Instituto Nacional de Matemática Pura e Aplicada

On Poincaré Series of Singularities of Curves over Finite Fields

Author: Jhon J. Mira A.

Adviser: Karl-Otto Stöhr

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A Sandra, Mariana y laura

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Resumo

Para cada par de ideais fracionários de um anel local em um ponto singular de uma curva algébrica, geometricamente integral e definida sobre um corpo finito, há associada uma serie de Poincaré em m variáveis, onde m é o número de ramos da singularidade da curva. Esta serie codifica as cardinalidades de certos conjuntos finitos de ideais fracionários e pode ser representada como uma integral no contexto da análise harmônica. Além disso, também permite estudar funções zeta locais. Neste trabalho desenvolvemos métodos para computar estas series e estudamos o comportamento das mesmas à respeito de mudança do corpo de constantes e de explosões do anel local. Como os anéis que resultam após estas operações não são anéis locais, embora semi locais, nós estendemos naturalmente a definição da serie de Poincaré multi-variáveis para anéis semi-locais e mostramos a relação entre as duas teorias. Além disso, provamos no caso semi-local algumas propriedades provadas no caso local. Neste trabalho, nós também mostramos que quando o anel local é residualmente racional o semi-grupo associado determina as series de Poincaré multi-variáveis. Em particular, para curvas algebroides planas, esta serie permite associar ao anel local da curva uma serie em m variáveis, que é um invariante completo da classe de equi-singularidade. Esta serie é também similar as series de Poincaré em varias variáveis associadas a germes de curvas algébricas singulares complexas.

Palavras chaves: Series de Poincaré, funções zeta e singularidades de curvas sobre corpos finitos.

Contents

Introduction

In 1949, André Weil [31] formulated his now famous conjectures concerning the number of solutions of polynomial equations over finite fields. These conjectures suggested a deep connection between the arithmetic of algebraic varieties defined over finite fields and the topology of algebraic varieties defined over the complex numbers. Weil was led to his conjectures by consideration of the zeta functions of some special varieties. In the zeta functions associated with algebraic curves over finite fields there are encoded properties of arithmetic nature of the curves. In the non-singular case the theory is well-known, and it culminates in the Hasse-Weil theorem about the Riemann hypothesis for curves and in Deligne's theorem about the Weil's conjectures for higher dimensional varieties. One of Weil's major pieces of work was the proof of the fact that his conjectures hold for curves, say the rationality and the functional equation of this zeta function, and the analogue to the Riemann hypothesis. In 1973 Galkin published Paper [14], which deals with a zeta function of orders in global fields that encodes the number of ideals with given norms and is defined in the half-plane. His zeta-function coincides with Schmidt's zeta function in the case of a non-singular curve but it satisfies a functional equation only in Gorenstein case. In 1989, by slightly modifying the zeta function introduced by Galkin, Green [15] obtained a new zeta function in terms of the index of non-zero fractional ideals. Green's zeta function satisfies a functional equation, but it is not uniquely determined by the curve. Finally, in [26] Stöhr introduced a zeta function of a local ring $\mathcal{O}$ of a possibly singular, complete, geometrically irreducible algebraic curve X define over a finite field $k = \mathbb{F}_q$ of q elements with rational function field K. This zeta function, which encodes the number of positive fractional ideals ($\mathcal{O}$-ideals) of given degrees, is defined in the half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$ by the absolutely convergent Dirichlet series

$$\zeta(\mathcal{O}, s) := \sum_{\mathfrak{d} \supseteq \mathcal{O}} \#\left(\mathfrak{d}/\mathcal{O}\right)^{-s}, \ \Re(s) > 0$$

where the sum is taken over the $\mathcal{O}$-ideals $\mathfrak{d}$ that contain the local ring $\mathcal{O}$. Moreover, this zeta function coincides with the zeta function of Galkin in the Gorenstein case. It is a rational function and always satisfies a functional equation. Stöhr also introduced and studied, for any non-zero fractional ideal $\mathfrak{a}$ of the local ring $\mathcal{O}$, the *local zeta function* defined by the Dirichlet series

$$\zeta\left(\mathfrak{a}, s\right) := \sum_{\mathfrak{d}\supseteq\mathfrak{a}} \#\left(\mathfrak{d}/\mathfrak{a}\right)^{-s},\ \Re(s) > 0$$

where the sum is taken over the $\mathcal{O}$-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$. By breaking up the set of $\mathcal{O}$-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$ according to finitely many ideal classes, it is obtained a partition of the series $\zeta\left(\mathfrak{a}, s\right)$ as a finite sum of the *partial local zeta functions*

$$\zeta\left(\mathfrak{a}, \mathfrak{b}, s\right) := \sum_{\mathfrak{d}\supseteq\mathfrak{a},\ \mathfrak{d}\sim\mathfrak{b}} \#\left(\mathfrak{d}/\mathfrak{a}\right)^{-s},\ \Re(s) > 0$$

where the sum is taken over all $\mathcal{O}$-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$ and that are equivalent to $\mathfrak{b}$. These partial series only depend on the ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. And these partial series can be written as power series $Z\left(\mathfrak{a}, \mathfrak{b}, t\right)$ in $t := q^{-s}$ with integer coefficients, which converge absolutely in the disk $|t| < 1$ (cf. [26]).

In a recent paper, Stöhr [27] introduced, for any local ring $\mathcal{O}$ of a curve X (complete, geometrically irreducible, algebraic curve defined over a finite field $k = \mathbb{F}_q$ of q elements with rational function field K), and any pair of $\mathcal{O}$-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$, the *multi-variable Poincaré series* defined to be the multi-variable power series

$$P(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) := \sum \eta_{\mathbf{n}}\left(\mathfrak{a}, \mathfrak{b}\right)\mathbf{t}^{\mathbf{n}} \in \mathbb{Z}[[t_1, \ldots, t_m]]$$

whose coefficients are the cardinalities

$$\eta_{\mathbf{n}}\left(\mathfrak{a}, \mathfrak{b}\right) := \#\{\mathcal{O}\text{-ideals } \mathfrak{d} \text{ satisfying } \mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b} \text{ and } \mathfrak{d}\cdot\widetilde{\mathcal{O}} = \mathfrak{a}\cdot\mathfrak{p}^{-\mathbf{n}}\}$$

where $\mathbf{t}^{\mathbf{n}} := t_1^{n_1}\cdots t_m^{n_m}$ for each $\mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m$ and m is the number of branches centered at the curve singularity (cf. [27], Definition 2.1.) This series only depends on the $\mathcal{O}$-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$ and it converges absolutely in the poly-disk $|t_1| < 1, \cdots, |t_m| < 1$. Moreover, it is a rational function

$$P(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m) = \frac{\Lambda(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)}{(1-t_1)\cdots(1-t_m)}$$

where $\Lambda(\mathfrak{a}, \mathfrak{b}; t_1, \ldots, t_m) \in \mathbb{Z}[t_1, \cdots, t_m]$ is a polynomial of multi-degree $\leq \mathbf{b}$, where $\mathbf{b} = (b_1, \cdots, b_m)$ is the multi-exponent of the fractional ideal $(\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b}\widetilde{\mathcal{O}}$ in the integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ in K. The polynomial $\Lambda(\mathfrak{a}, \mathfrak{b}; t_1, \ldots, t_m)$ satisfies the following *functional equation*

$$\Lambda(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m) = [U_{\mathfrak{b}:\mathfrak{a}} : U_{\mathfrak{b}}]q^{\dim(\mathfrak{b}:\mathfrak{a}/(\mathfrak{b}:\mathfrak{a}):\widetilde{\mathcal{O}})}t_1^{b_1}\cdots t_m^{b_m}\Lambda(\mathcal{O}, \mathfrak{a}\cdot\mathfrak{b}^*, \frac{1}{q^{r_1}t_1}, \cdots, \frac{1}{q^{r_m}t_m})$$

where $\mathfrak{b}^*$ is the dual $\mathcal{O}$-ideal of $\mathfrak{b}$, $\mathfrak{b} : \mathfrak{a}$ is the quotient between the two $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$, and $r_1, \cdots, r_m$ are the degrees of the branches centered at the singularity (cf. [27], Theorem 7.1.)

It is important to notice that the multi-variable Poincaré series $P(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ can be represented by an integral within the framework of harmonic analysis. The series $P(\mathcal{O}, \mathcal{O}, q^{-r_1}t_1, \cdots, q^{-r_m}t_m)$ is, up to a constant factor, equal to the series $P_g(t_1, \ldots, t_m)$ defined by Delgado and Moyano [11], which may be viewed as an analogue of a multi-variable Poincaré series for complex algebraic curve singularities (cf. [27], Theorems 5.2 and 6.3.). Even more, the partial zeta function can be expressed in terms of the Poincaré series as

$$Z(\mathfrak{a}, \mathfrak{b}, t) = t^{\dim(\mathfrak{a}\widetilde{\mathcal{O}}/\mathfrak{a}) - \dim(\mathfrak{b}\widetilde{\mathcal{O}}/\mathfrak{b})} P(\mathfrak{a}, \mathfrak{b}, t^{r_1}, \ldots, t^{r_m})$$

(cf. [27], Theorem 2.3). Thus, the multi-variable Poincaré series $P(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ furnishes a deeper discernment into the nature of local zeta functions.

The main objective of this thesis is to study the properties of the local zeta function $\zeta(\mathfrak{a}, s)$ and the multi-variable Poincaré series $P(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$. One of our purposes is to describe a procedure which is useful to determine the ideal classes of a local ring $\mathcal{O}$ and to compute the Poincaré series $P(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ for each pair of $\mathcal{O}$-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. Moreover, we give some examples of multi-variable Poincaré series of some curves where we show the behavior of them under constant field extensions.

In a natural way, we extend the definitions of zeta function, partial zeta function and multi-variable Poincaré series to a semilocal ring of a curve X (possibly singular, complete, geometrically irreducible algebraic curve X define over a finite field $k = \mathbb{F}_q$ of q elements with rational function field K). Let S be a proper semilocal subring of the function field $K \mid k$ of a curve X. We associate to each S-ideal $\mathfrak{a}$, as well as to each pair of S-ideal (non-zero fractional ideal of S) classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$, the zeta function $\zeta_S(\mathfrak{a}, s)$, the partial zeta function $Z_S(\mathfrak{a}, \mathfrak{b}, t)$ and the multi-variable Poincaré series $P_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m) \in \mathbb{Z}[t_1, \ldots, t_m]$ in m variables with integer coefficients (see 3.1, 3.2 and 29). Just as in the local case, the series $P_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$ converges absolutely in the unit poly-disk $|t_1| < 1, \cdots, |t_m| < 1$, where m is the number of places lying over S. We prove the link between the local and semilocal definitions by means of an Euler product identity (see 33 and 35) that provides us a way to prove, for $Z_S(\mathfrak{a}, \mathfrak{b}, t)$ and $P_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$, some similar properties to those proved by Stöhr in [26] and [27].

The extended definitions for semilocal rings are important because they permit us to study the behavior of the series $\zeta(\mathfrak{a}, s)$, $Z(\mathfrak{a}, \mathfrak{b}, t)$ and $P(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$ under constant field extensions (see Section 4.3): if $\mathcal{O}$ is a local ring of the geometrically integral algebraic curve defined over a field $k = \mathbb{F}_q$, whose function field in one variable is $K \mid k$, and if k' is a finite field extension of k, then $k' \cdot K \mid k'$ is also a function field in one variable and $k' \cdot \mathcal{O}$ is a semilocal subring of $k' \cdot K \mid k'$, where $k' \cdot \mathcal{O}$ just consists of all linear combination of elements of the local ring $\mathcal{O}$ with coefficients in the field k' (cf. [22] section 3). The extended definitions associated to semilocal rings are also important because they permit us to study the behavior of the series $\zeta(\mathcal{O}, s)$, $Z(\mathfrak{a}, \mathfrak{b}, t)$, and $P(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$, attached to the local ring $\mathcal{O}$ of the curve X, with respect to the blow-up of the local ring $\mathcal{O}$, since the blow-up of a local ring $\mathcal{O}$ with respect to its maximal ideal $\mathfrak{m}$ is a semilocal ring $\mathcal{O}^{\mathfrak{m}}$ (cf. [20] Chapter VIII). Furthermore, they permit us to associate to a geometrically integral algebraic curve X defined over a finite field $\mathbb{F}_q$ of q elements the multi-variable Poincaré series $P_S(S, S; t_1, \ldots, t_m)$, where S is a semilocal ring of the curve X which is contained in the semilocal ring defined as the intersection of all the local rings corresponding to singular points of X.

We also observe that the mentioned definitions of zeta function, partial zeta function and multi-variable Poincaré series associated to non-zero fractional ideals of a local ring $\mathcal{O}$ of the irreducible algebraic curve X, can also be defined for regular fractional ideals of a reduced local ring $\mathcal{O}$ of a possibly singular, complete, reduced algebraic curve X define over a finite field $k = \mathbb{F}_q$ (see Section 5.1).

Let $\mathcal{O}$ be the local ring at a singular point of a geometrically integral algebraic curve defined over a finite field $k = \mathbb{F}_q$. We prove that, if $\mathfrak{b}$ is an $\mathcal{O}$-ideal such that the ring $\mathfrak{b} : \mathfrak{b}$ is a local ring, then the Poincaré series $P(\mathcal{O}, \mathfrak{b}, \mathbf{t})$ is congruent modulo $(q-1)\,\mathbb{Z}[[t_1, \cdots, t_m]]$ with the series $\frac{(t_1-1)\cdots(t_m-1)}{t_1\cdots t_m-1} \sum_{\mathbf{n}\in\mathbb{Z}^m} \mathrm{length}_{\mathfrak{b}:\mathfrak{b}}\left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n+1}}\right) \mathbf{t}^{\mathbf{n}}$, which is a polynomial when $m \geq 2$. Hence, in particular, if the local ring $\mathcal{O}$ correspond to a rational point, then

$$P(\mathcal{O}, \mathcal{O}, \mathbf{t}) \equiv \frac{(t_1-1)\cdots(t_m-1)}{t_1\cdots t_m-1} \sum_{\mathbf{n}\in\mathbb{Z}^m} \dim_k\left(\mathcal{O} \cap \mathfrak{p}^{\mathbf{n}} / \mathcal{O} \cap \mathfrak{p}^{\mathbf{n+1}}\right) \mathbf{t}^{\mathbf{n}}$$

modulo $(q-1)\,\mathbb{Z}[[t_1, \cdots, t_m]]$. This establishes a link with the Poincaré series $P_{\mathcal{C}}(t_1, ..., t_m)$ defined by Campillo, Delgado and Gusein-Zade in [7], which is a series attached to a germ $(\mathcal{C}, 0) \subseteq (\mathbb{C}^2, 0)$, and is equal to the Alexander polynomial of the link $\mathcal{C} \cap S^3_\epsilon \subseteq S^3_\epsilon$ for sufficiently small $\epsilon > 0$.

We show that, if $\mathcal{O}$ is a residually rational local ring, then the multi-variable Poincaré series $P(\mathcal{O}, \mathcal{O}, t_1, \cdots, t_m)$ depends only on the semigroup $S(\mathcal{O})$ of $\mathcal{O}$ (see 37 and 4.3). Thus, if $\mathcal{O}$ is a residually rational local ring and the residue field k of $\mathcal{O}$ is not too small, then we can associate to $\mathcal{O} \otimes_k \overline{k}$ the multi-variable rational function

$$\overline{P}(\mathcal{O} \otimes_k \overline{k}, \mathcal{O} \otimes_k \overline{k}, T_1, \cdots, T_m) := P(\mathcal{O}, \mathcal{O}, T_1, \cdots, T_m) \mod (q-1)\,\mathbb{Z}[[t_1, \cdots, t_m]],$$

where $T_1, \cdots, T_m$ are indeterminates. The rational function $\overline{P}(\mathcal{O}\otimes_k\overline{k}, \mathcal{O}\otimes_k\overline{k}, T_1, \cdots, T_m)$ only depends on $S(\mathcal{O})$ and it is a polynomial when $m \geq 2$. A key ingredient is a result proved by Zuñiga in [36] (cf. Proposition 4.7, page 35). If $\mathcal{O}$ is a residually rational local ring, then there exists a unique finite field extension $k_0 \mid k$ such that for each finite field extension l of k_0 the semigroups $S(\mathcal{O}\otimes_k k_0)$ and $S(\mathcal{O}\otimes_k l)$ are the same and, hence, $S(\mathcal{O}\otimes_k k_0) = S\left(\mathcal{O}\otimes_k \overline{k}\right)$. In virtue of this fact we may assume that $\mathcal{O}$ is residually rational ring and $S(\mathcal{O}) = S\left(\mathcal{O}\otimes_k \overline{k}\right)$. Thus, the series $\overline{P}(\mathcal{O}\otimes_k\overline{k}, \mathcal{O}\otimes_k\overline{k}, T_1, \cdots, T_m)$ is well defined. We study, in particular, the multi-variable Poincaré series $P(\mathcal{O}, \mathcal{O}, t_1, \cdots, t_m)$ of the reduced local ring $\mathcal{O} := \mathbb{F}_q[[X,Y]]/(f(X,Y))$ of a plane algebroid curve totally defined over a finite field $\mathbb{F}_q$. If the residue field of the algebroid curve is not too small, $P(\mathcal{O}, \mathcal{O}, t_1, \cdots, t_m)$ becomes a complete invariant of the equisingularity class of the algebroid curve $\mathcal{O}$ (cf. [30], [33]). Finally, we study a relation between $\overline{P}(\mathcal{O}\otimes_k\overline{k}, \mathcal{O}\otimes_k\overline{k}, T_1, \cdots, T_m)$ and $\prod_\sigma (1 - T_1^{m^\sigma(f_1)} \cdots T_m^{m^\sigma(f_m)})^{\#(E_\sigma \setminus E_\sigma^0)-2}$, which is a series associated to the minimal embedded resolution of the algebroid curve defined by the series $f = f_1 \cdots f_m$, taken free from multiple factors. In this case, $D = \bigcup_{\sigma\in\Gamma} E_\sigma$ is the exceptional divisor, $E_\sigma^o \subseteq E_\sigma$ is the complement in E_σ of the intersection with all other components of the total transform and $m^\sigma(f_1), \cdots, m^\sigma(f_m)$ are the multiplicities along E_σ of the liftings of $f_1, \cdots, f_m$, respectively.

The organization of this Thesis and results is as follows.

In Chapter 1 we review the notion of complete, geometrically irreducible, algebraic curves defined over a field as well as their main properties and local duality. We also recall the definition and main properties of zeta functions and Stöhr's Poincaré series of local rings of algebraic curves defined over a finite field.

In Chapter 2 we give the definition and main properties of semilocal subrings of a function field $K \mid k$ of one variable with constant field k. In particular, the property that, given a proper semilocal subring of a function field, it may be expressed as intersection of a finite number of local rings, no two of which are contained in the same valuation ring. Then, we prove that each fractional ideal of that semilocal ring may be expressed as an intersection of fractional ideals of the several components of the semilocal ring. We prove also that its degree may be expressed as sum of the degrees of the several fractional ideal components. Furthermore, we study the connection between some objects associated to non-zero fractional ideals of the semilocal ring and the corresponding objects of the local rings of its decomposition. In the last part of this chapter we prove some properties on semigroups.

In Chapter 3 we introduce the extended definitions of the multi-variable Poincaré series $P_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$, the zeta function $Z_S(\mathfrak{a}, t)$ and the partial local zeta function $Z_S(\mathfrak{a}, \mathfrak{b}, t)$, associated to each pair of S-ideal $\mathfrak{a}$ and $\mathfrak{b}$ of a semilocal ring S of a possibly singular, complete, geometrically irreducible algebraic curve X defined over a finite field

$k = \mathbb{F}_q$ of q elements with rational function field K. After this, we prove the Euler product identity which gives the link between the local and semilocal theory. We finish Chapter 3 by studying some properties of the multi-variable Poincaré series attached to semilocal ring which are similar to that proved in [27].

In Chapter 4 we indicate a procedure which is useful to determine the ideal classes of a local ring $\mathcal{O}$ and to compute the Poincaré series $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t})$ for each pair of ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. Moreover, we discuss basic examples of these objects that show the behavior of them under constant field extensions. After this, we discuss the behavior of the Multi-variable Poincaré series under constant field extensions.

In Chapter 5 we observe that the preceding theory about zeta function and multi-variable Poincaré series can be extended to reduced curves and we indicate the necessary modifications in order to show, in this case, some mentioned results. Then we study the multi-variable Poincaré series of a class of algebroid plane curves totally defined over a finite field. We show that its multi-variable Poincaré series is a complete invariant of its equisingularity class, in the sense of Zariski. We associate to a local ring $\mathcal{O}$ the rational function $\overline{P}(\mathcal{O} \otimes_k \overline{k}, \mathcal{O} \otimes_k \overline{k}, T_1, \cdots, T_m)$. We finish Chapter 5 by studying for algebroid plane curves a relation between $\overline{P}(\mathcal{O} \otimes_k \overline{k}, \mathcal{O} \otimes_k \overline{k}, T_1, \cdots, T_m)$ and the series $\prod_\sigma (1 - T_1^{m^\sigma(f_1)} \cdots T_m^{m^\sigma(f_m)})^{\#(E_\sigma \setminus E_\sigma^0) - 2}$ (a similar relation to Formula 5.1, which was proved by Campillo, Delgado and Gusein-Zade for germs of complex plane curves). We prove it in some particular cases.

