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Doctoral Thesis

SOME RESULTS FOR CONSTANT MEAN CURVATURE SURFACES

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To my parents Lourival Lúcio Leandro and
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La recherche de la vérité doit être le but de notre activité; c'est la seule fin qui soit digne d'elle. – Henry Poincaré - La valeur de la science.

Abstract

We first obtain a height estimate for a compact embedded surface Σ with positive constant mean curvature H in a product space $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a Hadamard surface with curvature $K_{\mathbb{M}^2} \leq -\kappa \leq 0$ and we show that this estimate is optimal. After, we give a condition that implies Σ, a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ whose boundary is a convex planar curve, stays in a half-space.

Moreover we prove that if $\Gamma_n \subset Q = \mathbb{H}^2 \times \{0\}$ is a sequence of embedded curves converging to point q, and $\Sigma_n \subset \mathbb{H}^2 \times \mathbb{R}_+$ are embedded compact H-surfaces with $\Gamma_n = \partial\Sigma_n$, then there is a subsequence of Σ_n that either converges to point q or to the rotationally invariant constant mean curvature sphere $S^2_H \subset \mathbb{H}^2 \times \mathbb{R}_+$ tangent to Q at q.

Finally, we prove that isolated singularities of sections with precribed mean curvature of a Riemanninan submersion fibered by geodesics of a vertical Killing field, are removable. Also we obtain information on the growth of the difference of two sections $u, v : \Omega \to \bar{M}$, having the same prescribed mean curvature and $u = v$ on $\partial\Omega$.

Keywords: H-surfaces, Half-Space, removable singularities, Riemannian submersions

Resumo

Primeiramente obtemos uma estimativa de altura para superfície Σ compacta mergulhada com curvatura média constante H em $\mathbb{M}^2 \times \mathbb{R}$, onde $\mathbb{M}^2$ uma superfície de Hadamard com curvatura $K_{\mathbb{M}^2} \leq -\kappa \leq 0$ e mostramos que esta estimativa é ótima. Apresentamos, em seguida, uma condição que implica que uma H-superfície Σ compacta mergulhada cujo bordo é uma curva convexa plana, permaneça em um dos semiespaços.

Além disso provamos que se $\Gamma_n \subset Q = \mathbb{H}^2 \times \{0\}$ é uma sequência de curvas mergulhadas convergindo a q e $\Sigma_n \subset \mathbb{H}^2 \times \mathbb{R}_+$ são H-superfícies compactas com $\Gamma_n = \partial\Sigma_n$, então existe uma subsequência de Σ_n que converge a q ou a $S_H^2 \subset \mathbb{H}^2 \times \mathbb{R}_+$ tangente a Q.

Finalmente, provamos que singularidades isoladas de seções com curvatura média prescrita de uma submersão Riemanniana cujas fibras são geodésicas de um campo Killing são removíveis. Também obtemos informações sobre o crescimento da diferença de duas

Palavras-chave: H-superfície, semiespaço, singularidade removível, submersão Riemanniana

Contents

Introduction

In the last decade the geometry of surfaces in the three dimensional homogeneous manifolds has been actively studied. In 1978, when Thurston gave a course at Princeton University, whose subject was the geometry and topology of three dimensional manifolds, he showed that these spaces admit ″nice″ metrics, i.e, can be endowed with a complete metric with a large isometry group.

Indeed, in Scott's work [28], which is based on Thurston's results, we can see the classification of the homogeneous simply connected 3-dimensional manifolds. Such a manifold has an isometry group of dimension 3, 4 or 6. When the dimension of the isometry group is 6, then we have a space form. When the dimension of the isometry group is 4, these manifolds, $\mathbb{E}(\kappa, \tau)$, admit natural equivariant Riemannian submersions over 2-dimensional space forms $\mathbb{M}^2(\kappa)$; $\mathbb{M}^2(\kappa) = \mathbb{S}^2(\kappa)$ for $\kappa \geq 0$, $\mathbb{R}^2$ for $\kappa = 0$, and $\mathbb{H}^2(\kappa)$ for $\kappa < 0$ with 1-dimensional, totally-geodesic fibers and are classified, up to isometry, by the curvature κ of the base surface of the fibration and the bundle curvature τ, where κ and τ can be any real number satisfying $\kappa \neq 4\tau^2$. They are the Berger spheres, the Heisenberg group, the special linear group $Sl(2, \mathbb{R})$ and the Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$. When the dimension of the isometry group is 3, the manifold has the geometry of the Lie group Sol_3.

In recent years the theory of constant mean curvature surfaces in these three-dimensional homogeneous spaces has been rapidly developed by many mathematicians, see for example [1], [2], [7], [8], [12], [18], [20], [29]. One reason for this was the work of Abresch and Rosenberg [1], where they found a holomorphic quadratic differential on any constant mean curvature surface of a homogeneous Riemannian 3-manifold with isometry group of dimension 4.

In this thesis we make some contributions to this theory. First we will begin to obtain an a priori estimate for the height of a embedded compact constant mean curvature surface in $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \leq -\kappa \leq 0$, κ constant, with boundary belonging to the plane $\mathbb{M}^2 \times \{0\}$ and transverse to the plane. More specifically, if Σ is a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$, with boundary belonging to $Q = \mathbb{M}^2 \times \{0\}$, $\Gamma = \partial\Sigma$, and transverse to the plane Q, we denote $\Sigma^+ = \Sigma \cap \mathbb{M}^2 \times \mathbb{R}_+$ and $\Sigma^- = \Sigma \cap \mathbb{M}^2 \times \mathbb{R}_-$. We call Σ_1 the connected component of Σ^+ that contains Γ.

Let $\hat{\Sigma}_1$ be the symmetry of Σ_1 through the plane Q. Then $\hat{\Sigma}_1 \cup \Sigma_1$ is a compact surface with no boundary and bounds a domain U in $\mathbb{M}^2 \times \mathbb{R}$. Let U_1 be the intersection of U with the half-space above Q. Thus U_1 is a bounded domain in $\mathbb{M}^2 \times \mathbb{R}$, whose boundary, ∂U_1, consists of the smooth connected surface Σ_1, and the union Ω of finitely smooth, compact and connected surfaces in Q.

We define A^+ the area of Σ_1. Then, using this notation we get

> Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \leq -\kappa \leq 0$. Let Σ be a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$, with boundary belonging to $Q = \mathbb{M}^2 \times \{0\}$ and transverse to the plane Q. If h denotes the height of Σ with respect to Q, we have that
>
> $$h \leq \frac{HA^+}{2\pi} - \frac{\kappa Vol\,(U_1)}{4\pi} \tag{1}$$
>
> where A^+ is and U_1 are as defined above. The equality holds if, and only if, $K \equiv -\kappa$ inside U_1 and Σ is a rotational spherical cap.

The study of height estimates for a constant mean curvature surface started with Serrin [27]. Serrin observed that a compact constant mean curvature graph in $\mathbb{R}^3$, with curvature $H > 0$, with planar boundary has height at most $\frac{1}{H}$ above the plane, and by Alexandrov reflection technique [3], one has that a compact constant mean curvature surface with planar boundary cannot extend more than $\frac{2}{H}$. Observe that this estimate is optimal because it is attained by the hemisphere of radius $\frac{1}{H}$.

Later N. Korevaar, R. Kusner, W. Meeks and B. Solomon [16] gave an optimal bound for graphs and for compact embedded surfaces in the hyperbolic 3-space $\mathbb{H}^3$ with non zero constant mean curvature and boundary on a plane. On the other hand, for the product space $\mathbb{M}^2 \times \mathbb{R}$ of a Riemannian surface $\mathbb{M}^2$ and the real line $\mathbb{R}$, Aledo, Espinar and Gálvez [2] obtained the following

Theorem 1 (Aledo, Espinar, Gálvez)**.** *Let $\mathbb{M}^2$ be a Riemannian surface without boundary. Let Σ be a compact graph in a set $\Omega \subseteq \mathbb{M}^2 \times \mathbb{R}$, with constant mean curvature $H > 0$ and whose boundary belongs to $Q = \mathbb{M}^2 \times \{0\}$. Let c be the minimum of the Gauss curvature on $\Omega \subseteq \mathbb{M}^2 \times \mathbb{R}$. Then the maximum height that Σ can rise above Q is*

$$\frac{4H}{\sqrt{-4cH^2 - c^2}} \arcsin\left(\frac{\sqrt{-c}}{2H}\right) \text{ if } c < 0 \text{ and } H > \frac{\sqrt{-c}}{2},$$

$$\frac{1}{H} \text{ if } c = 0,$$

$$\frac{4H}{\sqrt{4cH^2 + c^2}} \arcsin\left(\frac{\sqrt{c}}{2H}\right) \text{ if } c > 0,$$

Moreover, if the equality holds, then Ω has constant Gauss curvature c and the Abresch-Rosenberg differential vanishes identically on Σ.

Observe that our estimate has a different nature from that of Aledo, Espinar and Gálvez [2]. Our estimate depends on area and volume and not on the size of H. The hypothesis $H > \frac{\sqrt{-c}}{2}$ is not necessary in our result. Our estimate is inspired by the paper [18].

Our second result is to give the following sufficient condition for a compact constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ to be contained in a halfspace

> Let Σ be a compact constant mean curvature H surface embedded into $\mathbb{H}^2 \times \mathbb{R}$, $H > \frac{1}{2}$, whose boundary is a convex planar curve contained in the plane $Q = \mathbb{H}^2 \times \{0\}$. If $h_\Sigma < \frac{h_{S_H}}{2}$, where h_Σ and h_{S_H} are the height of the surfaces Σ and the H-sphere (embedded rotationally invariant cmc sphere given by [1]), respectively. Then Σ stays in a halfspace determined by Q and is transverse to Q along the boundary. Moreover, Σ inherits the symmetries of its boundary.

For the constant mean curvature H surface $\Sigma \subset \mathbb{R}^3$ whose boundary $\Gamma = \partial\Sigma$ is a convex curve in a plane $P \subset \mathbb{R}^3$, Earp, Brito, Meeks, Rosenberg proved in [25] that if $\Sigma \subset \mathbb{R}^3$ is a embedded constant mean curvature surface whose boundary is a convex curve contained in a plane P and if Σ is transverse to P along its boundary, then Σ is containted in one of the halfspaces of $\mathbb{R}^3$ determined by P. Lopes and Montiel in [18] give other conditions: if $Area(\Sigma)H^2 \leq 2\pi$, where Σ is a constant mean curvature H surface in $\mathbb{R}^3$, then Σ stays in a halfspace determined by P.

One direct consequence of this last theorem is that:

> Let Σ be a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$, $H > \frac{1}{2}$, with convex planar boundary. If $h_\Sigma < \dfrac{h_{S_H}}{2}$, where h_Σ is the height of Σ and h_{S_H} is the height of the rotational H-sphere, then the surface Σ is a graph.

Thus using this result joined with the Theorem 1 we get: Let Σ be a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$, with convex planar boundary. Then Σ is a graph if, and only if, $h_\Sigma < \dfrac{h_{S_H}}{2}$, where h_Σ is the height of Σ and h_{S_H} is the height of the rotational H-sphere.

Another interesting problem we will consider is the convergence of a sequence of constant mean curvature H surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with boundary contained in the slice $Q = \mathbb{H}^2 \times \{0\}$. In this case we prove that:

> Let $\Sigma_n \subset \mathbb{H}^2 \times \mathbb{R}_+$ be H-surfaces and $\Gamma_n = \partial\Sigma_n \subset D(r_n) = \left\{(x_1, x_2, 0) \in \mathbb{H}^2 \times \mathbb{R}_+ / x_1^2 + x_2^2 \leq r_n\right\}$, with r_n a sequence converging to zero. Then there is a subsequence of Σ_n that either converges to the origin $O \in \mathbb{H}^2 \times \mathbb{R}_+$ or to the rotationally invariant constant mean curvature sphere $S_H^2 \subset \mathbb{H}^2 \times \mathbb{R}_+$ tangent to $\mathbb{H}^2 \times \{0\}$ at O. In the first case the surfaces converge as subsets and in the second the convergence is smooth on compact subsets of $\left\{\mathbb{H}^2 \times \mathbb{R}\right\} - \{O\}$.

