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Doctor of Philosophy Dissertation

Price Discrimination without the Single-Crossing

by

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Contents

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Chapter 1

Nonlinear Pricing

1.1 Introduction

Nonlinear pricing schemes are widely used in many markets such as mobile telephony, fixed telephony, electricity and gas supply, postal services, air and railroad transport, cable TV and internet, hardware and software, cars. For this reason, it has attracted a lot of attention from both theoretical and applied economists.

Our research is about the design of an optimal nonlinear pricing mechanism for a monopolist who has incomplete information about the characteristics of the customers. Although there are theoretical results available in the literature, some of them are valid only under restrictive conditions, for instance the Spence and Mirrlees condition (SMC). Our goal is to take some steps in relaxing some of these conditions.

For a deeper understanding of these problems, however, much more have to be done. We believe that a more general treatment requires new mathematical tools which are still to be developed. This seems to be especially true when we are in a multidimensional context.

In this chapter we make a brief survey of the literature and the objective is to place our research in the proper context. It is organized as follows. In Section 2 we present the model we are using. Then, in Section 3 we study the problem in a one-dimensional context. First we consider it under the Spence and Mirrlees condition (SMC), and we describe three different approaches for solving it. Then we present a model relaxing the SMC. In Section 4 we present some results from the literature of multidimensional screening problems. Finally, in Section 5 we present the directions undertaken by our research.

1.2 Model

The customer has a quasi-linear utility function depending on his type $\theta \in \Theta \subseteq \mathbb{R}^N$, which is private information:

$$V(q, t, \theta) = v(q, \theta) - t,$$

where $q \in Q \subseteq \mathbb{R}_+^M$ is the customer's decision and t is the monetary transference.

The firm is a profit-maximizing monopolist producing any $q \in Q \subseteq \mathbb{R}_+^M$ and incurring in a cost $C(q)$. It does not observe θ and has a prior distribution P over Θ. His revenue is also quasi-linear:

$$\Pi(q, t) = t - C(q).$$

We formulate the monopolist's problem as choosing a decision function $q : \Theta \to Q$ and a monetary transfer $t : \Theta \to \mathbb{R}$ that solves:

$$\max_{\{q(\cdot), t(\cdot)\}} \int_\Theta \{t(\theta) - C(q(\theta))\} dP(\theta), \tag{Π}$$

subject to the *individual rationality* constraints:

$$v(q(\theta), \theta) - t(\theta) \geq 0 \quad \forall \theta \in \Theta, \tag{IR}$$

and the *incentive compatibility* constraints:

$$\theta \in \arg\max_{\theta' \in \Theta} \{v(q(\theta'), \theta) - t(\theta')\}, \quad \forall \theta \in \Theta. \tag{IC}$$

The allocation rule represented by the pair (q, t) is called a contract. We say that a contract (q, t) is incentive compatible when it satisfies the IC constraints. The decision function q is implementable if there exists a monetary transfer t such that (q, t) is incentive compatible. Hammond [11] and Guesnerie [9] have showed that a contract is incentive compatible if and only if there exists a nonlinear tariff $T : Q \to \mathbb{R}$ such that:

$$q(\theta) \in \arg\max_{q \in Q} \{v(q, \theta) - T(q)\}, \quad \forall \theta \in \Theta.$$

This is known as the *'Taxation Principle'* and it easily follows from the fact that t depends on θ only through the decision $q(\theta)$.

One of the greatest difficulties related to the monopolist's problem is how to deal with the IC constraints. In general, the binding IC constraints may be determined only endogenously which makes it a rather difficult task. There are some technical results about the characterization of incentive compatible decisions in Rochet [25] which uses the concept of cyclical monotonicity and Carlier [6] who uses abstract convexity concepts. Without imposing further structure on the utility function $v(q, \theta)$, however, no much can be said about implementability.

In the unidimensional (N=M=1) case, for instance, there is a condition that provides a very simple characterization of implementable decisions:

Definition 1. ***(The Spence and Mirrlees Condition.)*** *Suppose that* $v : Q \times \Theta \to \mathbb{R}$ *is twice differentiable. Then* v *satisfies SMC if and only if only one of the two possibilities below is valid:*

$$\begin{cases} v_{q\theta} > 0 \text{ on } Q \times \Theta & (CS_+) \\ v_{q\theta} < 0 \text{ on } Q \times \Theta & (CS_-). \end{cases} \tag{SMC}$$

The SMC, also known as the single-crossing condition, is an standard assumption in the literature of monopolistic screening. Under the SMC a decision function is implementable if and only if it is monotonic.[1] The monotonicity of the decision function is related to the first- and second-order conditions of the customer's maximization problem (IC). Consequently, under the SMC the first- and second-order conditions are necessary and sufficient for implementability.[2]

In the multidimensional case, when the monopolist sells a single good and the customer has multidimensional type, there is the generalized single-crossing condition defined by McAfee and McMillan [17]:

Definition 2. ***(The Generalized Single-Crossing Condition.)*** *Suppose that* $v : Q \times \Theta \to \mathbb{R}$ *is twice differentiable. Then* v *satisfies the generalized single-crossing property if for all* q, θ_1 *and* θ_0 *there exists* $\lambda(q, \theta_1, \theta_0) > 0$ *such that:*

$$v_q(q, \theta_1) - v_q(q, \theta_0) = \lambda \nabla_\theta v_q(q, \theta_0) \cdot (\theta_1 - \theta_0). \tag{GSC}$$

[1] See Rochet [25].

[2] See Guesnerie and Laffont [10].

Under this condition, we also have that the first- and second-order conditions are necessary and sufficient for implementability. This condition is too restrictive as it imposes that the set of types who choose the same allocation is a linear subspace. In the third chapter, we find an example of optimal contract that does not satisfy the GSC.

1.3 Unidimensional Screening

In this section we present some methods for solving the monopolist's problem (Π) under the assumption of a single characteristic ($N = 1$) and a single instrument ($M = 1$). The type space is $\Theta = [\underline{\theta}, \overline{\theta}]$. The common strategy for all these methods is to derive a relaxed problem for the monopolist. Then, if we find a solution for the relaxed problem that satisfies the constraints of the original problem, then this solution is also a solution for the original problem.

1.3.1 Unidimensional Screening under the SMC

We present three approaches for monopolist's problem under the (SMC). We assume the following conditions:

H1. $v(q, \theta) \in C^3$,

H2. $v_{qq} < 0$ and $v_\theta > 0$,[3]

H3. $v_{q^2\theta} > 0$ and $v_{q\theta^2} > 0$,

H4. $C(0) = 0, C'(q) > 0$ and $C''(q) < 0$.

We also assume the (CS_+). In this case, a decision q is implementable if and only if it is increasing.[4]

Mussa and Rosen's approach:

One of the seminal papers in the field of nonlinear pricing is by Mussa and Rosen [20]. They study monopoly pricing problems involving a quality-differentiated spectrum of goods. We are going to reproduce their derivation for the monopolist's problem. Suppose that (q, t) is incentive compatible and define the informational rent by:

$$V(\theta) = v(q(\theta), \theta) - t(\theta).$$

[3] When the signal of v_θ is not constant and/or when the individual rationality (IR) constraints are type dependent, we have countervailing incentive problem. This problem was studied by Jullien [13]. In Araujo and Moreira [1] there is a discussion about the possibility of countervailing incentives due to the absence of the SMC.

[4] The term increasing refers to both weakly and strictly increasing functions.

If we plug V into the monopolist's objective function of problem (Π), we get:

$$\int_{\underline{\theta}}^{\overline{\theta}} \{v(q(\theta), \theta) - C(q(\theta)) - V(\theta)\} d\theta. \tag{1.1}$$

By the envelope theorem (see Milgrom and Segal [18]), we have that:

$$V'(\theta) = v_\theta(q(\theta), \theta).$$

Using assumption **H2** and the fundamental theorem of calculus, we get:[5]

$$V(\theta) = \int_{\underline{\theta}}^{\theta} v_\theta(q(\theta), s) ds. \tag{1.2}$$

Plugging equation (1.2) into equation (1.1) and using an integration by parts we finally get:

$$\int_{\underline{\theta}}^{\overline{\theta}} \{v(q(\theta), \theta) - C(q(\theta)) + \frac{P(\theta) - 1}{p(\theta)} v_\theta(q(\theta), \theta)\} p(\theta) d\theta. \tag{1.3}$$

Let us define:

$$f(q, \theta) = v(q(\theta), \theta) - C(q(\theta)) + \frac{P(\theta) - 1}{p(\theta)} v_\theta(q(\theta), \theta). \tag{1.4}$$

Dropping the global IC constraints, we can define a relaxed version of the monopolist's problem:

$$\max_{\{q(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} f(q(\theta), \theta) p(\theta) d\theta. \tag{Π_{MR}}$$

This is a classical problem in the calculus of variations.[6]The Euler's equation gives the necessary condition for an extremum of problem (Π_{MR}):

$$f_q(q, \theta) = 0. \tag{1.5}$$

Let us denote the solution of equation (1.5) by $q_{MR}(\theta)$. If $q_{MR}(\theta)$ is increasing, then it is the solution of the monopolist's problem (Π). If $q_{MR}(\theta)$ is not increasing, then we must use the *ironing procedure* on $q_{MR}(\theta)$ to find the solution.

In the Figure 1.1 we illustrate the *ironing procedure*. Notice that $q_{MR}(\theta)$ is decreasing in the interval $[\theta_M, \theta_m]$. The procedure consists in optimally choosing an interval $I = [\theta_1, \theta_2]$ such that the optimal decision $q^*(\theta) = q_b, \quad \forall \theta \in I$. In this solution, we have a bunching of

[5]By assumption **H2**, V is increasing and we can set $V(\underline{\theta}) = 0$, eliminating the IR constraints.

[6]We suggest Petrov [22] as a basic reference for calculus of variations and optimal control.

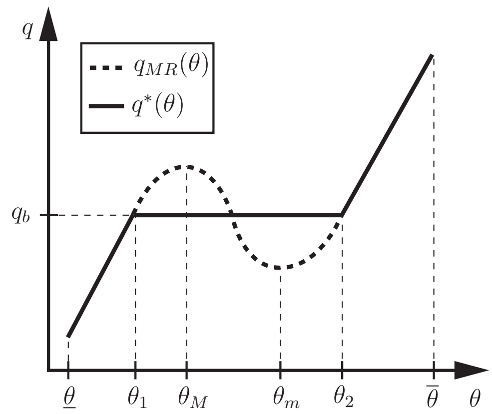

Figure 1.1: The ironing procedure.

customers choosing the same decision q_b. Using optimal control, the optimal bunching can be characterized by:

$$\int_{\theta_1}^{\theta_2} f_q(q_b, \theta)p(\theta)d\theta = 0.$$

Goldman, Leland and Sibley's approach:

The second approach we are going to present was introduced by Goldman, Leland and Sibley [8]. They study nonlinear pricing through quantity discounts. Remember that under the (CS_+), an incentive compatible decision q must be increasing so we can define its pseudo-inverse $\psi(q)$. It is called the type assignment function, as it assigns quantities q to the type $\theta = \psi(q)$ choosing q. Notice that $\psi(q)$ must be also increasing. In this approach, the monopolist's objective function will depend on $\psi(q)$. Then the monopolist has to optimally choose $\psi(q)$.

Consider an incentive compatible contract (q, t). Using the *taxation principle*, we have that $T(q(\theta)) = t(\theta)$. Using the fundamental theorem of calculus, and supposing that $T(0) = C(0) = 0$,[7] we can write the monopolist's objective function of problem (Π) as:

$$\int_{\underline{\theta}}^{\overline{\theta}} \{T(q(\theta)) - C(q(\theta))\}p(\theta)d\theta \;\; = \int_{\underline{\theta}}^{\overline{\theta}}\int_0^{q(\theta)} \{T'(q) - C'(q)\}p(\theta)dqd\theta,$$

and using Fubinis' theorem and the customer's first-order condition:

$$\int_{\underline{\theta}}^{\overline{\theta}}\int_0^{q(\theta)} \{T'(q) - C'(q)\}p(\theta)dqd\theta \;\; = \;\; \int_0^{\overline{q}}\int_{\psi(q)}^{\overline{\theta}} \{v_q(q, \psi(q)) - C'(q)\}p(\theta)d\theta dq.$$

Let us define:

$$F(q, \psi) = \int_{\psi}^{\overline{\theta}} \{v_q(q, \psi) - C'(q)\}p(\theta)d\theta.$$

[7] This is a simplifying hypothesis.

Dropping the global IC constraints, now related to the monotonicity of the type assignment function $\psi(q)$, we can define a relaxed version of the monopolist's problem:

$$\max_{\{\psi(\cdot)\}} \int_0^{\overline{q}} F(q, \psi(q))dq. \tag{Π_{GR}}$$

The Euler's equation for the problem (Π_{GR}) is given by:

$$F_\psi(q, \psi(q)) = 0. \tag{1.6}$$

Let us denote by $\psi_{GR}(q)$ the solution of equation (1.6). If $\psi_{GR}(q)$ is increasing, then it is the solution of the monopolist's problem (Π). If $\psi_{GR}(q)$ is not increasing, then the bunching condition for the decision function q translates into an optimal jump condition for the optimal type assignment function ψ_*:[8]

$$F(q_b, \psi^*(q_b+)) = F(q_b, \psi^*(q_b-)) = \max_{\{\theta \in [\underline{\theta}, \overline{\theta}]\}} F(q_b, \theta).$$

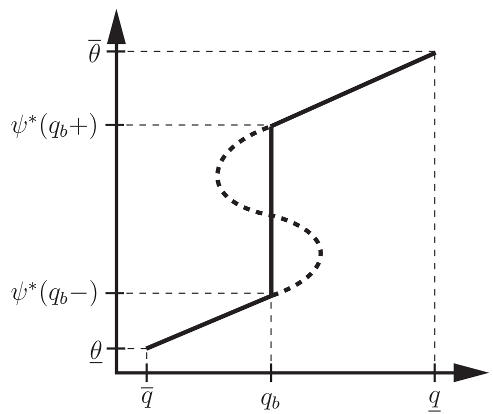

Figure 1.2: The optimal jump condition.

We will see that this approach provides a very convenient way to deal with the monopolist's problem without the SMC. The difference is that we will have two type assignment functions, because the decision functions may not be monotonic.

Demand Profile approach:

The demand profile approach was introduced by Brown and Sibley [5]. The derivation we do here, however, can be found in Wilson [29], which provides a detailed description of this

[8]This result was established by Nˆldeke and Samuelson [21].

approach, with many examples and applications to nonlinear pricing via quantity discounts. This approach results in an optimization problem depending on the marginal tariff $T'(q)$.

The demand profile is a function representing the fraction of customers purchasing at least q units when the monopolist chooses the tariff T:

$$\mathcal{N}(T(\cdot), q) = \Pr\left[\theta \in [\underline{\theta}, \overline{\theta}] | \exists x \in arg \max_{x \geq 0} \{v(x,\theta) - T(x)\} \text{ and } x \geq q\right].$$

In general $\mathcal{N}(T(\cdot), q)$ may depend on the entire tariff $T(\cdot)$ and this could make the problem rather complicated. However, it is possible to simplify the demand profile by assuming an aditional condition.

Definition 3. ***(Single-Crossing Tariff.)*** [9] *A tariff is said to be single-crossing at q if, for any θ such that $v_q(q,\theta) \geq T'(q)$, $v_q(q',\theta) \geq T'(q')$ for $q' \leq q$. If $T(q)$ is single-crossing for all q, we term it a single-crossing tariff.*

When we have a single-crossing tariff, the demand profile depends only on the marginal tariff $T'(q)$. In this case it is denoted by $N(T'(q), q)$ and we have that:

$$N(T'(q), q) = \Pr\left[\theta \in \Theta | v_q(q,\theta) \geq T'(q)\right].$$

The derivation of the monopolist's optimization problem resembles the previous one from Goldman, Leland and Sibley. Using the fundamental theorem of calculus, Fubini's theorem and supposing that $T(0) = C(0) = 0$, we have that:

$$\begin{aligned}
\int_{\underline{\theta}}^{\overline{\theta}} \{T(q(\theta)) - C(q(\theta))\} p(\theta) d\theta &= \int_{\underline{\theta}}^{\overline{\theta}} \int_0^{q(\theta)} \{T'(q) - C'(q)\} dq p(\theta) d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \int_0^{\infty} \{T'(q) - C'(q)\} 1_{\{q \leq q(\theta)\}} dq p(\theta) d\theta \\
&= \int_0^{\infty} \{T'(q) - C'(q)\} \Pr(q \leq q(\theta)) dq.
\end{aligned}$$

Finally, if T is a single-crossing tariff, then $\Pr(q \leq q(\theta)) = N(T'(q), q)$, so we can write the monopolist's objective function as:

$$\int_0^{\infty} \{T'(q) - C'(q)\} N(T'(q), q) dq.$$

[9] This definition was taken from [8].

Again, we define the relaxed version of the monopolist's problem:

$$\max_{\{T'(\cdot)\}} \int_0^\infty \{T'(q) - C'(q)\} N(T'(q), q) dq. \tag{Π_{DR}}$$

The Euler's equation for the problem (Π_{DR}) is given by:

$$N(T'(q), q) + \{T'(q) - C'(q)\} N_1(T'(q), q) = 0, \tag{1.7}$$

where N_1 denotes the derivative of N with respect to its first variable.

We have to check *ex-post* whether the optimal marginal tariff $T'(q)$ is single-crossing. In this formulation, bunching emerges as the result of jumps in the optimal marginal tariff $T'(q)$ obtained from equation (1.7).

1.3.2 Unidimensional Screening without the SMC

Without the SMC the problem becomes rather complex. Unlike the previous case, a decision function satisfying the first- and second-order conditions of the customer's maximization problem may not be implementable.

Araujo and Moreira [1] introduced a model where the SMC is relaxed to:

AM1 $v_{q\theta}(\xi, \theta) = 0$ *defines a decreasing function* $q_0 : \theta \to \mathbb{R}_+$ *such that:*

$$\forall \theta \in \Theta \quad v_{q\theta}(\xi, \theta) \geq 0 \Leftrightarrow \xi \geq q_0(\theta). \tag{1.8}$$

So the curve q_0 divides the (θ, q) plane in two regions: CS_+, where $v_{q\theta} > 0$ and CS_-, where $v_{q\theta} < 0$.