Chapter 1

Preliminaries

This chapter contains some preliminary definitions and results about singular curves, zeta functions and Poincaré series needed in the sequel (cf. [25], [26], [27] and [22]).

1.1 Singular curves

In this section we present the basic facts about singular curves, for a treatment of them we followed as main reference [25].

Let X be a complete, geometrically irreducible, algebraic curve defined over a field k and let K be the rational functions on X. This means that K is the functions field in one variable with the constant field k and that X is (the index set of) a set $\{\mathcal{O}_P\}_{P\in X}$ of local k-algebras, properly contained in K with quotient field K, satisfying the two properties:

i. For almost all $P \in X$, the local ring $\mathcal{O}_P$ is discrete valuation ring.

ii. For each discrete valuation ring R of $K \mid k$ there is an unique $P \in X$ such that $\mathcal{O}_P \subseteq R$.

(In the schemes language X has one more point, namely its generic point whose local ring is the function field K.) Thus, by the first condition the number of singular points of X is finite. By the second condition there exists a morphism $\pi : \widetilde{X} \longrightarrow X$ where $\widetilde{X}$ denote the set of all discrete valuation rings of $K \mid k$ and it is called the *non-singular model* of X (also named the normalizations of X over k.) For each $P \in X$ the elements of the fiber $\pi^{-1}(P)$ are called the *branches of X centered at P*. By the extension theorem of valuation theory, the morphism $\pi : \widetilde{X} \longrightarrow X$ is surjective. Furthermore, the number of branches centered at P is finite, because the branches centered at P are precisely the zeros of each rational function vanishing at P.

By a *divisor* of X we mean a coherent fractional ideal sheaf or, equivalently, a formal product

$$\mathfrak{a} = \prod_{P \in X} \mathfrak{a}_P$$

where for each P the P-component $\mathfrak{a}_P$ (i.e the stalk of $\mathfrak{a}$ at P) is a non-zero fractional ideal of $\mathcal{O}_P$ and $\mathfrak{a}_P = \mathcal{O}_P$ for almost all P. We say that $\mathfrak{a}_P$ is an *$\mathcal{O}_P$-ideal.* The set of divisors of X is denoted by $Div(X)$. A divisor $\mathfrak{a}$ is *called locally principal* (or a *Cartier divisor*) if each P-component $\mathfrak{a}_P$ is a principal $\mathcal{O}_P$-ideal.

Given two divisors $\mathfrak{a}$ and $\mathfrak{b}$ it is defined the *product* $\mathfrak{a} \cdot \mathfrak{b}$ and the *quotient* $\mathfrak{a} : \mathfrak{b}$ by setting:

$$(\mathfrak{a} \cdot \mathfrak{b})_P = \mathfrak{a}_P \cdot \mathfrak{b}_P$$

and

$$(\mathfrak{a} : \mathfrak{b})_P = \mathfrak{a}_P : \mathfrak{b}_P$$

where $\mathfrak{a}_P \cdot \mathfrak{b}_P$ is the $\mathcal{O}_P$-ideal generated by the products ab with $a \in \mathfrak{a}_P$ and $b \in \mathfrak{b}_P$, and $\mathfrak{a}_P : \mathfrak{b}_P = \{z \in K : z\mathfrak{b}_P \subseteq \mathfrak{a}_P\}$

In $Div(X)$, it is defined a partial order by: $\mathfrak{a} \geq \mathfrak{b}$ if and only if $\mathfrak{a}_P \supseteq \mathfrak{b}_P$ for all $P \in X$. Hence, a divisor $\mathfrak{a}$ is called *positive* (or *effective*) if $\mathfrak{a} \geq \mathcal{O}$, where $\mathcal{O} := \prod_{P \in X} \mathcal{O}_P$ is the *structure divisor* of X. It is common in the literature but it would be inconvenient in our approach, to invert the ordering.

The *degree* of a divisor is uniquely defined by the properties:

i. $\deg(\mathcal{O}) := 0$ and

ii. $\deg(\mathfrak{a}) - \deg(\mathfrak{b}) = \sum_{P \in X} \dim(\mathfrak{a}_P / \mathfrak{b}_P)$ whenever $\mathfrak{a} \geq \mathfrak{b}$.

For each non-zero rational function $z \in K \backslash \{0\}$ let $\operatorname{div}(z)$ be its *principal divisor* defined by

$$\operatorname{div}(z) := \prod_{P \in X} z^{-1} \mathcal{O}_P.$$

For each $\mathfrak{a}$ divisor of X let

$$L(\mathfrak{a}) := \bigcap_{P \in X} \mathfrak{a}_P = \{z \in K : \operatorname{div}(z) \cdot \mathfrak{a} \geq \mathcal{O}\}$$

be the *k-vector space of global sections of* $\mathfrak{a}$ (Also denoted by $H^0(X,\mathfrak{a})$) and let

$$\Lambda(\mathfrak{a}) := \prod_{P\in X} \widehat{\mathfrak{a}}_P$$

be the *paralleletope of* $\mathfrak{a}$, where $\widehat{\mathfrak{a}}_P$ is the the completion of the P-component $\mathfrak{a}_P$ of a divisor $\mathfrak{a}$. It is well known (cf. [22] and [25]) that $L(\mathfrak{a}) = \Lambda(\mathfrak{a}) \cap K$ and that $\Lambda(\mathfrak{a})$ is contained in the k-algebra $A_{K|k}$ of adeles of $K \mid k$ defined to be the restricted product of the local fields $\widehat{K}_Q$ of the branches $Q \in \widetilde{X}$. Moreover, the two dimensions $l(\mathfrak{a}) := \dim L(\mathfrak{a})$ and $i(\mathfrak{a}) := \dim(A_{K|k}/\Lambda(\mathfrak{a})+K)$ are finite. In this way, the Riemann-Roch Theorem for functions field was generalized by Rosenlicht to curves with singularities, that is, each divisor $\mathfrak{a}$ of X satisfies $l(\mathfrak{a}) = \deg(\mathfrak{a}) + 1 - g + i(\mathfrak{a})$, where $g := i(\mathcal{O})$ is called the arithmetic genus of X. Thus, the degree of $\mathfrak{a}$ only depends on the *linear equivalence* class

$$\{\operatorname{div}(z)\cdot\mathfrak{a} : z \in K, z \neq 0\}$$

of the divisor $\mathfrak{a}$. By the Riemann-Roch theorem it is gotten the *genus formula*

$$g = \widetilde{g} + \sum_{P\in X} \delta_P$$

where $\widetilde{g}$ is the *geometric genus* of X defined to be the genus of the non-singular model $\widetilde{X}$ (cf. [22] and [25]).

A (*Weil*) *differential* of X is defined as a k-linear functional $A_{K|k} \longrightarrow k$ vanishing on $\Lambda(\mathfrak{a})+K$ for some divisor $\mathfrak{a}$ of X. Since $\Lambda(\mathfrak{a} : \widetilde{\mathcal{O}}) \subseteq \Lambda(\mathfrak{a}) \subseteq \Lambda(\mathfrak{a}\cdot\widetilde{\mathcal{O}})$ this notion only depends on the non-singular model $\widetilde{X}$. The k-vector space of all differentials vanishing on $\Lambda(\mathfrak{a})$ is denoted by $\Omega(\mathfrak{a})$. Thus, $i(\mathfrak{a}) = \dim\Omega(\mathfrak{a})$.

Let λ be a non-zero differential. Among the paralleletopes where λ vanishes there is a largest one, say $\Lambda(\mathfrak{c})$ (cf. [25]). The divisor $\operatorname{div}(\lambda) := \mathfrak{c}$ is called the *divisor of* λ on X. Observe that the divisor of λ on the non-singular model $\widetilde{X}$ corresponds to the divisor $\mathfrak{c} : \widetilde{\mathcal{O}}$ of X which is the largest $\widetilde{\mathcal{O}}$-divisor smaller than or equal to $\mathfrak{c}$.

Since the space $\Omega_{K|k}$ of differentials is one-dimensional vector apace over the function field K (cf. [21]), the linear equivalence class of the divisor $\mathfrak{c} = \operatorname{div}(\lambda)$ does not depend on the choice of the non-zero differential λ, and it is called the *canonical class.* It is deduced that $\Omega(\mathfrak{a}) = L(\mathfrak{c} : \mathfrak{a})\,\lambda$ and, therefore, that $i(\mathfrak{a}) = l(\mathfrak{c} : \mathfrak{a})$ for each divisor $\mathfrak{a}$ of X. In particular it follows that

$$l(\mathfrak{c}) = i(\mathcal{O}) = \dim\Omega(\mathcal{O}) = g.$$

Furthermore, by applying the Riemann-Roch theorem,

$$\deg(\mathfrak{c}) = 2g - 2$$

The mentioned definitions and results are the main ingredient in the study of the Dirichlet series

$$\zeta(\mathcal{O}_X, s) := \sum_{\mathfrak{a} \geq \mathcal{O}} q^{-s \deg(\mathfrak{a})}$$

where $\mathfrak{a}$ ranges over the positive divisors of X. In the next section we present some known facts about this important series.

We finish this section given some local definitions and properties of the curve X which are the essential tools to study the local factors of the zeta function $\zeta(\mathcal{O}_X, s)$ as well as the Poincaré series $P(\mathfrak{a}_P, \mathfrak{b}_P, \mathfrak{t})$. Let P be a point of X, let $\mathcal{O}_P$ be the local ring of X at P and let $\widetilde{\mathcal{O}}_P$ be the integral closure of $\mathcal{O}_P$ in K. The *degree of singularity of X at P* is defined as

$$\delta_P := \dim(\widetilde{\mathcal{O}}_P / \mathcal{O}_P).$$

Since K is a function field in one variable with constant field k, each integral k-algebra A with quotient field K has finite k-codimension in its integral closure $\widetilde{A}$ (cf. [22]). Therefore δ_P is an integer number. Thus, the *degree of singularity of* X, defined by

$$\delta := \sum_{P \in X} \delta_P,$$

is well defined as well as the *local degree function* $\deg_P$ defined by the properties:

i. $\deg_P(\mathcal{O}_P) := 0$ and

ii. $\deg_P(\mathfrak{a}_P) - \deg_P(\mathfrak{b}_P) = \dim(\mathfrak{a}_P/\mathfrak{b}_P)$ whenever $\mathfrak{a}_P \supseteq \mathfrak{b}_P$.

The Local Duality Theorem was also generalized to singular curves (cf. [25]).

Theorem 1 ***(Local Duality)*** *Let $\mathfrak{a}_P$, $\mathfrak{b}_P$ be $\mathcal{O}_P$-ideals such that $\mathfrak{a}_P \supseteq \mathfrak{b}_P$. Then there is an isomorphism of k-vector spaces*

$$(\mathfrak{c}_P : \mathfrak{b}_P)/(\mathfrak{c}_P : \mathfrak{a}_P) \xrightarrow{\sim} \hom_k(\mathfrak{a}_P/\mathfrak{b}_P, k) \tag{1.1}$$

defined by $\overline{c} \mapsto (\overline{a} \mapsto \lambda_P(ac))$, where $\mathfrak{c} = \operatorname{div}(\lambda)$, λ_P is the P-component of the differential λ and λ_P is defined to be the composition homomorphism:

$$\lambda_P : K \hookrightarrow \widehat{K}_{Q_1} \times \cdots \times \widehat{K}_{Q_m} \hookrightarrow A_{K|k} \xrightarrow{\lambda} k.$$

It follows from the local duality that $\deg_P(\mathfrak{a}_P) + \deg_P(\mathfrak{c}_P : \mathfrak{a}_P)$ does not depend on the $\mathcal{O}_P$-ideal (fractional) $\mathfrak{a}_P$ and therefore

$$\deg_P(\mathfrak{c}_P : \mathfrak{a}_P) = \deg_P(\mathfrak{c}_P) - \deg_P(\mathfrak{a}_P). \tag{1.2}$$

Thus, from local duality and from the previous equality, it was proved the following reciprocity formula (cf. [25]):

Corollary 2 ***(Reciprocity formula)*** *For each divisor* $\mathfrak{a}$ *of* X*,*

$$\mathfrak{c} : (\mathfrak{c} : \mathfrak{a}) = \mathfrak{a} \tag{1.3}$$

In particular $\mathfrak{c} : (\mathfrak{c} : \mathcal{O}) = \mathcal{O}$, that is, $\mathfrak{c} : \mathfrak{c} = \mathcal{O}$. Observe that each divisor $\mathfrak{c}$ satisfying $l(\mathfrak{c}) \geq g$ and $\deg(\mathfrak{c}) = 2g-2$ is a canonical divisor. The divisor $\mathfrak{c}$ is uniquely determined up equivalence by Property 1.3, that is, a divisor $\mathfrak{d}$ satisfies $\mathfrak{d} : (\mathfrak{d} : \mathfrak{a}) = \mathfrak{a}$ for each divisor $\mathfrak{a}$ if and only if $\mathfrak{d} = \mathfrak{b} \cdot \mathfrak{c}$ for some locally principal divisor $\mathfrak{b}$ i.e. for each point P there is $z_P \in K\backslash\{0\}$ such that $\mathfrak{d}_P = z_P^{-1}\mathfrak{c}_P$ (cf. [19]).

In the set of $\mathcal{O}_P$-ideals it is defined the following equivalence relation: for each $\mathcal{O}_P$-ideals $\mathfrak{d}_P$ and $\mathfrak{b}_P$, $\mathfrak{d}_P \sim \mathfrak{b}_P$ if and only if $\mathfrak{d}_P = z_P^{-1}\mathfrak{b}_P$ for some $z_P \in K\backslash\{0\}$. For instance, $\mathfrak{d}_P$ is an $\widetilde{\mathcal{O}}_P$-ideal if and only if $\mathfrak{d}_P \sim \widetilde{\mathcal{O}}_P$. The ring $\mathfrak{b}_P : \mathfrak{b}_P$ and its group of units

$$U_{\mathfrak{b}_P} := \{u \in K\backslash\{0\} : \mathfrak{b}_P = u^{-1}\mathfrak{b}_P\},$$

which is a multiplicative subgroup of $K\backslash\{0\}$, only depend on the $\mathcal{O}_P$-class $[\mathfrak{b}_P]$.

By the reciprocity, the assignment

$$\mathfrak{a}_P \longmapsto \mathfrak{a}_P^* := \mathfrak{c}_P : \mathfrak{a}_P$$

defines an anti-monotonous permutation between the $\mathcal{O}_P$-ideals and $\mathfrak{a}_P^*$ is called *dual ideal* of $\mathfrak{a}_P$. It satisfies the following properties (cf. [27]):

Proposition 3 *Let* $\mathfrak{a}_P$ *and* $\mathfrak{b}_P$ *be* $\mathcal{O}_P$*-ideals. Then*

1. $\dim(\mathfrak{b}_P^*/\mathfrak{a}_P^*) = \dim(\mathfrak{a}_P/\mathfrak{b}_P)$
2. $\deg_P(\mathfrak{a}_P^*) - \deg(\mathfrak{b}_P^*) = \deg_P(\mathfrak{b}_P) - \deg_P(\mathfrak{a}_P)$
3. $\mathfrak{a}_P^* : \mathfrak{b}_P^* = \mathfrak{b}_P : \mathfrak{a}_P$
4. $(\mathfrak{b}_P \cap \mathfrak{a}_P)^* = \mathfrak{a}_P^* + \mathfrak{b}_P^*$ *and* $(\mathfrak{b}_P + \mathfrak{a}_P)^* = \mathfrak{a}_P^* \cap \mathfrak{b}_P^*$
5. $(\mathfrak{a}_P \cdot \mathfrak{b}_P^*)^* = \mathfrak{b}_P : \mathfrak{a}_P$ *and* $(\mathfrak{a}_P : \mathfrak{b}_P)^* = \mathfrak{b}_P^* \cdot \mathfrak{a}_P$.

In particular, if $\mathfrak{a}_P$ is an $\widetilde{\mathcal{O}}_P$-ideal say $\mathfrak{a}_P = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}$ then $\mathfrak{a}_P^* = \widetilde{\mathcal{O}}_P^* : \mathfrak{a}_P = \mathfrak{a}_P^{-1} \cdot \widetilde{\mathcal{O}}_P^*$ i.e. $\left(\mathfrak{p}_1^{n_1} \cdot \cdots \cdot \mathfrak{p}_m^{n_m}\right)^* = \left(\mathfrak{p}_1^{-n_1} \cdot \cdots \cdot \mathfrak{p}_m^{-n_m}\right) \cdot \widetilde{\mathcal{O}}_P^*$.

Proposition 4 *Let $\mathfrak{b}_P$ be an $\mathcal{O}_P$-ideal. Then*

1. $(\mathfrak{b}_P \cdot \widetilde{\mathcal{O}}_P)^* = \mathfrak{b}_P^* : \widetilde{\mathcal{O}}_P$ *and* $(\mathfrak{b}_P : \widetilde{\mathcal{O}}_P)^* = \mathfrak{b}_P^* \cdot \widetilde{\mathcal{O}}_P$

2. $\mathfrak{b}_P^* : \mathfrak{b}_P^* = \mathfrak{b}_P : \mathfrak{b}_P$, $U_{\mathfrak{b}_P^*} = U_{\mathfrak{b}_P}$ *and* $(\mathfrak{b}_P^* : \widetilde{\mathcal{O}}_P) : (\mathfrak{b}_P^* \cdot \widetilde{\mathcal{O}}_P) = (\mathfrak{b}_P : \widetilde{\mathcal{O}}_P) : (\mathfrak{b}_P \cdot \widetilde{\mathcal{O}}_P)$. *This means that the ring $\mathfrak{b}_P : \mathfrak{b}_P$, the group $U_{\mathfrak{b}_P}$ and the $\widetilde{\mathcal{O}}_P$-ideal $(\mathfrak{b}_P : \widetilde{\mathcal{O}}_P) : (\mathfrak{b}_P \cdot \widetilde{\mathcal{O}}_P)$, which only depend on the ideal class $[\mathfrak{b}_P]$, remain unchanged if $\mathfrak{b}_P$ is replaced by the dual ideal $\mathfrak{b}_P^*$.*

3. $\dim(\mathfrak{b}_P^* \cdot \widetilde{\mathcal{O}}_P/\mathfrak{b}_P^*) = \dim(\mathfrak{b}_P/\mathfrak{b}_P : \widetilde{\mathcal{O}}_P)$ *and* $\dim(\mathfrak{b}_P^*/\mathfrak{b}_P^* : \widetilde{\mathcal{O}}_P) = \dim(\mathfrak{b}_P \cdot \widetilde{\mathcal{O}}_P/\mathfrak{b}_P)$. *This means that the dimensions* $\dim(\mathfrak{b}_P \cdot \widetilde{\mathcal{O}}_P/\mathfrak{b}_P)$ *and* $\dim(\mathfrak{b}_P/\mathfrak{b}_P : \widetilde{\mathcal{O}}_P)$, *which only depend on the ideal class $[\mathfrak{b}_P]$, are interchanged if $\mathfrak{b}_P$ is replaced by the dual ideal $\mathfrak{b}_P^*$.*

By using the previous proposition, it follows that the sum and the product of $\dim((\mathfrak{b}_P \cdot \widetilde{\mathcal{O}}_P)/\mathfrak{b}_P)$ and $\dim(\mathfrak{b}_P/(\mathfrak{b}_P : \widetilde{\mathcal{O}}_P))$ remain unchanged if $\mathfrak{b}_P$ is replaced by $\mathfrak{b}_P^*$.

The $\mathcal{O}_P$-ideal $\mathfrak{f}_P := \mathcal{O}_P : \widetilde{\mathcal{O}}_P$ is called the *conductor ideal*. Observe that $\mathfrak{f}_P$ is an $\widetilde{\mathcal{O}}_P$-ideal too. The one-dimensional local ring $\mathcal{O}_P$ is called *Gorenstein ring* if $\dim(\mathcal{O}_P/\mathfrak{f}_P) = \delta_P$. An algebraic curve X is called a *Gorenstein curve* if all its local rings are Gorenstein rings. It is well known the following result due to Rosenlicht (cf. [22] and [25]). The curve X is a Gorenstein curve if and only if its canonical divisors are locally principal.

Let $Q_1, \cdots, Q_m \in \widetilde{X}$ be the branches centered at P and let $\mathcal{O}_{Q_1}, \cdots, \mathcal{O}_{Q_m}$ be the corresponding local rings at these points. Since the function field of X and $\widetilde{X}$ are the same and $\widetilde{X}$ is a non-singular curve, the local rings $\mathcal{O}_{Q_1}, \cdots, \mathcal{O}_{Q_m}$ are precisely the valuation rings of $K \mid k$ over $\mathcal{O}_P$. The integral closure of $\mathcal{O}_P$ is $\widetilde{\mathcal{O}}_P = \mathcal{O}_{Q_1} \cap \cdots \cap \mathcal{O}_{Q_m}$ and $\widetilde{\mathcal{O}}_P$ is a semi-local principal ideal domain whose maximal ideals, say $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$, correspond bijectively to the branches $Q_1, \cdots, Q_m$, that is, for each $i = 1, \cdots, m$

$$\mathfrak{p}_i = \{z \in \widetilde{\mathcal{O}}_P : v_i(z) \geq 1\}$$

where $v_i = ord_{Q_i}$ is the corresponding valuation of the function field $K \mid k$. Thus the divisors of the non-singular model $\widetilde{X}$ correspond bijectively to the divisors of X whose P-components are non-zero fractional ideals of $\widetilde{\mathcal{O}}_P$ too. The completion of the local ring $\mathcal{O}_P$ is denoted by $\widehat{\mathcal{O}}_P$ and the completion of the semilocal ring $\widetilde{\mathcal{O}}_P$ with respect to its Jacobson ideal is denoted by $\widehat{\widetilde{\mathcal{O}}}_P$. Since $\mathcal{O}_P$ has finite k-codimension in $\widetilde{\mathcal{O}}_P$, that is, $\delta_P < \infty$, by the Artin Rees Lemma the topology of $\mathcal{O}_P$ is induced by the topology of $\widetilde{\mathcal{O}}_P$, and so $\widehat{\mathcal{O}}_P$ is a closed subring of $\widehat{\widetilde{\mathcal{O}}}_P$ of k-codimension δ_P. By applying the Chinese remainder theorem to the residue ring $\widetilde{\mathcal{O}}_P$ and passing to the projective limit is obtained

$$\widehat{\widetilde{\mathcal{O}}}_P = \widehat{\mathcal{O}}_{Q_1} \times \cdots \times \widehat{\mathcal{O}}_{Q_m}$$

which is contained in the product $\widehat{K}_{Q_1} \times \cdots \times \widehat{K}_{Q_m}$.

