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Equilibrium fluctuations for a nongradient energy conserving stochastic model

TESE

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A Isabella

"De amor es el cuento, que quiero contarte,
un día cualquiera o en este momento ...
Mi querida niña, de pequeño talle,
de graciosos ojos y de dulce aliento ...
Quisiera ... que toda tu vida de amor,
sea un cuento ... Y tus días felices ...
Y a medida que crezcas, ese cuento de amor,
en ti, sea cierto ... "

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Contents

0.1 Resumo

Em trabalhos recentes (*e.g.* [BO], [Be],[FNO]) um modelo microscópico para a condução de calor em sólidos tem sido considerado. Neste modelo, os átomos interagem como um sistema de osciladores harmônicos perturbado por medio de um ruído que conserva energia. Dito ruído troca energia cinética entre vizinhos mais próximos.

Mais precisamente, no caso de condições de contorno periódicas, os átomos são marcados com $x \in \mathbb{T}_N = \{1, \cdots, N\}$. O espaço de estados é definido por $\Omega^N = (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$ onde, para uma configuração típica $(p_x, r_x)_{\mathbb{T}_N} \in \Omega^N$, r_x representa a distância entre as partículas x e $x+1$, e p_x a velocidade da partícula x. O gerador formal do sistema $\mathcal{L}_N$ escreve-se como a soma dos operadores,

$$\mathcal{A}_N = \sum_{x \in \mathbb{T}_N} \{(p_{x+1} - p_x)\partial_{r_x} + (r_x - r_{x-1})\partial_{p_x}\} , \tag{0.1.1}$$

e

$$\mathcal{S}_N = \frac{1}{2} \sum_{x \in \mathbb{T}_N} X_{x,x+1}[X_{x,x+1}] , \tag{0.1.2}$$

onde $X_{x,y} = p_y \partial_{p_x} - p_x \partial_{p_y}$. Aqui $\mathcal{A}_N$ representa o operador de Liouville correspondente a um sistema de osciladores harmônicos e $\mathcal{S}_N$ representa o operador de ruído.

O presente trabalho concentra-se no operador de ruído $\mathcal{S}_N$, que atua somente em velocidades. Assim, o espaço de configurações pode-se restringir à $\mathbb{R}^{\mathbb{T}_N}$. A energia total da configuração $(p_x)_{x \in \mathbb{T}_N}$ é definida como

$$\mathcal{E} = \frac{1}{2} \sum_{x \in \mathbb{T}_N} p_x^2 . \tag{0.1.3}$$

É fácil verificar que $\mathcal{S}_N(\mathcal{E}) = 0$, *i.e* a energia total é constante no tempo.

A dinâmica induzida por $\mathcal{S}_N$ resulta ser um *sistema gradiente*, isto é, a corrente microscópica instantânea de energia entre x e $x+1$ pode-se exprimir como o gradiente de um função local, de fato, $W_{x,x+1} = \frac{1}{2}(p_{x+1} - p_x)$.

De um ponto de vista físico, os *sistemas não gradiente* fornecem modelos mais realistas, e de um ponto de vista matemático, os *sistemas gradiente* formam um conjunto de dimensão pequena no espaço de modelos estocásticos reversíveis que apresentam leis de conservação locais (veja [W] e referências ali citadas).

Neste momento, o único método para lidar com *sistemas não gradiente* é o desenvolvido por S.R.S Varadhan (cf. [V]) onde, *grosso modo*, a idéia é obter uma decom-

posição da corrente como a soma de um termo dissipativo mais um termo de flutuação.

Por analogia com [V] introduzimos não homogeneidades na difusão gerada por (0.2.2),

$$\mathcal{L}_N = \frac{1}{2} \sum_{x \in \mathbb{T}_N} X_{x,x+1}[a(p_x, p_{x+1}) X_{x,x+1}] \ , \qquad (0.1.4)$$

onde $a(x,y)$ é uma função diferenciável satisfazendo $0 < c \leq a(x,y) \leq C < \infty$, com derivadas de primeira ordem contínuas e limitadas. A introdução desta função, implica na perda da estrutura gradiente de (0.1.2).

O comportamento coletivo do sistema pode ser descrito através da *medida empírica de energia*, que é definida como

$$\pi_t^N(\omega, du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} p_x^2(t) \delta_{\frac{x}{N}}(du) \ , \qquad (0.1.5)$$

onde δ_z representa a medida de Dirac concentrada em z.

Denote por Y_t^N o campo de flutuações da medida empírica de energia, que atua sobre as funções suaves $H : \mathbb{T} \to \mathbb{R}$ como

$$Y_t^N(H) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} H(x/N) \{p_x^2(t) - y^2\} \ .$$

O resultado principal deste trabalho é a convergência em distribuição de $Y_t^N(H)$ para um processo de Ornstein-Uhlenbeck generalizado. A tese divide-se em dois capítulos.

Adaptando o método introduzido em [V] identificamos o coeficiente de difusão (Seção 1.3), o que nos permite derivar o princípio de Boltzmann-Gibbs (Seção 1.4). Este é o ponto chave para mostrar que o campo de flutuações da energia converge, no sentido das distribuições finito dimensionais, para um processo de Ornstein-Uhlenbeck generalizado (Seção 1.2). Além disso, usando novamente o princípio de Boltzmann-Gibbs, prova-se a rigidez do campo de flutuações em um certo espaço de Sobolev (Seção 1.5). Desses dois últimos fatos decorre a convergência em distribuição.

O segundo capítulo é dedicado à detalhes mais técnicos. Na Seção 2.3 definimos e damos uma caracterização do espaço vetorial $\mathcal{H}_y$, que será central na prova do Teorema 5. Esta caracterização baseia-se em condições de integrabilidade de um sistema de Poisson e numa estimativa do buraco espectral para o gerador do processo. Estes dois assuntos são tratados na seções 2.1 e 2.2, respectivamente. Finalmente, na Seção 2.4 enunciamos, sem dar uma prova, um resultado de equivalência dos conjuntos.

A robustez do método de Varadhan para lidar com *sistemas não gradiente* tem sido amplamente corroborada em modelos nos quais as quantidades conservadas são funcionais lineares das coordenadas do sistema, em outras palavras, os conjuntos invariantes são hiperplanos. A partir de (0.1.3) podemos ver que esta propriedade não é satisfeita no modelo que nos interessa (aqui os conjuntos invariantes são hiperesferas). Este fato introduz dificuldades adicionais de natureza geométrica, já que o método de Varadhan envolve manipulações em campos de vetores definidos sobre os hiperplanos invariantes, ou mais geralmente, sobre as hipersuperfícies invariantes . Estas dificuldades vão aparecer nas secções 1.3, 2.1 e 2.3, onde explicações mais detalhadas serão dadas.

0.2 Introduction

A central problem in classical statistical mechanics is to provide a bridge between the macroscopic (thermodynamics) and microscopic (classical mechanics) description of the different phenomena observed in physical systems.

The ideal approach would be to start from a microscopic model of many components interacting with realistic forces and evolving with Newtonian dynamics, and then to derive some collective phenomena like the hydrodynamical regime or fluctuation-dissipation relations. For the moment, a rigorous mathematical derivation from deterministic microscopic models seems outside the range of the mathematical techniques.

During the last decades the study of stochastic lattice systems, where particles interact randomly, has been largely exploited, basically because such systems may exhibit phenomena analogous to that of real physical systems (*e.g.* hydrodynamical equations, fluctuation-dissipation relations and metastable states) with precise mathematical properties as counterpart of such phenomena (law of large numbers, central limit theorem and large deviation results).

A major breakthrough in the study of hydrodynamics for stochastic lattice systems appears with the work of Guo, Papanicolau, and Varadhan [GPV] where, by mean of intensive uses of ideas of large deviations, the authors derive the hydrodynamic behavior of the Ginzburg-Landau model. They consider the discrete torus $\mathbb{T}_N = \{1, \cdots, N\}$ and represent by p_i the charge at site i. The evolution in time of the vector $(p_1, \cdots, p_N)$ is given by a diffusion with infinitesimal generator

$$\mathcal{L}_N = \frac{1}{2} \sum_{x \in \mathbb{T}_N} D^2_{x,x+1} - (\phi'(p_x) - \phi'(p_{x+1})) D_{x,x+1},$$

where $D_{x,x+1} = \partial_{p_x} - \partial_{p_{x+1}}$ and ϕ is a continuously differentiable function such that $\int_{-\infty}^{\infty} e^{-\phi(x)} dx = 1$, $\int_{-\infty}^{\infty} e^{\lambda x - \phi(x)} dx < \infty$ for all $\lambda \in \mathbb{R}$ and $\int_{-\infty}^{\infty} e^{\sigma|\phi'(x)| - \phi(x)} dx < \infty$ for all $\sigma > 0$.

The generator $\mathcal{L}_N$ defines a diffusion process with invariant measure $\otimes_{x \in \mathbb{T}_N} e^{-\phi(p_x)} dp_x$, which is not ergodic because for all $\alpha \in \mathbb{R}$ the hyperplanes $\sum_{x \in \mathbb{T}_N} p_x = N\alpha$ of average charge α are invariant sets. Nevertheless, the restriction of the diffusion to each of such hyperplanes is nondegenerate and ergodic.

The arguments used in [GPV] provide a robust method to derive the hydrodynamic behavior for a large class of systems. In the beginning it seemed that this method could only be applied to the so-called *gradient systems*, that is, systems in which the instantaneous current can be expressed as the discrete gradient of a local function. While this simplifies the proofs considerably, it is not a natural condition. The study of *nongradient systems* (systems without this microscopic gradient condition) is of great interest because from a physical point of view they provide more realistic models, and from the mathematical point of view *gradient system* form only a set of low dimension in the space of stochastic reversible lattice models with local conservation laws (see [W] and references therein).

In a later paper Varadhan [V] managed to apply the method for the following non-gradient perturbation of the Ginzburg-Landau model

$$\mathcal{L}_N = \frac{1}{2}\sum_{i\in\mathbb{T}_N}\{D_{i,i+1}[a(x_i,x_{i+1})D_{i,i+1}] - (\phi'(x_i)-\phi'(x_{i+1}))a(x_i,x_{i+1})D_{i,i+1}\},$$

where $a(r,s)$ is a function satisfying $0 < c \le a(r,s) \le C$ with bounded continuous first derivatives.

The arguments used in [V] permit to extend the entropy method to reversible non-gradient systems, provided the generator of the system restricted to a cube of size l has a spectral gap that shrinks as l^{-2}.

At this time, the more general method to deal with *nongradient systems* is the one developed in [V], where roughly speaking, the idea is to approximate the current by a gradient plus a fluctuation term.

On the other hand, in recent works, a microscopic model for heat conduction in solids had been considered (*c.f.* [BO], [Be],[FNO]). In this model nearest neighbor atoms interact as coupled oscillators forced by an additive noise which exchange kinetic energy between nearest neighbors.

More precisely, in the case of periodic boundary conditions, atoms are labeled by $x \in \mathbb{T}_N = \{1,\cdots,N\}$. The configuration space is defined by $\Omega^N = (\mathbb{R}\times\mathbb{R})^{\mathbb{T}_N}$, where for a typical element $(p_x,r_x)_{x\in\mathbb{T}_N} \in \Omega^N$, r_x represents the distance between particles x and $x+1$, and p_x the velocity of particle x. The formal generator of the system reads as $\mathcal{L}_N = \mathcal{A}_N + \mathcal{S}_N$, where

$$\mathcal{A}_N = \sum_{x\in\mathbb{T}_N}\{(p_{x+1}-p_x)\partial_{r_x} + (r_x - r_{x-1})\partial_{p_x}\}\ , \tag{0.2.1}$$

and

$$\mathcal{S}_N = \frac{1}{2}\sum_{x\in\mathbb{T}_N} X_{x,x+1}[X_{x,x+1}]\ , \tag{0.2.2}$$

where $X_{x,x+1} = p_{x+1}\partial_{p_x} - p_x\partial_{p_{x+1}}$. Here $\mathcal{A}_N$ is the Liouville operator of a chain of interacting harmonic oscillators and $\mathcal{S}_N$ is the noise operator.

Denote by $\{(p(t), r(t)), t \geq 0\}$ the Markov process generated by $N^2\mathcal{L}_N$ (the factor N^2 corresponds to an acceleration of time). Let $C(\mathbb{R}_+, \Omega^N)$ be the space of continuous trajectories. Fixed a time $T > 0$, and for a given measure μ^N on Ω^N, the probability measure on $C([0,T], \Omega^N)$ induced by this Markov process starting in μ^N will be denoted by $\mathbb{P}_{\mu^N}$. As usually, expectation with respect to $\mathbb{P}_{\mu^N}$ will be denoted by $\mathbb{E}_{\mu^N}$.

In this work we focus on the noise operator $\mathcal{S}_N$ which acts only on velocities, so we restrict the configuration space to $\mathbb{R}^{\mathbb{T}_N}$. The total energy of the configuration $(p_x)_{x\in\mathbb{T}_N}$ is defined by

$$\mathcal{E} = \frac{1}{2}\sum_{x\in\mathbb{T}_N} p_x^2 \ . \tag{0.2.3}$$

It is easy to check that $\mathcal{S}_N(\mathcal{E}) = 0$, *i.e* total energy is constant in time.

The generator $\mathcal{S}_N$ defines a diffusion process with invariant measures given by $\nu_y^N(dp) = \otimes_{x\in\mathbb{T}_N}\frac{1}{\sqrt{2\pi y}}e^{-p_x^2/2y^2}dp_x$ for all $y > 0$. The process is not ergodic with respect to these measures, in fact, for all $\beta > 0$ the hyperspheres $p_1^2 + \cdots + p_N^2 = N\beta$ of average kinetic energy β are invariant sets. Nevertheless, the restriction of the diffusion to each of such hyperspheres is nondegenerate and ergodic.

The dynamics induced by $\mathcal{S}_N$ turns to be a *gradient system*, in fact, the microscopic instantaneous current of energy between x and $x+1$ can be expressed as $W_{x,x+1} = \frac{1}{2}(p_{x+1}^2 - p_x^2)$.

In analogy to [V] we introduce a coefficient into the generator (0.2.2) to break the gradient structure, namely

$$\mathcal{L}_N = \frac{1}{2}\sum_{x\in\mathbb{T}_N} X_{x,x+1}[a(p_x, p_{x+1})X_{x,x+1}] \ , \tag{0.2.4}$$

where $a(r,s)$ is a differentiable function satisfying $0 < c \leq a(r,s) \leq C < \infty$ with bounded continuous first derivatives.

The collective behavior of the system can be described thanks to empirical measures. The energy empirical measure associated to the process is defined by

$$\pi_t^N(\omega, du) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} p_x^2(t)\delta_{\frac{x}{N}}(du) \ , \tag{0.2.5}$$

where δ_z represents the Dirac measure concentrated on z.

To investigate equilibrium fluctuations of the empirical measure π^N we fix $y > 0$ and consider the system in the equilibrium ν_y^N. Denote by Y_t^N the empirical energy fluctuation field acting on smooth functions $H : \mathbb{T} \to \mathbb{R}$ as

$$Y_t^N(H) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} H(x/N)\{p_x^2(t) - y^2\} \ .$$

The main result of this work is the convergence of the energy fluctuation field $Y_t^N(H)$, as N goes to infinity, to a generalized Ornstein-Uhlenbeck process characterized by its covariances. This covariances are given in terms of the diffusion coefficient $\hat{a}(y)$ (*see* (1.1.6)).

This diffusion coefficient is given in terms of a variational formula which is equivalent to the Green-Kubo formula (*c.f.* [Sp] p.180). The main task of this work is to establish rigorously this variational formula.

Intuitively, non conserved quantities fluctuate in a much faster scale than conserved ones. Therefore, the only part of the fluctuations field of a non conserved quantity which should persist, when considering the scale in which the conserved quantity fluctuates, is their projection on the fluctuation field corresponding to the conserved quantity. This is the content of the Boltzmann-Gibbs principle. Indeed, the diffusion coefficient is the coefficient of that projection.

In order to study the equilibrium fluctuations of interacting particle systems, Brox and Rost [BR] introduced the Boltzmann-Gibbs principle and proved their validity for attractive zero range processes. Chang and Yau [CY] proposed an alternative method to prove the Boltzmann-Gibbs principle for *gradient systems*. This approach was extended to *nongradient systems* by Lu [L] and Sellami [Se]. Once the diffusion coefficient is determined, we will essentially follow their approach.

Now we describe the main features of the model we consider.

The model is not gradient. This difficulty already appears in the work of Bernardin [Be], where there are two conserved quantities (total deformation and total energy). The energy current is not the gradient of a local function. To overcome this problem they obtain an exact fluctuation-dissipation relation, that is, they write the current as a gradient plus a fluctuation term. On the other hand, in [FNO] Fritz *et al* studied the equilibrium fluctuations for the model given in [Be] . The exact fluctuation-dissipation relation mentioned above plays a central role in the proofs of the hydrodynamic limit and the equilibrium fluctuations.

Systems in which exists an exact fluctuation-dissipation relation are called *almost gradient systems*. For this kind of systems one can find the minimizer in the variational formula of the diffusion coefficient.

In our setting we do not have such an exact relation, so we use the nongradient Varadhan's method.

The only conserved quantity (total energy) is not a linear function of the coordinates of the system. In other words, the invariant surfaces are not hyperplanes, in fact, in our case invariant surfaces are hyperspheres.

Having a characterization of the space over which is taken the infimum in the variational problem defining the diffusion coefficient, is central in the nongradient Varadhan's method. Some results related to differential forms on spheres and integration over spheres are needed in order to obtain such characterization.

We do not have good control when dealing with large velocities. This makes it necessary to introduce a cutoff in the proof of the characterization mentioned above. The introduction of this cutoff is justified by the strong law of large numbers.

This lack of control also difficult the estimation of exponential moments. In [Be] the author manages to overcome this difficulty by adopting a microcanonical approach. Estimation of exponential moments arises in our case when trying to do the usual proof of tightness. Using the microcanonical approach mentioned before lead us to an identity equivalent to the one conjectured by Bernardin ([Be], lemma 6.3), that we are unable to prove. To avoid this difficulty we follow a strategy proposed by Chang *et al.* in [CLO] which exploits the fact that Boltzmann-Gibbs principle can be interpreted as an asymptotic gradient condition. In this way we avoid the exponential estimate.

Let us end this introduction by saying how this thesis is organized. By adapting the method introduced in [V] we identify the diffusion term (Section 1.3), which allows us to derive the Boltzmann-Gibbs principle (Section 1.4). This is the key point to show that the energy fluctuation field converges in the sense of finite dimensional distributions to a generalized Ornstein-Uhlenbeck process (Section 1.2). Moreover, using again the Boltzmann-Gibbs principle we also prove tightness for the energy fluctuation field in a specified Sobolev space (Section 1.5), which together with the finite dimensional convergence implies the convergence in distribution to the generalized Ornstein-Uhlenbeck process mentioned above.

The second chapter is devoted to more technical details. In Section 2.3 a characterization of the space involved in the variational problem defining the diffusion coefficient is given. This characterization relies on a sharp spectral gap estimate for the generator of the process. Such estimate is obtained in Section 2.2 by comparing the Dirichlet form of our model with the Dirichlet form of the Kac's model, and then using the spectral gap result given in [J] for that model. Some integrability conditions for Poisson systems are also needed, and are studied in Section 2.1. For the sake of completeness we state without proof an equivalence of ensembles result in Section 2.4.

Chapter 1

Equilibrium Fluctuations

1.1 Notations and Results

We will now give a precise description of the model. We consider a system of N particles in one dimension evolving under an interacting random mechanism. It is assumed that the spatial distribution of particles is uniform, so that the state of the system is given by specifying the N velocities.

Let $\mathbb{T} = (0, 1]$ be the 1-dimensional torus, and for a positive integer N denote by $\mathbb{T}_N$ the lattice torus of length N : $\mathbb{T}_N = \{1, \cdots, N\}$. The configuration space is denoted by $\Omega^N = \mathbb{R}^{\mathbb{T}_N}$ and a typical configuration is denoted by $p = (p_x)_{x\in\mathbb{T}_N}$, where p_x represents the velocity of the particle in x. The velocity configuration p changes with time and, as a function of time undergoes a diffusion in $\mathbb{R}^N$.

The diffusion mentioned above have as infinitesimal generator the following operator

$$\mathcal{L}_N = \frac{1}{2} \sum_{x\in\mathbb{T}_N} X_{x,x+1}[a(p_x, p_{x+1})X_{x,x+1}], \tag{1.1.1}$$

where $X_{x,z} = p_z\partial_{p_x} - p_x\partial_{p_z}$ and $a : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function satisfying $0 < c \leq a(x, y) \leq C < \infty$ with bounded continuous first derivatives. Of course, all the sums are taken *modulo* N. Observe that the total energy defined as $\mathcal{E}^N = \frac{1}{2}\sum_{x\in\mathbb{T}_N} p_x^2$ satisfies $\mathcal{L}_N(\mathcal{E}^N) = 0$, *i.e* total energy is a conserved quantity.

Let us consider for every $y > 0$ the Gaussian product measure ν_y^N on Ω^N with density relative to the Lebesgue measure given by

$$\nu_y^N(dp) = \prod_{x\in\mathbb{T}_N} \frac{e^{-\frac{p_x^2}{2y^2}}}{\sqrt{2\pi y}} dp_x,$$

where $p = (p_1, p_2, \ldots, p_N)$.

