Absolutely continuous invariant measures for non-uniformly expanding skew-products

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Abstract

We prove that for certain partially hyperbolic skew-products on the cylinder, non-uniform hyperbolicity along the leaves implies existence of absolutely continuous invariant probability measures. The main technical tool is an extension for sequences of maps of a result of de Melo and van Strien relating hyperbolicity to recurrence properties of orbits. As a consequence of our main result, we obtain extensions of Keller's theorem guaranteeing the existence of absolutely continuous invariant measures for non-uniformly hyperbolic one dimensional maps.

A mi familia:
Vin, Jaimito, Eduardo y Willy.

Todo ser nuevo que encontramos viene de otro relato
y es el puente que une dos leyendas y dos mundos.
Ursúa, William Ospina.

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Introduction

An invariant measure reflects the asymptotic behavior of almost every point with respect to this measure. For any continuous dynamical system on a compact manifold it is known that invariant finite measures do exist. In general, a dynamical system may have a great abundance of invariant measures. For example, if the dynamical system has periodic points, then the measure concentrated on the orbit of each periodic point is an invariant measure; in this case, almost every point just means for every point of the orbit. One natural interest is to choose inside the set of invariant measures those which are more relevant for understanding the dynamics, that means to define a criterion which says what are the measures that describe the dynamics in a significant way. On Riemannian manifolds the more natural choice is the volume induced by the Riemannian metric. One may ask for invariant measures that describe the asymptotics of almost every, or at least, a positive volume measure set of trajectories. This is the case for instance, if the invariant measure is absolutely continuous with respect to the volume and ergodic (by the Birkhoff's ergodic theorem).

Thus, in this work we focus on the problem of existence of absolutely continuous invariant measures.

It is a classical fact (see Mañé, [M]) that uniformly expanding smooth maps on compact manifolds admit a unique ergodic absolutely continuous invariant measure, and it describes the asymptotics of almost every point. Moreover, see Bowen [B], uniformly hyperbolic diffeomorphisms also have a finite number of such *physical measures*, describing also the asymptotics of almost every point. Actually, in this case, the physical measures are absolutely continuous only along certain directions, namely, the expanding ones.

The present work is motivated by the question of knowing, to what extent, weaker forms of hyperbolicity are still sufficient for the existence of such measures. A precise statement in this direction is:

Conjecture (Viana,[V2])**.** *If a smooth map has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits some physical measure.*

Two main results provide some evidence in favor of this conjecture. The older one is the remarkable theorem of Keller [Ke] stating that *for unimodal maps of the interval with negative Schwarzian derivative, existence of absolutely continuous invariant probability is equivalent to positive Lyapunov exponents*, i.e.

$$\limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)| > 0 \tag{$*$}$$

on a positive measure set of points x.

Then, more recently, Alves, Bonatti, Viana [ABV] proved that *every non-uniformly* *expanding local diffeomorphism on any compact manifold admits a finite number of ergodic absolutely continuous invariant measures describing the asymptotics of almost every point*. This notion of non-uniform expansion means that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\|^{-1} \geq c > 0 \tag{**}$$

almost everywhere. This result has been improved by several authors. In particular, Pinheiro [P] shows that one may replace lim inf by lim sup; conform (*). Alves, Bonatti and Viana [ABV] also give a version of this result for maps with singularities, that is, which fail to be a local diffeomorphism on some subset $\mathcal{S}$ of the ambient manifold. In their work they introduce and make an extensive use of the notion of *hyperbolic times*. A hyperbolic time k for a point x is, basically, an iterate such that f^k has the behavior of a uniformly expanding map for a certain neighborhood of the point x. However, due to the presence of singularities they need an additional hypotheses (of slow recurrence to the singular set $\mathcal{S}$) which is often difficult to verify. Given that Keller's theorem has no such hypotheses (in his case $\mathcal{S} = \{\text{critical points}\}$), one may ask to what extent this condition is really necessary.

Here we aim to extend both results mentioned previously to a setting of higher dimensional transformations with (non-empty) singular set.

Motivated by a family of maps introduced by Viana [V1] and studied by several other authors (see for example [A, AV, BST, P]) we consider transformations of the form $\varphi : \mathbb{T}^1 \times I_0 \to \mathbb{T}^1 \times I_0$, $(\theta, x) \mapsto (g(\theta), f(\theta, x))$, where g is a uniformly expanding circle map, each $f(\theta, \cdot)$ is a smooth interval map with negative Schwarzian derivative, and φ is partially hyperbolic with vertical central direction:

$$|\partial_\theta g(\theta)| > |\partial_x f(\theta, x)| \qquad \text{at all points .}$$

We prove that if φ is non-uniformly expanding then it admits some absolutely continuous invariant probability.

The Viana maps [V1] correspond to the case when g is affine, $g(\theta) = d\theta \pmod 1$ with $d >> 1$, and f has the form $f(\theta, x) = a_0 + \alpha \sin(2\pi\theta) - x^2$ (actually, [V1] deals also with arbitrary small perturbations of such maps). It was shown in [V1] that Viana maps are indeed non-uniformly expanding. Moreover, Alves [A] proved that they have a unique physical measure, which is absolutely continuous and ergodic. Their methods hold even for a whole open set of maps not necessarily of skew-product form. In fact, the argument of Alves [A] rely on a proof of slow recurrence to the critical set which in that case is the circle $\mathbb{T}^1 \times \{0\}$.

Our method is completely different. We view φ as a family of smooth maps of the interval, namely, its restrictions to the vertical fibers $\{\theta\} \times I_0$. Thus, our main technical tool is an extension for such families of maps of a result proved by de Melo, van Strien [MvS, Theorem V.3.2, page 371] for individual unimodal maps saying, in a few words, that positive Lyapunov exponents manifest themselves at a macroscopic level: intervals that are mapped diffeomorphically onto large domains under iterates of the map. This, in turn, allows us to make use of the hyperbolic times technique similar to the introduced by Alves, Bonatti, Viana [A, ABV].

This paper is organized as follows. In section 1 we give the precise statement of the main results. In section 2 we introduce a few preliminary facts, mostly well-known, which will be useful in the sequel. There are two subsections: the first one concerns one dimensional dynamics; in the second one we recall the statement of Alves, Bonatti, Viana [ABV] for maps with singularities, and also we recall the definitions of hyperbolic times and the condition of slow recurrence to a singular set. In section 3 we prove our Theorem B, which is the extension of [MvS, Theorem V.3.2, page 371] mentioned before. The section 4 contains the proof of some extensions of Keller's theorem to different settings. In the proof of these results we need some recurrence properties for interval maps, which are contained on Appendix A. In section 5 we prove another key result (Proposition 5.3): for each interval which is mapped diffeomorphically onto a large domain under an iterate of the skew-product, there exists an open set containing this interval which is sent diffeomorphically onto its image under the same iterate; moreover, this map has bounded distortion and the measure of the image is bounded away from zero. We call these iterates *hyperbolic-like times*.

In section 6 we combine Lemma 3.1, which is the main lemma for the proof of Theorem B, with the Pliss Lemma to conclude that the set of points with infinitely many (and even positive density of) hyperbolic-like times has positive Lebesgue measure. The construction of the absolutely continuous invariant measure for the skew-product φ then follows along well-known lines, as we explain in subsection 6.3.

1 Statement of the results

1.1 Non-uniformly expandig skew-products

Let I_0 be an interval and $\mathbb{T}^1$ the circle. We will consider C^3 partially hyperbolic skew-products defined on $\mathbb{T}^1 \times I_0$, which present an expanding behavior in the horizontal direction and critical points in the vertical direction.

We call a C^1 mapping $\varphi : M \to M$ *partially hyperbolic* endomorphism if there are constants $0 < a < 1$, $C > 0$ and a continuous decomposition of the tangent bundle $TM = E^c \oplus E^u$ such that

(a) $\|D\varphi^n|_{E^u(z)}\| > C^{-1}a^{-n}$

(b) $\|D\varphi^n|_{E^c(z)}\| < Ca^n\|D\varphi^n|_{E^u(z)}\|$

for all $z \in M$ and $n \geq 0$. Observe that we do not ask in the definition that these subbundles be invariant. The subbundle E^c is called *central* and the E^u is called *unstable*. In our case, the mappings are precisely

$$\begin{array}{rcccc} \varphi & : & \mathbb{T}^1 \times I_0 & \to & \mathbb{T}^1 \times I_0 \\ & & (\theta, x) & \to & (g(\theta), f(\theta, x)) \end{array}$$

where g is an uniformly expanding smooth map on $\mathbb{T}^1$ and

$$\begin{array}{rccc} f_\theta : & I_0 & \to & I_0 \\ & x & \to & f(\theta, x) \end{array}$$

is a smooth map with critical points for every $\theta \in \mathbb{T}^1$. The central subbundle is given by the vertical direction and the unstable one by the horizontal. Notice that for the partial hyperbolicity property in this skew product context we must have,

$$\frac{\prod_{i=0}^{n-1} |\partial_x f(\varphi^i(\theta, x))|}{|\partial_\theta g^n(\theta)|} \leq Ca^n \tag{1}$$

for all $(\theta, x) \in \mathbb{T}^1 \times I_0$.

In the result of Alves, Bonatti and Viana (Theorem 2.7), the set $\mathcal{S}$ of singular points of φ satisfies the non-degenerate singular set conditions. These conditions allow the co-existence of critical points and points with $|\det D\varphi| = \infty$. We will only admit critical points.

We denote by $\mathscr{C}$ the set of critical points and by $\mathscr{C}_\theta$ the set of critical points contained in the θ-vertical leaf. By $\mathrm{dist}_{\mathrm{vert}}$ we denote the distance induced by the Riemmanian metric in the vertical leaf, i.e, if $z = (\theta, x)$ for some x, $\mathrm{dist}_{\mathrm{vert}}(z, \mathscr{C}) = \mathrm{dist}(z, \mathscr{C}_\theta)$. Let us explain the conditions of the theorem.

Let $M = \mathbb{T}^1 \times I_0$ and $\mathscr{C} \subset M$. We consider a C^3 skew product map $\varphi : M \to M$ which is a local C^3 diffeomorphism in the whole manifold except in a critical set $\mathscr{C}$ such that,

(F_1) $p = \sup \#\mathscr{C}_\theta < \infty$;

there exists $B > 0$ such that, for every $z \in M \setminus \mathscr{C}$, $w \in M$ with $\mathrm{dist}(z, w) < \mathrm{dist}_{\mathrm{vert}}(z, \mathscr{C})/2$,

(F_2) $$\left|\log |\partial_x f(z)| - \log |\partial_x f(w)|\right| \leq \frac{B}{\mathrm{dist}_{\mathrm{vert}}(z, \mathscr{C})} \mathrm{dist}(z, w).$$

and for all $\theta \in \mathbb{T}^1$,

(F_3) $Sf(\theta, x) < 0$, for $x \in I_0$ where this quantity is defined.

When $M = I_0$, if f satisfies the one dimensional definition of non-flatness and $Sf < 0$, then it automatically satisfies these conditions.

Remark 1.1.

1. Note that (F_2) implies that for any $z \in M$, $\mathrm{dist}(z, \mathscr{C}) \geq \frac{\mathrm{dist}_{\mathrm{vert}}(z,\mathscr{C})}{2}$.
2. If $C_\theta = \emptyset$ for some $\theta \in \mathbb{T}^1$ then, as a consequence of (F_2), $C_\theta = \emptyset$ for every $\theta \in \mathbb{T}^1$. This case is covered by [ABV, Corollary D], but also follows from (a simple version of) our arguments. For completeness we define $\mathrm{dist}(z, \emptyset) = 1$.

In order to obtain the conclusions of the theorem of this section:

3. We may replace in the condition (F_2), $\mathrm{dist}_{\mathrm{vert}}$ by dist, if there was $\Xi > 0$ such that $\mathrm{dist}(z, \mathscr{C}) \geq \Xi \, \mathrm{dist}_{\mathrm{vert}}(z, \mathscr{C})$ for all $z \in M$.
4. In the condition (F_2) we may put $\mathrm{dist}_{\mathrm{vert}}(z, \mathscr{C})^\gamma$ (with $\gamma > 1$) instead of $\mathrm{dist}_{\mathrm{vert}}(z, \mathscr{C})$, if we had a better domination for φ, namely, if for all $(\theta, x) \in \mathbb{T}^1 \times I_0$,

$$\frac{\prod_{i=0}^{n-1} |\partial_x f(\varphi^i(\theta, x))|^\gamma}{|\partial_\theta g^n(\theta, x)|} \leq Ca^n.$$

Let us state now our principal result

Theorem A. *Assume that $\varphi : \mathbb{T}^1 \times I_0 \to \mathbb{T}^1 \times I_0$ is a C^3 partially hyperbolic skew product satisfying (F_1), (F_2) and (F_3). If φ is non-uniformly expanding, i.e, for Lebesgue almost every $z \in \mathbb{T}^1 \times I_0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|D\varphi(\varphi^j(z))^{-1}\| < 0 \tag{2}$$

Then φ admits an absolutely continuous invariant measure.

For the proof of Theorem A, we will study the dynamics in the vertical foliation of M and the result in this direction that we will use is Theorem B, which will be stated in the next subsection. Actually, for the proof of Theorem A, we use Lemma 3.1, which is also the main lemma in the proof of Theorem B.

1.2 Sequences of smooth one dimensional maps

The general strategy that we will use to construct the absolutely continuous invariant measure is to consider the push-forward of Lebesgue measure by the iterates of the map. We define the Cesaro sums and take a weak limit.

In order to prove the absolute continuity it is fundamental to control the distortion. In the setting defined by Theorem A, we will show that the distortion can be controlled, if we bound the distortion along the vertical direction. When we focus on the dynamics of the skew product restricted to the orbit of any vertical leaf, we can take as inspiration the one dimensional dynamics methods.

Given $f : I_0 \to I_0$ a smooth map and $x \in I_0$, for every $n \in \mathbb{N}$, let us consider $T_n(x)$, the maximal interval containing x where f^n is a diffeomorphism; and $r_n(x)$, the length of the smallest component of $f^n(T_n(x)) \setminus f^n(x)$. The Koebe Principle guarantee distortion bounds in the orbit of a point x, if the respective $r_n(x)$ are not too small, whenever $Sf < 0$ or we have bounds for the cross-ratio operator.

Of course, a lower bound on $r_n(x)$ is also a lower bound on $f^n(T_n(x))$ so that, in this case, the images of the monotonicity intervals are not too small.

In their proof of Keller's theorem [MvS], de Melo and van Strien show that the hypotheses about the positiveness of the Lyapunov exponent implies a control of the $r_n(x)$:

$$\limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)| > 0 \quad \Longrightarrow \quad \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} r_i(x) > 0$$

$$\text{Lebesgue a.e. } x \qquad\qquad \text{Lebesgue a.e. } x$$

In order to prove Theorem A, we need an extension of this fact for C^1 compact families of interval maps. For every $n \in \mathbb{N}$, the map f^n will be replaced by the composition of n smooth maps, possibly with critical points. The precise statement is given in Theorem B below. In the setting of Theorem A, the result is applied to the restrictions of φ to the orbits of the vertical leaves.

First we give some definitions and notations.

Let I_0 be an interval. Let us consider a sequence $\{f_k\}_{k\geq 0}$ of C^1 maps $f_k : I_0 \to I_0$. Every map in the sequence has a set of critical points. Let us denote by $\mathscr{C}_k$ the set of critical points of f_k, for every $k \geq 0$. $\mathscr{C}_k$ could be an empty set for any $k \in \mathbb{N}$. We are interested on the study of the dynamics given by the compositions of maps in the sequence. Thus, we define for $i \geq 1$ and $x \in I_0$,

$$f^i(x) = f_{i-1} \circ \ldots \circ f_1 \circ f_0(x)$$

and we denote $f^0(x) = x$ for $x \in I_0$.

Based on the definitions of $T_i(x)$ and $r_i(x)$ on the case that there are just iterates of a function, we define for $i \in \mathbb{N}, x \in I_0$,

$$\begin{aligned} T_i\left(\{f_k\}, x\right) &:= \text{Maximal interval contained in } I_0, \text{containing } x, \\ &\quad\; \text{such that } f^j(T_i(x)) \cap \mathscr{C}_j = \emptyset \;\text{ for } 0 \le j < i. \\ L_i\left(\{f_k\}, x\right), R_i\left(\{f_k\}, x\right) &:= \text{Connected components of } \; T_i\left(\{f_k\}, x\right) \setminus \{x\}. \\ r_i\left(\{f_k\}, x\right) &:= \min\left\{ \left| f^i\left(L_i\left(\{f_k\}, x\right)\right)\right|, \left| f^i\left(R_i\left(\{f_k\}, x\right)\right)\right| \right\}. \end{aligned}$$

When it does not lead to confusion, we denote these functions by $T_i(x)$, $L_i(x)$, $R_i(x)$, $r_i(x)$. In this subsection and in the proof of the results of this subsection, we will use this simplified notation, since the sequence $\{f_k\}$ is fixed.