We let $\widehat{\mathcal{O}}_{P\mathfrak{P}_j}$ $(j = 1, \cdots, m)$ denote the completion of the local ring $\widehat{\mathcal{O}}_P$ with respect to its minimal prime $\mathfrak{P}_j$ and we let $\psi : \widehat{\mathcal{O}}_P \longrightarrow \widehat{\mathcal{O}}_{P\mathfrak{P}_1} \times \cdots \times \widehat{\mathcal{O}}_{P\mathfrak{P}_m}$ denote the diagonal homomorphism. Since $\widehat{\mathcal{O}}_P$ is a reduced ring, ψ is injective. We have the following commutative diagram:

$$\begin{array}{ccc}
\widehat{\widetilde{\mathcal{O}}}_P & \longrightarrow & \widehat{\mathcal{O}}_{Q_1} \times \cdots \times \widehat{\mathcal{O}}_{Q_m} \\
\uparrow & & \uparrow \\
\widehat{\mathcal{O}}_P & \longrightarrow & \widehat{\mathcal{O}}_{P\mathfrak{P}_1} \times \cdots \times \widehat{\mathcal{O}}_{P\mathfrak{P}_m}
\end{array}$$

By the Cohen 's structure theorem for regular complete local rings each $\widehat{\mathcal{O}}_{Q_j}$ is isomorphic to $k_j[[t_j]]$, where $k_j = \widehat{\mathcal{O}}_{Q_j}/\widehat{\mathfrak{p}}_j$ is the residue field of $\widehat{\mathcal{O}}_{Q_j}$,that is, each $k_j = \widetilde{\mathcal{O}}_P/\mathfrak{p}_j$. Thus

$$\begin{aligned}
\widehat{\widetilde{\mathcal{O}}}_P \simeq & \widehat{\mathcal{O}}_{Q_1} \times \cdots \times \widehat{\mathcal{O}}_{Q_m} \\
\simeq & k_1[[t_1]] \times \cdots \times k_m[[t_m]].
\end{aligned}$$

This isomorphism is an important tool to determine the ideal classes of a local ring $\mathcal{O}$ and to compute the mentioned series.

1.2 Zeta functions

Let X be a complete, geometrically irreducible, algebraic curve defined over a finite field $k = \mathbb{F}_q$ of q elements and let K be the rational functions on X. The *zeta-function* of X is defined to be the Euler product

$$\zeta(X, s) := \prod_{P \in X} \frac{1}{1 - q^{-s \deg(P)}}$$

when the real part $\Re(s)$ of $s \in \mathbb{C}$ is larger than 1. The zeta-function $\zeta(X, s)$ is important in algebraic geometry because it satisfies the well known identity

$$\zeta(X, s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} N_n t^n\right)$$

where $N_n := \#X(\mathbb{F}_{q^n})$ is the number of rational points over the extension field $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ of degree n and $t := q^{-s}$ (see [23]). The zeta-function $\zeta(X,s)$, except for possible new zeros on the imaginary axis $\Re(s) = 0$, has the same zeros as the zeta-function $\zeta(\widetilde{X}, s)$ of the non-singular model of X. By the Riemann hypothesis for non-singular curves, $\zeta(\widetilde{X}, s)$ may be written

$$\zeta(\widetilde{X}, s) = \frac{L(\widetilde{X}, t)}{(1-t)(1-qt)}$$

where $L(\widetilde{X}, t)$ is an polynomial with integer coefficients in $t = q^{-s}$ of degree $2\widetilde{g}$ whose zeros are on the circle $|t| = q^{-1/2}$ (or equivalently on the line $\Re(s) = \frac{1}{2}$ in the s-plane) and it satisfies the global functional equation

$$L(\widetilde{X}, t) = q^{\widetilde{g}} t^{2\widetilde{g}} L(\widetilde{X}, 1/qt).$$

Thus

$$\zeta(X,s) = \frac{L(X,t)}{(1-t)(1-qt)},$$

where

$$L(X,t) := L(\widetilde{X}, t) \cdot \prod_{P \in X_{sing}} \frac{\prod_{Q|P}\left(1 - t^{\deg(Q)}\right)}{\left(1 - t^{\deg(P)}\right)},$$

P ranges over the singular points of X and the symbol "$Q \mid P$" indicate that Q ranges over the branches centered at P. Since $\deg(P)$ divide $\deg(Q)$ whenever Q is a branch centered at P, $L(X,t)$ is a polynomial in $t = q^{-s}$ with integer coefficients (see [25]). The zeta function $\zeta(X,s)$ is compared with the Dirichlet series

$$\zeta(\mathcal{O}_X, s) := \sum_{\mathfrak{a} \geq \mathcal{O}_X} q^{-s\deg(\mathfrak{a})}$$

where $\mathfrak{a}$ ranges over the positive divisors of X, and it satisfies the following functional equation: the function $q^{s(g-1)}\zeta(\mathcal{O}_X, s)$ is invariant when s is replaced by $1-s$.

The Dirichlet series $\zeta(\mathcal{O}_X, s)$ may be expanded as

$$\zeta(\mathcal{O}_X, s) = \prod_{P \in X} \zeta(\mathcal{O}_P, s). \tag{1.4}$$

Formula 1.4 establishes the link between the local and the global theory.

Since $\zeta(\mathcal{O}_P, s) = \frac{1}{1-q^{-s\deg(P)}}$ whenever P is a non-singular point of the curve X, it follows that the zeta-function $\zeta(X,s)$ coincide with the Dirichlet series $\zeta(\mathcal{O}_X, s)$ whenever X is a non-singular curve. In particular, the zeta function $\zeta(\widetilde{X}, s)$ and the Dirichlet series $\zeta(\mathcal{O}_{\widetilde{X}}, s)$ of the non-singular model of the curve X are the same.

Let $\mathcal{O}_P$ be a local ring of the curve X. Let us omit the subindex P, so that we write $\mathcal{O}$ instead of $\mathcal{O}_P$. For each $\mathcal{O}$-ideal $\mathfrak{a}$ the Dirichlet series $\zeta(\mathfrak{a}, s)$, with $\Re(s) > 0$, was introduced by Stöhr modifying the definitions of Galkin and Green in order to obtain a zeta function canonically associated to the local ring $\mathcal{O}$ which always satisfies the functional equations and which in the Gorenstein case coincides with Galkin 's zeta functions. By using the assignment $\mathfrak{d} \longmapsto \mathfrak{c} : \mathfrak{d}$, which defines a bijection between the set of $\mathcal{O}$-ideals that contain $\mathfrak{a}$ and the set of $\mathcal{O}$-ideals that are contained in $\mathfrak{a}$, it is obtained the connection between the local zeta function $\zeta(\mathfrak{a}, s)$ and Green's zeta function (cf. [14], [15] and [26]).

Let $\mathfrak{a}$ be an $\mathcal{O}$-ideal. For each $\mathcal{O}$-ideal $\mathfrak{d}$ containing $\mathfrak{a}$,

$$\#(\mathfrak{d}/\mathfrak{a}) = q^{\dim_k(\mathfrak{d}/\mathfrak{a})}.$$

Thus the series $\zeta(\mathfrak{a}, s)$ writes as follows as power series in $t = q^{-s}$ with integer coefficients

$$Z(\mathfrak{a}, t) := \sum_{\mathfrak{d} \supseteq \mathfrak{a}} t^{\dim_k(\mathfrak{d}/\mathfrak{a})}$$

where the sum is taken over the $\mathcal{O}$-ideals $\mathfrak{d}$ containing $\mathfrak{a}$. The series $Z(\mathfrak{a}, t)$ encodes the number of $\mathcal{O}$-ideals that admit $\mathfrak{a}$ as subspace of given codimension. Moreover the series $Z(\mathcal{O}, t)$ is a rational function

$$Z(\mathcal{O}, t) = \frac{L(\mathcal{O}, t)}{\prod_{i=1}^{m}(1 - t^{r_i})}$$

where $L(\mathcal{O}, t) \in \mathbb{Z}[t]$ is a polynomial with integer coefficient of degree 2δ in t, which satisfies the following *functional equation*

$$t^{-\delta} L(\mathcal{O}, t) = \left(\frac{1}{qt}\right)^{-\delta} L(\mathcal{O}, \frac{1}{qt}).$$

(cf. [26], Theorem 3.10.)

It is well known that the integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ in K is a principal ideal domain, then each $\mathcal{O}$-ideal $\mathfrak{d}$ is equivalent to some one $\mathcal{O}$-ideal $\mathfrak{b}$ satisfying $\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and, hence, $\mathcal{O} : \widetilde{\mathcal{O}} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$. This property permits to decompose the series $Z(\mathfrak{a}, t)$ in the following way:

$$Z(\mathfrak{a}, t) = \sum_{\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}} \left(\sum_{\mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b}} t^{\dim_k(\mathfrak{d}/\mathfrak{a})} \right)$$

where $\mathfrak{b}$ varies over the finitely many $\mathcal{O}$-ideals satisfying $\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and $\mathfrak{d}$ varies over $\mathcal{O}$-ideals that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$. On the other hand, for each pair of

$\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ the partial zeta-function $\zeta(\mathfrak{a}, \mathfrak{b}, s)$, with $\Re(s) > 0$, may be written as a power series in $t = q^{-s}$ with integer coefficients as follows:

$$Z(\mathfrak{a}, \mathfrak{b}, t) := \sum_{\mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b}} t^{\dim_k(\mathfrak{d}/\mathfrak{a})},\ |t| < 1$$

where the sum is taken over all $\mathcal{O}$-ideals that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$. Therefore,

$$Z(\mathfrak{a}, t) = \sum_{[\mathfrak{b}]} Z(\mathfrak{a}, \mathfrak{b}, t)$$

where $\mathfrak{b}$ ranges in (a complete system of representatives of) the ideal class semigroup of $\mathcal{O}$.

1.3 Poincaré series

In this section we present the main facts about the multi-variable Poincaré series $P(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$. We use as reference [27].

Let $\mathcal{O}$ be a local ring of a geometrically irreducible algebraic curve defined over a finite field $k = \mathbb{F}_q$ with rational function field K, and let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathcal{O}$-ideals. The maximal ideal, say $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$, of the integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ in K correspond bijectively to the valuations $v_1 = ord_{\mathfrak{p}_1}, \cdots, v_m = ord_{\mathfrak{p}_m}$ in the function field $K \mid k$. Each $\widetilde{\mathcal{O}}$-ideal is just of the form

$$\mathfrak{p}^{\mathbf{n}} := \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}, \text{ where } \mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m.$$

It is defined its *multi-exponent* by $\mathbf{v}(\mathfrak{p}^{\mathbf{n}}) := \mathbf{n}$. And for each non-zero rational function $z \in K \backslash \{0\}$ is abbreviated

$$\mathbf{v}(z) := \mathbf{v}(z\widetilde{\mathcal{O}}) = (v_1(z), \cdots, v_m(z)) \in \mathbb{Z}^m.$$

Definition 5 *The multi-variable Poincaré series associated to a pair of $\mathcal{O}$-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$ is defined to be the multi-variable power series*

$$P(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) := \sum \eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b})\, \mathbf{t}^{\mathbf{n}} \in \mathbb{Z}[[t_1, \ldots, t_m]],$$

whose coefficients are the cardinalities

$$\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b}) := \#\{\mathcal{O}\text{-ideals } \mathfrak{d} \text{ satisfying } \mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b} \text{ and } \mathfrak{d} \cdot \widetilde{\mathcal{O}} = \mathfrak{a} \cdot \mathfrak{p}^{-\mathbf{n}}\}$$

where $\mathbf{t}^{\mathbf{n}} := t_1^{n_1} \cdots t_m^{n_m}$ for each $\mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m$ (cf. [27] Definition 2.1).

It was proved that this cardinalities are really finite an that the convergence domain of this multi-variable series is the unit poly-disk (see [27] Theorem 3.2 (ii) and (iii)). This series only depends on the $\mathcal{O}$-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. Moreover, it may be expressed in the form

$$P(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \sum_{\mathfrak{d} \supset \mathfrak{a}, \mathfrak{d} \sim \mathfrak{b}} \mathbf{t}^{\mathbf{v}(\mathfrak{a} \cdot \widetilde{\mathcal{O}}) - \mathbf{v}(\mathfrak{d} \cdot \widetilde{\mathcal{O}})} \in \mathbb{Z}[[t_1, \ldots, t_m]]$$

where the sum is taken over all $\mathcal{O}$-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$.

Given that each $\widetilde{\mathcal{O}}$-ideal is equivalent to the $\mathcal{O}$-ideal $\widetilde{\mathcal{O}}$ and that the $\widetilde{\mathcal{O}}$-ideals containing $\mathfrak{a}$ are precisely of the form $\mathfrak{a} \cdot \mathfrak{p}^{-\mathbf{n}}$, where $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{N}^m$ and $\mathbb{N}$ stands for the additive semi-group of non-negative integers, it follows that, if $\mathfrak{b}$ is both $\mathcal{O}$-ideal and $\widetilde{\mathcal{O}}$-ideal, then

$$P(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \sum_{(n_1, \cdots, n_m) \in \mathbb{N}^m} t_1^{n_1} \cdots t_m^{n_m} = \frac{1}{(1 - t_1) \cdots (1 - t_m)} \text{ for any } \mathcal{O}\text{-ideal } \mathfrak{a}.$$

Theorem 6 *The following identity holds;*

$$Z(\mathfrak{a}, \mathfrak{b}, t) = t^{\dim_k(\mathfrak{a} \cdot \widetilde{\mathcal{O}}/\mathfrak{a}) - \dim_k(\mathfrak{b} \cdot \widetilde{\mathcal{O}}/\mathfrak{b})} P(\mathfrak{a}, \mathfrak{b}, t^{r_1}, \ldots, t^{r_m})$$

where $r_1 := \dim_k(\widetilde{\mathcal{O}}/\mathfrak{p}_1), \cdots, r_m := \dim_k(\widetilde{\mathcal{O}}/\mathfrak{p}_m)$ *are the degrees of the residue fields of* $\widetilde{\mathcal{O}}$ *over the constant field* k.

This means that the partial local-zeta function may be expressed in terms of the multi-variable Poincaré series. The previous theorem was proved by using the following property

Lemma 7 *For each non-zero rational function* $z \in K \backslash \{0\}$ *and for each* $\mathcal{O}$*-ideal* $\mathfrak{a}$,

$$\deg(\mathfrak{a}) - \deg(z\mathfrak{a}) = \mathbf{r} \cdot \mathbf{v}(z) = \sum_{i=1}^{m} r_i v_i(z).$$

In particular,

$$\deg(z\mathcal{O}) = -\mathbf{r} \cdot \mathbf{v}(z).$$

In this case, $\mathbf{r} \cdot \mathbf{n} := \sum_{i=1}^{m} r_i n_i$ *for each* $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$.

For each $\mathcal{O}$-ideal $\mathfrak{b}$ was defined the vector with integer coordenates

$$\mathbf{b} := \mathbf{v}((\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b} \cdot \widetilde{\mathcal{O}}),$$

the set

$$S(\mathfrak{b}) := \{\mathbf{v}(z) - \mathbf{v}(\mathfrak{b}\cdot\widetilde{\mathcal{O}}) : z \in \mathfrak{b}\backslash\{0\}\}$$

and

$$\mathfrak{b}_{\mathbf{n}} = \{z \in \mathfrak{b} : \mathbf{v}(z) = \mathbf{n}\}, \text{ for each integer vector } \mathbf{n} \in \mathbb{Z}^m.$$

Both the vector $\mathbf{b}$ and the set $S(\mathfrak{b})$ only depend on the ideal class $[\mathfrak{b}]$. Moreover, they satisfy the following properties:

$$\mathbf{b}+\mathbb{N}^m \subseteq S(\mathfrak{b}) \subseteq \mathbb{N}^m$$

and

$$S(\mathcal{O}) + S(\mathfrak{b}) \subseteq S(\mathfrak{b}).$$

In particular, $S(\mathcal{O})$ is a semigroup intermediate between $\mathbf{f}+\mathbb{N}^m$ and $\mathbb{N}^m$. It is called the *semi-group associated to the local ring* $\mathcal{O}$. Since

$$(\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b} = (\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b}\cdot\widetilde{\mathcal{O}} = (\mathfrak{b} : \widetilde{\mathcal{O}}) \cdot (\mathfrak{b}\cdot\widetilde{\mathcal{O}})^{-1}$$

and

$$\mathfrak{f} \subseteq (\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b}\cdot\widetilde{\mathcal{O}} \subseteq \widetilde{\mathcal{O}},$$

where $\mathfrak{f} := \mathcal{O} : \widetilde{\mathcal{O}}$ is the conductor ideal of $\mathcal{O}$ in its integral closure $\widetilde{\mathcal{O}}$ (see [27] Lemma 3.1), it follows that

$$\mathbf{b} := \mathbf{v}((\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b}\cdot\widetilde{\mathcal{O}}) = \mathbf{v}(\mathfrak{b} : \widetilde{\mathcal{O}}) - \mathbf{v}(\mathfrak{b}\cdot\widetilde{\mathcal{O}})$$

and

$$\mathbf{0} \leq \mathbf{b} \leq \mathbf{f} := \mathbf{v}(\mathfrak{f})$$

where "$\leq$" stands for the natural partial ordering of the Cartesian product $\mathbb{Z}^m$. Thus, if $\mathfrak{b}\cdot\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ then $(\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b} = \mathfrak{b} : \widetilde{\mathcal{O}}$, $\mathfrak{f} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$, $\mathbf{b} = \mathbf{v}(\mathfrak{b} : \widetilde{\mathcal{O}})$ and, hence, $\mathbf{b}$ is the smallest vector in the partial ordering of $\mathbb{Z}^m$ such that $\mathfrak{p}^{\mathbf{b}} \subseteq \mathfrak{b}$.

The vector $\mathbf{b}$ and the set $S(\mathfrak{b})$ are important because they provide important informations about the coefficients of the multi-variable Poincaré series (see [27]).

Theorem 8 *The coefficients of the Poincaré series* $P(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) := \sum \eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b})\, \mathbf{t}^{\mathbf{n}}$ *satisfies:*

1. $\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b}) = \frac{\#\left((\mathfrak{b}:\mathfrak{a})_{\mathbf{j}}/U_{\mathcal{O}}\right)}{[U_{\mathfrak{b}}:U_{\mathcal{O}}]}$ *where* $\mathbf{j} = \mathbf{n} - \mathbf{v}(\mathfrak{a}\cdot\widetilde{\mathcal{O}}) + \mathbf{v}(\mathfrak{b}\cdot\widetilde{\mathcal{O}})$ *for each* $\mathbf{n} \in \mathbb{Z}^m$

2. $\eta_{\mathbf{n}}(\mathfrak{a},\mathfrak{b}) > 0$ *if and only if* $\mathbf{n} \in S(\mathfrak{b})$

3. $0 \le \eta_{\mathbf{n}}(\mathfrak{a},\mathfrak{b}) \le [U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}}]$ *for each* $\mathbf{n} \in \mathbb{Z}^m$

4. $\mathbf{b}$ *is the smallest vector in the partial ordering of* $\mathbb{N}^m$ *with the following property: if* $\mathbf{n} \ge \mathbf{b}$ *then* $\eta_{\mathbf{n}}(\mathfrak{a},\mathfrak{b}) = [U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}}]$.

The multi-variable Poincaré series $P(\mathfrak{a},\mathfrak{b},\mathbf{t})$ can be represented by an integral within the framework of harmonic analysis as follows (see [27]):

Let $\mathcal{R} := \prod_{i=1}^{m} \widehat{K}_{v_i}$ be the locally compact total ring of fractions of the completion $\widehat{\mathcal{O}}$ of the local ring $\mathcal{O}$, and let $U_{\mathcal{R}} := \prod_{i=1}^{m} \widehat{K}^*_{v_i}$ be its group of units. The homomorphism $\mathbf{v}: K^* \longrightarrow \mathbb{Z}^m$ extends naturally to the group homomorphism $\mathbf{v}: U_{\mathcal{R}} \longrightarrow \mathbb{Z}^m$ that maps each unity $u := (u_1, \cdots, u_m)$ in $U_{\mathcal{R}}$ to the integer vector $\mathbf{v}(u) = (\widehat{v}_1(u_1), \cdots, \widehat{v}_m(u_m))$ in $\mathbb{Z}^m$. Moreover, there exists a Haar measure $\widehat{\mu}$ on the additive group of the locally compact $\mathbb{F}_q$-algebra $\mathcal{R}$, normalized so that $\widehat{\mu}_S(\widehat{\mathcal{O}}) = 1$. Thus,

Theorem 9 $P(\mathfrak{a},\mathfrak{b},\mathbf{t}) = \frac{q^{\varrho}}{[U_{\mathfrak{b}}:U_S](q^{\varrho}-1)} \int_{(\widehat{\mathfrak{b}}:\widehat{\mathfrak{a}})\cap U_{\mathcal{R}}} q^{\mathbf{r}\cdot\mathbf{v}(z)} \mathbf{t}^{\mathbf{v}(z)} d\widehat{\mu}(z)$ *in the unit poly-disk* $|t_1| < 1, \cdots, |t_m| < 1$, *where* $\varrho := \dim_k(\mathcal{O}/\mathfrak{m})$ *is the degree of the residue field of* $\mathcal{O}$ *over the constant field* k.