Denote by $L^2(\nu_y^N)$ the Hilbert space of functions f on Ω^N such that $\nu_y^N(f^2) < \infty$. $\mathcal{L}_N$ is formally symmetric on $L^2(\nu_y^N)$. In fact, is easy to see that for smooth functions f and g in a core of the operator $\mathcal{L}_N$, we have for all $y > 0$

$$\int_{\mathbb{R}^N} X_{x,z}(f) g \nu_y^N(dp) = -\int_{\mathbb{R}^N} X_{x,z}(g) f \nu_y^N(dp),$$

and therefore,

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{L}_N(f) g \nu_y^N(dp) &= \frac{1}{2}\sum_{x\in\mathbb{T}_N}\int_{\mathbb{R}^N} X_{x,x+1}[a(p_x,p_{x+1})X_{x,x+1}(f)] g \nu_y^N(dp)\\ &= -\frac{1}{2}\sum_{x\in\mathbb{T}_N}\int_{\mathbb{R}^N} X_{x,x+1}(f)[a(p_x,p_{x+1})X_{x,x+1}(g)]\nu_y^N(dp)\\ &= \int_{\mathbb{R}^N} f\mathcal{L}_N(g)\nu_y^N(dp).\end{aligned}$$

In particular, the diffusion is reversible with respect to all the invariant measures ν_y^N.

On the other hand, for every $y > 0$ the Dirichlet form of the diffusion with respect to ν_y^N is given by

$$\begin{aligned}\mathcal{D}_{N,y}(f) &= \langle -\mathcal{L}_N(f), f\rangle_y\\ &= \frac{1}{2}\sum_{x\in\mathbb{T}_N}\int_{\mathbb{R}^N} a(p_x,p_{x+1})[X_{x,x+1}(f)]^2\nu_y^N(dp)\ ,\end{aligned} \tag{1.1.2}$$

where $\langle\cdot,\cdot\rangle_y$ stands for the inner product in $L^2(\nu_y^N)$.

Denote by $\{p(t), t \geq 0\}$ the Markov process generated by $N^2\mathcal{L}_N$ (the factor N^2 correspond to an acceleration of time). Let $C(\mathbb{R}_+, \Omega^N)$ be the space of continuous trajectories on the configuration space. Fixed a time $T > 0$ and for a given measure μ^N on Ω^N, the probability measure on $C([0,T], \Omega^N)$ induced by this Markov process starting in μ^N will be denoted by $\mathbb{P}_{\mu^N}$. As usual, expectation with respect to $\mathbb{P}_{\mu^N}$ will be denoted by $\mathbb{E}_{\mu^N}$.

The diffusion generated by $N^2\mathcal{L}_N$ can also be described by the following system of stochastic differential equations

$$dp_x(t) = \frac{N^2}{2}\{X_{x,x+1}[a(p_x,p_{x+1})]p_{x+1} - X_{x-1,x}[a(p_{x-1},p_x)]p_{x-1} - p_x[a(p_x,p_{x+1}) \\ + a(p_{x-1},p_x)]\}dt + N[p_{x-1}\sqrt{a(p_{x-1},p_x)}dB_{x-1,x} - p_{x+1}\sqrt{a(p_x,p_{x+1})}dB_{x,x+1}],$$

where $\{B_{x,x+1}\}_{x\in\mathbb{T}_N}$ are independent standard Brownian motion.

Then, by Itô's formula we have that

$$dp_x^2(t) = N^2[W_{x-1,x} - W_{x,x+1}]dt + N[\sigma(p_{x-1},p_x)dB_{x-1,x}(s) - \sigma(p_x,p_{x+1})dB_{x,x+1}(s)], \tag{1.1.3}$$

where,

$$W_{x,x+1} = a(p_x,p_{x+1})(p_x^2 - p_{x+1}^2) - X_{x,x+1}[a(p_x,p_{x+1})]p_xp_{x+1} \ , \tag{1.1.4}$$

and,

$$\sigma(p_x,p_{x+1}) = 2p_xp_{x+1}\sqrt{a(p_x,p_{x+1})} \ . \tag{1.1.5}$$

We can think of $W_{x,x+1}$ as being the instantaneous microscopic current of energy between x and $x+1$. Observe that the current $W_{x,x+1}$ cannot be written as the gradient of a local function, neither by an exact fluctuation-dissipation equation, *i.e* as the sum of a gradient and a dissipative term of the form $\mathcal{L}_N(\tau_x h)$. That is, we are in the *nongradient* case.

The collective behavior of the system is described thanks to empirical measures. With this purpose let us introduce the energy empirical measure associated to the process defined by

$$\pi_t^N(\omega,du) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} p_x^2(t)\delta_{\frac{x}{N}}(du) \ ,$$

where δ_u represents the Dirac measure concentrated on u.

To investigate equilibrium fluctuations of the empirical measure π^N we fix once for all $y>0$ and consider the system in the equilibrium ν_y^N. Denote by Y_t^N the empirical energy fluctuation field acting on smooth functions $H:\mathbb{T}\to\mathbb{R}$ as

$$Y_t^N(H) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{T}_N} H(x/N)\{p_x^2(t) - y^2\} \ .$$

On the other hand, let $\{Y_t\}_{t\geq 0}$ be the stationary generalized Ornstein-Uhlenbeck process with zero mean and covariances given by

$$\mathbb{E}[Y_t(H_1)Y_s(H_2)] = \frac{4y^4}{\sqrt{4\pi(t-s)\hat{a}(y)}} \int_{\mathbb{T}} du \int_{\mathbb{R}} dv \bar{H}_1(u) \exp\left\{-\frac{(u-v)^2}{4(t-s)\hat{a}(y)}\right\} \bar{H}_2(v) \ , \tag{1.1.6}$$

for every $0 \le s \le t$. Here $\bar{H}_1(u)$ (resp $\bar{H}_2(u)$) is the periodic extension to the real line of the smooth function H_1 (resp H_2), and $\hat{a}(y)$ is the diffusion coefficient determined later in Section 1.3.

Consider for $k > \frac{3}{2}$ the Sobolev space $\mathcal{H}_{-k}$, whose definition will be given at the beginning of Section 1.5. Denote by $\mathbb{Q}_N$ the probability measure on $C([0,T],\mathcal{H}_{-k})$ induced by the energy fluctuation field Y_t^N and the Markov process $\{p^N(t), t \ge 0\}$ defined at the beginning of this section, starting from the equilibrium probability measure ν_y^N. Let $\mathbb{Q}$ be the probability measure on the space $C([0,T],\mathcal{H}_{-k})$ corresponding to the generalized Ornstein-Uhlenbeck process Y_t defined above.

We are now ready to state the main result of this work.

Theorem 1. *The sequence of probability measures $\{\mathbb{Q}_N\}_{N\ge1}$ converges weakly to the probability measure $\mathbb{Q}$.*

The proof of Theorem 1 will be divided into two parts. On the one hand, in Section 1.5 we prove tightness of $\{\mathbb{Q}_N\}_{N\ge1}$, where also a complete description of the space $\mathcal{H}_{-k}$ is given. On the other hand, in Section 1.2 we prove the finite-dimensional distribution convergence. These two results together imply the desired result. Let us conclude this section with a brief description of the approach we follow.

Given a smooth function $G : \mathbb{T} \times [0,T] \to \mathbb{R}$, we have after (1.1.3) that

$$\begin{aligned} \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} G\big(\frac{x}{N},t\big)p_x^2(t) &= \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} G\big(\frac{x}{N},0\big)p_x^2(0) + \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \partial_s G\big(\frac{x}{N},0\big)p_x^2(s)ds \\ &+ \int_0^t N^{\frac{3}{2}} \sum_{x\in\mathbb{T}_N} \big[G\big(\frac{x+1}{N},s\big) - G\big(\frac{x}{N},s\big)\big] W_{x,x+1}(s)ds \\ &+ \int_0^t \sqrt{N} \sum_{x\in\mathbb{T}_N} \big[G\big(\frac{x+1}{N},s\big) - G\big(\frac{x}{N},s\big)\big] \sigma(p_x,p_{x+1}) dB_{x,x+1}(s) \ . \end{aligned}$$

Thus,

$$\begin{aligned} M_N^G(t) = Y_t^N(G_t) - Y_t^N(G_0) - \int_0^t Y_t^N(\partial_s G_s)ds \\ + \int_0^t \sqrt{N} \sum_{x\in\mathbb{T}_N} \nabla_N G\big(\frac{x}{N}, s\big) W_{x,x+1}(s)ds \ , \end{aligned} \tag{1.1.7}$$

where the left hand side is the martingale

$$\frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \int_0^t \nabla_N G\big(\frac{x}{N}, s\big) \sigma(p_x, p_{x+1}) dB_{x,x+1}(s) \ ,$$

whose quadratic variation is given by

$$\langle M_N^G \rangle(t) = \frac{1}{N} \sum_{x\in\mathbb{T}_N} \int_0^t \big|\nabla_N G\big(\frac{x}{N}, s\big)\big|^2 a(p_x, p_{x+1}) p_x^2 p_{x+1}^2 ds \ .$$

Here ∇_N denotes the discrete gradient of a function defined in $\mathbb{T}_N$. Recall that if G is a smooth function defined on $\mathbb{T}$ and ∇ is the continuous gradient, then

$$\nabla_N G(\frac{x}{N}) = N\big[G\big(\frac{x+1}{N}\big) - G\big(\frac{x}{N}\big)\big] = (\nabla G)(\frac{x}{N}) + o(N^{-1}).$$

In analogy, Δ_N denotes the discrete Laplacian, which satisfies

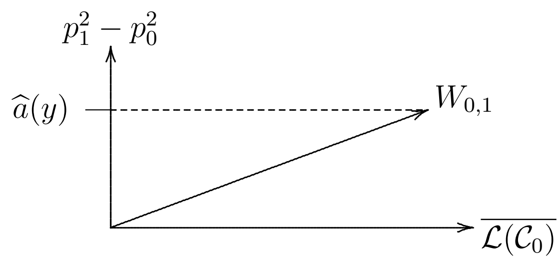
$$\Delta_N G(\frac{x}{N}) = N^2\big[G\big(\frac{x+1}{N}\big) - 2G\big(\frac{x}{N}\big) + G\big(\frac{x-1}{N}\big)\big] = (\Delta G)(\frac{x}{N}) + o(N^{-1}),$$

with Δ being the continuous Laplacian.

To close the equation for the martingale $M_N^G(t)$ we have to replace the term involving the microscopic currents in (1.1.7) with a term involving Y_t^N. Roughly speaking, what makes possible this replacement is the fact that non-conserved quantities fluctuates faster than conserved ones. Since the total energy is the unique conserved quantity of the system, it is reasonable that the only surviving part of the fluctuation field represented by the last term in (1.1.7) is its projection over the conservative field Y_t^N. This is the content of the Boltzmann-Gibbs Principle (see [BR]).

Recall that in fact we are in a *nongradient* case. Therefore, in order to perform the replacement mentioned in the previous paragraph, we follow the approach proposed by Varadhan in [V], which is briefly described in the following lines.

Denote by $\mathcal{C}_0$ the space of cylinder functions with zero mean with respect to all canonical measures. On $\mathcal{C}_0$ and for each $y > 0$ a semi-inner product $\ll \cdot \gg_y$ is defined. The current $W_{0,1}$ is seen to belong to the space generated by two orthogonal subspaces, namely, the linear space generated by the function $p_1^2 - p_0^2$ and the closure of the subspace $\mathcal{L}(\mathcal{C}_0)$. Here $\mathcal{L}$ stands for the natural extension of $\mathcal{L}_N$ to $\mathbb{Z}$.



More precisely, there exists $\hat{a}(y) \in \mathbb{R}$ such that for all $\delta > 0$ there exists $f \in \mathcal{C}_0$ such that

$$\ll W_{0,1} + \hat{a}(y)[p_1^2 - p_0^2] - \mathcal{L}(f) \gg_y \leq \delta \ .$$

The key point is that such a decomposition allows to study separately the diffusive part of the current and the part coming from a fluctuation term.

1.2 Convergence of the finite-dimensional distributions

To investigate equilibrium fluctuations of the empirical measure π^N, we fix once and for all $y > 0$ and we denote by Y_t^N the empirical energy fluctuation field, a linear functional which acts on smooth functions $H : \mathbb{T} \to \mathbb{R}$ as

$$Y_t^N(H) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} H(x/N)\{p_x^2(t) - y^2\} \ .$$

We state the main result of the section.

Theorem 2. *The finite dimensional distributions of the fluctuation field Y_t^N converge, as N goes to infinity, to the finite dimensional distributions of the generalized Ornstein-Uhlenbeck process Y defined in (1.1.6).*

In this setting convergence of finite dimensional distributions means that given a positive integer k, for every $\{t_1, \cdots, t_k\} \subset [0, T]$ and every collection of smooth functions

$\{H_1, \cdots, H_k\}$, the vector $\big(Y^N_{t_1}(H_1), \cdots, Y^N_{t_k}(H_k)\big)$ converges in distribution to the vector $(Y_{t_1}(H_1), \cdots, Y_{t_k}(H_k))$.

Recall that from Itô's formula we have

$$Y^N_t(H) = Y^N_0(H) - \int_0^t \sqrt{N} \sum_{x\in\mathbb{T}_N} \nabla_N H(x/N) W_{x,x+1}(s) ds$$
$$- \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H(x/N) \sigma(p_x(s), p_{x+1}(s))\, dB_{x,x+1}(s) \ .$$

Let us focus on the integral in the first line of the above expression. A central idea in the nongradient method proposed by Varadhan in [V], is to consider the current $W_{x,x+1}$ as an element of a Hilbert space generated by two orthogonal subspaces, one of which is the space generated by the gradient (in our case $p_x^2 - p_{x+1}^2$). Since non conserved quantities fluctuates in a much faster scale than conserved ones, it is expected that just the component corresponding to the projection over the energy fluctuation field survives after averaging over space and time. Recall that total energy is the only conserved quantity of the evolution.

The idea is to use the last observations, together with the fact that $\sum_{x\in\mathbb{T}_N} \Delta_N H(x/N) = 0$, in order to replace the integral term corresponding to the current $W_{x,x+1}$ by an expression involving the empirical energy fluctuation field, namely $\int_0^t Y^N_s(\Delta_N H) ds$.

Firstly, consider the empirical field Y^N_t acting now on time dependent smooth functions $H : \mathbb{T} \times [0, T] \to \mathbb{R}$ as

$$Y^N_t(H) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} H_t(x/N)\{p_x^2(t) - y^2\} \ ,$$

where $H_t(u) = H(u, t)$. We will denote by Y^N_t both, the field acting on time dependent and time independent functions. To distinguish, it suffices to check in what kind of function is being evaluated the field.

Again from the Itô's formula we obtain

$$Y^N_t(H_t) = Y^N_0(H_0) + \int_0^t Y^N_s(\partial_s H_s) ds \ - \int_0^t \sqrt{N} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) W_{x,x+1}(s) ds$$
$$- \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \sigma(p_x(s), p_{x+1}(s))\, dB_{x,x+1}(s) \ . \qquad (1.2.1)$$

Now we proceed to rewrite the last expression as

$$Y_t^N(H_t) = Y_0^N(H_0) + \int_0^t Y_s^N(\partial_s H_s + \hat{a}(y)\Delta_N H_s)ds \; - I^1_{N,F}(H^{\cdot t}) - I^2_{N,F}(H^{\cdot t}) \\ - M^1_{N,F}(H^{\cdot t}) - M^2_{N,F}(H^{\cdot t}) \; , \tag{1.2.2}$$

where F is a fixed smooth local function and

$$\begin{aligned} I^1_{N,F}(H^{\cdot t}) &= \int_0^t \sqrt{N} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \left[W_{x,x+1} - \hat{a}(y)[p^2_{x+1} - p^2_x] - \mathcal{L}_N \tau^x F\right] ds \; , \\ I^2_{N,F}(H^{\cdot t}) &= \int_0^t \sqrt{N} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \mathcal{L}_N \tau^x F ds \; , \\ M^1_{N,F}(H^{\cdot t}) &= \int_0^t \frac{2}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \tau^x \sqrt{a(p_0,p_1)} \left[p_0 p_1 + X_{0,1}\big(\sum_{i\in\mathbb{T}_N} \tau^i F\big)\right] dB_{x,x+1} \; , \\ M^2_{N,F}(H^{\cdot t}) &= \int_0^t \frac{2}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \sqrt{a(p_x,p_{x+1})} X_{x,x+1}\big(\sum_{i\in\mathbb{T}_N} \tau^i F\big) \; dB_{x,x+1}(s) \; . \end{aligned}$$

Here τ^x represents translation by x, and the notation $H^{\cdot t}$ stressed the fact that functionals depend on the function H through times in the interval $[0,t]$. Let us now explain the reason to rewrite expression (1.2.1) in this way.

In Section 1.3 (see (1.3.13) and (1.3.14)) the following variational formula for the diffusion coefficient $\hat{a}(y)$ is obtained:

$$\hat{a}(y) = y^{-4} \inf_F a(y,F),$$

where the infimum is taken over all local smooth functions, and

$$a(y,F) = \mathbf{E}_{\nu_y}[a(p_0,p_1)\{p_0 p_1 + X_{0,1}(\sum_{x\in\mathbb{Z}} \tau^x F)\}^2] \; .$$

Let $\{\frac{1}{2}F_k\}_{k\geq 1}$ be a minimizing sequence of local functions belonging to the Schwartz space. That is,

$$\lim_{k\to\infty} a(y, \frac{1}{2}F_k) = y^4 \hat{a}(y) \; . \tag{1.2.3}$$

With this notation we have the following result.

Theorem 3 (**Boltzmann-Gibbs Principle).** *For the sequence* $\{F_k\}_{k\geq 1}$ *given above* *and every smooth function $H : \mathbb{T} \times [0,T] \to \mathbb{R}$, we have*

$$\lim_{k\to\infty} \lim_{N\to\infty} \mathbb{E}_{\nu_y^N} \left[(I^1_{N,F_k}(H^{\cdot t})^2 \right] \;=\; 0 \,. \tag{1.2.4}$$

On the other hand, a judicious choice of the function H will cancel the second term in the right hand side of (1.2.2). Let us firstly note that we can replace $\Delta_N H_s$ by ΔH_s. In fact, the smoothness of H implies the existence of a constant $C > 0$ such that

$$|Y_s^N(\Delta H_s) - Y_s^N(\Delta_N H_s)| \leq \frac{C}{\sqrt{N}} \left(\frac{1}{N} \sum_{x \in \mathbb{T}_N} p_x^2(s) \right) \,,$$

uniformly in s.

Denote by $\{S_t\}_{t\geq 0}$ the semigroup generated by the Laplacian operator $\hat{a}(y)\Delta$. Given $t > 0$ and a smooth function $H : \mathbb{T} \to \mathbb{R}$, define $H_s = S_{t-s} H$ for $0 \leq s \leq t$. As is well known, the following properties are satisfied :

$$\partial_s H_s \;+\; \hat{a}(y)\Delta H_s \;=\; 0 \,, \tag{1.2.5}$$

$$\langle\, H_s \,,\, \hat{a}(y)\Delta H_s \,\rangle \;=\; -\frac{1}{2}\frac{d}{ds} \langle\, H_s \,,\, H_s \,\rangle \,, \tag{1.2.6}$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $L^2(\mathbb{T})$.

In this way we obtain for all smooth functions $H : \mathbb{T} \to \mathbb{R}$

$$Y_t^N(H) = \; Y_0^N(S_t H) \,+\, O(\frac{1}{N}) - I^1_{N,F}(H^{\cdot t}) - I^2_{N,F}(H^{\cdot t}) - M^1_{N,F}(H^{\cdot t}) - M^2_{N,F}(H^{\cdot t}) \,, \tag{1.2.7}$$

where $O(\frac{1}{N})$ denotes a function whose L^2 norm is bounded by C/N for a constant C depending just on H.

The following two lemmas concern the remaining terms.

Lemma 1. *For every smooth function $H : \mathbb{T} \times [0,T] \to \mathbb{R}$ and local function F in the Schwartz space,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_y^N} \left[\sup_{0\leq t\leq T} \left(I^2_{N,F}(H^{\cdot t}) + M^2_{N,F}(H^{\cdot t}) \right)^2 \right] \;=\; 0 \,. \tag{1.2.8}$$

Lemma 2. *The process M^1_{N,F_k} converges in distribution as k increases to infinity after*

N, to a generalized Gaussian process characterized by

$$\lim_{k\to\infty}\lim_{N\to\infty} \mathbb{E}_{\nu_y^N}\left[M^1_{N,F_k}(H_1^{\cdot t})M^1_{N,F_k}(H_2^{\cdot t})\right] \;=\; 4y^4\int_0^t\int_{\mathbb{T}} \hat{a}(y)H_1'(x,s)H_2'(x,s)\,dxds\;, \tag{1.2.9}$$

for every smooth function $H_i:\mathbb{T}\times[0,T]\to\mathbb{R}$ for $i=1,2$.

The proofs of Lemma 1 and Lemma 2 are postponed to the end of this section. The proof of Theorem 3 is considerably more difficult, and Section 1.4 is devoted to it .

Before entering in the proof of Theorem 2 we state some remarks. Firstly, the convergence in distribution of $Y_0^N(H)$ to a Gaussian random variable with mean zero and variance $2y^4\langle H,H\rangle$ as N tends to infinity, follows directly from the Lindeberg-Feller theorem.

Property (1.2.6) together with Lemma 2 imply the convergence in distribution as k increases to infinity after N of the martingale $M^1_{N,F_k}(H^{\cdot t})$ to a Gaussian random variable with mean zero and variance $2y^4\langle H,H\rangle - 2y^4\langle S_tH,S_tH\rangle$.

