Instituto Nacional de Matemática Pura e Aplicada

Global Geometry of Second Order Differential Equations

Hernan Maycol Falla Luza

Advisor: Jorge Vitório Pereira

Tese submetida ao Programa de Pós-graduação da Associação Instituto Nacional de Matemática Pura e Aplicada como requisito parcial para a obtenção do grau de Doutor em Matemática.

IMPA – Rio de Janeiro – June 2010

A mi madre Sandra, mis hermanos Edson y Sebastian y a la memoria de mi padre Hernán.

AGRADECIMENTOS

Gostaria de agradecer nestas linhas às pessoas e instituições que me deram apoio no desenvolvimento deste trabalho.
Primeiramente quero agradecer aos meus professores do IMCA no Perú, Renato, Félix e Percy pois muito provavelmente eu não tería conhecido o IMPA se não fosse por eles.
Gostaria de agradecer ao pessoal de folheações do IMPA, professores Alcides Lins Neto, Hossein Movasati, César Camacho, Paulo Sad e Jorge Vitório, pelas conversas valiosas que ajudaram a minha formação na área.
Quero agradecer especialmente ao professor Jorge Vitório, meu orientador, por muitas razões. Primeiro por ter aceito orientar minha tese, mesmo sabendo do grande trabalho que esto significava; por ter me ensinado o belo e difícil caminho de virar um pesquisador (e ter tido a paciência de não dessistir); pela disponibilidade e pela grande exigencia que ele teve comigo, a pesar da qual, ele foi um grande amigo e me deu o seu apoio em todos os momentos necesarios, eu aprendi dele como deve ser um verdadeiro profissional e muitas outras coisas importantes para minha vida, sou muito grato mesmo.
Ao professor Paulo Sad, por ter sido um amigo muito próximo e por ter me mostrado as folheações desde um outro ponto de vista. Minha estadia no IMPA foi muito mais agradável pelos muitos momentos que compartilhei com ele, incluindo conversas não matematicas e a caminhada à "pedra do sino" (onde conheci o fabuloso *caldo de piranha*) junto com o amigo Carlo.
Ao professor Eduardo Esteves, por me apresentar a Geometría Algébrica e pela boa disposição com as minhas dúvidas.
Aos membros da banca os profesores Alcides Lins Neto, César Camacho, Jorge Vitório, Israel Vainsencher, Marcio Soares e Paulo Sad pelas sugestões e correções para melhorar o texto. Agradeço especialmente aos professores Paulo Sad e Israel Vainsencher pela leitura minuciosa e comentários que foram muito úteis.
A mis amigos en Rio: Daniel Barrera, Miguel Orrillo e Junior (Acir) por todos los muy buenos momentos compartidos en el apartamento de Gloria. A Hernán Ianello por su amistad y confianza. A Freddy Hernandez, Mostafa Nasri y Pablo Guarino, con quienes, en diferentes momentos de mi doctorado, tuve el placer de divivir la oficina, tener grandes conversaciones y ratos muy agradables.

A José Manuel por los buenos churrascos, el buen café y sobre todo por la amistad. A Ricardo Pastrán, por los buenos momentos de fútbol, los jueves a la 1 a.m. en el aterro de Flamengo. A Jonathan (pato) por la amistad y los buenos juegos de Monopolio, Crash y War II. A Carlo Perrone, con quien pasé grandes momentos durante su paso por Rio, por las muchas discusiones sobre diversos temas. A Cristian Ortiz, Fabio XP, Gabriela Fernández, Thiago Fassarella, Thiago Drummond e Viviana Ferrer, por las valiosas conversaciones matemáticas durante mi doctorado en el IMPA.
A Mauricio Brito y su esposa Ángela, a Gustavo Abelha, su hermana Alessandra, su madre Sueli y su padre Antonio Carlos, excelentes amigos brasileros que me hicieron sentir como en casa siempre, y me trataron como si fuera parte de su familia.
A mi amigo/primo Arturo Fernández, con quien además del gusto por las foliaciones holomorfas, compartí muchos momentos buenos y difíciles. A su hermano "del medio" Miguel, por ser un gran amigo también.
A Carolina Parra, cuya compañía hizo mucho mejores los días difíciles, por su gran apoyo, paciencia y cariño en los momentos de presión y nerviosismo, y su alegría y entusiasmo en los momentos felices.
A mi familia que está lejos. En Perú: a mi abuelita Rita por su apoyo durante muchas etapas de mi vida. A mis tíos Rudy, Alfonso y Lita por su aliento y preocupación en todo momento. A mis abuelitos Marina y don Fallita y mis primos Alfonsito y David (Ata y Wavid respectivamente) por siempre estar presentes. En Argentina: a mi tía Carmen, Jorge y mi hermano Sebas (Alonso Falla, también conocido como Palizo), porque hicieron que mis visitas a Buenos Aires sean muy felices.
Muy especialmente quiero agradecer a mi familia, porque ellos fueron (y son) la fuente de mis esfuerzos y la alegría de mis victorias: A mi mamá Sandra, porque su apoyo siempre fué incondicional, por sus fuerzas y ganas de vivir feliz e ir siempre hacia adelante. A mi hermano Edson (Alonso Falla, tambíen conocido como Checo, o Arantes) por ir conmigo al estadio en los partidos del Flamengo, por su compañía y su aliento, porque él es gran parte de la inspiración que tengo para dar firme mis pasos. Agradezco a mi familia por compartir conmigo estos años tan importantes de mi vida.
Finalmete gostaría de agradecer ao IMPA pelo ótimo ambiente, e ao CNPq e a FAPERJ pelo auxílios concedidos.

Contents

Introdução

O objetivo deste trabalho é estudar equações diferenciais de segunda ordem. Vamos direcionar a nossa atenção para questões globais como a (in)existência de soluções algébricas.

É clássico na teoria das equações diferenciais a introdução de um novo parâmetro que representa a primeira derivada; assim transformamos uma equação diferencial de segunda ordem em um sistema de equações diferenciais de primeira ordem. Se considerarmos a variedade $M = \mathbb{P}(TS)$ que parametriza pontos e direções tangentes de uma superfície S, a chamada variedade de contato, podemos pensar em equações diferenciais de segunda ordem como os campos vetoriais (ou folheações de dimensão um) em M tangentes à distribuição de contacto.

Iremos nos concentrar principalmente no caso de equações diferenciais de segunda ordem em $\mathbb{P}^2$. O mais básico invariante destas é o chamado bigrau , que é um par ordenado (a, b) de números inteiros canonicamente associado a ela. Uma vez fixado este par (a, b) iremos estudar o espaço de equações diferenciais de segunda ordem com bigrau (a, b). Usando a geometria da distribuição de contacto obtemos algumas fórmulas de interseção que, em particular, fornecem significado geométrico do bigrau.

Lembramos que para equações diferenciais de segunda ordem, dado um ponto genérico p e uma direção tangente $v \in T_pS$, existe uma única solução passando por p e tangente à v. Portanto, podemos pensar em um pencil de folheações como um caso particular de equação diferencial de segunda ordem. Olhando para um sistema linear bidimensional de curvas em $\mathbb{P}^2$ como um pencil de folheações, vamos aplicar as fórmulas de interseção obtidas para limitar o número de fibras completamente decomponíveis.

Teorema 1. *Seja $\mathcal{P} = \mathbb{P}(\langle F, G, H\rangle) \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ um sistema linear bidimensional formado por curvas de grau d tal que F, G e H não possuem fator comum. Denote por $\Sigma \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ a subvariedade de curvas de grau d completamente decomponíveis. Então*

1. *Se $\mathcal{P} \cap \Sigma = C$, onde C é uma curva, então $deg(C) \leq 3$.*
2. *Se $\mathcal{P} \cap \Sigma = \{p_1, \dots, p_k\}$ é um conjunto de pontos, e o número de retas contidas em fibras de $\mathcal{P}$ é finito então $k \leq 31$.*

Podemos encontrar em [21] e [25] resultados semelhantes para o caso de pencil de curvas.

Também estamos interessados no estudo de webs, isto é, equações diferenciais de primeira ordem e grau arbitrário. É um resultado bem conhecido de Jouanolou que uma folheação genérica de grau pelo menos 2 não admite solução algébrica (ver [13]). Pode-se perguntar se isso ainda é verdade para webs e equações diferenciais de segunda ordem. Os resultados principais desta tese fornecem respostas positivas para estas perguntas.

Teorema 2. *Uma k-web de grau d genérica em $\mathbb{P}^2$ não possui curva algébrica invariante se $d \geq 2$.*

Teorema 3. *Um equação diferencial de segunda ordem de bigrau (a, b) genérica não possui solução algébrica se $a \geq 3$.*

Ambos os resultados acima são optimais: toda web de grau menor do que 2 admite uma reta invariante, e o mesmo vale para equações diferenciais de segunda ordem com $a < 3$. Nós também obtemos resultados semelhantes (porém não optimais) para webs e equações diferenciais de segunda ordem em superfícies projetivas arbitrárias.

Esta tese está dividida em três capítulos. A seguir apresentamos um breve resumo de cada um deles.

Capítulo 1. Preliminares. No capítulo 1 reunimos alguns fatos básicos da geometria de $M = \mathbb{P}(TS)$ (a variedade de contato) onde S é uma superficie complexa. Descrevemos os anéis de cohomologia $H^*(M)$ e introduzimos uma distribuição de codimensão um em M, a distribuição de contacto $\mathcal{D}$. Nos concentramos no caso particular $S = \mathbb{P}^2$, no qual podemos ver M como a variedade de incidência de pontos e retas em $\mathbb{P}^2$ e descrevemos $H^*(M)$ em termos de h e $\check{h}$, geradores da cohomologia de $\mathbb{P}^2$ e $\check{\mathbb{P}}^2$ respectivamente. Usando estes geradores explicitamos algumas classes em M, como a classe de K_M (o fibrado canônico de M) e os fibrados normal e determinante da distribuição de contato.

Capítulo 2. Equações diferenciais de segunda ordem. No capítulo 2 introduzimos o objeto principal deste trabalho: a equação diferencial de segunda ordem. Vemos estes objetos como folheações em M tangentes à distribuição de contato. Também definimos o bigrau de uma equação diferencial de segunda ordem $\mathcal{F}$ como sendo o par ordenado (a, b) tal que $T^*\mathcal{F} = \mathcal{O}_M(ah + b\check{h})$. Exibimos alguns exemplos de equações diferenciais de segunda ordem, as analisamos sob este ponto de vista e calculamos seus fibrados cotangentes.

Mostramos que a folheação vertical e o pencil de webs são as únicas equações diferenciais de segunda ordem com infinitas superfícies invariantes.

Damos algumas fórmulas relativas à intersecção de $T\mathcal{F}$ com curvas e superfícies não invariantes e também uma fórmula para o divisor de tangência entre duas equações diferenciais de segunda ordem. Usamos estas fórmulas para entender o significado geométrico do bigrau e dar cotas para o número de retas invariantes por webs planares.

Terminamos este capítulo estudando equações diferenciais de segunda ordem definidas por redes de curvas planas, ou seja, um sistema linear de dimensão dois de curvas em $\mathbb{P}^2$. Nós usamos as fórmulas de interseção para limitar o número de fibras completamente decomponíveis da rede, provando assim o Teorema 1.

Capítulo 3. Inexistência de soluções algébricas. O capítulo 3 é dedicado ao estudo de $\mathcal{E}(a, b) = \mathbb{P}H^0(M, \mathcal{D} \otimes \mathcal{O}_M(a, b))$, o espaço de equações diferenciais de segunda ordem com bigrau (a, b). Nós damos uma descrição explícita para os casos $(-2 \le a \le 0, 1 \le b)$ e $(1 \le a, -2 \le b \le 0)$, calculamos a dimensão deste espaço para todo bigrau, e mostramos que ele é o join de dois subespaços lineares que nós descrevemos.

Após estes preliminares o Capítulo 3 dedica-se as demonstrações dos Teoremas 2 e 3 enunciados acima. Lá eles são, respectivamente, Teoremas 3.15 e 3.9. O capítulo termina com análogos em superfícies projetivas arbitrárias destes resultados.

Introduction

The aim of this work is to study second order differential equations. We will restrict our attention to global issues such as the (in)existence of algebraic solutions.

It is classical in the theory of differential equations to introduce a new parameter representing the first derivative. The second order differential equation is converted into a system of first order differential equations. If we consider the variety $M = \mathbb{P}(TS)$ which parameterizes points and tangent directions of a surface S, the so-called contact variety, we can think on second order differential equations as vector fields (or one dimensional foliations) on M tangent to the contact distribution.

We will focus on the case of differential equations on $\mathbb{P}^2$. The most basic invariant of these is the so-called bidegree, which is an ordered pair (a, b) of integer numbers canonically associated to them. We will study the space of second order differential equations with fixed bidegree (a, b). Using the geometry of the contact distribution we obtain some intersection formulas which, in particular, provide a geometrical meaning to the bidegree.

We recall that for second order differential equations, given a generic point p and a tangent direction $v \in T_pS$, there is a unique solution passing through p and tangent to v. Therefore we can think in a pencil of foliations as a particular case of second order differential equation. Looking at a 2-dimensional linear system of curves in $\mathbb{P}^2$ as a pencil of foliations, we will apply the intersection formulas mentioned above to bound the number of completely decomposable elements.

Theorem 1. *Let $\mathcal{P} = \mathbb{P}(\langle F, G, H\rangle) \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ be a linear system of dimension 2 formed by curves of degree d and such that F, G and H do not have a common factor. Denote by $\Sigma \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ the subvariety of completely decomposable curves of degree d. Therefore*

1. *If $\mathcal{P} \cap \Sigma = C$, where C is a curve, then $deg(C) \leq 3$.*
2. *If $\mathcal{P} \cap \Sigma = \{p_1, \ldots, p_k\}$, a set of points, and the number of lines contained in fibers of $\mathcal{P}$ is finite then $k \leq 31$.*

We can find in [21] and [25] similar results for pencils of plane curves.

We are also interested in the study of webs, that is, first order differential equations of arbitrary degree. It is a well known result of Jouanolou that a generic foliation of degree at least 2 does not admit any algebraic solution (see [13]). One may ask whether this is still true for webs and for second order differential equations. The main results of this thesis give positive answers to these questions.

Theorem 2. *If $d \geq 2$ then a generic k-web of degree d in $\mathbb{P}^2$ does not admit any invariant algebraic curve.*

Theorem 3. *A generic second order differential equation of bidegree (a, b) with $a \geq 3$ has no algebraic solutions.*

Both results are sharp: every web of degree less than 2 admits an invariant line, and the same is true for second order differential equations with $a < 3$. We also obtain similar results (no longer sharp) for webs and second order differential equations on arbitrary projective surfaces.

This thesis is divided in three chapters. The following is a brief summary of each one.

Chapter 1. Preliminaries. In chapter 1 we have compiled some basic facts of the geometry of $M = \mathbb{P}(TS)$ (the contact variety) where S is a complex surface. We describe the cohomology rings $H^*(M)$ and introduce a codimension one distribution on M, the contact distribution $\mathcal{D}$. We focus on the particular case $S = \mathbb{P}^2$ in which we can see M as the incidence variety of points and lines in $\mathbb{P}^2$ and describe $H^*(M)$ in terms of h and $\check{h}$, the generators of the cohomology of $\mathbb{P}^2$ and $\check{\mathbb{P}}^2$ respectively. We express the classes of K_M (the canonical bundle of M), and of the normal and determinant bundles of the contact distribution in terms of these generators.

Chapter 2. Second order differential equations. In chapter 2 we introduce the main object of this work: second order differential equations. We see these objects as foliations on M tangent to the contact distribution. We also define the bidegree of a second order differential equation $\mathcal{F}$ as the pair (a, b) such that $T^*\mathcal{F} = \mathcal{O}_M(ah + b\check{h})$. Then we give some examples of second order differential equations, analyze them from this point of view and compute their cotangent bundles.

We show that the vertical foliation and the pencil of webs are the only second order differential equations with infinitely many invariant surfaces.

We give some formulas concerning the intersection of $T\mathcal{F}$ with non invariant curves and surfaces and also give a formula for the tangency divisor between two second order differential equations. We use these formulas to understand the geometrical meaning of the bidegree and to give bounds for the number of invariant lines of planar webs.

We finish this chapter studying second order differential equations defined by nets of plane curves, that is, a linear system of dimension two of curves on $\mathbb{P}^2$. We use the intersection formulas to bound the number of completely decomposable fibers of the net, proving then Theorem 1.

Chapter 3. Inexistence of algebraic solutions. Chapter 3 is devoted to study $\mathcal{E}(a, b) = \mathbb{P}H^0(M, \mathcal{D} \otimes \mathcal{O}_M(a, b))$, the space of second order differential equations with bidegree (a, b). We give an explicit description for the cases $(-2 \le a \le 0, 1 \le b)$ and $(1 \le a, -2 \le b \le 0)$, compute the dimension of this space for every bidegree, and show that it is the join of two linear subspaces which we describe.

After these preliminaries, Chapter 3 is dedicated to the proofs of Theorems 2 and 3 listed above. There they are, respectively, Theorems 3.15 and 3.9. The chapter ends with the analogue of these results in arbitrary projective surfaces .

Chapter 1

Preliminaries

In this first chapter we collect the basic facts about the projectivization of the tangent bundle of projective surfaces and its contact distribution. All the results presented here are well known and can be found in the literature, see for instance in [10, 24].

1.1 The Projectivization of TS

Let S be a complex manifold and let E be a holomorphic vector bundle of rank r over S. We denote by $\pi : \mathbb{P}(E) \to S$ the associated projective bundle. The convention adopted here is that over a point z the fiber $\pi^{-1}(z)$ parametrizes the one-dimensional subspaces of E_z.

Definition 1.1. The tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$ is the subbundle of the pullback bundle $\pi^*(E) \to \mathbb{P}(E)$ whose fiber at any point $(z, [v]) \in \mathbb{P}(E)$ is the line in $\pi^*(E)_{(z,[v])} = E_z$ represented by $[v]$:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(E)}(-1) \arrow[r, hook] \arrow[dr] & \pi^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\
& \mathbb{P}(E) \arrow[r] & S
\end{tikzcd}$$

The dual of the tautological bundle is denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Remark 1.2. The bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is not determined by the abstract projective bundle $\mathbb{P}(E) \to S$ alone, if $L \to S$ is any line bundle, we have that

$\mathbb{P}(E) \cong \mathbb{P}(E \otimes L)$ but (see [23], pg. 82)

$$\mathcal{O}_{\mathbb{P}(E\otimes L)}(-1) \cong \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^*(L).$$

There are two basic exact sequences on $\mathbb{P}(E)$

1. the Euler sequence (see [9], Appendix B, B.5.8)

$$0 \to \mathcal{O}_{\mathbb{P}(E)} \to \pi^*(E) \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \to T_{\mathbb{P}(E)|S} \to 0 \tag{1.1}$$

   where $T_{\mathbb{P}(E)|S}$ is the relative tangent bundle of $\mathbb{P}(E)$ over S; and

2. the defining sequence of the relative tangent bundle

$$0 \to T_{\mathbb{P}(E)|S} \to T\mathbb{P}(E) \to \pi^*(TS) \to 0 \tag{1.2}$$

   where the last map is given by the differential of π.