Our interest is to show that positive Lyapunov exponents (Lebesgue almost every point) implies that the average of the r_i is positive (Lebesgue almost every point).

We take a sequence $\{f_k\}$ satisfying the next condition: there exists $\lambda > 0$ such that

$$(P) \qquad \liminf_{n\to\infty} \frac{1}{n} \log |Df^n(x)| > 2\lambda$$

Lebesgue almost every $x \in I_0$.

We ask some uniformity of the continuity and the boundedness of the sequence in the C^1 topology. Recall that a sequence $\{f_k\}_k$ of C^1 maps $f_k : I_0 \to I_0$ is said to be C^1-*uniformly equicontinuous* if, given $\zeta > 0$, there exists $\epsilon > 0$ such that

$$|x - y| < \epsilon \quad \text{implies} \quad \begin{cases} |f_k(x) - f_k(y)| < \zeta \\ |Df_k(x) - Df_k(y)| < \zeta \end{cases} \tag{3}$$

for all $k \in \mathbb{N}$. Recall also that a sequence $\{f_k\}_k$ of C^1 maps $f_k : I_0 \to I_0$ is said to be C^1-*uniformly bounded* if there exists $\Gamma > 0$ such that for every $x \in I_0$,

$$|f_k(x)|\,,\; |Df_k(x)| \le \Gamma \tag{4}$$

for all $k \in \mathbb{N}$.

Our main result in this setting is the following.

Theorem B. *Let $\{f_k\}$ be a C^1-uniformly equicontinuous and C^1-uniformly bounded sequence of smooth maps $f_k : I_0 \to I_0$ for which $p = \sup_k \#\mathscr{C}_k < \infty$, and (P) holds for some $\lambda > 0$. Then, there exists $\varsigma > 0$ such that*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_i(x) \ge \varsigma \tag{5}$$

for Lebesgue almost every $x \in I_0$.

Remark 1.2. We do not require that f_k be multimodal, for any $k \ge 0$; and we do not make assumptions about the Schwarzian derivative of the maps in the sequence.

In a similar way we can prove the same statement but replacing lim inf by lim sup.

Corollary I. *Let $\{f_k\}$ be a C^1-uniformly equicontinuous and C^1-uniformly bounded sequence of smooth maps $f_k : I_0 \to I_0$ for which $p = \sup_k \#\mathscr{C}_k < \infty$, and there exists $\lambda > 0$ such that*

$$(P') \qquad \limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)| > 2\lambda$$

for Lebesgue almost every $x \in I_0$. Then, there exists $\varsigma > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_i(x) \geq \varsigma$$

for Lebesgue almost every $x \in I_0$.

In the case that the sequence $\{f_k\}_{k\in\mathbb{N}}$ is constant, i.e. $f_k = f$, for all $k \geq 0$, we also obtain the following consequence dealing with absolutely continuous invariant measures for unimodal maps. The statement has the same conclusions as the Keller's theorem, but we do not assume negative Schwarzian derivative. Instead, we just ask the non existence of neutral periodic points.

Corollary II. *Let $f : I_0 \to I_0$ be a C^3 multimodal map with non-flat critical point. Assume that f does not have neutral periodic points. If there exists $\lambda > 0$ such that for Lebesgue almost every point $x \in I_0$,*

$$(Q') \qquad \limsup_{n\to+\infty} \tfrac{1}{n} \log |Df^n(x)| > 2\lambda$$

Then there exists an absolutely continuous invariant measure.

From the method of proof of Theorem A we also obtain another extension of Keller's theorem. Here we do not assume the non-flatness of the critical points.

Corollary III. *Let $f : I_0 \to I_0$ be a C^3 multimodal map with $Sf < 0$. If there exists $\lambda > 0$ such that for Lebesgue almost every point $x \in I_0$,*

$$(Q') \qquad \limsup_{n\to+\infty} \tfrac{1}{n} \log |Df^n(x)| > 2\lambda$$

Then there exists an absolutely continuous invariant measure.

2 Preliminaries

In the one dimensional dynamics there are many tools that allows the construction of absolutely continuous invariant measures under varied hypotheses. In the first part of this section we describe some of the most important. We also state the Keller's theorem, which guarantee the existence of absolutely continuous invariant measures for unimodal maps with positive Lyapunov exponents assuming negative Schwarzian derivative. In the second part, we state the theorem of [ABV] for maps with singularities on manifolds of any dimension.

2.1 One dimensional dynamics

Let $f : I \to I$ be a piecewise monotone endomorphism with a finite number of turning points, i.e, points in the interior of I where f has a local extremum. Specifically, there exist $c_1 < c_2 < \ldots < c_s$ in the interior of I such that f is strictly monotone in each of the $s+1$ intervals $[a, c_1)$, (c_1, c_2), $\ldots$, $(c_s, b]$, where $I = [a, b]$. We say such a map is *s-modal* if f has exactly s turning points and if $f(\partial I) \subset \partial I$, when s is not specified we say the map is *multimodal*. In particular, we say that f is *unimodal* if f has precisely one turning point. We will always assume our maps are at least C^1 and sometimes we will ask they to be C^2 or moreover C^3, in these cases it will be specified.
When the multimodal map is C^1 and $c_i \in I$ is one of its critical points, then $Df(c_i) = 0$ and Df changes sign at this point. We say that this critical point c_i is C^n *non-flat of order* $l_i > 1$ if there exist ϕ_i a local C^n diffeomorphism with $\phi_i(c_i) = 0$, such that near c_i, f can be written as

$$f(x) = \pm|\phi_i(x)|^{l_i} + f(c_i).$$

The critical point is C^n *non-flat* if it is C^n non-flat of some order $l > 1$. In all that follows, we will simply say that c is a non-flat critical point of a C^n multimodal map f if c is a C^n non-flat critical point. For example if f is C^∞ and some derivative at c is non-zero then c is a non-flat critical point.
For unimodal maps with non-flat critical point of order $l > 1$, there exist $L > 1$ such that

$$\frac{|x-c|^{l-1}}{L} \leq |Df(x)| \leq L|x-c|^{l-1}$$

for $x \in [-1, 1]$. For multimodal maps with all critical points non-flat of the same order $l > 1$, there exists $L > 1$ such that

$$\frac{|x-\mathscr{C}|^{l-1}}{L} \leq |Df(x)| \leq L|x-\mathscr{C}|^{l-1}$$

for $x \in [-1, 1]$, where $\mathscr{C}$ denotes the set of critical points.
When the map f is C^3 (or three times differentiable) we can define

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

for x such that $f'(x) \neq 0$. This quantity is called the *Schwarzian derivative* of f at the point x. There are many results for one dimensional dynamics that are only known for those maps which have negative Schwarzian derivative.

Remark 2.1. There is a condition (it does not need assume the function f be C^3) which is enough to show many results for unimodal maps that are proved using the negative Schwarzian derivative condition; this condition is the convexity of the function $|Df|^{-1/2}$

on the connected components of $I \setminus \{c\}$ (see for example [NS]).

By a simple calculation we can verify a rule for the Schwarzian derivative of the composition of functions,

$$S(f \circ g)(x) = Sf(g(x))(g'(x))^2 + Sg(x)$$

when $f, g : [-1, 1] \to [-1, 1]$ and the quantities above are well defined. In particular, this implies that the negative Schwarzian derivative condition is inherited for the iterates of a function with this condition, i.e, if $Sf < 0$ then $S(f^n) < 0$ for all $n \in \mathbb{N}$.
There is an operator, known as *cross ratio operator*, acting in subintervals of a fixed interval which has connection with the Schwarzian derivative. Let $J \subset T$ be open intervals and L, R the connected components of $T \setminus J$, we define

$$b(T, J) = \frac{|J||T|}{|L||R|}$$

and if f is monotone continuous, we define

$$B(f, T, J) = \frac{b(f(T), f(J))}{b(T, J)},$$

when $f^n : I \to I$ is monotone then

$$B(f^n, T, J) = \prod_{i=0}^{n-1} B(f, f^i(T), f^i(J)).$$

In the case that f is C^3 with non-flat critical points, there exists $C > 0$ such that if $f^n : T \to f^n(T)$ is monotone

$$B(f^n, T, J) \geq \exp\Big(-C \sum_{i=0}^{n-1} |f^i(T)|^2\Big).$$

We can characterize the functions with zero Schwarzian derivative in terms of this operator: $Sf = 0$ if and only if $B(f, T^*, J^*) = 1$ for all $J^* \subset T^* \subset T$ (where T is the domain of f). In fact, according to the sign of the Schwarzian derivative, the operator is less, equal, or greater than 1. Specifically, it holds the next result which is the most important property for maps with negative Schwarzian derivative.

Lemma 2.1. *Let f be a C^3 map with $Sf < 0$ and M an interval such that f restricted to M is a diffeomorphism. Then*

$$B(f, M, J) \geq 1$$

for any $J \subset M$.

The fact that the operator B be greater than 1 is a great advantage to get control of distortion for one dimensional maps, since in certain conditions, this property implies bounded distortion. That is the content of the Koebe Principle.
If $U \subset V$ are intervals, we say that V is an *ϵ-scaled* neighborhood of U if both components of $V \setminus U$ have length $\epsilon|U|$.

Proposition 2.1 (**Koebe Principle**). *Let $J \subset T$ be intervals and $C_0 \in (0, 1]$ a constant. Assume $h : T \to h(T)$ is a C^1 diffeomorphism. Also assume that for any intervals $J^* \subset T^* \subset T$, we have*

$$B(h, T^*, J^*) \geq C_0 > 0$$

If $h(T)$ contains a τ-scaled neighborhood of $h(J)$, then

$$\frac{1}{K} \leq \frac{|Dh(x)|}{|Dh(y)|} \leq K$$

for all $x, y \in J$, where $K = (1 + \tau)^2/C_0^6\tau^2$. In particular, K does neither depend on the intervals $J \subset T$, nor on h.

Whenever h has negative Schwarzian derivative, to obtain bounded distortion in a certain interval J, we just need to guarantee the existence of a bigger interval T ($J \subset T$) such that, for some τ, $h(T)$ contains a τ-scaled neighborhood of $h(J)$, since by lemma 2.1 the condition about the operator B is satisfied with $C_0 = 1$.

Having in mind the Koebe Principle, one necessary step to control the distortion, when we do not assume negative Schwarzian derivative, is to bound the cross ratio operator. In the case that all periodic points are repelling, the problem is solved by the theorem that we now state. Recall that a periodic point p of period k is *repelling* if $|Df^k(p)| > 1$, *attracting* if $|Df^k(p)| < 1$ and *neutral* if $|Df^k(p)| = 1$. The proof of the result follows from a theorem of Kozlovski [Ko, Theorem B] for the unimodal case, and a theorem of van Strien & Vargas [SV, Theorem C] for the multimodal case. The hypotheses of these theorems are less restrictive than ours.

Theorem 2.1. *Let $f : [-1, 1] \to [-1, 1]$ be a C^3 multimodal map with non-flat critical points. Assume that the periodic points of f are repelling. Then, there exists $C > 0$ such that if $I \subset M$ are intervals, it holds*

$$B(f^n, M, I) \geq \exp(-C|f^n(M)|^2).$$

Remark 2.2. Whenever f has negative Schwarzian derivative and p is a neutral periodic point, p must be an attractor (see for example [CE]). Hence, if $Sf < 0$ and the Lyapunov exponents are positive, the periodic points of f must be repelling.

The bounded distortion property is a fundamental condition in the proofs of existence of absolutely continuous invariant measures, no matter what is the setting. A measure μ defined on the Borel σ-algebra of I is called *absolutely continuous* if it is absolutely continuous with respect to the Lebesgue measure on I, i.e, given any $A \subset I$ Borelian, if the Lebesgue measure of A is zero then $\mu(A)$ must be zero.

We are particularly interested on the existence of this kind of measures (absolutely continuous invariant) in the setting that we have defined before. For unimodal maps with non-flat critical point and with positive Lyapunov exponents, there is a well known theorem of Keller about the existence of these measures, under the negative Schwarzian derivative assumption (see [Ke]).

Theorem 2.2 (Keller). *Let $f : [-1, 1] \to [-1, 1]$ be a C^3 unimodal map with non-flat critical point and $Sf < 0$. If for Lebesgue almost every point $x \in [-1, 1]$,*

$$\limsup_{n\to+\infty} \frac{1}{n} \log |Df^n(x)| > 0, \tag{6}$$

then there exists an absolutely continuous invariant measure.

As a consequence of one of our results, we will prove the same conclusion without any assumption on the Schwarzian derivative. We already saw in Lemma 2.1 that this assumption gives immediately the boundedness of the operator B. We will use a result of Kozlovski to obtain boundedness of this operator for some applications induced (that we will define after) by the map f, and using the Koebe lemma the bounded distortion will follow.

One standard way to prove the existence of absolutely continuous invariant measures for f is to associate to it a Markov map and take advantage of the known fact of the existence of this kind of measures for Markov maps.

Definition 2.1. We call a map $F : I \to I$ *Markov* if there exists a enumerable family of disjoint open intervals $\{I_i\}_{i\in\mathbb{N}}$ with $\mathrm{Leb}(I \setminus \cup I_i) = 0$, such that

(M_1). There exists $K > 0$ such that for every $n \in \mathbb{N}$ and every J such that $F^j(J)$ is contained in some I_i for $j = 0, 1, \ldots, n$, it holds

$$\frac{|DF^n(x)|}{|DF^n(y)|} \leq K$$

for $x, y \in J$.

(M_2). If $F(I_k) \cap I_j \neq \emptyset$ then $I_j \subset F(I_k)$.

(M_3). There exists $r > 0$ such that $|F(I_i)| \geq r$ for all i.

And the result about existence of absolutely continuous invariant measures for Markov maps is the following (see [MvS, Theorem V.2.2]):

Theorem 2.3. *Let $F : I \to I$ be a Markov map and let $\cup I_i$ be corresponding partition. Then there exists an absolutely continuous F-invariant probability measure.*

We say that $F : \cup I_i \hookleftarrow$ is *induced* by $f : J \to J$, if $\cup I_i$ is a partition (modulo a zero Lebesgue measure set) of some interval $I \subset J$ and $F_{|I_i} = f^{k(i)}{}_{|I_i}$ for all $i \in \mathbb{N}$, where $k(i) \in \mathbb{N}$.
Under certain conditions, the measure for the induced Markov map will define, in a natural way, an absolutely continuous invariant measure for f.

Theorem 2.4. *Let $F : \cup I_i \to I$ be a Markov map induced by $f : J \to J$ and let $k(i)$ be such that $F_{|I_i} = f^{k(i)}{}_{|I_i}$. If ν is an absolutely continuous invariant probability of F and*

$$\sum_{i=1}^{\infty} k(i)\nu(I_i) < \infty,$$

then f has an absolutely continuous invariant probability defined by

$$\mu = \sum_{i=1}^{\infty} \sum_{j=0}^{k(i)-1} f_*^j \nu_i \,,$$

where ν_i denotes ν restricted to I_i, i.e, $\nu_i(A) = \nu(A \cap I_i)$.

One of the most important results proved first for maps with negative Schwarzian derivative and extended later for maps with non-flat critical points is about the non existence of wandering intervals. An interval $J \subset I$ is called a *wandering interval* if:

1. the intervals $J, f(J), \ldots$ are pairwise disjoint;

2. the images $f^n(J)$ do not converge to a periodic attractor when $n \to \infty$.

Theorem 2.5 (de Melo, van Strien, Martens). *If $f : I \to I$ is a C^2 map with non-flat critical points then f has no wandering interval.*

For the proof of this result see [SV, Corollary of the proof of Theorem A] or [MvS, Theorem A, Chapter IV]. There are another facts about unimodal maps that can be proved without any assumption in the Schwarzian derivative, assuming in general the non-flatness of the critical point. We put some of them (those which are useful in the construction of absolutely continuous invariant measures) together in one theorem. Recall that the Lebesgue measure is said to be *ergodic* for $f : J \to J$, if for each $X \subset J$ such that $f^{-1}(X) = (X)$, X or $\complement X$ have full Lebesgue measure.

Theorem 2.6. *Let $f : I \to I$ be a C^3 multimodal map with non-flat critical points and with all the periodic points repelling (i.e, f does not have either attracting periodic points or neutral periodic points). Then*

1. *f does not have homtervals, i.e, open intervals J such that $f^n_{/J}$ is homeomorphism onto its image, for all $n \geq 1$.*

2. *The set of preimages of the critical set $\mathscr{C}$ is dense in I.*

3. *For Lebesgue almost every $x \in I$, $\omega(x)$ contains a critical point.*

4. *If f is unimodal, the critical point is approximated by periodic or preperiodic points.*

5. *There are finitely many forward invariant sets $X_1, \ldots, X_k$ such that $\cup B(X_i)$ has full measure in I, and $f_{|B(X_i)}$ is ergodic with respect to the Lebesgue measure. Here, $B(X_i) = \{y; \omega(y) = X_i\}$ is the basin of X_i. In the unimodal case, $k = 1$, which implies that f is ergodic with respect to Lebesgue measure.*

For items 1. to 4., one just need f to be C^2. The proof of item 5. is contained in the proof of Theorem E of [SV]. However, in the unimodal case, a simpler proof follows combining Lemma 7.4 of [Ko] with canonical arguments, as the used for instance on the proof of Theorem V.1.2 of [MvS].