This kind of problem was proposed by Wente [30] when Γ_n is an arbitrary Jordan curve in $\mathbb{R}^3$ converging to a point p and Σ_n is an immersed topological disc bounded by Γ_n which minimizes area among disks bounding a fixed algebraic volume. Later Ros and Rosenberg [23] solved this problem when Γ_n is a sequence of embedded (perhaps nonconnected) curves converging to a point p and Σ_n is a sequence of H-surfaces in $\mathbb{R}^3$ whose boundary is contained in the plane $P \subset \mathbb{R}^3$. We use ideas in [23] to prove our result above.

Finally, we work with Riemannian submersions $\pi : \bar{M}^{n+1} \to M^n$ whose vertical fibers are given by flow lines of a unit Killing field. Observe that among the ambients for which this Riemannian submersion applies, we have the the Heisenberg spaces and the $Sl(2, \mathbb{R})$.

In these Riemannian submersions we will prove that

> Let $\Omega \subset M$ be a domain and $p \in \Omega$. Let $u : \Omega - \{p\} \to \mathbb{R}$ be a function whose Killing graph has prescribed mean curvature H. Then u extends smoothly to a solution at p.

This kind of theorem was first proved for minimal graphs in $\mathbb{R}^3$ by L. Bers [6]. R. Finn generalized this to prove the constant mean curvature equation has removable singularities [13]. We refer the reader to the paper of Nitsche [21] for some history.

The last result of this thesis is the following theorem

> Let $\Omega \subset M^n$ be a domain, M^n the base of a Killing submersion, such that Ω intersects the boundary of each geodesic ball centered at a fixed point in a region whose volume is bounded by a constant times the radius, and let u, v be two $C^2(\Omega)$ functions such that their Killing graphs have the same mean curvature, $H(u) = H(v)$ in Ω and $u|\partial\Omega$ and $v|\partial\Omega$ are piecewise differentiable and coincide in the points of continuity. Let $M(r) = \sup_{\Lambda_r} |u - v|$ where $\Lambda_r = \Omega \cap \{x \in M; dist(x, a) = r\}$. Then $\liminf_{r\to\infty} \dfrac{M(r)}{\log r} > 0$ if $u \neq v$. If the volume of Λ_r is uniformly bounded then $\liminf_{r\to\infty} \dfrac{M(r)}{r} > 0$.

Pascal Collin and Romain Krust studied graphs u, v over non compact domains $\Omega \subset \mathbb{R}^2$, that have the same mean curvature and with $u = v$ on $\partial\Omega$. They proved that when $u \neq v$, then $|u - v|$ must grow at least like $log(r)$, r radial distance in $\mathbb{R}^2$.

This theorem of Collin and Krust, and its technique of proof, have had many applications and generalizations.

We showed that their techniques apply to sections $u, v : \Omega \subset M \to \bar{M}$, provided the volume of Ω intersected with the geodesic spheres of M, grows at most linearly in the radius of the sphere. For example, when $\bar{M}$ = *Heisenberg Space* and M = the flat $\mathbb{R}^2$, this is always the case.

We conclude that for graphs of prescribed mean curvature in Heisenberg space, over domains $\Omega \subset \mathbb{R}^2$, there is at most one bounded solution of the mean curvature equation over Ω, with given boundary values. In particular, a bounded entire minimal graph is constant.

The organization of this thesis is as follows.

In Chapter 1, we first obtain a height estimate for a compact embedded surface Σ with positive constant mean curvature H in a product space $\mathbb{M}^2\times\mathbb{R}$ and we show that this estimate is optimal. After, we give a condition that implies Σ, a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ whose boundary is a convex planar curve, stays in a half-space. We conclude using these two results that if Σ is a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ with convex planar boundary and the height of the surface Σ is less than or equal to the height of a hemisphere of a complete rotational sphere with constant mean curvature H, then Σ is a graph. The equality holds if and only if Σ is the rotational H-hemisphere.

In Chapter 2, we will prove that if $\Gamma_n \subset Q = \mathbb{H}^2 \times \{0\}$ is a sequence of embedded curves converging to point q, and $\Sigma_n \subset \mathbb{H}^2 \times \mathbb{R}_+$ are embedded compact H-surfaces and $\Gamma_n = \partial\Sigma_n$, then there is a subsequence of Σ_n that either converges to point q or to the rotationally invariant constant mean curvature sphere $S^2_H \subset \mathbb{H}^2 \times \mathbb{R}_+$ tangent to Q at q. In the first case the surfaces converge as subsets and in the second the convergence is smooth on compact subsets of $\mathbb{H}^2 \times \mathbb{R} - \{q\}$.

Finally, in Chapter 3, we will prove that isolated singularities of sections with precribed mean curvature of a Riemanninan submersion fibered by geodesics of a vertical Killing field, are removable. Also we obtain information on the growth of the difference of two sections $u, v : \Omega \to \bar{M}$, having the same prescribed mean curvature and $u = v$ on $\partial\Omega$. This generalizes Theorem 2 of [9].

CHAPTER 1

Height estimates for cmc surfaces in $\mathbb{M}^2 \times \mathbb{R}$

1.1 Introduction

We first obtain a height estimate for a compact embedded surface Σ with positive constant mean curvature H in a product space $\mathbb{M}^2 \times \mathbb{R}$. We suppose that Σ has planar boundary $\Gamma = \partial\Sigma$, i. e., $\Gamma \subset \mathbb{M}^2 \times \{0\}$, and $\mathbb{M}^2$ is a Hadamard surface (complete, simply connected, and has everywhere non-positive sectional curvature) with curvature $K_{\mathbb{M}^2} \leq -\kappa \leq 0$. Height estimates, when they exist, usually depend only on H. Our estimates depend also on area and volume. We show that this estimate is optimal and the equality holds if and only if Σ is a rotational spherical cap.

We then prove that if Σ is a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ whose boundary, $\Gamma = \partial\Sigma$, is a convex planar curve contained in $Q = \mathbb{H}^2 \times \{0\}$ and the height of the surface Σ is less than or equal the height of the hemisphere of the rotational sphere with constant mean curvature H, then Σ stays in a half-space determined by Q and is transverse to Q along the boundary and inherits the same symmetries of its boundary.

We conclude using these two results that if Σ is a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ with convex planar boundary and the height of the surface Σ is less than or equal to the height of the hemisphere of the rotational sphere with constant mean curvature H, then Σ is a graph. The equality holds if and only if Σ is the rotational H-hemisphere.

1.2 The height estimates

In this section we first obtain an priori estimate for the height of a compact constant mean curvature surface embedded in $\mathbb{M}^2 \times \mathbb{R}$ with planar boundary, where $\mathbb{M}^2$ is a Hadamard surface with Gaussian curvature $K_{\mathbb{M}^2} \leq -\kappa \leq 0$. The existence of height estimates reveals, in general, important properties on the geometric behaviour of these surfaces, as we can see in [15], for example. Our estimate has a different nature from those of Serrin [27] and Aledo, Espinar and Gálvez [2]. Our estimate is inspired by the paper [18].

We then give a condition that implies if $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ is a H-surfaces with $\partial\Sigma \subset Q = \mathbb{H}^2 \times \{0\}$ then Σ lies on one side of its boundary plane. Now we give some notation.

Let $\mathbb{M}^2 \times \mathbb{R}$ be a 3-dimensional ambient space where $\mathbb{M}^2$ is a Hadamard surface with Gaussian curvature $K_{\mathbb{M}^2} \leq -\kappa \leq 0$. Let us denote by $g_{\mathbb{M}^2}$ the metric of $\mathbb{M}^2$. Thus the canonical metric product of $\mathbb{M}^2 \times \mathbb{R}$ is given by $g_{\mathbb{M}^2} + dt^2$.

Let γ be a horizontal complete geodesic in $\mathbb{M}^2 \times \{0\}$, then $\mathbb{M}^2 - \{\gamma\}$ has two connected components. We will distinguish them using the following notation.

Definition 1.2.1. Let J be the standard counter-clockwise rotation operator of $\mathbb{M}^2$. We call exterior set of γ in $\mathbb{M}^2$, $ext_{\mathbb{M}^2}(\gamma)$, the connected component of $\mathbb{M}^2 - \{\gamma\}$ towards wich $J\gamma'$ points. The other connected component of $\mathbb{M}^2 - \{\gamma\}$ is called the interior set of γ in $\mathbb{M}^2$ and denoted by $int_{\mathbb{M}^2}(\gamma)$.

Definition 1.2.2. Given a complete oriented geodesic γ in $\mathbb{M}^2 \times \{0\}$, we will call $\gamma \times \mathbb{R}$ a vertical plane of $\mathbb{M}^2 \times \mathbb{R}$ and we will call a slice $\mathbb{M}^2 \times \{\tau\}$ a horizontal plane.

Note that a vertical plane is isometric to $\mathbb{R}^2$ and horizontal plane is isometric to $\mathbb{M}^2$. Throughout this paper we will denote vertical planes by P and horizontal planes by $Q(\tau) = \mathbb{M}^2 \times \{\tau\}$, $Q(0) = Q$.

The notion of interior and exterior domain of a horizontal oriented geodesic extend naturally to vertical planes.

Definition 1.2.3. For a complete oriented geodesic γ in $\mathbb{M}^2 \times \{0\} \equiv \mathbb{M}^2$ we call, respectively interior and exterior of a vertical plane $P = \gamma \times \mathbb{R}$ the sets

$$int_{\mathbb{M}^2\times\mathbb{R}}(P) = int_{\mathbb{M}^2}(\gamma) \times \mathbb{R}, \quad ext_{\mathbb{M}^2\times\mathbb{R}}(P) = ext_{\mathbb{M}^2}(\gamma) \times \mathbb{R}.$$

We will often use the foliations by vertical plane of $\mathbb{M}^2 \times \mathbb{R}$. We use now make this precise.

Definition 1.2.4. Let P be a vertical plane in $\mathbb{M}^2 \times \mathbb{R}$ and $\gamma(t)$ be an oriented horizontal geodesic in $\mathbb{M}^2 \times \{0\}$ with t arc length along γ, $\gamma(0) = p_0 \in P$, $\gamma'(0)$ orthogonal to P at p_0 and $\gamma(t) \in ext_{\mathbb{M}^2\times\mathbb{R}}(P)$ for $t > 0$. We define the oriented foliation of vertical planes along γ, denoted by $P_\gamma(t)$ to be vertical planes orthogonal to $\gamma(t)$ with $P = P_\gamma(0)$.

Let $P_\gamma(t)$ be the foliation of vertical planes along γ with $P_\gamma(0) = P$ and $q \in P(t_0)$. Let ξ denote the horizontal Killing field of $\mathbb{M}^2 \times \mathbb{R}$ generated by translation along γ (ξ is tangent to each $\mathbb{M}^2 \times \{\tau\}$ and is translation along $\gamma \times \{\tau\}$), ξ is orthogonal to the planes $P_\gamma(t)$.

Remark: Throughout this paper a H-surface Σ means a compact constant mean curvature (equal to H) surface in $\mathbb{M}^2 \times \mathbb{R}$ with $\partial\Sigma \subset \mathbb{M}^2 \times \{0\}$.

1.3 The Main Result

Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \leq -\kappa \leq 0$ and Σ be a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$, with boundary belonging to $Q = \mathbb{M}^2 \times \{0\}$. Let Γ be the boundary of Σ, $\Gamma = \partial\Sigma$. We assume Σ is transverse to Q along Γ.

We denote $\Sigma^+ = \Sigma \cap \left(\mathbb{M}^2 \times \mathbb{R}_+\right)$ and $\Sigma^- = \Sigma \cap \left(\mathbb{M}^2 \times \mathbb{R}_-\right)$. So there is a connected component of Σ^+ or Σ^- that contains Γ. We can assume, without loss of generality, that $\Gamma \subset \partial\Sigma^+$. We call Σ_1 the connected component of Σ^+ that contains Γ.

Let $\hat{\Sigma}_1$ be the symmetry of Σ_1 through the plane Q. Then $\hat{\Sigma}_1 \cup \Sigma_1$ is a compact embedded surface with no boundary, with corners along $\partial\Sigma_1$, and bounds a domain U in $\mathbb{M}^2 \times \mathbb{R}$. Let U_1 the intersection of U with the half-space above Q. Thus U_1 is a bounded domain in $\mathbb{M}^2 \times \mathbb{R}$, whose boundary, ∂U_1, consists of the smooth connected surface Σ_1, and the union Ω of finitely smooth, compact and connected surfaces in Q. We define A^+ to be the area of Σ_1.