Finally, observe that the martingale $M^1_{N,F_k}(H^{\cdot t})$ is independent of the initial filtration $\mathcal{F}_0$.

Proof of Theorem 2. For simplicity and concreteness in the exposition we will restrict ourselves to the two-dimensional case $(Y_t^N(H_1),Y_0^N(H_2))$. Similar arguments may be given to show the general case.

In order to characterize the limit distribution of $(Y_t^N(H_1),Y_0^N(H_2))$ it is enough to characterize the limit distribution of all the linear combinations $\theta_1Y_t^N(H_1)+\theta_2Y_0^N(H_2)$.

From Lemma 1, Theorem 3 and expression 1.2.7 it follows

$$\begin{aligned}\theta_1Y_t^N(H_1)+\theta_2Y_0^N(H_2) \;&=\; \theta_1Y_0^N(S_tH_1) \;+\; \theta_1I^N(H_1,F_k)-\theta_1M^1_{N,F_k}(H_1^{\cdot t})+\theta_2Y_0^N(H_2)\\ &=\; Y_0^N(\theta_1S_tH_1+\theta_2H_2) \;+\; \theta_1I^N(H_1,F_k)-\theta_1M^1_{N,F_k}(H_1^{\cdot t})\;,\end{aligned}$$

where $I^N(H,F_k)$ denotes a function whose L^2 norm tends to zero as k increases to infinity after N.

Thus the random variable $\theta_1Y_t^N(H_1)+\theta_2Y_0^N(H_2)$ tends, as N goes to infinity, to a Gaussian random variable with mean zero and variance $2y^4\{\theta_1^2\langle H_1,H_1\rangle+2\theta_1\theta_2\langle S_tH_1,H_2\rangle+\theta_2^2\langle H_2,H_2\rangle\}$. This in turn implies

$$\mathbb{E}_{\nu_y^N}[Y_t(H_1)Y_0(H_2)] \;=\; 2y^4\langle S_tH_1,H_2\rangle,$$

which coincide with (1.1.6) as can be easily verified by using the explicit form of $S_t H$ in terms of the heat kernel. □

Now we proceed to give the proofs of Lemma 1 and Lemma 2.

Proof of Lemma 1. Let us define

$$\zeta_{N,F}(t) = \frac{1}{N^{3/2}} \sum_{x\in\mathbb{T}_N} \nabla_N H_t(x/N) \tau^x F(p(t)) \ .$$

From the Itô's formula we obtain

$$\zeta_{N,F}(t) - \zeta_{N,F}(0) = \frac{1}{2} I^2_{N,F}(H^{\cdot t}) + \int_0^t \frac{1}{N^{3/2}} \sum_{x\in\mathbb{T}_N} \partial_t \nabla_N H_s(x/N) \tau^x F(p) ds$$
$$+ \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \sum_{z\in\mathbb{T}_N} \sqrt{a(p_z, p_{z+1})} X_{z,z+1}(\tau^x F)\ dB_{z,z+1}(s) \ .$$

Then $(I^2_{N,F}(H^{\cdot t}) + M^2_{N,F}(H^{\cdot t}))^2$ is bounded above by 6 times the sum of the following three terms

$$(\zeta_{N,F}(t) - \zeta_{N,F}(0))^2,$$

$$\left(\int_0^t \frac{1}{N^{3/2}} \sum_{x\in\mathbb{T}_N} \partial_t \nabla_N H_s(x/N) \tau^x F(p) ds \right)^2 ,$$

$$\left(\frac{1}{2} M^2_{N,F}(H^{\cdot t}) - \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \sum_{z\in\mathbb{T}_N} \sqrt{a(p_z, p_{z+1})} X_{z,z+1}(\tau^x F)\ dB_{z,z+1}(s) \right)^2 .$$

Since F is bounded and H is smooth, the first two terms are of order $\frac{1}{N}$. Using additionally the fact that F is local, we can prove that the expectation of the $\sup_{0\le t\le T}$ of the third term is also of order $\frac{1}{N}$. In fact, if $M \in \mathbb{N}$ is such that $F(p) = F(p_0, \cdots, p_M)$, we are considering the difference between

$$\int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \sqrt{a(p_x, p_{x+1})} \sum_{j=1}^{M} X_{x,x+1}(\tau^{x-j} F)\ dB_{x,x+1}(s) \ ,$$

and

$$\int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H_s(x/N) \sum_{j=0}^{M+1} \sqrt{a(p_{x-1+j}, p_{x+j})} X_{x-1+j,x+j}(\tau^x F)\ dB_{x-1+j,x+j}(s) \ .$$

After rearrangement of the sum, last line can be written as

$$\int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \sqrt{a(p_x,p_{x+1})} \sum_{j=0}^{M+1} \nabla_N H_s(x-j+1/N) X_{x,x+1}(\tau^{x-j+1}F)\, dB_{x,x+1}(s) \;.$$

The proof is then concluded by using Doob's inequality. □

Proof of Lemma 2. Using basic properties of the stochastic integral and the stationarity of the process, we can see that the expectation appearing in the left side of (1.2.9) is equal to

$$4\int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} \nabla_N H_{1,s}(x/N) \nabla_N H_{2,s}(x/N) \mathbb{E}_{\nu_y^N} \left[\tau^x a(p_0,p_1) \left(p_0 p_1 + X_{0,1}(\sum_{i\in\mathbb{T}_N} \tau^i F_k) \right)^2 \right] ds \;.$$

Translation invariance of the measure ν_y^N lead us to

$$4\,\mathbb{E}_{\nu_y^N} \left[a(p_0,p_1) \left(p_0 p_1 + X_{0,1}(\sum_{i\in\mathbb{T}_N} \tau^i F_k) \right)^2 \right] \int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} \nabla_N H_{1,s}(x/N) \nabla_N H_{2,s}(x/N) ds \;,$$

and as N goes to infinity we obtain

$$4\,\mathbb{E}_{\nu_y} \left[a(p_0,p_1) \left(p_0 p_1 + X_{0,1}(\sum_{i\in\mathbb{Z}} \tau^i F_k) \right)^2 \right] \int_0^t \int_{\mathbb{T}} H_1'(u,s) H_2'(u,s)\, du\, ds \;.$$

Finally from (1.2.3), taking the limit as k tends to infinity we obtain the desired result. □

1.3 Central Limit Theorem Variances and Diffusion Coefficient

The aim of this section is to identify the diffusion coefficient $\hat{a}(y)$. Roughly speaking, the direction of the gradient is the only survival term when averaging the current $W_{x,x+1}$ over time and space. The coefficient of the component in this direction is the diffusion coefficient. In other words, the constant $\hat{a}(y)$ will have the property

$$\limsup_{N\to\infty}\frac{1}{2N}\limsup_{t\to\infty}\frac{1}{t}\mathbf{E}_{\nu_y}\left[\left(\int_0^t\sum_{-N\le x\le x+1\le N}W_{x,x+1}-\hat{a}(y)(p_{x+1}^2-p_x^2)\ \ ds\right)^2\right]=0. \tag{1.3.1}$$

Here we are considering the process generated by the natural extension of $\mathcal{L}_N$ to the infinite product space $\Omega=\mathbb{R}^{\mathbb{Z}}$, and ν_y denotes the product measure on Ω defined by $d\nu_y=\prod_{-\infty}^{\infty}\frac{\exp(\frac{-p_x^2}{2y^2})}{\sqrt{2\pi}y}dp$.

The form of the limit with respect to t appearing in (1.3.1) leads us to think in the central limit theorem for additive functionals of Markov processes. Let us begin by introducing some notations and stating some general results for continuous time Markov processes.

Consider a continuous time Markov process $\{Y_s\}_{s\ge 0}$, reversible and ergodic with respect to invariant measure π. Denote by $\langle\ ,\ \rangle_\pi$ the inner product in $L^2(\pi)$ and let us suppose that the infinitesimal generator of this process $\mathcal{L}: D(\mathcal{L})\subset L^2(\pi)\to L^2(\pi)$ is a negative operator.

Let $V\in L^2(\pi)$ be a mean zero function on the state space of the process. The central limit theorem proved by Kipnis and Varadhan in [KV] for

$$X_s=\int_0^t V(Y_s)ds\ ,$$

states that if V is in the range of $(-\mathcal{L})^{-\frac{1}{2}}$, then there exists a square integrable martingale $\{M_t\}_{t\ge 0}$ with stationary increments such that

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}\sup_{0\le s\le t}|X_s-M_s|=0\ .$$

This result in turn implies that the limiting variance defined by

$$\sigma^2(V,\pi)=\lim_{t\to\infty}\frac{1}{t}E[X_t^2]\ , \tag{1.3.2}$$

is equal to

$$2\langle V\ ,(-\mathcal{L})^{-1}V\ \rangle_\pi\ . \tag{1.3.3}$$

Observe that in the particular case when $V=\mathcal{L}U$ for some $U\in D(\mathcal{L})$ we have that

$\sigma^2(V,\pi)$ is equal to

$$2\langle U\ ,(-\mathcal{L})U\ \rangle_\pi\ , \tag{1.3.4}$$

which corresponds to twice the Dirichlet form associated to the generator $\mathcal{L}$ and the measure π, evaluated in U.

By standard arguments we can extend $\sigma^2(V,\pi)$ to a symmetric bilinear form $\sigma^2(V,Z,\pi)$ for V and Z in the range of $(-\mathcal{L})^{-\frac{1}{2}}$. This bilinear form represents limiting covariances, and analogous expressions to (1.3.3) and (1.3.4) can be easily obtained.

Now we introduce the Sobolev spaces $\mathscr{H}_1$ and $\mathscr{H}_{-1}$. For that, we can consider a stationary (not necessarily ergodic) measure π. We introduce $\mathscr{H}_1$ and $\mathscr{H}_{-1}$ in this section because (1.3.3) and (1.3.4) correspond to norms in these spaces, although their properties will be mainly used in Section 1.4.

Define for $f \in D(\mathcal{L}) \subset L^2(\pi)$,

$$||f||_1^2\ =\ \langle f,(-\mathcal{L}_N)f\rangle_\pi\ . \tag{1.3.5}$$

It is easy to see that $||\cdot||_1$ is a norm in $D(\mathcal{L})$ that satisfies the parallelogram rule, and therefore, that can be extended to an inner product in $D(\mathcal{L})$. We denote by $\mathscr{H}_1$ the completion of $D(\mathcal{L})$ under the norm $||\cdot||_1$, and by $\langle\ ,\ \rangle_1$ the induced inner product. Now define

$$||f||_{-1}^2\ =\ \sup_{g\in D(\mathcal{L})}\{2\,\langle f,g\rangle_\pi - \langle g,g\rangle_1\}\ , \tag{1.3.6}$$

and denote by $\mathscr{H}_{-1}$ the completion with respect to $||\cdot||_{-1}$ of the set of functions in $L^2(\pi)$ satisfying $||f||_{-1} < \infty$. Later we state some well known properties of these spaces.

Lemma 3. *For $f \in L^2(\pi)\cap\mathscr{H}_1$ and $g \in L^2(\pi)\cap\mathscr{H}_{-1}$, we have*

i) $||g||_{-1}\ =\ \sup_{h\in D(\mathcal{L})\setminus\{0\}} \frac{\langle h,g\rangle_\pi}{||h||_1}\ ,$

ii) $|\,\langle f,g\rangle_\pi\,|\ \leq\ ||f||_1||g||_{-1}\ .$

Property *i*) implies that $\mathscr{H}_{-1}$ is the topological dual of $\mathscr{H}_1$ with respect to $L^2(\pi)$, and property *ii*) entails that the inner product $\langle\ ,\ \rangle_\pi$ can be extended to a continuous bilinear form on $\mathscr{H}_{-1}\times\mathscr{H}_1$. The preceding results remain in force when $L^2(\pi)$ is replaced by any Hilbert space.

Observe that we can express the central limit theorem variance $\sigma^2(V,\pi)$ in terms of the norms defined above. Indeed, $\sigma^2(V,\pi) = 2||V||_{-1}$ for V in the range of $(-\mathcal{L})^{-\frac{1}{2}}$ and $\sigma^2(V,\pi) = 2||U||_1$ if $V = \mathcal{L}U$ for some $U \in D(\mathcal{L})$.

Now we return to our context. Let L_N be the generator defined by

$$L_N(f) = \frac{1}{2} \sum_{-N \le x \le x+1 \le N} X_{x,x+1}[a(p_x, p_{x+1}) X_{x,x+1}(f)] \ .$$

Here the sum is no longer periodic, as in the definition of $\mathcal{L}_N$.

Let $\mu_{N,y}$ be the uniform measure on the sphere

$$\left\{ (p_{-N}, \cdots, p_N) \in \mathbb{R}^{2N+1} \ : \sum_{i=-N}^{N} p_i^2 = (2N+1) y^2 \right\} \ ,$$

and $D_{N,y}$ the Dirichlet form associated to this measure and L_N, which is given by

$$D_{N,y}(f) = \frac{1}{2} \sum_{-N \le x \le x+1 \le N} \int a(p_x, p_{x+1}) [X_{x,x+1}(f)]^2 \mu_{N,y}(dp) \ .$$

Is not difficult to see that the measures $\mu_{N,y}$ are ergodic for the process with generator L_N. We are interested in the asymptotic behavior of the variance

$$\sigma^2(B_N + \widehat{a}(y) A_N - H_N^F \ , \ \mu_{N,y}) \ , \tag{1.3.7}$$

where,

$$\begin{aligned} A_N(p_{-N}, \cdots, p_N) &= p_N^2 - p_{-N}^2 \ , \\ B_N(p_{-N}, \cdots, p_N) &= \sum_{-N \le x \le x+1 \le N} W_{x,x+1} \ , \\ H_N^F(p_{-N}, \cdots, p_N) &= \sum_{-N \le x-k \le x+k \le N} L_N(\tau^x F) \ , \end{aligned}$$

with $F(x_{-k}, \cdots, x_k)$ a smooth function of $2k+1$ variables. Observe that these three classes of functions are sums of translations of local functions, and have mean zero with respect to every $\mu_{N,y}$. We introduce $\Delta_{N,y}$ to denote variances and covariances, for instance, $\Delta_{N,y}(B_N, B_N) = \sigma^2(B_N \ \mu_{N,y})$ and $\Delta_{N,y}(A_N, H_N^F) = \sigma^2(A_N, H_N^F \ , \ \mu_{N,y})$. The inner product in $L^2(\mu_{N,y})$ is denoted by $\langle \ , \ \rangle_{N,y}$.

Observe that the functions B_N and H_N^F belong to the range of L_N, in fact

$$B_N(p_{-N},\cdots,p_N) = L_N(\sum_{x=-N}^{N} x p_x^2) \ , \tag{1.3.8}$$

$$H_N^F(p_{-N},\cdots,p_N) = L_N(\psi_N^F) \ , \tag{1.3.9}$$

where,

$$\psi_N^F \ = \sum_{-N\le x-k\le x+k\le N} \tau^x F \ .$$

This in particular implies that the central limit theorem variances and covariances involving B_N and H_N^F exist. After (1.3.4) they are also easily computable, which is not the case for A_N.

In the remaining of this section we often use integration with respect to $\mu_{N,y}$. Let us state some classical results (Lemma 4 and Lemma 5) referring to integration over spheres. The proofs of these and other interesting results can be founded in [Ba].

Lemma 4. *Given* $p = (p_1,\cdots,p_n), \boldsymbol{a} = (a_1,\cdots,a_n), a = \sum_{k=1}^n a_k$ *and* $S^{n-1}(r) = \{p \in \mathbb{R}^n : |p| = r\}$ *, define*

$$E(p,\boldsymbol{a}) = \prod_{k=1}^{n} (x_k^2)^{a_k} \quad \textit{and} \quad S_n(\boldsymbol{a},r) = \int_{S^{n-1}(r)} E(p,\boldsymbol{a}) \ d\sigma_{n-1}$$

then,

$$S_n(\boldsymbol{a},r) = \frac{2\prod_{k=1}^n \Gamma(a_k+\frac{1}{2})}{\Gamma(a+\frac{n}{2})} r^{2a+n-1} \ .$$

Where $d\sigma_{n-1}$ *denotes* $(n-1)$*-dimensional surface measure and* Γ *is gamma function.*

Corollary 1. *There exist a constant* C *depending on* y *and the lower bound of* $a(\cdot,\cdot)$ *such that, for every* $u \in D(L)$

$$\left| \langle u, A_N \rangle_{N,y} \right| \le C(2N)^{\frac{1}{2}} D_{N,y}(u)^{\frac{1}{2}} \ .$$

Proof. Observe that

$$A_N(p_{-N},\cdots,p_N) = \sum_{x=-N}^{N-1} X_{x,x+1}(p_x p_{x+1}) \ ,$$

then,

$$
\begin{aligned}
\left|\langle u, A_N\rangle_{N,y}\right| &= \left|\sum_{x=-N}^{N-1}\int u(p)X_{x,x+1}(p_xp_{x+1})\mu_{N,y}(dp)\right| \\
&= \left|\sum_{x=-N}^{N-1}\int X_{x,x+1}(u)p_xp_{x+1}\mu_{N,y}(dp)\right| \\
&\le \int\sum_{x=-N}^{N-1}|\sqrt{a(p_x,p_{x+1})}X_{x,x+1}(u)|\frac{|p_xp_{x+1}|}{\sqrt{a(p_x,p_{x+1})}}\mu_{N,y}(dp) \\
&\le \int\left(\sum_{x=-N}^{N-1}a(p_x,p_{x+1})(X_{x,x+1}(u))^2\right)^{\frac{1}{2}}\left(\sum_{x=-N}^{N-1}\frac{|p_xp_{x+1}|^2}{a(p_x,p_{x+1})}\right)^{\frac{1}{2}}\mu_{N,y}(dp) \\
&\le \left(\int\sum_{x=-N}^{N-1}\frac{|p_xp_{x+1}|^2}{a(p_x,p_{x+1})}\mu_{N,y}(dp)\right)^{\frac{1}{2}}\left(\int\sum_{x=-N}^{N-1}a(p_x,p_{x+1})(X_{x,x+1}(u))^2\mu_{N,y}(dp)\right)^{\frac{1}{2}} \\
&\le C(2N)^{\frac{1}{2}}D_{N,y}(u)^{\frac{1}{2}}\ .
\end{aligned}
$$

□

This implies that the central limit theorem variances and covariances involving A_N exist. In spite of that, the core of the problem will be to deal with the variance of A_N which is not easily computable.

Corollary 2.

$$
\int\sum_{i=-N}^{N-1}p_i^2p_{i+1}^2\mu_{N,y}(dp) = \frac{2N(2N+1)^2}{(2N+3)(2N+1)}y^4\ .
$$

Proof. By the rotation invariance of $\mu_{N,y}$, the left hand side of last expression is equal to

$$
2N\int p_i^2p_{i+1}^2\mu_{N,y}(dp)\ ,
$$

which in turns, applying Lemma 4 with $a=\{1,\cdots,1\}$ and $r=y\sqrt{2N+1}$, is equal to

$$
2N\left(2\frac{\Gamma(\frac{1}{2})^{2N-1}\Gamma(\frac{3}{2})^2}{\Gamma(2+\frac{2N+1}{2})}r^{4+2N}\right)\left(2\frac{\Gamma(\frac{1}{2})^{2N+1}}{\Gamma(\frac{2N+1}{2})}r^{2N}\right)^{-1}\ .
$$

The proof is concluded by using well known properties of the Gamma function. □

Lemma 5. *(Divergence Theorem) Let* $B^n(r)=\{p\in\mathbb{R}^n:|p|\le r\}$ *and* $S^{n-1}(r)=\{p\in$

$\mathbb{R}^n : |p| = r\}$. *Then for every continuously differentiable function* $f : \mathbb{R}^n \to \mathbb{R}$ *we have,*

$$r \int_{B^n(r)} \frac{\partial f}{\partial p_i}(p)dp = \int_{S^{n-1}(r)} f(s_1, \cdots, s_n)s_i \, d\sigma_{n-1}.$$

Corollary 3. *Taking* $r = y\sqrt{2N+1}$ *in Lemma 5, we have for* $-N \le i, j \le N$

$$\int X_{i,j}(W)p_ip_j\mu_{N,y}(dp) = \frac{r}{|S^{2N}(r)|}\int_{B^{2N+1}(r)} \left(p_i\frac{\partial W}{\partial p_i} - p_j\frac{\partial W}{\partial p_j}\right) dp \; .$$

Proof. Recalling that $X_{i,j} = p_j\frac{\partial}{\partial p_i} - p_i\frac{\partial}{\partial p_j}$ and $\mu_{N,y}$ is the uniform probability measure over the sphere of radius $y\sqrt{2N+1}$ centered at the origin, we have that the left hand side in the preceding line is equal to

$$\frac{1}{|S^{2N}(r)|}\left(\int_{S^{2N}(r)} p_j\left(p_ip_j\frac{\partial W}{\partial p_i}\right)d\sigma_{2N} - \int_{S^{2N}(r)} p_i\left(p_ip_j\frac{\partial W}{\partial p_j}\right)d\sigma_{2N}\right).$$

From Lemma 5, the expression above is equal to

$$\frac{r}{|S^{2N}(r)|}\left(\int_{B^{2N+1}(r)}\left(p_i\frac{\partial W}{\partial p_i} - p_ip_j\frac{\partial W}{\partial p_i\partial p_j}\right)dp - \int_{B^{2N+1}(r)}\left(p_j\frac{\partial W}{\partial p_j} - p_ip_j\frac{\partial W}{\partial p_i\partial p_j}\right)dp\right),$$

concluding the proof. □

Corollary 3 is extremely useful for us, because it provides a way to perform telescopic sums over the sphere. In fact, it implies that given $-N \le i < j \le N$ we have

$$\int X_{i,j}(W)p_ip_j\mu_{N,y}(dp) = \int \sum_{k=i}^{j-1} X_{k,k+1}(W)p_kp_{k+1}\mu_{N,y}(dp) \; . \tag{1.3.10}$$

We should stress the fact that equality of the integrands is false.