Using the first- and second-order conditions, one can show that an implementable decision must be increasing in CS_+ and decreasing in CS_- so the typical decision function will have a U-shaped form. This shape allows that two distinct customers θ_1 and θ_2 choose the same decision q. Analysing the first-order conditions for the pooling types θ_1 and θ_2, we have another necessary condition for implementability imposing that their marginal utility from choosing q must be the same: $v_q(q, \theta_1) = v_q(q, \theta_1)$. This identity between the marginal utilities of pooling types is called the *U-condition* (UC). The derivation consists in incorporating the UC into the monopolist's objective function. The Goldman, Leland ans Sibley's approach is convenient for this task, because it allows the UC to be treated separetely for each q. The difference is that now we will have two type assignment functions $\psi_b(q)$ and $\psi_s(q)$. We are going to adapt their

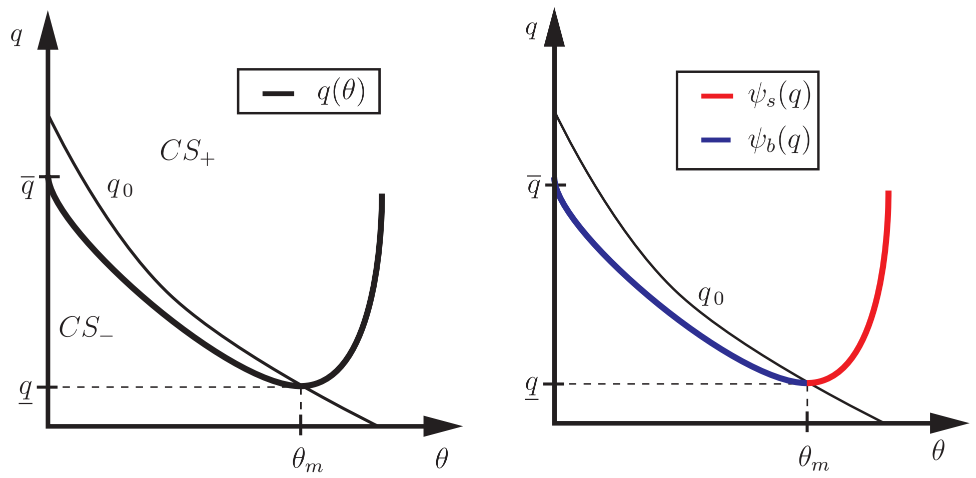

Figure 1.3: The decision function q and its type assignment functions.

derivation to Araujo and Moreira's setup. Considering and implementable decision q, we can write the monopolist's objective function as:

$$\int_{\underline{\theta}}^{\overline{\theta}} f(q(\theta),\theta)p(\theta)d\theta, \tag{1.9}$$

where $f(q,\theta)$ is define by equation (1.4).

Using the fundamental theorem of calculus and the Fubini's theorem, we can write:

$$\begin{aligned}\int_{\underline{\theta}}^{\overline{\theta}} f(q(\theta),\theta)p(\theta)d\theta &= \int_{\underline{\theta}}^{\overline{\theta}}\left\{\int_0^{q(\theta)} f_q(q,\theta)dq\right\}p(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} f(0,\theta)p(\theta)d\theta\\ &= \int_0^{\overline{q}}\{F(q,\psi_s(q)) - F(q,\psi_b(q))\}dq + \int_{\underline{\theta}}^{\overline{\theta}} f(0,\theta)p(\theta)d\theta,\end{aligned}$$

where F is defined as:

$$F(q,\psi) = \int_{\underline{\theta}}^{\psi} f_q(q,\theta)p(\theta)d\theta.$$

So we can define a relaxed version of the monopolist's optimization problem as:[10]

$$\begin{aligned}\max_{\{\psi_b(\cdot),\psi_s(\cdot)\}} &\int_0^{\overline{q}}\{F(q,\psi_s(q)) - F(q,\psi_b(q))\}dq + \int_{\underline{\theta}}^{\overline{\theta}} f(0,\theta)p(\theta)d\theta,\\ s.t. \quad & v_q(q,\psi_b(q)) - v_q(q,\psi_s(q)) \le 0.\end{aligned} \tag{Π_{AMR}}$$

With an abuse of notation the constraint in problem (Π_{AMR}) is called UC. The Euler's equation of problem above depends on whether the UC constraint is binding or not:[11]

[10] In Araujo and Moreira there are two distinct classes of solutions, one considering the constraint $v_q(q,\psi_b(q)) - v_q(q,\psi_s(q)) \le 0$ and the other considering $v_q(q,\psi_b(q)) - v_q(q,\psi_s(q)) \ge 0$. We restrict our description to the first kind of solution.

[11] We refer the reader to Araujo and Moreira [1] for more details.

- If UC is slack and $\underline{\theta} < \psi_b(q) < \psi_s(q) = \overline{\theta}$ then:

$$f_q(q, \psi_b(q)) = 0,$$

- If UC is binding and $\underline{\theta} < \psi_b(q) < \psi_s(q) < \overline{\theta}$ then:

$$\frac{f_q(q, \psi_s(q))}{v_{q\theta}(q, \psi_s(q))} = \frac{f_q(q, \psi_b(q))}{v_{q\theta}(q, \psi_b(q))},$$

- If the condition UC is binding and $\underline{\theta} < \psi_b(q) < \psi_s(q) = \overline{\theta}$ then:

$$v_q(q, \psi_b(q)) = v_q(q, \overline{\theta}).$$

The optimal solution will be obtained as a combination of the pointwise conditions above. In the appendix show that the tariff that implements the solutions in Araujo and Moreira [1] is single-crossing. This means that the demand profile approach could have been used extended to this non single-crossing setup. As far as we know, this methodology has never been used for this kind of problem. Under the demand profile approach, however, the drawback is that the computations are more complicated.

1.4 Multidimensional Screening

As we saw in the previous section, the solution of the monopolist's optimization problem is straightforward in the unidimensional case under the SMC. In the multidimensional case, however, the characterization of the optimal contract is a very complex problem. One of the main difficulties refers to the fact that the binding IC constraints may be determined only endogenously and we don't have a satisfatory way of incorporating them into the maximization problem. So there are only partial responses for this problem.[12]

Matthews and Moore [16] studied the monopolistic screening problem with a two-dimensional product space, representing the quality spectrum and the warranty. The customers are represented by a discrete parameter. Their goal was to extend Mussa and Rosen [20], and they observed that some global IC constraints may be binding in the optimal decision.

[12] We recommend the excellent survey in Multidimensional Screening by Rochet and Stole [27].

McAfee and McMillan [17] studied the problem of a multi-product monopolist who faces a customer with multidimensional characteristics with $(M \leq N)$. They introduced the *Generalized Single-Crossing* (GSC)[13], a condition under which the first- and second-order conditions for the customer's problem are necessary and sufficient for implementability. They also characterized the optimal contract when $(M = 1, N \geq 1)$ under the (GSC).

Armstrong [2] studied the problem of multiproduct monopolist who faces customer's with multidimensional characteristics $(N \geq 1, M \geq 1)$. He gave closed form solutions for some examples in this multidimensional context. One of his main contributions was to discover the exclusion property. This property states that the optimal contract leaves a positive measure set of customers excluded from consumption.

Rochet and ChonÈ [26] established the existence of the optimal contract and provided the characterization in the case of a multiproduct monopolist who faces a customer with multidimensional characteristics. They assume the number of products and characteristics to be the same $(N = M \geq 1)$ and also that the parametrization of customer's preferences is linear with types. They introduced the sweeping procedure as a generalization of the ironing procedure for dealing with bunching in the multidimensional context.

Basov [3] introduced the Hamiltonian approach as a tentative of generalizing Rochet and ChonÈ [26] to the case when the number of products and characteristics may be different $(N \geq 1, M \geq 1)$. Later on, these techniques were extended in his book [4] to deal with more general customer's preferences.

Parallel to this literature, there were some works analysing the existence of a solution for the monopolist's maximization problem (Π). We highlight the papers by Monteiro and Page [19] and by Carlier [6]. The former uses compactness properties resulting from budget constraints considerations and the latter using direct methods and concepts of abstract convexity. However, these approaches do not actually provide any recipe for the characterization of the optimal contract.

[13] The GSC was defined on Section 1.2.

1.5 Final Remarks

Our research concerns the problem of a single-product monopolist who faces a customer with two dimensions of heterogeneity. We assume neither the Spence and Mirrlees condition nor the Generalized Single-Crossing.

In the second chapter we study this problem in a particular context. In our model these two dimensions have antagonic effects in the decision q. We further assume that they are dependent in a way that we can reduce the problem into a one-dimensional screening model. The Spence and Mirrlees condition, however, is no longer valid because there is only one dimension to cope with the antagonic effects of the original parameters.

Comparing our results with Araujo and Moreira [1] we present some advances. The optimal decision we found Pareto dominates theirs and besides it is not implemented by a single-crossing tariff. We were able to incorporate a new global IC constraint into the monopolist's problem, and this gives rise to a new optimality condition for this problem. This condition is new in the literature of monopolistic screening.

In the third chapter we extend the techniques developed by Araujo and Moreira [1] to a bidimensional screening context. This chapter is divided in two main sections contemplating two cases.

In the first one, called discrete-continuous, one of the customer's characteristic is discrete and the other is continuous. We derived optimality conditions analogous to the ones in [1] and then we show how to use them to construct an optimal contract.

In the second one, called bicontinuous, both customers' characteristics are continuous. This case is close to McAfee and McMillan [17] generalization of Laffont, Maskin and Rochet [14]. The difference is that we did not assume their *Generalized Single-Crossing* (GSC). We solved an example where the (GSC) is not valid.

Appendix

Our goal is to prove that the optimal decisions in Araujo and Moreira [1] is implemented by a single-crossing tariff. In all that follows $\mathcal{R}(q)$ denotes the range of the function q. We begin defining the pseudoinverse of a monotonic function:

Definition 4. *Consider a monotonic function $q : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ along with its upper contour set $S^{q(\cdot)}(q) = \{\theta \in [\underline{\theta}, \overline{\theta}] : q(\theta) \geq q\}$.*
If q is decreasing, we define $\psi_s : \mathbb{R} \to [\underline{\theta}, \overline{\theta}]$ as:

$$\psi_s(q) = \begin{cases} \sup S(q), & \text{if } S(q) \neq \emptyset, \\ \underline{\theta}, & \text{if } S(q) = \emptyset. \end{cases}$$

If q is increasing, we define $\psi_b : \mathbb{R} \to [\underline{\theta}, \overline{\theta}]$ as:

$$\psi_b(q) = \begin{cases} \inf S(q), & \text{if } S(q) \neq \emptyset, \\ \overline{\theta}, & \text{if } S(q) = \emptyset. \end{cases}$$

In both cases, ψ_b and ψ_s are monotonic and they are denoted the pseudoinverses of q.

Then we establish the following result:

Lemma 1. *Consider a monotonic function $q : [a, b] \to \mathbb{R}$ along with its pseudoinverse $\psi(\cdot)$. For all x in the range of q, we have that:*

$$\begin{array}{ll} (i) & q(\theta) \geq x \Leftrightarrow \theta \geq \psi(x), \text{ if } q(\cdot) \text{ is increasing;} \\ (ii) & q(\theta) \geq x \Leftrightarrow \theta \leq \psi(x), \text{ if } q(\cdot) \text{ is decreasing.} \end{array}$$

Proof. We will only prove *(i)*:
($\Leftarrow$) It follows from the increasingness of q.
($\Rightarrow$) It follows from the definition of the pseudoinverse.
We can do the same for proving *(ii)*.

□

As a direct consequence of Lemma 1, we can state:

Lemma 2. *Consider $q : [a,b] \to \mathbb{R}$. Suppose that there exists $\theta_m \in (a,b)$, point of continuity of q, such that the restrictions, $q_b = q|_{[a,\theta_m]}$ and $q_s = q|_{[\theta_m,b]}$, are monotonic, with q_b decreasing and q_s increasing. Let ψ_b and ψ_s be their pseudoinverses. We have:*

$$\begin{aligned} (i) &\quad \text{If } x \in \mathcal{R}(q_b) \cap \mathcal{R}(q_s), \text{ then } q(\theta) \geq x \Leftrightarrow \theta \leq \psi_b(x) \text{ or } \theta \geq \psi_s(x), \\ (ii) &\quad \text{If } x \in \mathcal{R}(q_s) \cap \mathcal{R}(q_b)^c, \text{ then } q(\theta) \geq x \Leftrightarrow \theta \geq \psi_s(x), \\ (iii) &\quad \text{If } x \in \mathcal{R}(q_b)^c \cap \mathcal{R}(q_s), \text{ then } q(\theta) \geq x \Leftrightarrow \theta \leq \psi_b(x). \end{aligned}$$

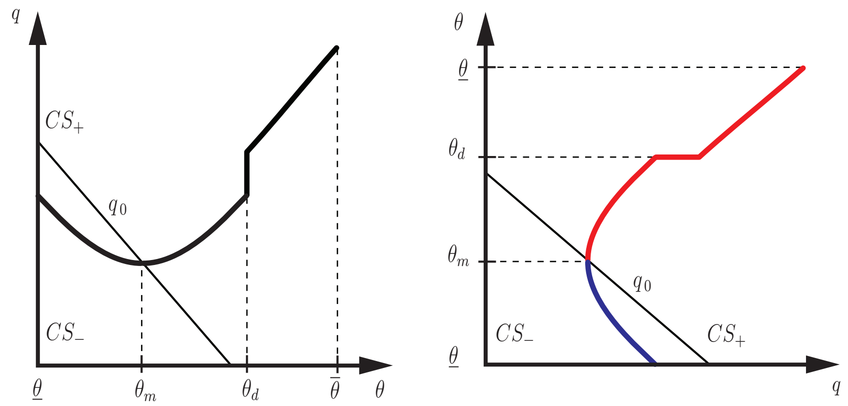

Figure 1.4: The (DD) decision function.

We divide the optimal decisions in Araujo Moreira in two kinds; the first one is denoted the discontinuous decision (DD) and the second one is denoted the continuous decision (CD).

The discontinuous decision (DD) is characterized by:

(1a) $q(\underline{\theta}) < q(\overline{\theta})$,

(1b) $\exists\ \theta_m \in (\underline{\theta}, \overline{\theta})$ continuity point of q, such that:

$$\begin{aligned} &(\theta, q(\theta)) \subseteq CS_- \text{ if } \underline{\theta} \leq \theta \leq \theta_m, \\ &(\theta, q(\theta)) \subseteq CS_+ \text{ if } \theta_m \leq \theta \leq \overline{\theta}, \\ &q_b = q|_{[\underline{\theta}, \theta_m]} \text{ is decreasing}, \\ &q_s = q|_{[\theta_m, \overline{\theta}]} \text{ is increasing}, \end{aligned}$$

(1c) q is discontinuous at $\theta_d \in (\theta_m, \overline{\theta})$,

(1d) $\psi_b : [q(\theta_m), q(\underline{\theta})] \to [\underline{\theta}, \theta_m]$ is continuous, (1e) $\psi_s : [q(\theta_m), q(\overline{\theta})] \to [\theta_m, \overline{\theta}]$ is continuous,

(1f) $v_q(q, \psi_b(q)) = v_q(q, \psi_s(q)) \quad \forall q \in [q(\theta_m), q(\underline{\theta})]$,

(1g) $v_q(q, \underline{\theta}) \leq v_q(q, \psi_s(q)) \quad \forall q \in (q(\underline{\theta}), q(\overline{\theta})]$.

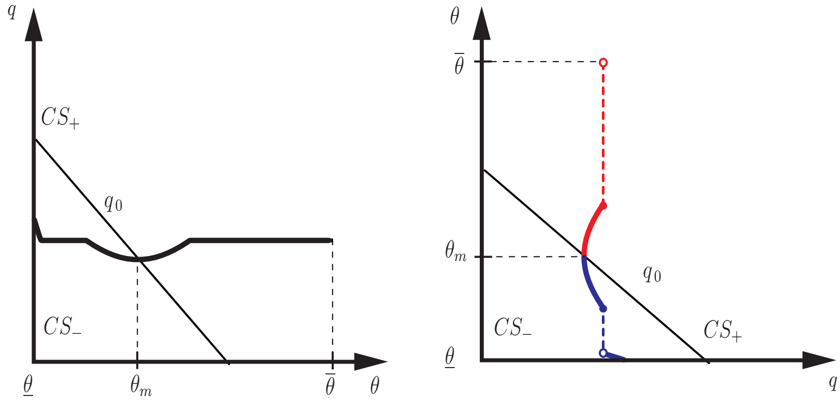

Figure 1.5: The (CD) decision function.

The second is denoted the continuous decision (CD), characterized by:

(2a) $q(\underline{\theta}) > q(\overline{\theta})$,

(2b) $\exists \ \theta_m \in (\underline{\theta}, \overline{\theta})$ continuity point of q, such that:

$$\begin{aligned} &(\theta, q(\theta)) \subseteq CS_- \ \textit{if} \ \underline{\theta} \leq \theta \leq \theta_m, \\ &(\theta, q(\theta)) \subseteq CS_+ \ \textit{if} \ \theta_m \leq \theta \leq \overline{\theta}, \\ &q_b = q|_{[\underline{\theta}, \theta_m]} \ \textit{ is decreasing}, \\ &q_s = q|_{[\theta_m, \overline{\theta}]} \ \textit{ is increasing}, \end{aligned}$$

(2c) q is continuous,

(2d) $\psi_b : [q(\theta_m), q(\underline{\theta})] \to [\underline{\theta}, \theta_m]$ is continuous except on $q(\overline{\theta})$,

(2e) $\psi_s : [q(\theta_m), q(\overline{\theta})] \to [\theta_m, \overline{\theta}]$ is continuous except on $q(\overline{\theta})$,

(2f) $v_q(q, \psi_b(q)) = v_q(q, \psi_s(q)) \quad \forall q \in [q(\theta_m), q(\overline{\theta})]$,

(2g) $v_q(q, \psi_b(q)) \leq v_q(q, \overline{\theta}) \quad \forall q \in (q(\overline{\theta}), q(\underline{\theta})]$.

We have the following result for the optimal decisions in Araujo and Moreira: For a DD solution, we have the followinf result.