It was also proved in [27] that:

Theorem 10 $P(\mathfrak{a},\mathfrak{b},\mathbf{t})$ *is a rational function*

$$P(\mathfrak{a},\mathfrak{b},t_1,\cdots,t_m) = \frac{\Lambda(\mathfrak{a},\mathfrak{b},t_1,\cdots,t_m)}{(1-t_1)\cdots(1-t_m)},$$

where $\Lambda(\mathfrak{a},\mathfrak{b};t_1,\ldots,t_m) \in \mathbb{Z}[t_1,\cdots,t_m]$ *is a polynomial of multi-degree* $\le \mathbf{b}$, *where* $\mathbf{b} = (b_1,\cdots,b_m)$ *is the multi-exponent of the fractional ideal* $(\mathfrak{b} : \widetilde{\mathcal{O}}) : \mathfrak{b}\widetilde{\mathcal{O}}$ *in the integral closure* $\widetilde{\mathcal{O}}$ *of* $\mathcal{O}$ *in* K, *which satisfies a* ***functional equations***

$$\Lambda(\mathfrak{a},\mathfrak{b},t_1,\cdots,t_m) = [U_{\mathfrak{b}:\mathfrak{a}} : U_{\mathfrak{b}}] q^{\dim(\mathfrak{b}:\mathfrak{a}/(\mathfrak{b}:\mathfrak{a}):\widetilde{\mathcal{O}})} t_1^{b_1} \cdots t_m^{b_m} \Lambda(\mathcal{O}, \mathfrak{a}\cdot\mathfrak{b}^*, \frac{1}{q^{r_1}t_1}, \cdots, \frac{1}{q^{r_m}t_m}),$$

where $\mathfrak{b}^*$ *is the dual* $\mathcal{O}$*-ideal of* $\mathfrak{b}$.

Furthermore, the multi-variable Poincaré series can be expressed in the following form:

Theorem 11

$$P(\mathcal{O},\mathfrak{b},\mathbf{t}) = \frac{q^{\delta-\deg(\mathfrak{b}\widetilde{\mathcal{O}})}}{[U_{\mathfrak{b}}:U_{\mathcal{O}}]\,(1-q^{-\varrho})}\frac{\prod_{i=1}^{m}(q^{r_i}t_i-1)}{q^{|\mathbf{r}|}t_1\cdots t_m-1}\sum_{\mathbf{n}\in\mathbb{Z}^m} q^{\mathbf{r}\cdot\mathbf{n}}(q^{\deg(\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}})}-q^{\deg(\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{1}})})\mathbf{t}^{\mathbf{n}}.$$

The previous identity is useful because it permit us to view the multi-variable Poincaré series as an analogue of a formula for Poincaré series of germs of complex curves, which was introduced by Campillo, Delgado and Gusein-Zade in [7].

Chapter 2

Semilocal rings and fractional ideals

The intersection of local rings of points on a singular curve is a semilocal ring; in this chapter we study this class of rings. After giving some results on semilocal rings in section 1, our main interest in section 2 is the decomposition of fractional ideals of a semilocal ring as intersection of fractional ideals of its several components. We use this decomposition for getting some results, which shall play a decisive role in the next chapter to extend to semilocal rings of a singular curve the definitions of zeta functions and multi-variable Poincaré series associated to local rings of a singular curve.

2.1 Semilocal rings

In this section we present the basic facts about semilocal subrings of a function field, and we use as main reference the Rosenlicht's paper [22].

A Noetherian ring is said to be a *semilocal ring* if it contains precisely only a finite number of maximal ideals. Let $K \mid k$ be a function field of one variable with constant field k and let R be a subring of K. The ring R is called *a subring of* $K \mid k$ if R contains the field k and the quotient field of R is K. We say that $\mathfrak{a}$ is an *R-ideal of* $K \mid k$ if $\mathfrak{a}$ is a non-zero fractional ideal of R.

The next theorem gives a characterization of semilocal subrings of a function field of one variable. It contains some equivalent properties which were proved in [22]. We add properties 4 and 5, which will be useful for us in this work.

Theorem 12 *Let $K \mid k$ be a function field in one variable with constant field k and* *let S be a subring of $K \mid k$. The following properties are equivalent:*

1. *S is a semilocal ring.*

2. *There exist $x, y_1, \cdots, y_r$ in S such that x is not constant and S is the set of of elements of the form $F(x, y_1, \cdots, y_r) / G(x)$ where F and G are polynomials in $k[T_0, T_1, \cdots, T_r]$ and $k[T_0]$, respectively, with coefficients in k and such that $G(0) \neq 0$.*

3. *There exist valuations $v_1, \cdots, v_m$ of the function field $K \mid k$ and an integer N such that S contains $\{z \in K : v_i(z) \geq N, 1 \leq i \leq m\}$.*

4. *There exist valuations $v_1, \cdots, v_m$ of the function field $K \mid k$ and integers $n_1, \cdots, n_m$ such that S contains $\{z \in K : v_i(z) \geq n_i, 1 \leq i \leq m\}$.*

5. *There exist valuations $v_1, \cdots, v_m$ of the function field $K \mid k$ and integers $f_1, \cdots, f_m$ such that*
$$(S : \widetilde{S}) = \{z \in K : v_i(z) \geq f_i, 1 \leq i \leq m\},$$
where $(S : \widetilde{S})$ is the conductor ideal of S in its integral closure $\widetilde{S}$.

6. *S is contained only in a finite number of valuation rings of the function field $K \mid k$.*

Proof. For the proof of $(1) \Longrightarrow (2)$, $(2) \Longrightarrow (3)$, and $(6) \Longrightarrow (1)$ see [22] Theorem 2.

$(3) \Longrightarrow (4)$ Let $n_i = N$ for each $i = 1, \cdots, m$. Then, by (3), the ring S contains the set $\{z \in K : v_i(z) \geq n_i, 1 \leq i \leq m\}$.

$(4) \Longrightarrow (5)$ Observe that, if v is a valuation of $K \mid k$ distinct from $v_1, \cdots, v_m$ then, by the approximation theorem, there exists $z \in K$ such that $v(z) < 0$ and $v_i(z) \geq n_i$ for each $i = 1, \cdots, m$, and hence $z \notin \mathcal{O}_v$ and $z \in S$. Thus any valuation of $K \mid k$ whose valuation ring contains the ring S is contained in $v_1, \cdots, v_m$. On the other hand, if the ring S is not contained in the valuation ring $\mathcal{O}_{v_m}$ then there exists $z_1 \in S$ with $v_m(z_1) < 0$. By the approximation theorem there exists $z_2 \in K$ such that $v_m(z_2) = n_m$ and $v_i(z_2)$ is very large for each $i = 1, \cdots, m-1$, so that $z_2 \in S$ and $v_i(z_1^{n_m} z_2) > 0$ for each $i = 1, \cdots, m-1$. Setting $y := z_1^{n_m} z_2 + 1$ we have $y \in S$, $v_i(y) = 0$ for each $i = 1, \cdots, m-1$ and $v_m(y) \leq -n_m$. Then, if $z \in K$ and $v_i(z) \geq n_i$ for each $i = 1, \cdots, m-1$, we have $zy^{-n} \in S$ for every sufficiently large integer n such that $v_m(zy^{-n}) \geq n_m$. Hence $z \in S$. Thus, if the valuation ring $\mathcal{O}_{v_m}$ does not contain the ring S, then S contains the set $\{z \in K : v_i(z) \geq n_i, 1 \leq i \leq m-1\}$. Therefore we can assume that the valuations $v_1, \cdots, v_m$ are precisely those whose valuation rings contain the ring S. Hence $\widetilde{S} = \cap_{i=1}^m \mathcal{O}_{v_i}$. Now, we set $(f_1, \cdots, f_m)$ to be the smallest vector in the partial ordering of $\mathbb{Z}^m$ such that S contains the set $\{z \in K : v_i(z) \geq f_i, 1 \leq i \leq m\}$. Since $z\widetilde{S} = \{y \in K : v_i(y) \geq v_i(z), 1 \leq i \leq m\}$ for any $z \in K \backslash \{0\}$, we conclude that $z \in (S : \widetilde{S})$ i.e. $z\widetilde{S} \subseteq S$ if and only if $v_i(z) \geq f_i, 1 \leq i \leq m$, proving (5).

$(5) \implies (6)$ If v is a valuation of $K \mid k$ distinct from $v_1, \cdots, v_m$ then, by the approximation theorem, there exists $z \in K$ such that $v(z) < 0$ and $v_i(z) \geq f_i$ for each $i = 1, \cdots, m$, that is, $z \in (S : \widetilde{S}) \subseteq S$ and $z \notin \mathcal{O}_v$. Thus, the ring S is only contained in the valuation rings corresponding to the valuations $v_1, \cdots, v_m$. ∎

Let S be a semilocal subring of a function field $K \mid k$. From (5), in Theorem 12, it follows that there exist valuations $v_1, \cdots, v_m$ of the function field $K \mid k$ and integers $f_1, \cdots, f_m$ such that

$$(S : \widetilde{S}) = \{z \in K : v_i(z) \geq f_i, 1 \leq i \leq m\}.$$

The valuations $v_1, \cdots, v_m$ are precisely those valuations of $K \mid k$ whose valuation rings contain S.

The next known corollary is consequence of the previous theorem:

Corollary 13 *Let S_1 and S_2 be subrings of the function field $K \mid k$.*

1. *If $S_1 \subseteq S_2$ and S_1 is semilocal, then S_2 is semilocal.*

2. *If S_1 and S_2 are semilocal then $S_1 \cap S_2$ is semilocal.*

Proof. Each valuation ring of $K \mid k$ containing S_2 contains the semilocal ring S_1, which is contained in only a finite number of valuation rings of $K \mid k$, and hence S_2 is semilocal.

The second sentence will follow from the fact that the ring $S_1 \cap S_2$ contains the intersection $(S_1 : \widetilde{S_1}) \cap (S_2 : \widetilde{S_2})$. Since the semilocal rings S_1 and S_2 satisfy Property (5) of Theorem 12, the ring $S_1 \cap S_2$ also satisfies Property (4) of that theorem. On the other hand, by the approximation theorem, we can choose a non zero rational function y in $(S_1 : \widetilde{S_1}) \cap (S_2 : \widetilde{S_2})$ and, hence, y is contained in $S_1 \cap S_2$. Then every rational function z in K may be expressed as the quotient $z = \frac{zy^n}{y^n}$, for every sufficiently large integer n, so that $zy^{n-1} \in \widetilde{S_1} \cap \widetilde{S_2}$ and, hence, $zy^n = (zy^{n-1})\, y \in S_1 \cap S_2$. Thus K is the quotient field of $S_1 \cap S_2$. Therefore we have proved that $S_1 \cap S_2$ is a semilocal subring of $K \mid k$ and the valuation rings of the function field $K \mid k$ containing $S_1 \cap S_2$ are precisely those containing S_1 or S_2. ■

Note that, by the first part of the preceding corollary, S_1 and S_2 are semilocal subrings of $K \mid k$ if and only if $S_1 \cap S_2$ is a semilocal subring of $K \mid k$.

Theorem 14 *Let S be a semilocal subring of the function field $K \mid k$ with $S \neq K$. Then S may be expressed, in one and only one way, as the intersection of a finite number of local subrings of $K \mid k$ such that there do not exist two of them which are contained in the same valuation ring of $K \mid k$.*

This theorem was proved in [22] using the following lemma (see [22] Theorem 3)

Lemma 15 *Let S_1 and S_2 be semilocal subrings of the function field $K \mid k$ such that S_1 and S_2 are not contained in the same valuation ring and $S_1 \cap S_2$ is contained in the local ring $\mathcal{O}$. Then $S_1 \subseteq \mathcal{O}$ or $S_2 \subseteq \mathcal{O}$.*

The preceding features have a geometric interpretation (see [22] Theorem 5). Indeed, let S be a semilocal subring of the function field $K \mid k$. By Theorem 12, there exist $x, y_1, \cdots, y_r$ in S such that x is not constant and S is the totality of elements of the form $F(x, y_1, \cdots, y_r)/G(x)$, where F and G are polynomials in $k[T_0, T_1, \cdots, T_r]$ with coefficients in k and $G(0) \neq 0$. Thus, the curve X whose non-homogeneus general point over k is $(x, y_1, \cdots, y_r)$ will have K as function field. The semilocal ring S is then the localization of the ring $k[x, y_1, \cdots, y_r]$ respect to the multiplicative closed subset $U = \{G(x) \in k[x, y_1, \cdots, y_r] : G(T_0) \in k[T_0],\ G(0) \neq 0\}$. Then, maximal ideals of the semilocal ring S contract in $k[x, y_1, \cdots, y_r]$ to maximal ideals that do not intersect the closed subset U. Indeed, let $P_1, \cdots, P_s$ be all the points of X which are at finite distance and for which $x = 0$. Then $P_1, \cdots, P_s$ are algebraic over k and form a set that is closed under conjugation over k. Let $\mathcal{O}_{X,P_i}$ be the local ring of P_i $(i = 1, \cdots, s)$. Since S is the totality of elements of the form $F(x, y_1, \cdots, y_r)/G(x)$ where F and G are polynomials in $k[T_0, T_1, \cdots, T_r]$ with coefficients in k and $G(0) \neq 0$, we have $S \subseteq \mathcal{O}_{X,P_1} \cap \cdots \cap \mathcal{O}_{X,P_s}$. But, every maximal ideal in S must be a prime ideal of the ideal Sx, generated by x, so that any quotient ring of S with respect to a maximal ideal of S is some $\mathcal{O}_{X,P_i}$, for some $i = 1, \cdots, s$. Thus $S = S_{\mathfrak{m}_1} \cap \cdots \cap S_{\mathfrak{m}_s} = \mathcal{O}_{X,P_1} \cap \cdots \cap \mathcal{O}_{X,P_s}$, where $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$ are the maximal ideals of S.

Therefore, if $\mathcal{O}_1, \cdots, \mathcal{O}_s$ are local subrings of the function field $K \mid k$, no two of which are contained in the same valuation ring of $K \mid k$, then the ring $S =: \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$

is a semilocal subring of $K \mid k$ and $\mathcal{O}_1, \cdots, \mathcal{O}_s$ are local rings of some curve X over k whose rational function field is K. Conversely, given a geometrically integral algebraic curve X defined over a field k with rational function K, we may define a semilocal subring of its function field $K \mid k$ to be the intersection S of finite local rings of the curve X. In particular, the intersection S of the local rings of the singular points of the curve X is a semilocal subring of $K \mid k$.

2.2 Decomposition of S-ideals

Given a proper semilocal subring of a function field of one variable, it may be expressed as intersection of a finite number of local rings, no two of which are contained in the same valuation ring. In this section we prove that each fractional ideal of that semilocal ring may be expressed as intersection of fractional ideals of the several components of the semilocal ring. We also prove that its degree may be expressed as the sum of the degrees of the several fractional ideal components. Furthermore, we study the connection between some objects associated to non-zero fractional ideals of the semilocal ring and the corresponding objects of its decomposition.

Let S be a semilocal subring of a function field $K \mid k$, with $S \neq K$, whose expression as intersection of local rings given by Theorem 14 is $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$. Given that S is a semilocal ring there exists only a finite number of valuations of $K \mid k$. Let $v_1, \cdots, v_m$ be all the valuations of $K \mid k$ whose corresponding valuation rings contain S. Then the integral closure of S is $\widetilde{S} = \mathcal{O}_{v_1} \cap \cdots \cap \mathcal{O}_{v_m}$. Hence $\widetilde{S}$ is a semilocal principal ideal domain and, from Theorem 14, it follows that the maximal ideals of it, say $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$, correspond bijectively to the valuations $v_1, \cdots, v_m$ and $\widetilde{S}_{\mathfrak{p}_i} = \mathcal{O}_{v_i}$ for $i = 1, \cdots, m$. Thus $\widetilde{S}$ is a Dedekind domain and each $\widetilde{S}$-ideal of $K \mid k$ is just of the form

$$\mathfrak{p}^{\mathbf{n}} := \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m} \text{ where } \mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m.$$

As in [27], we defined its *multi-exponent* by $\mathbf{v}(\mathfrak{p}^{\mathbf{n}}) := \mathbf{n}$. We abbreviate

$$\mathbf{v}(z) := \mathbf{v}(z\widetilde{S}) = (v_1(z), \cdots, v_m(z)) \in \mathbb{Z}^m$$

for each non-zero rational function $z \in K^*$ and we denote by $r_j := \dim_k(\widetilde{S}/\mathfrak{p}_j)$ the degree of the residue field of $\mathfrak{p}_j$ over the constant field k for $j = 1, \cdots, m$. Thus, by the Chinese remainder theorem,

$$\dim_k(\widetilde{S}/\mathfrak{p}^{\mathbf{n}}) = \mathbf{r} \cdot \mathbf{n} := \sum_{j=1}^{m} r_j n_j, \text{ for each } \mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{N}^m.$$

The *degree* $\deg_S(\mathfrak{a})$ of an S-ideal $\mathfrak{a}$ is defined by the properties:

i. $\deg_S(S) := 0$

ii. $\dim_k(\mathfrak{a}/\mathfrak{b}) = \deg_S(\mathfrak{a}) - \deg_S(\mathfrak{b})$ whenever $\mathfrak{a} \supseteq \mathfrak{b}$.

Let

$$\delta_S := \deg_S(\widetilde{S}) = \dim_k(\widetilde{S}/S)$$

be the *singularity degree* of the semilocal ring S. Then

$$\deg_S(\mathfrak{p}^{\mathbf{n}}) = \delta_S - \mathbf{r} \cdot \mathbf{n}, \text{ for each integer vector } \mathbf{n} \in \mathbb{Z}^m.$$

From (5) in Theorem 12, it follows that there exist integers $f_1, \cdots, f_m$ such that the conductor ideal $\mathfrak{f} := (S : \widetilde{S})$ is

$$\mathfrak{f} = \{z \in K : v_i(z) \geq f_i, 1 \leq i \leq m\} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_m^{f_m} = \mathfrak{p}^{\mathbf{f}}$$

where $\mathbf{f} := (f_1, \cdots, f_m) \in \mathbb{Z}^m$.

We observe that the multi-exponent and the degree of an $\widetilde{S}$-ideal $\mathfrak{p}^{\mathbf{n}}$ may be expressed in terms of those of the integral closure of its local rings in the decomposition given by Theorem 14. Indeed, if $v_{i1}, \cdots, v_{im_i}$ are all the valuations of $K \mid k$ whose valuation rings contain $\mathcal{O}_i$ for $i = 1, \cdots, s$, then the valuations v_{ij}, $1 \leq i \leq s$, $1 \leq j \leq m_i$, are precisely the valuations of $K \mid k$ whose valuation rings contain S. So the integral closure of S is $\widetilde{S} = \cap \mathcal{O}_{v_{ij}} = \widetilde{\mathcal{O}_1} \cap \cdots \cap \widetilde{\mathcal{O}_s}$. Its maximal ideals , say $\mathfrak{p}_{ij}$ $1 \leq i \leq s$, $1 \leq j \leq m_i$, correspond bijectively to the valuations v_{ij} $1 \leq i \leq s$, $1 \leq j \leq m_i$ and $\widetilde{S}_{\mathfrak{p}_{ij}} = \mathcal{O}_{v_{ij}}$ $1 \leq i \leq s$, $1 \leq j \leq m_i$. Thus each $\widetilde{S}$-ideal of $K \mid k$ may be expressed in terms of those of the several components as

$$\mathfrak{p}^{\mathbf{n}} := \mathfrak{P}_1^{\mathbf{n}_1} \cdots \mathfrak{P}_s^{\mathbf{n}_s}$$

where $\mathfrak{P}_i^{\mathbf{n}_i} := \mathfrak{p}_{i1}^{n_{i1}} \cdots \mathfrak{p}_{im}^{n_{im_i}}$, $\mathbf{n}_i := (n_{i1}, \cdots, n_{im_i}) \in \mathbb{Z}^{m_i}$, $1 \leq i \leq s$, and $\mathbf{n} := (\mathbf{n}_1, \cdots, \mathbf{n}_s) \in \mathbb{Z}^{m_1} \times \cdots \times \mathbb{Z}^{m_s}$. Its multi-exponent may be expressed by

$$\mathbf{v}(\mathfrak{p}^{\mathbf{n}}) = (\mathbf{v}_1(\mathfrak{p}_1^{\mathbf{n}_1}), \cdots, \mathbf{v}_s(\mathfrak{p}_s^{\mathbf{n}_s}))$$

where $\mathfrak{p}_i^{\mathbf{n}_i} := \mathfrak{P}_i^{\mathbf{n}_i}\widetilde{\mathcal{O}_i}$, $\mathbf{v}_i(\mathfrak{p}_i^{\mathbf{n}_i}) := \mathbf{n}_i$ $1 \leq i \leq s$, and for each non-zero rational function $z \in K^*$,

$$\mathbf{v}(z) := \mathbf{v}(z\widetilde{S}) = (\mathbf{v}_1(z), \cdots, \mathbf{v}_s(z)) \in \mathbb{Z}^{m_1} \times \cdots \times \mathbb{Z}^{m_s}$$

with $\mathbf{v}_i(z) = (v_{i1}(z), \cdots, v_{im_i}(z)) \in \mathbb{Z}^{m_i}$ for $i = 1, \cdots, s$. Therefore,

$$\dim_k(\widetilde{S}/\mathfrak{p}^{\mathbf{n}}) = \sum_{i=1}^{s} \dim_k(\widetilde{\mathcal{O}_i}/\mathfrak{p}_i^{\mathbf{n}_i}) = \sum_{i=1}^{s} \mathbf{r}_i \cdot \mathbf{n}_i = \sum_{i=1}^{s} \sum_{j=1}^{m_i} r_{ij} n_{ij}$$

where $r_{ij} := \dim_k(\widetilde{S}/\mathfrak{p}_{ij}) = \dim_k(\widetilde{\mathcal{O}_i}/\mathfrak{p}_{ij}\widetilde{\mathcal{O}_i})$, $1 \leq i \leq s$, $1 \leq j \leq m_i$.

