{n}{n-2}} - nU^{\frac{2}{n-2}}\phi_i - \tfrac{n-2}{2}h_{\hat{g}_i}(U+\phi_i)\right\}.$$

Observe that

$$
\begin{aligned}
(L_{\hat{g}_i} - \Delta)(y) &= (\hat{g}_i^{kl} - \delta^{kl})(y)\partial_k\partial_l + (\partial_k \hat{g}_i^{kl})(y)\partial_l \\
&\qquad - \frac{n-2}{4(n-1)} R_{\hat{g}_i}(y) + \frac{\partial_k \sqrt{\det \hat{g}_i}}{\sqrt{\det \hat{g}_i}} \hat{g}_i^{kl}\partial_l \\
&= (g_i^{kl} - \delta^{kl})(\epsilon_i y)\partial_k\partial_l + \epsilon_i(\partial_k g_i^{kl})(\epsilon_i y)\partial_l \\
&\qquad - \frac{n-2}{4(n-1)} \epsilon_i^2 R_{g_i}(\epsilon_i y) + O(\epsilon_i^N |y|^{N-1})\partial_l \,,
\end{aligned}
$$

where N can be taken arbitrarily large since we are using conformal Fermi coordinates. Hence, setting $N = n-3$,

$$
\begin{aligned}
Q_i(y) &= -\Lambda_i^{-1}\{(g_i^{kl} - \delta^{kl})(\epsilon_i y)\partial_k\partial_l(U+\phi_i) + \epsilon_i(\partial_k g_i^{kl})(\epsilon_i y)\partial_l(U+\phi_i) \\
&\quad - \frac{n-2}{4(n-1)}\epsilon_i^2 R_{g_i}(\epsilon_i y)(U+\phi_i) + \Delta\phi_i(y)\} + O(\Lambda_i^{-1}\epsilon_i^{n-3}|y|^{n-4}(1+|y|)^{1-n}) \\
&= O\left(\Lambda_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2(1+|y|)^{2-n}\right) + O(\Lambda_i^{-1}\epsilon_i^{n-3}(1+|y|)^{-3})\,, \quad (3.5.4)
\end{aligned}
$$

where we have used the identities (3.4.1) and (3.4.2) and Lemma 1.4.3.

Observe that

$$
\begin{aligned}
(n-2)&\hat{f}_i^{-\tau_i}(U+\phi_i)^{p_i} - (n-2)U^{\frac{n}{n-2}} - nU^{\frac{2}{n-2}}\phi_i \\
&= (n-2)\left(\hat{f}_i^{-\tau_i}(U+\phi_i)^{p_i} - (U+\phi_i)^{\frac{n}{n-2}}\right) + O(U^{\frac{4-n}{n-2}}\phi_i^2) \\
&= (n-2)\hat{f}_i^{-\tau_i}\left((U+\phi_i)^{p_i} - (U+\phi_i)^{\frac{n}{n-2}}\right) \\
&\qquad + (n-2)(\hat{f}_i^{-\tau_i} - 1)(U+\phi_i)^{\frac{n}{n-2}} + O(U^{\frac{4-n}{n-2}}\phi_i^2)\,.
\end{aligned}
$$

Using

$$
\begin{aligned}
&U^{\frac{4-n}{n-2}}\phi_i^2 = O(\epsilon_i^2|h_{kl}(0)|^2(1+|y|)^{2-n}), \\
&h_{\hat{g}_i}(y)(U+\phi_i)(y) = O(\epsilon_i^2|\partial^2 g_i|(1+|y|)^{3-n}), \\
&\hat{f}_i^{-\tau_i}\left((U+\phi_i)^{p_i} - (U+\phi_i)^{\frac{n}{n-2}}\right) = O(\tau_i(U+\phi_i)^{\frac{n}{n-2}}\log(U+\phi_i)) = O(\tau_i(1+|y|)^{1-n}), \\
&(\hat{f}_i^{-\tau_i} - 1)(U+\phi_i)^{\frac{n}{n-2}} = O(\tau_i \log(f_i)(U+\phi_i)^{\frac{n}{n-2}}) = O(\tau_i(1+|y|)^{-n}),
\end{aligned}
$$

where in the second line we used Proposition 1.4.2, we obtain

$$
\bar{Q}_i(\bar{y}) = O\left(\Lambda_i^{-1}\epsilon_i^2(|\partial^2 g_i| + |\partial g_i|^2)(1+|\bar{y}|)^{3-n}\right) + O\left(\Lambda_i^{-1}\tau_i(1+|\bar{y}|)^{1-n}\right)\,. \quad (3.5.5)
$$

Moreover,

$$
b_i(y) \to nU^{\frac{2}{n-2}}\,, \quad \text{in } C^2_{loc}(\mathbb{R}^n)\,, \quad (3.5.6)
$$

and

$$b_i(y) \leq C(1+|y|)^{-2}\,, \quad \text{for } |y| \leq \delta\epsilon_i^{-1}\,. \tag{3.5.7}$$

The estimate (3.5.7) follows from Remark 3.3.18.

Since $|w_i| \leq |w_i(y_i)| = 1$, we can use standard elliptic estimates to conclude that $w_i \to w$, in $C^2_{loc}(\mathbb{R}^n_+)$, for some $w \in C^2_{loc}(\mathbb{R}^n_+)$, choosing a subsequence if necessary. From the identities (3.5.2), (3.5.4), (3.5.5) and (3.5.6), w satisfies

$$\begin{cases} \Delta w = 0\,, & \mathbb{R}^n_+\,, \\ \frac{\partial w}{\partial y_n} + nU^{\frac{2}{n-2}} w = 0\,, & \partial\mathbb{R}^n_+\,. \end{cases} \tag{3.5.8}$$

Claim. $w(y) = O((1+|y|)^{-1})$, for $y \in B^+_{\frac{1}{2}\delta\epsilon_i}$.

Choosing $\delta > 0$ small enough, we can consider the Green's function G_i for the conformal Laplacian $L_{\hat{g}_i}$ in $B^+_{\delta\epsilon_i^{-1}}$ subject to the boundary conditions $B_{\hat{g}_i}G_i = 0$, on $\partial' B^+_{\delta\epsilon_i^{-1}}$, and $G_i = 0$, on $\partial^+ B^+_{\delta\epsilon_i^{-1}}$, where η_i is the unit normal to $\partial^+ B^+_{\delta\epsilon_i^{-1}}$ pointing inwards. Then the Green's formula gives

$$\begin{aligned} w_i(y) = &- \int_{B^+_{\delta\epsilon_i^{-1}}} G_i(x,y) Q_i(x)\, dv_{\hat{g}_i}(x) + \int_{\partial^+ B^+_{\delta\epsilon_i^{-1}}} \frac{\partial G_i}{\partial \eta_i}(x,y) w_i(x)\, d\sigma_{\hat{g}_i}(x) \\ &+ \int_{\partial' B^+_{\delta\epsilon_i^{-1}}} G_i(x,y) \left(b_i(x) w_i(x) - \overline{Q}_i(x) \right) d\sigma_{\hat{g}_i}(x)\,. \end{aligned} \tag{3.5.9}$$

Using the estimates (3.5.4), (3.5.5) and (3.5.7) in the equation (3.5.9), we obtain

$$\begin{aligned}|w_i(y)| \leq & C\Lambda_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2 \int_{B^+_{\delta\epsilon_i^{-1}}} |\xi - y|^{2-n}(1+|\xi|)^{2-n} d\xi \\ & + C\Lambda_i^{-1}\epsilon_i^{n-3} \int_{B^+_{\delta\epsilon_i^{-1}}} |\xi - y|^{2-n}(1+|\xi|)^{-3} d\xi \\ & + C \int_{\partial' B^+_{\delta\epsilon_i^{-1}}} |\bar\xi - y|^{2-n}(1+|\bar\xi|)^{-2} d\bar\xi \\ & + C\Lambda_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2 \int_{\partial' B^+_{\delta\epsilon_i^{-1}}} |\bar\xi - y|^{2-n}(1+|\bar\xi|)^{3-n} d\bar\xi \\ & + C\Lambda_i^{-1}\tau_i \int_{\partial' B^+_{\delta\epsilon_i^{-1}}} |\bar\xi - y|^{2-n}(1+|\bar\xi|)^{1-n} d\bar\xi \\ & + C\Lambda_i^{-1}\epsilon_i^{n-2} \int_{\partial^+ B^+_{\delta\epsilon_i^{-1}}} |\theta - y|^{1-n} d\theta \,,\end{aligned}$$

for $|y| \leq \delta\epsilon_i^{-1}/2$. Here, we have used the fact that $|G_i(x,y)| \leq C|x-y|^{2-n}$ for $|y| \leq \delta\epsilon_i^{-1}/2$ and, since $v_i(y) \leq CU(y)$, $|w_i(y)| \leq C\Lambda_i^{-1}\epsilon_i^{n-2}$ for $|y| = \delta\epsilon_i^{-1}$. Hence, using Lemma C-2,

$$w_i(y) \leq C\left((1+|y|)^{-1} + \Lambda_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2 + \Lambda_i^{-1}\epsilon_i^{n-3} + \Lambda_i^{-1}\tau_i\right).$$

Now the Claim follows from the hypothesis (3.5.2).