Theorem 1.3.1. *Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \leq -\kappa \leq 0$. Let Σ be a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$, with boundary belonging to $Q = \mathbb{M}^2 \times \{0\}$ and transverse to Q. If h denotes the height of Σ with respect to Q, we have that*

$$h \leq \frac{HA^+}{2\pi} - \frac{\kappa Vol\,(U_1)}{4\pi} \tag{1.1}$$

where A^+ and U_1 are as defined above. The equality holds if, and only if, $K \equiv -\kappa$ inside U_1 and Σ is a rotational spherical cap.

Proof. From the surface Σ we obtain the surface Σ_1, the bounded domain $U_1 \subset \mathbb{M}^2 \times \mathbb{R}$ and the union Ω of finitely smooth, compact and connected surfaces in Q, as described above. Denote $\vec{H}$ the mean curvature vector of Σ_1 and we take the unit normal N of Σ_1 to point inside U_1. Let $\pi_1 : \mathbb{M}^2 \times \mathbb{R} \to \mathbb{M}^2$ and $\pi_2 : \mathbb{M}^2 \times \mathbb{R} \to \mathbb{R}$ be the usual projections. If we denote by $h_1 : \Sigma_1 \to \mathbb{R}$ the height function of Σ_1, that is, $h_1(p) = \pi_2(p)$ and $\nu = \left\langle N, \frac{\partial}{\partial t} \right\rangle$, we can write

$$\frac{\partial}{\partial t} = T + \nu N \tag{1.2}$$

where T is a tangent vector field on Σ_1. Since $\frac{\partial}{\partial t}$ is the gradient in $\mathbb{M}^2 \times \mathbb{R}$ of the function t, it follows that T is the gradient of h_1 on Σ_1.

If H is zero then h, the height of Σ, is a harmonic function and by the maximum principle we get that $\Sigma \subset \mathbb{M}^2 \times \{0\}$. So, we suppose that $H > 0$.

Let $A(t)$ be the area of $\Sigma_t = \{p \in \Sigma_1;\ h_1(p) \geq t\}$ and $\Gamma(t) = \{p \in \Sigma_1;\ h_1(p) = t\}$. By [26], theorem 5.8, we get

$$A'(t) = -\int_{\Gamma(t)} \frac{1}{\|\nabla h_1\|} ds_t, \quad t \in \mathcal{O},$$

where $\mathcal{O}$ is the set of all regular values of h_1.

If we denote by $L(t)$ the length of the planar curve $\Gamma(t)$, the Schwartz inequality yields

$$L^2(t) \leq \int_{\Gamma(t)} \|\nabla h_1\| ds_t \int_{\Gamma(t)} \frac{1}{\|\nabla h_1\|} ds_t = -A'(t) \int_{\Gamma(t)} \|\nabla h_1\| ds_t, \quad t \in \mathcal{O}. \tag{1.3}$$

But we have from (1.2) that, along the curve $\Gamma(t)$,

$$\|\nabla h_1\|^2 = 1 - \nu^2 = \left\langle \eta^t, \frac{\partial}{\partial t} \right\rangle^2.$$

where η^t is the inner conormal of Σ_t along $\partial \Sigma_t$. Since Σ_t is above the plane $Q(t)$ we know that $\left\langle \eta^t, \frac{\partial}{\partial t} \right\rangle \geq 0$. Hence

$$\|\nabla h_1\| = \left\langle \eta^t, \frac{\partial}{\partial t} \right\rangle.$$

Thus (1.3) can be rewritten as follows

$$L^2(t) \leq -A'(t) \int_{\Gamma(t)} \left\langle \eta^t, \frac{\partial}{\partial t} \right\rangle ds_t. \tag{1.4}$$

Now we recall the Flux Formula. Let Σ_t and $\Omega(t)$ be two compact, smooth, embedded not necessarily connected surfaces in $\mathbb{M}^2 \times \mathbb{R}$ such that their boundaries coincide. Assume that there exists a compact domain $U(t)$ in $\mathbb{M}^2 \times \mathbb{R}$ such that the boundary of $U(t)$ is $\partial U(t) = \Sigma_t \cup \Omega(t)$ and it is orientable. Notice that the boundary of $U(t)$ is smooth except perhaps along $\partial\Sigma_t = \partial\Omega(t)$.

Let $N_{\Sigma_t}, N_{\Omega(t)}$ be the unit normal fields to Σ_t and $\Omega(t)$, respectively, that point inside $U(t)$. Denote by η^t the unit conormal to Σ_t along $\partial\Sigma_t$, pointing inside Σ_t. Finally assume that Σ_t is a compact surface with constant mean curvature $H = \left\langle \vec{H}, N_{\Sigma_t} \right\rangle > 0$. Let Y be a Killing vector field in $\mathbb{M}^2 \times \mathbb{R}$. Then by the Flux Formula (Proposition 3 in [15]).

$$\int_{\partial\Sigma_t} \left\langle Y, \eta^t \right\rangle = 2H \int_{\Omega(t)} \left\langle Y, N_{Q(t)} \right\rangle . \tag{1.5}$$

Using (1.5), take $Y = \dfrac{\partial}{\partial t}$, we obtain

$$\int_{\Gamma(t)} \left\langle \frac{\partial}{\partial t}, \eta^t \right\rangle = 2H\|\Omega(t)\|$$

where $\|\Omega(t)\|$ is the area of the planar region $\Omega(t)$. Thus if we substitute in (1.4), we have

$$L^2(t) \leq -2HA'(t)\|\Omega(t)\|, \quad \text{for almost every } t \geq 0,\ t \in \mathcal{O} \tag{1.6}$$

Now we will show that

$$L^2(t) \geq 4\pi\|\Omega(t)\| + \kappa\|\Omega(t)\|^2. \tag{1.7}$$

We denote by $\Omega(t) = \displaystyle\bigcup_{i=1}^{n_t} \Omega_i(t)$, where $\Omega_1(t), \ldots, \Omega_{n_t}(t)$ are bounded domains which are determined in the plane $Q(t)$ by the closed curve $\Gamma(t)$, and $\|\Omega_i(t)\|$ (with $i = 0 \ldots, n_t$) the area of the corresponding $\Omega_i(t)$. Then $\|\Omega(t)\| = \displaystyle\sum_{i=1}^{n_t} \|\Omega_i(t)\|$. We know by [4] that the equation (1.7) is true if $n_t = 1$. Suppose that the result is true for $n_t = m$. We will prove this is true to $m+1$.

Let $\widetilde{L}(t)$ be the length of $\widetilde{\Omega}(t) = \displaystyle\bigcup_{i=1}^{m} \Omega_i(t)$. We know that

$$\begin{aligned} \widetilde{L}^2(t) &\geq 4\pi\|\widetilde{\Omega}(t)\| + \kappa\|\widetilde{\Omega}(t)\|^2, \quad \text{by hypothesis of induction.} && (1.8) \\ L^2_{m+1}(t) &\geq 4\pi\|\Omega_{m+1}(t)\| + \kappa\|\Omega_{m+1}(t)\|^2 \quad \text{by [4]}. && (1.9) \end{aligned}$$

The equation (1.8) and (1.9) implies, respectively,

$$\begin{array}{rcl} \widetilde{L}(t) & \geq & \sqrt{\kappa}\|\widetilde{\Omega}(t)\| \\ L_{m+1}(t) & \geq & \sqrt{\kappa}\|\Omega_{m+1}(t)\|. \end{array}$$

Thus

$$\widetilde{L}(t)L_{m+1}(t) \geq \kappa\|\widetilde{\Omega}(t)\|\|\Omega_{m+1}(t)\| \Rightarrow 2\tilde{L}(t)L_{m+1}(t) \geq 2\kappa\|\tilde{\Omega}(t)\|\|\Omega_{m+1}(t)\| \quad (1.10)$$

Adding (1.8), (1.9) and (1.10) we get

$$(\widetilde{L}(t) + L_{m+1}(t))^2 \geq 4\pi(\|\widetilde{\Omega}(t)\| + \|\Omega_{m+1}(t)\|) + \kappa(\|\widetilde{\Omega}(t)\| + \|\Omega_{m+1}(t)\|)^2,$$

and this prove (1.7).

Using (1.6) and (1.7) we have

$$\begin{array}{l} 4\pi\|\Omega(t)\| + \kappa\|\Omega(t)\|^2 \leq -2HA'(t)\|\Omega(t)\| \\ 4\pi\|\Omega(t)\| + \kappa\|\Omega(t)\|^2 + 2HA'(t)\|\Omega(t)\| \leq 0 \\ (4\pi + 2HA'(t) + \kappa\|\Omega(t)\|)\,\|\Omega(t)\| \leq 0 \\ 4\pi + 2HA'(t) + \kappa\|\Omega_i(t)\| \leq 0 \end{array}$$

Integrating this inequality from 0 to $h = \max_{p\in\Sigma} h_1(p) \geq 0$, one gets

$$4\pi h + 2H\,(A(h) - A(0)) + \kappa Vol(U_1) \leq 0,$$

then

$$A^+ = A(0) \geq \frac{2\pi h}{H} + \frac{\kappa Vol(U_1)}{2H},$$

which is the inequality that we looked for.

If the equality holds, then all the inequalities above become equalities. In particular, by [4], $\Gamma(t)$ is the boundary of a geodesic disk in $\mathbb{M}^2 \times \{t\}$, for every $t \geq 0$, and $K_{\mathbb{M}^2}(p) \equiv -\kappa$ for all $p \in U$.

Let $D \subset \mathbb{M}^2 \times \{0\}$ be the geodesic disk such that $\partial D = \partial\Sigma$ and $p \in D$ the center of D. Let γ be a horizontal complete oriented geodesic passing through the point p with $\gamma(0) = p$ and $P_\gamma(t)$ the oriented foliation of vertical planes along the γ given by definition 1.2.4. Let $P_\gamma(t_1)$ be a vertical plane in this horizontal foliation such that does not touch Σ. Then, doing Alexandrov reflection with the planes $P_\gamma(t)$, starting at $t = t_1$, and decrease t we obtain, by the symmetries of ∂D, that Σ is symmetric with respect to $P_\gamma(0)$. Since γ is an arbitrary horizontal complete geodesic passing through the point p, we have that Σ is a rotational spherical cap. $\square$

Corollary 1.3.2. *Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \leq -\kappa \leq 0$. If Σ is a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$ without boundary and area A. Let U be the compact domain bounded by Σ, then Σ lies in a horizontal slab with height less than $\dfrac{HA}{2\pi} - \dfrac{\kappa Vol\,(U)}{4\pi}$. One has equality if, and only if, Σ is a sphere of revolution, in which case the thinnest slab has height exactly $\dfrac{HA}{2\pi} - \dfrac{\kappa Vol\,(U)}{4\pi}$.*

Corollary 1.3.3. *Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $K_{\mathbb{M}} \leq -\kappa \leq 0$. If Σ is a compact H-surface embedded in $\mathbb{M}^2 \times \mathbb{R}$ with boundary in a plane Q and transverse to Q, then*

$$\kappa Vol(U_1) < 2\pi H A^+.$$

where A^+ and U_1 are as defined in the previous theorem.

1.4 Horizontal H-cylinders in $\mathbb{H}^2 \times \mathbb{R}$

Now we use a translation invariant H-hypersurfaces given by P. Bérard and R. Sa Earp in [5] and we give some conditions that implies that Σ lies above $Q = \mathbb{H}^2 \times \{0\}$ when $\partial\Sigma \subset Q$. We recall some ideas here.

Let γ_1 be a geodesic passing through $0 \in \mathbb{H}^2 \times \{0\}$ in $Q = \mathbb{H}^2 \times \{0\}$ and let $P_1 = \gamma_1 \times \mathbb{R} = \left\{(\gamma_1(s), t)\,,\ (s,t) \in \mathbb{R}^2\right\}$ the vertical plane , where s is the signed hyperbolic distance to 0 on γ_1.