Proposition 1. *Let q be a decision and $T : \mathcal{R}(q) \to \mathbb{R}$ be the tariff that implements it. If q is (DD) or (CD), then T is a single-crossing tariff.*

Proof. We will only prove the case (DD). The other case is analogous. If we consider the agent's maximization problem, the first-order conditions gives:

$$T'(q) = v_q(q, \psi_s(q)).$$

Assumption (AM1), items (1f) and (1g) imply that:

$$\begin{aligned} \forall q \in [q(\theta_m), q(\underline{\theta})] \quad v_q(q,\theta) \geq T'(q) \quad &\Leftrightarrow \quad \theta \leq \psi_b(q) \text{ or } \theta \geq \psi_s(q), \\ \forall q \in (q(\underline{\theta}), q(\overline{\theta})] \quad v_q(q,\theta) \geq T'(q) \quad &\Leftrightarrow \quad \theta \geq \psi_s(q). \end{aligned}$$

Finally, by Lemma 2, we conclude that:

$$q(\theta) \geq q \Leftrightarrow v_q(q,\theta) \geq T'(q)$$

And the result follows.

□

Chapter 2

Nonlineard Pricing Without a Single-Crossing Tariff

2.1 Introduction

Nonlinear pricing schemes are widely used in many markets such as mobile telephony, fixed telephony, electricity and gas supply, postal services, air and railroad transport, cable TV and internet. For this reason, it has attracted a lot of attention over the past decades from economists. One of the seminal papers in this field is by Mussa and Rosen [20], who analysed nonlinear prices for the provision of quality-differentiated goods. Other contributions were made by Goldman, Leland and Sibley [8], who studied price discrimination via quantity discounts and by Maskin and Riley [15] who observed that all these models belong to the same class of principal-agent problems.

On the other hand, the demand profile approach is an alternative formulation for the monopolistic screening problem, introduced in the literature by Brown and Sibley [5] and then thoroughly expounded in Wilson [29][1]. The basis for this approach is the requirement that the marginal tariff $T'(q)$ cuts the customer's demand curve once from below. When satisfied, we have that the customer's maximization problem is quasiconcave so its demand behavior can be

[1]In a survey on multidimensional screening, Rochet and Stole [27] make a distinction between the approach adopted by Mussa and Rosen, and Maskin and Riley, called parameter utility approach and the other one called demand profile approach.

deduced completely from the marginal benefit of consumption. In Goldman, Leland and Sibley, a tariff satisfying this requirement is called a single-crossing tariff, a terminology that we will also use in this chapter.

All these methods work fine and equivalently when the Spence and Mirrlees condition (SMC) is satisfied. The main reason is that it provides a natural ordering for the customer's demand curves. As a consequence, the monopolist is able to induce any monotonic decision for the customers by selecting a compatible tariff.

We study a nonlinear pricing model where the monopolist has incomplete information about the slope and the intercept of the customer's demand curve, represented by α and β. In our model these two dimensions have antagonic effects in the decision q which should increase with α and decrease with β. We further assume that these two dimensions are dependent in a way that we can reduce it into a one-dimensional screening model with parameter θ. Now a bigger θ will represent at the same time as a bigger α and β and the monopolist does not know which of the antagonic effects will dominate the other. By having only one dimension summarizing the effects of those two parameters, it ends up in a model where the SMC is not satisfied anymore.

Without the SMC the problem becomes rather complex, in part due to the absence of this natural ordering. As a consequence the decision q may be nonmonotonic. Besides the local incentive compatibility conditions may not be sufficient as new global conditions arise. This problem without the SMC was studied by Araujo and Moreira [1]. Although they relax the Spence and Mirrlees condition, the optimal contract proposed there is compatible with the demand profile approach, because it is implemented by a single-crossing tariff and consequently the customers face a quasiconcave problem.

Our goal is to show that in our monopolistic screening model without the SMC, the contract resulting from a single-crossing tariff, compatible with the demand profile approach may be suboptimal for the monopolist. We present another contract that provides a larger profit for the monopolist. This contract is not compatible with the demand profile approach because the tariff is not single-crossing. Moreover the optimal contract is discontinuous and induces a separation in the quality spectrum between two groups: one who buys low quality goods, and the other who buys high quality goods. The tariff should be adjusted in a way that one group does not envy the other.

The chapter is organized as follows. In Section 2.2 we define the model we are using. Then, in Section 2.3 we present some results related to the incentive compatibility condition. In Section 2.4 we derive the monopolist's optimization problem using an approach adapted from Goldman, Leland and Sibley, we establish the pointwise optimality conditions for the monopolist's problem and finally we solve some examples. In Section 2.6 we present the conclusions. All proofs are given in the Appendix.

2.2 Model

We use the Principal-Agent framework to analyse the monopolistic screening problem. In this model, each customer has a quasi-linear preference:

$$V(q,t,\theta) = v(q,\theta) - t,$$

where t represents the monetary transfer. The parameter θ is a random variable with a positive and continuous density representing the customer's type. The firm is a profit-maximizing monopolist which can produce any quality $q \in Q \subseteq \mathbb{R}_+$ incurring in a cost $C(q)$. Q represents the quality spectrum. The monopolist's revenue is given by:

$$\Pi(q,t) = t - C(q).$$

Using the *'Revelation Principle'*[2] the monopolist's problem can be stated as choosing the allocation rule $(q,t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ that solves:

$$\max_{\{q(\cdot),t(\cdot)\}} \int_{\Theta} \Pi(q(\theta),t(\theta))p(\theta)d\theta, \tag{2.1}$$

subject to the *individual-rationality* constraints:

$$v(q(\theta),\theta) - t(\theta) \geq 0 \quad \forall \theta \in \Theta, \tag{IR}$$

and the *incentive compatibility* constraints:

$$\theta \in \arg\max_{\theta' \in \Theta} \{v(q(\theta'),\theta) - t(\theta')\}, \quad \forall \theta \in \Theta. \tag{IC}$$

[2]The *'Revelation Principle'* has been enunciated in Gibbard [7].

Remark 1. *The 'Taxation Principle'*[3] *states that any allocation* (q, t) *satisfying the IC constraints can be implemented by a nonlinear tariff* $T : Q = q(\Theta) \to \mathbb{R}$ *where:*

$$T(q(\theta)) = t(\theta), \quad \forall \theta \in \Theta.$$

2.2.1 The Model Setup

We consider a monopolist who produces a single good and is uncertain about the slope and the intercept of the customer's demand curves. The utility function of each customer is parametrized according to:

$$v(q, \alpha, \beta) = \alpha q - \beta \frac{q^2}{2},$$

and the individual inverse demand curve is given by:

$$p(q) = \alpha - \beta q.$$

The cross derivatives of v satisfy: $v_{q\alpha} > 0$ and $v_{q\beta} \leq 0$. These conditions imply that an incentive compatible decision q must increase with α and decrease with β (see Rochet [25]). Hence the effects of these parameters on q are antagonic.

We analyse the case when the random variables α and β are dependent in the following manner:

$$\begin{cases} \alpha = \theta, \\ \beta = b\theta^2, \end{cases}$$

with θ uniformly distributed on $[\underline{\theta}, \overline{\theta}] = [a, a+1]$. Now we have then only one parameter θ to cope with the effects of α and β. A bigger θ means bigger α and β, so the decision will increase or decrease depending on which effect is dominant. This results in the following setup:

$$\begin{cases} v(q, \theta) &= \theta q - b\theta^2 \frac{q^2}{2}, \\ C(q) &= c\frac{q^2}{2}, \end{cases}$$

and we assume that a, b and c are nonnegative constants.

The inverse demand function is now given by

$$p(q) = \theta - b\theta^2 q,$$

[3] This principle can be found in Guesnerie [9], Hammond [11] and Rochet [24].

and the elasticity of demand is

$$\epsilon(q, \theta) = 1 - \frac{1}{b\theta q}.$$

The demand function for types θ_1 and θ_2 are drawn in Figure 2.1. For θ-type customer, the ratio $q = 1/(b\theta)$ measures the market size and the intercept θ represents the minimum price such that this customer's demand is zero.

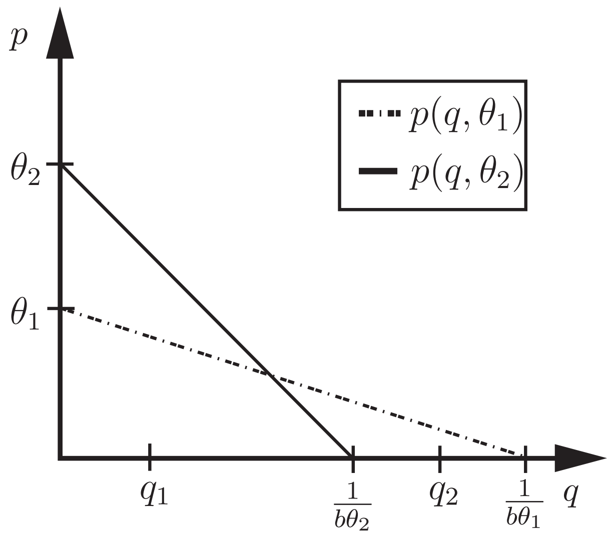

Figure 2.1: Demand curves for θ_1 and θ_2 customers.

When $b = 0$ the specification of this model is the same as Mussa and Rosen [20], but when $b > 0$ the Spence and Mirrlees condition[4]is no longer satisfied and the solution of $v_{q\theta}(q, \theta) = 0$ determines a decreasing function given by

$$q_0(\theta) = \frac{1}{2b\theta}. \tag{2.2}$$

This function gives a limit curve that divides the (θ, q) plane in two regions denoted by CS_+, where $v_{q\theta} > 0$ and CS_-, where $v_{q\theta} < 0$. The consequence is the lack of an ordering of the demand curves as we can see in Figure 2.1, where we have the following inequalities for the marginal utility: $v_q(q_1, \theta_1) < v_q(q_1, \theta_2)$ and $v_q(q_2, \theta_1) > v_q(q_2, \theta_2)$.

Unlike the SMC case, it is not possible to know *a priori* which customer will consume more, as this is determined endogenously as a response to the nonlinear tariff adopted by the monopolist. The decision now may be nonmonotonic, increasing in CS_+ and decreasing in CS_- and these conditions suggest a bell-shaped form for the decision function q.

2.3 Incentive Compatibility Conditions

Without the Spence and Mirrlees condition there is not a simple rule for characterizing an incentive compatible allocation and the problem becomes very complex. There are some necessary conditions, however, that should be taken into account.

In all that follows, we assume that the allocation rule (q, t) is bounded and incentive compatible. The informational rent $V : \Theta \to \mathbb{R}_+$ is given by:

$$V(\theta) = v(q(\theta), \theta) - t(\theta), \tag{2.3}$$

and $T : q(\Theta) \to \mathbb{R}$ is the tariff resulting from the *'Revelation Principle'*.

We begin by establishing a basic property regarding the Lipschitz continuity of both T and V.

Lemma 3. *(**Differentiability Condition**.) The tariff T and the informational rent V are Lipschitz continuous.*

Notice that Lemma 3 guarantees that both T and V are absolute continuous and then a.e. differentiable and now we are able to deduce the following envelope conditions involving their derivatives.

Lemma 4. *(**Envelope Conditions for** V **and** T.)*

(i) If V is differentiable at $\theta \in int(\Theta)$ and $q \in q(\theta)$, then

$$V'(\theta) = v_\theta(q, \theta). \tag{2.4}$$

(ii) If T is differentiable at $q \in q(\theta) \cap int(q(\Theta))$, then

$$T'(q) = v_q(q, \theta). \tag{2.5}$$

Notice that Lemma 4 (ii) can be understood as the first order condition of the θ-customer maximization problem:

$$\max_{q \in Q} \{v(q, \theta) - T(q)\}.$$

[4] The Spence and Mirrlees condition states that the cross derivative $v_{q\theta}$ does not change its signal. When satisfied the incentive compatibility is equivalent to the monotonicity of the decision function q. See Rochet [25], for instance.

With the SMC a decision is incentive compatible if and only if is monotonic. However, without the SMC we may have a nonmonotonic incentive compatible decision. The next result is related to the local monotonicity conditions on the decision q.

Lemma 5. *(**Local Monotonicity Conditions.**) Suppose that q is continuous at θ_0. Then:*

(i) If $q(\theta_0) > q_0(\theta_0)$ then q is increasing at θ_0,

(ii) If $q(\theta_0) < q_0(\theta_0)$ then q is decreasing at θ_0,

where q_0 is defined in equation (2.2).

The new possibilities for q coming from Lemma 5 motivates us to define the bell-shaped decisions.

Definition 5. *(**Bell-Shaped Decisions.**) The decision $q : \Theta \to \mathbb{R}_+$ is bell-shaped if there exists a continuity point $\theta_m \in \Theta$ such that the restrictions:*

$$q\big|_{[\underline{\theta},\theta_m]} \text{ is increasing and } \; q\big|_{[\theta_m,\overline{\theta}]} \text{ is decreasing.}$$

For a bell-shaped decision, we may have a discrete pooling situation, when types θ_1 and θ_2 choose the same decision q. Using Lemma 4 (ii), we can easily deduce a new necessary condition for incentive compatibility relating the marginal valuations of the pooled types.

Corollary 1. *(**Bell-Shaped Condition.**) Suppose that T is differentiable at $q \in q(\theta_1) \cap q(\theta_2)$ an interior point of $Q = q(\Theta)$. Then*

$$v_q(q, \theta_1) = v_q(q, \theta_2). \tag{BS}$$

This marginal utility equality in Corollary 1 will be denoted BS-condition, and it says that pooled types must pay the same marginal tariff for the chosen decision q.

2.4 The Monopolist's Problem

Suppose first that the alocation rule (q, t) is incentive compatible. Let us now deduce the monopolist's maximization problem, using the same derivation as Mussa and Rosen [20]. From the definition of the informational rent V (equation (2.3)) we can write the monetary transfer as $t(\theta) = v(q(\theta), \theta) - V(\theta)$ and then substitute it in equation (2.1). The result is the following problem:

$$\max_{\{q(\cdot)\}} \int_{\Theta} \{v(q(\theta), \theta) - C(q(\theta)) - V(\theta)\} d\theta. \tag{2.6}$$

Now using Lemma 4 and an integration by parts procedure, we can rewrite the problem above as[5]:

$$\max_{\{q(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} f(q(\theta), \theta) d\theta, \tag{Π_R}$$

where $f(q, \theta)$ is given by:

$$f(q, \theta) = v(q(\theta), \theta) - C(q(\theta)) + (\theta - a - 1) v_\theta(q(\theta), \theta).$$

This is called the relaxed version of the monopolist's maximization problem, because the monotonicity constraints are not considered in this problem. The inconvenience of the derivation above is that in many situations the solution of Problem (Π_R) is far from being incentive compatible. One reason is that it does not consider the (BS) condition. We are going to derive the monopolist's optimization problem in a way that the (BS) necessary condition given by Corollary 1 can be taken into account. We need the following definitions:

Definition 6. *Consider a monotonic function* $q : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ *along with its upper contour set* $S(q) = \{\theta \in [\underline{\theta}, \overline{\theta}] : q(\theta) \geq q\}$.

If q *is decreasing, we define* $\psi_s : \mathbb{R} \to [\underline{\theta}, \overline{\theta}]$ *as:*

$$\psi_s(q) = \begin{cases} \sup S(q), & \text{if } S(q) \neq \emptyset, \\ \underline{\theta}, & \text{if } S(q) = \emptyset. \end{cases}$$

If q *is increasing, we define* $\psi_b : \mathbb{R} \to [\underline{\theta}, \overline{\theta}]$ *as:*

$$\psi_b(q) = \begin{cases} \inf S(q), & \text{if } S(q) \neq \emptyset, \\ \overline{\theta}, & \text{if } S(q) = \emptyset. \end{cases}$$

In both cases, ψ_b *and* ψ_s *are monotonic and they are denoted the pseudoinverses of* q.

[5] We are implicitly assuming that $V'(\theta) \geq 0$. For a more general case we refer to Jullien [13].

For a bell-shaped decision, using a procedure similar to Goldman, Leland and Sibley [8], we can reformulate the problem (Π_R). First we use the fundamental theorem of calculus to write:

$$\int_{\underline{\theta}}^{\overline{\theta}} f(q(\theta),\theta)d\theta = \int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{q}}^{q(\theta)} f_q(q,\theta)dqd\theta.$$

Then, we use Fubini's theorem, and the following problem results:

$$\max_{\{\psi_b,\psi_s\}} \int_{\underline{q}}^{\overline{q}} \{F(q,\psi_s(q)) - F(q,\psi_b(q))\}dq + \int_{\underline{\theta}}^{\overline{\theta}} f(\underline{q},\theta)d\theta. \tag{2.7}$$

F is given by the following equation:

$$F(q,\psi) = \int_{\underline{\theta}}^{\psi} f_q(q,\theta)d\theta, \tag{2.8}$$

and besides ψ_b is the pseudoinverse of the increasing part of q and ψ_s is the inverse of the decreasing part of q. They are denoted type assignment functions, because they relate q with the type who purchases it. Notice that the integration variable on the left hand side of equation (2.7) is q, and now we can treat the constraint imposed by the condition (**BS**) condition for each q separately.

2.4.1 Single-Crossing Tariff

First let us define a single-crossing tariff[6]:

Definition 7. *(**Single-Crossing Tariff.**) A tariff is said to be single-crossing at q if, for any θ such that $v_q(q,\theta) \geq T'(q)$, $v_q(q',\theta) \geq T'(q')$ for $q' \leq q$. If $T(q)$ is single-crossing for all q, we term it a single-crossing tariff schedule.*

If the tariff is single-crossing and continuous, then the customer's maximization problem is quasiconcave.

Using the concept of a single-crossing tariff, we will be able to define the following optimization problem for the monopolist, when $q(\underline{\theta}) < q(\overline{\theta})$[7]:

$$\begin{cases} \max\limits_{\{\psi_b,\psi_s\}} \int_{\underline{q}}^{\overline{q}} \{F(q,\psi_s(q)) - F(q,\psi_b(q))\}dq + \int_{\underline{\theta}}^{\overline{\theta}} f(\underline{q},\theta)d\theta, \\ \text{s.t.} \quad v_q(q,\psi_b(q)) - v_q(q,\psi_s(q)) \leq 0. \end{cases} \tag{Π_{BS}}$$

[6]This definition was taken from Goldman, Leland and Sibley [8].

[7]The other case, when $q(\underline{\theta}) > q(\overline{\theta})$, is treated analogously, and we only have to reverse the inequality in problem Π_{BS}.

Notice that the problem Π_{BS} is coherent with a single-crossing tariff T. In this case, when $q(\underline{\theta}) < q(\overline{\theta})$ and $q < q(\overline{\theta})$, then $v_q(q, \psi_b(q)) \leq v_q(q, \psi_s(q))$, because $T'(q) = v_q(q, \psi_b(q))$ and $v_q(q, \overline{\theta}) \geq T'(q)$.