Now we return to the study of A_N, B_N and H_N^F. The next proposition entails the asymptotic behavior, as N goes to infinity, of the central limit theorem variances and covariances involving B_N or H_N^F, besides an estimate in the case of A_N.

Theorem 4. *The following limits hold locally uniformly in $y > 0$.*

$$
\begin{array}{rrcl}
i) & \lim_{N\to\infty} \frac{1}{2N}\Delta_{N,y}(B_N, B_N) & = & \boldsymbol{E}_{\nu_y}[a(p_0,p_1)(2p_0p_1)^2] \ , \\
ii) & \lim_{N\to\infty} \frac{1}{2N}\Delta_{N,y}(H_N^F, H_N^F) & = & \boldsymbol{E}_{\nu_y}[a(p_0,p_1)[X_{x,x+1}(\widetilde{F})]^2] \ , \\
iii) & \lim_{N\to\infty} \frac{1}{2N}\Delta_{N,y}(B_N, H_N^F) & = & -\boldsymbol{E}_{\nu_y}[2p_0p_1a(p_0,p_1)X_{x,x+1}(\widetilde{F})] \ , \\
iv) & \lim_{N\to\infty} \frac{1}{2N}\Delta_{N,y}(A_N, B_N) & = & -\boldsymbol{E}_{\nu_y}[(2p_0p_1)^2] \ , \\
v) & \lim_{N\to\infty} \frac{1}{2N}\Delta_{N,y}(A_N, H_N^F) & = & 0 \ , \\
vi) & \limsup_{N\to\infty} \frac{1}{2N}\Delta_{N,y}(A_N, A_N) & \le & C \ ,
\end{array}
$$

where $\widetilde{F}$ is formally defined by

$$
\widetilde{F}(p) = \sum_{j=-\infty}^{\infty} \tau^j F(p)
$$

and C is a positive constant depending uniformly on y. Although $\widetilde{F}$ does not really make sense, the gradients $X_{i,i+1}(\widetilde{F})$ are all well defined.

Proof. *i)* From (1.3.8) and (1.3.4) we have that

$$
\begin{aligned}
\Delta_{N,y}(B_N, B_N) &= 2D_{N,y}\left(\sum_{x=-N}^{N} x p_x^2\right) \\
&= 4\int \sum_{x=-N}^{N-1} a(p_x,p_{x+1})(p_xp_{x+1})^2 \mu_{N,y}(dp) \ .
\end{aligned}
$$

Since $\mu_{N,y}$ is rotation invariant we have

$$
\frac{1}{2N}\Delta_{N,y}(B_N, B_N) = 4\int a(p_0,p_1)p_0^2p_1^2\mu_{N,y}(dp) \ ,
$$

and the desired result comes from the *equivalence of ensembles* stated in Section 2.4.

ii) From (1.3.9) and (1.3.4) we have that

$$\begin{aligned}\Delta_{N,y}(H_N^F, H_N^F) &= 2D_{N,y}(\psi_N^F)\\ &= \int \sum_{x=-N}^{N-1} a(p_x, p_{x+1})[X_{x,x+1}(\psi_N^F)]^2 \mu_{N,y}(dp)\ .\end{aligned}$$

The sum in the last line can be broken into two sums, the first one considering the indexes in $\{-N+2k, \cdots, N-2k-1\}$ and the second one considering the indexes in the complement with respect to $\{-N, \cdots, N-1\}$. From the conditions imposed over F, when divided by N, the term corresponding to the second sum tends to zero as N goes to infinity. Then,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2N}\Delta_{N,y}(H_N^F, H_N^F) &= \lim_{N\to\infty} \int \frac{1}{2N} \sum_{x=-N}^{N-1} a(p_x, p_{x+1})[X_{x,x+1}(\psi_N^F)]^2 \mu_{N,y}(dp)\\ &= \lim_{N\to\infty} \int \frac{1}{2N} \sum_{x=-N}^{N-1} \tau^x g(p) \mu_{N,y}(dp)\ ,\end{aligned}$$

where $g(p) = a(p_0, p_1)[X_{x,x+1}(\widetilde{F})]^2$ and

$$\widetilde{F}(p) = \sum_{j=-\infty}^{\infty} \tau^j F(p).$$

One more time, the desired result comes from the rotation invariance of $\mu_{N,y}$ and the equivalence of ensembles.

iii) From (1.3.9), (1.3.4) and the fact that $W_{x,x+1} = -X_{x,x+1}[p_x p_{x+1} a(p_x, p_{x+1})]$, we have

$$\begin{aligned}\Delta_{N,y}(B_N, H_N^F) &= 2 \sum_{x=-N}^{N-1} \int X_{x,x+1}[p_x p_{x+1} a(p_x, p_{x+1})] \psi_N^F \mu_{N,y}(dp)\\ &= -2 \sum_{x=-N}^{N-1} \int p_x p_{x+1} a(p_x, p_{x+1}) X_{x,x+1}[\psi_N^F] \mu_{N,y}(dp)\ ,\end{aligned}$$

where the last equality is due to an integration by parts. Then, we divide the sum appearing in the last line into two sums in the very same way as was done for *ii*), obtaining

$$\lim_{N\to\infty}\frac{1}{2N}\Delta_{N,y}(B_N, H_N^F) = -2\int \sum_{x=-N+2k}^{N-2k-1} \tau^x h(p)\mu_{N,y}(dp)\ ,$$

where $h(p) = a(p_0, p_1)p_0p_1X_{x,x+1}(\widetilde{F})$.

Again, the result comes from rotation invariance of $\mu_{N,y}$ and the equivalence of ensembles.

iv) From (1.3.8), (1.3.4) and the fact that $A_N = X_{N,-N}(p_Np_{-N})$, we have

$$\begin{aligned}
\Delta_{N,y}(A_N, B_N) &= -2\int X_{N,-N}(p_Np_{-N})\left(\sum_{x=-N}^{N} xp_x^2\right)\mu_{N,y}(dp)\\
&= 2\int p_Np_{-N}X_{N,-N}[Np_N^2 - Np_{-N}^2]\mu_{N,y}(dp)\\
&= 8N\int p_N^2p_{-N}^2\mu_{N,y}(dp)\\
&= \frac{8N(2N+1)^2y^4}{(2N+3)(2N+1)}\ .
\end{aligned}$$

The second and fourth equalities come from integration by parts and corollary 2, respectively. Thus, we have

$$\lim_{N\to\infty}\frac{1}{2N}\Delta_{N,y}(A_N, B_N) = 4y^4.$$

v) By the same arguments used in the preceding items and using the telescoping sum obtained in (1.3.10) we have

$$\begin{aligned}
\Delta_{N,y}(A_N, H_N^F) &= -2\int X_{N,-N}(p_Np_{-N})\psi_N^F\mu_{N,y}(dp)\\
&= 2\int p_Np_{-N}X_{N,-N}(\psi_N^F)\mu_{N,y}(dp)\\
&= 2\int \sum_{i=-N}^{N-1} p_ip_{i+1}X_{i,i+1}(\psi_N^F)\mu_{N,y}(dp)\ .
\end{aligned}$$

From rotation invariance of $\mu_{N,y}$ and the equivalence of ensembles we obtain

$$\lim_{N\to\infty}\frac{1}{2N}\Delta_{N,y}(A_N, H_N^F) = \mathbf{E}_{\nu_y}[p_0p_1X_{0,1}(\widetilde{F})] = 0\ .$$

The last equality comes from an integration by parts.

vi) By duality $\Delta_{N,y}(A_N, A_N) = 2c^2$ where c is the smallest constant such that for every $u \in D(L)$,

$$\left| \langle u, A_N \rangle_{N,y} \right| \leq c D_{N,y}(u)^{\frac{1}{2}} \ . \tag{1.3.11}$$

Recall that Corollary 1 ensures the existence of a constant C depending locally uniformly on y, such that $C(2N)^{\frac{1}{2}}$ satisfies (1.3.11) for every $u \in D(L)$. Therefore, c is smaller than $C(2N)^{\frac{1}{2}}$ and

$$\frac{1}{2N} \Delta_{N,y}(A_N, A_N) \leq 2C^2 \ ,$$

which concludes the proof of Theorem 4. □

We proceed now to calculate the only missing limit variance (the one corresponding to A_N) in an indirect way, as follows.

Using the basic inequality

$$|\Delta_{N,y}(A_N, B_N - H_N^F)|^2 \leq \Delta_{N,y}(A_N, A_N) \Delta_{N,y}(B_N - H_N^F, B_N - H_N^F) \ ,$$

we obtain,

$$\liminf_{N\to\infty} \frac{\frac{|\Delta_{N,y}(A_N, B_N - H_N^F)|^2}{(2N)^2}}{\frac{\Delta_{N,y}(B_N - H_N^F, B_N - H_N^F)}{2N}} \leq \liminf_{N\to\infty} \frac{\Delta_{N,y}(A_N, A_N)}{2N} \ ,$$

which in view of Theorem 4 implies

$$\frac{(4y^4)^2}{\mathbf{E}_{\nu_y}[a(p_0, p_1)\{2p_0p_1 + X_{0,1}(\widetilde{F})\}^2]} \leq \liminf_{N\to\infty} \frac{\Delta_{N,y}(A_N, A_N)}{2N} \ . \tag{1.3.12}$$

Let us define $\hat{a}(y)$ by the relation

$$\hat{a}(y) = y^{-4} \inf_F a(y, F) \ , \tag{1.3.13}$$

where the infimum is taken over all local smooth functions in Schwartz space, and

$$a(y, F) = \mathbf{E}_{\nu_y}[a(p_0, p_1)(p_0p_1 + X_{0,1}(\widetilde{F}))^2] \ . \tag{1.3.14}$$

Since the limit appearing in (1.3.12) does not depend of F, we have

$$\liminf_{N\to\infty} \frac{\Delta_{N,y}(A_N, A_N)}{2N} \geq \frac{4y^4}{\widehat{a}(y)} \ . \tag{1.3.15}$$

Moreover, this limit is locally uniform in y.

We are now ready to state the main result of this section.

Theorem 5. *The function $\widehat{a}(y)$ is continuous in $y > 0$ and*

$$\lim_{N\to\infty} \frac{1}{2N} \Delta_{N,y}(A_N, A_N) = \frac{4y^4}{\hat{a}(y)} \ . \tag{1.3.16}$$

Proof. Let us define

$$l(y) = \limsup_{y'\to y} \limsup_{N\to\infty} \frac{1}{2N} \Delta_{N,y'}(A_N, A_N) \ . \tag{1.3.17}$$

By definition, $\widehat{a}(y)$ and $l(y)$ are upper semicontinuous functions.

In order to prove (1.3.16) it is enough to verify the following inequality

$$l(y) \le \frac{4y^4}{\hat{a}(y)} \ . \tag{1.3.18}$$

In fact, by the upper semicontinuity of $\hat{a}(y)$ and the lower bound in (1.3.15) we obtain

$$\begin{aligned} \frac{4y^4}{\hat{a}(y)} &\le \frac{4y^4}{\limsup_{y'\to y} \hat{a}(y')} \\ &\le \limsup_{y'\to y} \frac{4{y'}^4}{\hat{a}(y')} \\ &\le \limsup_{y'\to y} \limsup_{N\to\infty} \frac{\Delta_{N,y'}(A_N, A_N)}{2N}, \end{aligned}$$

from which we have the equality

$$l(y) = \frac{4y^4}{\hat{a}(y)}. \tag{1.3.19}$$

From the definition of $l(y)$, it is clear that

$$\limsup_{N\to\infty} \frac{1}{2N} \Delta_{N,y}(A_N, A_N) \le l(y) \ ,$$

which together with (1.3.15) proves (1.3.16). Moreover, equality (1.3.19) together with the upper semicontinuity of $l(y)$ gives the lower semicontinuity of $\widehat{a}(y)$. Ending the proof of Theorem 5.

Therefore it remains to check the validity of inequality (1.3.18), which is equivalent to prove that for every $\theta \in \mathbb{R}$,

$$l(y) > \theta \qquad \Longrightarrow \qquad \theta \leq \frac{4y^4}{\hat{a}(y)} \ . \tag{1.3.20}$$

Suppose that $l(y) > \theta$. Then, there exist a sequence $y_N \to y$ such that

$$\lim_{N\to\infty} \frac{1}{2N} \Delta_{N,y_N}(A_N, A_N) = A > \theta \ .$$

By i) in Lemma 3 we have

$$\{\Delta_{N,y_N}(A_N, A_N)\}^{1/2} \ = \sup_{h \in D(L_N)\setminus\{0\}} \frac{\langle h, A_N \rangle_{N,y_N}}{D_{N,y}(h)^{\frac{1}{2}}} \ .$$

Then there exist a sequence of smooth functions $\{w_N\}_{N\geq 1}$ such that $w_N \in Dom(L_N)$ and

$$\langle w_N, A_N \rangle_{N,y_N} > \sqrt{N\theta} D_{N,y_N}(w_N)^{\frac{1}{2}} \ .$$

We can suppose without loss of generality that

$$\langle w_N \rangle_{N,y_N} = 0 \ . \tag{1.3.21}$$

Taking $\beta_N = \frac{1}{2N} D_{N,y_N}(w_N)$ and $v_N = \beta_N^{-\frac{1}{2}} w_N$, we obtain a sequence of functions $\{v_N\}_{N\geq 1}$ such that

$$\liminf_{N\to\infty} \frac{1}{2N} \langle v_N, A_N \rangle_{N,y_N} \geq \sqrt{\frac{\theta}{2}} \ ,$$

and,

$$D_{N,y_N}(v_N) = 2N \ .$$

We can renormalize again by taking $\gamma_N = \frac{1}{y_N^2} \frac{1}{2N} \langle v_N, A_N \rangle_{N,y_N}$ and $u_N = \gamma_N^{-1} v_N$, obtaining a new sequence of functions $\{u_N\}_{N\geq 1}$ satisfying

$$\frac{1}{2N} \int X_{N,-N}(u_N) p_N p_{-N} \mu_{N,y_N}(dp) = y_N^2 \ , \tag{1.3.22}$$

and

$$\limsup_{N\to\infty} \frac{1}{2N} \int \sum_{-N \leq x \leq x+1 \leq N} a(p_x, p_{x+1}) [X_{x,x+1}(u_N)]^2 \mu_{N,y_N}(dp) \leq \frac{4y^4}{\theta} \ . \tag{1.3.23}$$

The aim of Lemma 6, Lemma 8 and Lemma 10 stated and proved below, is to use (1.3.22) and (1.3.23) in order to obtain a function ξ satisfying

$$\begin{aligned} &i) && \mathbf{E}_{\nu_y}[\xi] = 0 \ , \\ &ii) && \mathbf{E}_{\nu_y}[p_0 p_1 \xi] = 0 \ , \\ &iii) && \mathbf{E}_{\nu_y}[a(p_0,p_1)\{p_0p_1+\xi\}^2] \le \frac{4y^4}{\theta} \ , \end{aligned}$$

besides an additional condition concerning $X_{i,i+1}\tau^j\xi - X_{j,j+1}\tau^i\xi$.

Condition *iii*) obviously implies

$$\theta \le \frac{4y^4}{\mathbf{E}_{\nu_y}[a(p_0,p_1)\{p_0p_1+\xi\}^2]} \ . \tag{1.3.24}$$

Rather less obvious is the fact that *i*), *ii*) and the extra condition on $X_{i,i+1}\tau^j\xi - X_{j,j+1}\tau^i\xi$, imply that ξ belongs to the closure in $L^2(\nu_y)$ of the set over which the infimum in the definition of $\widehat{a}(y)$ is taken (see (1.3.13)). The proof of this fact is the content of Section 2.3.

In short, supposing $l(y) > \theta$ we will find a function ξ such that $\mathbf{E}_{\nu_y}[a(p_0,p_1)\{p_0p_1 + \xi\}^2] \le \frac{4y^4}{\theta}$. Additionally we will see that such a function belongs to the closure of $\{X_{0,1}(\widetilde{F}) : F \text{ is a local smooth function}\}$. These two facts imply the left hand side of (1.3.20), finishing the proof of Theorem 5. □

Now we state and prove the lemmas concerning the construction of the function ξ endowed with the required properties.

Lemma 6. *Given $\theta > 0$, $k \in \mathbb{N}$ and a convergent sequence of positive real numbers $\{y_N\}_{N\ge1}$ satisfying (1.3.22) and (1.3.23), there exists a sequence of functions $\{u_N^{(k)}\}_{N\ge1}$ depending on the variables $p_{-k}, \cdots, p_k$ such that*

$$\frac{1}{2k}\int X_{k,-k}(u_N^{(k)})p_k p_{-k}\mu_{N,y_N}(dp) = y_N^2 \ , \tag{1.3.25}$$

and,

$$\limsup_{N\to\infty} \frac{1}{2k}\int \sum_{-k\le x\le x+1\le k} a(p_x,p_{x+1})[X_{x,x+1}(u_N^{(k)})]^2\mu_{N,y_N}(dp) \le \frac{4y^4}{\theta} \ , \tag{1.3.26}$$

where y is the limit of $\{y_N\}_{N\ge1}$.

Proof. Define for $-N \leq x \leq x+1 \leq N$

$$\alpha^N_{x,x+1} = y_N^{-2}\mathbf{E}_{\mu_{N,y_N}}[p_x p_{x+1} X_{x,x+1}(u_N)]$$

and

$$\beta^N_{x,x+1} = y_N^{-4}\mathbf{E}_{\mu_{N,y_N}}[a(p_x,p_{x+1})[X_{x,x+1}(u_N)]^2] \ .$$

Where $\mathbf{E}_{\mu_{N,y_N}}$ denotes integration with respect to μ_{N,y_N} and u_N is a function satisfying (1.3.22) and (1.3.23).

Thanks to (1.3.22) and the telescopic sum obtained in (1.3.10) we have

$$\alpha^N_{-N,-N+1} + \cdots + \alpha^N_{N-1,N} = 2N \ .$$

After (1.3.23) for every $\epsilon > 0$ there exist N_0 such that

$$\beta^N_{-N,-N+1} + \cdots + \beta^N_{N-1,N} \leq 2N\Big(\frac{4}{\theta} + \epsilon y_N^{-4}\Big),$$

for all $N \geq N_0$.

By using Lemma 7 stated and proved below, we can conclude the existence of a block $\Lambda_{N,k}$ of size $2k$ contained in $\{-N,\cdots,N\}$ such that

$$\gamma_N \left(\sum_{x,x+1\in\Lambda_{N,k}} \alpha^N_{x,x+1}\right)^2 \geq 2k \sum_{x,x+1\in\Lambda_{N,k}} \beta^N_{x,x+1} \ , \tag{1.3.27}$$

with $\gamma_N = \frac{4}{\theta} + \epsilon y_N^{-4}$.

Let us now introduce some notation. Denote by R_N the rotation of axes defined as

$$\begin{array}{rccc} R_N : & \mathbb{R}^{2N+1} & \to & \mathbb{R}^{2N+1} \\ & (p_{-N},\cdots,p_N) & \to & (p_{-N+1},\cdots,p_N,p_{-N}) \ . \end{array}$$

For an integer $i > 0$ we denote by R_N^i the composition of R_N with itself i times, for $i < 0$ the inverse of R_N^{-i} by R_N^i , and R_N^0 for the identity function. As usually, given a function $u : \mathbb{R}^{2N+1} \to \mathbb{R}$ we define $R_N^i u = u \circ R_N^i$.

Let us define

$$w_N = R_N^i u_N \ ,$$

where i is an integer satisfying $\Lambda_{N,k} = \{i-k,\cdots,i+k\}$.

Now we proceed to check that (1.3.25) and (1.3.26) are satisfied by the sequence of functions $\{u_N^k\}_{N\geq 1}$ defined as

$$u_N^k = 2k \left(\sum_{x,x+1 \in \Lambda_{N,k}} \alpha_{x,x+1}^N \right)^{-1} \mathbf{E}_{\mu_{N,y_N}} [w_N \mid \Lambda_k] \ ,$$

where $\Lambda_k = \{p_{-k}, \cdots, p_k\}$. Because of the invariance under axes rotation of the measure μ_{N,y_N}, together with the relation

$$X_{x,x+1}(w_N) = R^{-i} X_{x+i,x+i+1}(u_N) \ ,$$

which follows directly from the definition of w_N, we have

$$\mathbf{E}_{\mu_{N,y_N}} [p_x p_{x+1} X_{x,x+1}(u_N^k)] = 2k \left(\sum_{x,x+1 \in \Lambda_{N,k}} \alpha_{x,x+1}^N \right)^{-1} \mathbf{E}_{\mu_{N,y_N}} [p_{x+i} p_{x+i+1} X_{x+i,x+i+1}(u_N)] \ ,$$

for all x such that $\{p_x, p_{x+1}\} \subset \Lambda_k$. Then, summing over x we obtain that the left hand side of (1.3.25) is equal to

$$\left(\sum_{x,x+1 \in \Lambda_{N,k}} \alpha_{x,x+1}^N \right)^{-1} \mathbf{E}_{\mu_{N,y_N}} \big[\sum_{x,x+1 \in \Lambda_{N,k}} p_x p_{x+1} X_{x,x+1}(u_N) \big],$$

which in turns is equal to y_N^2, proving (1.3.25).