Now, we can use the claim above and Lemma 1.3.3 to see that

$$w(y) = \sum_{j=1}^{n-1} c_j \frac{\partial U}{\partial y_j}(y) + c_n\left(\frac{n-2}{2}U(y) + y^b\partial_b U(y)\right),$$

for some constants $c_1, ..., c_n$. It follow from the identity (3.4.3) that $w_i(0) = \frac{\partial w_i}{\partial y_j}(0) = 0$ for $j = 1, ..., n-1$. Thus we conclude that $c_1 = ... = c_n = 0$. Hence, $w \equiv 0$. Since $w_i(y_i) = 1$, we have $|y_i| \to \infty$. This contradicts the claim above and concludes the proof of Lemma 3.5.2. □

Lemma 3.5.3. *There exists $C > 0$ such that*

$$\tau_i \leq C\max\{(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2, \epsilon_i^{n-3}\}.$$

Proof. Suppose, by contradiction, the result is false. Then we can suppose that

$$\tau_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2\,,\;\; \tau_i^{-1}\epsilon_i^{n-3} \to 0 \tag{3.5.10}$$

and, by Lemma 3.5.2, there exists $C > 0$ such that

$$|v_i - (U + \phi_i)|(y) \leq C\tau_i\,,$$

for $y \in B^+_{\delta\epsilon_i^{-1}}$. Define

$$w_i(y) = \tau_i^{-1}(v_i - (U + \phi_i))(y)\,, \quad \text{for } y \in B^+_{\delta\epsilon_i^{-1}}\,.$$

Then w_i satisfies the equations (3.5.3) with

$$\begin{aligned}
&b_i = (n-2)\hat{f}_i^{-\tau_i}\tfrac{v_i^{p_i} - (U+\phi_i)^{p_i}}{v_i - (U+\phi_i)}\,,\\
&Q_i = -\tau_i^{-1}\left\{(L_{\hat{g}_i} - \Delta)(U + \phi_i) + \Delta\phi_i\right\},\\
&\overline{Q}_i = -\tau_i^{-1}\left\{(n-2)\hat{f}_i^{-\tau_i}(U + \phi_i)^{p_i} - (n-2)U^{\frac{n}{n-2}} - nU^{\frac{2}{n-2}}\phi_i - \tfrac{n-2}{2}h_{\hat{g}_i}(U + \phi_i)\right\}.
\end{aligned}$$

Similarly to the estimates (3.5.4) and (3.5.5) we have

$$|Q_i(y)| \leq C\tau_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2(1 + |y|)^{2-n} + C\tau_i^{-1}\epsilon_i^{n-3}(1 + |y|)^{-3}\,, \tag{3.5.11}$$

$$|\overline{Q}_i(y)| \leq C\tau_i^{-1}(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2(1 + |y|)^{3-n} + C(1 + |y|)^{1-n} \tag{3.5.12}$$

and b_i satisfies the estimate (3.5.7).

By definition, $w_i \leq C$ and, by elliptic standard estimates, we can suppose that $w_i \to w$, in $C^2_{loc}(\mathbb{R}^n_+)$ for some $w \in C^2_{loc}(\mathbb{R}^n_+)$. By the identity (3.5.6) and the estimates (3.5.11) and (3.5.12) we see that w satisfies the equations (3.5.8). Recall that $\psi_n(y) = \frac{n-2}{2}U(y) + x^b\partial_b U(y)$ also satisfies the equations (3.5.8) (see Section 1.3).

Let η_i be the inward unit normal vector to $\partial^+ B^+_{\delta\epsilon_i^{-1}}$. Using the Green's formula, we have

$$\begin{aligned}
\int_{\partial' B^+_{\delta\epsilon_i^{-1}}} \psi_n(B_{\hat{g}_i}w_i + b_i w_i)\,d\sigma_{\hat{g}_i} &= \int_{\partial' B^+_{\delta\epsilon_i^{-1}}} (B_{\hat{g}_i}\psi_n + b_i\psi_n)w_i\,d\sigma_{\hat{g}_i}\\
&\quad + \int_{\partial^+ B^+_{\delta\epsilon_i^{-1}}} \left(\frac{\partial\psi_n}{\partial\eta_i}w_i - \psi_n\frac{\partial w_i}{\partial\eta_i}\right) d\sigma_{\hat{g}_i}\\
&\quad + \int_{B^+_{\delta\epsilon_i^{-1}}} \Big(w_i(L_{\hat{g}_i}\psi_n) - \psi_n(L_{\hat{g}_i}w_i)\Big)\,dv_{\hat{g}_i}\,.
\end{aligned} \tag{3.5.13}$$

It follows from the estimate (3.4.2) and the hypothesis (3.5.10) that

$$\lim_{i\to\infty}\int_{\partial^+ B^+_{\delta\epsilon_i^{-1}}}\left(\frac{\partial\psi_n}{\partial\eta_i}w_i-\psi_n\frac{\partial w_i}{\partial\eta_i}\right)d\sigma_{\hat{g}_i}=0\,. \tag{3.5.14}$$

Using the equations (3.5.3), the estimate (3.5.11) and again the hypothesis (3.5.10), we have

$$\lim_{i\to\infty}\int_{B^+_{\delta\epsilon_i^{-1}}}\psi_n(L_{\hat{g}_i}w_i)\,dv_{\hat{g}_i}=\lim_{i\to\infty}\int_{B^+_{\delta\epsilon_i^{-1}}}\psi_n Q_i\,dv_{\hat{g}_i}=0\,. \tag{3.5.15}$$

We will now derive a contradiction. First observe that

$$\psi_n(y)=\frac{n-2}{2}\frac{1-r^2}{(1+r^2)^{\frac{n}{2}}}\,,\quad \text{for } y_n=0\,. \tag{3.5.16}$$

Here, $r^2=y_1^2+...+y_{n-1}^2$. Then

$$\begin{aligned}\int_{\partial\mathbb{R}^n_+}\psi_n U^{\frac{n}{n-2}}&=\frac{n-2}{2}\sigma_{n-2}\int_0^\infty\frac{1-r^2}{(1+r^2)^n}r^{n-2}dr\\&=\frac{n-2}{2}\sigma_{n-2}\left(\int_0^1\frac{1-r^2}{(1+r^2)^n}r^{n-2}dr+\int_1^\infty\frac{1-r^2}{(1+r^2)^n}r^{n-2}dr\right)=0\,,\end{aligned}$$

where in the last equality we change variables $s=r^{-1}$. Now, observe that

$$\lim_{i\to\infty}\tau_i^{-1}\left(\hat{f}_i^{-\tau_i}(y)U^{p_i}(y)-U^{\frac{n}{n-2}}(y)\right)=-\left(\log f(0)+\log U(y)\right)U^{\frac{n}{n-2}}(y)\,,$$

and, similarly to the estimate (3.5.12), we have

$$\begin{aligned}\left|\overline{Q}_i(y)-(n-2)\tau_i^{-1}\left(\hat{f}_i^{-\tau_i}(y)(U+\phi_i)^{p_i}(y)-(U+\phi_i)^{\frac{n}{n-2}}(y)\right)\right|&\\ \leq C\tau_i^{-1}(|\partial^2 g_i|+|\partial g_i|^2)\epsilon_i^2(1+|y|)^{3-n}\,.&\end{aligned}$$

Therefore, since $\int_{\partial\mathbb{R}^n_+}\psi_n U^{\frac{n}{n-2}}\,d\sigma=0$,

$$\lim_{i\to\infty}\int_{\partial' B^+_{\delta\epsilon_i^{-1}}}\psi_n\bar{Q}_i\,d\sigma_{\hat{g}_i}=-(n-2)\int_{\partial\mathbb{R}^n_+}\psi_n\log(U)U^{\frac{n}{n-2}}\,d\sigma\,, \tag{3.5.17}$$

where we have used the hypothesis (3.5.10).

Claim. $\int_{\partial\mathbb{R}^n_+}\psi_n\log(U)U^{\frac{n}{n-2}}\,d\sigma>0$.

By the identity (3.5.16),

$$\int_{\partial\mathbb{R}^n_+} \psi_n(\log U)U^{\frac{n}{n-2}}\,d\sigma = -\frac{(n-2)^2}{4}\sigma_{n-2}\int_0^\infty \frac{1-r^2}{(1+r^2)^n}\log(1+r^2)r^{n-2}dr\,.$$

Changing variables $s = r^{-1}$, we get

$$\int_0^\infty \frac{1-r^2}{(1+r^2)^n}\log(1+r^2)r^{n-2}dr = 2\int_1^\infty \frac{1-r^2}{(1+r^2)^n}\log(r)r^{n-2}dr < 0\,,$$

which concludes the proof of the Claim.

On the other hand, the equation (3.5.13) together with the equations (3.5.3), (3.5.8), (3.5.14) and (3.5.15) gives

$$\begin{aligned}\lim_{i\to\infty}\int_{\partial' B^+_{\delta\epsilon_i^{-1}}} \psi_n\bar{Q}_i\,d\sigma_{\hat{g}_i} &= \lim_{i\to\infty}\int_{\partial' B^+_{\delta\epsilon_i^{-1}}} w_i(B_{\hat{g}_i}\psi_n + b_i\psi_n)\,d\sigma_{\hat{g}_i}\\ &\quad + \lim_{i\to\infty}\int_{B^+_{\delta\epsilon_i^{-1}}} w_i(L_{\hat{g}_i}\psi_n)\,dv_{\hat{g}_i} \qquad (3.5.18)\\ &= \int_{\partial\mathbb{R}^n_+} w\left(\frac{\partial\psi_n}{\partial y_n} + nU^{\frac{2}{n-2}}\psi_n\right)d\sigma + \int_{\mathbb{R}^n_+} w\Delta\psi_n\,dx = 0\,.\end{aligned}$$

Here, we have used the fact that, by the identity (3.5.17), this limit should be independent of $\delta > 0$ arbitrarily small. By the previous claim, this contradicts the identity (3.5.17). □

Proposition 3.5.4. *There exist $C, \delta > 0$ such that*

$$|v_i - (U+\phi_i)|(y) \leq C\max\{(|\partial^2 g_i| + |\partial g_i|^2)\epsilon_i^2, \epsilon_i^{n-3}\}\,,$$

for $y \in B^+_{\delta\epsilon_i^{-1}}$.