Once we find the optimal type assignment functions ψ_b and ψ_s we can invert them and find the optimal decision q.

Now let us establish the optimality conditions for Π_{BS}.

Theorem 1. *The solution of Π_{BS} is characterized by the following conditions:*

(i) If the condition BS is not binding and $\underline{\theta} < \psi_b(q) < \psi_s(q) = \overline{\theta}$ then:

$$f_q(q, \psi_b(q)) = 0, \tag{2.9}$$

(ii) If the condition BS is binding and $\underline{\theta} < \psi_b(q) < \psi_s(q) < \overline{\theta}$ then:

$$\frac{f_q(q, \psi_s(q))}{v_{q\theta}(q, \psi_s(q))} = \frac{f_q(q, \psi_b(q))}{v_{q\theta}(q, \psi_b(q))}, \tag{2.10}$$

(iii) If the condition BS is binding and $\underline{\theta} < \psi_b(q) < \psi_s(q) = \overline{\theta}$ then:

$$v_q(q, \psi_b(q)) = v_q(q, \overline{\theta}). \tag{2.11}$$

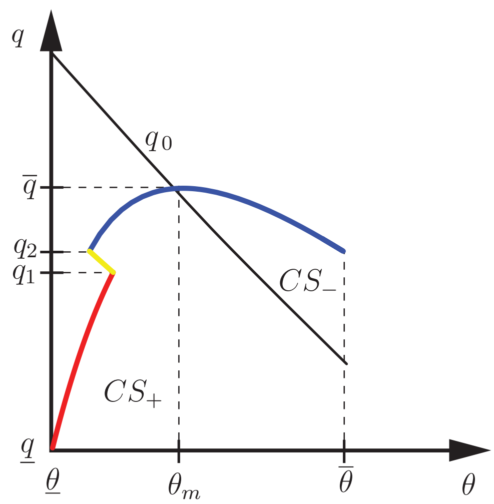

Figure 2.2: A typical solution for Π_{BS}.

In Figure 2.2, we show the typical solution for the optimization problem Π_{BS} resulting from Theorem 1. For $q < q_1$, we use condition (i), for $q \in (q_1, q_2)$ we use condition (iii) and finally for

$q \in (q_2, \overline{q})$ we use condition (ii). As we can see there is a problem related with the monotonicity condition resulting from Lemma 5.

We have two alternatives for fixing this lack of monotonicity. The first one is a vertical ironing compatible with a single-crossing tariff T and the next proposition shows how restrictive this condition is.

Proposition 2. *If the tariff T is single-crossing then the decision q is convex valued.*

As a consequence of Proposition 2, if the decision q jumps at θ_d, then the marginal tariff for $q \in [q(\theta_d-), q(\theta_d+)]$ is given by $T'(q) = v_q(q, \theta_d)$ and this affects the decision for customers with types θ close to $\overline{\theta}$. The reason is simple: with a single-crossing tariff, the marginal net benefit determines the customer's choice and when a jump occurs, the marginal tariff is automatically defined for all the intermediate quantities and of course this may affect some customer's decisions.

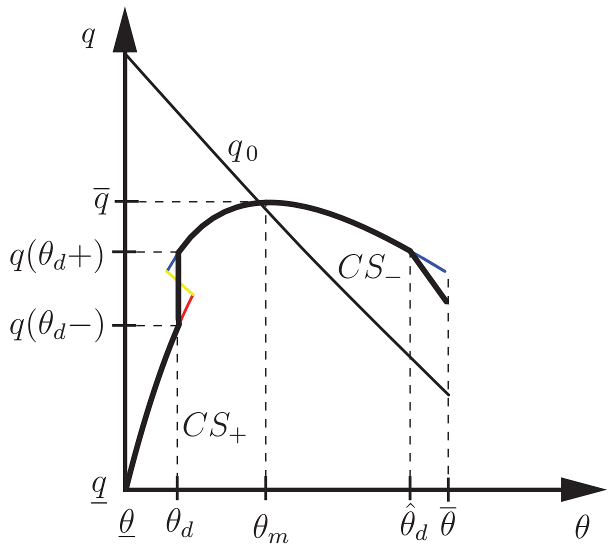

Figure 2.3: The vertical ironing procedure.

The decision function that emerges from this vertical ironing is given by:

$$q_{\theta_d}(\theta) = \begin{cases} q_R(\theta) & , \text{ if } \theta < \theta_d, \\ q_B(\theta) & , \text{ if } \theta_d \leq \theta \leq \hat{\theta}_d, \\ q_C(\theta, \theta_d) & , \text{ if } \hat{\theta}_d \leq \theta \leq \overline{\theta}, \end{cases} \tag{2.12}$$

where q_R and q_B result from Theorem 1 (i) and (ii), respectively. On the other hand q_C results from solving the equation:

$$v_q(q, \theta_d) = v_q(q, \theta).$$

Notice that the customers with type $\theta \geq \hat{\theta}_d$ choose $q \in [q(\theta_d-), q(\theta_d+)]$ and their decision is affected by the fact that for this q the marginal tariff is $T'(q) = v_q(q, \theta_d)$. The function q_C captures this effect. Finally, we get $\hat{\theta}_d$ from solving $q_B(\theta_d) = q_B(\hat{\theta}_d)$.

The point θ_d must be chosen optimally and is characterized by the following:

Theorem 2. *The first order condition for the optimal θ_d is given by:*

$$f(q_B(\theta_d), \theta_d) - f(q_R(\theta_d), \theta_d) = \int_{\hat{\theta}_d}^{\overline{\theta}} f_q(q_C(\theta, \theta_d), \theta) \frac{\partial q_C}{\partial \theta_d}(\theta, \theta_d) d\theta. \tag{2.13}$$

Notice that this vertical ironing fixes the monotonicity problem by keeping the type assignment functions ψ_b and ψ_s continuous. This kind of solution was also proposed in Araujo and Moreira [1]. It can also be obtained by using the demand profile approach of Wilson, but the computations are more difficult than the one we presented here.

2.4.2 Non Single-Crossing Tariff

Now we propose an alternative for this vertical ironing where ψ_b and ψ_s may be discontinuous. This is illustrated in Figure 2.4. More than that, our procedure will result in a non single-crossing tariff.

We will impose the global incentive compatibility constraint relating θ_d and $\overline{\theta}$ customers. We are asking that the $\overline{\theta}$-type customer does not envy θ_d-type customer. In other words we impose:

$$v(q(\overline{\theta}), \overline{\theta}) - t(\overline{\theta}) \geq v(q(\theta_d), \overline{\theta}) - t(\theta_d). \tag{2.14}$$

The following result will be useful to restate the condition above[8]:

Lemma 6. *Consider an allocation rule $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$. If (q, t) is incentive compatible then:*

$$\int_{\hat{\theta}}^{\theta} \int_{q(\hat{\theta})}^{q(\tilde{\theta})} v_{q\theta}(\tilde{q}, \tilde{\theta}) d\tilde{q} d\tilde{\theta} \geq 0, \quad \forall \theta, \hat{\theta} \in \Theta. \tag{2.15}$$

[8]This condition is actually equivalent to incentive compatibility, as shown by Araujo and Moreira [1].

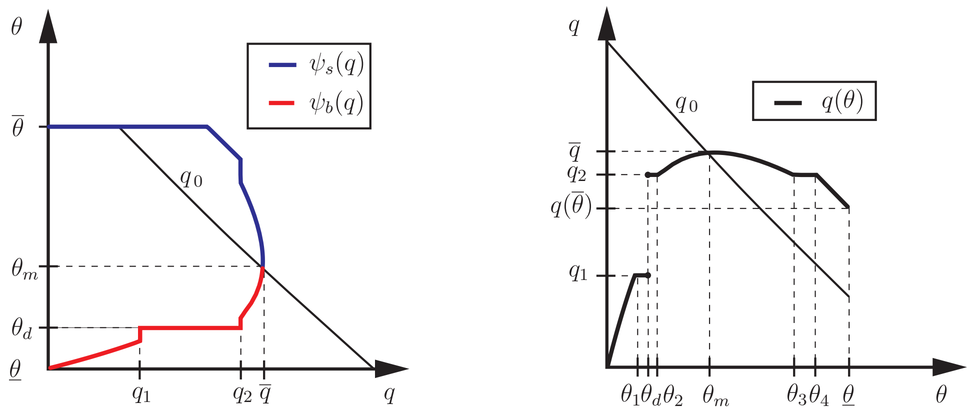

Figure 2.4: An alternative ironing procedure.

Notice that Lemma 6 relates incentive compatibility with an area condition represented by a double integral.

The same procedure used in the proof of Lemma 6 allows us to write the inequality (2.14) as:

$$\int_{\theta_d}^{\overline{\theta}} \int_{q(\theta_d)}^{q(\tilde{\theta})} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta} \geq 0. \tag{2.16}$$

Notice that:

$$\int_{\theta_d}^{\overline{\theta}} \int_{q(\theta_d)}^{q(\tilde{\theta})} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta} = \int_{\theta_d}^{\overline{\theta}} \int_{q_1}^{q_2} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta} + \int_{\theta_d}^{\overline{\theta}} \int_{q_2}^{q(\tilde{\theta})} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta}. \tag{2.17}$$

Using Fubini's theorem, we can write the second integral on the right hand side of equation (2.17) as:

$$\begin{aligned} \int_{\theta_d}^{\overline{\theta}} \int_{q_2}^{q(\tilde{\theta})} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta} &= \int_{q_2}^{\overline{q}} \int_{\psi_b(q)}^{\psi_s(q)} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{\theta}d\tilde{q} \\ &= \int_{q_2}^{\overline{q}} \{v_q(q,\psi_s(q)) - v_q(q,\psi_b(q))\}dq. \end{aligned}$$

However $v_q(q,\psi_s(q)) = v_q(q,\psi_b(q))$ for $q \in (q_2,\overline{q}]$ which implies that this part is null.

Using Fubini's theorem again we can write the first integral on the right hand side of equation (2.17) as:

$$\int_{\theta_d}^{\overline{\theta}} \int_{q_1}^{q_2} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta} = \int_{q_1}^{q_2} \{v_q(q,\psi_s(q)) - v_q(q,\theta_d)\}dq. \tag{2.18}$$

and the condition in equation (2.16) will be given by:

$$\int_{q_1}^{q_2} \{v_q(q, \psi_s(q)) - v_q(q, \theta_d)\} dq \geq 0. \tag{ISO}$$

In the interval $[q_1, q_2]$, we will optimally choose $\psi_s(q)$ such that the condition (ISO) is fulfilled. So we have the following isoperimetric problem:

$$\begin{cases} \max_{\{\psi_s\}} \int_{q_1}^{q_2} \{F(q, \psi_s(q)) - F(q, \theta_d)\} dq \\ \text{s.t.(ISO).} \end{cases} \tag{Π_{ISO}}$$

Theorem 3. *The solution of Π_{ISO} is characterized by the following condition:*

$$f_q(q, \psi_s(q)) + \lambda v_{q\theta}(q, \psi_s(q)) = 0, \tag{2.19}$$

where λ is chosen to satisfy the condition (ISO) with equality.

Now we use conditions given by Theorem 1 and Theorem 3 to build a candidate for the monopolist's optimization problem. The cited theorems give the pointwise conditions for the three pieces that constitute the candidate we are proposing. Looking again at Figure 2.4, we see that the only change is for $q \in [q_1, q_2]$, where we are doing this alternative ironing. The values q_1, q_2 and θ_d are endogenously determined as a result of an optimization process. Let us describe the type assignment functions ψ_b and ψ_s:

$$\psi_s(q) = \begin{cases} \psi_{sr}(q) & \text{, if } \underline{q} \leq q \leq q_1, \\ \psi_{si}(q) & \text{, if } q_1 \leq q \leq q_2, \\ \psi_{sc}(q) & \text{, if } q_2 \leq q \leq \overline{q}, \end{cases} \tag{2.20}$$

where we have that $\psi_{sr} = \overline{\theta}$, $\psi_{si}(q)$ is given by equation (2.19) and ψ_{sc} is given by equation (2.10).

$$\psi_b(q) = \begin{cases} \psi_{br}(q) & \text{, if } \underline{q} \leq q \leq q_1, \\ \psi_{bi}(q) & \text{, if } q_1 \leq q \leq q_2, \\ \psi_{bc}(q) & \text{, if } q_2 \leq q \leq \overline{q}, \end{cases} \tag{2.21}$$

where $\psi_{bi} = \theta_d$, we define ψ_{br} from equation (2.9) and ψ_{bc} from equation (2.10).

Now the monopolist's problem will depend on the variables q_1, q_2 and θ_d, and we can write its expected profit as $\Pi(q_1, q_2, \theta_d)$ where q_1, q_2 and θ_d satisfy the condition (ISO). Our last step is to optimally choose these parameters:

Theorem 4. *The optimal choices for the parameters q_1, q_2 and θ_d are characterized by:*

$$\text{(i)}\quad \frac{F(q_1,\psi_{sr}(q_1)) - F(q_1,\psi_{br}(q_1)) + F(q_1,\theta_d) - F(q_1,\psi_{si}(q_1))}{v_q(q_1,\psi_{si}(q_1)) - v_q(q_1,\theta_d)} = \lambda,$$

$$\text{(ii)}\quad \frac{F(q_2,\psi_{si}(q_2)) - F(q_2,\theta_d) + F(q_2,\psi_{bc}(q_2)) - F(q_2,\psi_{sc}(q_2))}{v_q(q_2,\theta_d) - v_q(q_2,\psi_{si}(q_2))} = \lambda,$$

$$\text{(iii)}\quad \int_{q_1}^{q_2} \{F_\theta(q,\theta_d) + \lambda v_{q\theta}(q,\theta_d)\}dq = 0,$$

where λ is the Lagrangian multiplier associated with the isoperimetric constraint.

Remark 2. *Considering the isoperimetric condition, we have four equations and four parameters: q_1, q_2, θ_d and λ. We get the optimal candidate by solving this system of equations.*

2.5 An Example

This example illustrates the difference between the two ironing procedures used for fixing the monotonicity problem. The first one is compatible with the demand profile approach because the resulting tariff is single-crossing. The second one uses the isoperimetric condition and results in a non single-crossing tariff.

Let us choose the following values for the parameters in our model: $a = 2, b = 1$ and $c = 6$. With these parameters, we have that θ is uniformly distributed in $[2, 3]$ and:

$$\begin{cases} v(q,\theta) &= \theta q - \theta^2 \dfrac{q^2}{2}, \\ C(q) &= 3q^2. \end{cases}$$

Using Theorem 1, we can derive the solution for Π_{BS}:

$$\psi_b(q) = \begin{cases} \frac{3q+1-\sqrt{-9q^2-3q+1}}{3q}, & \text{if } \frac{1}{6} \le q \le \frac{1}{102}\left(15+\sqrt{21}\right), \\ \frac{1-3q}{q}, & \text{if } \frac{1}{102}\left(15+\sqrt{21}\right) \le q \le \frac{1}{66}\left(9+\sqrt{15}\right), \\ \frac{3q-\sqrt{3}\sqrt{q^2-24q^4}}{6q^2}, & \text{if } \frac{1}{66}\left(9+\sqrt{15}\right) \le q \le \frac{1}{2\sqrt{6}}. \end{cases}$$

$$\psi_s(q) = \begin{cases} 3, & \text{if } \frac{1}{6} \le q \le \frac{1}{102}\left(15+\sqrt{21}\right), \\ 3, & \text{if } \frac{1}{102}\left(15+\sqrt{21}\right) \le q \le \frac{1}{66}\left(9+\sqrt{15}\right), \\ \frac{3q+\sqrt{3}\sqrt{q^2-24q^4}}{6q^2}, & \text{if } \frac{1}{66}\left(9+\sqrt{15}\right) \le q \le \frac{1}{2\sqrt{6}}. \end{cases}$$

As we can see in Figure 2.5 ψ_b is nonmonotonic. We are going to derive the solutions using the two ironing procedures developed before.

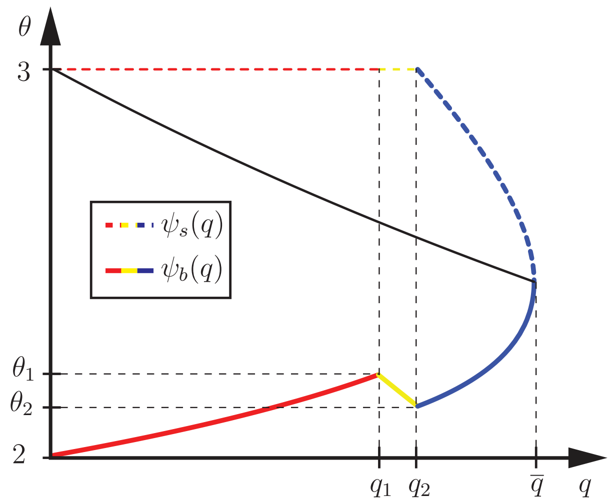

Figure 2.5: The solutions ψ_b and ψ_s for Π_{BS}.

a) Standard Ironing Procedure:

The first step is to write the solution depending on the parameter θ_d, as we did in equation (2.12):

$$q_{\theta_d}(\theta) = \begin{cases} q_R(\theta) = \frac{3-2\theta}{-3\theta^2+6\theta-6} & \text{, if } \theta < \theta_d, \\ q_B(\theta) = \frac{3\theta+\sqrt{3\theta^2-12}}{2(3\theta^2+6)} & \text{, if } \theta_d \leq \theta \leq \hat{\theta}_d, \\ q_C(\theta, \theta_d) = \frac{1}{\theta+\theta_d} & \text{, if } \hat{\theta}_d \leq \theta \leq \overline{\theta}, \end{cases}$$

where $\hat{\theta}_d = 2\theta_d - \sqrt{3}\sqrt{\theta_d^2 - 4}$.

Using Theorem 2, we can find the optimal θ_d^* and $\hat{\theta}_d^*$:

$$\begin{aligned} \theta_d^* &\approx 2.164282, \\ \hat{\theta}_d^* &\approx 2.89596. \end{aligned}$$

The decision fucntion resulting from this ironing procedure is denoted $q_{SC}(\theta)$. We can find the tariff $T_{SC}(q)$ that implements the decision by integrating the marginal tariff given by:

$$T'_{SC}(q) = v_q(q, \psi_b(q)),$$

where ψ_b is the pseudoinverse of $q_{SC}(\theta)|_{[\underline{\theta},\sqrt{6}]}$. In Figure 2.6, we plot the decision function and the marginal tariff. Notice that the marginal tariff is continuous so the tariff is continuously differentiable.