The conductor ideal of S in $\widetilde{S}$ may be expressed in terms of those of its components as $\mathfrak{f} = \mathfrak{P}_1^{\mathbf{f}_1} \cdots \mathfrak{P}_s^{\mathbf{f}_s}$, where $\mathfrak{P}_i^{\mathbf{f}_i}\widetilde{\mathcal{O}_i} = \mathfrak{p}_{i1}^{f_{i1}} \cdots \mathfrak{p}_{im_i}^{f_{im_i}}\widetilde{\mathcal{O}_i} = (\mathcal{O}_i : \widetilde{\mathcal{O}_i})$ is the conductor ideal of $\mathcal{O}_i$ in $\widetilde{\mathcal{O}_i}$ and $\mathbf{f}_i := (f_{i1}, \cdots, f_{im_i}) \in \mathbb{Z}^{m_i}$ for each $i = 1, \cdots, s$.

Lemma 16 *Let S_1 and S_2 be semilocal subrings of the function field $K \mid k$ such that S_1 and S_2 are not contained in the same valuation ring. Then*

$$\dim_k\left(\frac{\mathfrak{a}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{b}_2}\right) = \dim_k\left(\frac{\mathfrak{a}_1}{\mathfrak{b}_1}\right) + \dim_k\left(\frac{\mathfrak{a}_2}{\mathfrak{b}_2}\right),$$

where $\mathfrak{a}_i$ and $\mathfrak{b}_i$ are S_i-ideals such that $\mathfrak{b}_i \subseteq \mathfrak{a}_i$ for $i = 1, 2$.

Proof. We have

$$\dim_k\left(\frac{\mathfrak{a}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{b}_2}\right) = \dim_k\left(\frac{\mathfrak{b}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{b}_2}\right) + \dim_k\left(\frac{\mathfrak{a}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{a}_2}\right).$$

So we must prove that $\dim_k\left(\frac{\mathfrak{a}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{a}_2}\right) = \dim_k\left(\frac{\mathfrak{a}_1}{\mathfrak{b}_1}\right)$ and $\dim_k\left(\frac{\mathfrak{b}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{b}_2}\right) = \dim_k\left(\frac{\mathfrak{a}_2}{\mathfrak{b}_2}\right)$. In fact, the k-linear application

$$\frac{\mathfrak{a}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{a}_2} \longrightarrow \frac{\mathfrak{a}_1}{\mathfrak{b}_1},\ x + \mathfrak{b}_1 \cap \mathfrak{a}_2 \mapsto x + \mathfrak{b}_1$$

is one to one. To prove that this application is surjective, we consider both the S_1-ideal $\mathfrak{b}_1 : \widetilde{S}_1$, which is the largest $\widetilde{S}_1$-ideal contained in $\mathfrak{b}_1$, and the S_2-ideal $\mathfrak{a}_2 : \widetilde{S}_2$, which is the largest $\widetilde{S}_2$-ideal contained in $\mathfrak{a}_2$. Thus, if $y \in \mathfrak{a}_1$ then, by the approximation theorem, there exists $x \in K$ such that $\mathbf{v}_1(x - y) \geq \mathbf{v}_1(\mathfrak{b}_1 : \widetilde{S}_1)$ and $\mathbf{v}_2(x) \geq \mathbf{v}_2(\mathfrak{a}_2 : \widetilde{S}_2)$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the multi-exponents of the semilocal rings S_1 and S_2, respectively. Thus $x - y \in (\mathfrak{b}_1 : \widetilde{S}_1) \subseteq \mathfrak{b}_1 \subseteq \mathfrak{a}_1$ and $x \in (\mathfrak{a}_2 : \widetilde{S}_2) \subseteq \mathfrak{a}_2$, hence, $x = y + (x - y) \in \mathfrak{a}_1 \cap \mathfrak{a}_2$ and $x - y \in \mathfrak{b}_1$. Therefore, $\dim_k\left(\frac{\mathfrak{a}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{a}_2}\right) = \dim_k\left(\frac{\mathfrak{a}_1}{\mathfrak{b}_1}\right)$. In a similar way we obtain the second identity $\dim_k\left(\frac{\mathfrak{b}_1 \cap \mathfrak{a}_2}{\mathfrak{b}_1 \cap \mathfrak{b}_2}\right) = \dim_k\left(\frac{\mathfrak{a}_2}{\mathfrak{b}_2}\right)$. ■

Proposition 17 *Let S be a semilocal subring of the function field $K \mid k$, with $S \neq K$, whose expression as intersection of local rings is $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$. Then, for each S-ideal $\mathfrak{a}$*

$$\mathfrak{a} = \mathfrak{a}\mathcal{O}_1 \cap \cdots \cap \mathfrak{a}\mathcal{O}_s$$

Moreover,

1. $\dim_k(\mathfrak{a}/\mathfrak{b}) = \sum_{i=1}^{s} \dim_k(\mathfrak{a}\mathcal{O}_i/\mathfrak{b}\mathcal{O}_i)$ *whenever* $\mathfrak{b}$ *is an* S*-ideal with* $\mathfrak{b} \subseteq \mathfrak{a}$

2. $\dim_k(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{p}^{\mathbf{n}}) = \sum_{i=1}^{s} \dim_k(\mathfrak{a}\mathcal{O}_i/\mathfrak{a}\mathcal{O}_i \cap \mathfrak{p}_i^{\mathbf{n}_i})$ *for each* $\mathbf{n} := (\mathbf{n}_1, \cdots, \mathbf{n}_s) \in \mathbb{Z}^{m_1} \times \cdots \times \mathbb{Z}^{m_s}$.

Proof. Let $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$ be the maximal ideals of the semilocal ring S. Since $\mathfrak{a}$ is contained in $\mathfrak{a}\mathcal{O}_i$ for each $i = 1, \cdots, s$ it follows that $\mathfrak{a} \subseteq \mathfrak{a}\mathcal{O}_1 \cap \cdots \cap \mathfrak{a}\mathcal{O}_s$.

Conversely, if $x \in \mathfrak{a}\mathcal{O}_1 \cap \cdots \cap \mathfrak{a}\mathcal{O}_s$ then x may be expressed as the quotient $x = \frac{a_i}{b_i}$, where $a_i \in \mathfrak{a}$ and $b_i \in S - \mathfrak{m}_i$, for each $i = 1, \cdots, s$ and $x = \frac{a}{b}$, where a, b are elements of S and $b \neq 0$. Given that $\mathfrak{a}$ is an S-ideal, there exists a non zero $y \in S$ such that $y\mathfrak{a}$ is an ideal (integral) of S. So $by\mathfrak{a}$ is an ideal of S and it has the primary decomposition $by\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_d$, where each $\mathfrak{q}_j$ is a $\mathfrak{m}_j$-primary ideal of S. Then $yab_j = ya_jb \in \mathfrak{q}_j$ for $j = 1, \cdots, d$, and hence $ya \in \mathfrak{q}_j$ for $j = 1, \cdots, d$ i.e. $ya \in by\mathfrak{a}$ i.e. $a \in b\mathfrak{a}$ i.e $x \in \mathfrak{a}$.

From item (1) and from Lemma 16 we have

$$\begin{aligned} \dim_k(\mathfrak{a}/\mathfrak{b}) &= \dim_k\left(\bigcap_{i=1}^{s} \mathfrak{a}\mathcal{O}_i / \bigcap_{i=1}^{s} \mathfrak{b}\mathcal{O}_\mathrm{i}\right) \\ &= \sum_{i=1}^{s} \dim_k(\mathfrak{a}\mathcal{O}_i/\mathfrak{b}\mathcal{O}_\mathrm{i}). \end{aligned}$$

This proves the second statement.

From the item (1) it follows that

$$\begin{aligned} \mathfrak{a} \cap \mathfrak{p}^{\mathbf{n}} &= (\mathfrak{a}\mathcal{O}_1 \cap \mathfrak{p}^{\mathbf{n}}) \cap \cdots \cap (\mathfrak{a}\mathcal{O}_s \cap \mathfrak{p}^{\mathbf{n}}) \\ &= (\mathfrak{a}\mathcal{O}_1 \cap \mathfrak{p}_1^{\mathbf{n}_1}) \cap \cdots \cap (\mathfrak{a}\mathcal{O}_s \cap \mathfrak{p}_s^{\mathbf{n}_s}), \end{aligned}$$

and, hence, that

$$\begin{aligned} \dim_k(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{p}^{\mathbf{n}}) &= \dim_k\left(\bigcap_{i=1}^{s} \mathfrak{a}\mathcal{O}_i / \bigcap_{i=1}^{s} \mathfrak{a}\mathcal{O}_i \cap \mathfrak{p}_i^{\mathbf{n}_i}\right) \\ &= \sum_{i=1}^{s} \dim_k(\mathfrak{a}\mathcal{O}_i/\mathfrak{a}\mathcal{O}_i \cap \mathfrak{p}_i^{\mathbf{n}_i}) \end{aligned}$$

■

Corollary 18 *For each* S*-ideal* $\mathfrak{a}$

$$\deg_S(\mathfrak{a}) = \sum_{i=1}^{s} \deg_{\mathcal{O}_i}(\mathfrak{a}\mathcal{O}_i).$$

In particular,

$$\deg_S(\widetilde{S}) = \sum_{i=1}^{s} \deg_{\mathcal{O}_i}(\widetilde{\mathcal{O}_i}) = \sum_{i=1}^{s} \delta_{\mathcal{O}_i}$$

where each $\deg_{\mathcal{O}_i}$ *and* $\delta_{\mathcal{O}_i}$ *are the degree and the singularity degree of the local ring* $\mathcal{O}_i$*, respectively.*

Proposition 19 *The assignment* $\mathfrak{a} \longmapsto (\mathfrak{a}\mathcal{O}_j)_{j=1,\cdots,s}$ *defines a one-to-one monotonous bijection between the partially ordered set of* S*-ideals and the partially ordered direct product of* $\mathcal{O}_j$*-ideals,* $j = 1, \cdots, s$*, and it satisfies:*

1. *If* $\mathfrak{c}$ *is a dualizing ideal of the semilocal ring* S*, then* $\mathfrak{c}\mathcal{O}_j$ *is a dualizing ideal of the local ring* $\mathcal{O}_j$ *for each* $j = 1, \cdots, s$*.*

2. *If* $\mathfrak{c}_j$ *is a dualizing ideal of the local ring* $\mathcal{O}_j$ *for each* $j = 1, \cdots, s$*, then there exists an* S*-ideal* $\mathfrak{c}$ *such that* $\mathfrak{c}$ *is a dualizing ideal of the ring* S *and* $\mathfrak{c}\mathcal{O}_j = \mathfrak{c}_j$ *for* $j = 1, \cdots, s$*.*

Proof. Since $\mathfrak{a}\mathcal{O}_j$ is an $\mathcal{O}_j$-ideal $j = 1, \cdots, s$ for each S-ideal $\mathfrak{a}$ it follows that the assignment $\mathfrak{a} \longmapsto (\mathfrak{a}\mathcal{O}_j)_{j=1,\cdots,s}$ is well defined. By Proposition 17 $\mathfrak{a} = \mathfrak{a}\mathcal{O}_1 \cap \cdots \cap \mathfrak{a}\mathcal{O}_s$, then this function is one-to-one. It is well known that for every prime ideal $\mathfrak{p}$ of the ring S the assignment $\mathfrak{q} \longmapsto \mathfrak{q}^e = \mathfrak{q}S_{\mathfrak{p}}$ gives a bijection between the set of primary ideals with radical $\mathfrak{p}$ in S and the set of proper ideals in the local ring $S_{\mathfrak{p}}$ (cf. [2]). Let $(\mathfrak{b}_j)_{j=1,\cdots,s}$ be an s-tuple of $\mathcal{O}_j$-ideals $j = 1, \cdots, s$. Then for each $j = 1, \cdots, s$ there exists a non-zero rational function x_j in $\mathcal{O}_j$ such that $x_j\mathfrak{b}_j \subseteq \mathcal{O}_j$ and $x_j\mathfrak{b}_j$ is an ideal of $\mathcal{O}_j$. Since K is the quotient field of S we can assume that each $x_j \in S$, so, by setting $x := x_1 \cdots x_s \in S$, we have $x\mathfrak{b}_j \subseteq \mathcal{O}_j$ and $x\mathfrak{b}_j$ is an ideal of $\mathcal{O}_j$ for each $j = 1, \cdots, s$. Thus, for each $j = 1, \cdots, s$ there exists an ideal $\mathfrak{q}_j$ of the ring S such that $\mathfrak{q}_j^e = \mathfrak{q}_j\mathcal{O}_j = x\mathfrak{b}_j$, where $\mathfrak{q}_j$ is either the $\mathfrak{m}_j$-primary ideal of S given by the assignment $\mathfrak{q} \longmapsto \mathfrak{q}^e$, if $x\mathfrak{b}_j$ is a proper ideal of $\mathcal{O}_j$, or $\mathfrak{q}_j$ is equal to S, if $x\mathfrak{b}_j = \mathcal{O}_j$. In either case $x^{-1}\mathfrak{q}_j$ is an S-ideal, $(x^{-1}\mathfrak{q}_j)\mathcal{O}_j = \mathfrak{b}_j$ and $(x^{-1}\mathfrak{q}_j)\mathcal{O}_i = x^{-1}\mathcal{O}_i$ for each $i \neq j$. Then $\mathfrak{a} := x^{-1}\mathfrak{q}_1 \cap \cdots \cap x^{-1}\mathfrak{q}_s$ is an S-ideal and $\mathfrak{a}\mathcal{O}_j = (x^{-1}\mathfrak{q}_1 \cap \cdots \cap x^{-1}\mathfrak{q}_s)\,\mathcal{O}_j = (x^{-1}\mathfrak{q}_1)\,\mathcal{O}_j \cap \cdots \cap (x^{-1}\mathfrak{q}_s)\,\mathcal{O}_j = \mathfrak{b}_j \cap x^{-1}\mathcal{O}_j = \mathfrak{b}_j$ for each $j = 1, \cdots, s$.

In the proof of (1) and (2) we will use the following result: if N and P are submodules of an S-module M and P is finitely generated, then $U^{-1}(N : P) = (U^{-1}N : U^{-1}P)$, where U is a multiplicative closed subset of S (cf. [2] page. 43). Thus, if $\mathfrak{c}$ is a dualizing ideal of the ring S, then, for each S-ideal $\mathfrak{a}$, it follows that $\mathfrak{c}\mathcal{O}_j : (\mathfrak{c}\mathcal{O}_j : \mathfrak{a}\mathcal{O}_j) = \mathfrak{a}\mathcal{O}_j$ for each $j = 1, \cdots, s$. Given that for each $\mathcal{O}_j$-ideal $\mathfrak{b}$ we can choose an S-ideal $\mathfrak{a}$ such that

$\mathfrak{a}\mathcal{O}_j = \mathfrak{b}$ we have $\mathfrak{c}\mathcal{O}_j : (\mathfrak{c}\mathcal{O}_j : \mathfrak{b}) = \mathfrak{b}$ for each $\mathcal{O}_j$-ideal $\mathfrak{b}$ i.e. $\mathfrak{c}\mathcal{O}_j$ is a dualizing ideal of the local ring $\mathcal{O}_j$.

By the first part of this proposition there exists an S-ideal $\mathfrak{c}$ such that $\mathfrak{c}\mathcal{O}_j = \mathfrak{c}_j$, where $\mathfrak{c}_j$ is a dualizing ideal of $\mathcal{O}_j$ for each $j = 1, \cdots, s$. If $\mathfrak{a}$ is an S-ideal, then

$$\mathfrak{c} : (\mathfrak{c} : \mathfrak{a}) = \bigcap_{j=1}^{s} (\mathfrak{c} : (\mathfrak{c} : \mathfrak{a}))\, \mathcal{O}_j = \bigcap_{j=1}^{s} (\mathfrak{c}\mathcal{O}_j : (\mathfrak{c}\mathcal{O}_j : \mathfrak{a}\mathcal{O}_j)) = \bigcap_{j=1}^{s} \mathfrak{a}\mathcal{O}_j = \mathfrak{a}.$$

Thus $\mathfrak{c}$ is the dualizing ideal of S. ■

The index $[\mathfrak{a} : \mathfrak{b}]$ of any two S-ideals $\mathfrak{a}$ and $\mathfrak{b}$, is defined by considering the following cases:

i. If $\mathfrak{b} \subseteq \mathfrak{a}$, then $[\mathfrak{a} : \mathfrak{b}]$ is the index of $\mathfrak{b}$ in $\mathfrak{a}$, that is, the order of the quotient group $\mathfrak{a}/\mathfrak{b}$.

ii. $[\mathfrak{a} : \mathfrak{b}] := [\mathfrak{a} : \mathfrak{d}]/[\mathfrak{b} : \mathfrak{d}]$ for any S-ideal $\mathfrak{d}$ contained in $\mathfrak{a}$ and $\mathfrak{b}$.

It is proved that this definition is independent on the choice of $\mathfrak{d}$ and it extends to a definition of index given when $\mathfrak{b} \subseteq \mathfrak{a}$. One can see that $[\mathfrak{a} : \mathfrak{b}] = [\mathfrak{b} : \mathfrak{a}]^{-1}$, $[\mathfrak{a} : \mathfrak{b}] = [\mathfrak{a} : \mathfrak{d}][\mathfrak{d} : \mathfrak{b}]$ for all S-ideals $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{d}$. The S-norm of any S-ideal $\mathfrak{a}$ can be expressed in terms of the index as $\|\mathfrak{a}\|_S = [\mathfrak{a} : S]$, so that the norm of any non-zero rational function z is $\|z\| = \|zS\|_S$ (cf. [14]). From the properties of the index we observe that, if S is a semilocal subring of the function field $K \mid k$, with $S \neq K$, and $k = \mathbb{F}_q$ is a finite field, then $\log_q \|S\|_S = 0$ and $\dim(\mathfrak{a}/\mathfrak{b}) = \log_q \|\mathfrak{a}\|_S - \log_q \|\mathfrak{b}\|_S$ whenever $\mathfrak{b} \subseteq \mathfrak{a}$. Thus, in this case, $\|\mathfrak{a}\|_S = q^{\deg_S(\mathfrak{a})}$ for each S-ideal $\mathfrak{a}$. We observe that, if S is a semilocal subring of the function field $K \mid k$, with $S \neq K$, and $k = \mathbb{F}_q$ is a finite field, then for each pair of S-ideals $\mathfrak{a}$ and $\mathfrak{b}$ we have:

1. $[\mathfrak{a} : \mathfrak{b}] = \prod_{j=1}^{s} [\mathfrak{a}\mathcal{O}_j : \mathfrak{b}\mathcal{O}_j]$ and
2. $\|\mathfrak{a}\|_S = \prod_{j=1}^{s} \|\mathfrak{a}\mathcal{O}_j\|_{\mathcal{O}_j}$.

Indeed, by the definition of index for S-ideals, $[\mathfrak{a} : \mathfrak{b}] = [\mathfrak{a} : \mathfrak{d}]/[\mathfrak{b} : \mathfrak{d}]$, where $\mathfrak{d}$ is any S-ideal contained in $\mathfrak{a}$ and $\mathfrak{b}$. Thus

$$[\mathfrak{a} : \mathfrak{b}] = q^{\dim(\mathfrak{a}/\mathfrak{d}) - \dim(\mathfrak{b}/\mathfrak{d})}$$

where $\mathfrak{d}$ is any S-ideal contained in $\mathfrak{a}$ and $\mathfrak{b}$. Since $\mathfrak{d}\mathcal{O}_j$ is an $\mathcal{O}_j$-ideal contained in $\mathfrak{a}\mathcal{O}_j$ and $\mathfrak{b}\mathcal{O}_j$, it follows

$$[\mathfrak{a}\mathcal{O}_j : \mathfrak{b}\mathcal{O}_j] = [\mathfrak{a}\mathcal{O}_j : \mathfrak{d}\mathcal{O}_j]/[\mathfrak{b}\mathcal{O}_j : \mathfrak{d}\mathcal{O}_j] = q^{\dim(\mathfrak{a}\mathcal{O}_j/\mathfrak{d}\mathcal{O}_j)-\dim(\mathfrak{b}\mathcal{O}_j/\mathfrak{d}\mathcal{O}_j)}.$$

Now, by applying Proposition 17, it follows that $[\mathfrak{a} : \mathfrak{b}] = \prod_{j=1}^{s} [\mathfrak{a}\mathcal{O}_j : \mathfrak{b}\mathcal{O}_j]$ for each pair of S-ideals $\mathfrak{a}$ and $\mathfrak{b}$. So the second item follows from the first one.