Using Jensen's inequality, and an analogous argument as the one used in the preceding lines, we obtain that

$$\mathbf{E}_{\mu_{N,y_N}} \big[a(p_x, p_{x+1}) [X_{x,x+1}(u_N^k)]^2 \big]$$

is bounded above by

$$4k^2 \left(\sum_{x,x+1 \in \Lambda_{N,k}} \alpha_{x,x+1}^N \right)^{-2} \mathbf{E}_{\mu_{N,y_N}} \big[a(p_{x+i}, p_{x+i+1}) [X_{x+i,x+i+1}(u_N)]^2 \big],$$

for all x such that $\{p_x, p_{x+1}\} \subset \Lambda_k$. This implies (1.3.26) after adding over x, using relation (1.3.27) and taking the superior limit as N goes to infinity. □

Now we state and proof the technical result used to derive 1.3.27.

Lemma 7. *Let* $\{a_i\}_{i=1}^m$ *and* $\{b_i\}_{i=1}^m$ *two sequences of real and positive real numbers, respectively, satisfying*

$$\sum_{i=1}^m a_i = m \qquad \text{and} \qquad \sum_{i=1}^m b_i \leq m\gamma \ , \tag{1.3.28}$$

for fixed constants $m \in \mathbb{N}$, $\gamma > 0$ *and* $k << m$. *Then, there exists a block* Λ *of size* $2k$ *contained in the discrete torus* $\{1, \cdots, m\}$ *such that*

$$\gamma \left(\sum_{i \in \Lambda} a_i \right)^2 \geq 2k \sum_{i \in \Lambda} b_i \ . \tag{1.3.29}$$

Proof. It is enough to check the case where $2k$ is a factor of m. In fact, in the opposite case we can consider periodic sequences of size $2km$ instead of the originals ones $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$.

Therefore we can suppose that $m = 2kl$ for some integer l, and define for $i \in \{1, \cdots, l\}$

$$\alpha_i = \sum_{x \in \Lambda_i} a_x \qquad \text{and} \qquad \beta_i = \sum_{x \in \Lambda_i} b_x \ ,$$

where $\Lambda_i = \{2k(i-1), \cdots, 2ki\}$.

We want to conclude that (1.3.29) is valid for at least one of the Λ_i's. Let us argue by contradiction.

Suppose that $\sqrt{2k\beta_i} > \alpha_i \gamma^{\frac{1}{2}}$ for every $i = 1, \cdots, l$. Adding over i and using the first part of hypothesis (1.3.28), we obtain

$$\sum_{i=1}^l \sqrt{2k\beta_i} > m\gamma^{\frac{1}{2}} \ .$$

By squaring both sides of the last inequality we have,

$$\sum_{i=1}^l \beta_i > m\gamma \ ,$$

which is in contradiction with the second part of hypothesis (1.3.28). □

Now we proceed to take, for each positive integer k, a weak limit of the sequence

$\{u_N^{(k)}\}_{N\geq 1}$ obtained in Lemma 6.

Lemma 8. *For each positive integer k there exists a function $\widetilde{u}_k$ depending on the variables $p_{-k}, \cdots, p_k$ such that*

$$\frac{1}{2k}\boldsymbol{E}_{\nu_y}[X_{k,-k}(\widetilde{u}_k)p_k p_{-k}] = y^2 \ , \tag{1.3.30}$$

and

$$\frac{1}{2k}\boldsymbol{E}_{\nu_y}\left[\sum_{-k\leq x\leq x+1\leq k} a(p_x, p_{x+1})[X_{x,x+1}(\widetilde{u}_k)]^2\right] \leq \frac{4y^4}{\theta} \ . \tag{1.3.31}$$

Proof. Consider the linear functionals $\Lambda_{i,i+1}^N$ defined for $-k \leq i \leq i+1 \leq k$ by

$$\begin{array}{rccc} \Lambda_{i,i+1}^N : & L^2(\mathbb{R}^{2k+1};\nu_y) & \to & \mathbb{R} \\ & w & \to & \mathbf{E}_{\mu_{N,y_N}}[X_{i,i+1}(u_N^{(k)})w] \ . \end{array}$$

Let $\mathcal{P}^k$ be an enumerable dense set of polynomials in $L^2(\mathbb{R}^{2k+1};\nu_y)$. From (1.3.31) and the Cauchy-Schwartz inequality we obtain the existence of a constant C such that

$$|\Lambda_{i,i+1}^N(w)| \leq C\left(\int w^2 d\mu_{N,y_N}\right)^{\frac{1}{2}} \ ,$$

for every $w \in \mathcal{P}^k$.

By a diagonal argument we can draw a subsequence for which the limits of $\Lambda_{i,i+1}^N(w)$ exist for all $w \in \mathcal{P}^k$. Moreover, passing to the limit and extending to the whole space $L^2(\mathbb{R}^{2k+1};\nu_y)$, we get linear functionals $\Lambda_{i,i+1}$ satisfying

$$|\Lambda_{i,i+1}(w)| \leq C\left(\int w^2 d\nu_y\right)^{\frac{1}{2}} . \tag{1.3.32}$$

On the other hand, consider the linear functionals Λ^N defined by

$$\begin{array}{rccc} \Lambda^N : & L^2(\mathbb{R}^{2k+1};\nu_y) & \to & \mathbb{R} \\ & w & \to & \mathbf{E}_{\mu_{N,y_N}}[u_N^{(k)}w] \ . \end{array}$$

Because of (1.3.31), (1.3.21) and Poincaré's inequality we have

$$\mathbf{E}_{\mu_{N,y_N}}\left[(u_N^{(k)})^2\right] \leq \frac{4y^4 C}{\theta} \ ,$$

for a constant C depending only on k. Then, by the very same arguments used above, we get a linear functional Λ satisfying

$$|\Lambda(w)| \leq \sqrt{C}\left(\int w^2 d\nu_y\right)^{\frac{1}{2}} . \tag{1.3.33}$$

Finally, it follows from (1.3.32) and (1.3.33) the existence of a function $\widetilde{u}_k$ satisfying

$$\Lambda_{i,i+1}(w) = \mathbf{E}_{\nu_y}[X_{i,i+1}(\widetilde{u}_k)w] ,$$

$$\Lambda(w) = \mathbf{E}_{\nu_y}[\widetilde{u}_k w] ,$$

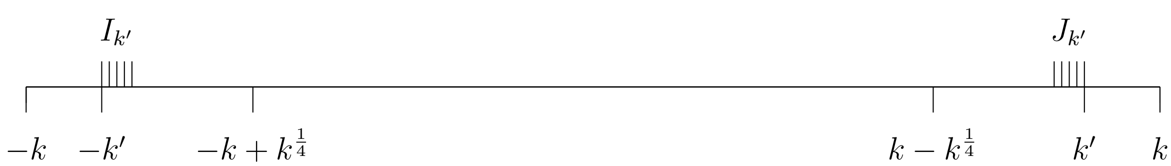
and therefore, satisfying (1.3.30) and (1.3.31). □

Now using the sequence $\{\widetilde{u}_k\}_{k\in\mathbb{N}}$ we construct a sequence of functions $\{u_{k'}\}_{k'\in M}$ indexed on an infinite subset of $\mathbb{N}$, each one depending on the variables $p_{-k'}, \cdots, p_{k'}$. This sequence will satisfy, besides (1.3.30) and (1.3.31), an additional condition regarding the contribution of the terms near the boundary of $\{-k', \cdots, k'\}$ to the total Dirichlet form.

Lemma 9. *There exist a sequence of functions $\{u_{k'}\}_{k'\in M}$ indexed on an infinite subset of $\mathbb{N}$, each one depending on the variables $p_{-k'}, \cdots, p_{k'}$, satisfying (1.3.30), (1.3.31) and*

$$\boldsymbol{E}_{\nu_y}[(X_{x,x+1}(u_{k'}))^2] = O(k^{7/8}) ,$$

for $\{x, x+1\} \subset I_{k'} \cup J_{k'}$. The blocks $I_{k'} = [-k', -k' + (k')^{1/8}]$ and $J_{k'} = [k' - (k')^{1/8}, k']$ are illustrated in the following figure.



Proof. Given $k > 0$ divide each interval $[-k, -k + k^{1/4}]$ and $[k - k^{1/4}, k]$ into $k^{1/8}$ blocks of size $k^{1/8}$, and consider the sequence $\{\widetilde{u}_k\}_{k\in\mathbb{N}}$ obtained in Lemma 8.

Because of (1.3.31), for every $k > 0$ there exist $k' \in [k - k^{1/8}, k]$ such that

$$\mathbf{E}_{\nu_y}\left[\sum_{\{x,x+1\}\in I_{k'}\cup J_{k'}} (X_{x,x+1}(\widetilde{u}_k))^2\right] = O(k^{7/8}) .$$

Define for each $k>0$ the function $u_{k'} = \frac{1}{C_{y,k,k'}}\mathbf{E}_{\nu_y}\left[\widetilde{u}_k \mid \mathfrak{F}^{k'}_{-k'}\right]$, where

$$C_{y,k,k'} = \frac{1}{2k'y^2}\left\{2ky^2 - \mathbf{E}_{\nu_y}\left[\sum_{\{x,x+1\}\in I_{k'}\cup J_{k'}} p_x p_{x+1} X_{x,x+1}(\widetilde{u}_k)\right]\right\}$$

and $\mathfrak{F}^{k'}_{-k'}$ denotes the σ-field generated by $p_{-k'},\cdots,p_{k'}$.

Is easy to see that the sequence $\{u_{k'}\}_{k'}$ satisfies the desired conditions. □

Finally, we obtain the weak limit used in the proof of Theorem 5.

Lemma 10. *There exist a function ξ in $L^2(\nu_y)$ satisfying*

$$\boldsymbol{E}_{\nu_y}[\xi] = 0, \tag{1.3.34}$$

$$\boldsymbol{E}_{\nu_y}[p_0p_1\xi] = 0, \tag{1.3.35}$$

$$\boldsymbol{E}_{\nu_y}[a(p_0,p_1)[p_0p_1+\xi]^2] \le \frac{4y^4}{\theta}, \tag{1.3.36}$$

and the integrability conditions

$$X_{i,i+1}(\tau^j\xi) = X_{j,j+1}(\tau^i\xi) \quad \text{if} \quad \{i,i+1\}\cap\{j,j+1\} = \emptyset, \tag{1.3.37}$$

$$p_{i+1}[X_{i+1,i+2}(\tau^i\xi) - X_{i,i+1}(\tau^{i+1}\xi)] = p_{i+2}\tau^i\xi - p_i\tau^{i+1}\xi \quad \text{for} \quad i\in\mathbb{Z}. \tag{1.3.38}$$

Proof. For all integer $k>0$ define

$$\zeta_k = \frac{1}{2k'}\sum_{i=-k'}^{k'-1} X_{i,i+1}(u_{k'})(\tau^{-i}).$$

It is clear from the definition of ζ_k that

$$\mathbf{E}_{\nu_y}[\zeta_k] = 0,$$

and

$$p_0p_1\zeta_k(\omega) = \frac{1}{2k'}\sum_{i=-k'}^{k'-1}\{p_ip_{i+1}X_{i,i+1}(u_{k'})\}(\tau^{-i}).$$

Then, from (1.3.30) it follows

$$\mathbf{E}_{\nu_y}[p_0p_1\zeta_k] = y^4.$$

On the other hand, from (1.3.31) we have

$$\begin{aligned}
\mathbf{E}_{\nu_y}[a(p_0,p_1)\zeta_k^2] &= \mathbf{E}_{\nu_y}\left[a(p_0,p_1)\left(\frac{1}{2k'}\sum_{i=-k'}^{k'-1}[X_{i,i+1}(u_{k'})](\tau^{-i}\omega)\right)^2\right] \\
&\le \mathbf{E}_{\nu_y}\left[a(p_0,p_1)\frac{1}{2k'}\sum_{i=-k'}^{k'-1}[X_{i,i+1}(u_{k'})]^2(\tau^{-i}\omega)\right] \\
&= \frac{1}{2k}\mathbf{E}_{\nu_y}\left[\sum_{i=-k'}^{k'-1}a(p_i,p_{i+1})[X_{i,i+1}(u_{k'})]^2\right] \\
&\le \frac{4y^4}{\theta}.
\end{aligned}$$

Now consider the sequence $\{\xi_k\}_{k\ge1}$ defined by

$$\xi_k = \zeta_k - p_0p_1.$$

Since the preceding sequence is uniformly bounded in $L^2(\nu_y)$, there exists a weak limit function $\xi \in L^2(\nu_y)$. Obviously, the function ξ satisfies (1.3.34),(1.3.34) and (1.3.36).

In addition, an elementary calculation shows that 1.3.37 and 1.3.38 are satisfied by ξ_k up to an error coming from a small number of terms near the edge of $[-k',k']$. Then, in view of Lemma 9, the final part of the lemma is satisfied as well. □

1.4 Boltzmann-Gibbs Principle

The aim of this section is to provide a proof for Theorem 3. In fact, we will prove a stronger result that will be also useful in the proof of tightness. Namely,

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t\sqrt{N}\sum_{x\in\mathbb{T}_N}\nabla_N H_s(x/N)[V_x(p(s))-\mathcal{L}_N\tau^xF_k(p(s))]\,ds\right)^2\right] \;=\; 0\,,$$

where,

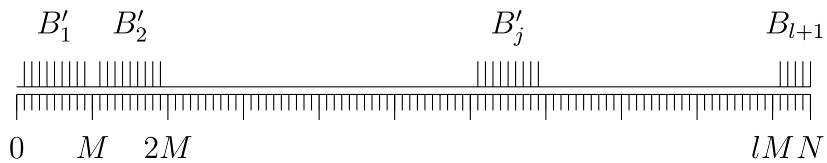
$$V_x(p) = W_{x,x+1}(p) - \hat{a}(y)[p_{x+1}^2 - p_x^2]\,.$$

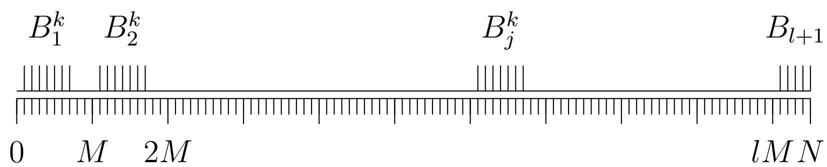
We begin localizing the problem. Fix an integer M that shall increase to infinity after N. Being l and r the integers satisfying $N = lM + r$ with $0 \le r < M$, define for

$j = 1, \cdots, l$

$$B_j = \{(j-1)M+1, \cdots, jM\} ,$$
$$B'_j = \{(j-1)M+1, \cdots, jM-1\} ,$$
$$B^k_j = \{(j-1)M+1, \cdots, jM-s_k\} ,$$

where s_k is the size of the block supporting F_k. Define the remaining block as $B_{l+1} = \{lM+1, \cdots, N\}$. The blocks B'_j and B^k_j are illustrated in the following figures,





With this notation we can write

$$\sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N H_s(x/N)[V_x(p(s)) - \mathcal{L}_N \tau^x F_k(p(s))] \;=\; V_1 + V_2 + V_3 \,, \qquad (1.4.1)$$

with,

$$V_1 = \sqrt{N} \sum_{j=1}^{l} \sum_{x \in B'_j} \nabla_N H_s(x/N) V_x - \sqrt{N} \sum_{j=1}^{l} \sum_{x \in B^k_j} \nabla_N H_s(x/N) \mathcal{L}_N \tau^x F_k \,,$$
$$V_2 = \sqrt{N} \sum_{x \in B_{l+1}} \nabla_N H_s(x/N) V_x \;-\; \sqrt{N} \sum_{x \in B_{l+1}} \nabla_N H_s(x/N) \mathcal{L}_N \tau^x F_k \,,$$
$$V_3 = \sqrt{N} \sum_{j=1}^{l} \nabla_N H_s(jK/N) V_{jM} - \sqrt{N} \sum_{j=1}^{l} \sum_{x \in B_j \setminus B^k_j} \nabla_N H_s(x/N) \mathcal{L}_N \tau^x F_k \,.$$

Observe that V_1 is a sum of functions which depends on disjoint blocks, and contains almost all the terms appearing in the left hand side of (1.4.1), therefore, V_2 and V_3 can be considered error terms. In order to prove Theorem 3 it suffices to show

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t V_i\,ds\right)^2\right] \;=\; 0\;, \tag{1.4.2}$$

for each V_i separately.

The following is a very useful estimate of the time variance in terms of the $\mathscr{H}_{-1}$ norm defined in (1.3.6).

Proposition 1. *Given $T>0$ and a mean zero function $V\in L^2(\pi)\cap\mathscr{H}_{-1}$,*

$$\mathbb{E}_{\pi}\left[\sup_{0\le t\le T}\left(\int_0^t V(p_s)\,ds\right)^2\right] \;\le\; 24T||V||_{-1}^2\;.$$

See Lemma 2.4 in [KmLO] or Proposition 6.1 in [KL] for a proof .

Remark 1. *A slightly modification in the proof given in [KmLO] permit to conclude that, for every smooth function $h:[0,T]\to\mathbb{R}$,*

$$\mathbb{E}_{\pi}\left[\sup_{0\le t\le T}\left(\int_0^t h(s)V(p_s)\,ds\right)^2\right] \;\le\; C_h||V||_{-1}\;,$$

where,

$$C_h \;=\; 6\{4||h||_\infty^2T^2+||h'||_\infty^2T^3\}\;.$$

Moreover, in our case we have

$$\mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t\sum_{j=1}^l h_j(s)V_{B_j}(p_s)\,ds\right)^2\right] \;\le\; \sum_{j=1}^l C_{h_j}||V_{B_j}||_{-1}\;,$$

for functions $\{V_{B_j}\}_{j=1}^l$ depending on disjoint blocks.

The proof of (1.4.2) will be divided in three lemmas.

Lemma 11.

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t V_2\,ds\right)^2\right] \;=\; 0\;.$$

Proof. By Proposition 1, the expectation in the last expression is bounded above by the sum of the following three terms

$$\frac{3C_H|B_{l+1}|}{N}\sum_{x\in B_{l+1}}\left\langle W_{x,x+1}\,,\,(-\mathcal{L}_N)^{-1}W_{x,x+1}\right\rangle\ , \tag{1.4.3}$$

$$\frac{3\hat{a}(y)^2C_H|B_{l+1}|}{N}\sum_{x\in B_{l+1}}\left\langle p_{x+1}^2-p_x^2\,,\,(-\mathcal{L}_N)^{-1}p_{x+1}^2-p_x^2\right\rangle\ , \tag{1.4.4}$$

$$\frac{3C_H|B_{l+1}|}{N}\sum_{x\in B_{l+1}}\langle(-\mathcal{L}_N)\tau^xF_k,\tau^xF_k\rangle\ . \tag{1.4.5}$$

Here C_H represents a constant depending on H and T, that can be multiplied by a constant from line to line.

Using the variational formula for the $\mathscr{H}_{-1}$ norm (see (1.3.6)) we can see that the expression in (1.4.3) is equal to

$$\frac{C_H|B_{l+1}|}{N}\sum_{x\in B_{l+1}}\sup_{g\in L^2(\nu_y^N)}\left\{2\left\langle W_{x,x+1},g\right\rangle+\langle g,\mathcal{L}_{x,x+1}g\rangle\right\}\ ,$$

where,

$$\mathcal{L}_{x,x+1}=\frac{1}{2}X_{x,x+1}[a(p_x,p_{x+1})X_{x,x+1}]\ .$$

From the definition given in (1.1.4) we have $W_{x,x+1}=-X_{x,x+1}[a(p_x,p_{x+1})p_xp_{x+1}]$. Performing integration by parts in the two inner products, we can write the quantity inside the sum as

$$\frac{1}{2}\sup_{g\in L^2(\nu_y^N)}\left\{4\left\langle a(p_x,p_{x+1})p_xp_{x+1},X_{x,x+1}g\right\rangle-\left\langle X_{x,x+1}g,a(p_x,p_{x+1})X_{x,x+1}g\right\rangle\right\}\ ,$$

which by the elementary inequality $2ab\le A^{-1}a^2+Ab^2$, is bounded above by

$$2\left\langle a(p_x,p_{x+1})p_x^2p_{x+1}^2\right\rangle\ .$$

Then the expression in (1.4.3) is bounded above by

$$\frac{C_H|B_{l+1}|^2}{N}\ .$$

The same is true for the term corresponding to (1.4.4), which coincides with (1.4.3) if we take $a(r,s)\equiv 1$.