Proof. This result follows from Lemmas 3.5.2 and 3.5.3. □

Now, we are able to prove Proposition 3.5.1.

Proof of Proposition 3.5.1. Define

$$w_i(y) = (v_i - (U+\phi_i))(y)\,, \quad \text{for } y \in B^+_{\delta\epsilon_i^{-1}}\,.$$

Then w_i is uniformly bounded in compacts (by Proposition 3.5.4) and satisfies the equations (3.5.3) with

$$
\begin{aligned}
b_i &= (n-2)\hat{f}_i^{-\tau_i}\tfrac{v_i^{p_i}-(U+\phi_i)^{p_i}}{v_i-(U+\phi_i)}\,,\\
Q_i &= -\Big\{(L_{\hat{g}_i}-\Delta)(U+\phi_i)+\Delta\phi_i\Big\},\\
\overline{Q}_i &= -\Big\{(n-2)\hat{f}_i^{-\tau_i}(U+\phi_i)^{p_i}-(n-2)U^{\frac{n}{n-2}}-nU^{\frac{2}{n-2}}\phi_i-\tfrac{n-2}{2}h_{\hat{g}_i}(U+\phi_i)\Big\}.
\end{aligned}
$$

Observe that b_i satisfies the estimate (3.5.7). Similarly to the estimates (3.5.4), (3.5.5) we have

$$
\begin{aligned}
|Q_i(y)| &\le C\epsilon_i^2(|\partial^2 g_i|+|\partial g_i|^2)(1+|y|)^{2-n}+C\epsilon_i^{n-3}(1+|y|)^{-3}\,, && (3.5.19)\\
|\overline{Q}_i(y)| &\le C\epsilon_i^2(|\partial^2 g_i|+|\partial g_i|^2)(1+|y|)^{3-n}+C\tau_i(1+|y|)^{1-n}\\
&\le C\epsilon_i^2(|\partial^2 g_i|+|\partial g_i|^2)(1+|y|)^{3-n}+C\epsilon_i^{n-3}(1+|y|)^{1-n}\,, && (3.5.20)
\end{aligned}
$$

where in the last inequality we used Lemma 3.5.3.

The Green's formula gives

$$
\begin{aligned}
w_i(y) = -\int_{B^+_{\delta\epsilon_i^{-1}}} G_i(x,y)Q_i(x)\,dv_{\hat{g}_i}(x) + \int_{\partial^+B^+_{\delta\epsilon_i^{-1}}} \frac{\partial G_i}{\partial\eta_i}(x,y)w_i(x)\,d\sigma_{\hat{g}_i}(x)\\
+\int_{\partial' B^+_{\delta\epsilon_i^{-1}}} G_i(x,y)\Big(b_i(x)w_i(x)-\overline{Q}_i(x)\Big)\,d\sigma_{\hat{g}_i}(x)\,. \qquad (3.5.21)
\end{aligned}
$$

where η_i is the inward unit normal vector to $\partial^+B^+_{\delta\epsilon_i^{-1}}$ and G_i is the Green's function G_i for the conformal Laplacian $L_{\hat{g}_i}$ in $B^+_{\delta\epsilon_i^{-1}}$ subject to the boundary conditions $B_{\hat{g}_i}G_i=0$, on $\partial' B^+_{\delta\epsilon_i^{-1}}$, and $G_i=0$, on $\partial^+B^+_{\delta\epsilon_i^{-1}}$. Using the estimates (3.5.7), (3.5.19), (3.5.20), Lemma 3.5.3 and Proposition 3.5.4 in equation (3.5.21), as in the proof of Lemma 3.5.2 we obtain

$$
|w_i(y)| \le C\epsilon_i^2(|\partial^2 g_i|+|\partial g_i|^2)(1+|y|)^{-1}+C\epsilon_i^{n-3}(1+|y|)^{-1}\,, \qquad (3.5.22)
$$

for $y\in B^+_{\frac{1}{2}\delta\epsilon_i^{-1}}$. If $n=5$, we have the result. If $n\ge 6$, we plug the inequality (3.5.22) in the Green's formula (3.5.21) until we reach

$$
|w_i(y)| \le C\epsilon_i^2(|\partial^2 g_i|+|\partial g_i|^2)(1+|y|)^{4-n}+C\epsilon_i^{n-3}(1+|y|)^{-1}\,.
$$

The derivative estimates follow from elliptic theory, finishing the proof. □

3.6 Local blow-up analysis

In this section we will prove the vanishing of the trace-free second fundamental form in an isolated simple blow-up point if $n \geq 7$. We will also prove a Pohozaev sign condition that will be used later in the study of the blow-up set. The basic tool here will be the Pohozaev-type the identity of Section 3.2 and the blow-up estimates of Section 3.5.

3.6.1 Vanishing of the trace-free 2nd fundamental form

The vanishing of π_{kl}, the trace-free 2nd fundamental form of the boundary, in an isolated simple blow-up point is stated as follows:

Theorem 3.6.1. *Suppose that $n \geq 7$. Let $x_i \to x_0$ be an isolated simple blow-up point for the sequence $\{u_i \in \mathcal{M}_i\}$. Then*

$$|\pi_{kl}(x_i)|^2 \leq C\epsilon_i \,.$$

In particular, $\pi_{kl}(x_0) = 0$.

Proof. In what follows we are using conformal Fermi coordinates centered at x_i. By Proposition 1.4.2, we can suppose that $h(0) = h_{,k}(0) = 0$. In particular, $\pi_{kl}(0) = h_{kl}(0)$. Recall that we use indices $1 \leq k, l \leq n-1$ and $1 \leq a, b \leq n$ when working with coordinates. In many parts of the proof we will use the identity (3.4.5).

We write the Pohozaev identity of Proposition 3.2.1 as

$$P(u_i, r) = F_i(u_i, r) + \bar{F}_i(u_i, r) + \frac{\tau_i}{p_i + 1} Q_i(u_i, r)\,, \tag{3.6.1}$$

where
$F_i(u, r) = -\int_{B_r^+}(x^b\partial_b u + \frac{n-2}{2}u)(L_{g_i} - \Delta)u\, dx$,
$\bar{F}_i(u, r) = \frac{n-2}{2}\int_{\partial' B_r^+}(\bar{x}^b\partial_b u + \frac{n-2}{2}u)h_{g_i}u\, d\bar{x}$,
$Q_i(u, r) = \frac{(n-2)^2}{2}\int_{\partial' B_r^+} f_i^{-\tau_i}u^{p_i+1}d\bar{x} - (n-2)\int_{\partial' B_r^+}(x^k\partial_k f)f_i^{-\tau_i-1}u^{p_i+1}d\bar{x}$.

Fix $r > 0$ small enough such that $Q_i(u_i, r) \geq 0$. For the term $\bar{F}_i$ we have,

$$\bar{F}_i(u_i, r) = \frac{n-2}{2}\epsilon_i^{-\frac{2}{(p_i-1)}+n-2}\int_{\partial' B^+_{r\epsilon_i^{-1}}}\left(\bar{y}^b\partial_b v_i + \frac{n-2}{2}v_i\right)\epsilon_i h_{g_i}(\epsilon_i\bar{y})v_i(\bar{y})d\bar{y}\,,$$

Since $h(0) = h_{,k}(0) = 0$ and the fact that, according to Proposition 3.3.17, $\lim_{i\to\infty} \epsilon_i^{-\frac{2}{p_i-1}+n-2} = \lim_{i\to\infty} \epsilon_i^{-(n-2)\frac{\tau_i}{p_i-1}} = 1$, we have

$$\begin{aligned}\bar{F}_i(u_i, r) &= (1 + o_i(1)) \int_{\partial' B^+_{r\epsilon_i^{-1}}} O((1 + |\bar{y}|)^{2-n}) O(\epsilon_i^3 |\partial^3 g_i| |\bar{y}|^2) O((1 + |\bar{y}|)^{2-n}) d\bar{y} \\ &\geq -C\epsilon_i^3 |\partial^3 g_i| \int_{\partial' B^+_{rM_i^{p_i-1}}} (1 + |\bar{y}|)^{6-2n} d\bar{y}\,. \qquad (3.6.2)\end{aligned}$$

Set $\tilde{u}_i(x) = \epsilon_i^{-\frac{1}{p_i-1}} (U + \phi_i)(\epsilon_i^{-1} x)$. Using the facts that $g_i^{nn} \equiv 1$ and $g_i^{kn} \equiv 0$ in Fermi coordinates, we have

$$\begin{aligned}F_i(u_i, r) &= -\int_{B_r^+} (x^b \partial_b u_i + \frac{n-2}{2} u_i)(L_{g_i} - \Delta) u_i dx \\ &= -\epsilon_i^{-\frac{2}{(p_i-1)}+n-2} \int_{B^+_{r\epsilon_i^{-1}}} (y^b \partial_b v_i + \frac{n-2}{2} v_i)(L_{\hat{g}_i} - \Delta) v_i dy\,,\end{aligned}$$

$$\begin{aligned}F_i(\tilde{u}_i, r) &= -\int_{B_r^+} (x^b \partial_b \tilde{u}_i + \frac{n-2}{2} \tilde{u}_i)(L_{g_i} - \Delta) \tilde{u}_i dx \\ &= -\epsilon_i^{-\frac{2}{(p_i-1)}+n-2} \int_{B^+_{r\epsilon_i^{-1}}} (y^b \partial_b (U + \phi_i) + \frac{n-2}{2} (U + \phi_i))(L_{\hat{g}_i} - \Delta)(U + \phi_i) dy\,.\end{aligned}$$