From the second formula we can prove the Euler product identity of Green's zeta function $\varsigma_{\mathcal{O}}(\mathfrak{d}, s)$ (see [15] page. 486). This zeta function was obtained by Green, by slightly modifying the zeta function introduced by Galkin in [14]. It was defined in terms of the index of non-zero fractional ideals as:

$$\varsigma_{\mathcal{O}}(\mathfrak{d}, s) = \sum\nolimits_{\mathfrak{a}\subseteq\mathfrak{d}} \|\mathfrak{a}\|_{\mathcal{O}}^{s}, \quad \Re(s) > 0$$

where the sum is taken over the $\mathcal{O}$-ideals $\mathfrak{a}$ that are contained in the $\mathcal{O}$-ideal $\mathfrak{d}$ and s is a complex variable, for an order $\mathcal{O}$ in a global field. Green's zeta function satisfies a functional equation, but it is not uniquely determined by the curve. By the reciprocity 2, the assignment $\mathfrak{a} \longmapsto \mathfrak{a}^* := \mathfrak{c} : \mathfrak{a}$, which defines an anti-monotonus permutation between the $\mathcal{O}$-ideals, indicates the connection between the local zeta function and Green's zeta function.

In the set of S-ideals is defined the following equivalence relation: $\mathfrak{a} \sim \mathfrak{b}$ if $\mathfrak{b} = z^{-1}\mathfrak{a}$ for some $z \in K\backslash\{0\}$. The equivalence classes form a semigroup (cf. [14]).

As in [27], for each S-ideal $\mathfrak{b}$, we consider the $\widetilde{S}$-ideal $(\mathfrak{b} : \widetilde{S}) : \mathfrak{b} \cdot \widetilde{S}$, which is the quotient of $\mathfrak{b} : \widetilde{S}$, the largest $\widetilde{S}$-ideal contained in $\mathfrak{b}$, and $\mathfrak{b}\widetilde{S}$, the smallest $\widetilde{S}$-ideal that contains $\mathfrak{b}$. This quotient only depends on the S-ideal class $[\mathfrak{b}]$. Moreover,

$$(\mathfrak{b} : \widetilde{S}) : \mathfrak{b} = (\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S} = (\mathfrak{b} : \widetilde{S}) \cdot (\mathfrak{b}\widetilde{S})^{-1}, \quad \mathfrak{f} \subseteq (\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S} \subseteq \widetilde{S}$$

and, hence,

$$\mathbf{0} \leq \mathbf{v}((\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S}) = \mathbf{v}(\mathfrak{b} : \widetilde{S}) - \mathbf{v}(\mathfrak{b}\widetilde{S}) \leq \mathfrak{f}$$

where $\mathbf{0} := (0, \cdots, 0)$ and $\mathfrak{f} := \mathbf{v}(\mathfrak{f}) = (f_1, \cdots, f_m)$. If $\mathfrak{b}\widetilde{S} = \widetilde{S}$ then $(\mathfrak{b} : \widetilde{S}) : \mathfrak{b} = (\mathfrak{b} : \widetilde{S})$ and $\mathbf{b} := \mathbf{v}((\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S})$ is the smallest vector in the partial ordering of $\mathbb{Z}^m$ such that $\mathfrak{p}^{\mathbf{b}} \subseteq \mathfrak{b}$. We also consider the set

$$S(\mathfrak{b}) := \{\mathbf{v}(z) - \mathbf{v}(\mathfrak{b}\widetilde{S}) \mid z \in \mathfrak{b}\backslash\{0\}\},$$

which only depends on the S-ideal class $[\mathfrak{b}]$. Moreover

$$\mathbf{b} + \mathbb{N}^m \subseteq S(\mathfrak{b}) \subseteq \mathbb{N}^m \text{ and } S(S) + S(\mathfrak{b}) \subseteq S(\mathfrak{b}).$$

In fact, let $\mathbf{n}$ be in $\mathbb{N}^m$. Since $\widetilde{S}$ is a principal ideal domain there exist $z_1, z_2 \in K^*$ such that $(\mathfrak{b} : \widetilde{S}) = \widetilde{S}z_1$ and $\mathfrak{p}^{\mathbf{n}} = \widetilde{S}z_2 \subseteq \widetilde{S}$. By setting $z := z_1 z_2$ we obtain $z \in \mathfrak{b}\backslash\{0\}$ such that $\mathbf{v}(z) = \mathbf{v}(\mathfrak{b} : \widetilde{S}) + \mathbf{n}$, that is, $\mathbf{b} + \mathbf{n} = \mathbf{v}(z) - \mathbf{v}(\mathfrak{b}\widetilde{S})$ for some $z \in \mathfrak{b}\backslash\{0\}$ i.e. $\mathbf{b} + \mathbf{n} \in S(\mathfrak{b})$. Similarly, we can prove that $S(S) + S(\mathfrak{b}) \subseteq S(\mathfrak{b})$.

In particular, $S(S)$ is an intermediate semigroup between $\mathfrak{f} + \mathbb{N}^m$ and $\mathbb{N}^m$, called the *semigroup associated* to the semilocal ring S.

In the same way, as it was observed by Stöhr (see [27]) in the local case, we observe that for each semilocal subring S of the function field $K \mid k$ and each S-ideal $\mathfrak{b}$, the set

$$U_{\mathfrak{b}} := \{u \in K\backslash\{0\} : \mathfrak{b} = u^{-1}\mathfrak{b}\}$$

is a group, it only depends on the class $[\mathfrak{b}]$ and it is equal to the group $U_{\mathfrak{b}:\mathfrak{b}}$ of units of the semilocal ring $\mathfrak{b} : \mathfrak{b}$, which is also an S-ideal.

Proposition 20 *Let S be a semilocal subring of the function field $K \mid k$, with $S \neq K$, whose expression as intersection of local rings is $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$ and let $\mathfrak{b}$ be an S-ideal. Then*

1. $U_{\mathfrak{b}} = U_{\mathfrak{b}\mathcal{O}_1} \cap \cdots \cap U_{\mathfrak{b}\mathcal{O}_s}$
2. *The application*
$$(\mathfrak{b}\backslash\{0\})/U_{\mathfrak{b}} \longrightarrow (\mathfrak{b}\mathcal{O}_1\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_1} \times \cdots \times (\mathfrak{b}\mathcal{O}_s\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_s} \tag{2.1}$$
defined by $zU_{\mathfrak{b}} \longmapsto (zU_{\mathfrak{b}\mathcal{O}_1}, \cdots, zU_{\mathfrak{b}\mathcal{O}_s})$, is a bijective function, where $(\mathfrak{b}\backslash\{0\})/U_{\mathfrak{b}}$ and $(\mathfrak{b}\mathcal{O}_i\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_i}$ $(i = 1, \cdots s)$ denote the quotient of $\mathfrak{b}\backslash\{0\}$ and $\mathfrak{b}\mathcal{O}_i\backslash\{0\}$ $(i = 1, \cdots s)$ by the action of the groups $U_{\mathfrak{b}}$ and $U_{\mathfrak{b}\mathcal{O}_i}$ $(i = 1, \cdots s)$ respectively.
3. *There is an isomorphism of multiplicative abelian groups*
$$U_{\mathfrak{b}}/U_S \xrightarrow{\sim} U_{\mathfrak{b}\mathcal{O}_1}/U_{\mathcal{O}_1} \times \cdots \times U_{\mathfrak{b}\mathcal{O}_s}/U_{\mathcal{O}_s}$$
defined by $uU_S \longmapsto (uU_{\mathcal{O}_1}, \cdots, uU_{\mathcal{O}_s})$.
4. $S(\mathfrak{b}) = S(\mathfrak{b}\mathcal{O}_1) \times \cdots \times S(\mathfrak{b}\mathcal{O}_s)$.

Proof. (1) From Proposition 17 (1), it follows that

$$\mathfrak{b} : \mathfrak{b} = (\mathfrak{b}\mathcal{O}_1 : \mathfrak{b}\mathcal{O}_1) \cap \cdots \cap (\mathfrak{b}\mathcal{O}_s : \mathfrak{b}\mathcal{O}_s)$$

and, hence,

$$U_{\mathfrak{b}:\mathfrak{b}} = U_{\mathfrak{b}\mathcal{O}_1:\mathfrak{b}\mathcal{O}_1} \cap \cdots \cap U_{\mathfrak{b}\mathcal{O}_s:\mathfrak{b}\mathcal{O}_s}.$$

(2) Let z, $w \in \mathfrak{b}\backslash\{0\}$. Then $(zU_{\mathfrak{b}\mathcal{O}_1}, \cdots, zU_{\mathfrak{b}\mathcal{O}_s}) = (wU_{\mathfrak{b}\mathcal{O}_1}, \cdots, wU_{\mathfrak{b}\mathcal{O}_s})$ if and only if $zU_{\mathfrak{b}\mathcal{O}_i} = wU_{\mathfrak{b}\mathcal{O}_i}$ for each $i = 1, \cdots, s$ i.e $z^{-1}w \in U_{\mathfrak{b}\mathcal{O}_i}$ for each $i = 1, \cdots, s$ i.e $z^{-1}w \in U_{\mathfrak{b}}$ i.e $zU_{\mathfrak{b}} = wU_{\mathfrak{b}}$. Thus the application 2.1 is well defined and it is injective. Let $(z_1U_{\mathfrak{b}\mathcal{O}_1}, \cdots, z_sU_{\mathfrak{b}\mathcal{O}_s}) \in (\mathfrak{b}\mathcal{O}_1\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_1} \times \cdots \times (\mathfrak{b}\mathcal{O}_s\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_s}$. For each $i = 1, \cdots, s$ we pick an element $w_i \in (\cap_{j\neq i}\mathfrak{m}_j)\backslash\mathfrak{m}_i$. Then $\mathbf{v}_i(w_i) = \mathbf{0}$ in $\mathcal{O}_i = S_{\mathfrak{m}_i}$ and $\mathbf{v}_j(w_i)$ has only positive coordinates whenever $j \neq i$. Let us choose $N_1, \cdots, N_s$ to be positive integers, and let $z := \sum_{j=1}^{s} z_j w_j^{N_j}$. Since $S \subseteq \mathcal{O}_i \subseteq (\mathfrak{b}\mathcal{O}_i : \mathfrak{b}\mathcal{O}_i)$, $w_i \in S$ and $z_i \in \mathfrak{b}\mathcal{O}_i$, it follows that $z_i w_i^{N_i} \in \mathfrak{b}\mathcal{O}_i$ and $w_i^{N_i} \in \mathfrak{b}\mathcal{O}_i : \mathfrak{b}\mathcal{O}_i$ for each $i = 1, \cdots, s$. On the other hand, if $j \neq i$ and N_j is sufficiently large, then $\mathbf{v}_i\left(z_j w_j^{N_j}\right)$ and $\mathbf{v}_i\left(z_i^{-1} z_j w_j^{N_j}\right)$ have sufficiently large positive coordinates. Thus, as $\mathbf{v}_i(w_i) = \mathbf{0}$, $\mathfrak{b}\mathcal{O}_i : \widetilde{\mathcal{O}_i} \subseteq \mathfrak{b}\mathcal{O}_i$, $\mathfrak{b}\mathcal{O}_i \subseteq \mathfrak{b}\mathcal{O}_i : \mathfrak{b}\mathcal{O}_i$, for each $i = 1, \cdots, s$, and $\mathfrak{b} = \cap_{i=1}^{s}\mathfrak{b}\mathcal{O}_i$, we have, by taking $N_1, \cdots, N_s$ sufficiently large, $z := \sum_{j=1}^{s} z_j w_j^{N_j} \in \mathfrak{b}$, $\mathbf{v}_i\left(z_i^{-1}z\right) = \mathbf{0}$ and $zz_i^{-1} \in \mathfrak{b}\mathcal{O}_i : \mathfrak{b}\mathcal{O}_i$. Whence, $zz_i^{-1} \in U_{\mathfrak{b}\mathcal{O}_i:\mathfrak{b}\mathcal{O}_i}$ i.e. $zU_{\mathfrak{b}\mathcal{O}_i} = z_iU_{\mathfrak{b}\mathcal{O}_i}$ for each $i = 1, \cdots, s$.

(3) The application

$$U_{\mathfrak{b}} \longrightarrow U_{\mathfrak{b}\mathcal{O}_1}/U_{\mathcal{O}_1} \times \cdots \times U_{\mathfrak{b}\mathcal{O}_s}/U_{\mathcal{O}_s}$$

defined by $u \longmapsto (uU_{\mathcal{O}_1}, \cdots, uU_{\mathcal{O}_s})$ is a homomorphism of groups whose kernel is equal to $U_S = U_{\mathcal{O}_1} \cap \cdots \cap U_{\mathcal{O}_s}$. It remains to prove that this homomorphism is surjective. Let $u_i \in U_{\mathfrak{b}\mathcal{O}_i} = U_{\mathfrak{b}\mathcal{O}_i:\mathfrak{b}\mathcal{O}_i}$ for each $i = 1, \cdots, s$. For each $i = 1, \cdots, s$ we pick an element $y_i \in (\cap_{j\neq i}\mathfrak{m}_j)\backslash\mathfrak{m}_i$. Then $\mathbf{v}_i(y_i) = \mathbf{0}$ in $\mathcal{O}_i = S_{\mathfrak{m}_i}$ and $\mathbf{v}_j(y_i)$ has only positive coordinates whenever $j \neq i$. Since $\mathfrak{b} : \mathfrak{b} = (\mathfrak{b}\mathcal{O}_1 : \mathfrak{b}\mathcal{O}_1) \cap \cdots \cap (\mathfrak{b}\mathcal{O}_s : \mathfrak{b}\mathcal{O}_s)$ and $\mathcal{O}_i \subseteq \mathfrak{b}\mathcal{O}_i : \mathfrak{b}\mathcal{O}_i \subseteq \widetilde{\mathcal{O}_i}$, for each $i = 1, \cdots, s$, by taking sufficiently large powers of y_i, the element $u := \sum_{i=1}^{s} u_i y_i^{N_i} \in U_{\mathfrak{b}:\mathfrak{b}} = U_{\mathfrak{b}}$ and $u_i^{-1}u \in U_{\mathcal{O}_i}$ for each $i = 1, \cdots, s$ (because we can choose sufficiently large powers of each y_j such that $u \in \mathfrak{b} : \mathfrak{b}$, $\mathbf{v}(u) = \mathbf{0}$, $u_i^{-1}u \in \mathcal{O}_i$ and $\mathbf{v}_i\left(u_i^{-1}u\right) = \mathbf{0}$).

(4) Since the set $S(\mathfrak{b})$ only depends on the class $[\mathfrak{b}]$, we can assume that $\mathfrak{b}\widetilde{S} = \widetilde{S}$, hence $(\mathfrak{b}\mathcal{O}_i)\widetilde{\mathcal{O}_i} = \widetilde{\mathcal{O}_i}$ $(i = 1, \cdots, s)$. It is clear that $S(\mathfrak{b}) \subseteq S(\mathfrak{b}\mathcal{O}_1) \times \cdots \times S(\mathfrak{b}\mathcal{O}_s)$. Conversely, if $\mathbf{n} := (\mathbf{n}_1, \cdots, \mathbf{n}_s) \in S(\mathfrak{b}\mathcal{O}_1) \times \cdots \times S(\mathfrak{b}\mathcal{O}_s)$, then there exists $z_i \in \mathfrak{b}\mathcal{O}_i$ such that $\mathbf{v}_i(z_i) = \mathbf{n}_i$ in $\mathcal{O}_i = S_{\mathfrak{m}_i}$ $(i = 1, \cdots, s)$. For each $i = 1, \cdots, s$ we pick an element $x_i \in (\cap_{j\neq i}\mathfrak{m}_j)\backslash\mathfrak{m}_i$. Then $\mathbf{v}_i(x_i) = \mathbf{0}$ in $\mathcal{O}_i = S_{\mathfrak{m}_i}$ and $\mathbf{v}_j(x_i)$ has only positive coordinates whenever $j \neq i$. Hence, taking sufficiently large powers of x_i, the element $z := \sum_{i=1}^{s} z_i x_i^{N_i} \in \mathfrak{b}$ and $\mathbf{v}(z) = (\mathbf{n}_1, \cdots, \mathbf{n}_s) = \mathbf{n}$. ■

Corollary 21 *For each S-ideal $\mathfrak{b}$,*

$$[U_{\mathfrak{b}} : U_S] = [U_{\mathfrak{b}\mathcal{O}_1} : U_{\mathcal{O}_1}] \cdot \cdots \cdot [U_{\mathfrak{b}\mathcal{O}_s} : U_{\mathcal{O}_s}]$$

and for each pair of S-ideals $\mathfrak{a}$ and $\mathfrak{b}$,

$$[U_{\mathfrak{b}:\mathfrak{a}} : U_{\mathfrak{b}}] = [U_{\mathfrak{b}\mathcal{O}_1:\mathfrak{a}\mathcal{O}_1} : U_{\mathfrak{b}\mathcal{O}_1}] \cdot \cdots \cdot [U_{\mathfrak{b}\mathcal{O}_s:\mathfrak{a}\mathcal{O}_s} : U_{\mathfrak{b}\mathcal{O}_s}].$$

2.3 Dimension formulae and some properties on semigroups

Let S be a semilocal subring of a function field $K \mid k$, with $S \neq K$, and let $\mathfrak{b}$ be an S-ideal. We have associated to the S-ideal $\mathfrak{b}$ both the set $S(\mathfrak{b})$ and the vector $\mathbf{b} := \mathbf{v}((\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S}) = (b_1, \cdots, b_m)$, which only depend on the ideal class $[\mathfrak{b}]$. They satisfy $\mathbf{0} \leq \mathbf{b} \leq \mathfrak{f}$, $\mathbf{b}$ is the smallest vector in the partial ordering of $\mathbb{Z}^m$ satisfying $S(\mathfrak{b}) \supseteq \mathbf{b} + \mathbb{N}^m$. Moreover, if $\mathfrak{b} \cdot \widetilde{S} = \widetilde{S}$, then $\mathbf{b}$ is the smallest vector in the partial ordering of $\mathbb{Z}^m$ such that $\mathfrak{p}^{\mathbf{b}} \subseteq \mathfrak{b}$.

Given the integer vector $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$, we observe that

$$\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} = \{z \in \mathfrak{b} : \mathbf{v}(z) \geq \mathbf{n} + \mathbf{v}(\mathfrak{b}\widetilde{S})\}$$

and

$$0 \longrightarrow \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i} \longrightarrow \mathfrak{b} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i} \longrightarrow \mathfrak{b} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} \longrightarrow 0$$

is an exact sequence for each $i = 1, \cdots, m$, where $\mathbf{e}_i \in \mathbb{Z}^m$ denotes the vector whose i-th coordinated is 1 while all other coordinates are 0. We denote by

$$l(\mathfrak{b}, \mathbf{n}) := \dim_k (\mathfrak{b} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}})$$

the codimension of the S-ideal $\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}}$ in $\mathfrak{b}$. Thus,

$$l(\mathfrak{b}, \mathbf{n} + \mathbf{e}_i) - l(\mathfrak{b}, \mathbf{n}) = \dim_k \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) \text{ for each } i = 1, \cdots, m.$$

In particular, if $\mathcal{O}$ is a local subring of the function field $K \mid k$, then $l(\mathcal{O}, \mathbf{0}) = 0$ and $l(\mathcal{O}, \mathbf{e}_i) = \dim_k (\mathcal{O}/\mathfrak{m})$ for each $i = 1, \cdots, m$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$.

Lemma 22 *Let S be a semilocal subring of a function field $K \mid k$, with $S \neq K$, and let $\mathfrak{b}$ be an S-ideal and let $\mathbf{n} \in \mathbb{Z}^m$. Then*

1.
$$0 \leq \dim_k \left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) \leq r_i \quad \textit{for each } i = 1, \cdots, m$$

where each r_i is the degree of the residue field of $\mathfrak{p}_i$ over the constant field k. In particular, if S is residually rational, then

$$0 \leq \dim_k \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) \leq 1 \quad \textit{for each} \ \ i = 1, \cdots, m.$$

2. $\dim_k \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) = r_i$ *for each $i = 1, \cdots, m$ whenever* $\mathbf{n} \geq \mathbf{b}$.

3. *If $n_j \geq b_j$ for some $j = 1, \cdots, m$ then* $\dim_k \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_j}\right) = r_j$.

4. $\dim_k \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{b}}\right) < r_i$ *for each $i = 1, \cdots, m$.*

Proof. We choose generators $\pi_1, \cdots, \pi_m$ of the maximal integral $\widetilde{S}$-ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$, respectively. By the weak approximation theorem, we can assume that $v_j(\pi_i - 1) \geq f_j$ whenever $j \neq i$. Then, we have an injective k-linear application

$$\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i} \hookrightarrow \mathfrak{b}\widetilde{S} / \mathfrak{b}\mathfrak{p}_i$$

defined by $x + \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i} \longmapsto x\pi_i^{-n_i} + \mathfrak{b}\mathfrak{p}_i$. Thus

$$\dim_k \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) \leq \dim_k(\mathfrak{b}\widetilde{S} / \mathfrak{b}\mathfrak{p}_i) = r_i,$$

so (1) is valid.