We are interested in the case where $E = TS$, where S is a projective surface. Throughout M will stand for $\mathbb{P}(TS)$.

If $\{U_i\}$ is an open covering of S with transition functions ψ_{ij} then we can realize $\mathbb{P}(TS)$ as the gluing of the local pieces $U_i \times \mathbb{P}^1$ and $U_j \times \mathbb{P}^1$ by the isomorphisms

$$\begin{array}{cccc} \Phi_{ij}: & (U_i \cap U_j) \times \mathbb{P}^1 & \longrightarrow & (U_i \cap U_j) \times \mathbb{P}^1 \\ & (z;[t:u]) & \mapsto & (\psi_{ij}(z);[D\psi_{ij}(z)(t,u)]). \end{array}$$

Example 1.3. In the case of $S = \mathbb{P}^2$, we can cover M by six affine coordinates $(x, y; p)$, $(x, y; p_1)$, $(u, v; q)$, $(u, v; q_1)$, $(r, s; t)$ and $(r, s; t_1)$, which are related in the following way

$$\begin{array}{ccc} p_1 = \frac{1}{p} & q_1 = \frac{1}{q} & t_1 = \frac{1}{t} \\ \\ u = \frac{1}{x} & v = \frac{y}{x} & q = y - xp \\ \\ r = \frac{x}{y} & s = \frac{1}{y} & t = \frac{p}{xp-y} \end{array}$$

Here we observe that, in each chart, for example $(x, y; p)$; the pair (x, y) represents the affine coordinates in $\mathbb{P}^2$, and the third component p represents the tangent direction (which is in some affine chart of $\mathbb{P}^1$).

1.2 The Cohomology Rings

In this section we describe the cohomology of M. We will use both the additive and multiplicative notation for the operation between bundles.

For a complex vector bundle $\pi : E \to S$ the pullback map

$$\pi^* : H^*(S) \to H^*(\mathbb{P}(E))$$

endows $H^*(\mathbb{P}(E))$ with an structure of $H^*(S)$-algebra. More precisely, (see [10, pg.606])

Proposition 1.4. *For any compact oriented C^∞ manifold S and any complex vector bundle $\pi : E \to S$ of rank r the cohomology ring $H^*(\mathbb{P}(E))$ is the $H^*(S)$-algebra generated by the Chern class of the tautological bundle*

$$\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))$$

with the single relation

$$\sum_{i=0}^{r}(-1)^i c_i(E)\xi^{r-i} = 0 \tag{1.3}$$

where $c_i(E)$ denotes the i-th Chern class of the vector bundle E.
(Note: Here and in the sequel, we omit pullbacks of classes when convenient and when no confusion should result)

When $E = TS$, $H^*(M)$ is a $H^*(S)$-algebra generated by $\xi = c_1(\mathcal{O}_M(-1))$ with the relation:

$$\xi^2 - c_1(TS)\xi + c_2(TS) = 0 \tag{1.4}$$

Lemma 1.5. *The canonical bundle of M is given by:*

$$K_M = 2\pi^*(K_S) + 2\xi$$

where K_S is the canonical bundle of S.

Proof. Taking determinants in sequences (1.1) and (1.2) we obtain

$$T_{\mathbb{P}(E)|S} = \pi^*(det(E)) + \mathcal{O}_{\mathbb{P}(E)}(r)$$

and

$$-K_{\mathbb{P}(E)} = T_{\mathbb{P}(E)|S} + \pi^*(-K_S).$$

Therefore

$$K_{\mathbb{P}(E)} = -\pi^*(det(E)) + \mathcal{O}_{\mathbb{P}(E)}(-r) + \pi^*(K_S).$$

To conclude the proof of the lemma we only need to notice that $det(E) = -K_S$ and $r = 2$. $\square$

Example 1.6. Let us give a description of the cohomology rings in the case where $S = \mathbb{P}^2$. First notice that the sequence (1.1) for $S = \{point\}$ and E =3-dimensional vector space, gives us the classical Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to T\mathbb{P}^2 \to 0. \tag{1.5}$$

Therefore

$$\begin{aligned} c(T\mathbb{P}^2) &= c(\mathcal{O}_{\mathbb{P}^2}(1))^3 \\ &= (1 + c_1(\mathcal{O}_{\mathbb{P}^2}(1)))^3 \end{aligned}$$

where $c(E)$ denotes the total Chern class of the fiber bundle E.

Denoting by $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ the hyperplane class, we obtain

$$c_1(T\mathbb{P}^2) = 3h \quad , \quad c_2(T\mathbb{P}^2) = 3h^2. \tag{1.6}$$

As h is the class of a line in $\mathbb{P}^2$, it satisfies $h^3 = 0$. From equation (1.3) we conclude the following description of the cohomology rings of M

$$H^*(M) = \frac{\mathbb{Z}[\xi, h]}{\langle h^3, \xi^2 - 3h\xi + 3h^2 \rangle}$$

Remark 1.7. Since $K_{\mathbb{P}^2} = -3h$, we obtain from lemma 1.5 that:

$$K_M = -6h + 2\xi.$$

There is another description of $H^*(M)$ (for $S = \mathbb{P}^2$) in terms of h and $\check{h}$ (which corresponds to a line in the dual plane $\check{\mathbb{P}}^2$) which we will present in Section 1.3.2.

1.3 The Contact Distribution

Let $M = \mathbb{P}(TS)$ be the projectivization of the tangent bundle of S, and $\pi : M \to S$ the natural projection. The three dimensional variety M is usually called **the contact variety**.

For each point $x = (z, [v]) \in M$,i.e. $z \in S$ and $v \in T_zS$, one has the plane $\mathcal{D}_x := (d\pi(x))^{-1}(\mathbb{C}v)$. We obtain in this way a two dimensional distribution $\mathcal{D}$ in M, the so called **contact distribution**. It fits into the exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow TM \longrightarrow N_{\mathcal{D}} \longrightarrow 0 \tag{1.7}$$

where $N_{\mathcal{D}}$ is the normal bundle of this distribution.

Let us describe the contact distribution in terms of a system of coordinates.

Remember from Section 1.1 that we can obtain Mas the gluing of $U_i \times \mathbb{P}^1$ and $U_j \times \mathbb{P}^1$ by the isomorphisms $\Phi_{ij}(z; [t : u]) = (\psi_{ij}(z); [D\psi_{ij}(z)(t, u)])$. Denoting by $((x_i, y_i); [t_i : u_i])$ the coordinates on $U_i \times \mathbb{P}^1$, and by A_{ij} the jacobian matrix of ψ_{ij}, we have

$$dx_i = a_{ij}dx_j + b_{ij}dy_j,$$
$$dy_i = c_{ij}dx_j + e_{ij}dy_j,$$

where

$$\begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} = A^t_{ij}.$$

We also notice that

$$\begin{pmatrix} t_i \\ u_i \end{pmatrix} = A^t_{ij} \begin{pmatrix} t_j \\ u_j \end{pmatrix}.$$

Now, take the local 1-forms on $U_i \times \mathbb{C}^2$

$$\eta_i := t_i dy_i - u_i dx_i.$$

Using the above observations we obtain

$$\begin{aligned}\eta_i &= (a_{ij}t_j + b_{ij}u_j)(c_{ij}dx_j + e_{ij}dy_j) - (c_{ij}t_j + e_{ij}u_j)(a_{ij}dx_j + b_{ij}dy_j) \\ &= \det(A_{ij})\eta_j.\end{aligned} \tag{1.8}$$

Hence the collection $\{\eta_i\}$ defines a morphism

$$r^*(det(T^*S)) \to T^*(TS), \tag{1.9}$$

where $r : TS \to S$ is the natural projection. Now consider the 1-forms

$$\alpha_i = \frac{1}{t_i}\eta_i, \ \ \beta_i = \frac{1}{u_i}\eta_i$$

over the open sets of $U_i \times \mathbb{P}^1$ defined by $t_i \neq 0$ and $u_i \neq 0$ respectively. As they satisfy

$$\alpha_i = \frac{u_i}{t_i}\beta_i, \tag{1.10}$$

the pair $\{\alpha_i, \beta_i\}$ defines a morphism

$$q_i^*\mathcal{O}_{\mathbb{P}^1}(-1) \to T^*(U_i \times \mathbb{P}^1) \tag{1.11}$$

where $q_i : U_i \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the second factor.

From equations (1.8) and (1.10) we infer that the collection of local 1-forms $\alpha = \{\alpha_i, \beta_i\}_i$ defines a codimension one distribution on M. We also observe that the 1-forms α_i, β_i have no singular points.

It is clear that the contact distribution $\mathcal{D}$ on M is the given by the collection $\alpha = \{\alpha_i, \beta_i\}_i$; i.e. locally $\mathcal{D}$ is defined by the kernel of the 1-forms

$$dy_i - \frac{u_i}{t_i}dx_i, \ \ \frac{t_i}{u_i}dy_i - dx_i.$$

We usually call α the contact form.

Lemma 1.8. *The normal bundle of the contact distribution $N_{\mathcal{D}}$ is given by*

$$N_{\mathcal{D}} = \pi^*(-K_S)) \otimes \mathcal{O}_M(1).$$

Proof. Follows from (1.9), (1.11) and the fact that

$$q_i^* \mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{O}_{U_i \times \mathbb{P}^1}(-1).$$

□

Remark 1.9. From the local definition of $\mathcal{D}$, we can compute $N_{\mathcal{D}}$ in the following alternative way. First notice that $N_{\mathcal{D}}$ is the cokernel of the inclusion of $\mathcal{D}$ in TM. Then consider the exact sequences (1.1) and (1.2) as in the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{D} \arrow[d] & \mathcal{O}_M(-1) \arrow[d] & \\
0 \arrow[r] & T_{M|S} \arrow[r] & TM \arrow[r, "d\pi"] \arrow[d] & \pi^*(TS) \arrow[r] \arrow[d] & 0 \\
 & & N_{\mathcal{D}} \arrow[d] & T_{M|S} \otimes \mathcal{O}_M(-1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

If we take local coordinates (x, y, p) in M – here, (x, y) represents the point in S and $p = [1 : p]$ a tangent direction–, we know that $\mathcal{D}$ is given in this chart by $dy - pdx$. Thus $\mathcal{D}$ is generated by the local vector fields

$$\frac{\partial}{\partial p}, \ \frac{\partial}{\partial x} + p\frac{\partial}{\partial y}.$$

If $v = a\frac{\partial}{\partial p} + b\frac{\partial}{\partial x} + pb\frac{\partial}{\partial y}$ is a vector tangent to $\mathcal{D}$ at the point $\tilde{z} = (x, y, p)$, we have

$$d\pi_{\tilde{z}}(v) = b\frac{\partial}{\partial x} + pb\frac{\partial}{\partial y} \in \mathcal{O}_M(-1)_{\tilde{z}} = \langle \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} \rangle.$$

Thus the map $\mathcal{D} \to \mathcal{O}_M(-1)$ is surjective over each point and its kernel is clearly $T_{M|S}$. Hence we can complete our previous diagram as below.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{M|S} \arrow[r] \arrow[d, equal] & \mathcal{D} \arrow[r] \arrow[d] & \mathcal{O}_M(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{M|S} \arrow[r] & TM \arrow[r, "d\pi"] \arrow[d] & \pi^*(TS) \arrow[r] \arrow[d] & 0 \\
 & & N_{\mathcal{D}} \arrow[d] & T_{M|S} \otimes \mathcal{O}_M(-1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Therefore

$$
\begin{aligned}
N_{\mathcal{D}} = det(TM) - det(\mathcal{D}) &= det(TM) + \mathcal{O}_M(1) - T_{M|S} \\
&= \pi^*(det(TS)) + \mathcal{O}_M(1) = \pi^*(-K_S) + \mathcal{O}_M(1).
\end{aligned}
$$

As an application of the above, one obtains.

Proposition 1.10. *The determinant of the contact distribution is given by*

$$det(\mathcal{D}) = \pi^*(-K_S) + \mathcal{O}_M(1)$$

Proof. From (1.7) and Lemma 1.5 we have

$$
\begin{aligned}
det(\mathcal{D}) &= -K_M - N_{\mathcal{D}} \\
&= -2\pi^*(K_S) + 2\mathcal{O}_M(1) + \pi^*(K_S) - \mathcal{O}_M(1) \\
&= \pi^*(-K_S) + \mathcal{O}_M(1).
\end{aligned}
$$

The lemma is proved. □

1.3.1 Lift of curves

There is a natural way to lift a curve $C \subseteq S$ to M. Over a nonsingular point $z \in C$, the fiber $\mathbb{P}(T_z S) \subseteq M$ parametrizes all the tangent directions of S at z, in particular we can take the point which represent the direction tangent to C at z. Varying z in the smooth locus of C we form a curve in M, the closure of this curve is called the first lift (or just the lift) of C, and will be denoted $C^{(1)}$ or $\widetilde{C}$.

It is not difficult to see that the contact distribution has the following properties.

1. **$\mathcal{D}$ is not integrable.** This can be verified locally. If we take coordinates $(x, y; p)$ in M, then $\mathcal{D}$ is given by $\alpha = dy - pdx$ and

$$\alpha \wedge d\alpha = (dy - pdx) \wedge (-dp \wedge dx) = -dx \wedge dy \wedge dp.$$

   By Frobenius theorem, we are done.

2. **The lift of a curve $C \subseteq S$ to M is tangent to $\mathcal{D}$.** If we take over a smooth point z of C a parametrization of the form $(t, \gamma(t))$ (we are assuming without loss of generality that the tangent direction of C at z is not vertical), then $\widetilde{C}$ is given in coordinates $(x, y, p) = (x, y, [1 : p])$ by $(t, \gamma(t), \gamma'(t))$. Therefore the tangent vector to $\widetilde{C}$ is of the form

$$v(t) = \frac{\partial}{\partial x} + \gamma'(t)\frac{\partial}{\partial y} + \gamma''(t)\frac{\partial}{\partial p}$$

   which allows to compute

$$\alpha(v(t)) = (dy - pdx)(\frac{\partial}{\partial x} + \gamma'(t)\frac{\partial}{\partial y} + \gamma''(t)\frac{\partial}{\partial p}) = 0$$

   The computations in the other coordinates of M are similar.

Actually, we have the following lemma.

Lemma 1.11. *Let $\widetilde{C} \subseteq M$ be an irreducible curve which is not a fiber of $\pi : M \to S$. Then $\widetilde{C}$ is tangent to $\mathcal{D}$ if and only if $\widetilde{C}$ is the lift of an irreducible curve $C \subseteq S$.*

Proof. Since $\widetilde{C}$ is not a vertical curve, we can assume that in the coordinates $(x, y; p)$ $\widetilde{C}$ is given in a neighborhood of a regular point, by a parametrization of the form: $(t, y(t), p(t))$. The condition of tangency with $\mathcal{D}$ gives us $p(t) = y'(t)$. This implies that $\widetilde{C}$ coincides with the lift of the irreducible curve $C = \pi(\widetilde{C})$. □

1.3.2 The case of $\mathbb{P}^2$

We shall work in the case where $S = \mathbb{P}^2$. In this case the variety M can be identified with the point-line incidence variety $\{(x, l) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 : x \in l\}$. We denote by $\check{\pi}$ the restriction to M of the projection onto $\check{\mathbb{P}}^2$.

$$\begin{tikzcd} & M = \mathbb{P}(T\mathbb{P}^2) \arrow[dl, "\pi"'] \arrow[dr, "\check{\pi}"] & \subseteq \mathbb{P}^2 \times \check{\mathbb{P}}^2 \\ \mathbb{P}^2 & & \check{\mathbb{P}}^2 \end{tikzcd}$$

Let $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ and $\check{h} = c_1(\mathcal{O}_{\check{\mathbb{P}}^2}(1))$ be the hyperplane classes on $\mathbb{P}^2$ and $\check{\mathbb{P}}^2$ respectively. We still denote by h and $\check{h}$ the respective pullbacks to M by π and $\check{\pi}$. Notice that in coordinates (x, y, p), $\check{h}$ is the class of the divisor $\{p = 0\}$, which can be thought of as a pencil of lines in $\mathbb{P}^2$. Let us also denote by $\mathcal{O}_M(a, b)$ the class of the line bundle $\mathcal{O}_M(ah + b\check{h})$.

Lemma 1.12. *The class $(\alpha) := \mathcal{O}_M((\alpha)_0 - (\alpha)_\infty)$ of the contact form α is given by*

$$(\alpha) = \mathcal{O}_M(-1, -1)$$

where (α_0) and (α_∞) represent the divisors of zeros and poles of α respectively.

Proof. We just need to write $\alpha = dy - pdx$ and change to the other affine coordinates by the coordinate changes given in example 1.3

$$dy - pdx, \qquad \tfrac{1}{p_1}(p_1 dy - dx), \qquad \tfrac{1}{u}(dv - qdu),$$

$$\tfrac{1}{uq_1}(q_1 dv - du), \qquad \tfrac{1}{s(rt-s)}(ds - tdr), \qquad \tfrac{1}{s(r-t_1 s)}(t_1 ds - dr).$$

and the lemma follows. □

From this lemma we infer that

$$N_{\mathcal{D}} = \mathcal{O}_M((\alpha)_\infty - (\alpha)_0) = \mathcal{O}_M(1, 1) \tag{1.12}$$

and therefore

Proposition 1.13. *The class of the tautological bundle is given by*

$$\xi = 2h - \check{h}.$$

Proof. It is enough to observe that $\mathcal{O}_M(1) = N_{\mathcal{D}} + \pi^*(K_{\mathbb{P}^2})$. $\square$

Now we obtain another description of $H^*(M)$.

Corollary 1. *The cohomology rings are given by*

$$H^*(M) = \frac{\mathbb{Z}[h, \check{h}]}{\langle h^3, h^2 - h\check{h} + \check{h}^2 \rangle}$$

Proof. Follows from Example 1.6 and Proposition 1.13. $\square$

One also has

Lemma 1.14. *The following relations hold true*

1. $h^2\check{h} = h\check{h}^2 = 1$,
2. $\check{h}^3 = 0$.

Proof. For the first equation observe that h^2 is the class of a fiber of π, then $h^2\check{h} = 1$. Now multiply the relation $h^2 - h\check{h} + \check{h}^2 = 0$ by h to obtain $h\check{h}^2 = 1$. The second equality follows from the fact that $\check{h}$ represents the class of a line on $\check{\mathbb{P}}^2$. $\square$

With this at hand, we recover some formulas for the bundles that we work with.