It follows from Proposition 3.5.1 that

$$\begin{aligned}|F_i(u_i, r) - F_i(\tilde{u}_i, r)| &\leq C\epsilon_i^3 (|\partial g_i| + |\partial^2 g_i|)(|\partial^2 g_i| + |\partial g_i|^2) \int_{B^+_{r\epsilon_i^{-1}}} (1 + |y|)^{5-2n} dy \\ &\quad + C\epsilon_i^{n-2} (|\partial g_i| + |\partial^2 g_i|) \int_{B^+_{r\epsilon_i^{-1}}} (1 + |y|)^{-n} dy\,. \qquad (3.6.3)\end{aligned}$$

We write

$$F_i(\tilde{u}_i, r) = (1 + o_i(1)) \left\{ R_i(U, U) + R_i(U, \phi_i) + R_i(\phi_i, U) + R_i(\phi_i, \phi_i) \right\}, \qquad (3.6.4)$$

where we have defined

$$R_i(w_1, w_2) = -\int_{B_{r\epsilon_i^{-1}}} (y^b \partial_b w_1 + \frac{n-2}{2} w_1)(L_{\hat{g}_i} - \Delta) w_2 dy\,.$$

Using the identities (3.6.2), (3.6.3) and (3.6.4) and the fact that $Q_i(u_i, r) \geq 0$ in the equality (3.6.1), we have

$$\begin{aligned} P(u_i, r) \geq (1 + o_i(1)) \Big\{ R_i(U, U) + R_i(U, \phi_i) + R_i(\phi_i, U) + R_i(\phi_i, \phi_i) \Big\} \\ - C(|\partial g_i||\partial^2 g_i| + |\partial g_i|^3 + |\partial^2 g_i|^2 + |\partial^3 g_i|)\, \epsilon_i^3 \\ - C(|\partial g_i| + |\partial^2 g_i|)\, \epsilon_i^{n-2} (\log \epsilon_i)(\log r)\,. \end{aligned} \tag{3.6.5}$$

By Lemma 1.4.3 and the estimate (3.4.2),

$$\begin{aligned} R_i(U, \phi_i) + R_i(\phi_i, U) = &- \int_{B_{r\epsilon_i^{-1}}} \Big(y^b \partial_b \phi_i + \frac{n-2}{2} \phi_i \Big) (L_{\hat{g}_i} - \Delta) U dy \\ &- \int_{B_{r\epsilon_i^{-1}}} \Big(y^b \partial_b U + \frac{n-2}{2} U \Big) (L_{\hat{g}_i} - \Delta) \phi_i dy \\ \geq &- \int_{B_{r\epsilon_i^{-1}}} \Big(y^b \partial_b \phi_i + \frac{n-2}{2} \phi_i \Big) (2\epsilon_i h_{kl}(0) y_n \partial_k \partial_l U) dy \\ &- \int_{B_{r\epsilon_i^{-1}}} \Big(y^b \partial_b U + \frac{n-2}{2} U \Big) (2\epsilon_i h_{kl}(0) y_n \partial_k \partial_l \phi_i) dy \\ &- C\epsilon_i^3 |h_{kl}(0)| (|\partial^2 g_i| + |\partial g_i|^2) \int_{B_{r\epsilon_i^{-1}}} (1 + |y|)^{5-2n} dy\,. \end{aligned}$$

Now we apply Proposition 3.4.2 to this inequality to ensure that

$$R_i(U, \phi_i) + R_i(\phi_i, U) \geq -C \Big(\epsilon_i^3 |h_{kl}(0)| (|\partial^2 g_i| + |\partial g_i|^2) + |h_{kl}(0)|^2 \epsilon_i^{n-2} r^{2-n} \Big)\,. \tag{3.6.6}$$

It follows from the estimate (3.4.2) that

$$R_i(\phi_i, \phi_i) = \epsilon_i^3 |h_{kl}(0)|^2 |\partial g_i| \int_{B_{r\epsilon_i^{-1}}} O((1 + |y|)^{5-2n}) dy\,. \tag{3.6.7}$$

We will now handle the term $R_i(U, U)$. Observe that

$$\begin{aligned} \partial_l U(y) &= -(n-2) \Big((1 + y_n)^2 + |\bar{y}|^2 \Big)^{-\frac{n}{2}} y_l\,, \\ \partial_k \partial_l U(y) &= (n-2) \Big((1 + y_n)^2 + |\bar{y}|^2 \Big)^{-\frac{n+2}{2}} \Big(n y_k y_l - ((1 + y_n)^2 + |\bar{y}|^2) \delta_{kl} \Big)\,, \\ y^b \partial_b U + \frac{n-2}{2} U &= -\frac{n-2}{2} \Big((1 + y_n)^2 + |\bar{y}|^2 \Big)^{-\frac{n}{2}} (|y|^2 - 1)\,. \end{aligned}$$

Using this we obtain

$$
\begin{aligned}
R_i(U,U) = \frac{(n-2)^2}{2} \int_{B^+_{r\epsilon_i^{-1}}} & \frac{|y|^2-1}{((1+y_n)^2+|y|^2)^{n+1}} \\
& \cdot (g_i^{kl}-\delta^{kl})(\epsilon_i y)\Big(ny_k y_l - ((1+y_n)^2+|\bar{y}|^2)\delta_{kl}\Big)\,dy \\
- \frac{(n-2)^2}{2} & \int_{B^+_{r\epsilon_i^{-1}}} \frac{|y|^2-1}{((1+y_n)^2+|y|^2)^{n}} \cdot \epsilon_i(\partial_k g^{kl})(\epsilon_i y) y_l dy \\
- \frac{(n-2)^2}{8(n-1)} & \int_{B^+_{r\epsilon_i^{-1}}} \frac{|y|^2-1}{((1+y_n)^2+|y|^2)^{n-1}} \cdot \epsilon_i^2 R_{g_i}(\epsilon_i y) dy\,.
\end{aligned}
$$

Using Lemma 1.4.3 and symmetry arguments, we have

$$
R_i(U,U) \geq \frac{(n-2)^2}{2}(A_1+A_2+A_3+A_4) - C(|\partial^2 g_i|+|\partial g_i|^2)\epsilon_i^{n-2} r^{2-n}\,,
$$

where

$$
\begin{aligned}
A_1 &= n \int_{y_n=0}^{\infty}\int_{s=0}^{\infty} \tfrac{s^2+y_n^2-1}{(s^2+(y_n+1)^2)^{n+1}} \left\{ \int_{S_s^{n-2}} (g_i^{kl}-\delta_i^{kl})(\epsilon_i y) y_k y_l \, d\sigma_s(y) \right\} ds dy_n, \\
A_2 &= -\int_{y_n=0}^{\infty}\int_{s=0}^{\infty} \tfrac{s^2+y_n^2-1}{(s^2+(y_n+1)^2)^{n}} \left\{ \int_{S_s^{n-2}} (g_i^{kl}-\delta_i^{kl})(\epsilon_i y) \delta_{kl} \, d\sigma_s(y) \right\} ds dy_n, \\
A_3 &= -\int_{y_n=0}^{\infty}\int_{s=0}^{\infty} \tfrac{s^2+y_n^2-1}{(s^2+(y_n+1)^2)^{n}} \left\{ \epsilon_i \int_{S_s^{n-2}} (\partial_k g_i^{kl})(\epsilon_i y) y_l \, d\sigma_s(y) \right\} ds dy_n, \\
A_4 &= \tfrac{-1}{4(n-1)} \int_{y_n=0}^{\infty}\int_{s=0}^{\infty} \tfrac{s^2+y_n^2-1}{(s^2+(y_n+1)^2)^{n-1}} \left\{ \epsilon_i^2 \int_{S_s^{n-2}} R_{g_i}(\epsilon_i y) \, d\sigma_s(y) \right\} ds dy_n.
\end{aligned}
$$

Using Lemmas 1.4.3 and 1.4.5 we see that

$$
\begin{aligned}
\int_{S_s^{n-2}} (g_i^{kl}-\delta_i^{kl})(\epsilon_i y) y_k y_l \, d\sigma_s &= \sigma_{n-2}\epsilon_i^2 \frac{y_n^2 s^n}{n-1} \cdot 2|h_{kl}(0)|^2 + \epsilon_i^3 |\partial^3 g_i| O(|(s,y_n)|^{n+3})\,, \\
\int_{S_s^{n-2}} (g_i^{kl}-\delta_i^{kl})(\epsilon_i y) \delta_{kl} \, d\sigma_s &= \sigma_{n-2}\epsilon_i^2 \cdot y_n^2 s^{n-2} \cdot 2|h_{kl}(0)|^2 \\
&\quad + \epsilon_i^3 |\partial^3 g_i| O(|(s,y_n)|^{n+1})\,, \\
\epsilon_i \cdot \int_{S_s^{n-2}} (\partial_k g_i^{kl})(\epsilon_i y) y_l \, d\sigma_s &= \epsilon_i^3 |\partial^3 g_i| O(|(s,y_n)|^{n+1})\,, \\
\epsilon_i^2 \cdot \int_{S_s^{n-2}} R_{g_i}(\epsilon_i y)\, d\sigma_s &= -\sigma_{n-2}\epsilon_i^2 \cdot s^{n-2} \cdot |h_{kl}(0)|^2 \\
&\quad + \epsilon_i^3 (|\partial^3 g_i| + |\partial^2 g_i||\partial g_i|) O(|(s,y_n)|^{n-1})\,,
\end{aligned}
$$