(2) Since $\mathfrak{b}\mathfrak{p}^{\mathbf{b}} = \mathfrak{b} : \widetilde{S} \subseteq \mathfrak{b}$, we have $\mathfrak{b}\mathfrak{p}^{\mathbf{n}} \subseteq \mathfrak{b}$ whenever $\mathbf{n} \geq \mathbf{b}$. Hence, if $\mathbf{n} \geq \mathbf{b}$ then $\dim_k \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) = \dim_k \left(\mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) = r_i$.

(3) If $\mathbf{s} \geq \mathbf{n}$, then $\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{s}} \subseteq \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}}$ and, hence, we have a k-linear application

$$\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{s}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{s}+\mathbf{e}_j} \longrightarrow \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_j}$$

defined by $x + \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{s}+\mathbf{e}_j} \longmapsto x + \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_j}$, which is injective if and only if $n_j = s_j$ or $\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{s}} = \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{s+\mathbf{e}_j}$. On the other hand, if $n_j \geq b_j$ for some $j = 1, \cdots, m$, then we can choose $\mathbf{s} \in \mathbb{N}^m$ such that $\mathbf{s} \geq \mathbf{n}$, $\mathbf{s} \geq \mathbf{b}$ and $s_j = n_j \geq b_j$. Thus (3) follows from (1) and (2).

(4) Let $i \in \{1, \cdots, m\}$. We observe that $\mathbf{b} := \mathbf{v}((\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S}) = (b_1, \cdots, b_m)$ satisfies $\pi^{\mathbf{b}-\mathbf{e}_i} \notin (\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S}$. Then there exists $z \in \mathfrak{b}\widetilde{S}$ such that $z\pi^{\mathbf{b}-\mathbf{e}_i} \notin \mathfrak{b}$. Let $y := z\pi^{\mathbf{b}}\pi_i^{-b_i}$. Then $y \in \mathfrak{b}\widetilde{S}$. On the other hand, if $x \in \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i}$, that is, if $x \in \mathfrak{b}$ and $\mathbf{v}(x) \geq \mathbf{b} - \mathbf{e}_i + \mathbf{v}(\mathfrak{b}\widetilde{S})$, it follows that $z\pi^{\mathbf{b}-\mathbf{e}_i} - x \in \mathfrak{b}\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i}$, $z\pi^{\mathbf{b}-\mathbf{e}_i} - x \notin \mathfrak{b}$ and, hence, $z\pi^{\mathbf{b}-\mathbf{e}_i} - x \notin \mathfrak{b} : \widetilde{S} = \mathfrak{b}\mathfrak{p}^{\mathbf{b}}$, so, $v_i\left(z\pi^{\mathbf{b}-\mathbf{e}_i} - x\right) = b_i - 1 + v_i(\mathfrak{b}\widetilde{S})$ i.e.

$v_i(z\pi^{\mathbf{b}}\pi_i^{-b_i} - x\pi_i^{-(b_i-1)}) = v_i(\mathfrak{b}\widetilde{S})$ i.e. $v_i(y - x\pi_i^{-(b_i-1)}) = v_i(\mathfrak{b}\widetilde{S})$, where $v_i\left(\mathfrak{b}\widetilde{S}\right)$ denotes the i-th entry of the vector $\mathbf{v}(\mathfrak{b}\widetilde{S})$. Thus the injective k-linear application $\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{b}} \hookrightarrow \mathfrak{b}\widetilde{S}/\mathfrak{b}\mathfrak{p}_i$ defined by $x+\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i} \longmapsto x\pi_i^{-(b_i-1)} + \mathfrak{b}\mathfrak{p}_i$ is not surjective. So (4) is valid. ■

Note that, if S is residually rational, then $l(\mathfrak{b},\mathbf{n}+\mathbf{e}_i) \leq l(\mathfrak{b},\mathbf{n})+1$ and equality holds if and only if $\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i} \subsetneqq \mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}}$, that is, if and only if there exists $\mathbf{s}\in S(\mathfrak{b})$ such that $\mathbf{s}\geq\mathbf{n}$ with $s_i=n_i$. Since $l(\mathfrak{b},\mathbf{n})=0$ when $\mathbf{n}=(0,\cdots,0)$, we see in this case, by induction, that the integers $l(\mathfrak{b},\mathbf{n})$ may be expressed in terms of the set $S(\mathfrak{b})$.

Proposition 23 *Let S be a semilocal subring of a function field $K \mid k$, with $S\neq K$, let $\mathfrak{b}$ be an S-ideal such that $\mathfrak{b}\cdot\widetilde{S}=\widetilde{S}$ and, let $\mathbf{n}\in\mathbb{Z}^m$. Then*

$$\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right)+\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right)\leq r_i$$

for each $i=1,\cdots,m$.

Proof. For each $\mathbf{s}\in\mathbb{Z}^m$ we can consider the quotient $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}+\mathbf{e}_i}$ as a k-vector subspace of $\widetilde{S}/\mathfrak{p}_i$ under the injective k-linear application

$$\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}+\mathbf{e}_i}\hookrightarrow\widetilde{S}/\mathfrak{p}_i$$

defined by $x+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}+\mathbf{e}_i}\longmapsto x\pi_i^{-s_i}+\mathfrak{p}_i$. Since $\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}}\right)<r_i$ we can choose a one-codimensional k-vector subspace V_i of $\widetilde{S}/\mathfrak{p}_i$ which contains the image of $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}}$. Let us to consider the k-bilinear application

$$\widetilde{S}/\mathfrak{p}_i\times\widetilde{S}/\mathfrak{p}_i\longrightarrow\frac{\widetilde{S}/\mathfrak{p}_i}{V_i}$$

defined by $(x,y)\longmapsto x\cdot y+V_i$, which is non-degenerated because the multiplication by a non-zero element of $\widetilde{S}/\mathfrak{p}_i$ defines a k-automorphism of $\widetilde{S}/\mathfrak{p}_i$. Let $\mathbf{n}\in\mathbb{Z}^m$. If $x=a\pi_i^{-n_i}+\mathfrak{p}_i$ and $y=b\pi_i^{-(b_i-n_i-1)}+\mathfrak{p}_i$ are elements of $\widetilde{S}/\mathfrak{p}_i$, where $a\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}$ and $b\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}$, then $x\cdot y=ab\pi_i^{-(b_i-1)}+\mathfrak{p}_i$ and $ab\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{e}_i}$, hence $x\cdot y\in V_i$. Thus, the image of $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}$ in $\widetilde{S}/\mathfrak{p}_i$ is contained in the orthogonal complement of the image of $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}$ in $\widetilde{S}/\mathfrak{p}_i$. Therefore,

$$\begin{aligned}\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right)&\leq\dim_k\left(\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right)^{\perp}\right)\\&\leq r_i-\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right).\end{aligned}$$

■

Lemma 24 *Let* $\mathbf{n} = (n_{1,} \cdots, n_m) \in \mathbb{Z}^m$*. Then the vector* $\mathbf{n}' := \mathbf{n} - \sum_{n_j<0} n_j \mathbf{e}_j \in \mathbb{Z}^m$*, where the sum is over the integers* $j = 1, \cdots, m$ *with* $n_j < 0$*, and the integer vector* $\mathbf{n}'' := \mathbf{n} - \sum_{n_j>b_j} n_j \mathbf{e}_j \in \mathbb{Z}^m$*, where the sum is over the integers* $j = 1, \cdots, m$ *with* $n_j > b_j$*, satisfy the following properties:*

1. *If* $\mathbf{n} < \mathbf{b}$ *then*

$$\mathbf{0} \leq \mathbf{n}' < \mathbf{b}, \ \dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) = \dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}'} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}'+\mathbf{e}_i}),$$

and

$$\dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right) = \dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}'-\mathbf{e}_i} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}'})$$

for each $i = 1, \cdots, m$ *such that* $n_i \geq 0$*.*

2. *If* $\mathbf{n} > \mathbf{b}$ *then*

$$\mathbf{n}'' \leq \mathbf{b}, \ \dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) = \dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}''} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}''+\mathbf{e}_i}),$$

and

$$\dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right) = \dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}''-\mathbf{e}_i} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}''})$$

for each $i = 1, \cdots, m$ *such that* $n_i \leq b_i$*.*

Proof. We claim:

(i) If $n_i \geq 0$ then

$$\dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) = \dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_j} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_j+\mathbf{e}_i}\right)$$

and

$$\dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right) = \dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i-\mathbf{e}_j} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}\right)$$

for each $j = 1, \cdots, m$ such that $n_j < 0$.

(ii) If $n_i \leq b_i$ then

$$\dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right) = \dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}-\mathbf{e}_j} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}-\mathbf{e}_j+\mathbf{e}_i}\right)$$

and

$$\dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right) = \dim_k\left(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}+\mathbf{e}_j}\right)$$

for each $j = 1, \cdots, m$ such that $n_j > b_j$.

(i) Since $\mathfrak{b}\cdot\widetilde{S}=\widetilde{S}$, it follows that $\mathfrak{b}\subseteq\widetilde{S}$. Hence, $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_j}$ and $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_j+\mathbf{e}_i}$ for each $j=1,\cdots,m$ such that $n_j<0$. So, the first sentence in (i) holds. Let $j\in\{1,\cdots,m\}$ such that $n_j<0$. It is clear that $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}=\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}\right)\cap\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}\right)$ and $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}\subseteq\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i-\mathbf{e}_j}$. We assert that $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i-\mathbf{e}_j}$. Indeed, let us to consider $\mathbf{s}=(s_{1,}\cdots,s_m)\in\mathbb{Z}^m$ defined by $s_j:=b_j-n_j-1$ and $s_k:=\max\{b_k-n_k,b_k\}$ for each $k=1,\cdots,m$ with $k\neq j$. Thus, $\mathbf{s}\geq\mathbf{b}-\mathbf{n}-\mathbf{e}_j$, hence $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}}\subseteq\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}$. Since $s_j:=b_j-n_j-1\geq b_j$, it follows from Lemma 22 that the homomorphism defined by $x+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}+\mathbf{e}_j}\longmapsto x+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}$, for any $x\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}}$, is an isomorphism between $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}+\mathbf{e}_j}$ and $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i-\mathbf{e}_j}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}$. Therefore, if $z\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i-\mathbf{e}_j}$ then there exists $x\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{s}}$ such that $z-x\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}$, in consequence we have that $z=(z-x)+x\in\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}$. Thus,

$$\begin{aligned}\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}} &= \mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}\right)\cap\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}\right)\\ &\simeq \left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}\right)/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j}\\ &= \mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i-\mathbf{e}_j}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_j},\end{aligned}$$

and so the second sentence in (i) holds.

(ii) Since $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}+\mathbf{e}_j}$ and $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j}$ for each $j=1,\cdots,m$ such that $n_j>b_j$, it follows that the second sentence in (ii) holds. Let $j\in\{1,\cdots,m\}$ such that $n_j>b_j$. As in (i), $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}=(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}-\mathbf{e}_j})\cap(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}})$ and $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}+\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}-\mathbf{e}_j}$, which implies the first sentence in (ii).

The Lemma will follow by repeated application of (i) and (ii). ■

Theorem 25 *Let S be a semilocal subring of a function field $K\mid k$, with $S\neq K$, and let $\mathfrak{b}$ be an S-ideal such that $\mathfrak{b}\cdot\widetilde{S}=\widetilde{S}$. Then*

$$\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right)+\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right)=r_i \tag{2.2}$$

for each $i=1,\cdots,m$ and for each $\mathbf{n}\in\mathbb{Z}^m$ if and only if

$$2\dim_k(\mathfrak{b}/\mathfrak{b}:\widetilde{S})=\dim_k(\widetilde{S}/\mathfrak{b}:\widetilde{S}).$$

Proof. We observe that, if $\left(\mathbf{n}^{(j)}\right)_{1\leq j\leq l}$ is a strictly increasing sequence in $\mathbb{Z}^m$ such that $\mathbf{n}^{(0)}=0$, $\mathbf{n}^{(l)}=\mathbf{b}$ and for each $j=1,\cdots,l$ there exists $i(j)\in\{1,\cdots,m\}$ satisfying $\mathbf{n}^{(j)}-\mathbf{n}^{(j-1)}=e_{i(j)}$, then $2\dim_k\left(\mathfrak{b}/\mathfrak{b}:\widetilde{S}\right)=\dim_k\left(\widetilde{S}/\mathfrak{b}:\widetilde{S}\right)$ if and only if

$$\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(j)}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(j)}+\mathbf{e}_{i(j+1)}})+\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j)}-\mathbf{e}_{i(j+1)}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j)}})=r_{i(j+1)}$$

for each $j = 0, \cdots, l-1$. Indeed,

$$\mathfrak{b} = \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(0)}} \supseteq \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(1)}} \supseteq \cdots \supseteq \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(l)}} = \mathfrak{b} : \widetilde{S}$$

and

$$\mathfrak{b} : \widetilde{S} = \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(0)}} \subseteq \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(1)}} \subseteq \cdots \subseteq \mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(l)}} = \mathfrak{b},$$

hence

$$\begin{aligned}\dim_k(\mathfrak{b}/\mathfrak{b} : \widetilde{S}) &= \sum_{j=0}^{l-1} \dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(j)}}/\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(j)}+\mathbf{e}_{i(j+1)}}) \\ &= \sum_{j=0}^{l-1} \dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j+1)}}/\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j+1)}+\mathbf{e}_{i(j+1)}}).\end{aligned}$$

Since $\mathbf{n}^{(j+1)} - \mathbf{n}^{(j)} = e_{i(j+1)}$ for each $j = 0, \cdots, l-1$, by Proposition 23, we have

$$\begin{aligned}2\dim_k\left(\mathfrak{b}/\mathfrak{b} : \widetilde{S}\right) &= \sum_{j=0}^{l-1} \dim_k\left(\frac{\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(j)}}}{\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(j)}+\mathbf{e}_{i(j+1)}}}\right) + \dim_k\left(\frac{\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j)}-\mathbf{e}_{i(j+1)}}}{\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j)}}}\right) \\ &\leq \sum_{j=0}^{l-1} r_{i(j+1)} \\ &= \sum_{i=1}^{m} r_i b_i \\ &= \dim_k(\widetilde{S}/\mathfrak{b} : \widetilde{S}).\end{aligned}$$

In this manner, the observation is proved.

Now, by choosing any sequence as in the previous observation, it follows that, if 2.2 holds for each $\mathbf{n}$ in $\mathbb{Z}^m$ and for each $i = 1, \cdots, m$, then $2\dim_k(\mathfrak{b}/\mathfrak{b} : \widetilde{S}) = \dim_k(\widetilde{S}/\mathfrak{b} : \widetilde{S})$.

Conversely, assume that we have the equality $2\dim_k(\mathfrak{b}/\mathfrak{b} : \widetilde{S}) = \dim_k(\widetilde{S}/\mathfrak{b} : \widetilde{S})$. Let $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$ and let $i \in \{1, \cdots, m\}$. If $\mathbf{n} = \mathbf{b}$, then, by Lemma 22 and by Proposition 23, 2.2 holds in this case. Thus we can assume that $\mathbf{n} \neq \mathbf{b}$. To prove that 2.2 holds for $\mathbf{n}$ and for i we consider three cases:

First case: assume that $\mathbf{0} \leq \mathbf{n} < \mathbf{b}$. Observe that we can choose a strictly increasing sequence $\left(\mathbf{n}^{(j)}\right)_{1 \leq j \leq l}$ in $\mathbb{Z}^m$ such that $\mathbf{n}^{(0)} = 0$, $\mathbf{n}^{(l)} = \mathbf{b}$ and for each $j = 1, \cdots, l$ there exists $i(j) \in \{1, \cdots, m\}$ satisfying $\mathbf{n}^{(j)} - \mathbf{n}^{(j-1)} = e_{i(j)}$, $\mathbf{n}^{(k)} = \mathbf{n}$ and $i = i(k+1)$ for some $k \in \{1, \cdots, l-1\}$. Thus, it follows from the previous observation, by taking $j = k$, that

$$\dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(k)}}/\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}^{(k)}+\mathbf{e}_{i(k+1)}}) + \dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(k)}-\mathbf{e}_{i(k+1)}}/\mathfrak{b} \cap \mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(k)}}) = r_{i(k+1)}$$

i.e

$$\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right)+\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}\right)=r_i.$$

Second case: assume that $\mathbf{n}<\mathbf{b}$ and $n_j<0$ for some $j\in\{1,\cdots,m\}$. If $n_i<0$, then $\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}$ and $b_i-n_i-1\geq b_i$. Hence 2.2 holds from Lemma 22. Now, we assume that $n_i\geq 0$. Consequently, from Lemma 24,

$$\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}}\right)+\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}}\right)=\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}'}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}'+\mathbf{e}_i}}\right)+\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}'-\mathbf{e}_i}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}'}}\right)=r_i.$$

Third case: assume that $\mathbf{n}>\mathbf{b}$ and $n_j>b_j$ for some $j\in\{1,\cdots,m\}$. If $n_i>b_i$, then 2.2 holds from Lemma 22. Now, we assume that $n_i\leq b_i$. Thus, from Lemma 24,

$$\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}}\right)+\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\mathbf{e}_i}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}}\right)=\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}''}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}''+\mathbf{e}_i}}\right)+\dim_k\left(\tfrac{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}''-\mathbf{e}_i}}{\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}''}}\right)=r_i.$$

■

Corollary 26 *Let S be a semilocal subring of a function field $K\mid k$, with $S\neq K$, and let $\mathfrak{b}$ be an S-ideal such that $\mathfrak{b}\cdot\widetilde{S}=\widetilde{S}$. Then for each $\mathbf{n}\in\mathbb{Z}^m$*

$$\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\sum_{i=1}^m\mathbf{e}_i})+\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\sum_{i=1}^m\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}})\leq\sum_{i=0}^m r_i. \tag{2.3}$$

The equality holds for each $\mathbf{n}\in\mathbb{Z}^m$ if and only if $2\dim_k(\mathfrak{b}/\mathfrak{b}:\widetilde{S})=\dim_k(\widetilde{S}/\mathfrak{b}:\widetilde{S})$.

Proof. Let $\left(\mathbf{n}^{(j)}\right)_{1\leq j\leq m}$ be a strictly increasing sequence in $\mathbb{Z}^m$ such that $\mathbf{n}^{(0)}=\mathbf{n}$, $\mathbf{n}^{(m)}=\mathbf{n}+\sum_{i=1}^m\mathbf{e}_i$. Then, for each $j=1,\cdots,m$ there exists a unique $i\,(j)\in\{1,\cdots,m\}$ satisfying $\mathbf{n}^{(j)}-\mathbf{n}^{(j-1)}=e_{i(j)}$. Thus,

$$\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(0)}}\supseteq\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(1)}}\supseteq\cdots\supseteq\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(m)}}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\sum_{i=1}^m\mathbf{e}_i}$$

and

$$\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(0)}}\subseteq\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(1)}}\subseteq\cdots\subseteq\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(m)}}=\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\sum_{i=1}^m\mathbf{e}_i}.$$

Hence,

$$\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\sum_{i=1}^m\mathbf{e}_i})=\sum_{j=0}^{m-1}\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(j)}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(j)}+\mathbf{e}_{i(j+1)}})$$

and

$$\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\sum_{i=1}^m\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}})=\sum_{j=0}^{m-1}\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j+1)}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j+1)}+\mathbf{e}_{i(j+1)}}).$$

Since $\mathbf{n}^{(j+1)}-\mathbf{n}^{(j)}=e_{i(j+1)}$ for each $j=0,\cdots,m-1$, it follows by Proposition 23 that

$$\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\sum_{i=1}^m\mathbf{e}_i})+\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}-\sum_{i=1}^m\mathbf{e}_i}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}})\leq\sum_{j=0}^{m-1}r_{i(j+1)}=\sum_{i=0}^{m}r_i.$$

Assume that we have the equality in 2.3 for each $\mathbf{n}\in\mathbb{Z}^m$. Let $\mathbf{n}\in\mathbb{Z}^m$ and let $i\in\{1,\cdots,m\}$. Let us consider a strictly increasing sequence $\left(\mathbf{n}^{(j)}\right)_{1\leq j\leq m}$ in $\mathbb{Z}^m$ such that $\mathbf{n}^{(0)}=\mathbf{n}$, $\mathbf{n}^{(1)}=\mathbf{n}+\mathbf{e}_i$, and $\mathbf{n}^{(m)}=\mathbf{n}+\sum_{i=1}^m\mathbf{e}_i$. Then, by de above formula, $\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(j)}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}^{(j)}+\mathbf{e}_{i(j+1)}})+\dim_k(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j+1)}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}^{(j+1)}+\mathbf{e}_{i(j+1)}})=r_{i(j+1)}$ for each $j=0,\cdots,m-1$. Thus, $\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right)+\dim_k\left(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}}/\mathfrak{b}\cap\mathfrak{p}^{\mathbf{b}-\mathbf{n}+\mathbf{e}_i}\right)=r_i$, and so the result follows from Theorem 25.