2.2 Non uniformly expanding dynamics

We state the theorem of [ABV] about existence of absolutely continuous invariant measures for maps with singularities defined in manifolds of any dimension.

Let M be a compact manifold, $\mathcal{S} \subset M$ a compact subset and $\varphi : M \setminus \mathcal{S} \to M$ a C^2 map on $M \setminus \mathcal{S}$. We say that $\mathcal{S} \subset M$ is a *non-degenerate singular set* for φ if the following conditions hold. The first one essentially says that φ *behaves like a power of the distance* to $\mathcal{S}$: there are constants $B > 1$ and $\beta > 0$ such that for every $x \in M \setminus \mathcal{S}$

$$(S_1) \qquad \frac{1}{B}\operatorname{dist}(x,\mathcal{S})^{\beta} \leq \frac{\|D\varphi(x)v\|}{\|v\|} \leq B\operatorname{dist}(x,\mathcal{S})^{-\beta} \text{ for all } v \in T_xM.$$

Moreover, we assume that the functions $\log|\det D\varphi(x)|$ and $\log\|D\varphi(x)^{-1}\|$ are locally Lipschitz at points $x \in M \setminus \mathcal{S}$ with Lipschitz constant depending on $\operatorname{dist}(x,\mathcal{S})$: for every $x, y \in M \setminus \mathcal{S}$ with $\operatorname{dist}(y,x) < \operatorname{dist}(x,\mathcal{S})/2$ we have

$$(S_2) \qquad \left|\log\|D\varphi(x)^{-1}\| - \log\|D\varphi(y)^{-1}\|\,\right| \leq \frac{B}{\operatorname{dist}(x,\mathcal{S})^{\beta}}\operatorname{dist}(x,y);$$

$$(S_3) \qquad \left|\log|\det D\varphi(x)| - \log|\det D\varphi(y)|\,\right| \leq \frac{B}{\operatorname{dist}(x,\mathcal{S})^{\beta}}\operatorname{dist}(x,y).$$

In this setting, one property that must be controlled to get absolutely continuous invariant measures, is the recurrence of the points of M to the singular set $\mathcal{S}$. Given $\delta > 0$ and $x \in M \setminus \mathcal{S}$ we define the δ-truncated distance from x to $\mathcal{S}$ by

$$\operatorname{dist}_{\delta}(x,\mathcal{S}) = \begin{cases} 1 & \text{if } \operatorname{dist}(x,\mathcal{S}) \geq \delta \\ \operatorname{dist}(x,\mathcal{S}) & \text{otherwise} \end{cases}$$

Definition 2.2. A subset $H \subset M$ has slow recurrence to the singular set $\mathcal{S}$ if, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(\varphi^j(x), \mathcal{S}) \leq \epsilon$$

for Lebesgue a.e. $x \in H$.

A probability measure μ on the Borel sets of M is said to be an *SRB measure* if there exists a positive Lebesgue measure set of points $z \in M$ for which

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} h(\varphi^j(z)) = \int h \, d\mu$$

for any continuous function $h : M \to \mathbb{R}$. The set of points $z \in M$ for which this holds is called the *basin* of μ.

Theorem 2.7 ([ABV, Theorem C]). *Let $\varphi : M \to M$ a C^2 map and $\mathcal{S}$ a non-degenerate singular set. Assume that φ is non-uniformly expanding, i.e, for Lebesgue almost every $x \in M$*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|D\varphi(\varphi^j(x))^{-1}\|^{-1} < 0 \tag{7}$$

and M has slow recurrence to the singular set. Then Lebesgue almost every point in M belongs to the basin of some ergodic absolutely continuous invariant measure.

The proof of this theorem is based in a similar idea to the one used in one dimensional dynamics when we construct induced maps from the original dynamics, i.e, to found neighborhoods related to some iterate of φ in such a way that this iterate of the map, restricted to the related neighborhood, has bounded distortion and the image of this map has volume bounded away from zero. However, in this case one does not construct a Markov map, the work consists in a detailed analysis of Cesaro averages of $\varphi^j_* \operatorname{Leb}_{W_j}$ where W_j are open sets chosen conveniently in the space. The choice of these open sets is related to the notion of hyperbolic times.

Definition 2.3. Fix $B > 1$ and $\beta > 0$ as in the hypotheses $(S1), (S2), (S3)$, and take b a constant such that $0 < b < \min\{1/2, 1/(2\beta)\}$. Given $\sigma < 1$ and $\delta > 0$, we say that n is a (σ, δ)-hyperbolic time for a point $x \in M$ if,

$$\prod_{j=n-k}^{n-1} \|D\varphi(\varphi^j(x))^{-1}\| < \sigma^k \quad \text{and} \quad \operatorname{dist}_\delta(\varphi^{n-k}(x), \mathcal{S}) > \sigma^{bk} \quad \text{for all } 1 \leq k \leq n$$

The following result (see [ABV, Lemma 5.2.]) gives the most important geometric properties of hyperbolic times.

Proposition 2.2. *Given $\sigma < 1$ and $\delta > 0$, there exists $\delta_1 > 0$ such that if n is a (σ, δ)-hyperbolic time for a point $x \in M \setminus \mathcal{S}$, then there exists a neighborhood $V_n(x)$ of x such that*

1. *φ^n maps $V_n(x)$ diffeomorphically and with bounded distortion onto the ball of radius δ_1 around $\varphi^n(x)$*

2. *for every $1 \leq k \leq n$ and $y, z \in V_n(x)$*

$$\operatorname{dist}(\varphi^{n-k}(y), \varphi^{n-k}(z)) \leq \sigma^{k/2} \operatorname{dist}(\varphi^n(y), \varphi^n(z)).$$

3 Compositions of smooth one dimensional maps

3.1 Proof of Theorem B

We begin by introducing some sets useful for the proof of the theorem. Recalling the definitions in subsection 1.2, for every $n \in \mathbb{N}$ and $\delta > 0$ we denote by,

$$A_n(\{f_k\}, \delta) := \left\{ x \in I_0 \,;\, \frac{1}{n} \sum_{i=1}^{n} r_i(x) < \delta^2, \;\; r_n(x) > 0 \right\}, \tag{8}$$

and given $\lambda > 0$, we define for $n \in \mathbb{N}$,

$$Y_n(\{f_k\}, \lambda) := \left\{ x, \frac{1}{n} \log |Df^n(x))| > \lambda \right\}. \tag{9}$$

When it does not lead to confusion, we denote these sets by $A_n(\delta)$ and $Y_n(\lambda)$. In fact, we will do it in all this section.

It is clear that (5) holds (with $\varsigma = \delta^2$) for Lebesgue almost every $x \in I_0$ if $\cap_{n \geq N}(\complement A_n(\delta) \cap Y_n(\lambda))$ converges to the Lebesgue measure of I_0, when $N \to \infty$. We claim that this in effect, happens. Indeed, for every $N \in \mathbb{N}$, it holds

$$\left(\bigcap_{n \geq N} Y_n(\lambda) \right) \cap \complement \left(\bigcup_{n \geq N} A_n(\delta) \cap Y_n(\lambda) \right) \subset \bigcap_{n \geq N} \complement A_n(\delta) \cap Y_n(\lambda).$$

Then $| \cap_{n \geq N} \complement A_n(\delta) \cap Y_n(\lambda)| \geq | \cap_{n \geq N} Y_n(\lambda)| - | \cup_{n \geq N} A_n(\delta) \cap Y_n(\lambda)|$. By (P), $| \cap_{n \geq N} Y_n(\lambda)|$ converges to the Lebesgue measure of I_0. Thus, to prove the claim we just need to prove that $| \cup_{n \geq N} A_n(\delta) \cap Y_n(\lambda)|$ converges to zero. For this purpose we will state the following result which is the main lemma for the proof of Theorem B.

Lemma 3.1. *Let $\{f_k\}$ be a C^1-uniformly equicontinuous and C^1-uniformly bounded sequence of smooth maps $f_k : I_0 \to I_0$ for which $p = \sup_k \#\mathscr{C}_k < \infty$. Then, given $\lambda > 0$, there exist $\delta > 0$ such that*

$$|A_n(\delta) \cap Y_n(\lambda)| \leq |I_0| \exp(-n\lambda/2) \tag{10}$$

for n big enough. Moreover, δ depends only on the modulus of continuity (3), the uniform bound Γ in (4) and the uniform bound p for the number of critical points.

The proof of this lemma relies on bounding the number of connected components of the set $A_n(\delta)$ whose intersection with $Y_n(\lambda)$ is non-empty. We do not know a way to count directly the components of $A_n(\delta)$, for this reason we will define some sets related to these components. The new sets seem easier to deal and to count than the components of $A_n(\delta)$.

For $\delta > 0$, $a_i \in \{0, 1\}$ for $i = 1, 2, \dots, n$,

$$\begin{aligned} C_\delta(a_1, a_2, \dots, a_n) :=& \{x \in I_0 \,;\, r_i(x) \geq \delta \text{ if } a_i = 1, \\ & 0 < r_i(x) < \delta \text{ if } a_i = 0\} \end{aligned}$$

Note that every connected component of $C_\delta(a_1, \dots, a_s, a_{s+1})$ is contained in a connected component of $C_\delta(a_1, \dots, a_s)$, and moreover, every connected component of $C_\delta(a_1, \dots, a_s)$ is a union of connected components (with its boundaries) of $C_\delta(a_1, \dots, a_s, a_{s+1})$. Also note that for every I connected component of $C_\delta(a_1, \dots, a_s)$, $I \subset T_s(x)$ for all $x \in I$. Recall the definition of $T_i(x)$ in subsection 1.2.

Given $x \in I_0$ and $n \in \mathbb{N}$, if $f^i(x) \notin \mathscr{C}_i$ for $0 \leq i < n$, we can associate to it a sequence $\{a_i(x)\}_{i=1}^n$, according to the last definition, in a natural way:

$$a_i(x) = \begin{cases} 0 & \text{if } 0 < r_i(x) < \delta \\ 1 & \text{if } \ r_i(x) \geq \delta \end{cases}$$

For this sequence the inequality $(a_1(x) + \ldots + a_n(x))\delta \leq \sum_{i=1}^n r_i(x)$ is satisfied. In particular, for every $x \in A_n(\delta)$, the associated sequence $\{a_i(x)\}_{i=1}^n$ is such that $a_1(x) + \ldots + a_n(x) < \delta n$. Therefore, if we define

$$C_n(\delta) := \bigcup_{a_1 + \ldots + a_n < \delta n} C_\delta(a_1, \dots, a_n),$$

we conclude that $A_n(\delta) \subset C_n(\delta)$.

But in fact, we are interested on the connected components of $A_n(\delta)$ which intersect the set $Y_n(\lambda)$. We will say that a connected component J of $A_n(\delta)$ is a connected component of $A'_n(\delta)$ if $J \cap Y_n(\lambda) \neq \emptyset$ and we will say that a connected component I of $C_\delta(a_1, a_2, \dots, a_n)$ is a connected component of $C'_\delta(a_1, a_2, \dots, a_n)$ if $I \cap Y_n(\lambda) \neq \emptyset$.

We can associate to every connected component of $A'_n(\delta)$, a connected component of $C'_\delta(a_1, a_2, \dots, a_n)$, where $a_1 + a_2 + \ldots + a_n < \delta n$: for J connected component of $A'_n(\delta)$, there exist $a_1, \dots, a_n$ (such that $a_1 + a_2 + \ldots + a_n < \delta n$) and I connected component of $C'_\delta(a_1, a_2, \dots, a_n)$, for which $J \cap I \neq \emptyset$. Indeed, we can consider $a_i = a_i(x)$ $(1 \leq i \leq n)$ for

$x \in J \cap Y_n(\lambda)$, and I the connected component of $C'_\delta(a_1, a_2, \ldots, a_n)$ which contains x. Thus, we associate to J, the component I.

We would like to bound the number of connected components of $A'_n(\delta)$ by the number of connected components of $C'_\delta(a_1, a_2, \ldots, a_n)$, varying $a_1, \ldots, a_n$ such that $a_1 + a_2 + \ldots + a_n < \delta n$. That is not possible, since every connected component of $C'_\delta(a_1, a_2, \ldots, a_n)$ (with $a_1 + a_2 + \ldots + a_n < \delta n$) could intersect more than one connected component of $A'_n(\delta)$. By this reason we define the following set

$$A''_n(\delta) := \bigcup_{J' \in A'_n(\delta)} J'',$$

where

$$J'' := J' \quad \cup \bigcup_{a_1+\ldots+a_n<\delta n} \{\text{ connected components of } C_\delta(a_1, \ldots, a_n) \text{ which intersect } J' \cap Y_n(\lambda)\}$$

Obviously, a connected component of $A''_n(\delta)$ could contain more than one connected component of $A'_n(\delta)$. However, such as happens with the connected components of $A'_n(\delta)$, the restriction of f^n to every connected component of $A''_n(\delta)$ is a diffeomorphism. Using this fact, we will show in the proof of Lemma 3.1 that in order to obtain (10), it is enough to estimate the number of connected components of $A''_n(\delta)$.

Since every component of $A''_n(\delta)$ intersect at least one component of $C'_\delta(a_1, \ldots, a_n)$, we conclude that

$$\# A''_n(\delta) \leq \sum \# C'_\delta(a_1, \ldots, a_n) \tag{11}$$

where the sum is over all $a_1, \ldots, a_n$ such that $a_1 + \ldots + a_n < \delta n$, and $\#X$ denotes the number of connected components of X.

As we said, Lemma 3.1 will be a consequence of the following result, which gives an estimate of the number of connected components of $A''_n(\delta)$.

Lemma 3.2. *There exists $\delta > 0$ such that the number of connected components of $A''_n(\delta)$ is less than* $\exp(n\lambda/2)$*.*

For the proof of Lemma 3.2 we will use several results that we now state. First we give some notations. Given $\epsilon > 0$, for every $k \geq 0$, we call $V_\epsilon \mathscr{C}_k$ a neighborhood of $\mathscr{C}_k$ defined as the union of all $B(x, \epsilon)$ (ball centered in x of ratio ϵ) varying $x \in \mathscr{C}_k$. In order to simplify the notation we say $f^j(x) \in V_\epsilon \mathscr{C}$ if $f^j(x) \in V_\epsilon \mathscr{C}_j$ for any $j \in \mathbb{N}$. The next lemma says that for points in $Y_n(\lambda)$, the frequence of visits to the neighborhood $V_\epsilon \mathscr{C}$ can be made arbitrarily small, if ϵ is chosen small enough.

Lemma 3.3. *Given $\gamma > 0$, there exists $\epsilon > 0$, such that for $x \in Y_n(\lambda)$,*

$$\frac{1}{n} \sum_{j=0}^{n-1} \chi_{V_\epsilon \mathscr{C}}(f^j(x)) < \gamma \tag{12}$$

Moreover, ϵ does not depend on n, it depends on λ, the modulus of continuity of $\{f_k\}$ and the uniform bound of $\{Df_k\}$.

Proof. Using the fact that the sequence $\{f_k\}_{k\geq 0}$ is C^1-uniformly equicontinuous, we conclude that given $\zeta > 0$, there exists $\epsilon = \epsilon(\zeta)$ such that

$$|x - \mathscr{C}_k| < \epsilon \quad \text{implies} \quad |Df_k(x)| < \zeta \quad \text{for all } k \geq 0 \tag{13}$$

Thus, if $f^j(x) \in V_\epsilon \mathscr{C}$ then $\log |Df_j(f^j(x))| < \log \zeta$. On the other hand, since $\{f_k\}_{k\geq 0}$ is C^1-uniformly bounded, $|Df_k(x)| \leq \Gamma$ for all $k \geq 0$ and $x \in I_0$.

By the definition of $Y_n(\lambda)$,

$$\lambda n < \sum_{j=0}^{n-1} \log |Df_j(f^j(x))|.$$

Therefore, if we assume that (12) is false, we conclude that,

$$\lambda n < \gamma n \log \zeta + (1 - \gamma) n \log \Gamma$$

However, the function $\log \zeta \to -\infty$ when $\zeta \to 0$. Then for some ζ_0 small enough,

$$\lambda \geq \gamma \log \zeta_0 + (1 - \gamma) \log \Gamma,$$

since $\lambda > 0$. For the corresponding $\epsilon = \epsilon(\zeta_0)$, (12) must be valid. Obviously, because of the way that we found ϵ, it does not depend on n. □

Using the fact that ϵ does not depend on n, we easily can conclude the next result

Corollary 3.1. *If for Lebesgue almost every $x \in I_0$*

$$\liminf_{n\to\infty} \frac{1}{n} \log |Df^n(x))| \geq \lambda > 0$$

Then, given $\gamma > 0$, there exists $\epsilon > 0$, such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{V_\epsilon \mathscr{C}}(f^j(x)) < \gamma$$

Lebesgue almost every $x \in M$.