The monopolist's expected profit from $q_{SC}(\theta)$ is:

$$\Pi(\theta_d) = 0.200272 \tag{2.22}$$

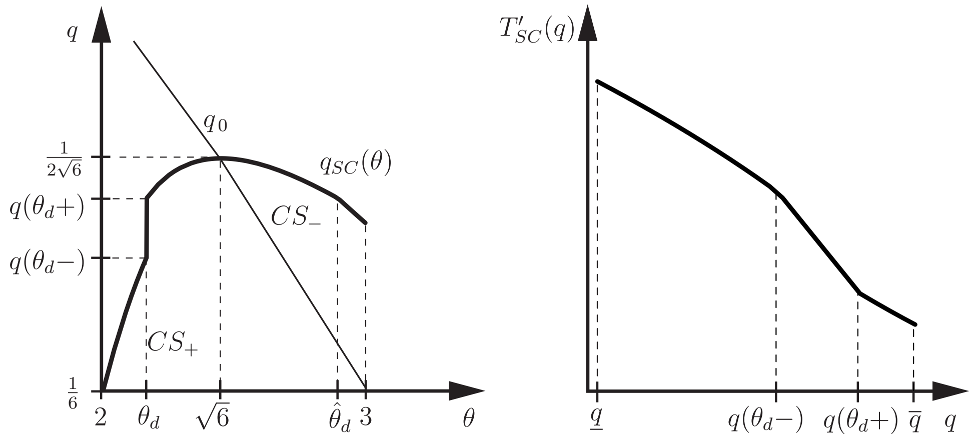

Figure 2.6: The decision function and the marginal tariff.

b) Isoperimetric Ironing Procedure:

Now we use the isoperimetric procedure. Let us describe the type assignment functions ψ_s and ψ_b according to equations (2.20) and (2.21):

$$\psi_s(q) = \begin{cases} \psi_{sr}(q) = 3 & , \text{ if } \frac{1}{6} \leq q \leq q_1, \\ \psi_{si}(q) = \frac{1-q(\lambda-3)+\sqrt{(\lambda^2-6\lambda-9)q^2+(\lambda-3)q+1}}{3q} & , \text{ if } q_1 \leq q \leq q_2, \\ \psi_{sc}(q) = \frac{3q+\sqrt{3}\sqrt{q^2-24q^4}}{6q^2} & , \text{ if } q_2 \leq q \leq \frac{1}{2\sqrt{6}}, \end{cases}$$

$$\psi_b(q) = \begin{cases} \psi_{br}(q) = \frac{3q+1-\sqrt{-9q^2-3q+1}}{3q} & , \text{ if } \frac{1}{6} \leq q \leq q_1, \\ \psi_{bi}(q) = \theta_d & , \text{ if } q_1 \leq q \leq q_2, \\ \psi_{bc}(q) = \frac{3q-\sqrt{3}\sqrt{q^2-24q^4}}{6q^2} & , \text{ if } q_2 \leq q \leq \frac{1}{2\sqrt{6}}. \end{cases}$$

Using Theorem 4 and the isoperimetric condition, we get the following values:

$$q_1 \approx 0.183112, \quad q_2 \approx 0.202431, \quad \theta_d \approx 2.201349, \quad \lambda \approx 0.067841.$$

When we invert the type assignment functions, we get the following decision function:

$$q_{NSC}(\theta) = \begin{cases} \frac{3-2\theta}{-3\theta^2+6\theta-6} & , \text{ if } \theta \leq \theta_1, \\ \frac{3-2\theta_1}{-3\theta_1^2+6\theta_1-6} & , \text{ if } \theta_1 \leq \theta \leq \theta_d, \\ \frac{3\theta_2+\sqrt{3\theta_2^2-12}}{2\left(3\theta_2^2+6\right)} & , \text{ if } \theta_d \leq \theta \leq \theta_2 \text{ or if } \theta_3 \leq \theta \leq \theta_4, \\ \frac{3\theta+\sqrt{3\theta^2-12}}{2(3\theta^2+6)} & , \text{ if } \theta_2 \leq \theta \leq \theta_3, \\ \frac{1-q(\lambda-3)+\sqrt{(\lambda^2-6\lambda-9)q^2+(\lambda-3)q+1}}{3q} & , \text{ if } \theta_4 \leq \theta \leq 3. \end{cases}$$

where:

$$\theta_1 \approx 2.117953, \quad \theta_2 \approx 2.286691, \quad \theta_3 \approx 2.653259, \quad \theta_4 \approx 2.863469.$$

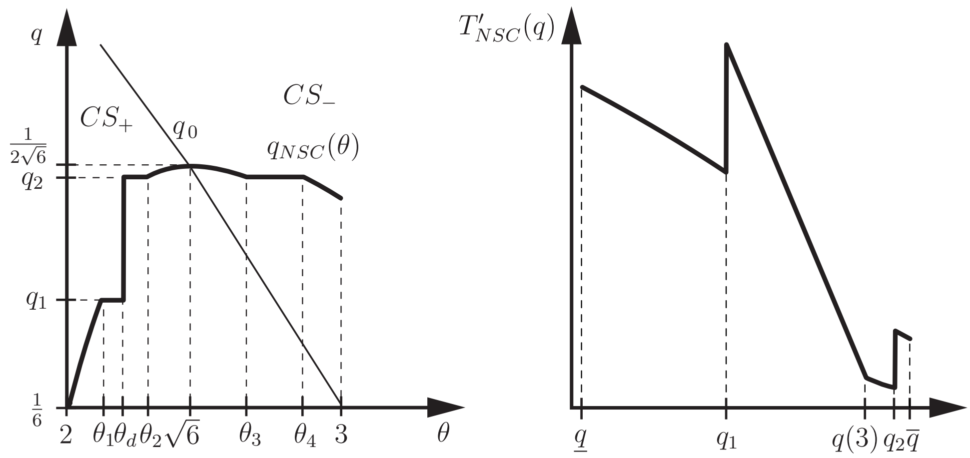

Figure 2.7: The decision function and the marginal tariff..

We can find the tariff $T_{NSC}(q)$ that implements the decision $q_{NSC}(\theta)$ by integrating the marginal tariff given by:

$$T'_{NSC}(q) = \begin{cases} v_q(q, \psi_{br}(q)), & \text{if } \frac{1}{6} \leq q \leq q_1, \\ v_q(q, \psi_{si}(q)), & \text{if } q_1 \leq q \leq q_2, \\ v_q(q, \psi_{bc}(q)), & \text{if } q_2 \leq q \leq \frac{1}{2\sqrt{6}}. \end{cases}$$

We can see in Figure 2.7 the decision function $q_{NSC}(\theta)$ and the marginal tariff $T'_{NSC}(q)$. Notice that jumps in the marginal tariff correspond to bunching in the decision function graphic.

The monopolist's expected profit from $q_{NSC}(\theta)$ is:

$$\Pi(q_1, q_2, \theta_d) = 0.200314 \tag{2.23}$$

Comparing the equations (2.22) and (2.23), we see that the isoperimetric ironing procedure results in a larger expected profit for the monopolist. We have an increase of approximately 0.021%. This shows the suboptimality the decision function $q_{NSC}(\theta)$.

In Figure 2.8 we make a superposition of the decision function and the marginal tariff for ironing procedures (a) and (b). Notice that the latter results in a smaller quality spectrum than the former.

The fundamental difference between the two decision functions is that $q_{SC}(\theta)$ is implemented by the single-crossing tariff $T_{SC}(q)$ and $q_{NSC}(\theta)$ is implemented by the non single-crossing tariff

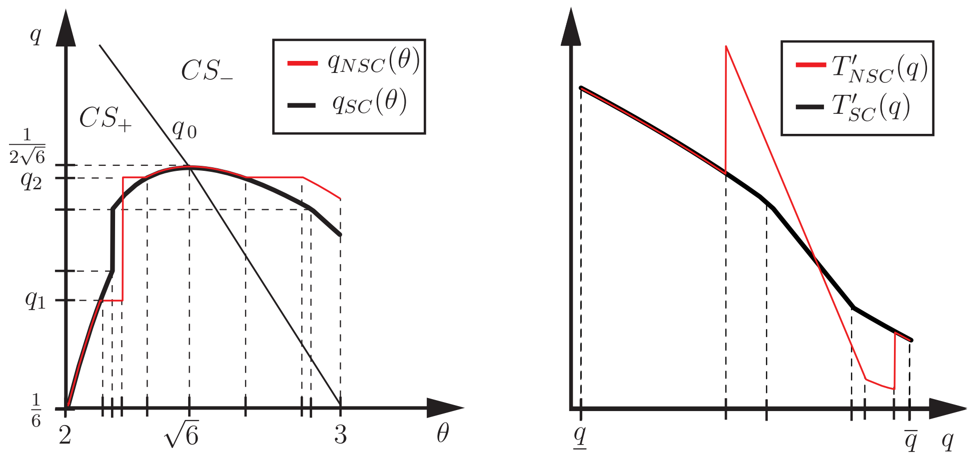

Figure 2.8: The decision an the marginal tariff for both procedures (a) and (b) .

$T_{NSC}(q)$. In fact, when the tariff is $T_{NSC}(q)$, the maximization problem for customer θ_d is not quasiconcave as it has two local maxima q_1 and q_2 so the decision $q_{NSC}(\theta)$ is not convex valued. In Figure 2.9 we plot the customer θ_d utility depending on his choice of q under the non single-crossing tariff $T_{NSC}(q)$.

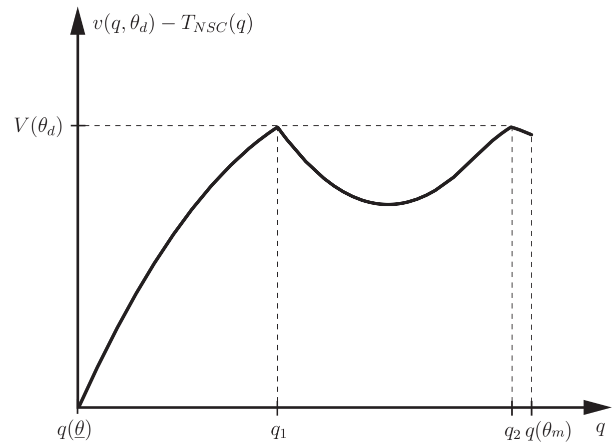

Figure 2.9: The customer θ_d's utility under the non single-crossing tariff $T_{NSC}(q)$.

2.6 Conclusion

In our model of nonlinear pricing without the Spence and Mirrlees condition, we showed that the basic assumption of the demand profile approach may not be valid and that the demand profile approach can lead to a suboptimal solution for the monopolist's optimization problem.

The reason is that the requirement of a single-crossing tariff is restrictive, as it imposes a convex valued decision.

In a multidimensional setting we also have the same situation described here, since there is not a natural ordering for the customer's demand curves. Then one has to investigate whether the restrictions imposed by the demand profile approach may affect or not the monopolist's welfare. Without a criterion based on the fundaments of the model that allows the monopolist to anticipate whether his maximization problem results or not in a single-crossing tariff, we cannot know if the contract resulting from the demand profile approach is a suboptimal contract. This poses a serious restriction in knowing which problems are adequate for this kind of approach.

Appendix

We need the concept of cyclical monotonicity[9]:

Definition 8. *A correspondence $q : \Theta \to Q$ is v-cyclic-monotone if and only if for all finite cycles $\theta_0, \theta_1, \ldots, \theta_{n+1} = \theta_0 \in \Theta$,*

$$\sum_{i=0}^{n} \{v(q(\theta_i), \theta_{i+1}) - v(q(\theta_i), \theta_i)\} \leq 0.$$

Theorem 5. *(Rochet, 1987) A decision $q : \Theta \to Q$ is implementable if and only if q is v-cyclic-monotone.*

We also need some abstract convexity results related to incentive compatibility that we will state here.[10]

Definition 9. *Let $V : \Theta \to \overline{\mathbb{R}}$. The v-subdifferential of V, $\partial^v V : \Theta \to Q$, is defined by:*

$$\partial^v V(\theta) = \left\{q \in Q : V(\theta') - V(\theta) \geq v(q, \theta') - v(q, \theta) \quad \forall \theta' \in \Theta\right\}.$$

Definition 10. *V is said to be v-subdifferentiable in Θ if and only if $\partial^v V(\theta) \neq \emptyset \quad \forall \theta \in \Theta$.*

Definition 11. *Let $V : \Theta \to \overline{\mathbb{R}}$. We define the v-conjugated of V by:*

$$V^v : Q \to \overline{\mathbb{R}} \quad V^v(q) = \sup_{\theta \in \Theta} \{v(q, \theta) - V(\theta)\};$$

and the v-biconjugate of V by:

$$V^{vv} : \Theta \to \overline{\mathbb{R}} \quad V^{vv}(\theta) = \sup_{q \in Q} \{v(q, \theta) - V^v(q)\}.$$

[9]This can be found in Rochet [25].
[10]These results can be found in Carlier [6] and Rachev and R¸schendorf [23].

Proposition 3. *Let* $V : \Theta \to \overline{\mathbb{R}}$ *, then:*

$$V \text{ is } v\text{-convex} \Leftrightarrow V = V^{vv}.$$

Proposition 4. *Let* $V : \Theta \to \overline{\mathbb{R}}$ *, then:*

$$q \in \partial^v V(\theta) \Leftrightarrow V(\theta) + V^v(q) = v(q, \theta).$$

Theorem 6. *(Carlier, 2001) A decision* $q : \Theta \to \mathbb{R}_+^M$ *is implementable if and only if there exists a* v*-convex and* v*-subdifferentiable function* $V : \Theta \to \mathbb{R}$ *such that:*

$$q(\theta) \in \partial^v V(\theta) \quad \forall \theta \in \Omega.$$

Remark 1. *So when the informational rent* V *is* v*-convex, the conjugate* V^v *plays the role of the nonlinear tariff* T *that implements the decision* q *and the correspondence* $\partial^v V(\theta)$ *is identified with* $q(\theta)$*.*

Proof. **of Lemma 3** The v-conjugation result allows us to write:

$$V(\theta) = \max_{q \in Q} \{v(q, \theta) - T(q)\},$$

and

$$T(q) = \max_{\theta \in \Theta} \{v(q, \theta) - V(\theta)\}. \tag{2.24}$$

Let us prove first that V is Lipschitz continuous. By the incentive compatibility constraint we have that:

$$v(q(\theta), \theta) - t(\theta) \geq v(q(\theta'), \theta) - t(\theta'). \tag{2.25}$$

Using the definition of V, we can rewrite equation (2.25) as:

$$V(\theta) - V(\theta') \geq v(q(\theta'), \theta) - v(q(\theta'), \theta'). \tag{2.26}$$

Changing the roles of θ and θ' in equation (2.26), we have that:

$$V(\theta') - V(\theta) \geq v(q(\theta), \theta') - v(q(\theta), \theta). \tag{2.27}$$

Equations (2.26) and (2.27) can be combined into:

$$v(q(\theta),\theta) - v(q(\theta),\theta') \geq V(\theta) - V(\theta') \geq v(q(\theta'),\theta) - v(q(\theta'),\theta').$$

As q is bounded and $v(q,\theta)$ is C^3, we conclude that there is $M > 0$ such that:

$$\left|V(\theta') - V(\theta)\right| \leq M\left|\theta - \theta'\right|.$$

Now let us prove that T is Lipschitz continuous. Supposing that $q_1 \in q(\theta_1)$ and $q_2 \in q(\theta_2)$, by Proposition 4 and equation (2.24), we can write:

$$T(q_1) = v(q_1,\theta_1) - V(\theta_1) \geq v(q_1,\theta_2) - V(\theta_2), \tag{2.28}$$

$$T(q_2) = v(q_2,\theta_2) - V(\theta_2) \geq v(q_2,\theta_1) - V(\theta_1). \tag{2.29}$$

Equations (2.28) and (2.29) results in:

$$v(q_1,\theta_1) - v(q_2,\theta_1) \geq T(q_1) - T(q_2) \geq v(q_1,\theta_2) - v(q_2,\theta_2).$$

As q is bounded and $v(q,\theta)$ is C^3, we conclude that there is $M > 0$ such that:

$$|T(q_1) - T(q_2)| \leq M|q_1 - q_2|.$$

□

Proof. **of Lemma 4(*i*)** We know that θ is an interior point of Θ, then by the differentiability of V at θ, we have that:

$$V(\theta + h) = V(\theta) + V'(\theta)\cdot h + o(h). \tag{2.30}$$

Using equation (2.26) we have that:

$$V(\theta + h) \geq V(\theta) + v(q(\theta),\theta + h) - v(q(\theta),\theta) = V(\theta). \tag{2.31}$$

Combining equations (2.30), (2.31) and using the differentiability of $v(q,\theta)$, we get:

$$V'(\theta)\cdot h - \frac{\partial v}{\partial \theta}(q(\theta),\theta)\cdot h \geq o(h). \tag{2.32}$$

Taking the limit in equation (2.32) first when $h \uparrow 0$ and then when $h \downarrow 0$ we get that:

$$V'(\theta) = \frac{\partial v}{\partial \theta}(q(\theta),\theta).$$

□

Proof. **of Lemma 4(*ii*)** We know that q is an interior point of Q, then by the differentiability of T at $q \in q(\theta)$, we have that:

$$T(q+h) = T(q) + T'(q) \cdot h + o(h). \tag{2.33}$$

And equation (2.24) gives us that:

$$T(q+h) \geq v(q+h,\theta) - V(\theta) = v(q+h,\theta) - v(q,\theta) + T(q), \tag{2.34}$$

and so:

$$T(q+h) - T(q) \geq v(q+h,\theta) - v(q,\theta). \tag{2.35}$$

Combining equations (2.33), (2.35) and using the differentiability of $v(q,\theta)$, we have that:

$$T'(q) \cdot h - \frac{\partial v}{\partial q}(q,\theta) \cdot h \geq o(h). \tag{2.36}$$

Taking the limit in equation (2.36) first when $h \uparrow 0$ and then when $h \downarrow 0$ we get that:

$$T'(q) = \frac{\partial v}{\partial q}(q,\theta).$$

□

Proof. **of Lemma 5** We will only prove (i). For (ii), the proof is analogous. The sets CS_+ is given by:

$$CS_+ \quad = \quad \{(\theta,q) \in \Theta \times \mathbb{R}_+ : q(\theta) < q_0(\theta)\}.$$

If q is continuous at θ_0 and $(\theta_0, q(\theta_0)) \in CS_+$, it is possible to build an interval $I = [\theta_0 - \delta, \theta_0 + \delta]$ such that $(\theta, q(\theta)) \in CS_+$ for all $\theta \in I$. Let us take $\theta_1 \in I$ and define:

$$\Delta(\theta_0,\theta_1) = [v(q(\theta_0),\theta_1) - v(q(\theta_0),\theta_0)] + [v(q(\theta_1),\theta_0) - v(q(\theta_1),\theta_1)].$$

By Theorem 5, we must have $\Delta(\theta_0,\theta_1) \leq 0$. Notice that we can write:

$$\Delta(\theta_0,\theta_1) = -\int_{\theta_0}^{\theta_1}\int_{q(\theta_0)}^{q(\theta_1)} v_{q\theta}(q,\theta)dqd\theta.$$

The region of integration is a subset of I, so the integrand $v_{q\theta}$ is always positive. This gives us that:

$$\Delta(\theta_0,\theta_1) \Leftrightarrow (q(\theta_1) - q(\theta_0))(\theta_1 - \theta_0) \geq 0.$$

So we conclude that q must be increasing in θ_0.