Take a geodesic γ_2 such that $\gamma_2(0) = \gamma_1(0)$, $\gamma_2'(0) \perp \gamma_1'(0)$. We consider the hyperbolic translation with respect to the geodesic γ_2. In the vertical plane P_1 we take the curve $\alpha(s) = (s, f(s))$, where f is a real function.

In $\mathbb{H}^2 \times \{f(s)\}$ we translate the point $\alpha(s)$ by the translations with respect to $\gamma_2 \times \{f(s)\}$ and we get the equidistant curves $(\gamma_2)_{\alpha(s)}$ passing through $\alpha(s)$, at distance s from $\gamma_2 \times \{f(s)\}$. The curve α then generates a translation surface $C = \bigcup\limits_s (\gamma_2)_{\alpha(s)}$ in $\mathbb{H}^2 \times \mathbb{R}$.

Principal Curvatures: The principal directions of curvature of C are tangent to the curve α in P_1 and the directions tangent to $(\gamma_2)_{\alpha(s)}$. The corresponding principal curvatures with respect to the unit normal pointing downwards are given by

$$\begin{aligned} k_{P_1} &= -f''(s)\left(1+(f'(s))^2\right)^{-\frac{3}{2}} \\ k_{(\gamma_2)_{\alpha(s)}} &= -f'(s)\left(1+(f'(s))^2\right)^{-\frac{1}{2}}\tanh(s). \end{aligned}$$

The first equality comes from the fact that P_1 is totally geodesic and flat. The second equality follows from the fact that $(\gamma_2)_{\alpha(s)}$ is at distance s from $\gamma_2 \times \{f(s)\}$ in $\mathbb{H}^2 \times \{f(s)\}$.

Mean Curvature: The mean curvature of the translation surface C associated with f is given by

$$\begin{aligned} 2H(s) &= -f''(s)\left(1+(f'(s))^2\right)^{-\frac{3}{2}} - f'(s)\left(1+(f'(s))^2\right)^{-\frac{1}{2}}\tanh(s) \\ 2H(s)\cosh(s) &= -f''(s)\left(1+(f'(s))^2\right)^{-\frac{3}{2}}\cosh(s) - f'(s)\left(1+(f'(s))^2\right)^{-\frac{1}{2}}\sinh(s) \\ 2H(s)\cosh(s) &= -\frac{d}{ds}\left(f'(s)\left(1+(f'(s))^2\right)^{-\frac{1}{2}}\cosh(s)\right) \end{aligned}$$

We assume that $H = \textit{constant}$. Observe that in our case $H > 0$. The generating curves of translation surfaces with mean curvature H are given by the differential equation

$$-f'(s)\left(1+(f'(s))^2\right)^{-\frac{1}{2}}\cosh(s) = 2H\sinh(s) + d_1$$

where d_1 is a constant.

We want that $f'(0) = 0$, thus we take $d_1 = 0$. Therefore

$$\begin{aligned} -f'(s)\left(1+(f'(s))^2\right)^{-\frac{1}{2}} &= 2H\tanh(s) \\ -f'(s) &= 2H\tanh(s)\left(1+(f'(s))^2\right)^{\frac{1}{2}} \\ (f'(s))^2 &= 4H^2\tanh^2(s)\left(1+(f'(s))^2\right) \\ (f'(s))^2 &= 4H^2\tanh^2(s) + (f'(s))^2 4H^2\tanh^2(s) \\ (f'(s))^2 &= \frac{4H^2\tanh^2(s)}{(1-4H^2\tanh^2(s))} \end{aligned}$$

We have two first-order linear ordinary differential equations given by

$$f_+'(s) = -\frac{2H\tanh(s)}{\sqrt{1-4H^2\tanh^2(s)}} \text{ and } f_-'(s) = \frac{2H\tanh(s)}{\sqrt{1-4H^2\tanh^2(s)}}$$

with $s \in (-s_H, s_H)$, where $s_H = \operatorname{arctanh}\left(\frac{1}{2H}\right)$.

Resolving the equations above we get, respectively

$$f_+(s) = -\frac{2H}{\sqrt{4H^2-1}} \arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) + d_2$$

and

$$f_-(s) = \frac{2H}{\sqrt{4H^2-1}} \arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) + d_3$$

where d_2 and d_3 are constant.

We want that $\lim\limits_{s\to\pm s_H} f_+(s) = \lim\limits_{s\to\pm s_H} f_-(s) = 0$, so we take $d_2 = -d_3 = \dfrac{H\pi}{\sqrt{4H^2-1}}$. Hence

$$f_+(s) = -\frac{2H}{\sqrt{4H^2-1}} \left(\arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) - \frac{\pi}{2}\right) \tag{1.11}$$

and

$$f_-(s) = \frac{2H}{\sqrt{4H^2-1}} \left(\arctan\left(\frac{\sqrt{4H^2-1}}{\sqrt{1-4H^2\tanh^2(s)}}\right) - \frac{\pi}{2}\right).$$

We have two curves: $\alpha_+(s) = (s, f_+(s))$ and $\alpha_-(s) = (s, f_-(s))$. The curve $\alpha = \alpha_+ \cup \alpha_-$ generates a complete embedded translation invariant H-surface, C_H. We call this surface an H-cylinder.

We observe that the height of C_H is given by

$$h_{C_H} = -\frac{4H}{\sqrt{4H^2-1}} \left(\arctan\left(\sqrt{4H^2-1}\right) - \frac{\pi}{2}\right).$$

Since $\arctan\dfrac{1}{x} = \dfrac{\pi}{2} - \arctan x$, $x > 0$, we get

$$h_{C_H} = \frac{4H}{\sqrt{4H^2-1}} \arctan\left(\frac{1}{\sqrt{4H^2-1}}\right).$$

But $\arctan x = \arcsin\left(\dfrac{x}{\sqrt{1+x^2}}\right)$, then

$$h_{C_H} = \frac{4H}{\sqrt{4H^2-1}} \arcsin\left(\frac{1}{2H}\right). \tag{1.12}$$

By Aledo, Espinar and Gálvez [2] we have that the height of the rotational H-sphere, S_H, is

$$\frac{8H}{\sqrt{4H^2-1}} \arcsin\left(\frac{1}{2H}\right),$$

therefore,

$$h_{C_H} = \frac{h_{S_H}}{2}.$$

Now, we use these C_H cylinders to prove the following.

Remark: In this subsection the height of a compact H-surface Σ embedded into $\mathbb{H}^2 \times \mathbb{R}$ is the height difference between its upper point and lower point.

Theorem 1.4.1. *Let Σ be a compact H-surface embedded into $\mathbb{H}^2 \times \mathbb{R}$, $H > \dfrac{1}{2}$, whose boundary is a convex planar curve contained in the plane $Q = \mathbb{H}^2 \times \{0\}$. Assume $2h_\Sigma < h_{S_H}$, where h_Σ and h_{S_H} are the height of the surfaces Σ and the H-sphere, respectively. Then Σ stays in a half-space determined by Q and is transverse to Q along the boundary. Moreover, Σ inherits the symmetries of its boundary.*

To prove this, we need the following lemma.

Lemma 1.4.2. *Let Σ be a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$, $H > \dfrac{1}{2}$, with planar boundary. If $2h_\Sigma < h_{S_H}$, where h_Σ and h_{S_H} are the height of the surfaces Σ and H-sphere, respectively, then the surface Σ lies inside the right vertical cylinder determined by the convex hull of its boundary.*

Proof. Suppose that there is some point of Σ projecting on a point $q_1 \in Q$ outside the convex hull V of the boundary of Σ, and choose $q_2 \in V$ minimizing the distance to q_1. Denote by γ_1 the geodesic of Q passing through q_1 and q_2, $\gamma_1(0) = q_2$, $\gamma_1(a) = q_1$, $a > 0$. Let $\gamma_2 \subset \mathbb{H}^2 \times \{0\}$ be a complete geodesic with $\gamma_2(0) = \gamma_1(0)$, $\gamma_2'(0) \perp \gamma_1'(0)$.

Consider C_H a horizontal constant mean curvature cylinder generated by $\alpha \subset P_1 = \gamma_1 \times \mathbb{R}$, as described above, with curvature H.

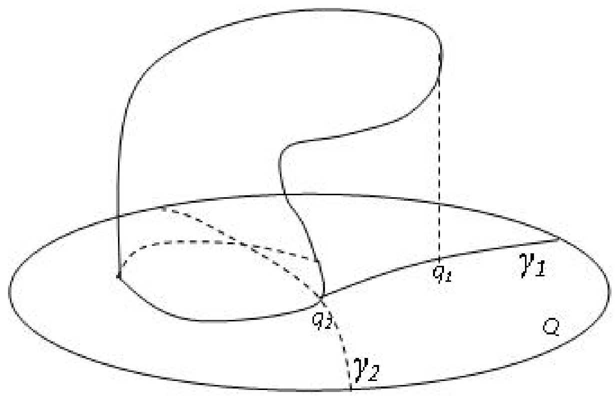

Figure 1.1:

We consider a half-cylinder C_{γ_1} generated by $\alpha(s)$, $s \in [0, s_H]$ or $s \in [-s_H, 0]$. We move C_{γ_1} by horizontal translation along γ_1 far enough so that it does not touch the surface Σ and we place its concave side in front of Σ.

The surface Σ is inside a slab B parallel to Q with height less than $\frac{h_{S_H}}{2}$. This slab is not necessarily symmetric with respect to Q but we may utilize half-cylinders with axis in the central plane of B, then making a vertical translation if necessary, we can suppose that B is symmetric with respect to Q, see figure 1.2.

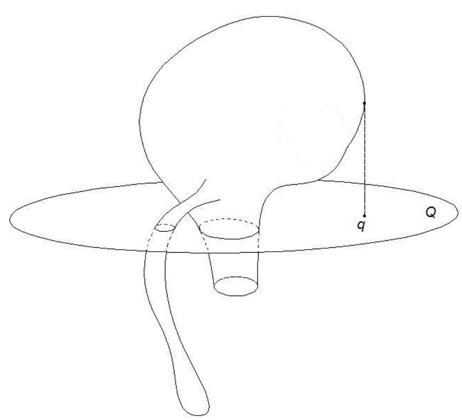

Figure 1.2:

Now we proceed to approach the half-cylinder C_{γ_1} to Σ by the horizontal translation along γ_1 and in this way we get a first (and so tangential) contact point between the two surfaces.

As γ_2 lies inside Q and there is a point of Σ projecting on the point q_1 outside the convex hull of the boundary, this contact point so obtained is a nonboundary point of the surface Σ. It is also an interior point of the half-cylinder C_{γ_1}, because Σ is inside the slab $B \subset \mathbb{H}^2 \times \left(-\frac{h_{S_H}}{2}, \frac{h_{S_H}}{2}\right)$. On the other hand, this half-cylinder has constant mean curvature H, with respect to the normal field pointing to its concave part. As we already know that Σ is in that concave part, by elementary comparison we have the same choice of normal at the contact point gives mean curvature H for Σ. But this is a contradiction to the maximum principle. As a consequence, all the points of the surface Σ must project on the convex hull of its boundary. □

Proof of Theorem 1.4.1. By the previous lemma we know that, if Ω is a compact convex domain in Q with $\partial\Omega = \partial\Sigma$, then $\Sigma \cap ext(\Omega) = \emptyset$. Then one can consider a hemisphere S under the plane Q whose boundary disc D is contained in Q and is large enough that $\Omega \subset int(D)$ and $S \cap \Sigma = \emptyset$. Thus $\Sigma \cup (D - \Omega) \cup (S - D)$ is a compact embedded surface in $\mathbb{H}^2 \times \mathbb{R}$ and so determines an interior domain, we call U. Choose a unit normal N for Σ in such a way that N point into U at each point. If there are points of the surface Σ in both half-spaces determined by Q, then N takes the same value at the points where the height function attains its maximum and minimum respectively. Reversing N if necessary, we can conclude that the normal of Σ (for which $H > 0$) takes the same value at the highest and the lowest points of the surface.

Lower a sphere S_H^2 to the highest point or pushing it up to the lowest one we obtain a contradiction using the interior maximum principle. Thus the surface lies in one of the half-spaces determined by the plane Q and rises in it less than $\frac{h_{S_H}}{2}$. Using again half-cylinders C_H with axis in a plane parallel to Q and height $\frac{h_{S_H}}{2}$, the boundary maximum principle shows us that the surface is transversal along its boundary.