Let us denote for $i, j \in \mathbb{N}$, and $x \in I_0$,

$$f_i^j(x) = f_{i+j-1} \circ \ldots f_{i+1} \circ f_i(x)$$

and $f_0^0(x) = x$. Note that $f_0^j(x) = f^j(x)$ for $j \geq 0$ and $x \in I_0$. Again by the C^1-uniform equicontinuity of the sequence $\{f_k\}$, we have the following property.

Lemma 3.4. *Given $\epsilon > 0$ and $l \in \mathbb{N}$, there exists $\delta = \delta(l)$ such that*

$$|x - y| \le 2\delta \quad \textit{implies} \quad |f_i^j(x) - f_i^j(y)| < \epsilon \tag{14}$$

for $0 \le j \le l$ and for all $i \ge 0$. Moreover, δ just depends (on l and) on the modulus of continuity of $\{f_k\}$.

Remark 3.1. When $l \to \infty$ then $\delta(l) \to 0$. Observe that we also have: given $\epsilon > 0$ and $\delta > 0$, there exists $l = l(\delta) \in \mathbb{N}$ such that (14) holds for $0 \le j \le l$.

In order to count the components whose intersection with $Y_n(\lambda)$ is non-empty, let us decompose this set in a convenient way. Given $\epsilon > 0$, $m \le n$, $\{t_1, \ldots, t_m\} \subset \{0, 1, \ldots, n-1\}$, we define

$$Y_{n,\epsilon}(t_1, \ldots, t_m) = \{z \in Y_n(\lambda);\ f^j(x) \in V_\epsilon \mathscr{C} \iff j \in \{t_1, \ldots, t_m\}\}$$

By Lemma 3.3 we conclude that given $\gamma > 0$, there exists $\epsilon > 0$ such that

$$Y_n(\lambda) = \cup_{m=0}^{\gamma n} \cup_{t_1, \ldots, t_m} Y_{n,\epsilon}(t_1, \ldots, t_m) \tag{15}$$

where the second union is over all subsets $\{t_1, \ldots, t_m\}$ of $\{0, 1, \ldots, n-1\}$. This together with (11) yields,

$$\#A_n''(\delta) \le \sum_{a_1, \ldots, a_n} \sum_{t_1, \ldots, t_m} \#\{I \subset C_\delta(a_1, \ldots, a_n); I \cap Y_{n,\epsilon}(t_1, \ldots, t_m) \ne \emptyset\} \tag{16}$$

where the first sum is over all $a_1, \ldots, a_n$ such that $a_1 + \ldots + a_n < \delta n$ and the second sum is over all subsets $\{t_1, \ldots, t_m\}$ of $\{0, 1, \ldots, n-1\}$ with $m < \gamma n$.

Proof of Lemma 3.2. To prove the lemma we just need to bound the double sum in (16). For this we will show some claims related to the number of connected components of the sets $C_\delta(a_1, \ldots, a_n)$. Recall that p is the maximum number of elements in any $\mathscr{C}_k$ (for $k \ge 0$). Given $I \subset I_0$ and $s \in \mathbb{N}$, we say $f^s(I) \cap \mathscr{C} = \emptyset (\ne \emptyset)$ if $f^s(I) \cap \mathscr{C}_s = \emptyset (\ne \emptyset)$.

Claim 3.1. *For any $a_1, a_2, \ldots, a_s$ with $a_j \in \{0, 1\}$ for all j,*

$$\#C_\delta(a_1, \ldots, a_s, 0) + \#C_\delta(a_1, \ldots, a_s, 1) \le 3(p+1)\#C_\delta(a_1, \ldots, a_s)$$

Claim 3.2. *Let $s, n \in \mathbb{N}$ and J be a component of $C_\delta(a_1, \ldots, a_s, 0)$. If $f^{s+i}(J) \cap \mathscr{C} = \emptyset$ for $1 \le i \le n$, then*

$$\#\{I \subseteq C_\delta(a_1, \ldots, a_s, 0^{i+1}), I \subseteq J\} \le i + 1.$$

for $1 \le i \le n$, where 0^{i+1} means that the last $i+1$ terms are equal to 0.

The next claim bounds the number of components whose intersection with $Y_{n,\epsilon}(t_1, \ldots, t_m)$ is non empty.

Claim 3.3. *Let $l \in \mathbb{N}$ and $\epsilon > 0$ be constants and let $\delta = \delta(l)$ be the number given by Lemma 3.4. For any $a_1, \ldots, a_s$ with $a_j \in \{0, 1\}$, $\{t_1, \ldots, t_m\} \subset \{0, 1, \ldots, n-1\}$, if $\{s+1, \ldots, s+i\} \cap \{t_1, \ldots, t_m\} = \emptyset$ and $i \leq l$, then*

$$\#\{I \subseteq C_\delta(a_1, \ldots, a_s, 0^{i+1}), I \cap Y_{n,\epsilon}(t_1, \ldots, t_m) \neq \emptyset\}$$
$$\leq (i+1)\#\{I \subseteq C_\delta(a_1, \ldots, a_s, 0), I \cap Y_{n,\epsilon}(t_1, \ldots, t_m) \neq \emptyset\}.$$

Assuming the claims, we will prove the lemma. We have basically four constants, namely, $\delta, \gamma, \epsilon, l$. It is very important the order in what they are chosen. First, we choose $l \in \mathbb{N}$ according to the equation (21), then we choose $\gamma > 0$ according to (22). Next, we find $\epsilon > 0$, using Lemma 3.3, in such a way that (15) holds. Finally, given ϵ and l, let $\delta > 0$ be the constant given by Lemma 3.4 and satisfying (23). In all that follows we consider n big enough.

First, given $m < n$, $a_1, \ldots, a_n$ with $a_i \in \{0, 1\}$ and $\{t_1, \ldots, t_m\} \subset \{0, \ldots, n-1\}$, we will count the components of $C_\delta(a_1, \ldots, a_n)$ whose intersection with $Y_{n,\epsilon}(t_1, \ldots, t_m)$ is non-empty. We can decompose the sequence $a_1 \ldots a_n$ in maximal blocks of 0's and 1's; we will write the symbol ξ in the j-th position if $a_j = 1$ or, $a_j = 0$ and $j = t_k$ for some $k \in \{1, \ldots, m\}$. In this way we have,

$$a_1 a_2 \ldots a_n = \xi^{i_1} 0^{j_1} \xi^{i_2} 0^{j_2} \ldots \xi^{i_h} 0^{j_h} \tag{17}$$

with $0 \leq i_k, j_k \leq n$ for $k = 1, \ldots, h$, $\sum_{k=1}^{h}(i_k + j_k) = n$ and $\sum_{k=1}^{h} i_k < m + \delta n$.

Lets us assume that $a_1, \ldots, a_n$ are as in (17). Let l, ϵ and δ be as in Lemma 3.4. Using claims 3.1 and 3.3 we have,

$$\begin{aligned}
&\#\{I \subset C_\delta(a_0, \ldots, a_n), I \cap Y_{n,\epsilon}(t_1, \ldots, t_m) \neq \emptyset\} \leq \\
&\quad\leq (3(p+1)(l+1)^{\frac{j_h}{l}+1}(3(p+1))^{i_h}) \ldots (3(p+1)(l+1)^{\frac{j_1}{l}+1}(3(p+1))^{i_1}) \\
&\quad\leq (3(p+1))^{\sum_{k=1}^{h} i_k}(3(p+1))^h (l+1)^{\frac{\sum_{k=1}^{h} j_k}{l}+h} \\
&\quad\leq (3(p+1))^{m+\delta n+h}(l+1)^{\frac{n}{l}+h}.
\end{aligned}$$

Let us remark some properties about the decomposition (17):

- If $m < \gamma n$ then, since $a_1 + a_2 + \ldots + a_n < \delta n$, we have that $\sum_{k=1}^{h} i_k < \gamma n + \delta n$,

- If $a_1 + a_2 + \ldots + a_n < \delta n$ and $m < \gamma n$, the number of blocks $\zeta^{i_t} 0^{j_t}$ is bounded by the sum of these quantities, i.e, $h < (\delta + \gamma)n + 1$.

Therefore, if $a_1 + a_2 + \ldots + a_n < \delta n$ and $m < \gamma n$ we conclude from the inequality above that for n big enough,

$$\begin{aligned}
&\#\{I \subset C_\delta(a_1,\ldots,a_n), I \cap Y_{n,\epsilon}(t_1,\ldots,t_m) \neq \emptyset\} \\
&\quad \leq (3(p+1))^{\gamma n+\delta n}(3(p+1))^{2(\delta+\gamma)n}(l+1)^{\frac{n}{l}+2(\delta+\gamma)n} \qquad (18)\\
&\quad \leq \exp(n\ \psi_0(l,\gamma,\delta))
\end{aligned}$$

where $\psi_0(l,\gamma,\delta) = 3(\delta+\gamma)\log(3(p+1)) + 2(\delta+\gamma+\frac{1}{l})\log(2l)$.

On the other hand, by the Stirling's formula, the number of subsets of $\{0,1,\ldots,n-1\}$ of size less than γn is bounded by $\exp(n(\psi_1(\gamma)))$ and $\psi_1(\gamma) \to 0$ when $\gamma \to 0$. Therefore, from this fact and (18), we conclude

$$\sum_{t_1,\ldots,t_m} \#\{I \subset C_\delta(a_1,\ldots,a_n); I \cap Y_{n,\epsilon}(t_1,\ldots,t_m) \neq \emptyset\} \leq \exp(n\ \psi_2(l,\gamma,\delta)) \qquad (19)$$

where the sum is over all $\{t_1,\ldots,t_m\}$ subset of $\{0,1,\ldots,n-1\}$ with $m < \gamma n$, and $\psi_2(l,\gamma,\delta) = \psi_0(l,\gamma,\delta) + \psi_1(\gamma)$.

Once again, using the Stirling's formula we can conclude that the number of sequences $a_1, a_2, \ldots, a_n$ of 0's and 1's such that $a_1 + a_2 + \ldots + a_n < \delta n$ is less or equal than $\exp(n\psi_3(\delta))$ with $\psi_3(\delta) \to 0$ when $\delta \to 0$. Hence, by (16) and (19), we have that whenever γ, ϵ satisfy (15),

$$\#A''_n(\delta) \leq \exp(n\ \psi_4(l,\gamma,\delta))$$

where

$$\psi_4(l,\gamma,\delta) = 3(\delta+\gamma)\log(3(p+1)) + 2(\delta+\gamma+\frac{1}{l})\log(2l) + \psi_1(\gamma) + \psi_3(\delta) \qquad (20)$$

Then we have to choose l such that

$$\frac{2}{l}\log(2l) < \frac{\lambda}{14} \qquad (21)$$

Next, let $\gamma > 0$ be such that

$$\left.\begin{aligned}
2\gamma\log(2l) &< \frac{\lambda}{14} \\
3\gamma\log(3(p+1)) &< \frac{\lambda}{14} \\
\psi_1(\gamma) &< \frac{\lambda}{14}
\end{aligned}\right\} \qquad (22)$$

Next, we find $\epsilon > 0$, using Lemma 3.3. Finally, given ϵ and l, let $\delta > 0$ be the constant given by Lemma 3.4 and satisfying

$$\left.\begin{aligned} 2\delta\log(2l) &< \frac{\lambda}{14} \\ 3\delta\log(3(p+1)) &< \frac{\lambda}{14} \\ \psi_3(\delta) &< \frac{\lambda}{14} \end{aligned}\right\} \tag{23}$$

With this choice, $\psi_4(l,\gamma,\delta) \le \frac{\lambda}{2}$. Hence Lemma 3.2 is proved, assuming the three claims.

It just remains to prove the claims.

Proof of Claim 3.1. Let I be a connected component of $C_\delta(a_1,\ldots,a_s)$.

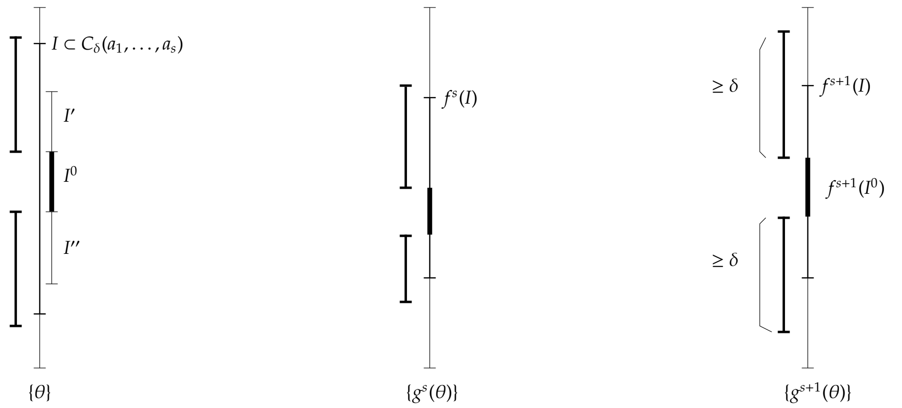

Figure 1: Forbidden situation in case that $f^s(I) \cap \mathcal{S} = \emptyset$: $I', I'' \subset C_\delta(a_1,\ldots,a_s,1)$ and $I_0 \in C_\delta(a_1,\ldots,a_s,0)$

Case 1. $f^s(I) \cap \mathscr{C} = \emptyset$. In this case, I is divided at most in 3 connected components of $C_\delta(a_1,\ldots,a_s,0) \cup C_\delta(a_1,\ldots,a_s,1)$. Indeed, since $I \subset T_{s+1}(x)$ for every $x \in I$, if $I' \subset I$ is a component of $C_\delta(a_1,\ldots,a_s,0)$, it can not exist one component of $C_\delta(a_1,\ldots,a_s,1)$ at each side of I'. Hence, the following situations can happen:

i) There are two components of $C_\delta(a_1,\ldots,a_s,0)$ in I, each of them has one extreme of I, and in the middle there is a component of $C_\delta(a_1,\ldots,a_s,1)$.

ii) There is exactly one component of $C_\delta(a_1,\ldots,a_s,0)$ in I. In this case there is one or none component of $C_\delta(a_1,\ldots,a_s,1)$ in I.

iii) There are no components of $C_\delta(a_1,\ldots,a_s,0)$ in I. In this case I is a component of $C_\delta(a_1,\ldots,a_s,1)$ in I.

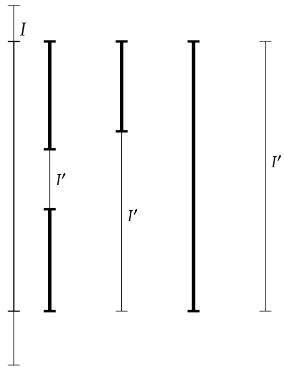

Figure 2: Possible cases when $f^s(I) \cap \mathcal{S} = \emptyset$: $I' \subset C_\delta(a_1, \dots, a_s, 1)$

Case 2. $f^s(I) \cap \mathscr{C} \neq \emptyset$. First I is divided at most in $p+1$ components, each one with at least one boundary which by f^s goes to $\mathscr{C}$. After that, following the same arguments used in case 1, we conclude that each one of these components is divided at most in 3 components. Hence, I is divided in at most $3(p+1)$ components of $C_\delta(a_1, \dots, a_s, 0) \cup C_\delta(a_1, \dots, a_s, 1)$. □

Proof of Claim 3.2. We will prove it by induction on i. For $i = 1$, it follows by the proof of claim 3.1. Let us assume that the statement is true for $j \leq i-1$. Let $I_1, \dots, I_t$ be the components of $C_\delta(a_1, \dots, a_s, 0^{(i-1)+1})$ contained in I. By the induction hypothesis $t \leq i$ and the assumption is that $f^i(I) \cap \mathscr{C} = \emptyset$. We claim that there exist at most one $k \in \{1, \dots, t\}$ such that I_k is divided in two components of $C_\delta(a_1, \dots, a_s, 0^{i+1})$ (the others I_k's generate one or none component of $C_\delta(a_1, \dots, a_s, 0^{i+1})$). Indeed, if I_{k_1} and I_{k_2} are divided in two components of $C_\delta(a_1, \dots, a_s, 0^{i+1})$, let $I^+_{k_1}$ and $I^-_{k_1}$ be the components of $C_\delta(a_1, \dots, a_s, 0^{i+1})$ and J_{k_1} the component of $C_\delta(a_1, \dots, a_s, 0^i, 1)$ contained on I_{k_1}. Analogously, let $I^+_{k_2}$, $I^-_{k_2}$, J_{k_2} be the corresponding for I_{k_2}. Two of the $I^*_{k_j}$ ($j \in \{1, 2\}, * \in \{+, -\}$) are between J_{k_1} and J_{k_2}, but that is a contradiction because for $x \in I^*_{k_j}$, $r_{s+i+1}(x) < \delta$, and for $x \in J_{k_1} \cup J_{k_2}$, $r_{s+i+1}(x) \geq \delta$. Hence, there are at most $i+1$ components of $C_\delta(a_1, \dots, a_s, 0^{i+1})$ contained in J. □

Proof of Claim 3.3. Let I be a connected component of $C_\delta(a_1, \dots, a_s, 0)$. Then we have $|f^{s+1}(I)| \leq 2\delta$, and by Lemma 3.4, $|f^{s+i}(I)| < \epsilon$ for $i \leq l+1$. If $f^{s+j}(I) \cap \mathscr{C} \neq \emptyset$ for some $j \leq i$, then for all $x \in I$, $f^{s+j}(x) \in V_\epsilon \mathscr{C}$. Since $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_m\} = \emptyset$, then $I \cap Y_{n,\epsilon}(t_1, \dots, t_m) = \emptyset$.