where in the last equality we used the fact that, by the Gauss equation, $R(0)+|h_{kl}(0)|^2=0$. Set $I=\int_0^\infty \frac{s^n}{(s^2+1)^n}ds$. Using Corollary B-3 and the four equalities above,

$$\begin{aligned}
A_1 &= \sigma_{n-2}\epsilon_i^2\cdot\frac{2n}{n-1}|h_{kl}(0)|^2\int_{y_n=0}^{\infty} y_n^2\left\{\int_{s=0}^{\infty}\frac{s^2+y_n^2-1}{(s^2+(y_n+1)^2)^{n+1}}s^n ds\right\}dy_n\\
&\qquad + \epsilon_i^3|\partial^3 g_i|\int_{B^+_{r\epsilon_i^{-1}}} O((1+|y|)^{5-2n})dy\\
&= \sigma_{n-2}\epsilon_i^2 I\cdot\frac{n+1}{n-1}|h_{kl}(0)|^2\int_{y_n=0}^{\infty} y_n^2(y_n+1)^{1-n}dy_n\\
&\quad + \sigma_{n-2}\epsilon_i^2 I\cdot|h_{kl}(0)|^2\int_{y_n=0}^{\infty} y_n^2(y_n^2-1)(y_n+1)^{-1-n}dy_n\\
&\qquad + \epsilon_i^3|\partial^3 g_i|\int_{B^+_{r\epsilon_i^{-1}}} O((1+|y|)^{5-2n})dy\,,
\end{aligned}$$

$$\begin{aligned}
A_2 &= -\sigma_{n-2}\epsilon_i^2\cdot 2|h_{kl}(0)|^2\int_{y_n=0}^{\infty} y_n^2\left\{\int_{s=0}^{\infty}\frac{s^2+y_n^2-1}{(s^2+(y_n+1)^2)^{n}}s^{n-2} ds\right\}dy_n\\
&\qquad + \epsilon_i^3|\partial^3 g_i|\int_{B^+_{r\epsilon_i^{-1}}} O((1+|y|)^{5-2n})dy\\
&= -\sigma_{n-2}\epsilon_i^2 I\cdot 2|h_{kl}(0)|^2\int_{y_n=0}^{\infty} y_n^2(y_n+1)^{1-n}dy_n\\
&\quad - \sigma_{n-2}\epsilon_i^2 I\cdot 2|h_{kl}(0)|^2\int_{y_n=0}^{\infty} y_n^2(y_n^2-1)(y_n+1)^{-1-n}dy_n\\
&\qquad + \epsilon_i^3|\partial^3 g_i|\int_{B^+_{r\epsilon_i^{-1}}} O((1+|y|)^{5-2n})dy\,,
\end{aligned}$$

$$A_3 = \epsilon_i^3|\partial^3 g_i|\int_{B^+_{r\epsilon_i^{-1}}} O((1+|y|)^{5-2n})dy\,,$$

$$
\begin{aligned}
A_4 &= \sigma_{n-2}\epsilon_i^2 \cdot \frac{1}{4(n-1)}|h_{kl}(0)|^2 \int_{y_n=0}^{\infty}\left\{\int_{s=0}^{\infty}\frac{s^2+y_n^2-1}{(s^2+(y_n+1)^2)^{n-1}}s^{n-2}ds\right\}dy_n \\
&\qquad + \epsilon_i^3(|\partial^3 g_i| + |\partial^2 g_i||\partial g_i|)\int_{B^+_{r\epsilon_i^{-1}}} O((1+|y|)^{5-2n})dy \\
&= \sigma_{n-2}\epsilon_i^2 I \cdot \frac{1}{2(n-3)}|h_{kl}(0)|^2 \int_{y_n=0}^{\infty}(y_n+1)^{3-n}dy_n \\
&\quad + \sigma_{n-2}\epsilon_i^2 I \cdot \frac{1}{2(n-1)}|h_{kl}(0)|^2 \int_{y_n=0}^{\infty}(y_n^2-1)(y_n+1)^{1-n}dy_n \\
&\qquad + \epsilon_i^3(|\partial^3 g_i| + |\partial^2 g_i||\partial g_i|)\int_{B^+_{r\epsilon_i^{-1}}} O((1+|y|)^{5-2n})dy\,.
\end{aligned}
$$

Set $J_k = \int_0^\infty \frac{y_n^k}{(1+y_n)^n}dy_n$. It follows from the above computations that

$$
\begin{aligned}
&R_i(U,U) \geq -C\epsilon_i^3(|\partial^3 g_i| + |\partial^2 g_i||\partial g_i|) - C\epsilon_i^{n-2}r^{2-n}(|\partial^2 g_i| + |\partial g_i|^2) \\
&+\sigma_{n-2}\epsilon_i^2 I \cdot \left\{\frac{n+1}{n-1}(J_3+J_2) + (J_3-J_2) - 2(J_3+J_2) - 2(J_3-J_2)\right\}|h_{kl}(0)|^2 \\
&+\sigma_{n-2}\epsilon_i^2 I \cdot \left\{\frac{1}{2(n-3)}(J_3+3J_2+3J_1+J_0) + \frac{1}{2(n-1)}(J_3+J_2-J_1-J_0)\right\}|h_{kl}(0)|^2 \\
&\quad = \sigma_{n-2}\epsilon_i^2 I \cdot (\alpha_3 J_3 + \alpha_2 J_2 + \alpha_1 J_1 + \alpha_0 J_0)\cdot |h_{kl}(0)|^2 \\
&\qquad - C\epsilon_i^3(|\partial^3 g_i| + |\partial^2 g_i||\partial g_i|) - C\epsilon_i^{n-2}r^{2-n}(|\partial^2 g_i| + |\partial g_i|^2)\,, \qquad (3.6.8)
\end{aligned}
$$

where $\alpha_3 = -2 + \frac{1}{2(n-3)} + \frac{5}{2(n-1)}$, $\alpha_2 = \frac{3}{2(n-3)} + \frac{5}{2(n-1)}$, $\alpha_1 = \frac{3}{2(n-3)} - \frac{1}{2(n-1)}$ and $\alpha_0 = \frac{1}{2(n-3)} - \frac{1}{2(n-1)}$.

By Lemma B-2, $J_2 = \frac{n-4}{3}J_3$, $J_1 = \frac{(n-4)(n-3)}{6}J_3$ and $J_0 = \frac{(n-4)(n-3)(n-2)}{6}J_3$. Then a directy computation shows that

$$
\alpha_0 J_0 + \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3 = \frac{n-6}{3}J_3\,.
$$

This, together with the inequality (3.6.8), implies

$$
\begin{aligned}
R_i(U,U) &\geq \sigma_{n-2}\epsilon_i^2\frac{n-6}{3}IJ_3|h_{kl}(0)|^2 - C\epsilon_i^{n-2}r^{2-n}(|\partial^2 g_i| + |\partial g_i|^2) \\
&\qquad - C\epsilon_i^3(|\partial^3 g_i| + |\partial^2 g_i||\partial g_i|)\,. \qquad (3.6.9)
\end{aligned}
$$

Hence, by the estimates (3.6.5), (3.6.6), (3.6.7) and (3.6.9),

$$
\begin{aligned}
P(u_i,r) &\geq (1+o_i(1))\sigma_{n-2}\epsilon_i^2\frac{n-6}{3}IJ_3|h_{kl}(0)|^2 - C\epsilon_i^{n-2}(\log\epsilon_i)r^{2-n}(|\partial g_i| + |\partial^2 g_i|) \\
&\qquad - C\epsilon_i^3(|\partial^3 g_i| + |\partial g_i||\partial^2 g_i| + |\partial^2 g_i|^2 + |\partial g_i|^3) \qquad (3.6.10)
\end{aligned}
$$

On the other hand, by Proposition 3.3.12 we can assume that $\epsilon_i^{-\frac{1}{p_i-1}} u_i$ converges in $C^2_{loc}(M\backslash\{x_0\})$. Hence, for $r > 0$ fixed, $\epsilon_i^{-\frac{2}{p_i-1}} P(u_i, r)$ converges and

$$P(u_i, r) \leq C\epsilon_i^{n-2}. \tag{3.6.11}$$

Then, for $r > 0$ small fixed, the estimate (3.6.10) together with the estimate (3.6.11) and our dimension assumption gives $|h_{kl}(0)|^2 \leq C\epsilon_i$. This proves Theorem 3.6.1, since under our assumptions $\pi_{kl}(x_i) = h_{kl}(0)$. □

3.6.2 Pohozaev sign condition

Now we will state and prove the Pohozaev sign condition.