Let γ be a horizontal complete oriented geodesic passing through the origin $O \in \mathbb{H}^2 \times \mathbb{R}$ and $P_\gamma(t_1)$ be a vertical plane such that $P_\gamma(t_1) \cap \Sigma = \emptyset$. We take the oriented foliation of vertical planes along γ given in definition 1.4, with $P = P_\gamma(0)$. Now, by applying Alexandrov reflection with these planes, starting $t = t_1$ and decreasing t, we obtain that Σ has all the symmetries of its boundary.

□

Corollary 1.4.3. *Let Σ be a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$, $H > \frac{1}{2}$, with convex planar boundary. Then Σ is a graph if, and only if, $h_\Sigma < \frac{h_{S_H}}{2}$, where h_Σ and h_{S_H} are the height of the surfaces Σ and the H-sphere, respectively.*

Proof. If the Σ is a graph the proof follow by [2], Theorem 2.1. Suppose now that $h_\Sigma < \frac{h_{S_H}}{2}$. By Theorem 1.4.1 we have that Σ must be contained in one of the half-spaces determined by the boundary plane and, moreover, by Lemma 1.4.2 Σ is inside the right vertical cylinder determined by the convex hull of its boundary. Using Alexandrov reflection with horizontal planes we get that Σ is a graph.

□

CHAPTER 2

CMC Surfaces in a Half-Space in $\mathbb{H}^2 \times \mathbb{R}_+$

2.1 Introduction

In this chapter we will consider compact embedded surfaces of constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}_+$ with boundary in $\mathbb{H}^2 \times \{0\}$. The structure of these surfaces seems far from being understood. We shall consider a simple situation concerning this problem. For this we use ideas of A. Ros and H. Rosenberg, see [23].

Antonio Ros and Harold Rosenberg showed in [23] that if Γ_n is a sequence of embedded (perhaps nonconnected) curves in $\mathbb{R}^3$, $\Gamma_n \subset \{x_3 = 0\}$, converging to a point q, and $\Sigma_n \subset \mathbb{R}^3_+$ is a sequence of embedded compact 1-surfaces ($H = 1$), with $\partial\Sigma_n = \Gamma_n$, then some subsequence of Σ_n converges either to q or to the unit sphere tangent to $\{x_3 = 0\}$ at q (the convergence being smooth on compact subsets of $\mathbb{R}^3 - \{p\}$). We will show that a similar result is true for $\mathbb{H}^2 \times \mathbb{R}_+$.

2.2 Notations

Here we will use the Poincaré disk model of $\mathbb{H}^2$ which is represented as the domain

$$\mathbb{D} = \left\{ z = (x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1 \right\}$$

endowed with the metric $g = \dfrac{4\,|dz|^2}{\left(1-|z|^2\right)^2}$.

The complete geodesics in this model are given by arcs of circles or straight lines which are orthogonal to the boundary at infinity

$$\mathbb{S}^1_\infty = \left\{z \in \mathbb{R}^2;\ |z| = 1\right\}.$$

We orient $\mathbb{H}^2$ so that its boundary at infinity is oriented counter-clockwise. Let γ be a complete oriented geodesic in $\mathbb{H}^2$, then

$$\partial_\infty \gamma = \{\gamma^-, \gamma^+\},$$

where $\gamma^- = \lim\limits_{t\to-\infty} \gamma(t)$ and $\gamma^+ = \lim\limits_{t\to+\infty} \gamma(t)$. Here t is arc length along γ. Then we can identify a geodesic γ with its boundary at infinity, writing

$$\gamma = \{\gamma^-, \gamma^+\}.$$

We observe that given an oriented geodesic $\gamma = \{\gamma^-, \gamma^+\}$ in $\mathbb{H}^2$ then $\mathbb{H}^2 - \{\gamma\}$ has two connected components. We will distinguish then using the following notation.

Definition 2.2.1. Let J be the standard counter-clockwise rotation operator. We call exterior set of γ in $\mathbb{H}^2$, $ext_{\mathbb{H}^2}(\gamma)$, the connected component of $\mathbb{H}^2 - \{\gamma\}$ towards which $J\gamma'$ points. The other connected component of $\mathbb{H}^2 - \{\gamma\}$ is called the interior set of γ in $\mathbb{H}^2$ and denoted by $int_{\mathbb{H}^2}(\gamma)$.

On the other hand, we consider the product space $\mathbb{H}^2 \times \mathbb{R}$ represented as the domain

$$\mathbb{H}^2 \times \mathbb{R} = \left\{(x, y, t) \in \mathbb{R}^3;\ (x, y) \in \mathbb{D}\right\}$$

endowed with the product metric.

Definition 2.2.2. Given a complete oriented geodesic γ in $\mathbb{H}^2 \times \{0\}$, we will call $\gamma \times \mathbb{R}$ a vertical plane of $\mathbb{H}^2 \times \mathbb{R}$ and we will call a slice $\mathbb{H}^2 \times \{\tau\}$ a horizontal plane.

Note that a vertical plane is isometric to $\mathbb{R}^2$ and horizontal plane is isometric to $\mathbb{H}^2$. Throughout this chapter we will denote vertical planes by P and horizontal planes by $Q(\tau) = \mathbb{H}^2 \times \{\tau\}$, $Q(0) = Q$.

The notion of interior and exterior domain of a horizontal oriented geodesic extend naturally to vertical planes.

Definition 2.2.3. For a complete oriented geodesic γ in $\mathbb{H}^2 \times \{0\} \equiv \mathbb{H}^2$ we call, respectively interior and exterior of a vertical plane $P = \gamma \times \mathbb{R}$ the sets

$$int_{\mathbb{H}^2\times\mathbb{R}}(P) = int_{\mathbb{H}^2}(\gamma) \times \mathbb{R}, \quad ext_{\mathbb{H}^2\times\mathbb{R}}(P) = ext_{\mathbb{H}^2}(\gamma) \times \mathbb{R}.$$

We will often use foliations by vertical planes of $\mathbb{H}^2 \times \mathbb{R}$. We now make this precise.

Definition 2.2.4. Let P be a vertical plane in $\mathbb{H}^2 \times \mathbb{R}$ and $\gamma(t)$ be an oriented horizontal geodesic in $\mathbb{H}^2 \times \{0\}$ with t arc length along γ, $\gamma(0) = p_0 \in P$, $\gamma'(0)$ orthogonal to P at p_0 and $\gamma(t) \in ext_{\mathbb{H}^2\times\mathbb{R}}(P)$ for $t > 0$. We define the oriented foliation of vertical planes along γ, denoted by $P_\gamma(t)$ to be the vertical planes orthogonal to $\gamma(t)$ with $P = P_\gamma(0)$.

Let $P_\gamma(t)$ be the foliation of vertical planes along γ with $P_\gamma(0) = P$ and $q \in P(t_0)$. Let ξ denote the horizontal Killing field of $\mathbb{H}^2 \times \mathbb{R}$ generated by translation along γ (ξ is tangent to each $\mathbb{H}^2 \times \{\tau\}$ and is translation along $\gamma \times \{\tau\}$), ξ is orthogonal to the planes $P_\gamma(t)$.

Definition 2.2.5. We call the surface Σ a horizontal graph over a domain $\Omega \subset P$ if Σ is a graph over Ω with respect to the orbits of ξ.

Remark 2.2.6. H-horizontal graphs are strongly stable. See remark 2.1, [20].

2.3 The main result

In the first part we consider the case $H > \frac{1}{2}$. After we give the analogous result to $H \leq \frac{1}{2}$.

Lemma 2.3.1. *Let $\{\Sigma_n\}$ be a sequence of H-horizontal graphs, $H > \frac{1}{2}$, in the $ext_{\mathbb{H}^2\times\mathbb{R}}(P)$ such that Σ_n is a graph over an open subset $\Omega_n \subset K \subset P$, K a compact. Then there exists a subsequence of Σ_n which converges on compact subsets of $ext_{\mathbb{H}^2\times\mathbb{R}}(P)$ to a H-horizontal graph (perhaps empty) in $ext_{\mathbb{H}^2\times\mathbb{R}}(P)$.*

Remark 2.3.2. We are not assuming any restriction on the open sets Ω_n. The assumption that Σ_n is a horizontal graph in $ext_{\mathbb{H}^2\times\mathbb{R}}(P)$ means that Σ_n is the graph of a positive function u_n on Ω_n, where u_n extends continuously to the closure of Ω_n with zero values. Note that the convergence above holds only in the interior. In general we cannot control the limit surface in P.

Proof. It follow from Espinar, Gálvez and Rosenberg, Theorem 6.2 in [11], that all the H-horizontal graphs Σ_n are contained in a fixed compact of $\mathbb{H}^2 \times \mathbb{R}$.

Let $P_\gamma(t)$ be the foliation as described above. We take $\epsilon > 0$ and $\Sigma_n^\epsilon = \Sigma \cap ext_{\mathbb{H}^2\times\mathbb{R}}\big(P_\gamma(\epsilon)\big)$. From Rosenberg, Souam and Toubiana [24], there exists a constant $C_1 = C_1(H, \epsilon)$ such that $\|A_{\Sigma_n^\epsilon}\| \leq C_1$, where $A_{\Sigma_n^\epsilon}$ is the second fundamental form of Σ_n^ϵ. Meeks and Tinaglia showed (in [19], Corollary 5.2): Suppose N^3 is a complete, simply connected three-manifold with absolute sectional curvature bounded by s_0 and suppose $\Sigma \subset N^3$ is a properly immersed CMC surface which is almost-embedded, has constant mean curvature H and satisfies $\|A_\Sigma\| \leq C_1$ for some constant $C_1 \geq 0$, then there is a constant $C_2 > 0$ depending on C_1, H, s_0 such that the area of Σ in ambient balls of radius one is at most C_2. This result implies, in our case, that there exist a constant $C_2 > 0$ such that $Area(\Sigma_n^\epsilon) \leq C_2$, $C_2 = C_2(H, \epsilon)$.

Standard compactness techniques yield a subsequence (which we also call) Σ_n^ϵ that converges on compact subsets of $ext_{\mathbb{H}^2\times\mathbb{R}}(P(\epsilon))$, see for instance [31] Theorem 3. The limit is either empty or a H-surface properly immersed Σ^ϵ.

Suppose the limit is not empty. Let $n = \langle N, \xi\rangle$, where N is the unit normal vector field of Σ^ϵ, we have that, since ξ is a Killing field, $\Delta n + (\|A_{\Sigma^\epsilon}\|^2 + Ric(N))n = 0$. As Σ^ϵ is a limit of graphs we get $n \leq 0$ and, so Gidas, Ni and Nirenberg [14] implies that $n < 0$ in Σ^ϵ, or $n \equiv 0$ on a connected component of Σ^ϵ. In the first case, we conclude that Σ^ϵ is a graph over $\Omega \subset P_\gamma(\epsilon)$.

In the second case, we know in this connected component, Σ_0^ϵ, $\partial\Sigma_0^\epsilon \subset P(\epsilon)$ and Σ_0^ϵ is compact, since Σ_n are contained in a fixed compact. Then let $q \in \Sigma_0^\epsilon$ be a furthest point from $P(\epsilon)$.

Let $P_\gamma(t)$ be the foliation of vertical plane along γ and $P_\gamma(t_1)$ outside the compact $\tilde{K}$. Now, take the foliation of the planes $P_\gamma(t)$, starting at $t = t_1$, and decrease t. There exists $t_0 \in \mathbb{R}$ such that $P_\gamma(t_0)$ is the first vertical plane that touch Σ_0^ϵ in a point q. We have in this point that $N(q) = -\xi(q)$. This is impossible because $\langle N, \xi\rangle = 0$ in Σ_0^ϵ. This proves the lemma.

□

This proof also works for H- vertical graphs, that is

Corollary 2.3.3. *Let $\{\Sigma_n\}$ be a sequence of H-vertical graphs, $H > \dfrac{1}{2}$, in $\mathbb{H}^2 \times \mathbb{R}_+$ such that Σ_n is a graph over an open subset $\Omega_n \subset K \subset Q = \mathbb{H}^2 \times \{0\}$, K compact. Then there exists a subsequence of Σ_n which converges on compact subsets of $\mathbb{H}^2 \times \mathbb{R}_+$ to a H-vertical graph (perhaps empty) in $\mathbb{H}^2 \times \mathbb{R}_+$.*

Now, we will prove the main result.