Corollary 2. *The canonical bundle of M is given by*

$$K_M = \mathcal{O}_M(-2, -2).$$

Proof. It is clear from remark 1.7 and proposition 1.13. $\square$

Corollary 3. *The determinant of the contact distribution is given by*

$$det(\mathcal{D}) = \mathcal{O}_M(1, 1).$$

Proof. It is a consequence of propositions 1.10 and 1.13. $\square$

Chapter 2

Second Order Differential Equations

In this chapter we present our first contributions. It starts with the definition of second order differential equations on projective surfaces and an analysis of a class of examples of such objects. Then we provide formulas for the intersection of the cotangent bundle of the second order differential equation with surfaces and smooth curves. These results are applied to study nets of curves on $\mathbb{P}^2$ and in particular to obtain bounds for the number of completely decomposable elements of a net.

2.1 Second Order Differential Equations vs Foliations Tangent to $\mathcal{D}$

Let us consider the following second order differential equation

$$y'' = \frac{A(x,y,y')}{B(x,y,y')} = \frac{a_0(x,y) + a_1(x,y)y' + \ldots + a_l(x,y)(y')^l}{b_0(x,y) + b_1(x,y)y' + \ldots + b_k(x,y)(y')^k} \tag{2.1}$$

where the coefficients $a_0(x,y), \ldots, a_l(x,y), b_0(x,y), \ldots, b_k(x,y)$ are polynomials.

Take coordinates $(x,y,p) = (x,y,[1:p])$ on $\mathbb{P}(T\mathbb{C}^2) \subseteq M = \mathbb{P}(T\mathbb{P}^2)$ and consider the vector field

$$X = B(x,y,p)\frac{\partial}{\partial x} + pB(x,y,p)\frac{\partial}{\partial y} + A(x,y,p)\frac{\partial}{\partial p} \tag{2.2}$$

in this chart.

Lemma 2.1. *The integral curves of X correspond, via π (the natural projection), to the solutions of (2.1).*

Proof. Let $\widetilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be an integral curve of X, so we can suppose that

$$\begin{cases} \gamma_1' &= B(\gamma_1, \gamma_2, \gamma_3) \\ \gamma_2' &= \gamma_3.B(\gamma_1, \gamma_2, \gamma_3) \\ \gamma_3' &= A(\gamma_1, \gamma_2, \gamma_3) \end{cases}$$

Set $(x, y) = \pi(\gamma_1, \gamma_2, \gamma_3) = (\gamma_1, \gamma_2)$, then by the relations above and the chain rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\gamma_2'}{\gamma_1'}\right) = \frac{1}{\gamma_1'}\frac{d}{dt}(\gamma_3) = \frac{\gamma_3'}{\gamma_1'} = \frac{A(\gamma_1, \gamma_2, \gamma_3)}{B(\gamma_1, \gamma_2, \gamma_3)} = \frac{A(x, y, \frac{dy}{dx})}{B(x, y, \frac{dy}{dx})}.$$

On the other hand, if $\gamma = (\gamma_1, \gamma_2)$ is a solution of (2.1) then the lift of this curve is parametrized by $\widetilde{\gamma} = (\gamma_1, \gamma_2, \frac{\gamma_2'}{\gamma_1'}) = (\gamma_1, \gamma_2, \gamma_3)$. Since γ is a solution of (2.1) we have

$$\frac{A(\gamma_1, \gamma_2, \gamma_3)}{B(\gamma_1, \gamma_2, \gamma_3)} = \frac{d^2y}{dx^2} = \frac{1}{\gamma_1'}\frac{d}{dt}\left(\frac{\gamma_2'}{\gamma_1'}\right),$$

therefore

$$\gamma_2' = \gamma_1'\gamma_3 \quad \text{and} \quad \gamma_3' = \gamma_1'\frac{A(\gamma_1, \gamma_2, \gamma_3)}{B(\gamma_1, \gamma_2, \gamma_3)}.$$

This clearly implies that $\widetilde{\gamma}$ is tangent to X. □

Notice that X is tangent to the contact distribution. Moreover every vector field, polynomial in (x, y, p), tangent to α, has this form. So X defines a 1-dimensional foliation in an open set of $\mathbb{P}(T\mathbb{C}^2) \subseteq M$ tangent to the contact distribution. Its leaves project to the graphs of the solutions of our initial second order differential equation.

Since X is a polynomial vector field this foliation extends to a 1-dimensional foliation $\mathcal{F}$ in M tangent to $\mathcal{D}$. If we denote by $T^*\mathcal{F}$ the cotangent bundle of $\mathcal{F}$, then the local expressions of X in local coordinates of M form a global section $X_{\mathcal{F}} \in H^0(M, \mathcal{D} \otimes T^*\mathcal{F})$. This motivates the following definitions.

Definition 2.2. A second order differential equation in S is a one dimensional foliation $\mathcal{F}$ in $M = \mathbb{P}(TS)$, tangent to $\mathcal{D}$. In other words, a foliation defined by an element $X_{\mathcal{F}} \in H^0(M, \mathcal{D} \otimes T^*\mathcal{F})$ for a suitable line bundle $T^*\mathcal{F}$ on M (called the cotangent bundle of $\mathcal{F}$). The solutions of this equation are the projections by π of the leaves of $\mathcal{F}_{sat}$, where $\mathcal{F}_{sat}$ is the saturated foliation associated to $\mathcal{F}$.

Definition 2.3. If we write $T^*\mathcal{F} = \mathcal{O}_M(a, b)$ for some integers a, b; we define the bidegree of the differential equation (or of the foliation $\mathcal{F}_X$ associated to X) to be the ordered pair (a, b).

Notice that we allow the foliation $\mathcal{F}$ to have a singular set of codimension one. Now we present two simple examples for $S = \mathbb{P}^2$.

Example 2.4. Let us consider the differential equation satisfied by the lines

$$y'' = 0.$$

The associated foliation in M will be denoted by $\mathcal{L}$. This foliation is given in coordinates (x, y, p) by the vector field

$$X_{\mathcal{L}} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y}.$$

If we make the change of coordinates in M (see example 1.3) we obtain

- $(x, y, p_1) = (x, y, \frac{1}{p})$,

$$X_{\mathcal{L},1}(x, y, p_1) = \frac{1}{p_1}\left[p_1\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right].$$

- $(u, v, q) = (\frac{1}{x}, \frac{y}{x}, y - xp)$,

$$X_{\mathcal{L},2}(u, v, q) = -u^2\left[\frac{\partial}{\partial u} + q\frac{\partial}{\partial v}\right].$$

- $(u, v, q_1) = (u, v, \frac{1}{q})$,

$$X_{\mathcal{L},3}(u, v, q_1) = \frac{-u^2}{q_1}\left[q_1\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right].$$

- $(r,s,t) = (\frac{x}{y}, \frac{1}{y}, \frac{p}{xp-y})$,

$$X_{\mathcal{L},4}(r,s,t) = \frac{-s^2}{rt-s}\left[\frac{\partial}{\partial r} + t\frac{\partial}{\partial s}\right].$$

- $(r,s,t_1) = (r,s,\frac{1}{t})$,

$$X_{\mathcal{L},5}(r,s,t_1) = \frac{-s^2}{r-st_1}\left[t_1\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right].$$

Since $T^*\mathcal{L} = \mathcal{O}_M\left((X_{\mathcal{L}})_\infty - (X_{\mathcal{L}})_0\right)$, we obtain

$$T^*\mathcal{L} = \mathcal{O}_M(-2,1).$$

Note that this foliation is tangent to the fibers of $\check{\pi} : M \to \check{\mathbb{P}}^2$.

Example 2.5. Let $\mathcal{V}$ be the foliation tangent to the fibers of the projection $\pi : M \to \mathbb{P}^2$. This foliation is given in the chart (x,y,p) by the vector field

$$X_{\mathcal{V}} = \frac{\partial}{\partial p},$$

so $\mathcal{V}$ is clearly tangent to $\mathcal{D}$. We can make again the change of coordinates in M to obtain

- $(x,y,p_1) = (x,y,\frac{1}{p})$,

$$X_{\mathcal{V},1}(x,y,p_1) = -p_1^2\frac{\partial}{\partial p_1}.$$

- $(u,v,q) = (\frac{1}{x}, \frac{y}{x}, y-xp)$,

$$X_{\mathcal{V},2}(u,v,q) = \frac{-1}{u}\frac{\partial}{\partial q}.$$

- $(u,v,q_1) = (u,v,\frac{1}{q})$,

$$X_{\mathcal{V},3}(u,v,q_1) = \frac{q_1^2}{u}\frac{\partial}{\partial q_1}.$$

- $(r, s, t) = (\frac{x}{y}, \frac{1}{y}, \frac{p}{xp-y})$,

$$X_{\mathcal{V},4}(r, s, t) = \frac{-(rt-s)^2}{s}\frac{\partial}{\partial t}.$$

- $(r, s, t_1) = (r, s, \frac{1}{t})$,

$$X_{\mathcal{V},5}(r, s, t_1) = \frac{(r-st_1)^2}{s}\frac{\partial}{\partial t_1}.$$

In this case we conclude

$$T^*\mathcal{V} = \mathcal{O}_M(1, -2).$$

Observe that tangent bundles of $\mathcal{L}$ and $\mathcal{V}$ fit into the following exact sequence

$$0 \longrightarrow T\mathcal{L} \longrightarrow \mathcal{D} \longrightarrow T\mathcal{V} \longrightarrow 0. \tag{2.3}$$

2.2 Pencil of webs

In this section we give an example of second order differential equation on $\mathbb{P}^2$, pencil of webs.

A global k-web $\mathcal{W}$ on a surface S is given by an open covering $\mathcal{U} = \{U_i\}$ of S and k-symmetric 1-forms $\omega_i \in \mathrm{Sym}^k\Omega^1_S(U_i)$ such that for each non-empty intersection $U_i \cap U_j$ of elements of $\mathcal{U}$ there exists a non-vanishing function $g_{ij} \in \mathcal{O}^*_S(U_i \cap U_j)$ such that $\omega_i = g_{ij}\omega_j$.

The transition functions g_{ij} determine a line bundle $\mathcal{N}$ over S and the k-symmetric 1-forms $\{\omega_i\}$ patch together to form a section of $\mathrm{Sym}^k\Omega^1_S \otimes \mathcal{N}$, that is, $\omega = \{\omega_i\}$ can be interpreted as an element of $H^0(S, \mathrm{Sym}^k\Omega^1_S \otimes \mathcal{N})$. The line bundle $\mathcal{N}$ will be called the normal bundle of $\mathcal{W}$.

Two global sections $\omega, \omega' \in H^0(S, \mathrm{Sym}^k\Omega^1_S \otimes \mathcal{N})$ determine the same web if and only if they differ by the multiplication by an element $g \in H^0(X, \mathcal{O}^*_S)$. If X is compact, or more generally if the only global sections of $\mathcal{O}^*_S$ are the non-zero constants, then a global k-web is nothing more than an element of

$\mathbb{P}H^0(S, \mathrm{Sym}^k\Omega^1_S \otimes \mathcal{N})$, for a suitable line bundle $\mathcal{N}$.

A k-web $\mathcal{W} \in \mathbb{P}H^0(S, \mathrm{Sym}^k\Omega^1_S \otimes \mathcal{N})$ is decomposable if there are global webs $\mathcal{W}_1$, $\mathcal{W}_2$ on S sharing no common subwebs such that $\mathcal{W} = \mathcal{W}_1 \boxtimes \mathcal{W}_2$, in other words if $\mathcal{W}$ is in the image of the natural map:

$$\mathbb{P}H^0(S, \mathrm{Sym}^{k_1}\Omega^1_S \otimes \mathcal{N}_1) \times \mathbb{P}H^0(S, \mathrm{Sym}^{k_2}\Omega^1_S \otimes \mathcal{N}_2) \to \mathbb{P}H^0(S, \mathrm{Sym}^k\Omega^1_S \otimes \mathcal{N})$$

for some k_1, k_2, $\mathcal{N}_1$, $\mathcal{N}_2$ such that $k_1 + k_2 = k$ and $\mathcal{N}_1 + \mathcal{N}_2 = \mathcal{N}$. A k-web $\mathcal{W}$ is completely decomposable if one can write $\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$ for k global foliations $\mathcal{F}_1 \ldots \mathcal{F}_k$ on S.

We are interested in webs defined in the projective plane $\mathbb{P}^2$. Let $\mathcal{W} = [\omega] \in \mathbb{P}H^0(\mathbb{P}^2, \mathrm{Sym}^k\Omega^1_{\mathbb{P}^2} \otimes \mathcal{N})$ be a k-web on $\mathbb{P}^2$. Analogously to the case of foliations, we define the degree of $\mathcal{W}$ as the number of tangencies, counted with multiplicities, of $\mathcal{W}$ with a line not everywhere tangent to $\mathcal{W}$, and we denote it by $deg(\mathcal{W})$. More precisely, if we have $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ a line on $\mathbb{P}^2$ then the image of i is everywhere tangent to $\mathcal{W}$ if and only if $i^*\omega$ vanishes identically. When this line is not invariant by $\mathcal{W}$ the points of tangency with $\mathcal{W}$ correspond to the zeroes of $[i^*\omega] \in \mathbb{P}H^0(\mathbb{P}^1, \mathrm{Sym}^k\Omega^1_{\mathbb{P}^1} \otimes i^*\mathcal{N})$. Observing that $i^*\mathcal{N}_{\mathcal{W}} = i^*\mathcal{O}_{\mathbb{P}^2}(deg(\mathcal{N}_{\mathcal{W}})) = \mathcal{O}_{\mathbb{P}^1}(deg(\mathcal{N}_{\mathcal{W}}))$ and $\mathrm{Sym}^k\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2k)$ we conclude that

$$deg(\mathcal{N}_{\mathcal{W}}) = d + 2k.$$

From now on we denote by $\mathbb{W}(k,d) = \mathbb{P}H^0(\mathbb{P}^2, \mathrm{Sym}^k\Omega^1_{\mathbb{P}^2}(d+2k))$ the space of k-webs of degree d in $\mathbb{P}^2$.

Let $\mathcal{W}$ be a $k-$web of degree d in $\mathbb{P}^2$. For a point $z \in \mathbb{P}^2$ we have k points (not necessarily different) $p_1(z), \ldots, p_k(z) \in \mathbb{P}(T_p\mathbb{P}^2)$ corresponding to the directions of $\mathcal{W}$ at this point. In this way we obtain a surface $S_{\mathcal{W}} \subseteq M$ which is the union of the lifts of the leaves of $\mathcal{W}$.

It is clear that $S_{\mathcal{W}}$ intersects a generic fiber of $\pi : M \to \mathbb{P}^2$ at k points (counting with multiplicities), so if we write $[S_{\mathcal{W}}] = ah + b\check{h}$ we have $k = [S_{\mathcal{W}}].h^2 = b$.

On the other hand we know that the class of the lift of a line $l \subseteq \mathbb{P}^2$ is $[\check{l}] = \check{h}^2$, and the points of $S_{\mathcal{W}} \cap \check{l}$ (for l generic) corresponds to points of $\mathbb{P}^2$

where $\mathcal{W}$ and l are tangent, so one concludes $d = [S_{\mathcal{W}}].\check{h}^2 = a$ and therefore

$$[S_{\mathcal{W}}] = dh + k\check{h} \tag{2.4}$$

Remark 2.6. To obtain (2.4) we have assumed that d is the number of tangencies between $\mathcal{W}$ and a generic line l in $\mathbb{P}^2$, but the formula remains true. In fact, if we consider the k-web $\mathcal{W}$ given by $f.\omega \in H^0(\mathrm{Sym}^k\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d + d_1 + 2k))$, where $f \in H^0(\mathbb{P}^2, \mathcal{O}(d_1))$ and the k-web $\mathcal{W}'$ given by $\omega \in H^0(\mathrm{Sym}^k\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d + 2k))$ has d tangencies with a generic line in $\mathbb{P}^2$, then

$$[S_{\mathcal{W}}] = [S_{W'}] + \pi^*[\{f = 0\}] = (d + d_1)h + k\check{h}.$$

Remark 2.7. If the k-web is given locally by the equation

$$F(x, y, y') = a_0(x, y) + \ldots + a_k(x, y)(y')^k = 0$$

where $a_0, \ldots, a_k$ are polynomials in (x, y), then the surface $S_{\mathcal{W}}$ is given in coordinates (x, y, p) by

$$S_{\mathcal{W}} = \{F(x, y, p) = 0\}.$$

The reader can see [24], lemma 3.2.3 and lemma 3.2.4 for a local argument to obtain equation (2.4).

The restriction of $\mathcal{D}$ to $S_{\mathcal{W}}$ defines a foliation (over the regular part of $S_{\mathcal{W}}$) whose leaves are the lifts of the leaves of $\mathcal{W}$. In fact, the lift of the leaves of $\mathcal{W}$ defines a foliation in $S_{\mathcal{W}}$ which is tangent to $\mathcal{D}$.

Take now two k-webs of degree d $\mathcal{W}_1$ and $\mathcal{W}_2$ given by

$$\omega_i \in H^0(\mathbb{P}^2, \mathrm{Sym}^k\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d + 2k)), \;\; i = 1, 2;$$

and we shall suppose that these elements are $\mathbb{C}$-linearly independent. Consider the pencil of webs $\{\mathcal{W}_t\}_{t\in\mathbb{P}^1}$ generated by $\mathcal{W}_1$ and $\mathcal{W}_2$, i.e. $\mathcal{W}_{[a:b]}$ is given by $a\omega_1 + b\omega_2$. Then we have a pencil of surfaces $\{S_{\mathcal{W}_t}\}$ in M and clearly this is the pencil generated by $S_{\mathcal{W}_1}$ and $S_{\mathcal{W}_2}$. Since one has a foliation over each $S_{\mathcal{W}_t}$, we obtain a one dimensional foliation $\mathcal{H}$ tangent to $\mathcal{D}$.

Since the classes $[S_{\mathcal{W}_1}], [S_{\mathcal{W}_2}]$ are $dh + k\check{h}$, the pencil $S_{\mathcal{W}_t}$ is given by an element $\eta \in H^0(M, \Omega^1_M \otimes \mathcal{L}_1)$ where $\mathcal{L}_1 = \mathcal{O}_M(2d, 2k)$ (η is locally the 1-form

$FdG - GdF$, where F and G are the local equations of $S_{\mathcal{W}_1}$ and $S_{\mathcal{W}_2}$). Then $\mathcal{H}$ is given by

$$\alpha \wedge \eta \in H^0(M, \Omega^2_M \otimes \mathcal{L})$$

where $\alpha \in H^0(M, \Omega^1_M \otimes N_{\mathcal{D}})$ is the contact form and

$$\mathcal{L} = \mathcal{L}_1 \otimes N_{\mathcal{D}} = \mathcal{O}_M(2d+1, 2k+1).$$

We define the second order differential equation associated to the pencil of webs $\{\mathcal{W}_t\}_{t \in \mathbb{P}^1}$ as the foliation $\mathcal{H}$ constructed above.