Set

$$P'(u, r) = \int_{\partial^+ B_r^+} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r \,.$$

Theorem 3.6.2. *Let $x_i \to x_0$ be a blow-up point for the sequence $\{u_i \in \mathcal{M}_i\}$. Assume that $\pi_{kl}(x_0) \neq 0$ and $n \geq 7$. We work in $B_\delta^+(0)$ using Fermi coordinates centered at x_i. For $0 < \tau_i \to 0$, set*

$$w_i(y) = \tau_i^{\frac{1}{p_i-1}} u_i(\tau_i y)$$

for $y \in B^+_{\delta\tau_i^{-1}}(0)$. Suppose that the origin 0 is an isolated simple blow-up point for the sequence $\{w_i\}$ and that $w_i(0) w_i \to G$ away from the origin, for some function G. Then

$$\lim_{r \to 0} P'(G, r) \geq 0 \,.$$

Proof. Observe that $|\pi_{kl}(0)| = |\pi_{kl}(x_i)| \geq \frac{1}{2}|\pi_{kl}(x_0)|$ for i large. We will restringe our analysis to $B_{\tilde\delta}^+(0) \subset B^+_{\delta\tau_i^{-1}}(0)$, for some $\tilde\delta > 0$ fixed. Set $\tilde\epsilon_i = \epsilon_i \tau_i^{-1}$. Hence, $w_i(0) = \tilde\epsilon_i^{-\frac{1}{p_i-1}}$ and $\tilde\epsilon_i \to 0$. Let $\tilde g_i$ be the metric on $B_{\tilde\delta}^+(0)$ with coefficients $(\tilde g_i)_{kl}(y) = (g_i)_{kl}(\tau_i y)$ and denote by $\tilde h_{kl}$ the corresponding 2nd fundamental form.

Similarly to the proof of Theorem 3.6.1, using conformal Fermi coordinates we have

$$\begin{aligned} P(w_i, r) \geq (1 + o_i(1))\sigma_{n-2}\tilde\epsilon_i^2 \frac{n-6}{3} I J_3 |\tilde h_{kl}(0)|^2 - C\tilde\epsilon_i^{n-2}(\log \tilde\epsilon_i) r^{2-n}(|\partial \tilde g_i| + |\partial^2 \tilde g_i|) \\ - C\tilde\epsilon_i^3(|\partial^3 \tilde g_i| + |\partial \tilde g_i||\partial^2 \tilde g_i| + |\partial^2 \tilde g_i|^2 + |\partial \tilde g_i|^3) \,. \end{aligned} \tag{3.6.12}$$

By the Young's inequality,

$$\tilde{\epsilon}_i^{n-2}(\log\tilde{\epsilon}_i)r^{2-n}|\partial\tilde{g}_i| \leq |\partial\tilde{g}_i|^2\tilde{\epsilon}_i^{n-2}(\log\tilde{\epsilon}_i)^2r^{2-2n} + \tilde{\epsilon}_i^{n-2}r^2\,.$$

Hence, writing the inequality (3.6.12) in terms of the metric g_i we have

$$\begin{aligned} P(w_i,r) \geq\ & (1+o_i(1))\sigma_{n-2}\epsilon_i^2\frac{n-6}{3}IJ_3|h_{kl}(0)|^2 \\ & - C\epsilon_i^2(|\partial g_i|^2 + |\partial^2 g_i|)\tilde{\epsilon}_i^{n-4}(\log\tilde{\epsilon}_i)^2r^{2-2n} \\ & \quad - C\epsilon_i^3(|\partial^3 g_i| + |\partial g_i||\partial^2 g_i| + \tau_i|\partial^2 g_i|^2 + |\partial g_i|^3) - C\tilde{\epsilon}_i^{n-2}r^2 \\ \geq\ & -C\tilde{\epsilon}_i^{n-2}r^2\,, \end{aligned}$$

for large i and $r>0$ small fixed. Here, we used our dimension assumption and the fact that $|\pi_{kl}(0)| \geq \frac{1}{2}|\pi_{kl}(x_0)| > 0$ in the last inequality. Hence, Proposition 3.3.17 implies

$$P'(G,r) = \lim_{i\to\infty}\tilde{\epsilon}_i^{-\frac{2}{p_i-1}}P(w_i,r) \geq -Cr^2\,.$$

This proves Theorem 3.6.2. □

3.7 Proof of Theorem 1.1.4

In this section we will prove Theorem 1.1.4.

The first proposition of this section states that every isolated blow-up point $x_i \to x_0$ is also simple, as long as π_{kl}, the boundary trace-free 2nd fundamental form, does not vanish at x_0.

Proposition 3.7.1. *Let $x_i \to x_0$ be a blow-up point for the sequence $\{u_i \in \mathcal{M}_i\}$. Assume that $\pi_{kl}(x_0) \neq 0$ and $n \geq 7$. We work in $B^+_\delta(0)$ using Fermi coordinates centered at x_i. If $0 < \tau_i \to 0$ or $\tau_i = 1$, set*

$$w_i(y) = \tau_i^{\frac{1}{p_i-1}}u_i(\tau_i y)$$

for $y \in B^+_{\delta\tau_i^{-1}}(0)$. Suppose that the origin 0 is an isolated blow-up point for the sequence $\{w_i\}$. Then it is also isolated simple.

Proof. Suppose that the origin is an isolated blow-up point for $\{w_i\}$ but is not simple. By definition, passing to a subsequence, there are at least two critical points of $r \mapsto r^{\frac{1}{p_i-1}}\overline{W}_i(r)$ in an interval $(0,\bar{\rho}_i)$, $\bar{\rho}_i \to 0$. Let

$r_i = R_i w_i(0)^{-(p_i-1)} \to 0$ and $R_i \to \infty$ as in Proposition 3.3.6. By the 2nd paragraph of Remark 3.3.11, there is exactly one critical point in the interval $(0, r_i)$. Let ρ_i be the second critical point. Then $\bar{\rho}_i > \rho_i \geq r_i$.

Set $v_i(z) = \rho_i^{\frac{1}{p_i-1}} w_i(\rho_i z)$, for $z \in B^+_{\delta\rho_i^{-1}}(0)$. Observe that, since $\rho_i \geq r_i$,

$$v_i(0)^{p_i-1} = \rho_i w_i(0)^{p_i-1} \geq R_i \to \infty\,.$$

Hence, $v_i(0) \to \infty$.

By the scaling invariance (see Remark 3.3.3), the origin is an isolated blow-up point for $\{v_i\}$. By the definitions, $r \mapsto r^{\frac{1}{p_i-1}}\bar{v}_i(r)$ has exactly one critical point in the interval $(0, 1)$ and

$$\frac{d}{dr}(r^{\frac{1}{p_i-1}}\bar{v}_i(r))|_{r=1} = 0\,. \tag{3.7.1}$$

Hence, the origin is an isolated simple blow-up point for $\{v_i\}$. It follows from Proposition 3.3.12 that $v_i(0)v_i$ is uniformly bounded in compacts of $\mathbb{R}^n_+\backslash\{0\}$. Using the equations (3.1.4) and the scaling invariance property stated in Section 3.1, we can suppose that

$$v_i(0)v_i(z) \to G(z) = a|z|^{2-n} + b(z)\,,$$

in $C^2_{loc}(\mathbb{R}^n_+\backslash\{0\})$. Here, b is harmonic on $\mathbb{R}^n_+$ with Neumann condition on $\partial\mathbb{R}^n_+$ and $a > 0$. Since $G > 0$, $\liminf_{|z|\to\infty} b(z) \geq 0$. By the Liouville's theorem, b is constant. By the equality (3.7.1),

$$\frac{d}{dr}(r^{\frac{n-2}{2}}h(r))|_{r=1} = 0\,,$$

which implies that $b = a > 0$. This contradicts the sign condition of Theorem 3.6.2. □

The next proposition ensures that the set $\{x_1, ..., x_N\} \subset \partial M$ of points obtained in Proposition 3.3.9 can only contain isolated blow-up points for any blow-up sequence $\{u_i \in \mathcal{M}_i\}$ as long as π_{kl} does not vanish at the blow-up point.

Proposition 3.7.2. *Assume that $n \geq 7$. Let $\beta > 0$ be small, $R > 0$ be large and consider $C_0 = C_0(\beta, R)$ and $C_1 = C_1(\beta, R)$ as in Proposition 3.3.9. Let $x_0 \in \partial M$ be a point such that $\pi_{kl}(x_0) \neq 0$. Then there is $\delta > 0$ such that, for any $u \in \mathcal{M}_p$ satisfying $\max_{\partial M} u \geq C_0$, the set $\partial' B^+_\delta(x_0) \cap \{x_1(u), ..., x_N(u)\}$ consists of at most one point. Here, $x_1(u), ..., x_N(u) \in \partial M$, with $N = N(u)$, are the points obtained in Proposition 3.3.9.*

Proof. Suppose the result is not true. Then there is a sequence $u_i \in \mathcal{M}_i$, $\max_{\partial M} u_i \geq C_0$, such that after relabeling the indices we have $x_1^{(i)}, x_2^{(i)} \to x_0$, as $i \to \infty$. Here, we have set $x_1^{(i)} = x_1(u_i), ..., x_{N_i}^{(i)} = x_{N_i}(u_i)$ and $N_i = N(u_i)$.

Define

$$s_i = d_{\bar{g}}(x_1^{(i)}, x_2^{(i)})^{-\frac{1}{2}} \to \infty.$$

Claim 1. There exist $1 \leq j_i \neq k_i \leq N_i$ such that $x_{j_i}^{(i)}, x_{k_i}^{(i)} \in \partial' B^+_{2s_i^{-1}}(x_1^{(i)})$,

$$\sigma_i := d_{\bar{g}}(x_{j_i}^{(i)}, x_{k_i}^{(i)}) \leq d_{\bar{g}}(x_1^{(i)}, x_2^{(i)}),$$

$$d_{\bar{g}}(x_l^{(i)}, x_m^{(i)}) \geq \frac{1}{2}\sigma_i\,, \quad \text{for all } x_l^{(i)}, x_m^{(i)} \in \partial' B^+_{s_i\sigma_i}(x_{j_i}^{(i)}),\ l \neq m\,.$$

Suppose that Claim 1 is false. Then there exist $x_{l_1}^{(i)}, x_{m_1}^{(i)} \in \partial' B^+_{s_i^{-1}}(x_1^{(i)})$, $l_1 \neq m_1$, with

$$\sigma_{1,i} := d_{\bar{g}}(x_{l_1}^{(i)}, x_{m_1}^{(i)}) < \frac{1}{2}\sigma_{0,i} := \frac{1}{2}s_i^{-2}.$$

If we repeat this procedure, we obtain sequences $x_{l_r}^{(i)}, x_{m_r}^{(i)} \in \partial' B^+_{s_i\sigma_{r-1,i}}(x_{l_{r-1}}^{(i)})$, $l_r \neq m_r$, with

$$\sigma_{r,i} = d_{\bar{g}}(x_{l_r}^{(i)}, x_{m_r}^{(i)}) < \frac{1}{2}\sigma_{r-1,i}.$$

Since $N_i < \infty$, this procedure has to stop and we reach a contradiction. This proves Claim 1.