Hence, if $I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset$ and $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_l\} = \emptyset$, then $f^{s+j}(I) \cap \mathscr{C} = \emptyset$ for all $1 \leq j \leq i$. Therefore, using the claim 3.2, we conclude this claim. □

This finishes the proof of Lemma 3.2. □

Now we will prove that Lemma 3.1 follows as a consequence of Lemma 3.2.

Proof of Lemma 3.1. Note that if J'' is a connected component of $A''_n(\delta)$ then f^n restricted to J'' is a diffeomorphism onto its image. The set $Y_n(\lambda)$ is an open in I_0 and then $Y_n(\lambda) \cap J$ is an open set, for every J connected component of $A'_n(\delta)$. Therefore, there exist at most countably many components $\{I_k\}_{k\in\mathbb{N}}$ of $Y_n(\lambda) \cap A'_n(\delta)$ on J'',

$$|I_k| < (\exp(-n\lambda))|f^n(I_k)|,$$

for all $k \in \mathbb{N}$, since for every $x \in I_k$, $|Df^n(x)| > \exp(\lambda n)$. Adding these inequalities ($k \in \mathbb{N}$),

$$| \cup_k I_k| < (\exp(-n\lambda)) \sum_k |f^n(I_k)| \leq (\exp(-n\lambda))|f^n(J'')|.$$

Then, since $|f^n(J'')|$ is bounded by $|I_0|$,

$$|(A_n(\delta) \cap J'') \cap Y_n(\lambda)| < |I_0| \exp(-n\lambda)$$

for every connected component J'' of $A''_n(\delta)$. To finish the proof of this lemma is enough to use the estimate of the number of components of $A''_n(\delta)$ given by Lemma 3.2.

On the other hand, observe that the choice of δ is given fundamentally by Lemmas 3.3 and 3.4. Namely, δ depends on: the constant λ in the definition of $Y_n(\lambda)$; the uniformity of ϵ (given $\zeta > 0$) on the equation (13); the uniform boundedness of $|Df_k|$ on the proof of Lemma 3.3; the uniformity of δ (given ϵ and l) on the equation (14); and the uniform boundedness of the number of critical points for f_k, where $k \geq 0$. So, δ depends only on the modulus of continuity (3), the uniform bound Γ in (4) and the uniform bound p for the cardinal of the set of critical points, as stated. This concludes the proof of the lemma. $\square$

Remark 3.2. Note that in the proof of Lemma 3.1 (that means, also in all the auxiliaries lemmas used in its proof) no hypotheses about the Lebesgue measure of $Y_n(\lambda)$ was used. All the lemmas are true no matter what is the measure of $Y_n(\lambda)$.

As we remarked at the beginning of this section, Lemma 3.1 clearly implies that

$$\bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \complement A_n(\delta) \cap Y_n(\lambda)$$

has full Lebesgue measure. Hence, (5) holds for $\varsigma = \delta^2$, where δ is the constant found on Lemma 3.1. This concludes the proof of Theorem B. $\square$

4 Some consequences of Theorem B

4.1 Proof of Corollary I

By (P'), for any $n \in \mathbb{N}$, $\cup_{k\geq n} Y_n(\lambda)$ has total Lebesgue measure in I_0. Thus, for any $n \in \mathbb{N}$,

$$\Big|\bigcap_{k\geq n} A_k(\delta) \cup \complement Y_k(\lambda)\Big| = \Big|\Big(\bigcap_{k\geq n} A_k(\delta) \cup \complement(Y_k(\lambda))\Big) \cap \bigcup_{k\geq n} Y_k(\lambda)\Big| \leq \sum_{k=n}^{\infty} |A_k(\delta) \cap Y_k(\lambda)|$$

and by Lemma 3.1, for any $\epsilon > 0$, this last sum is less than ϵ if $n \geq N(\epsilon)$. This implies that $| \cup_{n\geq N(\epsilon)} \cap_{k=n}^{\infty} A_k(\delta) \cup \complement Y_k(\lambda)| < \epsilon$, and thus $| \cap_{n\geq N(\epsilon)} \cup_{k=n}^{\infty} \complement A_k(\delta) \cap Y_k(\delta)| \geq |I_0| - \epsilon$. This means that

$$\Big\{x; \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_i(x) \geq \delta^2\Big\}$$

has Lebesgue measure $|I_0| - \epsilon$. Since that can be done for any $\epsilon > 0$, the corollary follows with $\varsigma = \delta^2$. □

4.2 Proof of Corollaries II and III

The proof that we will give is similar to the proof by de Melo and van Strien [MvS, Theorem V.3.2] for Keller's theorem. The proofs of both corollaries are basically the same, it changes in some points which will be highlighted during the proof. We will construct a Markov map F induced by f and we will prove that the invariant measure for F induces an invariant measure for f which also is absolutely continuous. To obtain bounded distortion for the induced maps we will make use of the Koebe Principle. For Corollary II, we will also need Theorem 2.1.

Proof of Corollaries II and III. By Corollary I, in the particular case in which the sequence $\{f_n\}_{n\geq 0}$ has just one function f, implies that

$$X = \Big\{x \in I_0;\ \limsup_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} r_i(x) \geq \varsigma\Big\}$$

has full Lebesgue measure for some $\varsigma > 0$.

Let us consider $\mathcal{P}$ a finite partition of I_0, with norm less than $\varsigma/4$ and such that the extremes of the elements of the partition are a forward invariant set. In the hypotheses of Corollary II, the existence of this partition follows from Theorem 2.6, item 4. and Lemma A.2 of the appendix. In the case of Corollary III, see Theorem A.2 in the appendix.

Let ς' be the minimum of the sizes of the elements of $\mathcal{P}$. For every $x \in I_0$, we will denote by $J(x)$ the element of the partition that contains x. And for every $J \in \mathcal{P}$, let us denote by J^- (respectively, J^+) the rightmost interval of the partition next to J (respectively, the leftmost interval of the partition next to J).

In the hypotheses of Corollary II, by Theorem 2.6, item 2, we can choose $N \in \mathbb{N}$ such that the intervals of monotonicity of f^n have size less than $\varsigma'/4$, for $n \geq N$. In the hypotheses of Corollary III, the same is obtained by Lemma A.1 of the appendix.

For every $x \in X$, there are infinitely many $k's$ such that $r_k(x) > \varsigma/2$. Let $k(x) \geq N$ be minimal such that

$$f^{k(x)}(T_{k(x)}(x)) \supset J(f^{k(x)}(x)) \cup J(f^{k(x)}(x))^+ \cup J(f^{k(x)}(x))^-, \tag{24}$$

and consider $I(x) \subset T_{k(x)}(x)$ such that

$$f^{k(x)}(I(x)) = J(f^{k(x)}(x)).$$

Obviously, for every $y \in I(x)$, $k(y) \leq k(x)$; and it is not difficult to see, using the forward invariance of the set of extremes of $\mathcal{P}$, that in fact, $k(y) = k(x)$ and $I(y) = I(x)$. Hence, we can define

$$F : \cup_{x \in X} I(x) \to \cup_{J \in \mathcal{P}} J$$

by $F_{|I(x)} = f^{k(x)}{}_{|I(x)}$. We claim that this map is Markov (recall Definition 2.1). Indeed, (M_3) is satisfied because $|F(I(x))| = |J(F(x))| \geq \varsigma'$. Since $I(x)$ does not contain extremes of $\mathcal{P}$ in its interior, $I(x)$ is completely contained on some element of $\mathcal{P}$. Thus, for any $J \in \mathcal{P}$, there exists $A_J \subset X$ such that $J = \cup_{x \in A_J} I(x)$. This implies that (M_2) holds.
By (24), $f^{k(x)}(T_{k(x)}(x))$ contains a neighborhood τ-scaled of $f^{k(x)}(I(x))$, where $\tau = 4\varsigma'/\varsigma$. On the other hand, by theorem 2.1, $B(f^{k(x)}, T, M) \geq K'$ for any $M \subset T \subset T_{k(x)}$. Hence, by the Koebe Principle (Proposition 2.1)

$$\frac{1}{K} \leq \frac{|DF(x)|}{|DF(y)|} \leq K$$

for $x, y \in I(x)$. To show this inequality for the iterates of F, given $x \in X$ and $s \in \mathbb{N}$; let $m(s, x) \in \mathbb{N}$ be such that $F^s(x) = f^{m(s,x)}(x)$ and $I_s(x)$ domain of F^s containing x. By the choice of N and since $m(s, x) \geq N$, $T_{m(s,x)}(x)$ is contained in at most two elements of $\mathcal{P}$. Using this and (24) we can prove inductively that for $x \in X$ and $s \geq 1$,

$$f^{m(s,x)}(T_{m(s,x)}(x)) \supset J(f^{m(s,x)}(x)) \cup J(f^{m(s,x)}(x))^+ \cup J(f^{m(s,x)}(x))^-,$$

and thus $f^{m(s,x)}(T_{m(s,x)}(x))$ contains a neighborhood τ-scaled of $f^{m(s,x)}(I_s(x))$ (with $\tau = 4\varsigma'/\varsigma$). Again by Theorem 2.1, we have $B(f^{m(s,x)}, ,) \geq K'$ and this implies the bounded distortion for the iterates of F, that is (M_1) holds. Hence, F is a Markov map like we claimed. By Theorem 2.3, there exists ν absolutely continuous invariant measure for F. In this setting the measure ν has at most finitely many ergodic components, then we can assume the measure itself is ergodic. According to Theorem 2.4, this measure induces an absolutely continuous invariant measure for f if,

$$\sum_{i=1}^{\infty} k(i)\nu(I_i) < \infty,$$

We will see that the last inequality is valid. Assume by contradiction that $\sum_{i=1}^{\infty} k(i)\nu(I_i) = \infty$. By the Birkhoff's Ergodic Theorem,

$$\frac{n_s(x)}{s} = \frac{k(x) + k(F(x)) + \ldots + k(F^s(x))}{s} \to \int k(x)d\nu(x) = \sum_{i=1}^{\infty} k(i)\nu(I_i) = \infty$$

for ν almost every point x. For every $x \in X$ and $i \in \mathbb{N}$, if $n_i(x) \leq n < n_{i+1}(x)$ and $r_n(x) > \varsigma/2$, then $n - n_i(x) < N$, since in this case, $f^n(T_n(x))$ covers one element of the partition and its two neighbours. Thus we have for $n_s(x) \leq n < n_{s+1}(x)$,

$$\frac{\#\{1 \leq i \leq n \,,\, r_i(x) > \varsigma/2\}}{n} \leq \frac{N(s+2)}{n_s(x)}$$

and then for $n_s(x) \leq n < n_{s+1}(x)$,

$$\frac{1}{n}\sum_{i=1}^{n} r_i(x) = \frac{1}{n}\sum_{i, r_i(x) > \varsigma/2} r_i(x) + \frac{1}{n}\sum_{i, r_i(x) \leq \varsigma/2} r_i(x) < \frac{N(s+2)}{n_s(x)}|I_0| + \varsigma/2$$

which implies that $\limsup_{n\to\infty} 1/n \sum_{i=1}^{n} r_i(x) < \varsigma$. Since it holds ν-a.e. x, it contradicts that X had full Lebesgue measure. Hence there exists μ absolutely continuous invariant measure for f. □

5 Hyperbolic-like times

This section does not depend either on the results of Theorem B or on Lemma 3.1. Our interest is to study the behavior of points in M such that $r_k \geq \sigma$, in order to do it, we need to adapt some notations from the subsection 1.2 to the setting defined by Theorem A.

First of all, for the skew-product φ, note that since $\varphi^i(\theta, y) = (g^i(\theta), f^i_\theta(y))$ (for $\theta \in \mathbb{T}^1, y \in I_0, i \in \mathbb{N}$), the dynamics of φ restricted to the vertical leaf $\theta \times I_0$ is described by the dynamics of the compositions of $f_\theta, f_{g(\theta)}, \ldots, f_{g^k(\theta)}, \ldots$.

For every $\theta \in \mathbb{T}^1$, let us denote by $T_i(\theta, x)$ the function $T_i(\{f_n\}, x)$ defined on subsection 1.2, considering the sequence $\{f_n\}_{n\geq 0}$ given by $f_n = f_{g^n(\theta)}$ for all $n \geq 0$. We proceed analogously with $L_i(\theta, x)$, $R_i(\theta, x)$, $r_i(\theta, x)$.

For every $z = (\theta, x) \in \mathbb{T}^1 \times I_0$ and every $i \in \mathbb{N}$, we define

$$\begin{aligned} T_i(z) &:= \{\theta\} \times T_i(\theta, x) \\ L_i(z),\, R_i(z) &:= \{\theta\} \times L_i(\theta, x),\, \{\theta\} \times R_i(\theta, x) \\ r_i(z) &:= r_i(\theta, x). \end{aligned}$$

We want to show that mixing the good behavior along the vertical direction ($r_k \geq \sigma$) with the behavior along the horizontal direction (see subsection 5.1), this last due to the partial hyperbolicity, we obtain neighborhoods which can be used to construct absolutely continuous invariant measures, i.e, neighborhoods as in Proposition 5.3. In all the results we assume that we are in the conditions of Theorem A.

5.1 Horizontal behavior of dominated skew-products

One important property of our mappings due to the domination condition is the preservation of the nearly horizontal curves. This means that the iterates of nearly horizontal curves are still nearly horizontal. We state it in a precise way.

Definition 5.1. We call $\hat{X} \subset \mathbb{T}^1 \times I_0$ an *α−curve* if there exists $J \subset \mathbb{T}^1$ and $X : J \to I_0$ such that $\hat{X} = \text{graph}(X)$ and

1. X is C^1
2. $|X'(\theta)| \leq \alpha$ for every $\theta \in J$.

There exists an analogous definition of Viana (see [V1], section 2.1), but he asks that the second derivative be also less than α and he calls the curves with these properties *admissible curves*. In his setting he proves that the admissible curves are preserved under iteration. In our setting, that is only true when the iterate is big enough. Specifically, the image by φ^n of an α-curve defined in a small interval is still an α-curve, for some α and for n big enough.

Proposition 5.1. *There exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that, if $\hat{X}$ is a α-curve and $\varphi^n(\hat{X})$ is the graph of a C^1 map, then $\varphi^n(\hat{X})$ is a α-curve, provided that $n \geq n_0$.*

Proof. Let $\hat{X} = \{(\theta, X(\theta)); \theta \in J\}$ be a C^1 curve with $|X'(\theta)| \leq \alpha$ for every $\theta \in J$. Let us define inductively for $n \geq 1$,

$$X_n(g^n(\theta)) = f(g^{n-1}(\theta), X_{n-1}(g^{n-1}(\theta))) \tag{25}$$

where $X_0 = X$. Then it holds for $n \geq 1$

$$\varphi^n(\hat{X}) := \hat{X}_n = \{(g^n(\theta), X_n(g^n(\theta))); \theta \in J\}$$

By the definition above we can prove inductively that for $n \geq 1$,

$$\varphi^n(\theta, X(\theta)) = (g^n(\theta), X_n(g^n(\theta))).$$

Using this relation and (25), we can show (also inductively) that the next equality is satisfied for $n \geq 1$,

$$X_n'(g^n(\theta))\partial_\theta g^n(\theta) = \partial_\theta f(\varphi^{n-1}(\theta, X(\theta)))\partial_\theta g^{n-1}(\theta) + \sum_{k=1}^{n-1}\prod_{i=1}^{k}\partial_x f(\varphi^{n-i}(\theta, X(\theta)))\partial_\theta f(\varphi^{n-(k+1)}(\theta, X(\theta)))\partial_\theta g^{n-(k+1)}(\theta) + \prod_{i=1}^{n}\partial_x f(\varphi^{n-i}(\theta, X(\theta)))X'(\theta)$$

Then,

$$|X_n'(g^n(\theta))| \leq \frac{|\partial_\theta f(\varphi^{n-1}(\theta, X(\theta)))\partial_\theta g^{n-1}(\theta)|}{|\partial_\theta g(g^{n-1}(\theta))\partial_\theta g^{n-1}(\theta)|} + \sum_{k=1}^{n-1}\frac{|\prod_{i=1}^{k}\partial_x f(\varphi^{n-i}(\theta, X(\theta)))\partial_\theta f(\varphi^{n-(k+1)}(\theta, X(\theta)))\partial_\theta g^{n-(k+1)}(\theta)|}{|\partial_\theta g^k(g^{n-k}(\theta))\partial_\theta g(g^{n-(k+1)}(\theta))\partial_\theta g^{n-(k+1)}(\theta)|} + \frac{|\prod_{i=1}^{n}\partial_x f(\varphi^{n-i}(\theta, X(\theta)))|}{|\partial_\theta g^n(\theta)|}|X'(\theta)|$$

Now, by (1), considering $L = \sup(\partial_\theta f/\partial_\theta g)$, we have that

$$|X_n'(g^n(\theta))| \leq L + \sum_{k=1}^{n-1} LC(a)^k + Ca^n\alpha \leq LCA + Ca^n\alpha \tag{26}$$

where $A = \sum_{k=0}^{\infty} a^k$. Hence for some α and n_0 big enough, $|X_n'(g^n(\theta))| \leq \alpha$ for all $n \geq n_0$. □

Let α, L, C, A be as in the proposition above. If we start with $\hat{X} = \{(\theta, X(\theta)); \theta \in J\}$ a C^1 curve with $|X'(\theta)| \leq \alpha$ for every $\theta \in J$, then there exists $\alpha' := LCA + C\alpha$, such that, for every $n \in \mathbb{N}$, $\varphi^n(\hat{X}) = \{(\theta, X_n(\theta)); \theta \in J_n\}$ is a curve with $|X_n'| \leq \alpha'$.