Since F_k is a local function supported in a box of size s_k and ν_y^N is translation invariant, we have for all $x, y \in \mathbb{T}_N$

$$\langle \tau^x F_k, (-\mathcal{L}_N)\tau^y F_k\rangle \le s_k \; ||X_{0,1}(\widetilde{F_k})||^2_{L^2(\nu_y^N)} \; ,$$

which implies that the expression in (1.4.5) is bounded by

$$\frac{C_H |B_{l+1}|^2}{N} s_k \; ||X_{0,1}(\widetilde{F_k})||^2_{L^2(\nu_y^N)} \; ,$$

ending the proof. □

Lemma 12.

$$\lim_{k\to\infty} \lim_{N\to\infty} \mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t V_3 \, ds\right)^2\right] \; = \; 0 \; .$$

Proof. The proof is similar to the preceding one. □

Lemma 13.

$$\lim_{k\to\infty} \lim_{N\to\infty} \mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t V_1 \, ds\right)^2\right] \; = \; 0 \; .$$

Proof. Recall that the expectation in the last expression is by definition

$$N\mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t \sum_{j=1}^{l}\left\{\sum_{x\in B_j'} \nabla_N H_s(x/N) V_x - \sum_{x\in B_j^k} \nabla_N H_s(x/N)\mathcal{L}_N \tau^x F_k \; ds\right\}\right)^2\right] .$$

The smoothness of the function H allows to replace $\nabla_N H_s(x/N)$ into each sum in the last expression by $\nabla_N H_s(x_j^*/N)$, where $x_j^* \in B_j$ (for instance, take $x_j^* = (j-1)K+1$), obtaining

$$N\mathbb{E}_{\nu_y^N}\left[\sup_{0\le t\le T}\left(\int_0^t \sum_{j=1}^{l} \nabla_N H_s(x_j^*/N)\left\{\sum_{x\in B_j'} V_x - \sum_{x\in B_j^k} \mathcal{L}_N \tau^x F_k \; ds\right\}\right)^2\right] .$$

By proposition 1 and Remark 1, the quantity in the preceding line is bounded above by

$$\frac{C_H}{N}\sum_{j=1}^{l}\left\langle \sum_{x\in B_j'} V_x - \sum_{x\in B_j^k}\mathcal{L}_N \tau^x F_k \; , \; (-\mathcal{L}_N)^{-1}\sum_{x\in B_j'} V_x - \sum_{x\in B_j^k}\mathcal{L}_N \tau^x F_k\right\rangle_{\nu_y} \quad ,$$

Using the variational formula for the $\mathscr{H}_{-1}$ norm given in (1.3.6) and the convexity of the Dirichlet form, we are able to replace $(-\mathcal{L}_N)^{-1}$ by $(-\mathcal{L}_{B'_j})^{-1}$ in the expression above. In addition, by translation invariance of the measure ν_y^N we can bound this expression by

$$C_H \frac{l}{N} \left\langle \sum_{x \in B'_1} V_x - \sum_{x \in B_1^k} \mathcal{L}_N \tau^x F_k \ , \ (-\mathcal{L}_{B'_1})^{-1} \sum_{x \in B'_1} V_x - \sum_{x \in B_1^k} \mathcal{L}_N \tau^x F_k \right\rangle_{\nu_y} .$$

By the equivalence of ensembles stated in Section 2.4 and the fact that $\frac{l}{N} \sim \frac{1}{M}$, the limit superior, as N goes to infinity, of the last expression is bounded above by

$$C_H \limsup_{M \to \infty} \frac{1}{M} \left\langle \sum_{x \in B'_1} V_x - \sum_{x \in B_1^k} \mathcal{L}_N \tau^x F_k \ , \ (-\mathcal{L}_{B'_1})^{-1} \sum_{x \in B'_1} V_x - \sum_{x \in B_1^k} \mathcal{L}_N \tau^x F_k \right\rangle_{\nu_{M,\sqrt{M}y}} .$$

The last line can be written as

$$C_H \limsup_{M \to \infty} \frac{1}{M} \Delta_{M,y}(B_M + \widehat{a}(y) A_M - H_M^{F_k}, \ B_M + \widehat{a}(y) A_M - H_M^{F_k}), \tag{1.4.6}$$

by using the notation introduced in Section 1.3. For that, it suffices to replace M by 2M+1 from the beginning of this section. Here B_M correspond to the current in a block, and is not to be confused with the notation used for the blocks themselves.

On the other hand, it is easy to check that the variance appearing in (1.4.6) is equal to

$$\begin{aligned} &(\widehat{a}(y))^2 \Delta_{M,y}(A_M, A_M) + \Delta_{M,y}(B_M, B_M) + \Delta_{M,y}(H_M^{F_k}, H_M^{F_k}) \\ &\qquad - 2\widehat{a}(y) \Delta_{M,y}(A_M, B_M) + \Delta_{M,y}(A_M, H_M^{F_k}) - \Delta_{M,y}(B_M, H_M^{F_k}) \ . \end{aligned}$$

Therefore, thanks to Theorem 4 and Theorem 5, if we divide by M and take the limit as M goes to infinity at both sides of last expression, we can conclude that

$$\lim_{M \to \infty} \frac{1}{M} \Delta_{M,y}(B_M + \widehat{a}(y) A_M - H_M^{F_k}) = 4a(y, 1/2F_k) - 4y^4 \widehat{a}(y) \ .$$

By the definition of the sequence $\{F_k\}_{k \geq 1}$ (see (1.2.3)), the limit as k goes to infinity of the last term is equal to zero. □

1.5 Tightness

Let us firstly introduce some notation in order to define a space in which fluctuations take place and in which we will be able to prove tightness. Let $-\Delta$ be the positive operator, essentially self-adjoint on $L^2([0,1])$ defined by

$$Dom(-\Delta) = C_0^2([0,1]),$$
$$-\Delta = -\frac{d^2}{dx^2} ,$$

where $C_0^2([0,1])$ denotes the space of twice continuously differentiable functions on $(0,1)$ which are continuous in $[0,1]$ and which vanish at the boundary. It is well known that its normalized eigenfunctions are given by $e_n(x) = \sqrt{2}\sin(\pi n x)$ with corresponding eigenvalues $\lambda_n = (\pi n)^2$ for every $n \in \mathbb{N}$, moreover, $\{e_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis of $L^2([0,1])$.

For any nonnegative real number k denote by $\mathcal{H}_k$ the Hilbert space obtained as the completion of $C_0^2([0,1])$ endowed with the inner product

$$\langle f, g\rangle_k = \langle f, (-\Delta)^k g\rangle ,$$

where $\langle , \rangle$ stands for the inner product in $L^2([0,1])$. We have from the spectral theorem for self-adjoint operators that

$$\mathcal{H}_k = \{f \in L^2([0,1]) : \sum_{n=1}^{\infty} n^{2k}\langle f, e_n\rangle < \infty\} , \tag{1.5.1}$$

and

$$\langle f, g\rangle_k = \sum_{n=1}^{\infty}(\pi n)^{2k}\langle f, e_n\rangle\langle g, e_n\rangle . \tag{1.5.2}$$

This is valid also for negative k. In fact, if we denote the topological dual of $\mathcal{H}_k$ by $\mathcal{H}_{-k}$ we have

$$\mathcal{H}_{-k} = \{f \in \mathcal{D}'([0,1]) : \sum_{n=1}^{\infty} n^{-2k} f(e_n)^2 < \infty\} . \tag{1.5.3}$$

The $\mathcal{H}_{-k}$-inner product between the distributions f and g can be written as

$$\langle f, g\rangle_{-k} = \sum_{n=1}^{\infty}(\pi n)^{-2k} f(e_n)g(e_n) , \tag{1.5.4}$$

Denote by $\mathbb{Q}_N$ the probability measure on $C([0,T],\mathcal{H}_{-k})$ induced by the energy fluctuation field Y_t^N and the Markov process $\{p^N(t), t \geq 0\}$ defined in Section 1.1, starting from the equilibrium probability measure ν_y^N. For notational convenience, we omit the dependence of k in $\mathbb{Q}_N$.

We are now ready to state the main result of this section, which proof is divided in lemmas.

Theorem 6. *The sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight in $C([0,T],\mathcal{H}_{-k})$ for $k > \frac{3}{2}$.*

In order to establish the tightness of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ of probability measures on $C([0,T],\mathcal{H}_{-k})$, it suffices to check the following two conditions (*c.f.* [KL] p.299),

$$\lim_{A\to\infty} \limsup_{N\to\infty} \mathbb{P}_{\nu_y} \left[\sup_{0\leq t\leq T} ||Y_t^N||_{-k} > A \right] = 0 \ , \tag{1.5.5}$$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}_{\nu_y} \left[w(Y^N,\delta) > \epsilon \right] = 0 \ , \tag{1.5.6}$$

where the modulus of continuity $w(Y,\delta)$ is defined by

$$w(Y,\delta) = \sup_{\substack{|t-s|<\delta \\ 0\leq s<t\leq T}} ||Y_t - Y_s||_{-k} \ .$$

Let us recall that for every function $H \in C^2(\mathbb{T})$ we have

$$Y_t^N(H) \ = \ Y_0^N(H) - Z_t^N(H) - M_t^N(H) \ , \tag{1.5.7}$$

where,

$$Z_t^N(H) = \int_0^t \sqrt{N} \sum_{x\in\mathbb{T}_N} \nabla_N H(x/N) W_{x,x+1}(s) ds \ ,$$

$$M_t^N(H) = \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{T}_N} \nabla_N H(x/N) \sigma(p_x(s), p_{x+1}(s)) \, dB_{x,x+1}(s) \ .$$

The quadratic variation of the martingale $\{M_t^N(H)\}_{t\geq 0}$ is given by

$$\langle M_t^N(H)\rangle(t) = \frac{1}{N} \sum_{x\in\mathbb{T}_N} \int_0^t |\nabla_N H(\frac{x}{N}, s)|^2 a(p_x, p_{x+1}) p_x^2 p_{x+1}^2 ds \ .$$

We begin by giving the following estimate.

Lemma 14. *There exist a constant $B = B(y,T)$ such that for every function $H \in C^2(\mathbb{T})$ and every $N \geq 1$*

$$\mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T}(Y_t^N(H))^2\right] \le B\left\{\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(x/N)^2 + \frac{1}{N}\sum_{x\in\mathbb{T}_N}(\nabla_N H(x/N))^2\right\} .$$

Proof. From the definition of the fluctuation field it is clear that

$$\mathbb{E}_{\nu_y}\left[(Y_0^N(H))^2\right] = 2y^4\frac{1}{N}\sum_{x\in\mathbb{T}_N} H(x/N)^2 , \tag{1.5.8}$$

and by Doob's inequality together with the fact that $a(\cdot,\cdot) \le C$ we have

$$\mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T}(M_t^N(H))^2\right] \le CTy^4\frac{1}{N}\sum_{x\in\mathbb{T}_N}(\nabla_N H(x/N))^2 . \tag{1.5.9}$$

From Proposition 1 of Section 1.4 and the variational formula given in (1.3.6) we obtain

$$\mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T}\left(Z_t^N(H)\right)^2\right] \le \frac{24T}{N}\sup_{g\in D(\mathcal{L})}\left\{\Big\langle 2\sum_{x\in\mathbb{T}_N}\nabla_N H(x/N)W_{x,x+1}g\Big\rangle_{\nu_y} + \langle g,\mathcal{L}g\rangle_{\nu_y}\right\} .$$

After integration by parts, the first term in the expression into braces can be written as

$$-2\Big\langle\sum_{x\in\mathbb{T}_N}\nabla_N H(x/N)a(p_x,p_{x+1})p_xp_{x+1}X_{x,x+1}(g)\Big\rangle_{\nu_y} ,$$

which by Schwartz inequality is bounded above by

$$2\Big\langle\sum_{x\in\mathbb{T}_N}(\nabla_N H(x/N))^2a(p_x,p_{x+1})p_x^2p_{x+1}^2\Big\rangle_{\nu_y} + \frac{1}{2}\Big\langle\sum_{x\in\mathbb{T}_N}a(p_x,p_{x+1})(X_{x,x+1}(g))^2\Big\rangle_{\nu_y} .$$

Thus,

$$\mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T}\left(Z_t^N(H)\right)^2\right] \le 48TCy^4\frac{1}{N}\sum_{x\in\mathbb{T}_N}(\nabla_N H(x/N))^2 .$$

□

Corollary 4. *Condition (1.5.5) is valid for $k > \frac{3}{2}$.*

Proof. From (1.5.4) and Lemma 14 we obtain

$$\limsup_{N\to\infty} \mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T} ||Y_t^N||_{-k}^2\right] \le \sum_{n=1}^{\infty}(\pi n)^{-2k} \limsup_{N\to\infty} \mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T} Y_t^N(e_n)^2\right]$$
$$\le B\sum_{n=1}^{\infty}(\pi n)^{-2k}(1+(\pi n)^2)\ .$$

The proof is then concluded by using Chebychev's inequality. □

In view of (1.5.4) and Lemma 14 we reduce the problem of equicontinuity as follows.

$$\limsup_{N\to\infty} \mathbb{E}_{\nu_y}\left[w(Y^N,\delta)\right] \le \sum_{n=1}^{\infty}(\pi n)^{-2k} \limsup_{N\to\infty} \mathbb{E}_{\nu_y}\left[\sup_{\substack{|t-s|<\delta\\0\le s<t\le T}} (Y_t^N(e_n)-Y_s^N(e_n))^2\right]$$
$$\le 4\sum_{n=1}^{\infty}(\pi n)^{-2k} \limsup_{N\to\infty} \mathbb{E}_{\nu_y}\left[\sup_{\substack{|t-s|<\delta\\0\le s<t\le T}} (Y_t^N(e_n))^2\right]$$
$$\le B\sum_{n=1}^{\infty}(\pi n)^{-2k}(1+(\pi n)^2)\ .$$

Therefore, the series appearing in the first line of the above expression is uniformly convergent in δ if $k>\frac{3}{2}$. Thus, in order to verify condition (1.5.6) it is enough to prove

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{E}_{\nu_y}\left[\sup_{\substack{|t-s|<\delta\\0\le s<t\le T}} (Y_t^N(e_n)-Y_s^N(e_n))^2\right] = 0\ ,$$

for every $n\ge 1$.

We analyze separately the terms corresponding to M_t^N and Z_t^N (see 1.5.7). In next lemma we state a global estimate for the martingale part.

Lemma 15. *For every function H and every $m\in\mathbb{N}$, there exists a constant C depending only on m such that*

$$\mathbb{E}_{\nu_y^N}[\,|M_t^N(H)|^{2m}\,] \le Cy^{2m}t^m\left\{\frac{1}{N}\sum_{x\in\mathbb{T}_N}\left|\nabla_N H(\frac{x}{N})\right|^2\right\}^m\ .$$

Proof. Denote the continuous martingale $M_t^N(H)$ by M_t, and let C_m be a constant depending only on m which can change from line to line.

Using the explicit expression for the martingale and applying Itô's formula we have

$$d(M_t)^{2m} = 2m(M_t)^{2m-1}dM_t + m(2m-1)(M_t)^{2m-2}Q_t dt,$$

where,

$$Q_t = \frac{1}{N}\sum_{x\in\mathbb{T}_N} |\nabla_N H(\frac{x}{N})|^2 a(p_x, p_{x+1}) p_x^2 p_{x+1}^2 \, .$$

Explicit calculations lead us to

$$\mathbb{E}_{\nu_y^N}[(Q_t)^m] \le C_m y^{2m} \left\{ \frac{1}{N}\sum_{x\in\mathbb{T}_N} \left|\nabla_N H(\frac{x}{N})\right|^2 \right\}^m \quad ,$$

thus, by stationarity and applying Hölder inequality for space and time we obtain

$$\mathbb{E}_{\nu_y^N}[(M_t)^{2m}] \le C_m y^2 t^{\frac{1}{m}} \frac{1}{N}\sum_{x\in\mathbb{T}_N} |\nabla_N H(\frac{x}{N})|^2 \left(\int_0^t \mathbb{E}_{\nu_y^N}[(M_s)^{2m}] ds \right)^{\frac{2m-2}{2m}} \quad . \tag{1.5.10}$$

In terms of the function $f(t) = \left(\int_0^t \mathbb{E}_{\nu_y^N}[(M_s)^{2m}]ds\right)^{\frac{1}{m}}$, inequality (1.5.10) reads

$$f'(t) \le C_m y^2 t^{\frac{1}{m}} \frac{1}{N}\sum_{x\in\mathbb{T}_N} \left|\nabla_N H(\frac{x}{N})\right|^2 ,$$

and integrating we obtain

$$f(t) \le C_m y^2 t^{1+\frac{1}{m}} \frac{1}{N}\sum_{x\in\mathbb{T}_N} \left|\nabla_N H(\frac{x}{N})\right|^2 .$$

The proof ends by using the last line to estimate the right hand side of (1.5.10). □

In order to pass from this global estimate to a local estimate, we will use the Garcia's inequality.

Lemma 16. *(Garcia-Rodemich-Rumsey) (cf [SV] p.47) Let* $p, \phi : [0,\infty] \to \mathbb{R}$ *continuous, strictly increasing functions such that* $p(0) = \psi(0) = 0$ *and* $\lim_{t\to\infty}\psi(t) = \infty$. *Given* $\phi \in C([0,T];\mathbb{R}^d)$ *such that*

$$\int_0^T \int_0^T \psi\left(\frac{|\phi(t)-\phi(s)|}{p(|t-s|)}\right) dsdt \le B < \infty,$$

then

$$|\phi(t) - \phi(s)| \leq 8 \int_0^{t-s} \psi^{-1}\left(\frac{4B}{u^2}\right) p(du),$$

for $0 \leq s < t \leq T$.

Lemma 17. *For every function* $H \in C^2(\mathbb{T})$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{E}_{\nu_y} \left[\sup_{\substack{|t-s|<\delta \\ 0 \leq s<t \leq T}} \left(M_t^N(H) - M_s^N(H)\right)^2 \right] = 0 \ .$$

Proof. Taking $p(u) = \sqrt{u}$ and $\psi(u) = u^6$ in Lemma 16 we get

$$|\phi(t) - \phi(s)| \leq CB^{1/6} |t-s|^{1/6} \ ,$$

where,

$$B = \int_0^T \int_0^T \frac{|\phi(t) - \phi(s)|^6}{|t-s|^3} dsdt \ . \tag{1.5.11}$$

Taking $\phi(t) = M_t^N(H)$ we obtain

$$\mathbb{E}_{\nu_y^N} \big[\sup_{\substack{|t-s|<\delta \\ 0 \leq s<t \leq T}} |\widehat{M}_N^H(t) - \widehat{M}_N^H(s)|^2 \big] \leq C y^2 \delta^{1/3} T^{2/3} \frac{1}{N} \sum_{x \in \mathbb{T}_N} (\nabla H(x/N))^2 \ ,$$

which implies the desired result.

Observe that the integral in (1.5.11) is finite, which permits to apply Lemma 16. In fact, as a consequence of Lemma 15 and Kolmogorov-Centsov theorem, we have α-Hölder continuity of paths for $\alpha \in [0, \frac{1}{2})$. □

The proof of Theorem 6 will be concluded by proving the following lemma.

Lemma 18. *For every function* $H \in C^2(\mathbb{T})$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{E}_{\nu_y} \left[\sup_{\substack{|t-s|<\delta \\ 0 \leq s<t \leq T}} \left(Z_t^N(H) - Z_s^N(H)\right)^2 \right] = 0$$

Proof. Recall that the expectation appearing above is by definition

$$\mathbb{E}_{\nu_y}\left[\sup_{\substack{|t-s|<\delta\\0\le s<t\le T}}\left(\int_s^t\sqrt{N}\sum_{x\in\mathbb{T}_N}\nabla_N H(x/N)W_{x,x+1}(s)ds\right)^2\right]. \tag{1.5.12}$$

Now we take advantage of the decomposition obtained for the current in the preceding sections, which allows to study separately the diffusive part of the current and the part coming from a fluctuation term. For this we add and subtract $\hat{a}(y)[p_{x+1}^2-p_x^2]+\mathcal{L}_N(\tau^x F_k(p))$ from $W_{x,x+1}$, obtaining that (1.5.12) is bounded above by 3 times the sum of the following terms,

$$4\mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T}\left(\int_0^t\sqrt{N}\sum_{x\in\mathbb{T}_N}\nabla_N H(x/N)[W_{x,x+1}(s)-\hat{a}(y)[p_{x+1}^2-p_x^2]-\mathcal{L}_N(\tau^x F_k(p))]ds\right)^2\right],$$

$$\hat{a}(y)^2\mathbb{E}_{\nu_y}\left[\sup_{\substack{|t-s|<\delta\\0\le s<t\le T}}\left(\int_s^t\sqrt{N}\sum_{x\in\mathbb{T}_N}\nabla_N H(x/N)[p_{x+1}^2-p_x^2]ds\right)^2\right],$$

$$\mathbb{E}_{\nu_y}\left[\sup_{\substack{|t-s|<\delta\\0\le s<t\le T}}\left(\int_s^t\sqrt{N}\sum_{x\in\mathbb{T}_N}\nabla_N H(x/N)\mathcal{L}_N(\tau^x F_k(p))ds\right)^2\right].$$

The first term tends to zero as k tends to infinity after N. In fact, this is the content of the Boltzmann-Gibbs Principle proved in Section 1.4.