Now, any nowhere vanishing local holomorphic section of K_M defines by contraction an isomorphism between local vector fields and local 2-forms generating $\mathcal{H}$, so that $K_M \cong Hom(T\mathcal{H}, \mathcal{L}^*) \cong T^*\mathcal{H} \otimes \mathcal{L}^*$. We use corollary 2 to conclude that

$$T^*\mathcal{H} = \mathcal{O}_M(2d-1, 2k-1). \tag{2.5}$$

Observe that $\mathcal{H}$ could have a singular set of codimension 1, which is given by the zero locus of $\alpha \wedge \eta$.

Example 2.8 (**Pencil of Foliations).** Let us consider the case where $\mathcal{W}_1$ and $\mathcal{W}_2$ are foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of the same degree d, i.e. $\mathcal{F}_i$ is given by $\omega_i \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(d+2))$, which is given in an affine chart by

$$\mathcal{F}_1 = [a_1 dx + b_1 dy = \eta_1 = 0],$$
$$\mathcal{F}_2 = [a_2 dx + b_2 dy = \eta_2 = 0].$$

Then locally $S_{\mathcal{F}_1} = \{F = a_1 + pb_1 = 0\}$, $S_{\mathcal{F}_2} = \{G = a_2 + pb_2 = 0\}$, so that

$$FdG - GdF = \left(F\frac{\partial G}{\partial x} - G\frac{\partial F}{\partial x}\right) dx + \left(F\frac{\partial G}{\partial y} - G\frac{\partial F}{\partial y}\right) dy + (a_1 b_2 - a_2 b_1) dp,$$

and then

$$\begin{aligned}\alpha \wedge \eta = [G\frac{\partial F}{\partial x} - F\frac{\partial G}{\partial x} - p(F\frac{\partial G}{\partial y} - G\frac{\partial F}{\partial y})] dx \wedge dy - p[a_1 b_2 - a_2 b_1] dx \wedge dp \\ + [a_1 b_2 - a_2 b_1] dy \wedge dp.\end{aligned}$$

If $\omega_1 \wedge \omega_2 = 0$ then $a_1 b_2 - a_2 b_1 = 0$ and in this case $\mathcal{H}$ is the "vertical" foliation $\mathcal{V}$ of the example 2.5 (with a singular set of codimension 1).

So, we will assume that $\omega_1 \wedge \omega_2 \neq 0$, that is $a_1b_2 - a_2b_1 \neq 0$. Observe that $a_1b_2 - a_2b_1$ only depends of x, y, then if $\alpha \wedge \eta$ has a singular set given by $\{H = 0\}$, H does not depend of p. The same argument holds if we use the coordinates $(x, y, p_1) = (x, y, \frac{1}{p})$. We conclude that if $\mathcal{H}$ has a singular set $S \subseteq M$ of codimension one, then $S = \pi^*(C)$ for a curve $C \subseteq \mathbb{P}^2$ and therefore $[S] = kh$, where $k = deg(C)$.

Consider now the curve $T \subseteq \mathbb{P}^2$ given by

$$T = \{\omega_1 \wedge \omega_2 = 0\} = \{\omega_s \wedge \omega_t = 0\},\ s \neq t \in \mathbb{P}^1. \tag{2.6}$$

Remark 2.9.

1. The degree of T is $2d + 1$ since

$$\omega_1 \wedge \omega_2 \in H^0(\mathbb{P}^2, K_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(2d + 4)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2d + 4 - 3))$$

2. $Sing(\mathcal{F}_t) \subseteq T$, $\forall t \in \mathbb{P}^1$.

3. Suppose that $\mathcal{F}_i$ has a singular set of codimension one $C_i = \{g_i = 0\}$ and denote by $\widetilde{\mathcal{F}_i}$ the foliation given by $\tilde{\omega}_i$ where $\omega_i = g_i\tilde{\omega}_i$, i=1,2. Then we define the tangency curve between $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$ as $tan(\widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2}) := \{\tilde{\omega}_1 \wedge \tilde{\omega}_2 = 0\}$. Observe also that

$$T = C_1 \cup C_2 \cup tang(\widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2}).$$

Proposition 2.10. *With the above notation, the curve C satisfies the following properties.*

1. *$C \subseteq T$.*

2. *Every irreducible component D of T which has multiplicity k_i on C_i divides C with multiplicity $k_1 + k_2$ if $k_1 = k_2$ and with multiplicity $k_1 + k_2 - 1$ otherwise.*

3. *An irreducible component E of T with multiplicity n divides C only if E is $\mathcal{F}_i$-invariant, $n = 1$ and some $\mathcal{F}_i$ has a radial singularity over E.*

Proof. To prove the first assertion take a point $z \notin T$, then we can choose coordinates around z where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by $\eta_1 = dy + dx$ and $\eta_2 = dy$. In this case $\eta = FdG - GdF = dp$ so that $\alpha \wedge \eta = pdx \wedge dp - dy \wedge dp$ has no singularities. We conclude that the curve C must be contained in T.

Now we are going to prove the second part. Let $D \subseteq T$ be an irreducible component of T which appears with multiplicity k_i on C_i, in particular it has multiplicity $m = k_1 + k_2$ on T. Then around the generic point of D we can choose a convenient coordinate system and we can write

$$\begin{aligned} D &= \{y = 0\}, \\ \eta_1 &= y^{k_1}(\tilde{a}_1 dx + \tilde{b}_1 dy), \\ \eta_2 &= y^{k_2}(\tilde{a}_2 dx + \tilde{b}_2 dy). \end{aligned}$$

Therefore $F = y^{k_1}\tilde{F}$, $G = y^{k_2}\tilde{G}$, where $\tilde{F} = \tilde{a}_1 + p\tilde{b}_1$ and $\tilde{G} = \tilde{a}_2 + p\tilde{b}_2$ and y does not divide $\tilde{F}\tilde{G}$. So, in these coordinates one has

$$\begin{aligned} \eta &= FdG - GdF \\ &= y^{k_1}\tilde{F}(k_2 y^{k_2-1}\tilde{G}dy + y^{k_2}d\tilde{G}) - y^{k_2}\tilde{G}(k_1 y^{k_1-1}\tilde{F}dy + y^{k_1}d\tilde{F}) \\ &= y^m(\tilde{F}d\tilde{G} - \tilde{G}d\tilde{F}) + (k_2 - k_1)y^{m-1}\tilde{F}\tilde{G}dy \end{aligned}$$

and

$$\begin{aligned} \alpha \wedge \eta &= [y^m(\tilde{G}\frac{\partial \tilde{F}}{\partial x} - \tilde{F}\frac{\partial \tilde{G}}{\partial x}) + py^m(\tilde{G}\frac{\partial \tilde{F}}{\partial y} - \tilde{F}\frac{\partial \tilde{G}}{\partial y}) - (k_2 - k_1)py^{m-1}\tilde{F}\tilde{G}]dx \wedge dy \\ &\quad -py^m[\tilde{a}_1\tilde{b}_2 - \tilde{a}_2\tilde{b}_1]dx \wedge dp + y^m[\tilde{a}_1\tilde{b}_2 - \tilde{a}_2\tilde{b}_1]dy \wedge dp. \end{aligned}$$

We can conclude the following.

1. If $k_1 = k_2$ then D appears with multiplicity $m = k_1 + k_2$ in C.
2. If $k_1 \neq k_2$ then D appears with multiplicity exactly $m - 1$ in C.

In the case where $k_1 \neq k_2$ we observe that

$$\frac{1}{y^{m-1}}(\alpha \wedge \eta) \wedge dy = py(\tilde{a}_1\tilde{b}_2 - \tilde{a}_2\tilde{b}_1)dx \wedge dy \wedge dp$$

in particular, $\pi^*(D)$ is invariant by the foliation induced by $\frac{1}{y^{m-1}}(\alpha\wedge\eta)$

In order to prove the last assertion take $E \subseteq T$ an irreducible component which appears in $tang(\mathcal{F}_1, \mathcal{F}_2)$ with multiplicity n. Observe that we have a map $\lambda : E \dashrightarrow \mathbb{P}^1$ such that for a generic point $p \in E$, $\eta_1(p) = \lambda(p)\eta_2(p)$.

Then $\lambda \equiv constant$ if and only if $E \subseteq Sing(\mathcal{F}_{[1:-\lambda]})$. In this case we proceed as in the previous case.

We assume now that $E \nsubseteq Sing(\mathcal{F}_t)$, $\forall t \in \mathbb{P}^1$, and take a generic point $p \in E$ which is a regular point for E and $\mathcal{F}_1$. We can assume that p is an isolated singularity of $\mathcal{F}_2$ (one replaces $\mathcal{F}_2$ by $\mathcal{F}_{[1:-\lambda(p)]}$).

If E is not $\mathcal{F}_1$-invariant (therefore is not $\mathcal{F}_2$-invariant), then we can choose a coordinate system around p and write

$$\begin{aligned} E &= \{x=0\}, p=(0,0), \\ \eta_1 &= dy, \\ \eta_2 &= x^n a dx + b dy \end{aligned}$$

where x does not divide $a.b$ and $b(0,0)=0$. So $F=p$ and $G=x^n a + pb$ and then

$$\begin{aligned} \alpha\wedge\eta = {} & [-p(nx^{n-1}a + x^n\frac{\partial a}{\partial x} + p\frac{\partial b}{\partial x}) - p^2(x^n\frac{\partial a}{\partial y} + p\frac{\partial b}{\partial y})]dx\wedge dy \\ & + px^n a dx\wedge dp - x^n a dy\wedge dp. \end{aligned}$$

We conclude that

$$\begin{aligned} x \ divides \ \alpha\wedge\eta &\Leftrightarrow x \ divides \ nx^{n-1}a + p\frac{\partial b}{\partial x} + p^2\frac{\partial b}{\partial y} \\ &\Leftrightarrow n \geq 2 \ and \, x \ divides \ \frac{\partial b}{\partial x} + p\frac{\partial b}{\partial y} \end{aligned}$$

but this is not possible, since $b(0,0)=0$ and x does not divides b. Then we conclude that E is not a component of C.

If E is $\mathcal{F}_1$-invariant (therefore is $\mathcal{F}_t$-invariant for all $t \in \mathbb{P}^1$), then we can choose a coordinate system around p and write

$$\begin{aligned} E &= \{y=0\}, p=(0,0), \\ \eta_1 &= dy, \\ \eta_2 &= y^n a dx + b dy \end{aligned}$$

where y does not divide $a.b$ and $b(0,0)=0$. So $F = p$ and $G = y^n a + pb$ and then

$$\alpha \wedge \eta = [-p(y^n \frac{\partial a}{\partial x} + p\frac{\partial b}{\partial x}) - p^2(ny^{n-1}a + y^n \frac{\partial a}{\partial y} + p\frac{\partial b}{\partial y})]dx \wedge dy$$
$$+ py^n a dx \wedge dp - y^n a dy \wedge dp.$$

In this case we conclude that

$$y \ \textit{divides} \ \alpha \wedge \eta \ \Leftrightarrow \ y \ \textit{divides} \ ny^{n-1}a + \frac{\partial b}{\partial x} + p\frac{\partial b}{\partial y}$$

If $n \geq 2$ this assertion implies that y divides $\frac{\partial b}{\partial x}$ and $\frac{\partial b}{\partial y}$, but this is not possible since $b(0,0)=0$ and y does not divide b.

If $n = 1$ the assertion implies that y divides $\frac{\partial b}{\partial x} + a$ and $\frac{\partial b}{\partial y}$, in particular p is a radial singularity. $\square$

2.3 Invariant subvarieties

The second order differential equations given by pencil of webs have an infinite number of invariant surfaces which are dominant over $\mathbb{P}^2$, more explicitly, the foliation $\mathcal{H}$ given by the pencil of webs $\{\mathcal{W}_t\}$ is tangent to the pencil of surfaces $\{S_{\mathcal{W}_t}\}$ in M. We shall say that an irreducible surface in M is horizontal if the restriction of the projection $\pi : M \to S$ to this surface is dominant, otherwise we say that the surface is vertical.

We have now the following general proposition. The proof is an easy adaptation of an argument of Ghys (see [11]).

Proposition 2.11. *Let $\mathcal{F}$ be a foliation by curves on a 3-dimensional projective manifold M. If $\mathcal{F}$ has an infinite number of invariant algebraic surfaces $S_1, S_2, \ldots$, then $\mathcal{F}$ has a rational first integral, i.e., there exists a rational map on M which is constant along the leaves of $\mathcal{F}$.*

Proof. Let us take $\mathcal{U} = \{U_i\}$ an open covering of M and on each U_i a vector field X_i and a algebraic function $f_i^{(k)}$ defining $\mathcal{F}$ and S_k on U_i respectively, and satisfying on each non-empty intersection $U_i \cap U_j$ the relations

$$\begin{aligned} X_i &= g_{ij} X_j, \\ f_i^{(k)} &= f_{ij}^{(k)} f_j^{(k)} \end{aligned}$$

where $g_{ij}, f_{ij}^{(k)} \in \mathcal{O}_M^*(U_i \cap U_j)$. Consider now $Div_{\mathcal{F}}(M) \subseteq Div(M)$ the subset of divisors such that every irreducible component is $\mathcal{F}$-invariant, and define the map

$$\begin{aligned} \varphi : Div_{\mathcal{F}} \otimes \mathbb{C} &\rightarrow H^1(M, \Omega^1_{M,c}) \\ \{f_{ij}^{(k)}\} &\mapsto \left\{ \frac{df_{ij}^{(k)}}{f_{ij}^{(k)}} \right\} \end{aligned}$$

where $\Omega^1_{M,c}$ denotes the sheaf of closed 1-forms in M.

Observe that since one has the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d} \Omega^1_{M,c} \longrightarrow 0$$

we have

$$H^1(M, \mathbb{C}) \longrightarrow H^1(M, \mathcal{O}) \longrightarrow H^1(M, \Omega^1_{M,c}) \longrightarrow H^2(M, \mathbb{C})$$

and then $dim H^1(M, \Omega^1_{M,c}) < \infty$. We conclude that $dim Ker \varphi = \infty$. Now, for every element $\sum_k \lambda_k f_{ij}^{(k)}$ of this kernel there are closed 1-forms β_i on U_i such that

$$\sum_k \lambda_k \frac{df_{ij}^{(k)}}{f_{ij}^{(k)}} = \beta_j - \beta_i$$

on $U_i \cap U_j$. Since $f_i^{(k)} = f_{ij}^{(k)} f_j^{(k)}$ this implies

$$\sum_k \lambda_k \frac{df_i^{(k)}}{f_i^{(k)}} + \beta_i = \sum_k \lambda_k \frac{df_j^{(k)}}{f_j^{(k)}} + \beta_j.$$

So we have a global meromorphic closed 1-form η (with poles on the S_k's that appears in the sum) given on U_i by

$$\eta = \sum_k \lambda_k \frac{df_i^{(k)}}{f_i^{(k)}} + \beta_i.$$

Now, using the relations on $U_i \cap U_j$ we have

$$\begin{aligned}
\frac{X_i(f_i^{(k)})}{f_i^{(k)}} &= g_{ij}\frac{X_j(f_{ij}^{(k)}f_j^{(k)})}{f_{ij}^{(k)}f_j^{(k)}} \\
&= g_{ij}\frac{X_j(f_j^{(k)})}{f_j^{(k)}} + g_{ij}\frac{X_j(f_{ij}^{(k)})}{f_{ij}^{(k)}}
\end{aligned}$$

and then

$$\begin{aligned}
\eta(X_i) &= \sum_k \lambda_k \frac{X_i(f_i^{(k)})}{f_i^{(k)}} + \beta_i(X_i) \\
&= \sum_k \lambda_k g_{ij}\frac{X_j(f_j^{(k)})}{f_j^{(k)}} + \sum_k \lambda_k g_{ij}\frac{X_j(f_{ij}^{(k)})}{f_{ij}^{(k)}} + g_{ij}\beta_i(X_j) \\
&= \sum_k \lambda_k g_{ij}\frac{X_j(f_j^{(k)})}{f_j^{(k)}} + g_{ij}[\sum_k \lambda_k \frac{df_{ij}^{(k)}}{f_{ij}^{(k)}} + \beta_i](X_j) \\
&= \sum_k \lambda_k g_{ij}\frac{X_j(f_j^{(k)})}{f_j^{(k)}} + g_{ij}\beta_j(X_j) \\
&= g_{ij}\eta(X_j).
\end{aligned}$$

Thanks to the $\mathcal{F}$-invariance of the S_k's, the local functions $\eta(X_i)$ are holomorphic and so one has an element $\{\eta(X_i)\} \in H^0(M, T^*\mathcal{F})$.

We have just constructed a map $\psi : \ker\varphi \to H^0(M, T^*\mathcal{F})$ and since $dim \ker\varphi = \infty$ we have closed meromorphic 1-forms η_1, η_2, η_3 with different set of poles and such that $\eta_r(X_i) \equiv 0$ $\forall i$ and $r = 1, 2, 3$ (in fact we have an infinite number of these forms).

If $\eta_1 \wedge \eta_2 \equiv 0$ there exists a rational non-constant function h such that $\eta_1 = h\eta_2$ and then

$$0 = d\eta_1 = dh \wedge \eta_2,$$

so dh defines the same foliation that η_2 and therefore is a first integral of $\mathcal{F}$.

If $\eta_1 \wedge \eta_2 \neq 0$ then $\eta_1 \wedge \eta_2 \wedge \eta_3 \equiv 0$ because this 3-form is zero on the tangent space of $\mathcal{F}$, in this case we have that $\eta_3 = f\eta_1 + g\eta_2$, and we can assume that f is not constant (f and g are rational functions). Therefore

$$\begin{aligned} & 0 = d\eta_3 = df \wedge \eta_1 + dg \wedge \eta_2 \\ \Rightarrow \quad & 0 = df \wedge \eta_1 \wedge \eta_2 \\ \Rightarrow \quad & df = f_1\eta_1 + g_1\eta_2 \end{aligned}$$

which implies that $df(X_i) \equiv 0$ and concludes the proof. □

As a consequence of this result we have

Corollary 4. *Let $\mathcal{F}$ be a second order differential equation on $\mathbb{P}^2$. If $\mathcal{F}$ admits infinitely many invariant horizontal surfaces, then $\mathcal{F}$ is the second order differential equation given by a pencil of webs.*

Proof. By the proposition we have that $\mathcal{F}$ is tangent to a pencil of surfaces in M and by hypothesis we can take two horizontal surfaces S_1, S_2 with classes $[S_i] = dh + k\check{h}$ as generators of this pencil. We conclude that $\mathcal{F}$ is the second order differential equation associated to the pencil of webs generated by the projection of the foliations on S_1 and S_2 given by $\mathcal{F}$. □

On the other hand we observe that the vertical foliation $\mathcal{V}$ of example 2.5 has infinitely many vertical invariant surfaces. As a consequence of proposition 2.11 we show that up to a singular set of codimension one, this is the only second order differential equation with this property (clearly any multiple of $\mathcal{V}$ also has infinitely many vertical invariant surfaces).