Remark 5.1. Given α of the last proposition, there exists $C_1 = C_1(L, C, A, \alpha)$ such that if $\hat{X}$ is an α-curve, then $\varphi^n(\hat{X})$ is an $C_1\alpha$-curve, for all n, provided that $\varphi^n(\hat{X})$ is a graph.

Since all the iterates of α-curves are almost horizontal (its graphs have derivatives bounded by $C_1\alpha$) then their lengths are given basically by the derivative of φ in the horizontal direction, i.e, the derivative of g. We state this in the following result.

Proposition 5.2. *There exists $K = K(\alpha) > 0$, such that if $\hat{X} = \{(\theta, X(\theta)); \theta \in J\}$ and $\varphi^k(\hat{X}) = \{(\theta, X_k(\theta)); \theta \in J_k\}$ are graphs with $|X'|, |X_k'| \leq C_1\alpha$, then for all $z, w \in \varphi^k(\hat{X})$,*

$$\mathrm{dist}_{\hat{X}}(\varphi^{-k}(z), \varphi^{-k}(w)) \leq K|\partial_\theta(g^k(\theta_k)|^{-1}\, \mathrm{dist}_{\varphi^k(\hat{X})}(z, w)$$

for some $\theta_k \in J$, where dist_C *is the distance induced by the metric over the curve C.*

Proof. Let us consider the norm in the tangent space given by the canonic internal product in a product manifold, i.e,

$$\|(v_1, v_2)\| = (|v_1|^2 + |v_2|^2)^{\frac{1}{2}}$$

where $v = (v_1, v_2) \in T_z(\mathbb{T}^1 \times I_0), v_1 \in T_\theta\mathbb{T}^1, v_2 \in T_x I_0$ and $z = (\theta, x)$.

We will denote the tangent vector to the curve $\hat{X}$ at the point $(\theta, X(\theta))$ by $(v_1(\theta), v_2(\theta))$. Let us consider $\theta_z, \theta_w \in J$ such that $\varphi^k(\theta_z, X(\theta_z)) = z$ and analogously for w. Then, since $|v_2(\theta)|/|v_1(\theta)| \leq C_1\alpha$,

$$\begin{aligned}
\mathrm{dist}_{\varphi^k(\hat{X})}(z, w) &= \int_{\theta_z}^{\theta_w} \|D\varphi^k(\theta, X(\theta))(v_1(\theta), v_2(\theta))\| d\theta \\
&\geq \int_{\theta_z}^{\theta_w} |\partial_\theta g^k(\theta)||v_1(\theta)| d\theta \\
&\geq \frac{1}{(1 + (C_1\alpha)^2)^{\frac{1}{2}}} \int_{\theta_z}^{\theta_w} |\partial_\theta g^k(\theta)|(|v_1(\theta)|^2 + |v_2(\theta)|^2)^{\frac{1}{2}} d\theta \\
&\geq \frac{1}{(1 + (C_1\alpha)^2)^{\frac{1}{2}}} |\partial_\theta g^k(\theta_k)|\, \mathrm{dist}_{\hat{X}}(\varphi^{-k}(z), \varphi^{-k}(w))
\end{aligned}$$

where θ_k is such that $|\partial_\theta g^k(\theta_k)| \leq |\partial_\theta g^k(\theta)|$ for $\theta \in [\theta_z, \theta_w]$. This means that we may take $K = (1 + (C_1\alpha)^2)^{\frac{1}{2}}$. □

5.2 Properties of the hyperbolic-like times

In this section we will prove a similar behavior of points with $r_k \geq \sigma$ (for some $\sigma > 0$) and points with k being one of its (σ', δ)-hyperbolic times (see definition 2.3). Specifically, if z is a point of the manifold and $k \in \mathbb{N}$ is such that $r_k(z) \geq \sigma$, there exists a neighborhood $V_k(z)$ of z such that $\varphi^k : V_k(z) \to \varphi^k(V_k(z))$ is a diffeomorphism with bounded distortion (independent of k and z). Moreover, the Lebesgue measure of $\varphi^k(V_k(z))$ is uniformly bounded away from zero. This fact can be concluded if z is a point with k like hyperbolic time (see proposition 2.2). Because of this, if $r_k(z) \geq \sigma$, we say k is a *σ-hyperbolic-like time* for $z \in M$. One of the differences between these two conditions is that for the hyperbolic times, we have contraction for all the inverse iterates in a certain neighborhood of $\varphi^k(z)$ and in the case of hyperbolic-like times we do not know if this property is verified.

We need to assume, to prove the bounded distortion of these neighborhoods associated with the hyperbolic-like times, that the functions f_θ in the definition of φ have negative Schwarzian derivative, for all $\theta \in \mathbb{T}^1$.

Proposition 5.3. *Let $\sigma > 0$ be a constant. Given $z \in M$ such that $r_k(z) \geq \sigma$ for some $k \in \mathbb{N}$, there exists a neighborhood $V_k(z)$ of z such that $\varphi^k : V_k(z) \to \varphi^k(V_k(z))$ is a diffeomorphism with bounded distortion (it depends on σ, but it is independent of z and of k).*

Proof. Let $T_k(z)$ be the maximal interval such that $\varphi^j(T_k(z)) \cap \mathscr{C} = \emptyset$ for $j < k$ and let $L_k(z)$, $R_k(z)$ be the components of $T_k(z) \setminus \{z\}$. By hypotheses $|\varphi^k(L_k(z))| \geq \sigma$ and $|\varphi^k(L_k(z))| \geq \sigma$. Let us consider $I_k(z) \subset T_k(z)$ such that every component of $\varphi^k(T_k(z)) \setminus \varphi^k(I_k(z))$ has Lebesgue measure equal to $\sigma/2$. In particular, we have that both components of $\varphi^k(I_k(z) \setminus \{z\})$ have measure greater or equal than $\sigma/2$. We will define some horizontal curves crossing by the points on $I_k(z)$, which under iteration for φ keep horizontal. For our purposes the arc length of these curves must not be very big.

Let $\rho > 0$ be a constant whose value will be made precise on (28), and α, C_1 the constants of the proposition 5.1 and remark 5.1. We consider ρ' the constant such that: given $J \subset \mathbb{T}^1$ interval with length ρ' and $X : J \to I_0$ a curve with $|X'| \leq C_1\alpha$, the arc length of graph(X) is less or equal than ρ.

Let $z = (\theta, x) \in \mathbb{T}^1 \times I_0$ be for some $\theta \in \mathbb{T}^1$, $x \in I_0$. Let us call z_- and z_+ the coordinates of the end points of $I_k(z)$, i.e, $I_k(z) = [z_-, z_+]$ (recall that $I_k(z)$ is contained in a vertical leaf). By the definition of φ, we know that the horizontal component of $\varphi^k(z)$ is $g^k(\theta)$. Let us consider $\eta_1 > 0$ and $\eta_2 > 0$ such that $g^k : (\theta - \eta_1, \theta + \eta_2) \to (g^k(\theta) - \rho', g^k(\theta) + \rho')$ is a diffeomorphism. Let us consider the horizontal curves passing by the points of $I_k(z)$. For every $w = (\theta, x_w) \in I_k(z)$, we will denote by C_w the line joining the points $(\theta - \eta_1, x_w)$ and $(\theta + \eta_2, x_w)$.

For $w \in I_k(z)$, we denote by C_w^j (for $j \leq k$) the curve image by φ^j of C_w, i.e, which satisfies $\varphi^j(C_w) = C_w^j$. Observe that $C_w^0 = C_w$ for any $w \in I_k(z)$. Note that by the choice of ρ', the arc length of C_w^k is less or equal than 2ρ, for any $w \in I_k(z)$.

In the same way we will denote by w^k the image by φ^k of the point $w = w^0$ and by T^j the set $\varphi^j(T_k(z))$ (since z and k are fixed along the proof, there is no confusion in omitting in the notation the dependence of T^j on z and k). In particular, we have that $[z_+^j, z_-^j] \subset T^j$ for $j \leq k$.

Claim 5.1. *There exists a neighborhood $V_k(z)$ of z such that $\varphi^k : V_k(z) \to \varphi^k(V_k(z))$ is a diffeomorphism and the volume of $\varphi^k(V_k(z))$ is bounded away from zero.*

Proof. We will use the bounded distortion of the map g. Namely, there exists $D > 0$ such that, if we have $J \subset \mathbb{T}^1$ and $n \in \mathbb{N}$ for which $g^n : J \to g^n(J)$ is a diffeomorphism, then

$$\frac{|\partial_\theta g^n(\theta)|}{|\partial_\theta g^n(\omega)|} \leq D \tag{27}$$

for all $\theta, \omega \in J$.

Recall the constants C, α, C_1 and K, specified in (1), Proposition 5.1, Remark 5.1 and Proposition 5.2, respectively. The constant ρ must satisfy the next condition

$$\left.\begin{array}{c} K\rho < (\sigma/8)(DC)^{-1} \\ \rho C_1 \alpha < \sigma/8 \end{array}\right\} \tag{28}$$

First we claim that $C_w^j \cap \mathscr{C} = \emptyset$ for $j < k$ and for any $w \in I_k(z)$. For a curve C, we will denote by $|C|$ its arc length.

For $j < k$, using Proposition 5.2 we know that $|C_w^{k-j}| \leq K|\partial_\theta g^j(\theta_j)|^{-1}2\rho$ for some $(\theta_j, x_j) \in C_w^{k-j}$ (then $K2\rho|\partial_\theta g^j(\theta_j)|^{-1}$-close to w^{k-j}).

On the other hand, for $j \leq k$, let us denote by I_w^{k-j} the component of $T^{k-j} \setminus \{w^{k-j}\}$ which has z_+^{k-j} in its boundary. By the mean value theorem, we have that $|I_w^{k-j}| \geq (\prod_{i=0}^{j-1} |\partial_x f(\varphi^i(\omega_j, y_j))|)^{-1}(\sigma/2)$ for some $(\omega_j, y_j) \in I_w^{k-j}$ and $|T^{k-j} \setminus I_w^{k-j}| \geq (\prod_{i=0}^{j-1} |\partial_x f(\varphi^i(\omega'_j, y'_j))|)^{-1}(\sigma/2)$, for some $(\omega'_j, y'_j) \in T^{k-j} \setminus I_w^{k-j}$. Two cases can happen:

(*i*) $|I_w^{k-j}| \leq |T^{k-j} \setminus I_w^{k-j}|$

(*ii*) $|I_w^{k-j}| > |T^{k-j} \setminus I_w^{k-j}|$

Let us assume that we have the case (*i*) (the other case is totally analogous), then combining (1) and (27), we have

$$|\partial_\theta g^j(\theta_j)|^{-1} < DC\, a^j \left(\prod_{i=0}^{j-1} \left| \partial_x f(\varphi^i(\omega_j, y_j)) \right| \right)^{-1},$$

and this, together with (28), gives

$$\begin{aligned} |C_w^{k-j}| \leq K\, |\partial_\theta g^j(\theta_j)|^{-1}\, 2\rho < a^j \left(\prod_{i=0}^{j-1} \left| \partial_x f(\varphi^i(\omega_j, y_j)) \right| \right)^{-1} (\sigma/4) \\ \leq a^j\, \frac{\mathrm{dist}_{\mathrm{vert}}(w^{k-j}, \mathscr{C})}{2} \end{aligned} \tag{29}$$

This last inequality, and the condition (F_2) which is satisfied by the skew product, implies that $C_w^{k-j} \cap \mathscr{C} = \emptyset$.

Now, let us define $B_k = \cup_{w \in I_k(z)} C_w$ and $B = \cup_{w \in I_k(z)} C_w^k$. By our definitions, we have that $\varphi^k(B_k) = B$. Observe that B_k contains complete vertical segments, which means that if it contains two points in a same vertical leaf then it contains all the points in this leaf between these two points; by the continuity of the functions $f(\theta, \cdot)$, this is also true for the set B.

We claim that $\varphi^k : B_k \to B$ is a diffeomorphism, which maps C_w to C_w^k, for all $w \in I_k(z)$. Indeed, since we already have proved that $C_w^j \cap \mathscr{C} = \emptyset$ for $w \in I_k(z)$ and $j < k$, then we have that $\varphi^k : B_k \to B$ is a local diffeomorphism. To conclude the proof of the claim we will show that the map is injective. If there exist (θ_1, x_1) and (θ_2, x_2) in B_k such that $\varphi^k(\theta_1, x_1) = \varphi^k(\theta_2, x_2) \in B$, first, since in the horizontal direction there is expansion ($\partial_\theta g > 1$), it must to have $\theta_1 = \theta_2$. Next, by the differentiability of the functions $f(\theta, \cdot)$, if $x_1 \neq x_2$, there must to be at least one point (θ_1, y) between (θ_1, x_1) and (θ_1, x_2) and $j < k$ such that this point is mapped by φ^j in a critical point. But this would imply that $C_y^j \cap \mathscr{C} \neq \emptyset$, a contradiction with what we already have proved. Hence $x_1 = x_2$, which implies that the map $\varphi^k : B_k \to B$ is injective.

Since, for $* \in \{+, -\}$, the curves $C_{z_*}^k$ are graphs,

$$C_{z_*}^k = \{(\omega, X_{k,z_*}(\omega)); \omega \in (g^k(\theta) - \rho', g^k(\theta) + \rho')\}$$

with $|X_{k,z_*}| \leq C_1\alpha$, for C_1 and α the constants of the Remark 5.1. It follows by the condition (28) that the set B contains

$$\Delta(\varphi^k(z), \sigma) = \cup_{a \in (y-(3\sigma)/8, y+(3\sigma)/8)} S_a$$

where S_a is the horizontal segment joining $(g^k(\theta) - \rho', a)$ and $(g^k(\theta) + \rho', a)$, $z = (\theta, x)$ and $\varphi^k(z) = (g^k(\theta), y)$ (see Figure 1).

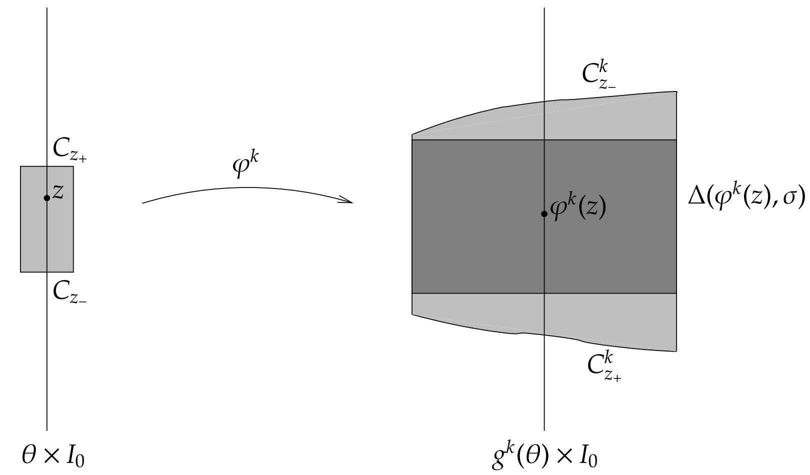

Figure 3: $\Delta(\varphi^k(z), \sigma)$

Calling $V_k(z) = B_k$, the claim follows. □

It just remains to prove that the transformation of the last claim has bounded distortion (independent of z, k). For any $J \subset M$ contained in the ω-vertical leaf we denote by $\mathbf{J} = \{x \in I_0; (\omega, x) \in J\}$. With this notation, $J = \{w\} \times \mathbf{J}$.