Performing a sum by parts and using Schwartz inequality together with the stationarity, we can see that the second term is bounded above by

$$\hat{a}(y)^2\delta T E_{\nu_y}\left[\left(\frac{1}{\sqrt{N}}\sum_{x\in\mathbb{T}_N}\Delta_N H(x/N)p_x^2\right)^2\right].$$

We can replace in the last line p_x^2 by $[p_x^2-y^2]$ (because of periodicity), obtaining that the expression is bounded above by

$$3y^4\hat{a}(y)^2\delta T\frac{1}{N}\sum_{x\in\mathbb{T}_N}(\Delta_N H(x/N))^2.$$

For the third term we add and subtract $M_{N,F_k}^2(H^{\cdot t})-M_{N,F_k}^2(H^{\cdot s})$ to the integral, where

$M^2_{N,F_k}(H^{\cdot t})$ is the martingale defined after equation (1.2.2). In that way we obtain that this term is bounded above by 2 times the sum of the following two terms,

$$\mathbb{E}_{\nu_y}\left[\sup_{\substack{|t-s|<\delta\\0\le s<t\le T}}\left(M^2_{N,F}(H^{\cdot t})-M^2_{N,F}(H^{\cdot s})\right)^2\right],$$

$$4\mathbb{E}_{\nu_y}\left[\sup_{0\le t\le T}\left(M^2_{N,F_k}(H^{\cdot t})+\int_0^t\sqrt{N}\sum_{x\in\mathbb{T}_N}\nabla_N H(x/N)\mathcal{L}_N(\tau^x F_k(p))ds\right)^2\right].$$

Since the functions F_k are local and belong to the Schwartz space, we can handle the first term in the same way as we did with $M^N_t(H)$ in Lemma 15 and Lemma 17 . The second term tends to zero as N goes to infinity, as stated in Lemma 1. □

Chapter 2

Technical Results

2.1 Some Geometrical Considerations

The aim of this section is to establish conditions over a given set of functions $\xi_{i,i+1} : \mathbb{R}^{n+1} \to \mathbb{R}$ for $1 \leq i \leq n$, which ensure the existence of a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying

$$X_{i,i+1}(g) = \xi_{i,i+1} \quad for \;\; 1 \leq i \leq n \, , \tag{2.1.1}$$

where $X_{x,y} = p_y \partial_{p_x} - p_x \partial_{p_y}$.

Observe that the vector fields $X_{i,j}$ act on spheres. In fact, we are interested in solving (2.1.1) over spheres. The results obtained in this section will be used in the proof of Theorem 8 in Section 2.3.

Let us remark that for $1 \leq i < j \leq n$ we have

$$[X_{i,i+1}, X_{j,j+1}] = \begin{cases} X_{i,j+1} & if \quad i+1 = j, \\ 0 & if \quad i+1 \neq j, \end{cases} \tag{2.1.2}$$

where $[,]$ stands for the Lie bracket. Thus, the existence of such a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying (2.1.1) give us some necessary conditions over the family $\{\xi_{i,i+1}\}$, namely

$$X_{i,i+1}(\xi_{j,j+1}) = X_{j,j+1}(\xi_{i,i+1}) \quad if \quad i+1 \neq j \, , \tag{2.1.3}$$

$$p_{i+1} X_{j,j+1}(\xi_{i,i+1}) - p_{i+1} X_{i,i+1}(\xi_{j,j+1}) = p_{j+1}\xi_{i,i+1} \; + \; p_i \xi_{j,j+1} \, , \tag{2.1.4}$$

for $1 \leq i < j \leq n$.

We state the main result of this section.

Theorem 7. *Let* $\xi_{i,i+1} : \mathbb{R}^{n+1} \to \mathbb{R}$ *for* $1 \leq i \leq n$ *be a a set of functions satisfying conditions (2.1.3) and (2.1.4). Then, for every* $r > 0$ *there exists a function* $g_r : S^n(r) \to \mathbb{R}$ *such that*

$$X_{i,i+1}(g_r) = \xi_{i,i+1} \quad on \quad S^n(r) \ .$$

The approach we adopt to prove this result consist in defining over each sphere $S^n(r)$ an *ad hoc* differential 1-form ω_r. Conditions (2.1.3) and (2.1.4) will imply the closeness of ω_r, which in view of Proposition 3 below implies the existence of the desired g_r.

Now we state two well known results in differential geometry.

Proposition 2. *Let* ω *be a* 1*-form on a differentiable manifold* M*, and* X, Y *differential vector fields on* M*, then*

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) \ ,$$

where $[\ ,]$ *denote the Lie bracket.*

Proposition 3. *Every closed differential 1-form on a n-dimensional sphere with* $n \geq 2$ *is exact.*

Proof of Theorem 7. Let $\{Y_1, \cdots, Y_n\}$ be a subset of n vector fields belonging to the Lie algebra generated by $\{X_{i,i+1}\}_{1\leq i\leq n}$. Consider the subset of the sphere A such that, for $p \in A \subset S^n(r)$ the set of vectors $\beta_p = \{Y_1(p), \cdots, Y_n(p)\}$ form a basis for the tangent space $T_pS^n(r)$, with associated dual basis denoted by $\beta_p^* = \{dY_1, \cdots, dY_n\}$.

Define on $A \subset S^n(r)$ the following differential 1-form

$$\omega = \sum_{k=1}^{n} \xi_k \ dY_k.$$

Here ξ_k is the corresponding component in terms of $\{\xi_{i,i+1}\}_{1\leq i\leq n}$. For instance, if $Y_k = X_{1,2}$ then $\xi_k = \xi_{1,2}$, or if $Y_k = [X_{1,2}, X_{2,3}]$ then $\xi_k = X_{1,2}(\xi_{2,3}) - X_{2,3}(\xi_{1,2})$.

It follows from (2.1.2) that the Lie algebra generated by the vector fields $\{X_{i,i+1}\}_{1\leq i\leq n}$ is of maximal dimension over each sphere. Thus, varying over all the subsets of size n of the Lie algebra, the sets A form a covering of the whole sphere. Moreover, any two differential forms defined in this way will coincide in their common domain of definition. This induces a differential 1-form ω_r well defined on the whole sphere $S^n(r)$.

In order to prove closeness of the differential 1-form ω_r , it suffices to prove closeness of each of the differential forms given above, which is reduce to prove $d\omega(Y_i, Y_j) = 0$

for $i, j \in \{1, \cdots, n\}$. The proof of this fact follows from conditions (2.1.3), (2.1.4) and Proposition 2. □

2.2 Spectral Gap

We investigate in this section the spectral gap for the dynamics induced by the infinitesimal generator given by

$$L_N(f) = \frac{1}{2} \sum_{x=1}^{N-1} X_{x,x+1}[a(p_x, p_{x+1}) X_{x,x+1}(f)] \;, \tag{2.2.1}$$

with associate Dirichlet form defined as

$$D_N(f) = \frac{1}{2} \sum_{x=1}^{N-1} \int_{\mathbb{R}^N} a(p_x, p_{x+1}) [X_{x,x+1}(f)]^2 \Phi_N dp, \tag{2.2.2}$$

where $\Phi_N(p) = \prod_{i=1}^{N} \frac{e^{-\frac{p_x^2}{2}}}{\sqrt{2\pi}}$.

The idea will be to relate our model with a similar one, known as the Kac's model, whose spectral gap is already known. Specifically, we find a relation between their Dirichlet forms and use it to obtain the desired spectral gap estimate for our model.

The generator of the Kac's model is defined for continuous functions as

$$\mathfrak{L}_N f = \frac{1}{C_2^N} \sum_{1 \le i \le j \le N} \frac{1}{2\pi} \int_0^{2\pi} [f(R_{i,j}^{\theta} x) - f(x)] d\theta \;, \tag{2.2.3}$$

where $R_{i,j}^{\theta}$ represents a clockwise rotation of angle θ on the plane i, j. It is easy to see that spheres are invariant under this dynamics.

To this generator is associated the following Dirichlet form

$$\mathfrak{D}_N(f) = \frac{1}{C_2^N} \sum_{1 \le i < j \le N} \frac{1}{2\pi} \int_0^{2\pi} \int_{S_r^{N-1}} [f(R_{i,j}^{\theta} x) - f(x)]^2 d\sigma_r(x) d\theta \;, \tag{2.2.4}$$

where S_r^{N-1} is the (N-1)-dimensional sphere of radius r centered at the origin and σ_r stands for the uniform measure over this sphere. In order to study the spectral gap is enough to treat with the unitary sphere, in which case we omit the subindex.

This dynamics was used by Kac as a model for the spatially homogeneous Boltzmann equation. A complete description of this model can be founded in [CCL].

Let us state the spectral gap estimate obtained in [J] for the Kac's model.

Lemma 19 (Janvresse)**.** *There exist a constant C such that for all $f \in L^2(S^{N-1})$ we have*

$$E_\sigma[f;f] \leq C\ N\ \mathfrak{D}_N(f)\ ,$$

where $E_\sigma[f;g]$ denotes the covariance between f and g with respect to σ.

A later work of Carlen (*et al*) [CCL] gives the exact spectral gap in the preceding setting and in other models within the so-called Kac's systems.

Now we proceed to establish the relation between (2.2.2) and (2.2.4). Firstly define,

$$B_{i,j}f(x) = \frac{1}{2\pi}\int_0^{2\pi}[f(R^\theta_{i,j}x) - f(x)]^2 d\theta\ .$$

Using the identity $\frac{\partial}{\partial\theta}[f(R^\theta_{i,j}x) - f(x)] = -X_{i,j}(f)(R^\theta_{i,j}x)$ and the Poincaré inequality on the interval $[0,2\pi]$ we obtain

$$B_{i,j}f(x) \leq \int_0^{2\pi}[X_{i,j}(f)(R^\theta_{i,j}x)]^2 d\theta\ ,$$

which implies,

$$\int_{S^{N-1}} B_{i,j}f(x)d\sigma(x) \leq 2\pi\int_{S^{N-1}}|X_{i,j}(f)|^2 d\sigma(x)\ .$$

Observe that in (2.2.2) just near neighbors interactions are involved, while in (2.2.4) long range interactions are also considered. This fact demands an additional argument in order to relate the two Dirichlet forms.

It follows from the definition of $B_{i,j}f$ that

$$B_{i,j}(f+g)(x) \leq 2B_{i,j}f(x) + 2B_{i,j}g(x)\ , \tag{2.2.5}$$

and

$$\int_{S^{N-1}} B_{i,j}f(x)d\sigma(x) \leq 4\int_{S^{N-1}} f^2(x)d\sigma(x)\ . \tag{2.2.6}$$

Denote by $S_{i,j}$ the exchange of coordinates i, j, and observe that

$$B_{i,i+k} = S_{i+1,i+k}\ B_{i,i+1}\ S_{i+1,j+k}\ .$$

Since the measure σ is invariant under exchange of coordinates, we have that $\int_{S^{N-1}} B_{i,i+k}f(x)d\sigma(x)$

is equal to

$$\int_{S^{N-1}} B_{i,i+1} S_{i+1,i+k} f(x) d\sigma(x) \ .$$

Adding and subtracting $B_{i,i+1} f(x)$ we can conclude after (2.2.5) and (2.2.6) that the expression in the last line is bounded above by

$$2 \int_{S^{N-1}} B_{i,i+1} f(x) d\sigma(x) + 8 \int_{S^{N-1}} [S_{j+1,i+k} f(x) - f(x)]^2 d\sigma(x) \ .$$

A telescopic argument permits to bound the second term in the last line by

$$16k \sum_{j=1}^{k-2} \int_{S^{N-1}} [S_{i+j,i+j+1} f(x) - f(x)]^2 d\sigma(x) \ ,$$

which in turns is bounded from above by

$$64k \sum_{j=1}^{k-2} \int_{S^{N-1}} B_{i+j,i+j+1} f(x) d\sigma(x) \ .$$

The last line is an immediate consequence of the inequality

$$\int_{S^{N-1}} [S_{i,j} f(x) - f(x)]^2 d\sigma(x) \leq 4 \int_{S^{N-1}} B_{i,j} f(x) d\sigma(x) \ , \tag{2.2.7}$$

which can be verified by adding and subtracting $\frac{1}{2\pi} \int_0^{2\pi} [f(R_{i,j}^\theta x)] d\theta$ into the square appearing in the left hand side of (2.2.7), and then using the invariance under rotations of the measure σ. Putting all together we finally obtain

$$\int_{s^{N-1}} B_{i,i+k} f(x) d\sigma(x) \leq 64k \sum_{j=0}^{k-2} \int_{s^{N-1}} B_{i+j,i+j+1} f(x) d\sigma(x) \ . \tag{2.2.8}$$

Now we state the main result of this section, which follows from the preceding inequality, Lemma 19, and the next well known formula

$$\int_{\mathbb{R}^N} f(p) \Phi_N dp = \int_0^\infty \left(\int_{S^{N-1}(\rho)} f(x) d\sigma(x) \right) \frac{\omega_N(\rho)}{(\sqrt{2\pi})^N} e^{-\frac{\rho^2}{2}} d\rho \ ,$$

where ω_N denotes the surface area of S^{N-1}.

Lemma 20. *There exists a positive constant C such that, for every $f \in L^2(\mathbb{R}^N)$ satis-*

fying for all $r > 0$,

$$\int_{S^{N-1}(r)} f d\sigma = 0 \ ,$$

we have

$$\int_{\mathbb{R}^N} f^2 \Phi_N dp \leq CN^2 \sum_{x=1}^{N-1} \int_{\mathbb{R}^N} [X_{x,x+1}(f)]^2 \Phi_N dp \ .$$

2.3 The Space $\mathcal{H}_y$

The aim of this section is to define the space $\mathcal{H}_y$ and prove the characterization that was used in the proof of Theorem 5. Let us begin by introducing some notation.

Let $\Omega = \mathbb{R}^{\mathbb{Z}}$ and $p = (\cdots, p_{-1}, p_0, p_1, \cdots)$ a typical element of this set. Define for $i \in \mathbb{Z}$ the shift operator $\tau^i : \Omega \to \Omega$ by $\tau^i(p)_j = p_{j+i}$, and $\tau^i f(p) = f(\tau^i p)$ for any function $f : \Omega \to \mathbb{R}$. We will consider the product measure ν_y on Ω given by $d\nu_y = \prod_{-\infty}^{\infty} \frac{\exp(\frac{-p_x^2}{2y^2})}{\sqrt{2\pi y}} dp$.

Let us define $\mathcal{A} = \cup_{k \geq 1} \mathcal{A}_k$, where $\mathcal{A}_k$ is the space of smooth functions F depending on $2k+1$ variables. Given $F \in \mathcal{A}_k$ we can consider the formal sum

$$\widetilde{F}(p) = \sum_{j=-\infty}^{\infty} \tau^j F(p) \ , \qquad (2.3.1)$$

and for $i \in \mathbb{Z}$ the well defined

$$\frac{\partial \widetilde{F}}{\partial p_i}(p) = \sum_{i-k \leq j \leq i+k} \frac{\partial}{\partial p_i} F(p_{j-k}, \cdots, p_{j+k}) \ .$$

The formal invariance $\widetilde{F}(\tau(p)) = \widetilde{F}(p)$ lead us to the precise covariance

$$\frac{\partial \widetilde{F}}{\partial p_i}(p) = \frac{\partial \widetilde{F}}{\partial p_0}(\tau^i p) \ . \qquad (2.3.2)$$

Recall that $X_{i,j} = p_j \partial_{p_i} - p_i \partial_{p_j}$. Given $F \in \mathcal{A}$ and $i \in \mathbb{Z}$, $X_{i,i+1}(\widetilde{F})$ is well defined and satisfies

$$X_{i,i+1}(\widetilde{F})(p) = \tau^i X_{0,1}(\widetilde{F})(p) \ .$$

Finally we define the following set

$$\mathcal{B}_y = \{X_{0,1}(\widetilde{F}) \in L^2(\nu_y) : F \in \mathcal{A}\}.$$

In terms of the notation introduced above, the variational formula obtained in (1.3.13) for the diffusion coefficient can be written as

$$\hat{a}(y) = y^{-4} \inf_{\xi \in \mathcal{B}_y} \mathbb{E}_{\nu_y}[a(p_0,p_1)(p_0p_1+\xi)^2] \,. \tag{2.3.3}$$

As is well known, if we denote by $\mathcal{H}_y$ the closure of $\mathcal{B}_y$ in $L^2(\nu_y)$, then

$$\hat{a}(y) = y^{-4} \inf_{\xi \in \mathcal{H}_y} \mathbb{E}_{\nu_y}[a(p_0,p_1)(p_0p_1+\xi)^2] \,,$$

and the infimum will be attained at a unique $\xi \in \mathcal{H}_y$.

At the end of the proof of Theorem 5 we used an intrinsic characterization of the space $\mathcal{H}_y$. In order to obtain such a characterization, we can first observe that defining $\xi = X_{0,1}(\widetilde{F})$ for $F \in \mathcal{A}$, the following properties are satisfied:

i) $\mathbb{E}_{\nu_y}[\xi] = 0$,

ii) $\mathbb{E}_{\nu_y}[p_0p_1\xi] = 0$,

iii) $X_{i,i+1}(\tau^j\xi) = X_{j,j+1}(\tau^i\xi) \quad$ if $\quad \{i,i+1\} \cap \{j,j+1\} = \emptyset$,

iv) $p_{i+1}[X_{i+1,i+2}(\tau^i\xi) - X_{i,i+1}(\tau^{i+1}\xi)] = p_{i+2}\tau^i\xi - p_i\tau^{i+1}\xi$ for $i \in \mathbb{Z}$.

Now we can claim the desired characterization.

Theorem 8. *If $\xi \in L^2(\nu_y)$ satisfies conditions i) to iv)* (the last two in a weak sense) *then $\xi \in \mathcal{H}_y$.*

The proof of Theorem 8 relies on the results obtained in Section 2.1 and Section 2.2. Additionally the introduction of a cut off function is required in order to control large energies.

Proof. The goal is to find a sequence $(F_N)_{N\geq 1}$ in $\mathcal{A}$, such that the sequence $\{X_{0,1}(\widetilde{F_N})\}_{N\geq 1}$ converges to ξ in $L^2(\nu_y)$. As is well known, the strong and the weak closure of a subspace of a Banach space coincide, therefore it will be enough to show that $\{X_{0,1}(\widetilde{F_N})\}_{N\geq 1}$ converges weakly to ξ in $L^2(\nu_y)$.

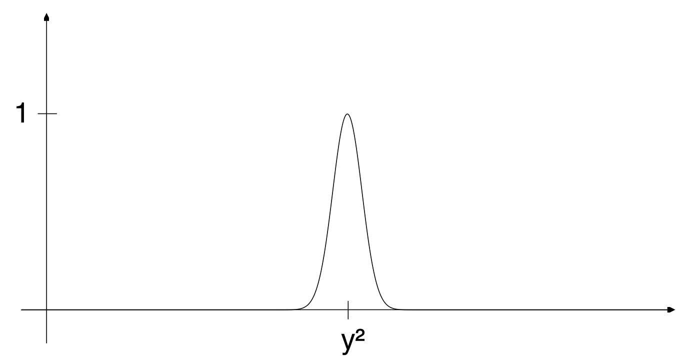
Firstly observe that for any smooth function $F(p_{-k},\cdots,p_k)$ we can rewrite $X_{0,1}(\widetilde{F})$, by using (2.3.2), as

$$\sum_{i=-k}^{k-1} X_{i,i+1}(F)(\tau^{-i}p) + \left(p_{k+1}\frac{\partial F}{\partial p_k}\right)(\tau^{-k}p) - \left(p_{-k-1}\frac{\partial F}{\partial p_{-k}}\right)(\tau^{k+1}p) \,. \tag{2.3.4}$$

Roughly speaking, the idea is to use the criteria obtained in section 2.1 to integrate the system (2.1.1) in order to find a function F such that ξ is approximated by the sum in the first term of (2.3.4), and then to control the border terms.
We define

$$\xi_{i,i+1}^{(2N)} = \mathbb{E}_{\nu_y}[\xi_{i,i+1}|\mathfrak{F}_{-2N}^{2N}]\varphi\left(\frac{1}{4N+1}\sum_{i=-2N}^{2N} p_i^2\right) \ , \tag{2.3.5}$$

where $\xi_{i,i+1}(p) = \tau_i\xi(p)$, $\mathfrak{F}_{-N}^{N}$ is the sub σ-field of Ω generated by $\{p_{-N}, \cdots, p_N\}$ and φ is a nonnegative smooth function with compact support such that $\varphi(y^2) = 1$.



We introduce this cutoff in order to do uniform bounds later.

Since ν_y is a product measure and the part corresponding to φ is radial, the set of functions $\{\xi_{i,i+1}^{(2N)}\}_{-2N\leq i\leq i+1\leq 2N}$ even satisfy conditions iii) and iv). Therefore, after Theorem 7 the system

$$X_{i,i+1}(g^{(N)}) = \xi_{i,i+1}^{(N)} \quad for \quad -2N \leq i \leq i+1 \leq 2N \tag{2.3.6}$$

can be integrated. Since $\mathbb{E}_{\nu_y}[g^{(N)}|p_{-2N}^2 + \cdots + p_N^2]$ is radial and the integration was performed over spheres, $\widetilde{g}^{(2N)} = g^{(2N)} - \mathbb{E}_{\nu_y}[g^{(2N)}|p_{-2N}^2 + \cdots + p_N^2]$ is still a solution of the system (2.3.6). Therefore, without lost of generality, we can suppose that $\mathbb{E}_{\nu_y}[g^{(N)}|p_{-2N}^2 + \cdots + p_{2N}^2 = y^2] = 0$ for every $y \in \mathbb{R}^+$. This will be useful when applying the *spectral gap* estimate.