Corollary 5. *Let $\mathcal{F}$ be a second order differential equation such that $\mathcal{F}_{red}$ (the reduced foliation associated to $\mathcal{F}$) is not the vertical foliation $\mathcal{V}$. Then the number of vertical invariant surfaces of $\mathcal{F}$ is finite.*

Proof. If $\mathcal{F}$ has infinitely many vertical invariant surfaces then $\mathcal{F}$ is tangent to a pencil of vertical surfaces. This pencil is given locally in M by a 1-form which depends only of (x, y) and then $\mathcal{F}$ is given by a 2-form which is locally of the form $F(x, y, p)dx \wedge dy$. This concludes the proof. □

2.4 Intersection formulas

In order to understand the geometric meaning of $T\mathcal{F}$ we give in this section some formulas concerning its intersection with curves and surfaces.

2.4.1 Generically transverse surfaces

Let $V \subseteq M$ be a compact surface, possibly singular, such that each irreducible-component of V is not $\mathcal{F}$-invariant. We define the *tangency curve* between $\mathcal{F}$ and V as the divisor on V given locally by

$$tang(\mathcal{F}, V) = \{X(F)|_V = 0\}$$

where $\{F = 0\}$ is a local equation of V and X is a local holomorphic vector field generating $\mathcal{F}$.

Observe that this divisor can be defined for any complex compact subvariety $V \subseteq M$ of codimension 1 and any foliation by curves $\mathcal{F}$ on a complex manifold M (we do not need here the condition of tangency with $\mathcal{D}$). Analogously to [2], proposition 2 on page 23, one has.

Proposition 2.12. *The tangency divisor between $\mathcal{F}$ and V is given by*

$$tang(\mathcal{F}, V) = T^*\mathcal{F}|_V + N_V.$$

Proof. We choose an open covering $\mathcal{U} = \{U_i\}$ of M, holomorphic vector fields X_i on U_i defining $\mathcal{F}$ and holomorphic functions F_i on U_i defining V. On the intersections $U_i \cap U_j$ we have

$$\begin{aligned} X_i &= g_{ij} X_j, \\ F_i &= F_{ij} F_j \end{aligned}$$

where $g_{ij}, F_{ij} \in \mathcal{O}_M^*(U_i \cap U_j)$. Hence

$$X_i(F_i) = g_{ij} F_{ij} X_j(F_j) + g_{ij} F_j X_i(F_{ij})$$

and then $\{X_i(F_i)|_V\}$ gives a section of $(T^*\mathcal{F} \otimes \mathcal{O}_M(V))|_V$. $\square$

Example 2.13 (The inflection curve of a web). Let $\mathcal{W}$ be a k-web of degree d on $\mathbb{P}^2$ and assume $\mathcal{W}$ has (at most) a finite number of invariant lines. Then every irreducible component of the surface $S_{\mathcal{W}} \subseteq M$ is not invariant by $\mathcal{L}$ (lines). Since we know that the class of $S_{\mathcal{W}}$ is $[S_{\mathcal{W}}] = dh + k\check{h}$, we have

$$\begin{aligned} [tang(\mathcal{L}, S_{\mathcal{W}})] &= (-2h + \check{h}).(dh + k\check{h}) + (dh + k\check{h})^2 \\ &= (d^2 - d)h^2 + (k^2 + k)\check{h}^2 + (2dk + d - 2k)h\check{h}. \end{aligned}$$

We define the inflection curve of $\mathcal{W}$ as the projection C of this divisor. Thus

$$deg(C) = k^2 + (2d - 1)k + d.$$

Observe that in the case $d = 1$,

$$deg(C) = k^2 + k + 1$$

which is the number of invariant lines of a generic k-web of degree 1.

On the other hand, in the particular case of foliations, k=1, we obtain the well-known bound for the number of invariant lines of a foliation of degree d,

$$deg(C) = 3d.$$

Example 2.14. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ of degree d with isolated and non-degenerate singularities. Consider $S_{\mathcal{F}} \subseteq M$ the surface associated to the foliation and $\mathcal{V}$ the vertical foliation. Then clearly $S_{\mathcal{F}}$ is not $\mathcal{V}$-invariant and then we can apply the proposition

$$\begin{aligned} [tang(\mathcal{V}, S_{\mathcal{F}})] &= (h - 2\check{h}).(dh + \check{h}) + (dh + \check{h})^2 \\ &= (d^2 + d + 1)h^2. \end{aligned}$$

Observe now that this divisor is formed by the fibers of π over the singular points of $\mathcal{F}$.

We recover the well-known fact that the number of singularities of $\mathcal{F}$ is

$$d^2 + d + 1.$$

2.4.2 An application

As in example 2.13, we can look for the inflection curve of a foliation in a complex surface S, that is, the curve where the foliation has high-order tangencies with a system of "lines".

We can adopt the following definition. Let S be a complex surface such that the universal covering $\widetilde{S}$ of S is an open set of $\mathbb{P}^2$ and the group of deck transformations acts on $\widetilde{S}$ by automorphisms of $\mathbb{P}^2$, then the lines on S will be the projection (by the universal covering map) of the lines on $\mathbb{P}^2$.

Let us take for example $S = (\mathbb{C}^*)^2$, in this case the universal covering is given by the map $\mathbb{C}^2 \to (\mathbb{C}^*)^2$, $(z, w) \mapsto (e^z, e^w)$ and then the lines on S are the leaves of the foliations $\mathcal{L}_{[\lambda:\mu]} = [\lambda y dx + \mu x dy = 0]$.

The inflection curve can be used to bound the order of Completely Decomposable Quasi Linear (CDQL) exceptional webs on this surface, see for instance [20, theorem 4], as follows. Let $\mathcal{W}$ be a linear completely decomposable k-web on S and $\mathcal{F}$ a nor-linear foliation. Then consider the rational map $P_{\mathcal{F}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ characterized by the property

$$P_{\mathcal{F}}^{-1}([\lambda : \mu]) = tang(\mathcal{F}, \mathcal{L}_{[\lambda:\mu]}).$$

The first step would be to show that if the web $\mathcal{W} \boxtimes \mathcal{F}$ is exceptional then the map $P_{\mathcal{F}}$ has at least k completely decomposable (that is, product of lines) fibers and thus we need to bound the number of such fibers. We will prove here that the number of completely decomposable fibers is at most 5.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ with isolated singularities and which is not formed by solutions of the foliations $\lambda y dx + \mu x dy$, $[\lambda : \mu] \in \mathbb{P}^1$.

Consider $\mathcal{H}$ the second order differential equation given by the pencil of foliations $\lambda y dx + \mu x dy$, $[\lambda : \mu] \in \mathbb{P}^1$, then we have $T^*\mathcal{H} = \mathcal{O}_M(1, 1)$ (see (2.5)). By our hypothesis the surface $S_{\mathcal{F}}$ is not $\mathcal{H}$ invariant, hence

$$[tang(\mathcal{H}, S_{\mathcal{F}})] = (deg(\mathcal{F})^2 + 4deg(\mathcal{F}) + 1)h^2 + (3deg(\mathcal{F}) + 3)\check{h}^2.$$

and then denoting by C the projection of this divisor we obtain

$$deg(C) = 3deg(\mathcal{F}) + 3.$$

With this at hand we get the following result.

Theorem 2.15. *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ a pencil of curves of degree d which is not formed by solutions of the pencil of foliations $\mathcal{P} = \{\lambda y dx + \mu x dy, [\lambda : \mu] \in \mathbb{P}^1\}$. Then the number of fibers invariant by $\mathcal{P}$ is at most 5.*

Proof. Let k be the number of fibers invariant by $\mathcal{P}$. If we denote by $\mathcal{F}$ the associated foliation, by $\widetilde{Q} = \prod \alpha_i^{m_i}$ the product of these fibers and by $Q = \prod \alpha_i$ the reduced polynomial with the same zero set, then we have

$$deg(\mathcal{F}) \leq 2d - 2 - deg(\frac{\widetilde{Q}}{Q}).$$

On the other hand it is clear that the zero set of Q is contained in C, hence $\widetilde{Q}$ divides $\frac{\widetilde{Q}}{Q}C$ and then

$$\begin{aligned} kd &\leq deg(\frac{\widetilde{Q}}{Q}) + deg(C) \\ &\leq deg(\frac{\widetilde{Q}}{Q}) + 3(2d - 2 - deg(\frac{\widetilde{Q}}{Q})) + 3 \\ &= 6d - 3 - 2deg(\frac{\widetilde{Q}}{Q}). \end{aligned}$$

Therefore

$$k \leq 6 - \frac{3}{d} - \frac{2}{d} deg(\frac{\widetilde{Q}}{Q}) < 6.$$

□

2.4.3 Non invariant curves

Let us consider now $\widetilde{C} \subseteq M$ a smooth compact curve, tangent to $\mathcal{D}$, such that each component of $\widetilde{C}$ is not $\mathcal{F}$-invariant and suppose that the codimension of the singular set of $\mathcal{F}$ is at least 2. Then we have the following diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & T\mathcal{F}|_{\widetilde{C}} \arrow[d] \arrow[dr, "\sigma"] & & \\ 0 \arrow[r] & T\widetilde{C} \arrow[r] & \mathcal{D}|_{\widetilde{C}} \arrow[r] & \mathcal{N}_{\widetilde{C}} \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{N}_{\widetilde{C}}$ is the normal bundle of $\widetilde{C}$ in $\mathcal{D}$.

Observe that the map σ vanishes exactly at the points where $\mathcal{F}$ is tangent to $\widetilde{C}$. We define the tangency index between $\mathcal{F}$ and $\widetilde{C}$ at a point $x \in \widetilde{C}$ as the vanishing order of the section induced by σ, $\mathcal{O}_{\widetilde{C}} \to T^*\mathcal{F}|_{\widetilde{C}} \otimes \mathcal{N}_{\widetilde{C}}$ (which we still denote by σ) at x

$$tang(\mathcal{F}, \widetilde{C}, x) = ord_x(\sigma).$$

Hence we can set

$$tang(\mathcal{F}, \widetilde{C}) = \sum_{x \in \widetilde{C}} tang(\mathcal{F}, \widetilde{C}, x).$$

In order to compute this number we have the following proposition.

Proposition 2.16. *The tangency index between $\mathcal{F}$ and a smooth compact curve $\widetilde{C}$ tangent to $\mathcal{D}$ is*

$$tang(\mathcal{F}, \widetilde{C}) = T^*\mathcal{F}.\widetilde{C} + det(\mathcal{D}).\widetilde{C} - \chi(\widetilde{C})$$

where $\chi(\widetilde{C})$ is the Euler characteristic of $\widetilde{C}$.

Proof. We just need to take Chern classes in the previous sequence. □

If one writes locally $\mathcal{F}$ induced by $X = B\frac{\partial}{\partial x} + pB\frac{\partial}{\partial y} + A\frac{\partial}{\partial p}$ and $\widetilde{C}$ parametrized by $(\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_1'\gamma_3 = \gamma_2'$, then the points of tangency are exactly the points where $B(\gamma_1, \gamma_2, \gamma_3)\gamma_3' - A(\gamma_1, \gamma_2, \gamma_3)\gamma_1'$ vanishes and the tangency index at these points is the order of vanishing of this function; in particular $tang(\mathcal{F}, \widetilde{C}, x) \geq 0$.

Now we can give a geometric interpretation of the bidegree of a second order differential equation.

Corollary 6. *Let $\mathcal{F}$ be a saturated second order differential equation of bidegree (a, b), which is neither $\mathcal{L}$ (lines) nor $\mathcal{V}$ (fibers) and denote by F and $\widetilde{l}$ a fiber of π and the lifting of a line in $\mathbb{P}^2$ respectively, then*

$$a = tang(\mathcal{F}, \widetilde{l}) + 1,$$
$$b = tang(\mathcal{F}, F) + 1.$$

In particular $a \geq 1$ and $b \geq 1$.

Proof. We have just to observe that $\chi(F) = \chi(\widetilde{l}) = 2$ and

$$[F] = h^2, \quad [\widetilde{l}] = \check{h}^2,$$

(because $\widetilde{l}$ is a fiber of $\check{\pi}$) and apply the proposition. $\square$

Example 2.17. We obtain (again) the class of the cotangent bundle of the second order differential equation $\mathcal{L}$ of the example 2.4.

Let us write $T^*\mathcal{L} = \mathcal{O}_M(a, b)$ and apply proposition 2.16 for $\widetilde{C} = F$, a fiber of π, to obtain

$$0 = tang(\mathcal{L}, F) = (ah + b\check{h}).h^2 + (h + \check{h}).h^2 - \chi(F) = b - 1.$$

On the other hand, observing that the tangency curve between $\mathcal{L}$ and $H = \pi^{-1}(l)$, where l is a line on $\mathbb{P}^2$, is exactly $\widetilde{l}$, the lifting of l, we can apply proposition (2.12) to obtain

$$\check{h}^2 = [tang(\mathcal{L}, H)] = (ah + \check{h}).h + h^2 = (a + 2)h^2 + \check{h}^2.$$

We have just proved that $T^*\mathcal{L} = \mathcal{O}_M(-2, 1)$.

Example 2.18. In a similar way we obtain the class of the cotangent bundle of the second order differential equation $\mathcal{V}$ of the example 2.5.

Let us write $T^*\mathcal{V} = \mathcal{O}_M(a, b)$ and apply proposition 2.16 for $\widetilde{C} = \widetilde{l}$, the lifting of a line in $\mathbb{P}^2$, to obtain

$$0 = tang(\mathcal{V}, \widetilde{l}) = (ah + b\check{h}).\check{h}^2 + (h + \check{h}).\check{h}^2 - \chi(\widetilde{l}) = a - 1.$$

On the other hand, observe that the tangency curve between $\mathcal{V}$ and $S = \check{\pi}^{-1}(\check{l})$, where $\check{l}$ is a line on $\check{\mathbb{P}}^2$, is exactly the fiber of π over the point in $\mathbb{P}^2$ corresponding to $\check{l}$, then we can apply proposition (2.12) to obtain

$$h^2 = [tang(\mathcal{V}, S)] = (h + b\check{h}).\check{h} + \check{h}^2 = h^2 + (b + 2)\check{h}^2.$$

Hence $T^*\mathcal{L} = \mathcal{O}_M(-2, 1)$.

2.4.4 Tangencies between differential equations

Take now $\mathcal{F}_1$ and $\mathcal{F}_2$ two second order differential equations on S (not necessarily $\mathbb{P}^2$). We define the *tangency divisor* between $\mathcal{F}_1$ and $\mathcal{F}_2$ as the divisor on M given locally by

$$tang(\mathcal{F}_1, \mathcal{F}_2) = \{X_1 \wedge X_2 = 0\}$$

where X_i is a local holomorphic vector field generating $\mathcal{F}_i$. Note that for an arbitrary pair of one dimensional foliations in M, the zero set of $X_1 \wedge X_2$ is not a divisor, but since second order differential equations are given by global sections $X_{\mathcal{F}_i} \in H^0(M, \mathcal{D} \otimes T^*\mathcal{F}_i)$ we have that

$$X_{\mathcal{F}_1} \wedge X_{\mathcal{F}_2} \in H^0(M, det(\mathcal{D}) \otimes T^*\mathcal{F}_1 \otimes T^*\mathcal{F}_2).$$

We have just proved the following proposition.

Proposition 2.19. *The tangency divisor between two second order differential equations is given by*

$$tang(\mathcal{F}_1, \mathcal{F}_2) = c_1(det(\mathcal{D}) \otimes T^*\mathcal{F}_1 \otimes T^*\mathcal{F}_2).$$

As an example we can give another proof of corollary 5.

Example 2.20. Let $\mathcal{F}$ be a second order differential equation such that $\mathcal{F}_{red}$ (the reduced foliation associated to $\mathcal{F}$) is not the vertical foliation $\mathcal{V}$ and with bidegree (a, b). Then $X_{\mathcal{F}} \wedge X_{\mathcal{V}} \neq 0$, where $X_{\mathcal{F}}$ is the vector field defining $\mathcal{F}$. Thus we can use the proposition to obtain

$$\begin{aligned} [tang(\mathcal{F}, \mathcal{V})] &= (h + \check{h}) + (ah + b\check{h}) + (h - 2\check{h}) \\ &= (a + 2)h + (b - 1)\check{h}. \end{aligned}$$

Clearly every vertical $\mathcal{F}$-invariant surface is contained in this divisor. This proves that the number of vertical invariant surfaces is finite.

2.5 Nets and Second Order Differential Equations

In [21] the authors study completely decomposable (i.e. product of hyperplanes not necessarily different) fibers of pencils of hypersurfaces on $\mathbb{P}^n$ and associated codimension one foliations. Their main result gives an upper bound for the number k of these fibers that depends only on n. In the particular case of pencil of curves on $\mathbb{P}^2$ they obtain the following theorem.

Theorem 2.21 ([21]). *If $\mathcal{P}$ is a pencil of curves of degree d on $\mathbb{P}^2$ with irreducible generic fiber and k is the number of completely decomposable fibers of $\mathcal{P}$ then $k \leq 5$ if $\mathcal{P}$ is not a pencil of lines.*

Proof. Let us assume that the pencil is not a pencil of lines and consider $\mathcal{F}$ the foliation on $\mathbb{P}^2$ associated to the pencil $\mathcal{P}$ and denote by $\widetilde{Q} = \prod \alpha_i^{m_i}$ the product of the completely decomposable fibers and by $Q = \prod \alpha_i$ the reduced polynomial with the same zero set. Observe that by our hypothesis the foliation $\mathcal{G}$ is not a pencil of lines and then we can apply example 2.13, thus if we denote by C the projection of the divisor $tang(\mathcal{L}, S_{\mathcal{F}})$ we know that

$$deg(C) = 3deg(\mathcal{F}) \leq 3(2d - 2 - deg(\frac{\widetilde{Q}}{Q})).$$

On the other hand it is clear that the zero set of Q is contained in C, hence $\widetilde{Q}$ divides $\frac{\widetilde{Q}}{Q}C$ and then

$$\begin{aligned} kd &\leq deg(\frac{\widetilde{Q}}{Q}) + deg(C) \\ &\leq deg(\frac{\widetilde{Q}}{Q}) + 3(2d - 2 - deg(\frac{\widetilde{Q}}{Q})) \\ &= 6d - 6 - 2deg(\frac{\widetilde{Q}}{Q}). \end{aligned}$$

Therefore

$$k \leq 6 - \frac{6}{d} - \frac{2}{d}deg(\frac{\widetilde{Q}}{Q}) < 6.$$

□

We remark that S. Yuzvinsky proves in [25] that actually $k \leq 4$.