□

Proof. **of Lemma 6** We must have:

$$v(q(\theta),\theta) - t(\theta) \geq v(q(\hat{\theta}),\theta) - t(\hat{\theta}).$$

Using the informational rent V, we can write the inequality above as:

$$V(\theta) - V(\hat{\theta}) \geq v(q(\hat{\theta}),\theta) - v(q(\hat{\theta}),\hat{\theta}). \tag{2.37}$$

By Lemma 4(i), we have that $V'(\theta) = v_\theta(q(\theta),\theta)$, and equation (2.37) becomes:

$$\int_{\hat{\theta}}^{\theta} v_\theta(q(\tilde{\theta}),\tilde{\theta})d\tilde{\theta} \geq \int_{\hat{\theta}}^{\theta} v_\theta(q(\hat{\theta}),\tilde{\theta})d\tilde{\theta}.$$

And using the fundamental theorem of calculus, we get the following equation:

$$\int_{\hat{\theta}}^{\theta}\int_{q(\hat{\theta})}^{q(\tilde{\theta})} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta} \geq 0.$$

□

Proof. **of Theorem 1**

We have a variational problem with inequality constraint. In this case the extremum is achieved on a composite curve of pieces of extremals of the unconstrained problem and pieces of extremals with binding constraints.[11]

(i) As the constraint (BS) is not binding, we have an unconstrained maximization problem for ψ_b. The Euler equation for problem Π_{BS} implies that:

$$F_\theta(q,\psi_b(q)) = 0,$$

and using the definition of F in equation (2.8) the result follows.

(ii) BS is binding and $\underline{\theta} < \psi_b(q) < \psi_s(q) < \overline{\theta}$. By introducing a Lagrangian multiplier λ for the constraint, we get:

$$L(q,\psi_b,\psi_s) = F(q,\psi_s(q)) - F(q,\psi_b(q)) + \lambda(q)(v_q(q,\psi_b(q)) - v_q(q,\psi_s(q))).$$

The first order condition for the Lagrangian L results in :

$$\lambda(q) = \frac{F_\theta(q,\psi_s(q))}{v_{q\theta}(q,\psi_s(q))} = \frac{F_\theta(q,\psi_b(q))}{v_{q\theta}(q,\psi_b(q))}.$$

And the result follows from the definition of F in equation (2.8).

[11] Petrov [22] derives the optimality conditions for this kind of problems.

(iii) BS is binding, but now we have a corner solution, as $\psi_s = \overline{\theta}$. So the characterization follows from solving the following constraint equation:

$$v_q(q, \psi_b(q)) = v_q(q, \overline{\theta}).$$

□

Proof. **of Theorem 2** The expected profit coming from the decision $q(\theta, s)$ is given by:

$$\Pi(s) = \int_{\underline{\theta}}^{s} f(q_R(\theta), \theta) d\theta + \int_{s}^{A(s)} f(q_B(\theta), \theta) d\theta + \int_{A(s)}^{\overline{\theta}} f(q_C(\theta, s), \theta) d\theta, \tag{2.38}$$

where $A(s)$ solves:

$$v_q(q_B(s), s) = v_q(q_B(A(s)), A(s)).$$

Differentiating equation (2.38) with respect to s gives us the result.

□

Proof. **of Proposition 2** Consider q_1, q_2 in $q(\theta)$ with $q_1 < q_2$. We have that:

$$\int_{q_1}^{q_2} \left\{ v_q(q, \theta) - T'(q) \right\} dq = v(q_2, \theta) - T(q_2) - (v(q_1, \theta) - T(q_1)) = 0. \tag{2.39}$$

As T is single crossing, if $v_q(q_2, \theta) - T'(q_2) \geq 0$ then:

$$v_q(q, \theta) - T'(q) \geq 0 \quad \forall q \in [q_1, q_2]. \tag{2.40}$$

Now, by equations (2.39) and (2.40) we conclude that

$$v_q(q, \theta) - T'(q) = 0 \quad \forall q \in [q_1, q_2] \text{ a.s.}$$

and remembering that for $q \in [q_1, q_2]$ we have

$$\int_{q_1}^{q} \left\{ v_q(x, \theta) - T'(x) \right\} dx = 0.$$

Then we have:

$$v(q, \theta) - T(q) = v(q_1, \theta) - T(q_1) \quad \forall q \in [q_1, q_2],$$

and so q is convex valued.

□

Proof. **of Theorem 3** This problem is known as an Isoperimetric Problem. It is solved by introducing a Lagrangian multiplier for the (ISO) condition, and appending the constraint with the multiplier to the original problem. Then the multiplier λ should be adjusted to cope with the isoperimetric constraint. The resulting problem is:[12]

$$\max_{\{\psi_s\}} \int_{\underline{q}}^{\overline{q}} \{F(q,\psi_s(q)) - F(q,\theta_d)\}dq + \lambda \int_{q_2}^{q_1} \{v_q(q,\psi_s(q)) - v_q(q,\theta_d)\}dq.$$

The Euler equation for the problem above is exactly:

$$F_\theta(q,\psi_s(q)) + \lambda v_{q\theta}(q,\psi_s(q)) = 0,$$

and remembering the definition of F in equation (2.8) we see that $F_\theta(q,\psi_s(q)) = f_q(q,\psi_s(q))$. Finally, we have to adjust λ in order that the condition (ISO) is satisfied.

□

Proof. **of Theorem 4** The way we have built our optimal candidate allows us to write the monopolist's expected profit as:

$$\begin{aligned}\Pi(q_1,q_2,\theta_d) = &\int_{\underline{q}}^{q_1} \{F(q,\psi_{sr}(q)) - F(q,\psi_{br}(q))\}dq + \\ &+ \int_{q_2}^{\overline{q}} \{F(q,\psi_{sc}(q)) - F(q,\psi_{bc}(q))\}dq + \\ + \int_{q_2}^{q_2} \{F(q,\psi_{si}(q)) - F(q,\theta_d) + \lambda[v_q(q,\psi_{si}(q)) - v_q(q,\theta_d)]\}dq.\end{aligned}$$

Differentiating $\Pi(q_1,q_2,\theta_d)$ with respect to q_1, q_2 and θ_d results in conditions (i), (ii) and (iii), respectively.

□

[12] We refer the interested reader to the book by Petrov [22]. It covers all the material from calculus of variations that we use here.

Chapter 3

Bidimensional Screening

3.1 Introduction

Most of the work available in the literature of hidden information is based on one-dimensional screening models. In this case there is a condition on the fundamentals, the Spence and Mirrlees condition (SMC), that allows an easy characterization of an incentive compatibility allocation rules. The reason is that the SMC provides a natural ordering in the type space which implies in an equivalence between monotonicity and implementability of the agent's decision function. Therefore the principal can restrict his attention to monotonic decision functions and then derive the optimal contract.

However, in the case of multidimensional screening models there is not an analogous condition[1] on the fundamentals. As a consequence, it is very difficult to deal with the incentive compatibility constraints. Unlike the previous case, it is not possible to know *a priori* the directions of the binding constraints. The result is that the principal may have too many constraints to take into account which makes the full characterization of the optimal contract a rather difficult problem.

This chapter is a tentative to extend the techniques developed by Araujo and Moreira [1] to a bidimensional screening context. We study a nonlinear pricing model where the monopolist sells a single product and the customer has a bidimensional characteristic. We study two cases.

[1]The generalized single-crossing property defined by McAfee and McMillan [17] is too restrictive.

In the first one, called discrete-continuous, one of the customer's characteristic is discrete and the other is continuous. In the second one, called bicontinuous, both customer's characteristics are continuous. In the latter case, this chapter is close to McAfee and McMillan [17] and Laffont, Maskin and Rochet [14]. But we do not assume the generalized single-crossing property as in [17], and we generalize the example solved by [14].

In both setups, we derive a new necessary condition for implementability, resulting from the customer's maximization problem. This condition states that if two customers choose the same decision they must pay the same marginal tariff which implies that their marginal utility must be the same. This condition will permit us to find the optimality conditions that we use for solving the monopolist's maximization problem.

The chapter is divided in two main sections, one treating the discrete-continuous setup, and the other one treating the bicontinuous setup.

3.2 Model

We use the Principal-Agent framework to analyse the bidimensional monopolistic screening problem. In this model, each customer has a quasi-linear preference:

$$V(q,t,a,b) = v(q,a,b) - t,$$

where $q \in \mathbb{R}_+$ is the customer's decision, t is the monetary transfer and $(a,b) \in \Theta \subseteq \mathbb{R}^2$ is the customer's characteristics or types which are private information. The distribution function $P(a,b)$ is common knowledge. We assume that the customer's utility function $v(q,a,b)$ is C^3.

The firm is a profit-maximizing monopolist which can produce any quality $q \in Q \subseteq \mathbb{R}_+$ incurring in a cost $C(q)$. The monopolist's revenue is given by:

$$\Pi(q,t) = t - C(q),$$

where $C(q)$ is a C^2, nonnegative, increasing and convex.

Using the *'Revelation Principle'*[2] the monopolist's problem can be stated as choosing the allocation rule $(q,t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ that solves:

$$\max_{\{q(\cdot),t(\cdot)\}} \int_\Theta \Pi(q(a,b), t(a,b)) dP(a,b), \tag{Π}$$

[2] The *'Revelation Principle'* has been enunciated in Gibbard [7].

subject to the *individual-rationality* constraints:

$$v(q(a,b),a,b) - t(a,b) \geq 0 \quad \forall (a,b) \in \Theta, \tag{IR}$$

and the *incentive compatibility* constraints:

$$(a,b) \in \arg \max_{(a',b') \in \Theta} \{v(q(a',b'),a,b) - t(a',b')\}, \quad \forall (a,b) \in \Theta. \tag{IC}$$

Remark 3. *An allocation rule (q,t) satisfying the IC constraints is called incentive compatible. A decision q is implementable if there exists a monetary transfer t such that the pair (q,t) is incentive compatible.*

Remark 4. *The 'Taxation Principle'*[3] *states that any allocation (q,t) satisfying the IC constraints can be implemented by a nonlinear tariff $T : Q = q(\Theta) \to \mathbb{R}$ where:*

$$T(q(a,b)) = t(a,b), \quad \forall (a,b) \in \Theta.$$

3.3 Discrete-Continuous Type

In this section we study a bidimensional monopolist's screening model where the customer's characteristics are represented by two parameters. One is a discrete variable and the other one is a continuous variable. This specification for the parameters can be understood as a limit case when their values are b_1 and b_2, as shown in Figure 3.1.

We have two distinct groups of customers, one with lower b and the other one with higher b. We assume that for a fixed b, the Spence and Mirrlees condition (SMC) is valid. By this assumption, an incentive compatible decision function must be monotonic in the variable a. However, the monotonicity is no longer a sufficient condition for implementability. The reason is that the ordering of the binding IC constraints among customers with lower and higher b is endogenous. In particular, there may exist two types (lower and higher b) choosing the same q.

From the customer's maximization problem, we derive a new necessary IC condition. If two customers choose the same decision they must pay the same marginal tariff. Consequently their marginal utility must be the same. We will refer to it as the *U-condition* (UC).

[3] This principle can be found in Guesnerie [9], Hammond [11] and Rochet [24].

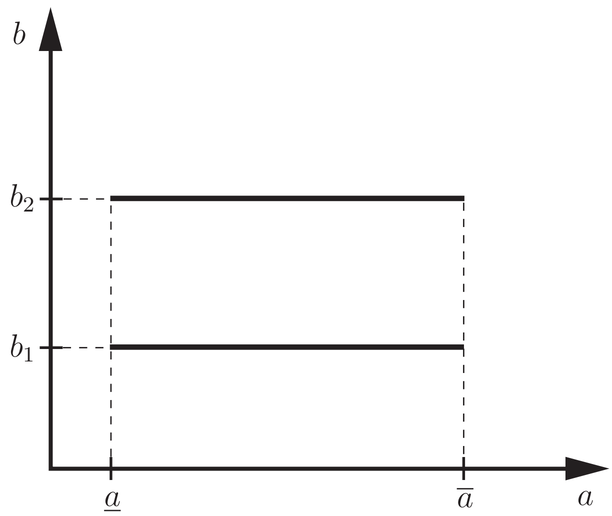

Figure 3.1: The parameter b may take the values b_1 and b_2.

We have two optimization problems: the relaxed problem and the UC problem which takes the *U-condition* into account. Consequently we have two kinds of optimal solutions, depending on whether the *U-condition* is binding or not. The solution we propose is a composite curve of pieces of the relaxed problem's solution and the UC problem's solution.

We assume that the variable a is distributed in $[\underline{a}, \overline{a}]$ with a continuous and positive density $p(a)$ and distribution function given by $P(a)$. The variable b is discrete, taking values in $\{b_1, b_2\}$, with a probability function given by $\Pr(b = b_i) = p_i$. These variables are independent and their joint distribution is given by $P(a, b_i) = p_i P(a)$. The type space is then given by $\Theta = [\underline{a}, \overline{a}] \times \{b_1, b_2\}$.

We assume the following condition:

Definition 12. ***(The Spence and Mirrlees condition.)*** *For a fixed $b = b_i$ the sign of the cross derivative of $v(q, a, b)$ with respect to the decision q and type a is positive:*

$$v_{qa}(q, a, b_i) > 0 \text{ on } \mathbb{R}_+ \times [\underline{a}, \overline{a}]. \tag{SMC}$$

With the SMC one can show that an incentive compatible decision must be increasing with a.[4] This condition, however, is no longer sufficient and we derive now another necessary condition for implementability. By Remark 3 for all incentive compatible allocation rule (q, t),

[4]The proof is exactly the same as the one used for the one-dimensional screening problem with the SMC. See Rochet [25].

we can find a tariff T that implements the decision q. This results in the following maximization problem for the customer (a, b_i):

$$\max_{q \in q(\Theta)} \{v(q, a, b_i) - T(q)\}.$$

The first-order condition for an interior solution of the problem above is:

$$v_q(q, a, b_i) = T'(q).$$

So we can state the following result as a direct consequence of the first-order condition.

Proposition 5. *(**U-condition**.) Suppose that T is differentiable at an interior point of $Q = q(\Theta)$, $q \in q(a, b_1) \cap q(a', b_2)$. Then*

$$v_q(q, a, b_1) = v_q(q, a', b_2). \tag{UC}$$

This marginal utility identity between the pooled customers (a, b_1) and (a', b_2) is called the *U-condition* (UC). It simply states that these customers must pay the same marginal tariff at their same decision. We can solve this marginal identity for a', which results in a relation between the higher and lower b customers that will be useful for deriving the optimality conditions when the UC is binding.

3.3.1 Optimality Conditions

In this subsection we derive the pointwise optimality conditions for the monopolist's problem. There are two maximization problems, one when the UC constraint is not binding and the other when it is binding. The former is the relaxed problem and the latter is the UC problem. The pointwise conditions derived give the relaxed solution and the UC solution q_R and q_{UC}, respectively.

Consider an incentive compatible allocation rule (q, t). Then we define the customer's informational rent by:

$$V(a, b_i) = v(q(a, b_i), a, b_i) - t(a, b_i).$$

We can plug the equation above into the monopolist's objective function in the problem (Π) and the result is:

$$\sum_{i=1}^{2} p_i \int_{\underline{a}}^{\overline{a}} \{v(q(a,b_i),a,b_i) - C(q(a,b_i)) - V(a,b_i)\}p(a)da, \tag{3.1}$$

Using the envelope theorem (see Milgrom and Segall [18]) and an integration by parts procedure we can rewrite the monopolist's objective function as:

$$\sum_{i=1}^{2} p_i \int_{\underline{a}}^{\overline{a}} g(q(a,b_i),a,b_i)da,$$

where g is defined as:

$$g(q,a,b_i) = p_i \left\{ v(q,a,b_i) - C(q) + \left(\frac{P(a)-1}{p(a)}\right) v_a(q,a,b_i) \right\} p(a). \tag{3.2}$$

By dropping the global incentive compatibility constraints we can define a relaxed version of problem (Π) which is called the monopolist's relaxed problem:

$$\max_{q(\cdot)} \sum_{i=1}^{2} p_i \int_{\underline{a}}^{\overline{a}} g(q(a,b_i),a,b_i)da. \tag{Π_R}$$

We have the following imediate result for the problem (Π_R):

Proposition 6. *(**The Relaxed Decision.**) The interior solution of (Π_R) must satisfy the following first-order condition:*

$$g_q(q,a,b_i) = 0. \tag{3.3}$$

Notice that this gives pointwise conditions for a solution of problem (Π_R). The relaxed decision $q_R(a,b_i)$ is exactly the solution of equation (3.3) for each q. The relaxed solution may not satisfy the necessary UC condition. So we may have to incorporate the UC constraint into the monopolist's problem.

Suppose that we have an interval $[r_1,r_2]$ of lower b customers such that the UC constraint is binding for them. By using the UC identity and the implicit function theorem we can define implicitly a function $a(q,r)$ by:

$$v_q(q,r,b_1) = v_q(q,a(q,r),b_2).$$

Let $\phi(r)$ denote the decision function associated with the (r, b_1) customer. Then we have:

$$v_q(\phi(r), r, b_1) = v_q(\phi(r), a(\phi(r), r), b_2). \tag{3.4}$$

Notice that for each $r \in [r_1, r_2]$, the customers (r, b_1) and $(a(\phi(r), r), b_2)$ choose the same decision $\phi(r)$.