Claim 5.2. *There exists $K_1 = K_1(\sigma) > 0$ such that for $z_1, z_2 \in I_k(z) \subset V_k(z)$,*

$$\frac{1}{K_1} \leq \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} \leq K_1$$

Proof. Let z_1 and z_2 be points in $I_k(z)$. Let θ be the horizontal component of z, i.e., $z = (\theta, x)$ for some $x \in I_0$. By definition, $I_k(z) \subset T_k(z)$.

Recalling the notation $f_\theta^k = f_{g^{k-1}(\theta)} \circ \ldots \circ f_{g(\theta)} \circ f_\theta$, where $f_\theta(x) = f(\theta, x)$ for $\theta \in \mathbb{T}^1$, $x \in I_0$, it holds the relation $\varphi^k(\theta, y) = (g^k(\theta), f_\theta^k(y))$ for any $y \in I_0$. Since $\varphi^j(T_k(z)) \cap \mathscr{C} = \emptyset$ for $j < k$, then $f_\theta^k : \mathbf{T_k(z)} \to f_\theta^k(\mathbf{T_k(z)})$ is a C^3 diffeomorphism.

By the form we have chosen $I_k(z)$ we know that every component of $f_\theta^k(\mathbf{T_k(z)}) \setminus f_\theta^k(\mathbf{I_k(z)})$ has Lebesgue measure equal to $\sigma/2$, then there exists $\tau > 0$ (depending only on σ), such that $f_\theta^k(\mathbf{T_k(z)})$ contains a τ-scaled neighborhood of $f_\theta^k(\mathbf{I_k(z)})$. Thus, by Koebe Principle (Proposition 2.1), there exists $K_1 > 0$ such that

$$\frac{1}{K_1} \leq \frac{|Df_\theta^k(y_1)|}{|Df_\theta^k(y_2)|} \leq K_1$$

for $y_1, y_2 \in \mathbf{I_k(z)}$.

Now, if $z_1 = (\theta, y_1)$ and $z_2 = (\theta, y_2)$, for $y_1, y_2 \in \mathbf{I_k(z)}$,

$$\frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} = \frac{\prod_{i=1}^k |\det D\varphi(\varphi^{k-i}(\theta, y_1))|}{\prod_{i=1}^k |\det D\varphi(\varphi^{k-i}(\theta, y_2))|} =$$

$$= \frac{\prod_{i=1}^k |\partial_\theta g(g^{k-i}(\theta))\partial_x f_{g^{k-i}(\theta)}(f_\theta^{k-i}(y_1))|}{\prod_{i=1}^k |\partial_\theta g(g^{k-i}(\theta))\partial_x f_{g^{k-i}(\theta)}(f_\theta^{k-i}(y_2))|} = \frac{|Df_\theta^k(y_1)|}{|Df_\theta^k(y_2)|}$$

Since K_1 just depends on σ, the claim follows. □

Now let us assume that $z_1 \in I_k(z)$ and $z_2 \in V_k(z)$ belong to the same horizontal leaf of z_1, i.e, z_1 and z_2 belong to C_y, where $z_1 = (\theta, y)$. By the definition of φ,

$$\left|\log \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|}\right| \leq \left|\log \frac{|\partial_\theta g^k(z_1)|}{|\partial_\theta g^k(z_2)|}\right| + \left|\log \frac{\prod_{i=1}^k |\partial_x f(\varphi^{k-i}(z_1))|}{\prod_{i=1}^k |\partial_x f(\varphi^{k-i}(z_2))|}\right|$$

Using the condition (F_2) satisfied by the skew product, together with (27),

$$\left|\log \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|}\right| \leq \log D + B \sum_{i=1}^k \frac{\operatorname{dist}(\varphi^{k-i}(z_1), \varphi^{k-i}(z_2))}{\operatorname{dist}_{\text{vert}}(\varphi^{k-i}(z_1), \mathscr{C})}$$

and by (29), we have

$$\left|\log \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|}\right| \leq \log D + B \sum_{i=1}^k a^i \leq B' \sum_{i=1}^\infty a^i = K_2'$$

Hence, for $z_1 \in I_k(z)$ and $z_2 \in V_k(z)$ in the same horizontal leaf of z_1,

$$\frac{1}{K_2} \leq \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} \leq K_2.$$

Finally, if $z_1 = (\theta_1, x)$, $z_2 = (\theta_2, y) \in V_k(z)$, let us consider $z_1', z_2' \in I_k(z)$ such that z_i' belongs to the same horizontal leaf than z_i ($i = 1, 2$). If $z_i \in I_k(z)$ we can consider $z_i' = z_i$ ($i = 1, 2$). Using the proved cases we have

$$\frac{1}{K_1 K_2^2} \leq \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} = \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_1')|}\frac{|\det D\varphi^k(z_1')|}{|\det D\varphi^k(z_2')|}\frac{|\det D\varphi^k(z_2')|}{|\det D\varphi^k(z_2)|} \leq K_1 K_2^2 \tag{30}$$

Since K_1 and K_2 do neither depend on the point z, nor on the iterate k, the distortion is bounded by a constant like we claimed and the proposition follows. □

5.3 Neighborhoods associated to hyperbolic-like times

For every $\sigma > 0$ and $i \in \mathbb{N}$, we will denote by $H_i(\sigma)$ the set of points $z \in M$ with $r_i(z) \geq \sigma$. In this section the goal is the proof of the following lemma, which will be very useful in the construction of the absolutely continuous invariant measure for φ.

Lemma 5.1. *Given $\sigma > 0$, there exists $\tau > 0$ such that for every $i \in \mathbb{N}$ there exists a finite set of points $x_1, \ldots, x_N$ in $H_i(\sigma)$, and neighborhoods of them $V_i'(x_1), \ldots, V_i'(x_N)$, which are two-by-two disjoint and their union $W_i = V_i'(z_1) \cup \ldots \cup V_i'(z_N)$ satisfies*

$$\mathrm{Leb}(W_i) \geq \tau\, \mathrm{Leb}(H_i(\sigma)).$$

Given $\sigma > 0$, the constants ρ, α, C_1 and ρ' appeared in the proof of Proposition 5.3; ρ was defined by (28), α and C_1 come from Proposition 5.1 and Remark 5.1. The constant ρ' was defined in the following way: given $J \subset \mathbb{T}^1$ interval with length ρ' and $X : J \to I_0$ a curve with $|X'| \leq C_1\alpha$, the arc length of graph(X) is less or equal than ρ. For $z = (\theta, x) \in M$, $\Delta(z, \sigma)$ will denote the set limited by: the horizontal segment from the point $(\theta - \rho', x - \sigma/8)$ to the point $(\theta + \rho', x - \sigma/8)$, the horizontal segment from the point $(\theta - \rho', x + \sigma/8)$ to the point $(\theta - \rho', x + \sigma/8)$, and the vertical segments joining the extremes of these two segments.

Given $b > 0$, $\Delta^b(z, \sigma)$ will denote the set above but with $b\rho'$ and $b\sigma/8$ in the place of ρ' and $\sigma/8$, respectively. Obviously for $0 < b < 1$, $\Delta^b(z, \sigma) \subset \Delta(z, \sigma)$.

Remark 5.2. Recall that $\Delta(\varphi^k(z), \sigma) \subset \varphi^k(V_k(z))$, where $V_k(z)$ is the neighborhood of z constructed in the last subsection for which $\varphi^k : V_k(z) \to \varphi^k(V_k(z))$ is a diffeomorphism with bounded distortion.

First, we state one property which is satisfied by any Borelian measure.

Claim 5.3. *If μ is a finite Borelian measure on M and $\Omega \subset M$ is measurable with $\mu(\Omega) > 0$. Then, given $a > 0$, there exists $w_1 \in M$ such that*

$$\mu(\Omega \cap \Delta(w_1, a)) > \frac{\mu(\Omega \cap \Delta(w, a))}{2}$$

for all $w \in M$.

Proof of Lemma 5.1. Since M is compact, there exists $l = l(\sigma) \in \mathbb{N}$ such that any set $\Delta(z, \sigma)$ (for any $z \in M$) can be covered by at most l sets $\Delta^{1/8}(z_i, \sigma)$, i.e, there exist $\{z_1, \dots, z_l\} \subset M$ such that $\Delta(z, \sigma) \subset \cup_{i=1}^{l} \Delta^{1/8}(z_i, \sigma)$.

Let us suppose that $\mathrm{Leb}(H_i(\sigma)) > 0$; otherwise, the lemma follows trivially. Let us consider $\Omega = \varphi^i(H_i(\sigma))$ and $\mu = \varphi^i_* \mathrm{Leb}$ in the Claim 5.3. Then there exists w_1 such that

$$\varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta(w_1, \sigma)) \geq \frac{\varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta(w, \sigma))}{2}$$

for all $w \in M$. Now, we can find a point $y_1 \in M$ such that $\Delta^{1/8}(y_1, \sigma)$ intersects $\Delta(w_1, \sigma)$ and

$$\varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta^{1/8}(y_1, \sigma)) \geq \frac{1}{l} \varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta(w_1, \sigma))$$

Then $\Delta^{1/8}(y_1, \sigma)$ contains some point $z_1 \in \varphi^i(H_i(\sigma))$. Using both inequalities above, we have

$$\begin{aligned} \varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta^{1/4}(z_1, \sigma)) &\geq \varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta^{1/8}(y_1, \sigma)) \geq \\ \frac{1}{l} \varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta(w_1, \sigma)) &\geq \frac{1}{2l} \varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta(z_1, \sigma)) \end{aligned} \tag{31}$$

Let us consider $\Omega_1 = \varphi^i(H_i(\sigma)) \setminus \Delta(z_1, \sigma)$. If Ω_1 has $\varphi^i_* \mathrm{Leb}$-measure zero, we take just the point z_1 and we finish. Otherwise, we may apply the same construction as before to Ω_1, and thus we find a point z_2. Obviously $\Delta^{1/4}(z_1, \sigma)$ and $\Delta^{1/4}(z_2, \sigma)$ are disjoint since $z_2 \notin \Delta(z_1, \sigma)$. Repeating this construction we find a sequence $\{z_n\}$ such that $\{\Delta^{1/4}(z_n, \sigma)\}$ are two-by-two disjoint. By compactness of M, the sequence must be finite. Let $\{z_n\}_{1 \leq n \leq N}$ be the sequence.

Using (31) and the fact that $\varphi^i(H_i(\sigma)) \subset \cup_{n=1}^{N} \Delta(z_n, \sigma)$, we have

$$\varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma))) \leq 2l \sum_{n=1}^{N} \varphi^i_* \mathrm{Leb}(\varphi^i(H_i(\sigma)) \cap \Delta^{1/4}(z_n, \sigma))$$

Let us take x_n the point in $H_i(\sigma)$ whose image is z_n. Let $V_i'(x_n)$ be the preimage associated to $\Delta^{1/4}(z_n, \sigma)$, i.e, such that $\varphi^i : V_i'(x_n) \to \Delta^{1/4}(z_n, \sigma)$ is a diffeomorphism with bounded distortion, with the constant of (30). Finally, let $W_i = V_i'(x_1) \cup \ldots \cup V_i'(x_N)$. By the inequality above, we have

$$\mathrm{Leb}(W_i) \geq \frac{1}{2l}\,\mathrm{Leb}(H_i(\sigma))$$

Considering $\tau = 1/2l$ the lemma is proved. □

6 The measure

6.1 Points with infinitely many hyperbolic-like times

As a consequence of Lemma 3.1, we will show that, for some $\varepsilon > 0$, the points with many ε-hyperbolic-like times are a positive Lebesgue measure set. In fact, for every point in this set, the density of ε-hyperbolic-like times is uniformly positive.

The idea is to use information about the vertical lines $\{\theta \times I_0\}_{\theta\in T}$ and to get from this, information about the whole manifold, using the absolute continuity of this foliation (the given by the vertical lines).

Recall that we denote $\mathbb{T}^1 \times I_0$ by M and the Lebesgue measure of M by m. For λ and $n \in \mathbb{N}$ we define,

$$Z_n(\lambda) = \left\{ z \in M, \frac{1}{n}\sum_{j=0}^{n-1} \log \|D\varphi(\varphi^j(z))^{-1}\|^{-1} > \lambda \right\}, \tag{32}$$

and for $\delta > 0$,

$$A_n(\delta) = \left\{ z \in M; \frac{1}{n}\sum_{i=1}^{n} r_i(z) < \delta^2,\ \ r_n(z) > 0 \right\}. \tag{33}$$

As we will see, these sets have relation with the sets defined in (9) and (8). For every $\theta \in \mathbb{T}^1$, let us consider the sequence $\{f_{g^n(\theta)}\}_{n\geq 0}$ of smooth maps. Recall that we denote by $r_i(\theta, x)$ the function $r_i(\{f_n\}, x)$ defined on subsection 1.2, considering the sequence $f_n = f_{g^n(\theta)}$ for $n \geq 0$. In the same way we denote by $A_n(\theta, \delta)$ the set $A_n(\{f_n\}, \delta)$ defined on (8), and by $Y_n(\theta, \lambda)$ the set $Y_n(\{f_n\}, \lambda)$ defined on (9), with $f_n = f_{g^n(\theta)}$ for $n \geq 0$.

Thus, we can conclude that

$$Z_n(\lambda) \subset \cup_{\theta\in T}(\theta \times Y_n(\theta, \lambda)) \qquad \text{and} \qquad A_n(\delta) = \cup_{\theta\in T}(\theta \times A_n(\theta, \delta)) \tag{34}$$

For every $\theta \in \mathbb{T}^1$, $\{f_{g^n(\theta)}\}$ is a C^1-uniformly equicontinuous and C^1 uniformly bounded sequence of smooth maps, since $\varphi(\theta, x) = (g(\theta), f(\theta, x))$ is a C^3 map. On the other hand, by the assumptions about the critical set $\mathscr{C}$ of φ, it holds that $p = \sup \#\mathscr{C}_{g^n(\theta)} < \infty$. Thus, we are in the context of Lemma 3.1. Moreover, fixed $\lambda > 0$, the constant δ found on Lemma 3.1 does not depend on θ, i.e., the constant δ is the same for any sequence $\{f_{g^n(\theta)}\}$. This happens because the modulus of continuity (3), the uniform bound Γ in (4) and the uniform bound p for the number of critical points, are the same for any sequence $\{f_{g^n(\theta)}\}$ (varying $\theta \in \mathbb{T}^1$). The last is true since φ is C^3 and (F_1) holds.

Proposition 6.1. *In the conditions of Theorem A, given* $0 < \vartheta < m(M)$*, there exist* $\varepsilon > 0$ *and* $N_0 \in \mathbb{N}$ *such that*

$$m\left(\left\{z\,;\,\sum_{i=1}^{n} r_i(z) \geq 2\varepsilon n, \textit{for all } n \geq N_0\right\}\right) \geq \vartheta/2 \tag{35}$$

Proof. By assumption of Theorem A, φ is non-uniformly expanding, then, given $0 < \vartheta < m(M)$, we can choose N and $\lambda > 0$ such that

$$m\Big(\cap_{n\geq N} Z_n(\lambda)\Big) \geq \vartheta.$$

For $\lambda, \delta > 0$ and every $N \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{T}^1}\int_{I_0} \chi_{\{\cap_{n=N}^{\infty} \complement A_n(\delta)\cap Z_n(\lambda)\}}(\theta,x)dm_{I_0}(x)dm_{\mathbb{T}^1}(\theta) \geq \\ \int_{\mathbb{T}^1}\int_{I_0} \chi_{\{\cap_{n=N}^{\infty} Z_n(\lambda)\}}(\theta,x)dm_{I_0}(x)dm_{\mathbb{T}^1}(\theta) \\ - \int_{\mathbb{T}^1}\int_{I_0} \chi_{\{\cup_{n=N}^{\infty} A_n(\delta)\cap Z_n(\lambda)\}}(\theta,x)dm_{I_0}(x)dm_{\mathbb{T}^1}(\theta)\end{aligned}$$

On the other hand, by lemma 3.1, there exists $\delta > 0$ such that for every $\theta \in \mathbb{T}^1$,

$$m_{I_0}\left(\bigcup_{n=N}^{\infty} A_n(\theta,\delta) \cap Y_n(\theta,\lambda)\right) \to 0,$$

when $N \to \infty$; and this together with (34) yield,

$$\begin{aligned}\int_{\mathbb{T}^1}\int_{I_0} \chi_{\{\cup_{n=N}^{\infty} A_n(\delta)\cap Z_n(\lambda)\}}(\theta,x)dm_{I_0}(x)dm_{\mathbb{T}^1}(\theta) \\ \leq \int_{\mathbb{T}^1}\int_{I_0} \chi_{\{\cup_{n=N}^{\infty} A_n(\theta,\delta)\cap Y_n(\theta,\delta)\}}(x)dm_{I_0}(x)dm_{\mathbb{T}^1}(\theta) \longrightarrow 0\end{aligned}$$

when $N \to \infty$. Hence, there exists N_0 such that

$$\int_{\mathbb{T}^1}\int_{I_0} \chi_{\{\cap_{n=N_0}^{\infty} \complement A_n(\delta)\cap Z_n(\lambda)\}}(\theta,x)dm_{I_0}(x)dm_{T}(\theta) \geq \vartheta/2.$$

Considering ε such that $2\varepsilon < \delta^2$, the proposition follows. □

This means that there is a positive measure set of points with many hyperbolic-like times.