Consider now a net on $\mathbb{P}^2$, that is, a linear system of dimension 2

$$\mathcal{P} = \mathbb{P}(\langle F, G, H \rangle) \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$$

formed by curves of degree d and such that F, G and H do not have a common factor. Denote by $\Sigma \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ the subvariety of completely decomposable curves of degree d. Observe that in this case, $\mathcal{P} \cap \Sigma$ could have components of dimension 0, 1 or 2. We shall assume that $\mathcal{P} \nsubseteq \Sigma$ (this is not always the case, for example $F = x^2$, $G = y^2$, $H = xy$).

Theorem 2.22. *In the previous situation we have*

1. *If $\mathcal{P} \cap \Sigma = C$, where C is a curve, then $deg(C) \leq 3$.*
2. *If $\mathcal{P} \cap \Sigma = \{p_1, \dots, p_k\}$ is a set of points, and the number of lines contained in fibers of $\mathcal{P}$ is finite then $k \leq 31$.*

Proof. Let us take the foliations (of degree 2d-2)

$$\mathcal{F}_1 = [\omega_1 = FdG - GdF = 0],$$
$$\mathcal{F}_2 = [\omega_2 = FdH - HdF = 0].$$

The fibers of $\mathcal{P}$ are exactly the leaves of the pencil of foliations generated by $\mathcal{F}_1$ and $\mathcal{F}_2$. To see this, it is enough to observe that for every $b, c \in \mathbb{C}$ we have

$$d\left(\frac{bG + cH}{F}\right) = \frac{1}{F^2}(b\omega_1 + c\omega_2).$$

Thus one can think on a net as a particular case of second order differential equation. Set $\mathcal{H}$ the second order differential equation associated to the linear system $\mathcal{P}$.

Recall from example 2.8 that $T^*\mathcal{H} = \mathcal{O}(4d-5-l, 1)$ where l appears due to the singular set.

By hypothesis we know that $\mathcal{H}$ is not the equation of the lines $\mathcal{L}$, thus we can apply proposition 2.19 to obtain

$$[tang(\mathcal{H}, \mathcal{L})] := [S] = (4d - 6 - l)h + 3\check{h}.$$

Suppose first that $\mathcal{P} \cap \Sigma = C \subseteq \mathbb{P}(\langle F, G, H\rangle)$ is a curve of degree k. We will prove that $k \leq 3$.

Consider the rational map $\phi = (F : G : H) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. If it is degenerate then $D = \overline{Im\phi}$ is an irreducible curve. For each point $p \in C$ we have the completely decomposable curve

$$C_p = \{a(p)F + b(p)G + c(p)H = 0\} = \phi^{-1}(L_p)$$

where L_p is the line $\{a(p)x + b(p)y + c(p)z = 0\} \subseteq \mathbb{P}^2$. Thus

$$\begin{aligned} C_p &= \phi^{-1}(L_p) \\ &= \phi^{-1}(L_p \cap D) \\ &= \phi^{-1}(q_1(p)) \cup \ldots \cup \phi^{-1}(q_r(p)). \end{aligned}$$

Varying p over C we obtain a curve in D such that the inverse image of the points of this curve are completely decomposable, but since D is irreducible, this implies that every fiber of $\mathcal{P}$ is completely decomposable, a contradiction.

Hence the map ϕ is dominant and thus for a generic point $p \in \mathbb{P}^2$, the line $l_p = \{[a : b : c] \in \mathbb{P}^2, aF(p) + bG(p) + cH(p) = 0\}$ intersects the curve C in exactly k points which correspond to fibers of $\mathcal{P}$ that are completely decomposable passing through p. Since p is generic, these curves have different directions in p. In this way we obtain a k-web $\mathcal{W}$ by lines formed by the completely decomposable fibers of our pencil. Thus it is clear that $S_{\mathcal{W}} \subseteq S$ and therefore $k \leq 3$.

Assume now that $\mathcal{P} \cap \Sigma = \{p_1, \ldots, p_k\}$, that is, there are exactly k fibers of $\mathcal{P}$ which are completely decomposable, and that the number of lines contained in fibers of $\mathcal{P}$ is finite.

In this case we have that S is not $\mathcal{L}$-invariant, otherwise S would be $\mathcal{H}$-invariant, contradicting our hypothesis. Thus we can apply proposition 2.12

$$[tang(\mathcal{L}, S)] = (4d - 6 - l)(4d - 8 - l)h^2 + (28d - 48 - 7l)h\check{h} + 12\check{h}^2.$$

Let E be the projection by π of this divisor, then we obtain that

$$deg(E) = 28d - 36 - 7l.$$

Denote by $\widetilde{Q} = \prod \alpha_i^{m_i}$ the product of the completely decomposable fibers and by $Q = \prod \alpha_i$ the reduced polynomial with the same zero set. It is clear that the zero set of Q is contained in E, hence $\widetilde{Q}$ divides $\frac{\widetilde{Q}}{Q}E$ and then

$$\begin{aligned} kd &\leq deg(\frac{\widetilde{Q}}{Q}) + deg(E) \\ &= 28d - 36 + (deg(\frac{\widetilde{Q}}{Q}) - 7l). \end{aligned}$$

On the other hand we know that

$$deg(\frac{\widetilde{Q}}{Q}) = \sum (m_i - 1).$$

If $m_i \geq 2$, since the number of fibers containing lines is finite then $\alpha_i^{m_i}$ is a factor of one fiber and in this case is easy to see that $\alpha_i^{m_i-1}$ divides $\omega_1 \wedge \omega_2$, thus

$$deg(\frac{\widetilde{Q}}{Q}) \leq 4d - 3.$$

So we have

$$kd \leq 32d - 39 \Rightarrow k \leq 32 - \frac{39}{d} < 32$$

and this concludes the proof of the theorem. □

Remark 2.23. We are imposing in the part two of the proposition the condition to have finitely many invariant lines to conclude that the surface of tangency S is not $\mathcal{L}$-invariant. We would like to have the result without this hypothesis but it is not clear for us that the fact of have finitely many completely decomposable fibers implies this condition.

Example 2.24. Consider a net in the space of conics in $\mathbb{P}^2$

$$\mathcal{P} \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathbb{P}^5.$$

Since we can see the conics as symmetric matrices and the decomposable ones are those with zero determinant, one has that the algebraic subset Σ of completely decomposable elements is a 4-dimensional variety of degree 3 which is the image of the natural map

$$\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5.$$

Thus if $\mathcal{P} \cap \Sigma = C$ is a curve, we have that $deg(C) = 3$.

Chapter 3

Inexistence of algebraic solutions

The study of the space of foliations on projective spaces was initiated by Jouanolou in [13]. One of the most important results of Jouanolou's monography states that a very generic holomorphic foliation of the projective plane, of degree at least 2, does not have any invariant algebraic curves. This result was extended in various ways, see [6], [14], [16] and [22]. For example, in [6] the authors prove that over a smooth complex projective variety of dimension greater than or equal to 2, a very generic holomorphic foliation of dimension one with sufficiently ample cotangent bundle has no proper invariant algebraic subvarieties of nonzero dimension. On the other hand, in [16] the author gives a different proof of Jouanolou's theorem following the ideas of [6] and restricting to $\mathbb{P}^2$.

Since we can think in holomorphic foliations as first order differential equations of degree one, we are tempted to believe that the same assertion holds true for first order differential equations of any degree (webs) or for higher order differential equations.

The main theorem of this thesis shows that the assertion is true for second order differential equations on $\mathbb{P}^2$, that is, we prove that a generic second order differential equation has no algebraic solutions when the bidegree (a, b) satisfies $a \geq 3$. Our result is sharp in the sense that the second order differential equations not satisfying the above mentioned conditions always have invariant algebraic solutions. We also obtain a similar result for second order differential equations on arbitrary projective surfaces.

We also prove an analogue of Jouanolou's for k-webs of degree d (first order differential equations of degree k) on $\mathbb{P}^2$ when $d \geq 2$; and for webs with sufficiently ample normal bundle on arbitrary projective surfaces.

3.1 The space of second order differential equations

We denote by $\mathcal{E}(a,b) = \mathbb{P}H^0(M, \mathcal{D} \otimes \mathcal{O}_M(a,b))$ the space of second order differential equations with bidegree (a,b) in $\mathbb{P}^2$. Let us consider the following maps

$$R_1(a,b) : \mathbb{P}H^0(M, \mathcal{O}_M(a,b)) \to \mathcal{E}(a-2, b+1), \quad R_1(F) = F.X_{\mathcal{L}},$$
$$R_2(a,b) : \mathbb{P}H^0(M, \mathcal{O}_M(a,b)) \to \mathcal{E}(a+1, b-2), \quad R_2(F) = F.X_{\mathcal{V}}$$

where $\mathcal{L}$ and $\mathcal{V}$ are the foliations of examples 2.4, 2.5 respectively.

Remark 3.1. Observe that the class of a surface in M, $[S] = ah + b\check{h}$ satisfies $a, b \geq 0$, because $a = S.\widetilde{l}$, $b = S.F$.

Using this observation one can describe the space $\mathcal{E}(a,b)$ for some bidegrees:

Proposition 3.2. *The map $R_1(a,b)$ (respectively $R_2(a,b)$) is an isomorphism for $0 \leq a \leq 2$, $b \geq 0$ (respectively for $0 \leq b \leq 2$, $a \geq 0$). In other words*

$$\mathcal{E}(a-2, b+1) = ImR_1(a,b) \quad \textit{for} \quad 0 \leq a \leq 2,\ b \geq 0;$$
$$\mathcal{E}(a+1, b-2) = ImR_2(a,b) \quad \textit{for} \quad 0 \leq b \leq 2,\ a \geq 0.$$

Proof. If one takes $\mathcal{F} \in \mathcal{E}(a-2, b+1)$, for some $0 \leq a \leq 2$, then by Remark 3.1 the bidegree of $\mathcal{F}_{red}$ is (c,d), with $c \leq 0$. Then by corollary 6 we conclude that $\mathcal{F}_{red} = \mathcal{L}$ (lines). The proof of the other cases is similar. □

The dimension of these spaces is given by the following proposition.

Proposition 3.3. *For any $a, b \geq 0$, the equalities*

$$h^0(\mathcal{O}_M(a,b)) = \frac{(a+1)(b+1)(a+b+2)}{2}$$

and

$$h^i(\mathcal{O}_M(a,b)) = 0, \text{ for } i \geq 1,$$

hold true.

Proof. Let us denote by X the product $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ and consider $M = \mathbb{P}(T\mathbb{P}^2)$ as the incidence variety on X. Then clearly we have

$$0 \longrightarrow \mathcal{O}_X(-1,-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_M \longrightarrow 0 \tag{3.1}$$

(here the definition of $\mathcal{O}_X(a,b)$ is the obvious one), and therefore

$$0 \longrightarrow \mathcal{O}_X(a-1,b-1) \longrightarrow \mathcal{O}_X(a,b) \longrightarrow \mathcal{O}_M(a,b) \longrightarrow 0.$$

To conclude is enough to observe that

$$h^0(\mathcal{O}_X(a,b)) = \frac{(a+1)(a+2)(b+1)(b+2)}{4}, \text{ for } a, b \geq -1$$

and

$$h^i(\mathcal{O}_X(a,b)) = 0, \text{ for } i \geq 1 \text{ and } a, b \geq -1.$$

which follows from

$$\begin{aligned} H^i(X, \mathcal{O}_X(a,b)) &= H^i(X, \pi^*(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes \check{\pi}^*(\mathcal{O}_{\check{\mathbb{P}}^2}(b))) \\ &= \bigoplus_{j+k=i} (H^j(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a)) \otimes H^k(\check{\mathbb{P}}^2, \mathcal{O}_{\check{\mathbb{P}}^2}(b))). \end{aligned}$$

□

For the other cases, one can also calculate the dimension of the space of differential equations.

Proposition 3.4. *For $a, b \geq 1$ we have*

$$\dim \mathcal{E}(a,b) = \frac{1}{2}(2a^2b + 2ab^2 + 3a^2 + 3b^2 + 12ab + 9a + 9b) - 1.$$

Proof. From sequence (2.3) one gets

$$0 \longrightarrow \mathcal{O}_M(a+2,b-1) \longrightarrow \mathcal{D}\otimes\mathcal{O}_M(a,b) \longrightarrow \mathcal{O}_M(a-1,b+2) \longrightarrow 0$$

and then

$$0 \to H^0(M,\mathcal{O}_M(a+2,b-1)) \to H^0(M,\mathcal{D}\otimes\mathcal{O}_M(a,b)) \to H^0(M,\mathcal{O}_M(a-1,b+2)) \to 0.$$

The proposition follows from proposition 3.3. □

When $a, b \geq 1$ one has the inclusions

$$\begin{aligned} ImR_1(a+2,b-1) &= A \subseteq \mathcal{E}(a,b), \\ ImR_2(a-1,b+2) &= B \subseteq \mathcal{E}(a,b). \end{aligned}$$

After counting dimensions, we conclude that $\mathcal{E}(a,b)$ is covered by the lines joining points of A and B, i.e. $\mathcal{E}(a,b) = Join(A,B)$.

Proposition 3.5. *Every element of $\mathcal{E}(a,b)$ is a linear combination of an element of A and an element of B.*

Proof. It is enough to observe that A and B are disjoint linear subspaces of $\mathcal{E}(a,b)$ and that

$$\begin{aligned} dim(A) + dim(B) &= h^0(\mathcal{O}_M(a+2,b-1)) - 1 + h^0(\mathcal{O}_M(a-1,b+2)) - 1 \\ &= \frac{(a+3)(b)(a+b+3)}{2} + \frac{(a)(b+3)(a+b+3)}{2} - 2 \\ &= dim\mathcal{E}(a,b) - 1. \end{aligned}$$

□

Remark 3.6. Since we have an identification between $\mathbb{P}(H^0(M,\mathcal{O}_M(d,k)))$ and the space of k-webs of degree d in $\mathbb{P}^2$ (see [24]), then in the particular case when a (respectively b) is equal to 1 one has that B (respectively A) can be identified with the space of $(b+2)$-webs of degree 0 (respectively curves of degree $a+2$).

Proposition 3.7. *The following assertions hold true when $a, b \geq 1$.*

1. *Every element of $\mathcal{E}(1,b)$ has a one parameter family of invariant curves which are lifts of lines on $\mathbb{P}^2$.*

2. *The generic element of* $\mathcal{E}(2,b)$ *has* $(b+2)^2+(b+2)+1$ *invariant curves which are lifts of lines on* $\mathbb{P}^2$.

3. *Every element of* $\mathcal{E}(a,1)$ *has a one parameter family of invariant curves which are lifts of lines on* $\check{\mathbb{P}}^2$ *(that is, fibers of* π*).*

4. *The generic element of* $\mathcal{E}(a,2)$ *has* $(a+2)^2+(a+2)+1$ *invariant curves which are lifts of lines on* $\check{\mathbb{P}}^2$.

Proof. Let us consider, for an element $\mathcal{F}$ in $\mathcal{E}(a,b)$ which is not in A or B, the following divisors in M

$$tang(\mathcal{F},\mathcal{L}) = (a-1)h + (b+2)\check{h}$$
$$tang(\mathcal{F},\mathcal{V}) = (a+2)h + (b-1)\check{h}.$$

Therefore if $a=1$ we obtain

$$tang(\mathcal{F},\mathcal{L}) = (b+2)\check{h}$$

which corresponds to a $(b+2)$-web $\mathcal{W}$ of degree 0 in $\mathbb{P}^2$. The lifting of each leaf of $\mathcal{W}$ is a leaf of $\mathcal{L}$ (because is the lifting of a line) and since is in the tangency divisor, it is also a leaf of $\mathcal{F}$. So $\mathcal{F}$ has a one-parameter family of lines which are solutions. Observe also that

$$A = ImR_1(3,b-1)$$
$$B = ImR_2(0,b+2)$$

so, the assertion is true for all the elements of $\mathcal{E}(1,b)$. This prove the first part of the proposition.

Consider now the case $a=2$. In this case

$$tang(\mathcal{F},\mathcal{L}) = h + (b+2)\check{h}$$

corresponds to a $(b+2)$-web $\mathcal{W}$ of degree 1 in $\mathbb{P}^2$. Since a generic $(b+2)$-web of degree 1 has $(b+2)^2+(b+2)+1$ invariant lines (see section 3.2), and the lifting of these lines are $\mathcal{L}$-invariant, we would like to say the same for the generic element of $\mathcal{E}(2,b)$ (note again that the assertion is true for the elements of A and B). We need the following lemma.

Lemma 3.8. *The following maps*

$$T_1 : H^0(M, \mathcal{D} \otimes \mathcal{O}_M(a,b)) \to H^0(M, O_M(a-1, b+2)), \quad T_1(X) = X \wedge X_{\mathcal{L}}$$
$$T_2 : H^0(M, \mathcal{D} \otimes \mathcal{O}_M(a,b)) \to H^0(M, O_M(a+2, b-1)), \quad T_2(X) = X \wedge X_{\mathcal{V}}$$

are surjective.

Proof. We do the proof only for T_1, the same argument works for T_2. Observe first that

$$ker(T_1) = \{F X_{\mathcal{L}} : F \in H^0(M, \mathcal{O}_M(a+2, b-1))\}.$$

Therefore, using propositions 3.3 and 3.4 we obtain

$$dim(Im T_1) = dim H^0(M, O_M(a-1, b+2))$$

and this conclude the proof. □

Using this lemma we deduce that the generic element of $\mathcal{E}(2,b)$ has $(b+2)^2 + (b+2) + 1$ invariant curves which are lifts of lines on $\mathbb{P}^2$, so we have the second assertion.

Since

$$tang(\mathcal{F}, \mathcal{V}) = (a+2)h + (b-1)\check{h}.$$

we can do the same analysis in the cases when b is equal to 1 or 2 and we are done. □

For the case $a \geq 3$ we have our main result (Theorem 3 of the introduction).

Theorem 3.9. *A generic second order differential equation of bidegree (a,b) with $a \geq 3$ has no invariant algebraic curves which are lifts of curves on $\mathbb{P}^2$. Moreover, when $a, b \geq 3$, the generic second order differential equation of bidegree (a,b) does not admit any algebraic solution.*

For the proof we need an analogous result for k-webs in $\mathbb{P}^2$ which we explain in the next section.