Considering equation (3.4), we can write the monopolist's objective function on this interval as:

$$p_1 \int_{r_1}^{r_2} g(\phi(r), r, b_1)dr + p_2 \int_{s_1}^{s_2} g(\psi(s), s, b_2)ds, \tag{3.5}$$

where:

$$\begin{cases} s & = a(\phi(r), r), \\ \psi(s) & = \phi(r), \\ s_1 & = a(\phi(r_1), r_1), \\ s_2 & = a(\phi(r_2), r_2). \end{cases} \tag{3.6}$$

Using the equations in (3.6) we can make a change of variables in expression (3.5), resulting in:

$$\int_{r_1}^{r_2} \{p_1 g(\phi(r), r, b_1) + p_2 g(\phi(r), a(\phi(r), r), b_2)[a_1(\phi(r), r)\phi'(r) + a_r(\phi(r), r)]\}dr.$$

Notice that the UC constraints are incorporated in the expression above. Let us define:

$$F(r, x, y) = p_1 g(x, r, b_1) + p_2 g(x, a(x, r), b_2)[a_1(x, r)y + a_r(x, r)].$$

Then we can define a less constrained version of the monopolist's maximization problem taking into account only the UC constraints:

$$\max_{\phi(\cdot)} \int_{r_1}^{r_2} F(r, \phi(r), \phi'(r))dr. \tag{3.7}$$

We then establish the pointwise optimality conditions for (3.7):

Proposition 7. *(**UC Decision.**) If ϕ^* is the optimal decision for problem (3.7), then:*

$$p_1 \frac{g_q(q, r, b_1)}{v_{qa}(q, r, b_1)} = -p_2 \frac{g_q(q, a(q, r), b_2)}{v_{qa}(q, a(q, r), b_2)}, \tag{3.8}$$

where $q = \phi^*(r)$.

This condition is analogous to the U-shaped condition in Araujo and Moreira [1] and to the Bell-shaped condition from Chapter 2.

Finally we suggest a method for determining the optimal decision. We construct a composite decision combining pieces of the relaxed and the UC decisions. It is important to optimally choose the point where you change from one to another. In what follows we solve an example that illustrates this method.

3.3.2 An Example

In this example, a is uniformly distributed in $[0, 1]$, $b_1 = 0$ and $b_2 = 1$, with $p_1 = p_2 = 1/2$. The customer's utility function is given by:

$$v(q, a, b_i) = aq - (1 + b_i)\frac{q^2}{2},$$

and for simplicity we assume that the monopolist's production costs are zero.

Hence:

$$g(x, a, b_i) = \frac{1}{2}\left\{(2a-1)q(a, b_i) - (b_i + 1)\frac{q(a, b_i)^2}{2}\right\}.$$

Using Proposition 6, we can find the relaxed decision:

$$\begin{cases} q_R(a, 0) = \{2a - 1\}^+, \\ q_R(a, 1) = \{\frac{2a-1}{2}\}^+. \end{cases}$$

The relaxed solution is not incentive compatible, because it does not satisfy the necessary UC condition.

Using the U-condition (UC), we have that:

$$a(q, r) = r + q,$$

and, using Proposition 7, we can find the UC decision:

$$\begin{cases} q_{UC}(a, 0) = \{4a - 2\}^+, \\ q_{UC}(a, 1) = \{\frac{4a-2}{5}\}^+. \end{cases}$$

In Figure 3.2 we plot the relaxed and the UC decision functions.

Our strategy now is to construct a decision function combining the relaxed and the UC decision, depending on the parameter a_U, which is the point where we change from one decision

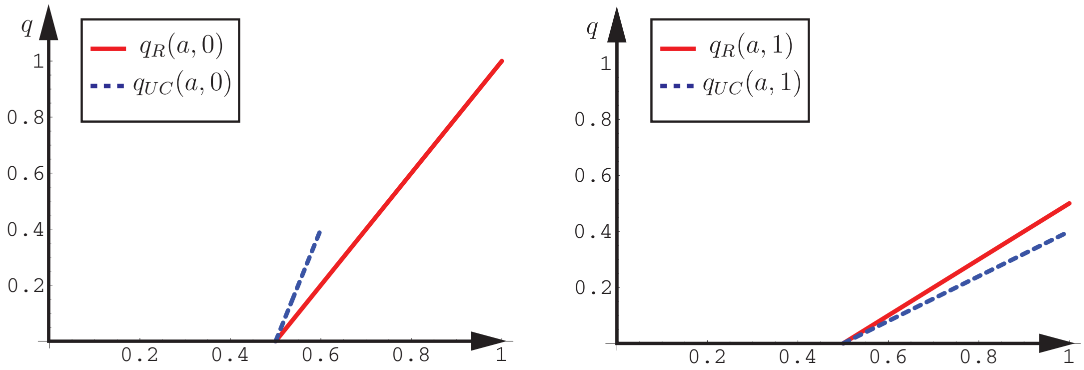

Figure 3.2: The relaxed and the UC decision functions.

to the other:

$$q(a,0,a_U) = \begin{cases} 0 & \text{if } 0 \le a \le \frac{1}{2}, \\ q_{UC}(a,0) & \text{if } \frac{1}{2} \le a \le a_U, \\ q_{UC}(a_U,0) & \text{if } a_U \le a \le a_R, \\ q_R(a,0) & \text{if } a_R \le a \le 1. \end{cases} \tag{3.9}$$

$$q(a,1,a_U) = \begin{cases} 0 & \text{if } 0 \le a \le \frac{1}{2}, \\ q_{UC}(a_U,1) & \text{if } \frac{1}{2} \le a \le \hat{a}_U, \\ q_R(\hat{a}_U,1) & \text{if } \hat{a}_U \le a \le 1, \end{cases} \tag{3.10}$$

where $\hat{a}_U = a(q_{UC}(a_U,0),a_U)$ and a_R is such that $q_{UC}(a_U,0) = q_R(a_R,0)$. In Figure 3.3 we plot this decision function which combines the relaxed decision and the UC decision.

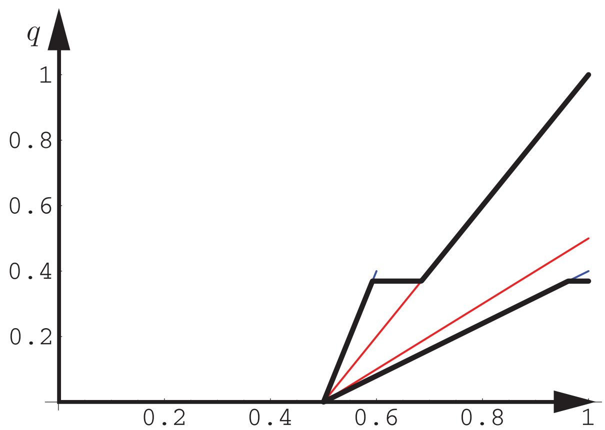

Figure 3.3: The composite solution in thicker black.

From the objective function in (Π_R), we can determine the monopolist profit as a function of a_U (see Figure 3.4):

$$\Pi(a_U) = \frac{28a_U^3}{3} - 18a_U^2 + \frac{23a_U}{2} - \frac{19}{8}.$$

Now we have to optimally choose the parameter a_U.[5] By solving the first-order condition

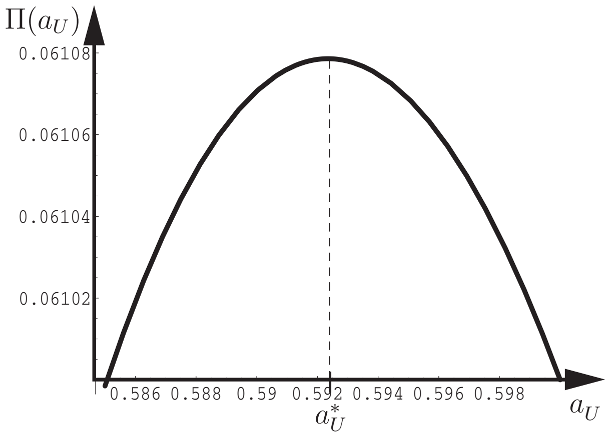

Figure 3.4: The expected profit depending as a function of a_U.

$\Pi'(a_U) = 0$, we find the optimal choice for a_U:

$$a_U^* = \frac{1}{28}(18 - \sqrt{2}).$$

So the solution we propose results from equations (3.9) and (3.10), choosing the parameter $a_U = a_U^*$. This solution is implemented by the following tariff:

$$T(q) = \begin{cases} \frac{1}{8}q(4-3q), & \text{if } 0 \le q \le \frac{1}{7}(4-\sqrt{2}), \\ \frac{1}{196}(-9+4\sqrt{2}+98q-49q^2), & \text{if } \frac{1}{7}(4-\sqrt{2}) \le q \le 1, \end{cases}$$

which is determined by integrating the marginal utility.

Comparing the relaxed decision $q_R(a,b)$ with $q(a,b,a_U^*)$, we have a decrease of approximately 2.3% in the expected profit.

3.4 Bicontinuous Type

In this section we show how to use the marginal utility condition to solve nonlinear pricing model where the customer has two dimensions of private information. Unlike the case treated

[5]The optimal a_U^* can also be found through an ironing procedure. See Araujo and Moreira [1].

in the previous section, where we have a continuous and a discrete parameter for the type, now we have that both dimensions are continuous, that is, the type space is the rectangle $\Theta = (0, \alpha) \times (0, \beta)$, with $\alpha, \beta > 0$.

As the decision q is a function defined from $\Theta \subseteq \mathbb{R}^2$ into $\mathbb{R}_+$, we expect that its level sets are one-dimensional curves with images contained in Θ. The level sets for q represents a continuous pooling of types, and the marginal condition states that pooling types must have identical marginal utility at the same chosen decision. The result is that both $q(a, b)$ and $v_q(q(a, b), a, b)$ should have common level sets. This principle may be expressed in a convenient way by using partial differential equation (PDE), as we will see in what follows.

Analysing the customer's maximization problem we derive a partial differential equation (PDE) whose solution can be interpreted as resolving this relation among the level sets. This is a quasilinear first-order partial differential equation and we use the method of characteristic curves to solve it. This method is adequate for our problem because the characteristic curves for this particular PDE is exactly the level sets of both $q(a, b)$ and $v_q(q(a, b), a, b)$. In the end it provides us a natural change of variables that reduces the problem from two dimensions to only one dimension. Then we apply variational calculus techniques to find the solution of this one-dimensional problem and finally we go back to the original variables.

The parameter representing the customer's type $(a, b) \in \Theta$ has a positive and continuous density $p(a, b)$. For simplicity, we assume that the monopolist's production costs are zero.

We also assume that the monetary transference t is positive. Consequently the tariff T is positive for all units consumed q. This means that the monopolist ever has positive profits when selling to any customer (a, b). Consider an incentive compatible allocation rule (q, t). From the (IC) condition, each customer has the following maximization problem:

$$\max_{(a',b')\in\Theta} \{v(q(a', b'), a, b) - t(a', b')\}.$$

The first-order conditions for the maximization problem above give us an analogous UC condition of the discrete-continuous problem[6]:

Proposition 8. ***(The Quasilinear Equation.)*** *Suppose that* $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ *is incentive*

[6] Actually the UC condition is equivalent to the PDE (3.11). The PDE formulation is more convenient to the bicontinuous model.

compatible and twice differentiable in an open set $\Omega \subset \Theta$. Then we have that:

$$v_{qa}(q(a,b),a,b)q_b(a,b) = v_{qb}(q(a,b),a,b)q_a(a,b). \tag{3.11}$$

Equation (3.11) is classified as a first-order partial differential equation (PDE)[7]. We can interpret it geometrically by observing that the gradients $\nabla_{a,b}v_q$ and ∇q have the same direction and, consequently, both q and v_q have the same level set.

Although Proposition 8 is a general result, we will work with a particular specification for the customer's utility function. This allow us to show how to use the PDE derived to solve the monopolist's problem. Hence we assume that the customer's utility function is given by:

$$v(q,a,b) = aq - (c + w(b))\frac{q^\gamma}{\gamma} - t,$$

where a and b are distributed independently and uniformly on Θ; $w(0) = 0$ and $w'(b) > 0$; $\gamma > 1$, q represents the consumption choice and t is the monetary transfer. The monopolist's production cost $C(q)$ is assumed to be zero.

For an incentive compatible allocation rule (q,t), the customer's informational rent is given by $V(a,b) = v(q(a,b),a,b) - t(a,b)$. As the production costs are now zero, if we plug the informational rent in equation (Π) we can rewrite the monopolist's objective function as:

$$\frac{1}{\alpha\beta}\int_0^\alpha\int_0^\beta \{v(q(a,b),a,b) - V(a,b)\}dadb.$$

Using the envelope theorem (see [18]), and an integration by parts procedure, we can derive the following result about the monopolist's expected profit:

Proposition 9. ***(The Monopolist's Expected Profit.)*** *Let q be an implementable decision. Then the monopolist's expected profit is given by:*

$$\frac{1}{\alpha\beta}\int_0^\alpha\int_0^\beta \{(2a-\alpha)q(a,b) - (c + w(b))\frac{q^\gamma}{\gamma}\}dadb. \tag{3.12}$$

[7] See Fritz John [12] for a complete analysis of this kind of PDE and the method of characteristic curves for solving it.

Let us define $G(q,a,b) = (2a-\alpha)q(a,b) - (c+w(b))\frac{q^\gamma}{\gamma}$.

From Proposition 8, we can establish the PDE (3.11) for our specific model:

$$w'(b)q^{\gamma-1}q_a + q_b = 0. \tag{3.13}$$

We now formulate the following Cauchy problem for this PDE:

$$\begin{cases} w'(b)q^{\gamma-1}q_a + q_b = 0, \\ q(r,0) = \phi(r), \end{cases} \tag{3.14}$$

where we assume that ϕ is a continuous and increasing function such that $\phi(0) = 0$.

We now use the method of characteristic curves to find the solution of (3.14). The method works in the following way: along the characteristic curves the PDE equation becomes an ordinary differential equation (ODE) with initial condition given by $q(r,0) = \phi(r)$. So, for each fixed r, we have an ODE with initial value and we then invoke the existence and uniquiness theorems to solve it. We choose the curve $(r,0)$ to give the initial value and it is important that the characteristic curves and the initial value curve to be not tangent. The idea is that we can find the characteristic curves and relate the value of q in Θ to the value of q in $(r,0)$. In Figure 3.5, the projections of the characteristic curves are plotted. As we can see in the figure, they are not tangent to the line $(r,0)$ and the value of q is constant along these curves.

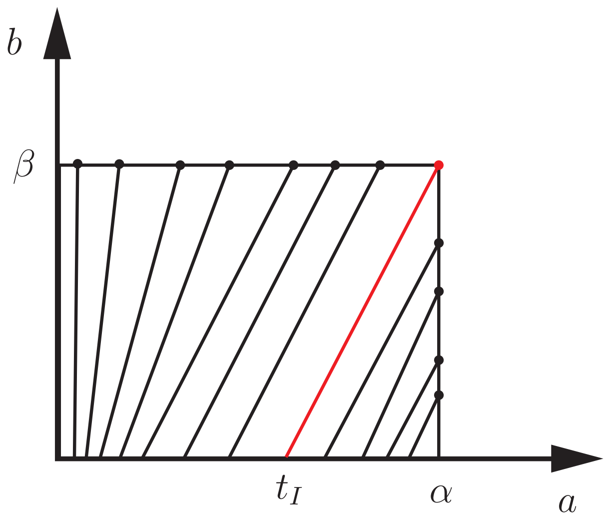

Figure 3.5: Characteristic curves in Θ.

So using the method of characteristic curves, we can establish the following result:

Proposition 10. *(**The Cauchy Problem**.) The solution for the Cauchy problem (3.14) is given by:*

$$q(a,b) = q(a(r,s), b(r,s)) = \phi(r(a,b)),$$

where we get r by inverting the following equations:

$$\begin{cases} a(r,s) = r + w(s)\phi(r)^{\gamma-1}, \\ b(r,s) = s. \end{cases}$$

For each r we have characteristic curves parametrized as $(a(r,s), b(r,s), \phi(r))$. With them we have a natural change of variables that we will plug in equation (3.12). This results in the following system of variables:

$$\begin{cases} a = A(x,r,s) = r + w(s)x^{\gamma-1}, \\ b = B(r,s) = s, \end{cases}$$

where the Jacobian is simply $A_1\phi'(r) + A_2$, such that $A_1 = \frac{\partial A}{\partial x}$ and $A_2 = \frac{\partial A}{\partial r}$.

Using these new variables, we can rewrite the monopolist's expected profit as

$$\frac{1}{\alpha\beta}\int_0^{\alpha}\int_0^{L(\phi(r),r)} \{G(\phi(r), A(\phi(r),r,s),s)\left[A_1\phi'(r) + A_2\right]\} ds dr,$$

where, for each r, L is defined as the s coordinate of the intersection between the characteristic curve and the boundary of Θ. Notice that there should exist an intermediate r_I such that for $r < r_I$, L is constant with $L = \beta$ and for $r > r_I$, L solves $A(x, r, L(x,r)) \equiv \alpha$. In Figure 3.5, L is represented by dot points.

Then we can define a less constrained version of the monopolist's maximization problem, by dropping the IC constraints and considering only the constraints following from the EDP given in Proposition 8:

$$\max_{\phi(\cdot)} \frac{1}{\alpha\beta}\int_0^{\alpha}\int_0^{L(\phi(r),r)} \{G(\phi(r), A(\phi(r),r,s),s)\left[A_1\phi'(r) + A_2\right]\} ds dr. \tag{3.15}$$

Now we can establish our main result:

Proposition 11. *The Euler's equation for the monopolist's optimization problem (3.15) is given by:*

$$\int_0^{L(\phi(r),r)} G_1(\phi(r), A(\phi(r),r,s),s) ds = 0. \tag{3.16}$$

3.4.1 Examples

Here we solve some examples of bidimensional contracts with one instrument.