Using Claim 1 and a relabeling of indices, we find $x_1^{(i)}, x_2^{(i)} \to x_0$ and $s_i \to \infty$ so that, if $\sigma_i = d_{\bar{g}}(x_1^{(i)}, x_2^{(i)})$, we have $s_i\sigma_i \to 0$ and

$$d_{\bar{g}}(x_l^{(i)}, x_m^{(i)}) \geq \frac{1}{2}\sigma_i\,, \quad \text{for all } x_l^{(i)}, x_m^{(i)} \in \partial' B^+_{s_i\sigma_i}(x_1^{(i)})\,,\ l \neq m\,.$$

By the item (3) of Proposition 3.3.9 we have $u_i(x_1^{(i)}), u_i(x_2^{(i)}) \to \infty$.

Now we use Fermi coordinates centered at $x_1^{(i)}$ and set

$$v_i(y) = \sigma_i^{\frac{1}{p_i-1}} u_i(\sigma_i y)\,, \quad \text{for } y \in B^+_{s_i}(0)\,.$$

If $x_l^{(i)} \in \partial' B^+_{s_i\sigma_i}(0)$ and we set $y_l^{(i)} = \sigma_i^{-1} x_l^{(i)}$ (in particular, $y_1^{(i)} = 0$), then each $y_l^{(i)}$ is a local maximum of v_i and by the item (3) of Proposition 3.3.9,

$$\min_{y_l^{(i)}}\{|y - y_l^{(i)}|^{\frac{1}{p_i-1}}\} v_i(y) \leq C\,, \quad \text{for } y \in \partial' B^+_{\frac{1}{2}s_i}(0)\,.$$

Furthermore $|y_2^{(i)}| = |y_1^{(i)} - y_2^{(i)}| = 1$ and $\min_{l\neq m} |y_l^{(i)} - y_m^{(i)}| \geq \frac{1}{2} + o_i(1)$.

Claim 2. $v_i(y_1^{(i)}), v_i(y_2^{(i)}) \to \infty$.

If $v_i(y_2^{(i)})$ stays bounded but $v_i(y_1^{(i)}) \to \infty$ then the blow-up at $v_i(y_1^{(i)})$ is isolated and hence isolated simple, while v_i remains uniformly bounded near $y_2^{(i)}$. It follows from Lemma C-3 and Proposition 3.3.12 that $v_i(y_2^{(i)}) \to 0$. This is a contradiction since the item (1) of Proposition 3.3.9 implies that

$$\sigma_i \geq \max\{Ru_i(x_1^{(i)})^{-(p_i-1)}, Ru_i(x_2^{(i)})^{-(p_i-1)}\},$$

thus

$$v_i(y_1^{(i)}), v_i(y_2^{(i)}) \geq R^{\frac{1}{p_i-1}}. \tag{3.7.2}$$

Of course the same argument holds if we exchange the roles of $v_i(y_1^{(i)})$ and $v_i(y_2^{(i)})$.

On the other hand, if both $v_i(y_1^{(i)})$ and $v_i(y_2^{(i)})$ remain bounded, we can suppose that any other $v_i(y_l^{(i)})$ also does, using the same argument above. Then after passing to a subsequence $v_i \to v > 0$ in $C^2_{loc}(\mathbb{R}^n_+)$, where v satisfies

$$\begin{cases} \Delta v = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial v}{\partial \eta} + (n-2)v^{p_0} = 0, & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

and $\partial_k v(0) = \partial_k v(y_2) = 0$ for $k = 1, ..., n-1$. Here $p_0 = \lim_{i\to\infty} p_i \in [\frac{n}{n-2} - \beta, \frac{n}{n-2}]$ and $y_2 = \lim_{i\to\infty} y_2^{(i)}$. Note that $|y_2| = 1$. Then Theorems 1.3.1 and 1.3.2 yield that $v \equiv 0$, which contradicts the inequalities (3.7.2). This proves Claim 2.

It follows from Claim 2 that $0 = y_1^{(i)}$ and $y_2^{(i)}$ are isolated blow-up points. Thus Proposition 3.7.1 implies that they are isolated simple.

Then

$$v_i(y_1^{(i)})v_i(y) \to G(y) := a_1|y|^{2-n} + a_2|y - y_2|^{2-n} + b(y)$$

in $C^2_{loc}(\mathbb{R}^n_+ - S)$, where S denotes the set of blow-up points for $\{v_i\}$, $b(y)$ is a harmonic function on $\mathbb{R}^n_+ - (S - \{0, y_2\})$ with Neumann boundary condition and $a_1, a_2 > 0$. By the maximum principle, $b(y) \geq 0$. Hence, for $|y|$ near 0,

$$G(y) = a_1|y|^{2-n} + b + O(|y|)$$

for some constant $b > 0$. This contradicts the sign condition of Theorem 3.6.2 and proves Proposition 3.7.2. □

Now we are able to prove Theorem 1.1.4.

Proof of Theorem 1.1.4. Suppose by contradiction that $x_i \to x_0$ is a blow-up point for a sequence $\{u_i \in \mathcal{M}_i\}$ and $\pi_{kl}(x_0) \neq 0$. Let $x_1(u_i), ..., x_{N(u_i)}(u_i)$ be the points obtained in Proposition 3.3.9. By the item (3) of this Proposition, we must have $d_g(x_i, x_{k_i}(u_i)) \to 0$ for some $1 \leq k_i \leq N(u_i)$. If $x_{k_i} = x_{k_i}(u_i)$, it is not difficult to see that $u_i(x_{k_i}) \to \infty$. Thus $x_{k_i} \to x_0$ is a blow-up point for $\{u_i\}$. It follows from Propositions 3.7.1 and 3.7.2 that $x_{k_i} \to x_0$ is isolated simple. This contradicts Theorem 3.6.1. □

CHAPTER 4

Appendices A, B and C

Appendix A

In this section we will use the results of Section 1.4 to calculate some integrals used in the computations of Chapter 2. We recall that all curvature coefficients are evaluated at $x_0 \in \partial M$ and we are making use of conformal Fermi coordinates centered at this point.

Lemma A-1. *We have*

$$\int_{S_r^{n-2}} (g^{ij} - \delta^{ij})(\epsilon y) y_i y_j d\sigma_r(y) = \sigma_{n-2}\epsilon^4 \frac{y_n^2 r^{n+2}}{(n+1)(n-1)} R_{ninj;ij} + \sigma_{n-2}\epsilon^4 \frac{y_n^4 r^n}{2(n-1)} (R_{ninj})^2 + O(\epsilon^5 |(r, y_n)|^{n+5}) .$$

Proof. By Lemma 1.4.3,

$$\int_{S_r^{n-2}} (g^{ij} - \delta^{ij})(\epsilon y) y_i y_j d\sigma_r(y) = \epsilon^4 \int_{S_r^{n-2}} \frac{1}{2} R_{ninj;kl} y_i y_j y_k y_l d\sigma_r(y) + O(\epsilon^5 |(r, y_n)|^{n+5}) + \epsilon^4 y_n^2 \int_{S_r^{n-2}} (\frac{1}{12} R_{ninj;nn} + \frac{2}{3} R_{nins} R_{nsnj}) y_i y_j d\sigma_r(y) .$$

Then we just use the identity (2.1.4), Lemma 1.4.5 and the fact that

$$\Delta^2(R_{ninj;kl}y_iy_jy_ky_l) = 16R_{ninj;ij}\,.$$

□

Lemma A-2. *We have*

$$\int_{S_r^{n-2}}(g^{ij}-\delta^{ij})(\epsilon y)R_{nknl}y_iy_jy_ky_l d\sigma_r(y) = \frac{2}{(n+1)(n-1)}\sigma_{n-2}\epsilon^2 y_n^2 r^{n+2}(R_{ninj})^2 + O(\epsilon^5|(r,y_n)|^{n+5})$$

Proof. As in Lemma A-1, the result follows from

$$\int_{S_r^{n-2}}(g^{ij}-\delta^{ij})(\epsilon y)R_{nknl}y_iy_jy_ky_l d\sigma_r(y) = \epsilon^2 y_n^2\int_{S_r^{n-2}}R_{ninj}R_{nknl}y_iy_jy_ky_l d\sigma_r(y) + O(\epsilon^5|(r,y_n)|^{n+5})\,,$$

the fact that $\Delta^2(R_{ninj}R_{nknl}y_iy_jy_ky_l) = 16(R_{ninj})^2$ and the identity (2.1.4). □

Lemma A-3. *We have*

$$\int_{S_r^{n-2}}R_g(\epsilon y)d\sigma_r(y) = \sigma_{n-2}\epsilon^2\left\{\frac{1}{2}y_n^2r^{n-2}R_{;nn} - \frac{1}{12(n-1)}r^n(\overline{W}_{ijkl})^2\right\} + O(\epsilon^3|(r,y_n)|^{n+1})\,.$$

Proof. As in Lemma A-1, the result follows from

$$\int_{S_r^{n-2}}R_g(\epsilon y)d\sigma_r(y) = \epsilon^2y_n^2\int_{S_r^{n-2}}\frac{1}{2}R_{;nn}d\sigma_r(y) + \epsilon^2\int_{S_r^{n-2}}\frac{1}{2}R_{;ij}y_iy_jd\sigma_r(y) + O(\epsilon^3|(r,y_n)|^{n+1})\,,$$

Lemma 1.4.5(x) and the identity (2.1.4). □

Appendix B

In this section we will perform some integrations by parts used in Chapters 2 and 3.