In order to construct the desired sequence firstly define

$$g^{(N,k)} = \frac{1}{2(N+k)y^4}\mathbb{E}_{\nu_y}[p_{-N-k-1}^2 p_{N+k+1}^2 g^{(2N)}|\mathfrak{F}_{-N-k}^{N+k}] \ ,$$

and,

$$\widehat{g}^N(p_{-7N/4},\cdots,p_{7N/4}) = \frac{4}{N}\sum_{k=N/2}^{3N/4} g^{(N,k)} \ .$$

Using (2.3.4) for $g^{(N,k)}$ and averaging over k we obtain that

$$X_{0,1}\left(\sum_{j=-\infty}^{\infty}\tau^j\widehat{g}^N\right) = \xi + y^{-4}\big\{I_N^1 + I_N^2 + I_N^3 + R_N^1 - R_N^2\big\},$$

where,

$$I_N^1 = \widehat{\sum_{k=N/2}^{3N/4}}\ \widehat{\sum_{i=-N-k}^{N+k}}\tau^{-i}\mathbb{E}_{\nu_y}\left[p^2_{N+k+1}p^2_{-N-k-1}(\xi^{(2N)}_{i,i+1}-\xi^{(N+k)}_{i,i+1})\varphi(r^2_{-2N,2N})|\mathfrak{F}^{N+k}_{-N-k}\right] \ ,$$

$$I_N^2 = \widehat{\sum_{k=N/2}^{3N/4}}\ \widehat{\sum_{i=-N-k}^{N+k}}\tau^{-i}\big\{(\xi^{(N+k)}_{i,i+1}-\xi_{i,i+1})\mathbb{E}_{\nu_y}\left[p^2_{N+k+1}p^2_{-N-k-1}\varphi(r^2_{-2N,2N})|\mathfrak{F}^{N+k}_{-N-k}\right]\big\} \ ,$$

$$I_N^3 = \widehat{\sum_{k=N/2}^{3N/4}}\ \widehat{\sum_{i=-N-k}^{N+k}}\xi(p)\tau^{-i}\mathbb{E}_{\nu_y}[p^2_{N+k+1}p^2_{-N-k-1}(\varphi(r^2_{-2N,2N})-1)|\mathfrak{F}^{N+k}_{-N-k}] \ ,$$

$$R_N^1 = \widehat{\sum_{k=N/2}^{3N/4}}\tau^{-N-k}\{p_{N+k+1}\frac{\partial}{\partial p_{N+k}}g^{(N,k)}\} \ ,$$

$$R_N^2 = \widehat{\sum_{k=N/2}^{3N/4}}\tau^{N+k+1}\{p_{-N-k-1}\frac{\partial}{\partial p_{-N-k}}g^{(N,k)}\} \ .$$

Here $\widehat{\sum_{k=N/2}^{3N/4}} = \frac{4}{N+4}\sum_{k=N/2}^{3N/4}$, that is, the hat over the sum symbol means that this sum is in fact an average. The notation $r^2_{-2N,2N}$ is just an abbreviation for $\frac{1}{4N+1}\sum_{i=-2N}^{2N}p_i^2$.

The proof of the theorem will be concluded in the following way. In Lemma 21 the convergence in $L^2(\nu_y)$ to zero of the middle terms I_N^1, I_N^2, I_N^3 is demonstrated. We stress the fact that weak convergence to zero of each border term is false. However, weak convergence to zero of the sequence $\{R_N^1 - R_N^2\}_{N\geq 1}$ is true, as ensured by Lemmas 22, 23 and 24.

Therefore

$$\left\{X_{0,1}\left(\sum_{j=-\infty}^{\infty}\tau^j\widehat{g}^N\right)\right\}_{N\geq 1} \ ,$$

is weakly convergent to ξ. □

Before entering in the proof of the lemmas, let us state two remarks.

Remark 2. *We know that* $\mathbb{E}_{\nu_y}[\xi_{0,1}|\mathfrak{F}_{-N}^{N}] \xrightarrow{L^2} \xi_{0,1}$*, i.e given* $\epsilon > 0$ *there exist* $N_0 \in \mathbb{N}$ *such that*

$$\mathbb{E}_{\nu_y}[|\xi_{0,1} - \xi_{0,1}^{(N)}|^2] \leq \epsilon \quad if \quad N \geq N_0 \ .$$

Moreover, by translation invariance we have

$$\mathbb{E}_{\nu_y}[|\xi_{i,i+1} - \xi_{i,i+1}^{(N)}|^2] \leq \epsilon \quad if \quad [-N_0 - i, N_0 + i] \subseteq [-N, N] \ .$$

Proof. Given $A \in \mathfrak{F}_{-N-i}^{N-i}$ we have

$$\begin{aligned}\int_A \xi_{i,i+1}^{(N)}(\tau^{-i}p)\nu_y(dp) &= \int_{\tau^{-i}(A)} \xi_{i,i+1}^{(N)}(p)\nu_y(dp) = \int_{\tau^{-i}(A)} \xi_{i,i+1}(p)\nu_y(dp)\\ &= \int_A \xi_{i,i+1}(\tau^{-i}(p))\nu_y(dp) = \int_A \xi_{0,1}(p)\nu_y(dp) \ .\end{aligned}$$

Since in addition we have $\xi_{i,i+1}^{(N)}(\tau^{-i}) \in \mathfrak{F}_{-N-i}^{N-i}$, then

$$\xi_{i,i+1}^{(N)}(\tau^{-i}) = \mathbb{E}_{\nu_y}[\xi_{0,1}|\mathfrak{F}_{-N-i}^{N-i}] \ ,$$

and therefore,

$$\mathbb{E}_{\nu_y}[|\xi_{i,i+1} - \xi_{i,i+1}^{(N)}|^2] = \mathbb{E}_{\nu_y}[|\xi_{0,1} - \xi_{i,i+1}^{(N)}(\tau^{-i})|^2] \leq \mathbb{E}_{\nu_y}[|\xi_{0,1} - \xi_{0,1}^{(N_0)}|^2].$$

□

Remark 3. *A strong law of large numbers is satisfied for* $(p_i^2)_{i\in\mathbb{Z}}$*. In fact we have*

$$\mathbb{E}_{\nu_y}\left[\left(\frac{1}{N}\sum_{i=1}^{N} p_i^2 - y^2\right)^2\right] \leq \frac{8y^8}{N} \ .$$

Lemma 21 (**Middle terms**)**.** *For* $i = 1, 2, 3$ *we have*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_y}[(I_N^i)^2] = 0 \ .$$

Proof. The convergence to zero as N tends to infinity of I_N^1 and I_N^2 in $L^2(\nu_y)$ follows directly from Schwartz inequality, Remark 2 and the fact that φ is a bounded function.

Using exchange symmetry of the measure, I_N^3 can be rewritten as

$$\xi(p)\widehat{\sum_{k=N/2}^{3N/4}}\widehat{\sum_{i=-N-k}^{N+k}}\mathbb{E}_{\nu_y}[\widehat{\sum_{j=1}^{N-k}}p^2_{N+k+j}p^2_{-N-k-j}(\varphi(r^2_{-2N,2N})-1)|\mathfrak{F}^{N+k}_{-N-k}](\tau^{-i}p)\ ,$$

and then we decompose it as $J_N^1 + y^2 J_N^2$, where

$$J_N^1(p)=\xi(p)\widehat{\sum_{k=N/2}^{3N/4}}\widehat{\sum_{i=-N-k}^{N+k}}\mathbb{E}_{\nu_y}[\widehat{\sum_{j=1}^{N-k}}\{p^2_{N+k+j}p^2_{-N-k-j}-y^4\}(\varphi(r^2_{-2N,2N})-1)|\mathfrak{F}^{N+k}_{-N-k}](\tau^{-i}p),$$

and,

$$J_N^2(p)=\xi(p)\widehat{\sum_{k=N/2}^{3N/4}}\widehat{\sum_{i=-N-k}^{N+k}}\mathbb{E}_{\nu_y}[\varphi(r^2_{-2N,2N})-1|\mathfrak{F}^{N+k}_{-N-k}](\tau^{-i}p).$$

Firstly observe that

$$|J_N^1|^2\le|\xi(p)|^2\widehat{\sum_{k=N/2}^{3N/4}}\widehat{\sum_{i=-N-k}^{N+k}}\mathbb{E}_{\nu_y}\left[\left(\widehat{\sum_{j=1}^{N-k}}\{p^2_{N+k+j}p^2_{-N-k-j}-y^4\}\right)^2\right].$$

Being the expectation in the last expression bounded by $\frac{8y^8}{N-k}$, we obtain

$$||J_N^1||^2_{L^2(\nu_y)}\le\frac{32y^4}{N}||\xi||^2_{L^2(\nu_y)}\ .$$

On the other hand, writing explicitly the conditional expectation appearing in J_N^2 we see that

$$|J_N^2(p)|^2\le|\xi(p)|^2\widehat{\sum_{k=N/2}^{3N/4}}\widehat{\sum_{i=-N-k}^{N+k}}\int\Big|\varphi\big(\frac{1}{4N+1}\sum_{|j|>N+k}q_j^2+\frac{1}{4N+1}\sum_{|j|\le N+k}p^2_{j+i}\big)-1\Big|^2\nu_y(dp)\ .$$

Rewrite the integral into last expression as

$$\int\Big|\varphi\big(\frac{1}{4N+1}\sum_{|j|>N+k}(q_j^2-y^2)+\frac{1}{4N+1}\sum_{|j|\le N+k}(p^2_{j+i}-y^2)+y^2\big)-1\Big|^2\nu_y(dp)\ .$$

Using the fact that φ is a Lipschitz positive function bounded from above by 1 and satisfying $\varphi(y^2) = 1$, we get that $|J_N^2|^2$ is bounded by

$$|\xi(p)|^2 \widehat{\sum_{k=N/2}^{3N/4}} \widehat{\sum_{i=-N-k}^{N+k}} 1 \wedge \int |\frac{1}{4N+1} \sum_{|j|>N+k} (q_j^2 - y^2) + \frac{1}{4N+1} \sum_{|j|\le N+k} (p_{j+i}^2 - y^2)|^2 \nu_y(dp) ,$$

where $a \wedge b$ denote the minimum of $\{a, b\}$.

Therefore, taking expectation and using the strong law of large numbers together with the dominated convergence theorem, the convergence to zero as N tends to infinity of I_N^3 in $L^2(\nu_y)$ is proved. □

Lemma 22 (**Bounding border terms**). *The sequences* $\{R_N^i\}_{N\ge 1}$ *are bounded in* $L^2(\nu_y)$ *for* $i = 1, 2$.

Proof. Recall that

$$R_N^1 = \widehat{\sum_{k=N/2}^{3N/4}} \frac{1}{2(N+k)} \tau^{-N-k} \{p_{N+k+1} \mathbb{E}_{\nu_y} [p_{-N-k-1}^2 p_{N+k+1}^2 \frac{\partial}{\partial p_{N+k}} g^{(2N)} | \mathfrak{F}_{-N-k}^{N+k}]\} .$$

Using the fact that $X_{N+k,N+k+1} = p_{N+k+1} \frac{\partial}{\partial p_{N+k}} - p_{N+k} \frac{\partial}{\partial p_{N+k+1}}$, we can rewrite last line as the sum of the following two terms.

$$\widehat{\sum_{k=N/2}^{3N/4}} \frac{1}{2(N+k)} \tau^{-N-k} \{p_{N+k+1} \mathbb{E}_{\nu_y} [p_{-N-k-1}^2 p_{N+k+1} X_{N+k,N+k+1} g^{(2N)} | \mathfrak{F}_{-N-k}^{N+k}]\}$$

and

$$\widehat{\sum_{k=N/2}^{3N/4}} \frac{1}{2(N+k)} \tau^{-N-k} \{p_{N+k} p_{N+k+1} \mathbb{E}_{\nu_y} [p_{-N-k-1}^2 p_{N+k+1} \frac{\partial}{\partial p_{N+k+1}} g^{(2N)} | \mathfrak{F}_{-N-k}^{N+k}]\} .$$

By Schwartz inequality and (2.3.6) we can see that the $L^2(\nu_y)$ norm of the first term is bounded by $\frac{y^3}{N} ||\xi||_{L^2(\nu_y)}$. After integration by parts, the second term can be written as

$$\widehat{\sum_{k=N/2}^{3N/4}} \frac{1}{2(N+k)y^2} \tau^{-N-k} \{p_{N+k} p_{N+k+1} \mathbb{E}_{\nu_y} [p_{-N-k-1}^2 (p_{N+k+1}^2 - y^2) g^{(2N)} | \mathfrak{F}_{-N-k}^{N+k}]\} . \quad (2.3.7)$$

Denote by $\sigma^{j,N+k+1}$ the interchange of coordinates p_j and p_{N+k+1}. Using exchange invariance of the measure, we can see that the conditional expectation appearing in last expression is equal to

$$\mathbb{E}_{\nu_y}[p^2_{-N-k-1}(p_j^2 - y^2)(g^{(2N)} \circ \sigma^{j,N+k+1})|\mathfrak{F}^{N+k}_{-N-k}] ,$$

for $j = N+k+1, \cdots, 2N$. This permits to introduce a telescopic sum which will serve later to obtain an extra $\frac{1}{N}$ in order to use a spectral gap estimate. Indeed, we decompose (2.3.7) as the sum of the following two terms.

$$\widehat{\sum_{k=N/2}^{3N/4}} \frac{1}{2(N+k)y^2} \tau^{-N-k} \mathbb{E}_{\nu_y}[p^2_{-N-k-1} \widehat{\sum_{j=N+k+1}^{2N}} (p_j^2 - y^2) g^{(2N)} |\mathfrak{F}^{N+k}_{-N-k}] , \tag{2.3.8}$$

and

$$\widehat{\sum_{k=N/2}^{3N/4}} \frac{1}{2(N+k)y^2} \tau^{-N-k} \mathbb{E}_{\nu_y}[p^2_{-N-k-1} \widehat{\sum_{j=N+k+1}^{2N}} (p_j^2 - y^2)(g^{(2N)} \circ \sigma^{j,N+k+1} - g^{(2N)}) |\mathfrak{F}^{N+k}_{-N-k}] . \tag{2.3.9}$$

By Schwartz inequality, the square of the conditional expectations appearing in last expressions are respectively bounded by

$$CN^{-1} y^{12} \mathbb{E}_{\nu_y}[(g^{(2N)})^2 |\mathfrak{F}^{N+k}_{-N-k}]$$

and

$$Cy^8 \mathbb{E}_{\nu_y}\big[\widehat{\sum_{j=N+k+1}^{2N}} (g^{(2N)} \circ \sigma^{j,N+k+1} - g^{(2N)})^2 |\mathfrak{F}^{N+k}_{-N-k}\big] ,$$

for a universal constant C.

Therefore, again by Schwartz inequality, we can see that the $L^2(\nu_y)$ norms of (2.3.8) and (2.3.9) are respectively bounded by

$$\frac{Cy^{10}}{N^3} \mathbb{E}_{\nu_y}\big[\big(\widehat{\sum_{k=N/2}^{3N/4}} p^2_{N+k}\big)(g^{(2N)})^2\big] , \tag{2.3.10}$$

and,

$$\frac{Cy^6}{N^2}\mathbb{E}_{\nu_y}\big[\big(\widehat{\sum_{k=N/2}^{3N/4}} p_{N+k}^2\big)\widehat{\sum_{j=3N/2+1}^{2N}}(g^{(2N)}\circ\sigma^{j,N+1}-g^{(2N)})^2\big] . \tag{2.3.11}$$

Observe that $\widehat{\sum_{k=N/2}^{3N/4}} p_{N+k}^2$ can be uniformly estimated because of the cutoff introduced in (2.3.5).

Using the spectral gap estimate obtained in Section 2.2 we can bound (2.3.10) by a constant, and thanks to the basic inequality

$$\mathbb{E}_{\nu_y}\big[(g^{(2N)}\circ\sigma^{j,j+1}-g^{(2N)})^2\big]\le C\mathbb{E}_{\nu_y}\big[(X_{j,j+1}g^{(2N)})^2\big] ,$$

we can see after telescoping, that (2.3.11) is also uniformly bounded. □

Lemma 23 (**Characterization of weak limits**). *Every weak limit function of the sequence $\{R_N^1-R_N^2\}_{N\ge1}$ is of the form cp_0p_1 for some constant c.*

Proof. Let us firstly consider the sequence $\{R_N^1\}_{N\ge1}$. In Lemma 22 we obtain a decomposition of R_N^1 as the sum of two terms, one of which converges to zero in $L^2(\nu_y)$. The other term, namely (2.3.7), is equal to $p_0p_1h_N^1(p_0,\cdots,p_{-7N/2})$ where

$$h_N^1 \;=\; \widehat{\sum_{k=N/2}^{3N/4}}\frac{1}{2(N+k)y^2}\tau^{-N-k}\mathbb{E}_{\nu_y}[p_{-N-k-1}^2(p_{N+k+1}^2-y^2)g^{(2N)}|\mathfrak{F}_{-N-k}^{N+k}] . \tag{2.3.12}$$

It was also proved that $\{p_0p_1h_N^1\}_{N\ge1}$ is bounded in $L^2(\nu_y)$, therefore it contains a weakly convergent subsequence, say $\{p_0p_1h_{N'}^1\}_{N'}$. By similar arguments as in the proof of Lemma 22, we can conclude that $\{h_N^1\}_{N\ge1}$ is bounded in $L^2(\nu_y)$, therefore $\{h_{N'}^1\}_{N'}$ contains a weakly convergent subsequence, whose limit will be denoted by h^1.

Applying the operator X_{i+i+1} in the two sides of (2.3.12) and using Schwartz inequality, is easy to see that

$$||X_{i,i+1}h_N^1||_{L^2(\nu_y)}\le\frac{C}{N}||\xi||_{L^2(\nu_y)}\qquad \textit{for}\qquad \{i,i+1\}\subseteq\{0,-1,-2,\cdots\} ,$$

which implies that $X_{i,i+1}h^1=0$ for $\{i,i+1\}\subseteq\{0,-1,-2,\cdots\}$. This, together with the fact that the function h^1 just depends on $\{p_0,p_{-1},p_{-2},\cdots\}$, permits to conclude that h^1 is a constant function, let's say c. Therefore $\{p_0p_1h_{N'}^1\}_{N'}$ converges weakly to cp_0p_1.

This proves that for every weakly convergent subsequence of $\{R_N^1\}_{N\ge1}$ there exists a constant c such that the limit is cp_0p_1. Exactly the same can be said about $\{R_N^2\}_{N\ge1}$.

Finally suppose that $\{R^1_{N'} - R^2_{N'}\}_{N'\geq 1}$ is a subsequence converging weakly to a function f. The boundness of $\{R^1_{N'}\}_{N'\geq 1}$ and $\{R^2_{N'}\}_{N'\geq 1}$ implies the existence of further subsequences $\{R^1_{N''}\}_{N''\geq 1}$ and $\{R^2_{N''}\}_{N''\geq 1}$ converging weakly to $c_1 p_0 p_1$ and $c_2 p_0 p_1$, respectively. Therefore, by unicity of the limit, we have $f = (c_1 - c_2)p_0 p_1$. □

Lemma 24 (**Convergence to zero**). *The sequence* $\{R^1_N - R^2_N\}_{N\geq 1}$ *converges weakly to zero.*

Proof. In view of the boundness of $\{R^1_{N'}\}_{N'\geq 1}$ and $\{R^2_{N'}\}_{N'\geq 1}$, it is enough to prove that every weak limit of the sequence $\{R^1_N - R^2_N\}_{N\geq 1}$ is equal to zero.

At the end of the proof of Lemma 23 we see that every weak limit of $\{R^1_N - R^2_N\}_{N\geq 1}$ is of the form $(c_1 - c_2)p_0 p_1$, where c_1 and c_2 are constants for which there exist further subsequences $\{R^1_{N'}\}_{N\geq 1}$ and $\{R^2_{N'}\}_{N\geq 1}$ converging weakly to $c_1 p_0 p_1$ and $c_2 p_0 p_1$, respectively.

On the other hand, recall that

$$X_{0,1}\left(\sum_{j=-\infty}^{\infty} \tau^j \widehat{g}^N\right) = \xi + y^{-4} I^1_N + y^{-4} I^2_N + y^{-4} I^3_N + y^{-4} R^1_N - y^{-4} R^2_N.$$

Let us multiply the two sides of the last equality by the function $p_0 p_1 \in L^2(\nu_y)$, and take expectation with respect to ν_y. Thanks to the orthogonality condition ii), namely $\mathbb{E}_{\nu_y}[p_0 p_1 \xi] = 0$, we have

$$0 \;=\; y^{-4}\mathbb{E}_{\nu_y}[p_0 p_1 (I^1_{N'} + I^2_{N'} + I^3_{N'})] + y^{-4}\mathbb{E}_{\nu_y}[p_0 p_1 (R^1_{N'} - R^2_{N'})].$$

Finally, taking the limit as N' tends to infinity we obtain that $0 = c_1 - c_2$. □

2.4 Equivalence of Ensembles

The classical result (usually attributed to Poincaré) of equivalence of ensembles in which we are interested, states that the first K coordinates of a point uniformly distributed over the N-dimensional sphere centered at the origin, are independent standard Gaussian variables in the limit as N increase to infinity.

In [DF] they get a sharp bound for the total variation distance (essentially $2K/N$) between this marginal and the law of K independent standard Gaussian variables. This permits to compare expectations of bounded functions with respect to these two measures.

In this work we need to consider equivalence of ensembles for unbounded functions. The same is required in [BBO] where, by means of a modification on the arguments in [DF], a proof of the following statement is given.

Lemma 25. *Let $\nu_{N,y\sqrt{N}}$ be the uniform measure on the sphere*

$$S^N(y\sqrt{N}) = \{(p_1, \cdots, p_N) \in \mathbb{R}^N : \sum_{i=1}^{N} p_i^2 = Ny^2\} \ ,$$

and ν_y^∞ the infinite product of Gaussian measures with mean zero and variance y^2. Given a function ϕ on $\mathbb{R}^K$, such that for some positive constants θ and C

$$|\phi(p_1, \cdots, p_K)| \leq C \left(\sum_{i=1}^{K} p_i^2 \right)^{\theta} \ ,$$

there exist a constant $C' = C'(C, \theta, K, y)$ such that

$$\limsup_{N \to \infty} N \left| E_{\nu_{N,y\sqrt{N}}}[\phi] - E_{\nu_y^\infty}[\phi] \right| \leq C' \ .$$

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