Theorem 2.3.4. *Let $\Sigma_n \subset \mathbb{H}^2 \times \mathbb{R}_+$ be an embedded H-surfaces with $H > \dfrac{1}{2}$ and $\Gamma_n = \partial\Sigma_n \subset D(r_n) = \left\{(x_1, x_2, 0) \in \mathbb{H}^2 \times \mathbb{R}_+;\ x_1^2 + x_2^2 \leq r_n\right\}$, with r_n a sequence converging to zero. Then there is a subsequence of Σ_n that either converges to* *the origin $O \in \mathbb{H}^2 \times \mathbb{R}_+$ or to the rotationally invariant constant mean curvature sphere $S_H^2 \subset \mathbb{H}^2 \times \mathbb{R}_+$ tangent to $\mathbb{H}^2 \times \{0\}$ at O. In the first case the surfaces converge as subsets and in the second the convergence is smooth on compact subsets of $\left\{\mathbb{H}^2 \times \mathbb{R}\right\} - \{O\}$.*

Proof. It follows from the Alexandrov reflection technique and the vertical height estimates, see [2] theorem 2.1, and the Espinar, Gálvez and Rosenberg, theorem 6.2 in [11] that all the Σ_n are contained in a fixed compact $\tilde{K}$ in $\mathbb{H}^2 \times \mathbb{R}_+$.

Let γ be a horizontal complete oriented geodesic passing through the origin $O \in \mathbb{H}^2 \times \mathbb{R}_+$ and $P_\gamma(t_1)$ be a vertical plane outside this compact $\tilde{K}$. We take the oriented foliation of vertical planes along γ given in definition 2.2.4, with $P = P_\gamma(0)$. Let $P_\gamma(t_1)$ be a vertical plane outside the compact $\tilde{K}$.

Ler $r > 0$ be a positive constant. For n large, $\partial\Sigma_n \subset D(r)$ where $D(r) = \left\{(x_1, x_2, 0) \in \mathbb{H}^2 \times \mathbb{R}_+;\ x_1^2 + x_2^2 \le r\right\}$ so, doing Alexandrov reflection with the planes $P_\gamma(t)$, starting at $t = t_1$, and decrease t, the symmetry of the part Σ_n in $ext_{\mathbb{H}^2\times\mathbb{R}}\left(P_\gamma(t)\right)$ does not touch Σ_n if $P_\gamma(t)$ does not touch $\partial D(r)$ and this part of Σ_n in $ext_{\mathbb{H}^2\times\mathbb{R}}\left(P_\gamma(t)\right)$ is a horizontal graph over a part of $P_\gamma(t)$.

Now, Alexandrov reflexion with horizontal planes and [2] give that the part of each Σ_n above $P(a)$, where $a = \frac{4H}{\sqrt{4H^2-1}} \arcsin\left(\frac{1}{2H}\right)$, is a vertical graph, so by Lemma 2.3.1 and Corollary 2.3.3 we get that Σ_n converges on compact subsets of $\left(\mathbb{H}^2 \times \mathbb{R}_+\right) - I$, where $I = 0 \times [0, a]$.

If outside of all compact subsets of $\left(\mathbb{H}^2 \times \mathbb{R}_+\right) - I$ the convergence is empty, then for n large, Σ_n is uniformly close to I. We take $\bar{H} \ge H$ and let $S_{\bar{H}}$ be the hemisphere of constant mean curvature $\bar{H}$ with $\partial S_{\bar{H}} \subset \mathbb{H}^2 \times \{a\}$. When we move $\partial S_{\bar{H}}$ from $\mathbb{H}^2 \times \{a\}$ to $\mathbb{H}^2 \times \{0\}$ down we obtain, by Comparison Theorem, that Σ_n is always below $S_{\bar{H}}$ (recall that $\partial\Sigma_n \subset D(r)$ with $r > 0$ small enough). Therefore, letting $\bar{H}$ go to infinity we conclude Σ_n converge to O.

Suppose that Σ_n converges to a surface Σ. This surface has constant mean curvature H and is properly embedded in $\left(\mathbb{H}^2 \times \mathbb{R}_+\right) - I$. Let γ a horizontal complete oriented geodesic passing through the origin $O \in \mathbb{H}^2 \times \mathbb{R}_+$ and $P_\gamma(t_1)$ be a vertical plane outside this compact $\tilde{K}$. For each $r > 0$, we know, doing Alexandrov reflection with the planes $P_\gamma(t)$, starting at $t = t_1$, and decrease t, the symmetry of the part Σ in $ext_{\mathbb{H}^2\times\mathbb{R}}\left(P_\gamma(t)\right)$ does not touch Σ if $P_\gamma(t) \cap D(r) = \emptyset$. Therefore the symmetry of the part Σ by these planes lies in the domain enclosed by Σ (since this hold for Σ_n, n large). So this works up until $r = 0$ by continuity and Σ is a rotational surface about the vertical line throught O and each component of Σ has multiplicity one, then Σ is

$S^2_H \subset \mathbb{H}^2 \times \mathbb{R}_+$.

Finally we show the convergence is uniform on compact subsets of $\left(\mathbb{H}^2 \times \mathbb{R}\right) - \{O\}$. Given $\epsilon > 0$, there exists $r > 0$ so that for n large

$$\Sigma_n \cap \left(D(r) \times \left(\frac{3}{2}a, \infty\right)\right) = \Sigma_n \cap D(r) \times (2a - \epsilon, 2a + \epsilon)$$

and this intersection is a graph above $D(r)$. Coming down with horizontal planes $Q(t) = \mathbb{H}^2 \times \{t\}$ from $t = 2a$ to $t = a$ we see that $\Sigma_n \cap (D(r) \times (\epsilon, 2a - \epsilon)) = \emptyset$. So we have uniform estimates for Σ_n on compact subsets of $\left(\mathbb{H}^2 \times \mathbb{R}\right) - \{O\}$, not just on compact subsets of $\left(\mathbb{H}^2 \times \mathbb{R}\right) - I$. □

Now we deal with H-surfaces with $H \in \left(0, \frac{1}{2}\right]$. We use here the H-cylinder, see section 1.4.

Lemma 2.3.5. *Let Σ be a compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$, $H \in \left(0, \frac{1}{2}\right]$, with planar boundary. Then the surface Σ lies inside the right vertical cylinder determined by the convex hull of its boundary.*

Proof. Suppose that there is some point of Σ projecting on a point $q_1 \in Q$ outside the convex hull V of the boundary of Σ, and choose $q_2 \in V$ minimizing the distance to q_1. Denote by γ_1 the geodesic of Q passing through q_1 and q_2, $\gamma_1(0) = q_2$, $\gamma_1(a) = q_1$, $a > 0$. Let $\gamma_2 \subset \mathbb{H}^2 \times \{0\}$ be a complete geodesic with $\gamma_2(0) = \gamma_1(0)$, $\gamma_2'(0) \perp \gamma_1'(0)$.

Consider $C_{\bar{H}}$ a horizontal constant mean curvature cylinder with curvature $\bar{H}$, $\bar{H} > \frac{1}{2}$, see section 1.4. We take the half-cylinder C_{γ_1} and we move C_{γ_1} by horizontal translation along γ_1 far enough so that it does not touch the surface Σ and we place its concave side in front of Σ.

Since the height $h_{C_{\bar{H}}}$ goes to infty when $\bar{H}$ goes to $\frac{1}{2}$, we can choose $\bar{H}$ such that $h_{C_{\bar{H}}} > 2h_\Sigma$, where h_Σ is the height of Σ. Observe that we not suppose that $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}_+$.

Now we proceed to approach the half-cylinder C_{γ_1} to Σ by the horizontal translation along γ_1 and in this way we get a first (and so tangential) contact point between the two surfaces.

As γ_2 lies inside Q and there is a point of Σ projecting on the point q_1 outside the convex hull of the boundary, this contact point so obtained is a nonboundary point of the surface Σ. It is also an interior point of the half-cylinder C_{γ_1}, because $h_{C_{\bar{H}}} > 2h_\Sigma$. On the other hand, this half-cylinder has constant mean curvature $\bar{H} > H$, with respect to the normal field pointing to its concave part. As we already know that Σ is in that concave part, by elementary comparison we have the same choice of normal at the contact point gives mean curvature H for Σ. But this is a contradiction to the maximum principle. As a consequence, all the points of the surface Σ must project on the convex hull of its boundary. □

Theorem 2.3.6. *Let Σ be a compact H-surface embedded into $\mathbb{H}^2 \times \mathbb{R}$, $H \in \left(0, \frac{1}{2}\right]$, whose boundary is a convex planar curve contained in the plane $Q = \mathbb{H}^2 \times \{0\}$. Then Σ stays in a half-space determined by Q and is transverse to Q along the boundary. Moreover, Σ is a graph and inherits the symmetries of its boundary.*

Proof. By the previous lemma we know that, if Ω is a compact convex domain in Q with $\partial\Omega = \partial\Sigma$, then $\Sigma \cap ext(\Omega) = \emptyset$ and using Alexandrov reflection with horizontal planes we get that Σ is a graph. Then one can consider a hemisphere S under the plane Q whose boundary disc D is contained in Q and is large enough that $\Omega \subset int(D)$ and $S \cap \Sigma = \emptyset$. Thus $\Sigma \cup (D - \Omega) \cup (S - D)$ is a compact embedded surface in $\mathbb{H}^2 \times \mathbb{R}$ and so determines an interior domain, we call U. Choose a unit normal N for Σ in such a way that N point into U at each point. If there are points of the surface Σ in both half-spaces determined by Q, then N takes the same value at the points where the height function attains its maximum and minimum respectively. Reversing N if necessary, we can conclude that the normal of Σ (for which $H > 0$) takes the same value at the highest and the lowest points of the surface.

By lower a slice $Q(t)$ to the highest point or pushing it up to the lowest one we obtain a contradiction using the interior maximum principle. Thus the surface lies in one of the half-spaces determined by the plane Q. Using half-cylinders $C_{\bar{H}}$, $\bar{H} > \frac{1}{2}$, with axis in a plane parallel to Q and the lowest point in Q, the boundary maximum principle shows us that the surface is transversal along its boundary.

Let γ be a horizontal complete oriented geodesic passing through the origin $O \in \mathbb{H}^2 \times \mathbb{R}$ and $P_\gamma(t_1)$ be a vertical plane such that $P_\gamma(t_1) \cap \Sigma = \emptyset$. We take the oriented foliation of vertical planes along γ given in definition 2.2.4, with $P = P_\gamma(0)$. Now, by applying Alexandrov reflection with these planes, starting $t = t_1$ and decreasing t, we obtain that Σ has all the symmetries of its boundary.

□

Using the Lemma 2.3.5 we can prove the following.

Theorem 2.3.7. *Let $\Sigma_n \subset \mathbb{H}^2 \times \mathbb{R}$ be an embedded H-surfaces with $H \in \left(0, \frac{1}{2}\right]$ and $\Gamma_n = \partial\Sigma_n \subset D(r_n) = \left\{(x_1, x_2, 0) \in \mathbb{H}^2 \times \mathbb{R};\ x_1^2 + x_2^2 \leq r_n\right\}$, with r_n a sequence converging to zero. Then Σ_n converges, as subset, to the origin $O \in \mathbb{H}^2 \times \mathbb{R}$.*

Proof. Let $r = r(\bar{H})$ be the diameter of $S_{\bar{H}}$, where $S_{\bar{H}}$ is the rotationally invariant constant mean curvature $\bar{H}$ sphere, $\bar{H} > \frac{1}{2}$. For n large $\partial\Sigma_n \subset D(r)$, where $D(r) = \left\{(x_1, x_2, 0) \in \mathbb{H}^2 \times \mathbb{R};\ x_1^2 + x_2^2 \leq r\right\}$, and by the lemma 2.3.5 we get that each Σ_n lies inside the right vertical cylinder determined by the convex hull of its boundary and thus $\Sigma_n \subset D(r) \times \mathbb{R}$.