3.2 Webs without algebraic leaves on $\mathbb{P}^2$

We recall that a global k-web $\mathcal{W}$ on a surface S is given by an open covering $\mathcal{U} = \{U_i\}$ of S and k-symmetric 1-forms $\omega_i \in \mathrm{Sym}^k\Omega^1_S(U_i)$ such that for each non-empty intersection $U_i \cap U_j$ of elements of $\mathcal{U}$ there exists a non-vanishing function $g_{ij} \in \mathcal{O}^*_S(U_i \cap U_j)$ such that $\omega_i = g_{ij}\omega_j$.

In the case of $\mathbb{P}^2$ we know that $\mathbb{W}(k,d) = \mathbb{P}(W(k,d))$, where $W(k,d) = H^0(\mathbb{P}^2, \mathrm{Sym}^k\Omega^1_{\mathbb{P}^2}(d+2k))$, is the space of k-webs of degree d.

3.2.1 Invariant algebraic curves

Let $\mathcal{W}$ be a k-web of degree d in $\mathbb{P}^2$ defined by ω and let $C \subseteq \mathbb{P}^2$ be an irreducible algebraic curve. As in the case of lines we say that C is $\mathcal{W}$-invariant if $i^*\omega \equiv 0$ where i is the inclusion of the smooth part of C into $\mathbb{P}^2$.

Observe that we have an isomorphism between $\mathbb{P}H^0(\mathbb{P}^2, \mathrm{Sym}^k T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(d-k))$ and $\mathbb{W}(k,d)$ giving by the contraction with the volume form of $\mathbb{P}^2$, locally described as follows. For a local k-symmetric vector field

$$X = \sum a_{ij} \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j$$

we associate the local k-symmetric form

$$\omega = \sum a_{ij} \left(i_{\frac{\partial}{\partial x}}(dx \wedge dy)\right)^i \left(i_{\frac{\partial}{\partial y}}(dx \wedge dy)\right)^j .$$

Therefore the web $\mathcal{W}$ is defined by an element $X_{\mathcal{W}} \in \mathbb{P}H^0(\mathbb{P}^2, \mathrm{Sym}^k T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(d-k))$.

Let us assume that C is given by the irreducible homogenous polynomial F of degree r. Then C is $\mathcal{W}$-invariant if and only if there exist a homogenous polynomial H of degree $d + k(r-1) - r$ such that

$$X_{\mathcal{W}}(F) = FH \tag{3.2}$$

where $X_{\mathcal{W}}(F)$ is the application of dF to $X_{\mathcal{W}}$.

Remark 3.10. An important fact about equation (3.2) is that it still works for reducible curves, i.e. if the decomposition of the curve is $F = F_1^{n_1} \ldots . F_k^{n_k}$, then the equation (3.2) holds true if and only if each F_j defines a $\mathcal{W}$-invariant curve.

Consider now the following set:

$$\mathcal{C}(r) = \{\mathcal{W} \in \mathbb{W}(k,d) / \exists \, curve \, of \, degree \, r \, \mathcal{W} - invariant\}.$$

We have the following proposition.

Proposition 3.11. *The set $\mathcal{C}(r)$ is an algebraic closed subset of $\mathbb{W}(k,d)$.*

Proof. Denote by $S_r = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$ and consider

$$\mathcal{Z}(r) \subseteq \mathbb{P}(W(k,d) \times S_{d+k(r-1)-r}) \times \mathbb{P}(S_r)$$

the subset defined by

$$\mathcal{Z}(r) = \{([(\mathcal{W}, H)], [F]) / X_{\mathcal{W}}(F) - FH = 0\}.$$

Observe that the rational map

$$\Pi : \mathbb{P}(W(k,d) \times S_{d+k(r-1)-r}) \times \mathbb{P}(S_r) \dashrightarrow \mathbb{W}(k,d) \times \mathbb{P}(S_r)$$

is regular over $\mathcal{Z}(r)$ and then takes it onto the closed set $\Sigma(r) \subseteq \mathbb{W}(k,d) \times \mathbb{P}(S_r)$ formed by the pairs $(\mathcal{W}, [F])$ such that the curve defined by F is invariant by $\mathcal{W}$. To conclude is enough to observe that $\mathcal{C}(r)$ is the image of $\Sigma(r)$ via the projection $\mathbb{W}(k,d) \times \mathbb{P}(S_r) \to \mathbb{W}(k,d)$. $\square$

3.2.2 Webs of degree 0 and 1

Given a projective curve $C \subseteq \mathbb{P}^2$ of degree k and a line $l_0 \in \check{\mathbb{P}}^2$ transverse to C there is a germ of k-web $\mathcal{W}_C(l_0)$ on $(\check{\mathbb{P}}^2, l_0)$ defined by the submersions $p_1, \ldots, p_k : (\check{\mathbb{P}}^2, l_0) \to C$ which describe the intersections of $l \in (\check{\mathbb{P}}^2, l_0)$ with C. These webs are called algebraic k-webs.

It is clear from the definition that the fiber of p_i through a point $l \in (\check{\mathbb{P}}^2, l_0)$ is contained in the set of lines that contain $p_i(l)$. Consequently the fibers of this submersion are contained in lines.

When C is a reducible curve with irreducible components $C_1, \ldots, C_m$ then $\mathcal{W}_C(l_0) = \mathcal{W}_{C_1}(l_0) \boxtimes \ldots \boxtimes \mathcal{W}_{C_m}(l_0)$.

If no irreducible component of C is a line then the leaves of $\mathcal{W}_C(l_0)$ through l are the hyperplanes passing through it and tangent to $\check{C}$ at some point $p \in \check{C}$.

Consider now the incidence variety $M \subseteq \mathbb{P}^2 \times \check{\mathbb{P}}^2$. One defines for every curve $C \subseteq \mathbb{P}^2$ its dual web $\mathcal{W}_C$ as the one defined by the surface $\pi^{-1}(C)$ seen as a multisection of $\check{\pi} : M \to \check{\mathbb{P}}^2$ (for more details see [18], section 1.3). It is easy to verify that the germification of this global web at a generic point $l_0 \in \check{\mathbb{P}}^2$ coincides with $\mathcal{W}_C(l_0)$ defined before. The following proposition can be found in [18], proposition 1.4.2.

Proposition 3.12. *If $C \subseteq \mathbb{P}^2$ is a projective curve of degree k, then $\mathcal{W}_C$ is a k-web of degree zero on $\check{\mathbb{P}}^2$. Reciprocally, if $\mathcal{W}$ is a k-web of degree zero on $\check{\mathbb{P}}^2$ then there exists a projective curve $C \subseteq \mathbb{P}^2$ of degree k such that $\mathcal{W} = \mathcal{W}_C$.*

Therefore one has the description of the webs of degree zero as one-parameter families of lines in the plane.

Given now a k-web $\mathcal{W}$ of degree 1 on $\mathbb{P}^2$, one can consider its lift $S_{\mathcal{W}}$ to M with class $[S_{\mathcal{W}}] = h + k\check{h}$. Observe first that the projection $\check{\pi}|_{S_{\mathcal{W}}} : S_{\mathcal{W}} \to \check{\mathbb{P}}^2$ is dominant and has degree one: a generic fiber of $\check{\pi} : M \to \check{\mathbb{P}}^2$ which is the lift of a line in $\mathbb{P}^2$ intersects $S_{\mathcal{W}}$ in the point corresponding to the unique tangency between this line and $\mathcal{W}$. Then one obtains a foliation $\mathcal{F}_{\mathcal{W}}$ on $\check{\mathbb{P}}^2$.

Since the tangencies between this foliation and a generic line in $\check{\mathbb{P}}^2$ corresponds to the intersection points between $S_{\mathcal{W}}$ and the lift by $\check{\pi}$, which is a fiber of π, this foliation has degree k. Observe that since $\mathcal{W}$ could be the product of a web of degree zero with a web of degree 1, $\mathcal{F}_{\mathcal{W}}$ could have a codimension 1 singular set.

Reciprocally, if we begin with a foliation $\mathcal{F}$ of degree k in $\check{\mathbb{P}}^2$, we obtain in the same way a k-web of degree one $\mathcal{W}_{\mathcal{F}}$ in $\mathbb{P}^2$. Moreover, as the reader can verify in [18], theorem 1.4.8, we have $\mathcal{W}_{\mathcal{F}_{\mathcal{W}}} = \mathcal{W}$, so this correspondence is in fact an isomorphism between $\mathbb{W}(k, 1)$ and the space of foliations of degree k in $\check{\mathbb{P}}^2$.

Assume now that we have a foliation $\mathcal{F}$ in $\check{\mathbb{P}}^2$ with a non-degenerate singularity at $l \in \check{\mathbb{P}}^2$. Then the fiber of $\check{\pi}$ over l is contained in $S_{\mathcal{F}}$. Since this fiber corresponds to the lift of the line that l represents we conclude that l is $\mathcal{W}_{\mathcal{F}}$-invariant. Using the fact that a generic foliation of degree k has k^2+k+1 non-degenerate singularities, one obtains the following proposition.

Proposition 3.13. *A generic k-web of degree one has k^2+k+1 invariant lines.*

Remark 3.14. Using proposition 3.11 we conclude that every k-web of degree 1 has at least one invariant line. Observe that these webs could also have an infinite number of invariant lines, for example in the case of the product of webs of degree one with webs of degree zero.

3.2.3 Webs of degree greater than 2

For webs of higher degree, we have the following theorem (Theorem 2 of the introduction).

Theorem 3.15. *A generic k-web of degree d in $\mathbb{P}^2$ does not admit any invariant algebraic curve if $d \geq 2$.*

Here by generic we mean that the set of webs that does not have any invariant curve is the complement of a countable union of algebraic closed proper subsets.

First we recall some facts for a generic k-web $\mathcal{W}$ of degree $d \geq 2$ on $\mathbb{P}^2$ (see [24]):

1. The surface $S_{\mathcal{W}} \subseteq M$ associated to $\mathcal{W}$ is smooth and its class is given by $[S_{\mathcal{W}}] = dh + k\check{h}$.

2. Let $\mathcal{F}_{\mathcal{W}}$ be the foliation on $S_{\mathcal{W}}$ given by the restriction of the contact distribution, or by the lifting of the leaves of $\mathcal{W}$; then the normal bundle of $\mathcal{F}_{\mathcal{W}}$ is given by $N\mathcal{F}_{\mathcal{W}} = \mathcal{O}_S(h_r + \check{h}_r)$, where h_r and $\check{h}_r$ are the restriction of h and $\check{h}$ to $S_{\mathcal{W}}$.

3. If we write the web in coordinates $(x, y) \in \mathbb{C}^2$ as

$$\omega = a_0(x,y)dx^k + a_1(x,y)dx^{k-1}dy + \ldots + a_k(x,y)dy^k$$

then $S_{\mathcal{W}}$ is given by the zero set of

$$F(x,y,p,q) = a_0(x,y)q^k + a_1(x,y)q^{k-1}p + \ldots + a_k(x,y)p^k$$

and the foliation $\mathcal{F}_{\mathcal{W}}$ is defined by the restriction of the vector field

$$X = (F_p\frac{\partial}{\partial x} - F_x\frac{\partial}{\partial p}) + (F_y\frac{\partial}{\partial q} - F_q\frac{\partial}{\partial y})$$

to $S_{\mathcal{W}}$. Then the singular set of $\mathcal{F}_{\mathcal{W}}$ is given in these coordinates by $\{F = F_q = F_p = qF_x + pF_y = 0\}$ which is a finite set.

Let us suppose that $\mathcal{W}$ has an algebraic invariant curve C and let $\widetilde{C}$ be its lifting to M, which is contained in $S_{\mathcal{W}}$.

Lemma 3.16. *We have that* $\widetilde{C} \cap sing(\mathcal{F}_{\mathcal{W}}) \neq \emptyset$.

Proof. Let us suppose that $\widetilde{C} \cap sing(\mathcal{F}_{\mathcal{W}}) = \emptyset$, then by Camacho-Sad formula $\widetilde{C}^2 = 0$ and therefore

$$N\mathcal{F}_{\mathcal{W}}.\widetilde{C} = \widetilde{C}^2 + Z(\mathcal{F}_{\mathcal{W}}, \widetilde{C}) = 0$$

(see [2]) which is not possible since $h_r.\widetilde{C} + \check{h}_r.\widetilde{C}$ is a positive number. □

Remember that we can identify the set of k-webs of degree d with the projective space $\mathbb{P}H^0(M, \mathcal{O}_M(d,k))$. Consider now the algebraic set

$$\mathcal{S} = \{(S_{\mathcal{W}}, z) \in \mathbb{W}(k,d) \times M : z \in sing(\mathcal{F}_{\mathcal{W}})\} \subseteq \mathbb{W}(k,d) \times M$$

and its projection to the second factor $\pi_2 : \mathcal{S} \to M$. Remark (3) implies that for each $z \in M$ the fiber $\pi_2^{-1}(z)$ is a linear subspace of $\mathbb{W}(k,d) \times \{z\}$.

Since for every $z_1,\ z_2 \in M$ there is a biholomorphism F of the form $F(p,[v]) = (f(p), Df(p)v)$, for some $f \in Aut(\mathbb{P}^2)$, sending z_1 to z_2, we conclude that all the fibers of π_2 are smooth, irreducible and isomorphic, which implies that $\mathcal{S}$ is irreducible.

Baum-Bott formula implies that $\sum BB(\mathcal{F}_{\mathcal{W}}, p) = (N\mathcal{F}_{\mathcal{W}})^2 = (h_r + \check{h}_r)^2 = 3(k+d)$, therefore the other projection $\pi_1 : \mathcal{S} \to \mathbb{W}(k,d)$ is a generically finite map. Thus $dim\mathcal{S} = dim\mathbb{W}(k,d)$.

Fix now a polynomial $\chi \in \mathbb{Q}[t]$ of degree one different from the Hilbert polynomial of a fiber of π and set $H = Hilb_\chi(M)$ the Hilbert scheme of M with respect to χ. If we denote by $H(\mathcal{D})$ the subset of H consisting of the subschemes of M tangent to $\mathcal{D}$ and with Hilbert polynomial χ, then we shall prove in the next section that $H(\mathcal{D})$ is a closed subset of H. Consider now the closed set $D \subseteq \mathbb{W}(k,d) \times M$ defined as

$$D = \{(S_{\mathcal{W}}, z) \in \mathbb{W}(k,d) \times M : \exists Y \in H(\mathcal{D}),\ z \in Y,\ Y \subset S_{\mathcal{W}}\}.$$

Let us assume the theorem for $(k-1)$-webs of degree $d \geq 2$ and suppose that $\pi_1(D) = \mathbb{W}(k,d)$; that is, every k-web of degree d has an algebraic invariant curve whose lifting has Hilbert polynomial χ. By lemma 3.16 π_1 sends a dense subset of $D \cap \mathcal{S}$ to a dense subset of $\mathbb{W}(k,d)$. Therefore $\pi_1(D \cap \mathcal{S}) = \mathbb{W}(k,d)$, but since $\mathcal{S}$ is irreducible and has the same dimension that $\mathbb{W}(k,d)$ we conclude that $D \cap \mathcal{S} = \mathcal{S}$.

To conclude the theorem we choose a generic $(k-1)$-web of degree d $\mathcal{W}_1$ with no algebraic invariant curves and a pencil of lines through a point $\mathcal{G}$ such that there exist a singularity z of $\mathcal{F}_{\mathcal{W}_1}$ which is not in $S_{\mathcal{G}}$ and take $\mathcal{W} = \mathcal{W}_1 \boxtimes \mathcal{G}$. Then we note that $S_{\mathcal{W}} = S_{\mathcal{W}_1} \cup S_{\mathcal{G}}$ and through z, which is a singularity of $\mathcal{F}_{\mathcal{W}}$ we do not have any invariant curve different from a fiber, which is a contradiction. Since there are only countable many Hilbert polynomials, we conclude the theorem.

Remark 3.17. We have a different proof for theorem 3.15 for k-webs of degree $d \geq 2$ when $(k,d) \neq (2,2)$ as follows.

Let us suppose that the assertion is true for $W_1 := \mathbb{W}(k,d)$, with $k \geq 1$ and $d \geq 3$, and we will show the theorem for $W_2 := \mathbb{W}(k+1,d)$.

By proposition 3.11 is enough to show that for each positive integer r, the set of $(k+1)$-webs of degree d having an invariant curve of degree r is a proper subset of $\mathbb{W}(k+1,d)$. So, for each positive integer r define the algebraic set

$$\Sigma(r) := \{([\eta], C) : C\ is\ \mathcal{W}_\eta - invariant\} \subseteq W_2 \times \mathbb{P}(S_r)$$

and assume that the projection over the second factor $\pi_2 : \Sigma(r) \to W_2$ is surjective. Then there exist an irreducible component $\Sigma \subseteq \Sigma(r)$ such that the restriction of π_2 to Σ, which we still denote by π_2, is also surjective. Consider now the map

$$\phi : W_1 \times \mathbb{P}^2 \to W_2,\ \phi([\omega], p) = [\omega.\omega_p]$$

where ω_p is the 1-form defining the radial foliation through p, and denote $\widetilde{W_1} = \phi(W_1 \times \mathbb{P}^2) \subseteq W_2$ which is an irreducible subvariety of W_2. Take

$$\eta = \alpha.C.\omega_p \in \widetilde{W_1}$$

where α defines a k-web of degree 1 having $k^2 + k + k$ lines as the unique invariant curves, C is a curve of degree $d-1$, and p is a point which is neither in the lines invariant by $\mathcal{W}_\alpha$ nor in the curve C.

We set

$$P = \{curves\, \mathcal{W}_\eta - invariant\, of\, degree\, r\, passing\, through\, p\} \subseteq \mathbb{P}(S_r)$$

and

$$Q = \{curves\, \mathcal{W}_\eta - invariant\, of\, degree\, r\, not\, passing\, through\, p\} \subseteq \mathbb{P}(S_r).$$

Observe that P is a closed subset of a linear subspace of $\mathbb{P}(S_r)$ while Q is a finite subset disjoint from P. Clearly we have

$$\pi_2^{-1}(\eta) \subseteq P \cup Q$$

and since η can be seen as the product of a k-web of degree 1 and a foliation of degree $d-1$, and $\pi_2^{-1}(\alpha.\beta) \subseteq Q$ for the generic $\beta \in \mathbb{W}(1, d-1)$ (because $d - 1 \geq 2$), one concludes that $\pi_2^{-1}(\eta) \cap Q \neq \emptyset$ and π_2 is generically finite (in particular dim Σ = dim W_2). Therefore we can set

$$\widetilde{\Sigma} := \Sigma \cap (\mathbb{W}(k+1, d) \times (\mathbb{P}(S_r) - P))$$

which is an open dense subset of Σ and has nonempty intersection with $\pi_2^{-1}(\eta)$. So we can choose an element $(\eta, F) \in \widetilde{\Sigma}$ and an analytic open neighborhood

$$(\eta, F) \in U \subseteq \widetilde{\Sigma}.$$

Since π_2 is generically finite and Σ is irreducible, $d\pi_2$ is an isomorphism at the generic point of U, then $\pi_2(U)$ contains an open set. Observe also that π_2

is a proper map, so $\pi_2(U)$ is a germ of analytic set, therefore $\pi_2(U)$ contains an open neighborhood of η.