Lemma B-1. *We have:*

(a) $\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m}{\alpha+1}\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m+1}}$ *, for* $\alpha+1<2m$*;*

(b) $\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m}{2m-\alpha-1}\int_0^\infty \frac{s^{\alpha}ds}{(1+s^2)^{m+1}}$ *, for* $\alpha+1<2m$*;*

(c) $\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m-\alpha-3}{\alpha+1}\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m}}$ *, for* $\alpha+3<2m$*.*

Proof. Integrating by parts,

$$\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m+1}} = \int_0^\infty s^{\alpha+1}\frac{s\,ds}{(1+s^2)^{m+1}} = \frac{\alpha+1}{2m}\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m},$$

for $\alpha+1<2m$, which proves the item (a).

The item (b) follows from the item (a) and from

$$\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \int_0^\infty \frac{s^\alpha(1+s^2)}{(1+s^2)^{m+1}}ds = \int_0^\infty \frac{s^\alpha ds}{(1+s^2)^{m+1}} + \int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^{m+1}}.$$

To prove the item (c), observe that, by the item (a),

$$\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^{m-1}} = \frac{2(m-1)}{\alpha+1}\int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^m},$$

for $\alpha+3<2m$. But, by the item (b), we have

$$\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^{m-1}} = \frac{2(m-1)}{2(m-1)-\alpha-1}\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m}.$$

□

Lemma B-2. *For* $m>k+1$*,*

$$\int_0^\infty \frac{t^k}{(1+t)^m}dt = \frac{k!}{(m-1)(m-2)...(m-1-k)}.$$

Proof. Integrating by parts,

$$\int_0^\infty t^{k-1}(1+t)^{1-m}dt = \frac{m-1}{k}\int_0^\infty t^k(1+t)^{-m}dt.$$

On the other hand,

$$\int_0^\infty t^{k-1}(1+t)^{1-m}dt = \int_0^\infty \frac{t^{k-1}(1+t)}{(1+t)^m}dt = \int_0^\infty \frac{t^k}{(1+t)^m}dt + \int_0^\infty \frac{t^{k-1}}{(1+t)^m}dt.$$

Hence,

$$\int_0^\infty \frac{t^k}{(1+t)^m}dt = \frac{k}{m-1-k}\int_0^\infty \frac{t^{k-1}}{(1+t)^m}dt\,.$$

Now the result follows observing that $\int_0^\infty \frac{1}{(1+t)^m}dt = \frac{1}{m-1}$. □

Corollary B-3. *Set* $I = \int_0^\infty \frac{s^n}{(s^2+1)^n}ds$ *. Then*

(i) $\int_0^\infty \frac{s^2+(t^2-1)}{(s^2+(t+1)^2)^{n+1}}s^n ds = I\left\{\frac{n+1}{2n}(t+1)^{1-n} + \frac{n-1}{2n}(t^2-1)(t+1)^{-1-n}\right\}$*;*
(ii) $\int_0^\infty \frac{s^2+(t^2-1)}{(s^2+(t+1)^2)^{n}}s^{n-2} ds = I\left\{(t+1)^{1-n} + (t^2-1)(t+1)^{-1-n}\right\}$*;*
(iii) $\int_0^\infty \frac{s^2+(t^2-1)}{(s^2+(t+1)^2)^{n-1}}s^{n-2} ds = I\left\{2\frac{n-1}{n-3}(t+1)^{3-n} + 2(t^2-1)(t+1)^{1-n}\right\}$*.*

Proof. By a change of variables we obtain

$$\int_0^\infty \frac{s^2+(t^2-1)}{(s^2+(t+1)^2)^{n+1}}s^n ds = (t+1)^{1-n}\int_0^\infty \frac{s^{n+2}}{(s^2+1)^{n+1}}ds + (t^2-1)(t+1)^{-1-n}\int_0^\infty \frac{s^n}{(s^2+1)^{n+1}}ds,$$
$$\int_0^\infty \frac{s^2+(t^2-1)}{(s^2+(t+1)^2)^{n}}s^{n-2} ds = (t+1)^{1-n}\int_0^\infty \frac{s^{n}}{(s^2+1)^{n}}ds + (t^2-1)(t+1)^{-1-n}\int_0^\infty \frac{s^{n-2}}{(s^2+1)^{n}}ds,$$
$$\int_0^\infty \frac{s^2+(t^2-1)}{(s^2+(t+1)^2)^{n-1}}s^{n-2} ds = (t+1)^{3-n}\int_0^\infty \frac{s^{n}}{(s^2+1)^{n-1}}ds + (t^2-1)(t+1)^{1-n}\int_0^\infty \frac{s^{n-2}}{(s^2+1)^{n-1}}ds.$$

Then we use Lemma B-1 to see that $\int_0^\infty \frac{s^{n+2}}{(s^2+1)^{n+1}} = \frac{n+1}{2n}I$, $\int_0^\infty \frac{s^{n}}{(s^2+1)^{n+1}} = \frac{n-1}{2n}I$, $\int_0^\infty \frac{s^{n-2}}{(s^2+1)^{n}} = I$, $\int_0^\infty \frac{s^{n}}{(s^2+1)^{n-1}} = 2\frac{n-1}{n-3}I$ and $\int_0^\infty \frac{s^{n-2}}{(s^2+1)^{n-1}} = 2I$. □

Appendix C

In this section we will state some technical results used in Chapter 3.

Our first result is a modification of Proposition 2.7 in [34]. The proof is similar.

Lemma C-1. *Let* (M,g) *be a Riemannian manifold with boundary* ∂M*. Let* $x \in \partial M$ *and* $\mathcal{U} \subset M$ *be an open set containing* x*. Let* u *be a a weak solution to*

$$\begin{cases} \Delta u = 0\,, & \text{in } \mathcal{U}\backslash\{x\} \\ (\frac{\partial}{\partial\eta} + \psi)u = 0\,, & \text{on } \mathcal{U}\cap\partial M\backslash\{x\}\,, \end{cases}$$

where η *is the inward unit normal vector to* ∂M*. Suppose that* $u \in L^q(\mathcal{U})$ *for some* $q > \frac{n}{n-2}$ *and* $u, \psi u \in L^1(\mathcal{U}\cap\partial M)$*. Then* u *is a weak solution to*

$$\begin{cases} \Delta u = 0\,, & \text{in } \mathcal{U}\,, \\ (\frac{\partial}{\partial\eta} + \psi)u = 0\,, & \text{on } \mathcal{U}\cap\partial M\,. \end{cases}$$

The proof of the following lemma is similar to the result in [28], p.150 (see also [5], p.108).

Lemma C-2. *Let $\rho > 0$ be small and suppose that $\rho \leq \beta \leq \beta - \rho \leq \alpha \leq n - \rho$. Then there is $C = C(n, \rho) > 0$ such that*

$$\int_{\mathbb{R}^n} |y - x|^{\beta - n}(1 + |x|)^{-\alpha} dx \leq C(1 + |y|)^{\beta - \alpha}$$

for any $y \in \mathbb{R}^{n+k} \supset \mathbb{R}^n$.

For the proof we decompose $\mathbb{R}^n$ in three regions
$\mathcal{A} := \{x \in \mathbb{R}^n;\, |x - y| \leq \frac{1}{2}|y| + \frac{1}{2}\}$,
$\mathcal{B} := \{x \in \mathbb{R}^n;\, |x - y| \geq \frac{1}{2}|y| + \frac{1}{2}, |x| \leq 2|y| + 1\}$,
$\mathcal{C} := \{x \in \mathbb{R}^n;\, |x| \geq 2|y| + 1\}$,
and perform the estimates in each one separately.

The following Harnack-type inequality is Lemma A.1 of [29]:

Lemma C-3. *Let L be an operator of the form*

$$Lu = \partial_a \left(\alpha_{ab}(x)\partial_a u + \beta_a(x)u\right) + \gamma_a(x)\partial_a u + \zeta(x)u$$

and assume that for some constant $\Lambda > 1$ the coefficients satisfy

$$\Lambda^{-1}|\xi|^2 \leq \alpha_{ab}\xi_a\xi_b \leq \Lambda|\xi|^2,$$

$$|\beta_a(x)| + |\gamma_a(x)| + |\zeta(x)| \leq \Lambda,$$

for all $x \in B_3^+ = B_3^+(0)$ and all $\xi \in \mathbb{R}^n$. If $|q(x)| \leq \Lambda$, for any $x \in \partial' B_3^+$, and $u \in C^2(B_3^+) \cap C^1(\overline{B_3^+})$ satisfies

$$\begin{cases} Lu = 0, \;\; u > 0, & \text{in } B_3^+, \\ \alpha_{nb}(x)\partial_b u = q(x)u, & \text{on } \partial' B_3^+, \end{cases}$$

then there exists $C = C(n, \Lambda) > 1$ such that

$$\max_{\overline{B_1^+}} u \leq C \min_{\overline{B_1^+}} u\,.$$

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