Let $S^+ = S_{\bar{H}} \cap \mathbb{H}^2 \times [0, \infty)$ be the superior hemisphere. We will prove that Σ_n is below S^+ for all n. Suppose not, so for some n the highest point of Σ_n is bigger than the height of S^+. Lift up S^+ to be above Σ_n, then move S^+ down towards Σ_n, we get the first tangent point of Σ_n. This is a interior point of S^+ since $\Sigma_n \subset D(r) \times \mathbb{R}$. But this is a contradiction by maximum principle. Using the same idea we obtain that Σ_n is above $S^- = S_{\bar{H}} \cap \mathbb{H}^2 \times [-\infty, 0)$. This prove the theorem since that r goes to 0 when $\bar{H}$ goes to infinity. □

CHAPTER 3

Removable singularities for sections of Riemannian submersions of prescribed mean curvature

3.1 Introduction

We will prove that isolated singularities of sections with precribed mean curvature of a Riemanninan submersion fibered by geodesics of a vertical Killing field, are removable. Also we obtain information on the growth of the difference of two sections $u, v : \Omega \to \bar{M}$, having the same prescribed mean curvature and $u = v$ on $\partial\Omega$. This generalizes theorem 2 of [9].

3.2 Killing Graphs

Let $\pi : \bar{M}^{n+1} \to M^n$ be Riemannian submersion such that the orbits of the vertical fibers are geodesics of a nonsingular unit Killing field denoted by $\xi \in \mathcal{X}(\bar{M})$. Let $\Omega \subset M$ be a domain. We assume that the integral curves ϕ_s of ξ in $\bar{M}_0 = \pi^{-1}(\Omega)$ are complete non compact.

We first derive a formula for the mean curvature of a section of $\bar{M}$.

Lemma 3.2.1. *Let Σ be a hypersurface of $\bar{M}$, transverse to the fibers of ξ. Let N be a unit normal vector field to Σ. Then*

$$div_M(\pi_* N) = n \left\langle \vec{H}, N \right\rangle_{\bar{M}} = nH$$

Proof. Let $\bar{x} \in \Sigma$, $\pi(\bar{x}) = x \in \Omega$, and let $X_1, \ldots, X_n$ be an orthonormal frame of M in a neighborhood of x. Let $\bar{X}_1, \ldots, \bar{X}_n$ be their horizontal lifts to $\bar{M}$.

Extend a neighborhood of $\bar{x}$ in Σ to a foliation of a neighborhood of $\bar{x}$ in $\bar{M}$, by the flow of ξ. Also extend N to this neighborhood as well (by ξ). It is well know that

$$div_{\bar{M}}(N) = n\left\langle N, \vec{H}\right\rangle = nH,$$

where $\vec{H}$ is the mean curvature vector of the leaves of the local foliation.

Write $N = N^h + \langle N, \xi\rangle\, \xi$. Then

$$div_{\bar{M}}(N) = \sum_{i=1}^{n}\left\langle \bar{\nabla}_{\bar{X}_i} N, \bar{X}_i\right\rangle + \langle \bar{\nabla}_{\xi} N, \xi\rangle .$$

Since $\xi\,\langle N, \xi\rangle = 0$, $\langle \bar{\nabla}_{\xi} N, \xi\rangle = -\langle N, \bar{\nabla}_{\xi}\xi\rangle = 0$.

We have $N = N^h + \langle N, \xi\rangle\, \xi$, so

$$\left\langle \bar{\nabla}_{\bar{X}_i} N, \bar{X}_i\right\rangle = \left\langle \bar{\nabla}_{\bar{X}_i} N^h, \bar{X}_i\right\rangle + \langle N, \xi\rangle \left\langle \bar{\nabla}_{\bar{X}_i}\xi, \bar{X}_i\right\rangle$$

By O'Neills formula [22], $\bar{\nabla}_{\bar{X}_i}\bar{X}_i$ is horizontal, so differentiating $\langle \xi, \bar{X}_i\rangle = 0$,

$$\left\langle \bar{\nabla}_{\bar{X}_i}\xi, \bar{X}_i\right\rangle = -\left\langle \xi, \bar{\nabla}_{\bar{X}_i}\bar{X}_i\right\rangle = 0$$

Again, by O'Neills formula,

$$\left\langle \bar{\nabla}_{\bar{X}_i} N^h, \bar{X}_i\right\rangle_{\bar{M}} = \left\langle \nabla_{X_i}\pi(N^h), X_i\right\rangle_M,$$

and this proves the lemma.

□

Now consider two section $u, v : \Omega \to \bar{M}$, transverse to ξ, such that the surfaces $\Sigma_u = u(\Omega)$ and $\Sigma_v = v(\Omega)$ have the same prescribed mean curvature at each $x \in \Omega$. We assume the mean curvature function H is continuous on Ω. Let X_u, X_v be the vector fields on Ω, the projection of the unit normals N_u, and N_v to the sections. Let $\varphi = u - v$ be the function on Ω, distance along the ξ orbits from $v(x)$ to $u(x)$, $x \in \Omega$.

It is not hard to see that

$$\langle \nabla\varphi, X_u - X_v\rangle_M = \left\langle \widetilde{\nabla\varphi}, N_u - N_v\right\rangle_{\bar{M}} \geq 0,$$

and one has equality precisely when φ is constant. It is usefull to have an explicit formular for the quantities involved, so we will prove the above statement using formulas in local coordinates derived in [10]. We will prove:

$$\langle \nabla\varphi, X_u - X_v \rangle_M = \left(\frac{W_u + W_v}{2}\right) \|N_u - N_v\|^2 ,$$

where $W_u = \dfrac{1}{\langle N_u, \xi\rangle}$, $W_v = \dfrac{1}{\langle N_v, \xi\rangle}$.

Consider a smooth embedding $\mathtt{i} : \Omega \to \bar{M}$, which is a section of the fibration, and assume $\Sigma_0 = \mathtt{i}(\Omega)$ is transverse to ξ.

Thus, the hypersurfaces $\Sigma_s = \phi_s(\Sigma_0)$ foliate $\bar{M}_0$ by isometric hypersurfaces.

Definition 3.2.2. The Killing graph $\Sigma = \Sigma_u$ of a function $u \in C^2(\Omega)$ is the hypersurface

$$\Sigma = \left\{\phi\left(u(p), p\right);\ p \in \Sigma_0\right\}$$

where u is seen as a function on Σ_0 by taking $u(p) = u(x)$ when $\pi(p) = x$.

3.3 The Mean Curvature Equation

Let $X_1, ..., X_n$ be a frame on Ω and $\sigma_{ij} = \left\langle X_i, X_j\right\rangle_M$. Let $\bar{Y}_1, ..., \bar{Y}_n$ be the corresponding local frame on Σ_0, i.e., $\bar{Y}_i(p) = \mathtt{i}_* X_i(x)$, where $x \in \Omega$ and $p = \mathtt{i}(x)$.

We extend $\bar{Y}_i$ by the flow:

$$\bar{Y}_i(\phi(s, p)) = (\phi)_*(\bar{Y}_i(p)),$$

$p \in \Sigma_0$.

Let $\bar{X}_1, ..., \bar{X}_n$ in $\bar{M}$ denote the horizontal lifts of $X_1, ..., X_n$. If $q = \phi(s, p)$ for $p \in \Sigma_0$, then $\pi(q) = \pi \circ \phi(s, p) = \pi(p)$. Therefore

$$\bar{X}_i(q) = \phi_*(s, p)\bar{X}_i(p)$$

since $\phi_*(s, p)\bar{X}_i(p)$ is horizontal and

$$\pi_*(q)\phi_*(s, p)\bar{X}_i(p) = (\pi \circ \phi)_*(s, p)\bar{X}_i(p) = \pi_*(p)\bar{X}_i(p).$$

Also

$$\left\langle \bar{X}_i, \bar{X}_j\right\rangle_{\bar{M}} = \left\langle X_i, X_j\right\rangle_M = \sigma_{ij}.$$

We denote by s, the function on $\bar{M}_0$ which is distance from q to Σ_0, along the integral curve of ξ from q to Σ_0. We calculate $\bar{X}_i(s)$.

Let $\bar{\nabla}s$ be the gradient of the function s. Using

$$\pi_*(q)\bar{Y}_i = \pi_*(p)\mathtt{i}_*X_i(x) = X_i(x) = \pi_*(q)\bar{X}_i$$

and

$$1 = \xi(s) = \langle\bar{\nabla}s, \xi\rangle_{\bar{M}}$$

we have that two frames in $\bar{M}$ are related by

$$\begin{cases} \bar{\nabla}s = \xi + \sigma^{ij}\bar{X}_j(s)\bar{X}_i, \\ \bar{Y}_i = \delta_i\xi + \bar{X}_i. \end{cases}$$

where σ^{ij} is the inverse of σ_{ij}. Then

$$\delta_i = \langle\bar{Y}_i(q), \xi(q)\rangle_{\bar{M}} = \Big\langle(\phi_s)_*(p)\bar{Y}_i(p), (\phi_s)_*(p)\xi(p)\Big\rangle_{\bar{M}} = \langle\bar{Y}_i(p), \xi(p)\rangle_{\bar{M}}$$

and

$$0 = \bar{Y}_j(s) = \Big\langle\bar{\nabla}s, \bar{Y}_j\Big\rangle_{\bar{M}} = \delta_j + \bar{X}_j(s).$$

Let Σ be a Killing graph. Consider Σ as given by the immersion

$$\mathbf{I}_u : x \in \Omega \subset M \mapsto \phi(u(x), \mathtt{i}(x)).$$

Its tangent bundle is spanned by the vector fields

$$\begin{aligned} (\mathbf{I}_u)_*X_i &= X_i(u)\phi_s + (\phi \circ \mathtt{i})_*X_i \\ &= X_i(u)\xi + \bar{Y}_i \end{aligned} \tag{3.1}$$

We may regard u as a function in $\bar{M}_0$ by means of the extension $u(q) = u(x)$ if $\pi(q) = x$. Thus $\xi(u) = 0$ and hence

$$\bar{X}_i(u) = \bar{Y}_i(u) - \delta_i\xi(u) = \bar{Y}_i(u) = X_i(u).$$

Therefore, we have using (1) that

$$\begin{aligned} (\mathbf{I}_u)_*X_i &= \bar{X}_i(u)\xi + \bar{Y}_i \\ &= \bar{X}_i(u)\xi + (\delta_i\xi + \bar{X}_i) \\ &= \bar{X}_i(u)\xi - \bar{X}_i(s)\xi + \bar{X}_i \\ &= \bar{X}_i(u - s)\xi + \bar{X}_i \end{aligned}$$

Thus it is easy see that the unit normal vector field to Σ pointing upwards is

$$N = \frac{1}{W}(\xi - \hat{u}^j\bar{X}_j)$$

where $\hat{u}^j = \sigma^{ij}\bar{X}_i(u-s)$ and $W^2 = 1 + \sigma_{ij}\hat{u}^i\hat{u}^j = 1 + \hat{u}^i\hat{u}_i$ for $\hat{u}_i = \sigma_{ij}\hat{u}^j$. We extend N to $\bar{M}_0$ by the flow ξ.

Observe that if we define

$$Gu = \hat{u}^j X_j = \sigma^{ij}\bar{X}_i(u-s)X_j$$

then $-\dfrac{Gu}{W} = N^h$ and Lemma 3.2.1 gives $div_M\left(\pi\left(-\dfrac{Gu}{W}\right)\right) = nH$.

Also

$$\begin{aligned} Gu &= \sigma^{ij}\bar{X}_i(u)X_j - \sigma^{ij}\bar{X}_i(s)X_j \\ &= \sigma^{ij}X_i(u)X_j - \sigma^{ij}\bar{X}_i(s)X_j \\ &= \nabla u - \sigma^{ij}\bar{X}_i(s)X_j. \end{aligned}$$

Notice that $\sigma^{ij}\bar{X}_i(s)X_j$ are defined om M since they are independent of s. Hence for two sections $u, v : G(u) - G(v) = \nabla u - \nabla v$.

Notice that $\sigma^{ij}\bar{X}_i(s)X_j$ are defined on M since they are independent of s. Hence if u and v are sections with the same mean curvature then $Gu - Gv = \nabla u - \nabla v$.

3.4 Some Results

We will now prove the removable singularities theorem. First, a lemma.