Example 1. ***(Laffont, Maskin and Rochet, 1987.)*** *This example was taken from [14]. We set $\alpha = \beta = c = 1$, $\gamma = 2$ and $w(b) = b$ in our model.*

We have that:

$$\begin{cases} G(q,a,b) &=& (2a-1)q - \frac{1}{2}(b+1)q^2, \\ A(x,r,s) &=& r + sx, \\ B(x,r,s) &=& s. \end{cases}$$

We can determine L as:

$$L(x,r) = \begin{cases} 1 & \text{, if } r < r_I, \\ \frac{1-r}{x} & \text{, if } r > r_I. \end{cases}$$

By Proposition 11, the Euler equation is given by:

$$\int_0^{L(\phi(r),r)} G_1(\phi(r), A(\phi(r),r,s),s) A_2(\phi(r),r,s) ds = 0.$$

This gives us:

$$\int_0^L \{(2(r+s\phi) - 1 - (s+1)\phi)\}\, ds = (2r - 1 - \phi)L + \frac{L^2}{2}\phi = 0.$$

Solving the equation above for ϕ results in:

$$\phi(r) = \begin{cases} 0 & \text{, if } 0 \le r \le \frac{1}{2}, \\ 4r-2 & \text{, if } \frac{1}{2} \le r \le \frac{3}{5}, \\ \frac{3r-1}{2} & \text{, if } \frac{3}{5} \le r \le 1. \end{cases}$$

Now we have to go back to the original variables. Solving the equation $A(\phi(r),r,s) = r + s\phi(r)$ for r in terms of a and b, we get:

$$r(a,b) = \begin{cases} a & \text{, if } a \le \frac{1}{2}, \\ \frac{a+2b}{1+4b} & \text{, if } \frac{1}{2} \le \frac{a+2b}{1+4b} \le \frac{3}{5}, \\ \frac{2a+b}{2+3b} & \text{, if } \frac{3}{5} \le \frac{2a+b}{2+3b} \le 1. \end{cases}$$

We have that $q(a,b) = \phi(r(a,b))$. Making the substitution we can find the optimal decision $q(a,b)$:

$$q(a,b) = \begin{cases} 0 & \text{, if } a \le \frac{1}{2}, \\ \frac{4a-2}{1+4b} & \text{, if } \frac{1}{2} \le \frac{a+2b}{1+4b} \le \frac{3}{5}, \\ \frac{3a-1}{2+3b} & \text{, if } \frac{3}{5} \le \frac{2a+b}{2+3b} \le 1, \end{cases}$$

and also the nonlinear tariff[8] T that implements q:

$$T(q) = \begin{cases} \frac{q}{2} - \frac{3q^2}{8} & \text{, if } q \leq \frac{2}{5}, \\ \frac{q}{3} - \frac{q^2}{6} + \frac{1}{30} & \text{, if } q \geq \frac{2}{5}. \end{cases}$$

Example 2. *This example is an slight variation of the previous one. The fundamental difference is that now the characteristic curves are not straight lines for* $r > 1/2$. *We set* $\alpha = \beta = c = 1$, $\gamma = 2$ *and* $w(b) = b^2$ *in our model.*

We have that:

$$\begin{cases} G(q,a,b) &= (2a-1)q - \frac{1}{2}(b^2+1)q^2, \\ A(x,r,s) &= r + s^2 x, \\ B(x,r,s) &= s. \end{cases}$$

We can determine L as:

$$L(x,r) = \begin{cases} 1 & \text{, if } r \leq r_I, \\ \sqrt{\frac{1-r}{x}} & \text{, if } r \geq r_I. \end{cases}$$

By Proposition 11, the Euler equation is given by:

$$\int_0^{L(\phi(r),r)} G_1(\phi(r), A(\phi(r),r,s), s) A_1(\phi(r),r,s) ds = 0.$$

This gives us:

$$\int_0^L \left\{ (2(r + s^2\phi) - 1 - (s^2+1)\phi) \right\} ds = (2r - 1 - \phi)L + \frac{L^3}{3}\phi = 0.$$

Solving the equation above for ϕ results in:

$$\phi(r) = \begin{cases} 0 & \text{, if } 0 \leq r \leq \frac{1}{2}, \\ \frac{6r-3}{2} & \text{, if } \frac{1}{2} \leq r \leq \frac{5}{8}, \\ \frac{5r-2}{3} & \text{, if } \frac{5}{8} \leq r \leq 1. \end{cases}$$

Now we have to go back to the original variables. And solving the equation $A(\phi(r),r,s) = r + s^2\phi(r)$ for r in terms of a and b, we get:

$$r(a,b) = \begin{cases} a & \text{, if } a \leq \frac{1}{2}, \\ \frac{2a+3b^2}{2+6b^2} & \text{, if } \frac{1}{2} \leq \frac{2a+3b^2}{2+6b^2} \leq \frac{5}{8}, \\ \frac{3a+2b^2}{3+5b^2} & \text{, if } \frac{5}{8} \leq \frac{3a+2b^2}{3+5b^2} \leq 1. \end{cases}$$

[8] We find the nonlinear tariff T by integrating the marginal tariff given by $v_q(q,a,0)$.

We have that $q(a,b) = \phi(r(a,b))$. Making the substitution we can find the optimal decision $q(a,b)$:

$$q(a,b) = \begin{cases} 0 & \text{, if } a \leq \frac{1}{2}, \\ \frac{6a-3}{2+6b^2} & \text{, if } \frac{1}{2} \leq \frac{2a+3b^2}{2+6b^2} \leq \frac{5}{8}, \\ \frac{5a-2}{3+5b^2} & \text{, if } \frac{5}{8} \leq \frac{3a+2b^2}{3+5b^2} \leq 1, \end{cases}$$

and also the nonlinear tariff T that implements q:

$$T(q) = \begin{cases} \frac{q}{2} - \frac{q^2}{3} & \text{, if } q \leq \frac{3}{8}, \\ \frac{2q}{5} - \frac{q^2}{5} + \frac{3}{160} & \text{, if } q \geq \frac{3}{8}. \end{cases}$$

Example 3. ***(Basov, 2005.)*** *This is a generalization of the first one and was proposed by Basov [4]. We set $w(b) = b$ and $\beta < 2c$ in our model.*

We have that:

$$\begin{cases} G(q,a,b) &= (2a-\alpha)q - \frac{1}{\gamma}(b+c)q^\gamma, \\ A(x,r,s) &= r + sx^{\gamma-1}, \\ B(x,r,s) &= s. \end{cases}$$

We can determine L as:

$$L(x,r) = \begin{cases} \beta & \text{, if } r \leq r_I, \\ \frac{\alpha-r}{x^{\gamma-1}} & \text{, if } r \geq r_I. \end{cases}$$

By Proposition 11, the Euler equation is given by:

$$\int_0^{L(\phi(r),r)} G_1(\phi(r), A(\phi(r),r,s), s) A_1(\phi(r),r,s) ds = 0.$$

This gives us:

$$\int_0^L \left\{(2(r + s\phi^{\gamma-1}) - \alpha - (s+c)\phi^{\gamma-1})\right\} ds = (2r - \alpha - c\phi^{\gamma-1})L + \frac{L^2}{2}\phi^{\gamma-1} = 0.$$

Solving the equation above for ϕ results in:

$$\phi(r) = \begin{cases} 0 & \text{, if } 0 \leq r \leq \frac{\alpha}{2}, \\ \left(\frac{4r-2\alpha}{2c-\beta}\right)^{\frac{1}{\gamma-1}} & \text{, if } \frac{\alpha}{2} \leq r \leq \alpha\frac{2c+\beta}{2c+3\beta}, \\ \left(\frac{3r-\alpha}{2c}\right)^{\frac{1}{\gamma-1}} & \text{, if } \alpha\frac{2c+\beta}{2c+3\beta} \leq r \leq \alpha. \end{cases}$$

Now we have to go back to the original variables. And solving the equation $A(\phi(r), r, s) = r + s^2\phi(r)$ for r in terms of a and b, we get:

$$r(a,b) = \begin{cases} a & ,\text{if } a \leq \frac{\alpha}{2}, \\ \frac{a(2c-\beta)+2\alpha b}{2c-\beta+4b} & ,\text{if } \frac{\alpha}{2} \leq \frac{a(2c-\beta)+2\alpha b}{2c-\beta+4b} \leq \alpha\frac{2c+\beta}{2c+3\beta}, \\ \frac{2ac+\alpha b}{2c+3b} & ,\text{if } \alpha\frac{2c+\beta}{2c+3\beta} \leq \frac{2ac+\alpha b}{2c+3b} \leq \alpha. \end{cases}$$

We have that $q(a,b) = \phi(r(a,b))$. Making the substitution we can find the optimal decision $q(a,b)$:

$$q(a,b) = \begin{cases} 0 & ,\text{if } a \leq \frac{\alpha}{2}, \\ \left(\frac{4a-2\alpha}{2c-\beta+4b}\right)^{\frac{1}{\gamma-1}} & ,\text{if } \frac{\alpha}{2} \leq \frac{a(2c-\beta)+2\alpha b}{2c-\beta+4b} \leq \alpha\frac{2c+\beta}{2c+3\beta}, \\ \left(\frac{3a-\alpha}{2c+3b}\right)^{\frac{1}{\gamma-1}} & ,\text{if } \alpha\frac{2c+\beta}{2c+3\beta} \leq \frac{2ac+\alpha b}{2c+3b} \leq \alpha, \end{cases}$$

and also the nonlinear tariff T that implements q:

$$T(q) = \begin{cases} \dfrac{\alpha q}{2} - \dfrac{(\beta+2c)q^\gamma}{4\gamma} & ,\text{if } q \leq \frac{2\alpha}{3\beta+2c}, \\ \dfrac{\alpha q}{3} - \dfrac{cq^\gamma}{3\gamma} + \dfrac{2\alpha^2}{18\beta+12c} - \dfrac{3\beta+2c}{12\gamma}\left(\dfrac{2\alpha}{3\beta+2c}\right)^\gamma & ,\text{if } q \geq \frac{2\alpha}{3\beta+2c}. \end{cases}$$

3.5 Conclusion

In this chapter we showed that the techniques by Araujo and Moreira [1] can be extended to the bidimensional context at least for some particular specifications. The marginal utility identity plays a fundamental role in our analysis. In a sense, it allows us to reduce the problem to only one dimension. Then we could derive the optimality conditions for the monopolist's maximization problem.

In the discrete-continuous setup we derived the pointwise optimality conditions, and then we gave an example illustrating how to use them to build a solution for the monopolist's maximization problem in a way similar to Araujo and Moreira [1]. In the bidimensional setup, we also derived optimality conditions and then we were able to solve examples from the literature, and to solve a new example that does not satisfy the generalized single-crossing property assumed by [17].

Appendix

Proof. **of Proposition 5** The customer (a, b_i) has the following maximization problem:

$$\max_{q \in q(\Theta)} \{v(q, a, b_i) - T(q)\}.$$

If $q = q(a, b_1) = q(a, b_2)$ is an interior point of $Q = q(\Theta)$ and T is differentiable at q, deriving the first-order condition for the maximization problem above results in:

$$v_q(q, a, b_1) = T'(q) = v_q(q, a, b_2).$$

□

Proof. **of Proposition 6** The Euler's equation[9] for problem (Π_R) is exactly $g_q(q, a, b_i) = 0$.

□

Proof. **of Proposition 7** Consider the following functional:

$$\int_{r_1}^{r_2} F(r, \phi(r), \phi'(r))dr.$$

The Euler's equation for the functional above is:

$$F_x(r, \phi(r), \phi'(r)) = \frac{d}{dr} F_y(r, \phi(r), \phi'(r)).$$

If we expand the Euler's equation, we get:

$$p1 g_q(\phi(r), r, b_1) + p2 g_q(\phi(r), a(\phi(r), r), b_2) a_r(\phi(r), r) = 0.$$

But the function $a(q, r)$ is implicitly defined as the solution of:

$$v_q(q, r, b_1) = v_q(q, a(q, r), b_1).$$

Differentiating the equation above with respect to q results in:

$$a_r(q, r) = \frac{v_{qa}(q, r, b_1)}{v_{qa}(q, a(q, r), b_2)},$$

and the result follows.

□

[9] See Petrov [22].

Proof. **of Proposition 8** For each customer (a, b), the first-order conditions from the (IC) give that:

$$t_a = v_q(q(a,b), a, b)q_a(a,b),$$
$$t_b = v_q(q(a,b), a, b)q_b(a,b).$$

By taking the cross derivatives in both equations above, we get:

$$t_{ab} = (v_{qq}(q,a,b)q_b + v_{qb}(q,a,b))q_a + v_q(q,a,b)q_{ab},$$
$$t_{ba} = (v_{qq}(q,a,b)q_a + v_{qa}(q,a,b))q_b + v_q(q,a,b)q_{ba}.$$

As t is twice differentiable at (a, b), using Schwarz' theorem we have that the cross derivatives are equal and the result follows.

□

Proof. **of Proposition 9** Consider an incentive compatible (IC) allocation rule (q, t). For each customer (a, b) we should have:

$$V(a,b) = v(q(a,b), a, b) - t(a,b) \geq v(q(a',b'), a, b) - t(a',b'), \quad \forall (a,b), (a',b'),$$

and using the envelope theorem (see Milgrom and Segall [18]), we have that:

$$V_a(a,b) = v_a(q(a,b), a, b) = q(a,b) \text{ and } V_b(a,b) = v_b(q(a,b), a, b) = -w'(b)\frac{q(a,b)^\gamma}{\gamma}.$$

The fundamental theorem of calculus allows us to write:

$$V(a,b) = V(0,b) + \int_0^a q(\theta, b) d\theta,$$

and

$$V(0,b) = V(0,0) - \int_0^b w'(\theta)\frac{q(0,\theta)^\gamma}{\gamma} d\theta.$$

We have that $v(q, 0, b) \leq 0$ and, by assumption, t is always positive. As a consequence we must have $q(0, b) = 0$. Otherwise, customer $(0, b)$ would have a negative utility. This results in:

$$V(0,b) = 0 \text{ and } V(a,b) = \int_0^a q(\theta, b) d\theta.$$

Using the informational rent we can replace $t(a, b)$ in the monopolist's objective function in the problem (Π):

$$\frac{1}{\alpha\beta}\int_0^\alpha\int_0^\beta \{v(q(a,b), a, b) - V(a,b)\} da db. \tag{3.17}$$

Using an integration by parts procedure we have that:

$$
\begin{aligned}
\int_0^\beta\int_0^\alpha V(a,b)dadb &= \int_0^\beta\int_0^\alpha\int_0^a q(\theta,b)d\theta dadb \\
&= \int_0^\beta \left[a\int_0^a q(\theta,b)d\theta\Big[_0^\alpha - \int_0^\alpha aq(\theta,b)da\right] \\
&= \int_0^\beta\int_0^\alpha (\alpha-a)q(a,b)dadb.
\end{aligned}
$$

Plugging the equation above in equation (3.17) we get the result.

□

Proof. **of Proposition 10** The characteristic curves for equation (3.11) emerge as the solution of the following system of ordinary differential equations[10]:

$$
\begin{cases}
\frac{da}{ds} = -v_{qb}(z(r,s),a(r,s),b(r,s)) = w'(b)z^{\gamma-1}, & \text{with } a(r,0)=r; \\
\frac{db}{ds} = v_{qa}(z(r,s),a(r,s),b(r,s)) = 1, & \text{with } b(r,0)=0; \\
\frac{dZ}{ds} = 0, & \text{with } z(r,0)=\phi(r),
\end{cases}
$$

and the solution for the system of ordinary differential equations above is given by:

$$
\begin{cases}
a(r,s) = r + w(s)\phi(r)^{\gamma-1}; \\
b(r,s) = s; \\
z(r,s) = \phi(r).
\end{cases}
$$

We have that $q(a,b) = z(r(a,b),s(a,b) = \phi(r(a,b))$ and we only have to invert the variables to find the solution.

□

Proof. **of Proposition 11** Define the function:

$$
J(\epsilon) = \int_0^\alpha\int_0^{L(\phi(r)+\epsilon h(r),r)} \{G(\phi(r)+\epsilon h(r), A(\phi(r)+\epsilon h(r),r,s),s)
$$

$$
[A_1(\phi(r)+\epsilon h(r),r,s),s)(\phi'(r)+\epsilon h'(r)) + A_2(\phi(r)+\epsilon h(r),r,s),s)]\}dsdr,
$$

[10]See Fritz John [12].

where $h(r)$ is an increasing function such that $h(0) = h(\alpha) = 0$. Differentiating J with respect to ϵ, and evaluating at $\epsilon = 0$, we get:

$$\int_0^{\alpha}\left\{\int_0^{L(\phi(r),r)} \big([G_1 + G_2A_1][A_1\phi' + A_2] + G[A_{11}\phi' + A_{21}]\big)\,ds + G(\phi, A(\phi, r, L), L)[A_1(\phi, r, L)\phi' + A_2(\phi, r, L)]\right\} h(r)dr + \int_0^{\alpha}\left\{\int_0^{L(\phi(r),r)} GA_1 ds\right\} h'(r)dr.$$

Rewriting the expression above and imposing that $J'(0) = 0$ results in:

$$\int_0^{\alpha} \big\{I_1(r)h(r) + I_2(r)h'(r)\big\}\, dr = 0.$$

Using a classical result from calculus[11], we conclude from equation above that:

$$I_1(r) = I_2'(r). \tag{3.18}$$

Using equation (3.18), we get:

$$\int_0^{L(\phi(r),r)} G_1A_2 ds + \overline{G}(\overline{A}_2L_1 - \overline{A}_1L_2) = 0, \tag{3.19}$$

where

$$\begin{cases} \overline{G} = G(\phi, A(\phi, r, L), L), \\ \overline{A}_1 = A_1(\phi, r, L), \\ \overline{A}_2 = A_2(\phi, r, L). \end{cases}$$

Now, we observe that if $r < r_I$, then $L(x, r) = \beta$ and so $L_1 = L_2 = 0$; if $r > r_I$, L solves $A(x, r, L(x, r)) \equiv 1$ and deriving this equation with respect to x and r, we get:

$$\left\{\ \overline{A}_1 + \overline{A}_3L_1 = 0, \overline{A}_2 + \overline{A}_3L_2 = 0, \right. \tag{3.20}$$

and this gives us:

$$\begin{cases} L_1 = -\overline{A}_1/\overline{A}_3, \\ L_2 = -\overline{A}_2/\overline{A}_3. \end{cases} \tag{3.21}$$

Finally, we conclude that $\overline{A}_2L_1 - \overline{A}_1L_2 = 0$ and we can simplify equation (3.19):

$$\int_0^{L(\phi(t),r)} G_1A_2 ds = 0, \tag{3.22}$$

and recalling that $A(x, r, s) = r + w(s)x^{\gamma-1}$, we have that $A_2 = 1$, and the result follows.

□

[11] See Troutman [28], pg. 98.

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