On the other hand, by our hypothesis on W_1 there are elements of the form $\omega.\omega_p \in \widetilde{W_1}$ close enough to η such that the algebraic curves invariant by these elements are product of lines passing through p. Clearly these elements can not be in $\pi_2(U)$, we arrive in a contradiction.

Since we know by Jouanolou's theorem that the assertion holds true for $\mathbb{W}(1, d)$, $d \geq 3$ (see [13], théorème 1.1, p. 158), we conclude the assertion for $\mathbb{W}(k, d)$ when $k \geq 1$ and $d \geq 3$. For the case when $d = 2$, one takes $\mathcal{W}$ a 2-web of degree k, with $k \geq 3$, without singular points and such that $\mathcal{W}$ does not have algebraic invariant curves. Then its dual web $\check{\mathcal{W}}$, that is, the projection on $\check{\mathbb{P}}^2$ of the foliation given by the contact form over the surface $S_{\mathcal{W}} \subseteq M$ associated to $\mathcal{W}$, is a k-web of degree 2. It is easy to verify that $\check{\mathcal{W}}$ does not have algebraic invariant curves and the proof of the theorem is complete when $(k, d) \neq (2, 2)$.

3.2.4 Webs on complex surfaces

We can obtain a similar result for webs on complex surfaces. Let S be a compact complex surface and we set $\mathbb{W}(k, \mathcal{N}) = \mathbb{P}H^0(S, \mathrm{Sym}^k \Omega^1_S \otimes \mathcal{N})$ the space of k-webs on S with normal bundle $\mathcal{N}$.

Theorem 3.18. *Let $\mathcal{N}$ be an ample line bundle. Then for $r \gg 0$ the k-web on S induced by a very generic element of $\mathbb{W}(k, \mathcal{N}^{\otimes r})$ has no algebraic invariant curves.*

Proof. Following the notation of the previous section, one can define the closed set $D = \{(S_{\mathcal{W}}, z) \in \mathbb{W}(k, \mathcal{N}) \times M : \exists Y \in H(\mathcal{D}),\ z \in Y,\ Y \subset S_{\mathcal{W}}\}$, where $M = \mathbb{P}(TS)$. Then for $r \gg 0$ we can choose integers $r_i \gg 0$ which add up to r and $\omega_i \in \mathbb{P}H^0(S, \Omega^1_S \otimes \mathcal{N}^{\otimes r_i})$ such that the foliation on S defined by ω_i has not algebraic invariant curves (see [6] theorem 1.1). Then is clear that $\omega = \omega_1 \ldots \omega_k$ is an element of $\mathbb{W}(k, \mathcal{N}^{\otimes r})$ which is not in the image of D by the first projection. □

Remark 3.19. Unlike the case of $\mathbb{P}^2$, where Theorem 3.15 is effective in the sense that the webs not satisfying its hypothesis always have an invariant curve, this result just works for webs with "big enough" normal bundle, but we do not know exactly how ample this bundle should be for the theorem be true and also we do not know if the webs with not very ample normal bundle always have algebraic curves.

3.3 Proof of Theorem 3.9

Let us consider a nonsingular distribution of codimension one D defined by a 1-form α over a complex manifold M.

Lemma 3.20. *Let $Y \subseteq M$ be an irreducible subvariety (not necessarily regular) of codimension k and let $\mathcal{I}$ be its ideal sheaf. Then Y is tangent to D if and only if for every $(f_1, \ldots, f_k) \in \mathcal{I}^{\oplus k}$, the equality*

$$\alpha \wedge df_1 \wedge \ldots \wedge df_k|_Y \equiv 0$$

holds true.

Proof. Let us suppose that Y is tangent to D, that is, Y is tangent to D at the smooth points, and take $f_1, \ldots, f_k \in \mathcal{I}$. Then, at a regular point $y \in Y$ we can take an open neighborhood $V \subseteq M$ and local coordinates $(x_1, \ldots, x_n)$ such that $Y \cap V = \{x_1 = \ldots x_k = 0\}$. Therefore we can write

$$f_i|_V = \sum_{j=1}^{k} a_{ij} x_j, \quad \alpha = b_1 dx_1 + \ldots + b_n dx_n$$

for some analytic functions a_{ij}, b_i. So, by hypothesis we have

$$b_{k+1}|_{V\cap Y} = \ldots = b_n|_{V\cap Y} = 0.$$

Since

$$\begin{aligned}\alpha \wedge df_1 \wedge \ldots \wedge df_k|_V &= h\alpha \wedge dx_1 \wedge \ldots \wedge dx_k|_V \\ &= h(b_{k+1}dx_{k+1} \wedge dx_1 \wedge \ldots \wedge dx_k + \ldots b_n dx_n \wedge dx_1 \wedge \ldots \wedge dx_k)\end{aligned}$$

(here h is an invertible function) we obtain that

$$Y \cap V \subseteq sing(\alpha \wedge df_1 \wedge \cdots \wedge df_k)$$

and this shows the assertion. The other implication is left to the reader. $\square$

Let us return to the case when $M = \mathbb{P}(T\mathbb{P}^2)$ is the contact variety and $D = \mathcal{D}$ is the contact distribution. Fix now a polynomial $\chi \in \mathbb{Q}[t]$ of degree one and set $H = Hilb_\chi(M)$ the Hilbert scheme of M with respect to χ. Define also $H(\mathcal{D})$, a subset of H, by

$$H(\mathcal{D}) = \{Y \in H : Y \; is \; tangent \; to \; \mathcal{D}\}.$$

Remark 3.21. It is easy to show that $\mathcal{D}$ has no invariant subvarieties of dimension two, so the condition on the degree of χ is not really necessary.

Proposition 3.22. *$H(\mathcal{D})$ is a closed subset of H.*

Proof. Let $\mathcal{U}$ be the universal family in $M \times H$

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \subseteq M \times H \\ M & & H \end{tikzcd}$$

Remember that for any $Y \subseteq M$ closed subscheme one has the following exact sequence (see [12], section 2.8)

$$\frac{\mathcal{I}}{\mathcal{I}^2} \xrightarrow{\delta} \Omega^1_M|_Y \longrightarrow \Omega^1_Y \longrightarrow 0$$

where $\delta(f) = df|_Y$ and $\mathcal{I}$ is the ideal sheaf of Y. This means, writing $\mathcal{K}_Y = Im(\delta)$ (the conormal sheaf of Y), that one has

$$0 \longrightarrow \mathcal{K}_Y \longrightarrow \Omega^1_M|_Y \longrightarrow \Omega^1_Y \longrightarrow 0\,.$$

Then we can consider $\mathcal{K}$ defined by

$$0 \longrightarrow \mathcal{K} \longrightarrow (\Omega^1_{M\times H|H})|_{\mathcal{U}} \longrightarrow \Omega^1_{\mathcal{U}|H} \longrightarrow 0\,.$$

By doing the exterior product by the contact form α we have a map

$$\bigwedge^2 \mathcal{K} \xrightarrow{\theta} (q_1^*\Omega^1_M) \otimes \mathcal{L}$$

for some line bundle $\mathcal{L}$. We conclude by lemma 3.20 that

$$H(\mathcal{D}) = \{Y \in H : \theta_Y = 0\}.$$

Let $\mathcal{M}$ be a very ample sheaf over $\mathcal{U}$. Give an integer $r \gg 0$ it follows by Serre's theorem that there exists a positive integer N and a surjective map $\beta : \mathcal{O}_{\mathcal{U}}^{\oplus N} \to \bigwedge^2 \mathcal{K} \otimes \mathcal{M}^{\otimes r}$. Denoting by σ the composition

$$\begin{tikzcd} \mathcal{O}_{\mathcal{U}}^{\oplus N} \arrow[r, "\beta"] \arrow[rr, bend right, "\sigma"'] & \bigwedge^2 \mathcal{K} \otimes \mathcal{M}^{\otimes r} \arrow[r, "\theta"] & (q_1^*\Omega_M^1) \otimes \mathcal{L} \otimes \mathcal{M}^{\otimes r} \end{tikzcd}$$

we have, since β is surjective, that

$$H(\mathcal{D}) = \{Y \in H : \sigma_Y = 0\}.$$

Consider now the following lemma, which is exactly lemma (2.2) of [6].

Lemma 3.23. *Let $p : \mathfrak{X} \to T$ be a projective morphism. Assume that $\mathcal{F}$ is a p-flat coherent $\mathcal{O}_{\mathfrak{X}}$-module such that $R^1p_*\mathcal{F} = 0$. If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_T$-modulo and $\sigma : p^*\mathcal{G} \to \mathcal{F}$ is a homomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules, then the set $\{t \in T : \sigma_t = 0\}$ is closed in T.*

In our case we have $p = q_2 : \mathcal{U} \to H$, $\mathcal{F} = (q_1^*\Omega_M^1) \otimes \mathcal{L} \otimes \mathcal{M}^{\otimes r}$, $\mathcal{G} = \mathcal{O}_S^{\oplus N}$ and $\sigma : p^*\mathcal{G} = \mathcal{O}_{\mathcal{U}}^{\oplus N} \to \mathcal{F}$. We use theorem 3.8.8 of [12] to obtain $R^1p_*\mathcal{F} = 0$ and thus we can apply the lemma to conclude the proof. □

As a consequence one concludes the following proposition.

Proposition 3.24. *The set*

$$\mathcal{Z}_\chi = \{X \in \mathcal{E}(a,b) : \mathcal{F}_X \textit{ has an invariant subscheme of Hilbert polynomial } \chi\}$$

is closed in $\mathcal{E}(a,b)$.

Proof. Let us denote $\Sigma = \mathbb{P}H^0(M, TM \otimes \mathcal{O}_M(a,b))$. By the proposition (2.1) of [6] we have that the set

$$Z = \{(X,Y) \in \Sigma \times H : Y \textit{ is } \mathcal{F}_X - \textit{invariant}\}$$

is a closed subset of $\Sigma \times H$. Observe that we have an inclusion $\mathcal{E}(a,b) \subseteq \Sigma$, so

$$Z \cap (\mathcal{E}(a,b) \times H(\mathcal{D})) \subseteq \mathcal{E}(a,b) \times H$$

is a closed subset. Now it is enough to note that $\mathcal{Z}_\chi$ is the natural projection of this closed set. □

We conclude now the proof of the theorem 3.9. Denoting by χ_0 to the Hilbert polynomial of a fiber (of π) F, and taking a $(b+2)$-web $\mathcal{W}$ of degree $a-1$, with $a \geq 3$ and $b \geq 1$, we have that $F.X_{\mathcal{V}}$ is an element of $\mathcal{E}(a,b)$, where F is the section corresponding to the surface $S_{\mathcal{W}}$. By Theorem 3.15 we can choose $\mathcal{W}$ with no algebraic invariant curves and then $F.X_{\mathcal{V}} \notin \mathcal{Z}_{\chi}$ for every $\chi \neq \chi_0$. Since there are only countable many Hilbert polynomials, we conclude the first part of the theorem.

When $a, b \geq 3$, we take a $(b-1)$-web $\mathcal{W}$ of degree $a+2$ and so $F.X_{\mathcal{L}}$ has bidegree (a,b) (here again F is the section corresponding to $S_{\mathcal{W}}$). We choose $\mathcal{W}$ without singular points and with no algebraic invariant curves. Since the curves which are lifting of curves on $\mathbb{P}^2$ have different Hilbert polynomial from χ_0, $F.X_{\mathcal{L}}$ is not in $\mathcal{Z}_{\chi_0}$. This finishes the proof.

3.4 Second order differential equations on complex surfaces

Let S be any complex compact surface and $M = \mathbb{P}(TS)$ the contact variety. We recall that $H^*(M)$ is generated as a $H^*(S)$-algebra by the Chern class of $\mathcal{O}_M(1)$. We denote by $\mathcal{E}(\mathcal{N}, k) = \mathbb{P}H^0(M, \mathcal{D} \otimes \pi^*(\mathcal{N}) \otimes \mathcal{O}_M(1)^{\otimes k})$ the space of second order differential equations on S with cotangent bundle $\pi^*(\mathcal{N}) \otimes \mathcal{O}_M(1)^{\otimes k}$.

Remark 3.25. For any $\mathcal{W} \in \mathbb{W}(k, \mathcal{N})$ which is given locally by $\omega = a_0(x,y) + a_1(x,y)\frac{dy}{dx} + \ldots + a_k(x,y)\frac{dy}{dx}^k$, the associated surface $S_{\mathcal{W}}$ is given by the zero set of $F(x,y,p) = a_0(x,y) + a_1(x,y)p + \ldots + a_k(x,y)p^k$, where (x,y) are local coordinates on S. Doing the change of coordinates we see that $[S_{\mathcal{W}}] = \pi^*(\mathcal{N}) \otimes \mathcal{O}_M(1)^{\otimes k}$.

Following the ideas of the main theorem we can get a same result for any surface:

Theorem 3.26. *Let $\mathcal{N}$ be an ample line bundle on S and $r \gg 0$. Then for any $k \geq 1$ the second order differential equation defined by a very generic element of $\mathcal{E}(\mathcal{N}^{\otimes r}, k-2)$ has no algebraic solutions.*

Proof. Using the notation of the previous section one can define the closed set $\mathcal{Z}_\chi$ as

$$\{X \in \mathcal{E}(\mathcal{N}^{\otimes r}, k-2) : \mathcal{F}_X \text{has an invariant subscheme of Hilbert polynomial } \chi\}.$$

To conclude the theorem we must show that this is a proper subset of $\mathcal{E}(\mathcal{N}^{\otimes r}, k-2)$. Take now $\mathcal{W} \in \mathbb{W}(k, \mathcal{N}^{\otimes r} \otimes K_S^*)$ with no algebraic invariant curves (here K_S is the canonical bundle of S). Notice that as a consequence of Euler sequence (1.1) we have $T^*\mathcal{V} = \pi^*(K_S) \otimes \mathcal{O}_M(-1)^{\otimes 2}$. Thus is clear that $F_{\mathcal{W}}.\mathcal{V} \in \mathcal{E}(\mathcal{N}^{\otimes r}, k-2)$ and since $F_{\mathcal{W}}.\mathcal{V}$ is not an element of $\mathcal{Z}_\chi$ we are done. $\square$

Bibliography

[1] V. I. Arnol'd, *Geometrical methods in the theory of ordinary differential equations*. Grundlehren der Mathematischen Wissenschaften 250. Springer-Verlag, 1988.

[2] M. Brunella, *Birational Geometry of Foliations*. First Latin American Congress of Mathematicians, IMPA, 2000.

[3] S. J. Colley, G. Kennedy. *The enumeration of simultaneous higher-order contacts between plane curves*. Compositio Math. 93 (1994), no. 2, 171-209.

[4] S.J. Colley, G. Kennedy. *Triple and quadruple contact of plane curves*. Contemp. Math., 123, Amer. Math. Soc., Providence, RI, 1991.

[5] S.J. Colley, G. Kennedy. *A higher-order contact formula for plane curves*. Comm. Algebra 19 (1991), no. 2, 479-508.

[6] S.C. Coutinho and J.V. Pereira, *On the density of algebraic foliations without algebraic invariant sets*. Journal für die Reine und Angewandte Mathematik, 594, pp. 117-136, 2006.

[7] G. Darboux, *Mémoire sur les équations différentielles algébriques du I^o ordre et du premier degré*. Bull. des Sc. Math., (Mélanges), 60-96, 123-144, 151-200, 1878.

[8] J. P. Demailly, *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*. Proceedings of Symposia in Pure Math., Vol. 62.2 (AMS Summer Institute on Algebraic Geometry, Santa Cruz, July 1995), ed. J. Kollár, R. Lazarsfeld, (1997), 285-360.

[9] W. Fulton, *Intersection theory*. Berlin, New York, Springer-Verlag, 1984.

[10] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley Classics Library Edition Published 1994.

[11] E. Ghys *À propos d'un théorème de J.-P. Jouanolou concernant les feuilles fermées des feuilletages holomorphes.* Rend. Circ. Mat. Palermo (2), 49 (2000), no. 1, 175-180.

[12] R. Hartshorne, *Algebraic geometry.* New York : Springer-Verlag, c1977.

[13] J. P. Jouanolou, *Equations de Pfaff algébriques.* Lect. Notes Math. 708, Springer, 1979.

[14] A. Lins Neto, *Algebraic solutions of polynomial differential equations and foliations in dimension two.* Lect. Notes Math. 1345 (1988), 192-232.

[15] I. Pan, *Quelques remarques sur les d-web des surfaces complexes et un problème proposé par D. Cerveau.* Bol. Asoc. Mat. Venez. 10 (2003), no. 1, 21-33.

[16] J. V. Pereira, *Sobre a densidade de Folheações sem soluções Algébricas.* Revista del Seminario Iberoamericano de Matemáticas Singularidades en Tordesillas 3, no. 4, 51-57, 2007.

[17] J. V. Pereira, *Vector fields, invariant varieties and linear systems.* Ann. Inst. Fourier (Grenoble) 51 (2001), no. 5, 1385-1405

[18] J.V. Pereira and L. Pirio, *An invitation to web geometry.* Publicações Matemáticas, IMPA, 2009.

[19] J.V. Pereira and L. Pirio, *Classification des tissus exceptionnels quasilinéaires complètement décomposables.* C. R. Math. Acad. Sci. Paris 346 (2008), no. 19-20, 1093-1098.

[20] J.V. Pereira and L. Pirio, *The Classification of Exceptional CDQL Webs on Compact Complex Surfaces.* International Mathematics Research Notices, Article ID rnp208.

[21] J.V. Pereira and S. Yuzvinsky, *Completely reducible hypersurfaces in a pencil.* Adv. Math. 219 (2008), no. 2, 672-688.

[22] M. G. Soares, *On algebraic sets invariant by one-dimensional foliations on* $\mathbb{C}P(3)$. Ann. Inst. Fourier 43 (1993), 143-162.

[23] C. Voisin, *Hodge theory and complex algebraic geometry*. Cambridge studies in advanced mathematics ; 76-77.

[24] J. N.A. Yartey, *Number of singularities of a generic web on the complex projective plane*. J. Dyn. Control Syst. 11 (2005), no. 2, 281–296.

[25] S. Yuzvinsky, *A new bound on the number of special fibers in a pencil of curves*. Proc. Amer. Math. Soc. 137 (2009), no. 5, 1641-1648.