Lemma 3.4.1. *Let u and v be functions in $C^2(\Omega)$. Then*

$$\left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v}\right\rangle_M = \left(\frac{W_u + W_v}{2}\right)\|N_u - N_v\|^2_{\bar{M}} \geq 0 \tag{3.2}$$

where $W_u^2 = 1 + \|Gu\|^2_M$ and $W_v^2 = 1 + \|Gu\|^2_M$, with equality at a point if and only if, $\nabla u = \nabla v$.

Proof. We know that $Gu = \hat{u}^i X_i$, $Gv = \hat{v}^j X_j$, $W_u^2 = 1 + \hat{u}_i\hat{u}^i$, $W_v^2 = 1 + \hat{v}_j\hat{v}^j$, $N_u = \dfrac{1}{W_u}\left(\xi - \hat{u}^i Y_i\right)$, $N_v = \dfrac{1}{W_v}\left(\xi - \hat{v}^j Y_j\right)$ and

$$\left\langle \bar{X}_i, \bar{X}_j\right\rangle_{\bar{M}} = \left\langle X_i, X_j\right\rangle_M.$$

Thus

$$\begin{aligned} \langle N_u - N_v, N_u - N_v\rangle_{\bar{M}} &= \langle N_u, N_u\rangle_{\bar{M}} + \langle N_v, N_v\rangle_{\bar{M}} - 2\langle N_u, N_v\rangle_{\bar{M}} \\ &= 2 - 2\langle N_u, N_v\rangle_{\bar{M}} \\ &= 2 - \frac{2}{W_u W_v}\left(1 + \hat{u}_i\hat{v}^i\right). \end{aligned}$$

Then

$$\hat{u}_i \hat{v}^i = \left(W_u W_v - \frac{W_u W_v}{2} \, \|N_u - N_v\|^2_{\bar{M}} \right) - 1. \tag{3.3}$$

Futher

$$\begin{aligned}
\left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle_M &= \frac{\hat{u}_i \hat{u}^i}{W_u} - \frac{\hat{u}_i \hat{v}^i}{W_v} - \frac{\hat{u}_i \hat{v}^i}{W_u} - \frac{\hat{v}_i \hat{v}^i}{W_v} \\
&= \frac{W_u^2 - 1}{W_u} - \hat{u}_i \hat{v}^i \left(\frac{1}{W_u} + \frac{1}{W_v} \right) + \frac{W_v^2 - 1}{W_v} \\
&= \frac{W_u + W_v}{2} \, \|N_u - N_v\|^2_{\bar{M}} \geq 0
\end{aligned}$$

by (3.3). Thus equality yields $N_u = N_v$ and this implies that $\nabla u = \nabla v$ since $G_u - G_v = \nabla u - \nabla v$

□

Theorem 3.4.2. *Let $u : \Omega - \{p\} \to \mathbb{R}$, $\Omega \subset M$, be a function whose Killing graph has prescribed mean curvature H. Then u extends smoothly to a solution at p.*

Proof. Let R be small so that there exists a smooth function v defined on $B_R(p)$, with:

$$\begin{cases} div_M \dfrac{Gv}{W} = nH, & \text{in} \quad B_R(p), \\ \qquad\quad v = u, & \text{in} \quad \partial B_R(p), \end{cases}$$

This exists by [10].

Let C be a positive constant. Define

$$\varphi = \begin{cases} u - v, & \text{if} \quad |u - v| < C, \\ \quad C, & \text{if} \quad |u - v| \geq C, \end{cases}$$

Then, φ is Lipschitz and $\nabla \varphi = \nabla u - \nabla v = Gu - Gv$ in the set $|u - v| < C$ and $\nabla \varphi = 0$ in the complement of this set. We have for $0 < r < R$

$$\int_{\partial A(r,R)} \varphi \left\langle \frac{Gu}{W_u} - \frac{Gv}{W_v}, \nu \right\rangle \leq 2C vol(S_r),$$

where $W_u = \sqrt{1 + \|Gu\|^2}$, $W_v = \sqrt{1 + \|Gv\|^2}$ and $vol(S_r)$ is the volume of $S_r = \partial B_r(p)$.

Since the Killing graphs of u and v have the same mean curvature, we have

$$\begin{aligned} div_M\, \varphi\left(\frac{Gu}{W_u}-\frac{Gv}{W_v}\right) &= \left\langle \nabla\varphi, \frac{Gu}{W_u}-\frac{Gv}{W_v}\right\rangle + \varphi div_M\left(\frac{Gu}{W_u}-\frac{Gv}{W_v}\right) \\ &= \left\langle \nabla\varphi, \frac{Gu}{W_u}-\frac{Gv}{W_v}\right\rangle \\ &= \left\langle \nabla u - \nabla v, \frac{Gu}{W_u}-\frac{Gv}{W_v}\right\rangle \\ &= \left\langle Gu - Gv, \frac{Gu}{W_u}-\frac{Gv}{W_v},\right\rangle. \end{aligned}$$

on $|u-v| < C$. By Stokes Theorem, we have

$$\int_{A(r,R)} div_M\, \varphi\left(\frac{Gu}{W_u}-\frac{Gv}{W_v}\right) = \int_{\partial A(r,R)} \varphi\left\langle \frac{Gu}{W_u}-\frac{Gv}{W_v}, \nu\right\rangle \leq 2Cvol(S_r). \qquad (3.4)$$

Futhermore, by Lemma 4.1, we get

$$\begin{aligned} div_M\, \varphi\left(\frac{Gu}{W_u}-\frac{Gv}{W_v}\right) &= \left\langle Gu - Gv, \frac{Gu}{W_u}-\frac{Gv}{W_v}\right\rangle_M \\ &= \frac{W_u+W_v}{2}\|N_u - N_v\|^2_{\bar{M}}, \end{aligned} \qquad (3.5)$$

when $|u-v| < C$ and $div_M\, \varphi\left(\frac{Gu}{W_u}-\frac{Gv}{W_v}\right) = 0$ when $|u-v| \geq C$. Thus we have, by (3.4) and (3.5) that

$$0 \leq \int_{A(r,R)\cap\{|u-v|<C\}} div_M\, \varphi\left(\frac{Gu}{W_u}-\frac{Gv}{W_v}\right) \leq 2Cvol(S_r)$$

As r decreases to zero we get that $N_u = N_v$ on the set $|u-v| < C$. Hence $Gu = Gv$ in the set $|u-v| < C$.

Since C was arbitrary, we have that $Gu = Gv$ in $A(0,R)$ and $u = v$ in $B_R(p) - \{p\}$. Thus $u = v$ in $B_R(p)$.

□

3.5 Asymptotic Properties of Sections $u, v : \Omega \to \bar{M}$ with the same prescribed mean curvature and equal on $\partial\Omega$.

In [9], Pascal Collin and Romain Krust studied graphs u, v over non compact domains $\Omega \subset \mathbb{R}^2$, that have the same mean curvature and with $u = v$ on $\partial\Omega$. They proved that when $u \neq v$, then $|u - v|$ must grow at least like $log(r)$, r radial distance in $\mathbb{R}^2$.

This theorem of Collin and Krust, and its technique of proof, have had many applications and generalizations.

We now show that their techniques apply to sections $u, v : \Omega \subset M \to \bar{M}$, provided the volume of Ω intersected with the geodesic spheres of M, grows at most linearly in the radius of the sphere. For example, when $\bar{M} =$ *Heisenberg Space* and $M =$ the flat $\mathbb{R}^2$, this is always the case.

Theorem 3.5.1. *Let $\Omega \subset M^n$ be a domain such that Ω intersects the boundary of each geodesic ball centered at a fixed point in a region whose volume is bounded by a constant times the radius, and u, v two $C^2(\Omega)$ functions such that their Killing graphs have the same mean curvature, $H(u) = H(v)$ in Ω and $u|\partial\Omega$ and $v|\partial\Omega$ are piecewise differentiable and coincide in the points of continuity. Let $M(r) = \sup_{\Lambda_r} |u - v|$ where $\Lambda_r = \Omega \cap \{x \in M; dist(x, a) = r\}$. Then $\liminf_{r\to\infty} \dfrac{M(r)}{\log r} > 0$ if $u \neq v$. If the volume of Λ_r is uniformly bounded then $\liminf_{r\to\infty} \dfrac{M(r)}{r} > 0$.*

Proof. First observe that if $\bar{\Omega}$ is compact then $u = v$ in Ω. For, in this case, one can move the graph Σ_u of u, by the flow ϕ_t of ξ, and $\phi_t(\partial\Sigma_u) \cap \partial\Sigma_u = \emptyset$ for $t \neq 0$. Choose a largest $|t| \neq 0$ such that $\phi_{|t|}(\Sigma_u) \cap \Sigma_v \neq \emptyset$. Then $\phi_{|t|}(\Sigma_u)$ and Σ_v touch at an interior point, hence they are equal by the maximum principle. This is a contradiction.

Recall that the unit normal to the graph Σ_u of u is written:

$$N_u = \frac{-Gu}{W_u} + \frac{1}{W_u}\xi,$$

where $-Gu$ is horizontal. Let X_u and X_v be the horizontal projections of $\dfrac{Gu}{W_u}$ and $\dfrac{Gv}{W_v}$ to M; also we will think of Gu and Gv as tangent to M.

We saw that

$$Gu - Gv = \nabla u - \nabla v$$

and

$$\left\langle Gu - Gv, \frac{Gu}{W_u} - \frac{Gv}{W_v} \right\rangle_{\bar{M}} = \langle \nabla u - \nabla v, X_u - X_v \rangle_M = \frac{(W_u + W_v)}{2} \|N_u - N_v\|^2_{\bar{M}}.$$

These equations are precisely what one needs to prove theorem 3.5.1 using the technique of Collin-Krust.

We begin the argument and we refer the reader to [9] for the completion of the proof.

Assuming $u \neq v$, we can suppose

$$A = \{x \in \Omega / u(x) > v(x)\}$$

is not bounded and connected.

Define $A_r = \{x \in A, dist(x,p) < r\}$ and $\Lambda_r = \{x \in A, dist(x,p) = r\}$. We will denote $vol(\Lambda_r)$ the volume of the Λ_r. Let r_0 such that $\mu = \displaystyle\int_{A_{r_0}} |X_u - X_v|^2 > 0$, where $X_u = \dfrac{Gu}{W_u}$, $X_v = \dfrac{Gv}{W_v}$ (μ exists since $u \neq v$ and $A_{r_0} \neq \emptyset$).

By Stokes theorem we have

$$\begin{aligned} \int_{\partial A_r} (u-v)\langle X_u - X_v, \nu\rangle &= \int_{A_r} div((u-v)(X_u - X_v)) \\ &= \int_{A_r} \langle \nabla u - \nabla v, X_u - X_v\rangle \\ &= \int_{A_r} \langle Gu - Gv, X_u - X_v\rangle . \end{aligned} \tag{3.6}$$

By Lemma 4.1 we get

$$\langle Gu - Gv, X_u - X_v\rangle_M = \frac{1}{2}(W_u + W_v)\,\|N_u - N_v\|^2_{\bar{M}} . \tag{3.7}$$

Since $\frac{1}{2}(W_u + W_v) \geq 1$. We have by (3.6) and (3.7) that

$$\int_{\Lambda_r} (u-v)\langle X_u - X_v, \nu\rangle = \int_{\partial A_r} (u-v)\langle X_u - X_v, \nu\rangle \geq \int_{A_r} |X_u - X_v|^2. \tag{3.8}$$

By (3.8) we have

$$\mu + \int_{A_r - A_{r_0}} |X_u - X_v|^2 \leq M(r)\eta(r), \tag{3.9}$$

where

$$\eta(r) = \int_{\Lambda_r} |X_u - X_v|.$$

By Schwartz's lemma, we have

$$\eta(r)^2 \leq vol(\Lambda_r) \int_{\Lambda_r} |X_u - X_v|^2. \tag{3.10}$$

Now the reader can read [9], for the completion of the argument.

□

Remark. For graphs of prescribed mean curvature in Heisenberg space, over domains $\Omega \subset \mathbb{R}^2$, we conclude there is at most one bounded solution of the mean curvature equation over Ω, with given boundary values. In particular, a bounded entire minimal graph is constant.

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