6.2 Positive density of the hyperbolic-like times

The following lemma will permit us to prove that, in fact, for every point in the set of (35), the density of hyperbolic-like times is uniformly positive. For the proof, see lemma 3.1 on [ABV].

Lemma 6.1 (Pliss Lemma). *Given $A \geq c_2 > c_1 > 0$, let $\zeta_0 = (c_2 - c_1)/(A - c_1)$. Then, given any real numbers $a_1, \ldots, a_N$ such that*

$$\sum_{j=1}^{N} a_j \geq c_2 N \qquad \text{and} \qquad a_j \leq A \ \text{for every} \ 1 \leq j \leq N,$$

there are $q \geq \zeta_0 N$ and $1 < n_1 < \ldots < n_q \leq N$ so that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n) \quad \text{for every} \quad 0 \leq n < n_i, \ \text{and} \ i = 1, \ldots, q.$$

For every $\varepsilon > 0$ and $n \in \mathbb{N}$, we will denote by $H_n(\varepsilon)$ the set of points $z \in M$ with $r_n(z) \geq \varepsilon$. The result about the density is the following

Lemma 6.2. *Given $\varepsilon > 0$, there exists $\zeta > 0$ such that*

$$\frac{\#\{1 \leq i \leq n; z \in H_i(\varepsilon)\}}{n} \geq \zeta$$

for any z such that

$$\sum_{i=1}^{n} r_i(z) \geq 2\varepsilon n \tag{36}$$

Proof. Considering $c_2 = 2\varepsilon$ and $c_1 = \varepsilon$, applying the Pliss lemma, there are $q \geq \zeta_0 N$ and $0 < n_1 < \ldots < n_q \leq n$ so that

$$\sum_{j=n+1}^{n_i} r_j(z) \geq \varepsilon(n_i - n) \quad \text{for every} \quad 0 \leq n < n_i, \ \text{and} \ i = 1, \ldots, q.$$

Observe that ζ does not depend on z neither on n, which means that for any z which satisfies (36), there exists $0 < n_1 < \ldots < n_q \leq n$ such that $r_{n_i}(z) \geq \varepsilon$ with $q/n \geq \zeta$. □

6.3 Construction of the measure: Proof of Theorem A

We consider the sequence

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \varphi_*^i \operatorname{Leb}_M$$

of averages of forward iterates of Lebesgue measure on M. The main idea is to decompose μ_n (for every n) as a sum of two measures, ν_n and η_n, such that ν_n is uniformly absolutely continuous and has total mass bounded away from zero. The measure ν_n will be the part of μ_n carried on the sets $\Delta(\cdot, \varepsilon)$ around points $\varphi^i(z)$, where z is a point which has i as ε-hyperbolic-like time.

Let us fix $\varepsilon > 0$ from Proposition 6.1. Let W_i be the set found on Lemma 5.1 for $\sigma = \varepsilon$. We consider the measures

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \varphi_*^i \operatorname{Leb}_{W_i}$$

and $\eta_n = \mu_n - \nu_n$. Now, we state and prove the main result of the section.

Proposition 6.2. *The measures ν_n are uniformly absolutely continuous and have total mass uniformly bounded away from zero for all large n.*

Proof. By the proposition 5.3, the measures ν_n are absolutely continuous and the densities are uniformly bounded from above. It just remains to prove the claim about the total mass. By lemma 5.1, we conclude that

$$\nu_n(M) \geq \tau \frac{1}{n} \sum_{i=1}^{n} \operatorname{Leb}(H_i(\varepsilon)).$$

So, it suffices to control the right side of the last expression. For this, let us consider π_n the measure in $\{1, 2, \ldots, n\}$ defined by $\pi_n(B) = \#(B)/n$, for every subset B. Using Fubini's theorem, we have

$$\frac{1}{n} \sum_{i=1}^{n} \operatorname{Leb}(H_i(\varepsilon)) = \int \int \chi(z,i) d \operatorname{Leb}(z) d\pi_n(i) = \int \int \chi(z,i) d\pi_n(i) d \operatorname{Leb}(z)$$

where $\chi(z,i) = 1$ if $z \in H_i(\varepsilon)$ and $\chi(z,i) = 0$ otherwise. By Lemma 6.2, it holds $\int \chi(z,i) d\pi_n(i) \geq \zeta$ if z is such that $\sum_{i=1}^{n} r_i(z) \geq 2\varepsilon n$. Hence

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} \operatorname{Leb}(H_i(\varepsilon)) \ \geq \ \zeta \operatorname{Leb}\left(\left\{ z;\ \sum_{i=1}^{n} r_i(z) \geq 2\varepsilon n \right\}\right) \geq \\ \geq \zeta \ \operatorname{Leb}\left(\left\{ z;\ \sum_{i=1}^{n} r_i(z) \geq 2\varepsilon n, \ \text{for all } n \geq N_0 \right\}\right) \end{aligned}$$

if $n \geq N_0$, where N_0 is the number found in proposition 6.1, i.e, the number such that (35) holds. In this way, we conclude that the total mass of the measure ν_n is bounded for $\tau\zeta(\vartheta/2)$ if $n \geq N_0$, and the proposition follows. $\square$

End of proof of Theorem A. It just remains to prove that we can choose our measure in such a way that it be invariant. Let us choose $\{n_k\}_k$ such that μ_{n_k}, ν_{n_k} and η_{n_k} converge and let μ, ν, η be the respective limit.

We can decompose $\eta = \eta^{ac} + \eta^s$ as the sum of an absolutely continuous measure η^{ac} and one singular measure η^s (with respect to Lebesgue measure). Then,

$$\mu = (\nu + \eta^{ac}) + \eta^s$$

gives one decomposition of μ as sum of one absolutely continuous and one singular measure. Since φ preserves the class of absolutely continuous measures and μ is invariant,

$$\mu = \varphi_*\mu = \varphi_*(\nu + \eta^{ac}) + \varphi_*\eta^s$$

gives another decomposition of μ as sum of one absolutely continuous and one singular measure. By the uniqueness of the decomposition we must have $\varphi_*(\nu + \eta^{ac}) = \nu + \eta^{ac}$. Hence, $\nu + \eta^{ac}$ is a non-zero absolutely continuous invariant measure for φ. Thus, the proof of Theorem A is complete. □

A Non wandering intervals

There are proofs of the non existence of wandering intervals for one dimensional maps on different generalities. Guckenheimer proved the result for unimodal maps with negative Schwarzian derivative and non degenerate critical points. De Melo and van Strien proved for unimodal maps with non-flat critical points. Block and Lyubich proved for smooth maps without inflection points and with non-flat critical points. De Melo, Martens and van Strien proved for smooth maps with non-flat critical points.

To the best of our knowledge there are no proof of non existence of wandering intervals for smooth maps (with turning and inflection points) with negative Schwarzian derivative. We prove these result in the case that the map has positive Lyapunov exponents.

Proposition A.1. *Let $f : I_0 \to I_0$ be a C^3 map with $Sf < 0$ and finite critical points. If there exists $\lambda > 0$ such that for Lebesgue almost every point $x \in I_0$,*

$$(P') \qquad \limsup_{n\to+\infty} \frac{1}{n} \log |Df^n(x)| > \lambda$$

Then f does not have wandering intervals.

Proof. Let us suppose, by contradiction, that f has a wandering interval J. Considering an iterate of J instead of J we may assume that no iterate of J contains critical points. Moreover, we may assume that J is a maximal wandering interval, i.e., that J is not strictly contained in some bigger wandering interval.

Let us consider T_n the maximal interval containing J such that $f^j(T_n) \cap \mathscr{C} = \emptyset$ for $1 \le j < n$. Denoting by L_n and R_n the connected components of $T_n \setminus J$, we can conclude that $|L_n|$ and $|R_n|$ goes to 0 when $n \to \infty$, otherwise, J would not be a maximal wandering interval.

By Corollary I we know that the set

$$X = \left\{ x \in I_0;\ \limsup_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} r_i(x) \geq \varsigma \right\}$$

has full Lebesgue measure for some $\varsigma > 0$. Let x_0 be some point in $J \cap X$. For every $n \in \mathbb{N}$, $T_n(x_0) = T_n$. Since $x_0 \in X$, there exists a sequence $\{n_k\}_k$ (depending on x_0) such that both components of $f^{n_k}(T_{n_k}) \setminus \{f^{n_k}(x_0)\}$ have length bigger than δ.

On the other hand, since J is a wandering interval,

$$|f^n(J)| \to 0 \qquad \text{when } n \to \infty.$$

Hence, given $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that for $n_k \geq N$,

$$f^{n_k}(T_{n_k}) \text{ contains an } \epsilon\text{-scaled neighborhood of } f^{n_k}(J)$$

Thus, by the Macroscopic Koebe Principle (see [MvS, Theorem IV.3.3]) this implies that T_{n_k} contains an $B_0(\epsilon)$-scaled neighborhood of J for $n_k \geq N$. But this contradicts the fact that $|L_{n_k}|$ and $|R_{n_k}|$ goes to zero when $k \to \infty$. Therefore f has no wandering intervals. □

Lemma A.1. *Let $f : I_0 \to I_0$ be a smooth map without wandering intervals and all periodic points are repelling. Then the set of preimages of the critical set $\mathscr{C}$ of f is dense in I_0.*

Proof. Let us suppose by contradiction, that there exists an interval I such that

$$f^j(I) \cap \mathscr{C} = \emptyset \quad \text{ for } j \in \mathbb{N}.$$

The intervals $\{f^j(I)\}_j$ may be disjoint or not. In case that they are disjoint, since there are no wandering intervals, J must converge to a periodic point. But this is a contradiction with the hypotheses about the Lyapunov exponents of f. Then, $\{f^j(I)\}_j$ are not disjoint. Hence, there exist an interval I^* and $m \in \mathbb{N}$, such that f^m maps I^* into itself diffeomorphically. But in this case, every non-periodic point of I^* is asymptotic to a periodic attractor. Once again, this is a contradiction with the hypotheses about the Lyapunov exponents. Therefore, the set $\cup_{n\in\mathbb{N}} f^{-n}(\mathscr{C})$ is dense in I_0. □

Recall that given $x \in I_0$ we denote by $T_n(x)$ the maximal interval containing x such that $f^j(T_n(x)) \cap \mathscr{C} = \emptyset$ for $1 \leq j < n$, where $\mathscr{C}$ is the set of critical points (zeroes of the derivative) of f. From the last corollary follows the next result.

Corollary A.1. *Let $f : I_0 \to I_0$ be a smooth map without wandering intervals and all periodic points are repelling. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $T_n(x)$ has length less than ϵ for $n \geq N$ and every $x \in I_0$.*

Let us denote by $\mathscr{C}_\Omega$ the set of critical points which are non wandering. We want to prove that the preimages of this set are also dense in I_0.

Lemma A.2. *Assume $f : I_0 \to I_0$ is a smooth map with finite critical points, without wandering intervals and all periodic points are repelling. Then the set of preimages of $\mathscr{C}_\Omega$ is dense in I_0.*

Proof. Let us assume by contradiction that the preimages of $\mathscr{C}_\Omega$ are not dense in I_0. For every $c \in \mathscr{C} \setminus \mathscr{C}_\Omega$, lets us consider V_c such that $f^n(V_c) \cap V_c = \emptyset$ for all $n \in \mathbb{N}$. Let $U \subset I_0$ be an open interval such that $(\cup_{n\in\mathbb{N}} f^{-n}(\mathscr{C}_\Omega)) \cap U = \emptyset$. Let $(a_0, b_0) \subset U$ some interval such that $f^{k_0}_{|(a_0,b_0)}$ is a diffeomorphism, $f^{k_0}(a_0) = c_0$ for some $c_0 \in \mathscr{C} \setminus \mathscr{C}_\Omega$ and $f^{k_0}((a_0, b_0)) \subset V_{c_0}$. This can be done considering $(a_0, b_0) := T_n(x_0)$ for some $x_0 \in U$ such that $x_0 \notin \cup_{n\in\mathbb{N}} f^{-n}(\mathscr{C})$ and n big enough (recall Corollary A.1). Let (a_1, b_1) be some subinterval of $f^{k_0}((a_0, b_0)) \setminus \{c_0\}$ such that $f^{k_1}_{|(a_1,b_1)}$ is a diffeomorphism, $f^{k_1}(a_1) = c_1$ for some $c_1 \in \mathscr{C} \setminus \mathscr{C}_\Omega$ and for some $k_1 \in \mathbb{N}$. We also assume that $f^{k_1}((a_1, b_1)) \subset V_{c_1}$. Thus, we define inductively one sequence of intervals: given (a_{n-1}, b_{n-1}) such that $f^{k_{n-1}}((a_{n-1}, b_{n-1})) \subset V_{c_{n-1}}$ for some $k_{n-1} \in \mathbb{N}$ and some $c_{n-1} \in \mathscr{C} \setminus \mathscr{C}_\Omega$, we define (a_n, b_n) as being one subinterval of $f^{k_{n-1}}((a_{n-1}, b_{n-1}))$ such that $f^{k_n}((a_n, b_n)) \subset V_{c_n}$ for some $c_n \in \mathscr{C} \setminus \mathscr{C}_\Omega$ and some $k_n \in \mathbb{N}$. This can be done by the results of corollaries A.1 and A.1.

But $\mathscr{C}$ is a finite set, then for some $m, n \in \mathbb{N}$, $c_n = c_{n+m}$, and this is contradictory with the choice of V_c. Therefore, there can not exist a set U as above, and the claim follows. □

The next claim will be useful to prove the existence of partitions of I_0 arbitrary small, whose boundaries are a forward invariant set.

Lemma A.3. *Assume $f : I_0 \to I_0$ is a smooth map with finite critical points. If c is a non wandering turning critical point then $c \in \overline{\mathrm{Per}(f)}$.*

Proof. For each turning point $c \in \mathscr{C}$, there exists a neighborhood V_c of c and a continuous function $\tau : V_c \to V_c$ such that $f(\tau(x)) = f(x)$ for every $x \in V_c$ and $\tau(x) \neq x$ for $x \in V_c \setminus \{c\}$. Then, given $\epsilon > 0$, let $\beta > 0$ be such that $\tau((c - \beta, c + \beta)) \subset (c - \epsilon, c + \epsilon)$.

We claim that there exist $y \in (c - \beta, c + \beta)$ and $m \in \mathbb{N}$ such that $f^m(y) = c$. Let us assume by contradiction that it does not happen.

Let us consider the first return map R_J to $J := (c - \beta, c + \beta)$. Observe that if $x \notin \partial J$ is a discontinuity point of R_J then R_J is continuous from one side, $R_J(x) \in \partial J$ and the limit from the other side belongs to $R_J(\partial J)$. On the other hand, if x is a turning point of R_J, then $f^n(x) = c_i$ for some $c_i \in \mathscr{C}$ and some $n \in \mathbb{N}$, and $R_J(x) = f^{n+m_i}(x) = f^{m_i}(c_i)$ for some $m_i \in \mathbb{N}$. Since $\mathscr{C}$ is a finite set, the values of R_J for the turning points is a finite set. Then the distance of the image of R_J to c is given by

$$\delta = \min\{\mathrm{dist}(c, f^{m_i}(c_i)), \mathrm{dist}(c, R_J(\partial J))\}$$

where $c_i \in \mathscr{C}$ and $f^{m_i}(c_i)$ is the first entry of c_i to J, when it is defined. Note that if ∂J is not in the domain of R_J, we do not consider the second term above. But this implies that $f^n((c - \delta, c + \delta)) \cap (c - \delta, c + \delta) = \emptyset$ for all $n \in \mathbb{N}$, and this contradicts that c is non wandering. Therefore, there exists $y \in (c - \beta, c + \beta)$ and $m \in \mathbb{N}$ such that $f^m(y) = c$. By definition of the function τ, $f^m(\tau(y)) = c$. Hence, by the continuity of f^m, there must exist $p \in (c - \epsilon, c + \epsilon)$ such that $f^m(p) = p$. Since ϵ is arbitrary, the claim follows. □

Combining Lemmas A.2 and A.3 we have the following result

Proposition A.2. *Assume $f : I_0 \to I_0$ is a smooth map with finite critical points, without wandering intervals and all periodic points are repelling. Given $\delta > 0$ there exists a finite partition $\mathcal{P}$ of I_0, with norm less than δ and such that the extremes of the elements of the partition are a forward invariant set.*

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