Conversely, assume that $2\dim_k(\mathfrak{b}/\mathfrak{b}:\widetilde{S})=\dim_k(\widetilde{S}/\mathfrak{b}:\widetilde{S})$. From the previous formula and Theorem 25 we conclude that the equality in 2.3 holds. ∎

In particular, by choosing $\mathfrak{b}=S$, we obtain the following corollary:

Corollary 27 *Let S be a semilocal subring of a function field $K\mid k$, with $S\neq K$. The following properties are equivalent:*

1. *S is a Gorenstein ring.*
2. *$2\dim_k(S/S:\widetilde{S})=\dim_k(\widetilde{S}/S:\widetilde{S})$.*
3. *$\dim_k\left(S\cap\mathfrak{p}^{\mathbf{n}}/S\cap\mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}\right)+\dim_k\left(S\cap\mathfrak{p}^{\mathbf{f}-\mathbf{n}-\mathbf{e}_i}/S\cap\mathfrak{p}^{\mathbf{f}-\mathbf{n}}\right)=r_i$ for each $i=1,\cdots,m$.*
4. *$\dim_k\left(S\cap\mathfrak{p}^{\mathbf{n}}/S\cap\mathfrak{p}^{\mathbf{n}+\sum_{i=1}^m\mathbf{e}_i}\right)+\dim_k\left(S\cap\mathfrak{p}^{\mathbf{f}-\mathbf{n}-\sum_{i=1}^m\mathbf{e}_i}/S\cap\mathfrak{p}^{\mathbf{f}-\mathbf{n}}\right)=\sum_{i=0}^{m}r_i$ where $\mathbf{f}=\mathbf{v}(S:\widetilde{S})$ is the multi-exponent of the conductor ideal of S in its integral closure $\widetilde{S}$.*

It is known (see [10], Proposition 1.2 page. 2942) that if $\mathcal{O}$ is a residually rational local subring of $K\mid k$ and the field k has more than m elements, then the semigroup $S(\mathcal{O})$ satisfies the following properties:

1. If $\mathbf{n}=(n_1,\cdots,n_m)$ and $\mathbf{s}=(s_1,\cdots,s_m)$ are elements of $S(\mathcal{O})$ then the vector whose coordinates are $\min\{n_i,s_i\}$ also belongs to $S(\mathcal{O})$
2. If $\mathbf{n}=(n_1,\cdots,n_m)$ and $\mathbf{s}=(s_1,\cdots,s_m)$ are elements of $S(\mathcal{O})$ and $n_i=s_i$ for some i, then there exists a vector $\mathbf{t}=(t_1,\cdots,t_m)$ in $S(\mathcal{O})$ such that $t_i>n_i$ and $t_j\geq\min\{n_j,s_j\}$ for each j and $t_j=\min\{n_j,s_j\}$ whenever $n_j\neq s_j$.

3. There exists $\mathbf{f} \in \mathbb{N}^m$ such that $S(\mathcal{O}) \supseteq \mathbf{f} + \mathbb{N}^m$.

4. $\mathbf{n} \in S(\mathcal{O})$ if and only if $\dim_k(\mathcal{O}/\mathcal{O} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}) = \dim_k(\mathcal{O}/\mathcal{O} \cap \mathfrak{p}^{\mathbf{n}}) + 1$ for each $i = 1, \cdots, m$.

Now we prove that some of these properties are also satisfied by the set $S(\mathfrak{b})$, when $\mathfrak{b}$ is an S-ideal of a semilocal ring S.

Proposition 28 *Let S be a semilocal subring of a function field $K \mid k$, with $S \neq K$, and let $\mathfrak{b}$ be an S-ideal. Assume that the field k has more than m elements. If $n = (n_1, \cdots, n_m)$ and $\mathbf{s} = (s_1, \cdots, s_m)$ are elements of $S(\mathfrak{b})$ then the vector whose coordinates are* $\min\{n_i, s_i\}$ *also belongs to* $S(\mathfrak{b})$.

Proof. Since $S(\mathfrak{b})$ only depends on the ideal class $[\mathfrak{b}]$ we can assume that $\mathfrak{b} \cdot \widetilde{S} = \widetilde{S}$ so that $\mathfrak{f} \subseteq \mathfrak{b} \subseteq \widetilde{S}$, where $\mathfrak{f} = (S : \widetilde{S})$ is the conductor ideal of S. Let x and y be elements of $\mathfrak{b}$ such that $\mathbf{n} = (v_1(x), \cdots, v_m(x))$ and $\mathbf{s} = (v_1(y), \cdots, v_m(y))$. If $n_i \neq s_i$ for each $i = 1, \cdots, m$, then $v_i(x+y) = \min\{v_i(x), v_i(y)\}$ for each $i = 1, \cdots, m$. This proves the proposition in this case. Assume (eventually renumbering the indexes) that $n_1 \neq s_1, \cdots, n_{j-1} \neq s_{j-1}$ and $n_j = s_j, n_{j+1} = s_{j+1}, \cdots, n_m = s_m$, for some $j > 1$. Since $\mathfrak{b} \subseteq \widetilde{S}$ and $x, y \in \mathfrak{b}$, it follows for each $i = j, \cdots, m$ that $x = \pi_i^{n_i} u_i \in \mathcal{O}_{v_i}$ and $y = \pi_i^{n_i} w_i \in \mathcal{O}_{v_i}$, where $\pi_i \in \mathcal{O}_{v_i}$ is a local parameter of $\mathcal{O}_{v_i}$ and $u_i, w_i \in \mathcal{O}_{v_i} \backslash \mathfrak{m}_{v_i}$. If the classes $u_i + \mathfrak{m}_{v_i}$ and $w_i + \mathfrak{m}_{v_i}$ are linearly independent over k then $u_i + \alpha w_i \notin \mathfrak{m}_{v_i}$ for all $\alpha \in k$. On the other hand if $u_i + \mathfrak{m}_{v_i}$ and $w_i + \mathfrak{m}_{v_i}$ are linearly dependent then $u_i + \alpha_i w_i \in \mathfrak{m}_{v_i}$ for some $\alpha_i \in k$. Thus, since $\#(k) > m$, we can choose a constant $\alpha \in k \backslash \{0\}$ such that $u_i + \alpha w_i \notin \mathfrak{m}_{v_i}$ for each $i = j, \cdots, m$ (by choosing $\alpha \in k \backslash \{0\}$ such that $\alpha \neq \alpha_i$ whenever $u_i + \alpha_i w_i \in \mathfrak{m}_{v_i}$). Hence, for each $i = j, \cdots, m$ we have $v_i(x + \alpha y) = v_i(\pi_i^{n_i}(u_i + \alpha w_i)) = n_i + v_i(u_i + \alpha w_i) = n_i = \min\{n_i, s_i\}$. Therefore, there exists $\alpha \in k \backslash \{0\}$ such that $v_i(x + \alpha y) = \min\{v_i(x), v_i(y)\}$ for each $i = 1, \cdots, m$. So the proposition is valid. ■

Observe that, if the field k has more than m elements, then $\mathbf{n} \in S(\mathfrak{b})$ if and only if $l(\mathfrak{b}, \mathbf{n}) < l(\mathfrak{b}, \mathbf{n} + \mathbf{e}_i)$ for each $i = 1, \cdots, m$, that is, $\dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}) > 0$ for each $i = 1, \cdots, m$. In fact, if $\mathbf{n} \in S(\mathfrak{b})$, then we can choose $x \in \mathfrak{b} \backslash \{0\}$ such that $\mathbf{v}(x) = \mathbf{n}$, hence $x \in \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}}$ and $x \notin \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}$ for each $i = 1, \cdots, m$, i.e. $\dim_k(\mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{e}_i}) > 0$ for each $i = 1, \cdots, m$. The converse follows from the minimum property in Proposition 28. In particular, if S is residually rational, then $S(\mathfrak{b})$ satisfies (4) too.

$S(\mathfrak{b})$ does not satisfy the previous property (2) without the assumption that the semilocal ring S is residually rational. For example, if

$$S(\mathcal{O}) = \{(0,0), (1,1)\} \cup ((2,1) + \mathbb{N} \times \{0\}) \cup ((2,2) + \mathbb{N} \times \mathbb{N})$$

(cf. See Example 48), then $(1,1)$ and $(2,1)$ belong to $S(\mathcal{O})$, but does not exists $(1,t_2) \in S(\mathcal{O})$ such that $t_2 > 1$. Nevertheless, when S is a residually rational semilocal subring of $K \mid k$, then $S(\mathfrak{b})$ satisfies (2). Indeed, since $S(\mathfrak{b})$ only depends on the ideal class $[\mathfrak{b}]$, we can assume that $\mathfrak{b} \cdot \widetilde{S} = \widetilde{S}$. Hence if $\mathbf{n} = (n_1, \cdots, n_m)$ and $\mathbf{s} = (s_1, \cdots, s_m)$ are elements of $S(\mathfrak{b})$, then there exist x and y in $\mathfrak{b}$ such that $n= (v_1(x), \cdots, v_m(x))$ and $\mathbf{s} = (v_1(y), \cdots, v_m(y))$. Assume that $n_i = s_i$ for some i, that is $v_i(x/y) = 0$ for some i. By the assumption, the ring S is residually rational. Then there exists $\alpha \in k$ such that $v_i(\alpha + x/y) > 0$. Hence, $v_i(x + \alpha y) = v_i(y(\alpha + x/y)) > v_i(y)$. From the properties of the valuation functions we also get that $v_j(x + \alpha y) \geq \min\{n_j, s_j\}$ for each $j \neq i$ (and that the equality holds if $n_j \neq s_j$).

Chapter 3

Zeta functions and Multi-variable Poincaré series associated to semilocal rings of a geometrically integral algebraic curve defined over a finite field.

3.1 Introduction

Throughout this chapter we start to treat one of our purpose of this work. In fact, we want to extend the definitions of the series $Z(\mathfrak{a}, t)$, $Z(\mathfrak{a}, \mathfrak{b}, t)$ and $P(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ to the series $Z_S(\mathfrak{a}, t)$, $Z_S(\mathfrak{a}, \mathfrak{b}, t)$ and $P_S(\mathfrak{a}, \mathfrak{b}, t_1, \ldots, t_m)$ associated to S-ideals $\mathfrak{a}$ and $\mathfrak{b}$, where S is a semilocal ring of a geometrically integral algebraic curve X defined over a field $\mathbb{F}_q$ of q elements. The extended definitions $Z_S(\mathfrak{a}, t)$, $Z_S(\mathfrak{a}, \mathfrak{b}, t)$ and $P_S(\mathfrak{a}, \mathfrak{b}, t_1, \ldots, t_m)$ are important because they permit us to study the behavior of these series under constant field extensions: if $\mathcal{O}$ is a local ring of a geometrically integral algebraic curve defined over a field $k = \mathbb{F}_q$ whose function field in one variable is $K \mid k$, and if k' is a finite field extension of k, then $k' \cdot K \mid k'$ is also a function field in one variable and $k' \cdot \mathcal{O}$ is a semilocal subring of $k' \cdot K \mid k'$, where $k' \cdot \mathcal{O}$ just consists of all linear combination of elements of the local ring $\mathcal{O}$ with coefficients in the field k' (cf. [22] section 3). The extended definitions of zeta functions and multi-variable Poincaré series are also important because they permit us to study the behavior of these series with respect to blow-up of the local ring $\mathcal{O}$, since the blow-up of a local ring $\mathcal{O}$ with respect to its maximal ideal $\mathfrak{m}$ is a semilocal ring $\mathcal{O}^{\mathfrak{m}}$ (cf. [20] Chapter VIII). Furthermore, they permit us to associate to a geometrically integral algebraic curve X defined over a finite field $\mathbb{F}_q$ of q elements the multi-variable Poincaré series $P_S(S, S; t_1, \ldots, t_m)$, where S is a semilocal ring which is contained in the *semilocal ring of the curve* X defined as the intersection of all the local rings corresponding to singular points of X.

3.2 Semilocal zeta functions and multi-variable Poincaré series

Let S be a semilocal ring of a geometrically integral algebraic curve X defined over a field $\mathbb{F}_q$ of q elements, that is, S is a semilocal subring of the function field $K \mid k$ whose expression as intersection of local rings given by Theorem 14 is $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$ where each $\mathcal{O}_j$ is a local ring of the curve X. Let $v_1, \cdots, v_m$ be the valuations of $K \mid k$ whose valuation rings contain S.

Following the same ideas as in [26] and [27], we consider for any S-ideal $\mathfrak{a}$ the *semi-local zeta function*

$$\zeta_S(\mathfrak{a}, z) := \sum_{\mathfrak{d} \supseteq \mathfrak{a}} \#(\mathfrak{d}/\mathfrak{a})^{-z}, \ \Re(z) > 0 \tag{3.1}$$

where the sum is taken over all S-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$. This series can be written as power series $Z_S(\mathfrak{a}, t) \in \mathbb{Z}[[t]]$ in $t := q^{-z}$ with integer coefficients. In a similar way, we consider the *partial semi-local zeta function*

$$\zeta_S(\mathfrak{a}, \mathfrak{b}, z) := \sum_{\mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b}} \#(\mathfrak{d}/\mathfrak{a})^{-z}, \quad \Re(z) > 0 \tag{3.2}$$

where the sum is taken over all S-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$. Those series only depend on the ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$ and they can be written as power series $Z_S(\mathfrak{a}, \mathfrak{b}, t) \in \mathbb{Z}[[t]]$ in $t := q^{-z}$, with integer coefficients, which converge in the unit disk $|t| < 1$. Moreover, as the local zeta functions, the semi-local zeta function may be expressed as a finite sum of the partial semi-local zeta functions.

Since the integral closure $\widetilde{S} = \mathcal{O}_{v_1} \cap \cdots \cap \mathcal{O}_{v_m}$ of the semilocal ring S is a semilocal principal ideal domain, whose maximal ideals, say $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$, correspond bijectively to the valuations $v_1, \cdots, v_m$, we have observed in the previous chapter that each $\widetilde{S}$-ideal may be expressed as $\mathfrak{p}^{\mathbf{n}} := \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}$, where $\mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m$, and that its multi-exponent is $\mathbf{v}(\mathfrak{p}^{\mathbf{n}}) := \mathbf{n}$. Then, similarly, we can extend to semilocal rings of geometrically irreducible curves Definition 5 of multi-variable Poincaré series defined in [27] for local rings of geometrically irreducible curves. For each non-zero rational function $z \in K^*$ and for each $\mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m$ we abbreviate by

$$\mathbf{v}(z) := \mathbf{v}(z\widetilde{S}) = (v_1(z), \cdots, v_m(z)) \in \mathbb{Z}^m$$

the multi-valuation, and by

$$\mathbf{t}^{\mathbf{n}} := t_1^{n_1} \cdots t_m^{m},$$

the Laurent monomial, respectively.

Definition 29 *Associated to the semi-local ring S and to the pair of S-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$ it is defined the multi-variable Poincaré series*

$$P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) := \sum_{\mathfrak{d} \supset \mathfrak{a}, \mathfrak{d} \sim \mathfrak{b}} \mathbf{t}^{\mathbf{v}(\mathfrak{a}\widetilde{\mathcal{O}}) - \mathbf{v}(\mathfrak{d}\widetilde{\mathcal{O}})}$$

where the sum is taken over all S-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$.

In order to prove the relation between partial semilocal zeta function and semilocal multi-variable Poincaré series as well as to prove the reduction to the case $\mathfrak{a} = S$, we observe that for each pair of S-ideals $\mathfrak{a}$ and $\mathfrak{b}$ the S-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$ are of the form $z^{-1}\mathfrak{b}$, where z varies over a complete system of representatives of $(\mathfrak{b} : \mathfrak{a}) \backslash \{0\}$ modulo the action of the group $U_{\mathfrak{b}}$, as in the local case (cf. [27] sections 2 and 3). Thus, from Definition 3.2 we obtain

$$Z_S(\mathfrak{a}, \mathfrak{b}, t) = \sum_{z \in \mathfrak{b}:\mathfrak{a} \backslash \{0\} / U_{\mathfrak{b}}} t^{\dim_k (z^{-1}\mathfrak{b}/\mathfrak{a})} \tag{3.3}$$

and from Definition 29 we deduce

$$P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \sum_{z \in \mathfrak{b}:\mathfrak{a} \backslash \{0\} / U_{\mathfrak{b}}} \mathbf{t}^{\mathbf{v}(\mathfrak{a} \cdot \widetilde{S}) - \mathbf{v}(\mathfrak{b} \cdot \widetilde{S}) + \mathbf{v}(z)}. \tag{3.4}$$

In the sequel, let

$$\mathfrak{d}_{\mathbf{n}} := \{z \in \mathfrak{d} : \mathbf{v}(z) = \mathbf{n}\},$$

where $\mathfrak{d}$ is an S-ideal and $\mathbf{n} \in \mathbb{Z}^m$ is an integral vector.

It follows that, if we restrict the action of $U_{\mathfrak{b}}$ to U_S in Formula 3.4, then we have to divide by the index $[U_{\mathfrak{b}} : U_S]$. Hence

$$P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \sum_{\mathbf{n} \in \mathbb{Z}^m} \frac{\#((\mathfrak{b} : \mathfrak{a})_{\mathbf{n}} / U_S)}{[U_{\mathfrak{b}} : U_S]} \mathbf{t}^{\mathbf{v}(\mathfrak{a} \cdot \widetilde{S}) - \mathbf{v}(\mathfrak{b} \cdot \widetilde{S}) + \mathbf{n}}. \tag{3.5}$$

Theorem 30 *The following identity holds;*

$$Z_S(\mathfrak{a}, \mathfrak{b}, t) = t^{\dim_k(\mathfrak{a}\cdot\widetilde{S}/\mathfrak{a}) - \dim_k(\mathfrak{b}\cdot\widetilde{S}/\mathfrak{b})} P_S(\mathfrak{a}, \mathfrak{b}, t^{r_1}, \ldots, t^{r_m})$$

where $r_1 := \dim_k(\widetilde{S}/\mathfrak{p}_1), \cdots, r_m := \dim_k(\widetilde{S}/\mathfrak{p}_m)$. *This means that the partial local-zeta function may be expressed in terms of the multi-variable Poincaré series.*

Proof. By Formula 3.3,

$$Z_S(\mathfrak{a}, \mathfrak{b}, t) = \sum_{z \in (\mathfrak{b}:\mathfrak{a})\setminus\{0\}/U_\mathfrak{b}} t^{\deg_S(z^{-1}\mathfrak{b}) - \deg_S(\mathfrak{a})}$$

and, by Formula 3.4,

$$P_S(\mathfrak{a}, \mathfrak{b}, t^{r_1}, \ldots, t^{r_m}) = \sum_{z \in (\mathfrak{b}:\mathfrak{a})\setminus\{0\}/U_\mathfrak{b}} t^{\mathbf{r}\cdot\mathbf{v}(\mathfrak{a}\cdot\widetilde{S}) - \mathbf{r}\cdot\mathbf{v}(\mathfrak{b}\cdot\widetilde{S}) + \mathbf{r}\cdot\mathbf{v}(z)}$$

where $\mathbf{r}\cdot\mathbf{n} := \sum_{i=1}^{m} r_i n_i$ for each $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$. The Chinese remainder theorem now yields

$$\dim_k(\widetilde{S}/\mathfrak{p}^{\mathbf{n}}) = \mathbf{r}\cdot\mathbf{n} \text{ for each } \mathbf{n} \in \mathbb{N}^m$$

and, hence,

$$\deg_S(\mathfrak{p}^{\mathbf{n}}) = \deg_S(\widetilde{S}) - \mathbf{r}\cdot\mathbf{n} \text{ for each } \mathbf{n} \in \mathbb{Z}^m.$$

We thus get

$$P_S(\mathfrak{a}, \mathfrak{b}, t^{r_1}, \ldots, t^{r_m}) = \sum_{z \in (\mathfrak{b}:\mathfrak{a})\setminus\{0\}/U_\mathfrak{b}} t^{-\deg_S(\mathfrak{a}\cdot\widetilde{S}) + \deg_S(\mathfrak{b}\cdot\widetilde{S}) + \mathbf{r}\cdot\mathbf{v}(z)}.$$

So, to prove the theorem we must prove

$$\dim_k(\mathfrak{a}\widetilde{S}/\mathfrak{a}) - \dim_k(\mathfrak{b}\widetilde{S}/\mathfrak{b}) - \deg_S(\mathfrak{a}\widetilde{S}) + \deg_S(\mathfrak{b}\widetilde{S}) + \mathbf{r}\cdot\mathbf{v}(z) = \deg_S(z^{-1}\mathfrak{b}) - \deg_S(\mathfrak{a})$$

i.e.

$$\deg_S(\mathfrak{b}) - \deg_S\left(z^{-1}\mathfrak{b}\right) = \mathbf{r}\cdot\mathbf{v}\left(z^{-1}\right).$$

But, the last equality follows from the local case (cf. Lemma 7) and from Corollary 18. ■

Theorem 31 ***(Reduction to the case*** $\mathfrak{a} = S$*) Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be* S*-ideals. Then*

$$P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = [U_{\mathfrak{b}:\mathfrak{a}} : U_\mathfrak{b}]\mathbf{t}^{\mathbf{v}((\mathfrak{b}:\widetilde{S}):\mathfrak{b}\cdot\widetilde{S}) - \mathbf{v}(((\mathfrak{b}:\mathfrak{a}):\widetilde{S}):(\mathfrak{b}:\mathfrak{a})\cdot\widetilde{S})} P_S(S, \mathfrak{b}:\mathfrak{a}, \mathbf{t})$$

and

$$\mathbf{0} \le \mathbf{b} - \mathbf{d} = \mathbf{v}((\mathfrak{b}:\mathfrak{a})\cdot\widetilde{S}) + \mathbf{v}(\mathfrak{a}\cdot\widetilde{S}) - \mathbf{v}(\mathfrak{b}\cdot\widetilde{S}) \le \mathbf{b}$$

where $\mathfrak{d} := \mathfrak{b} : \mathfrak{a}$, $\mathbf{b} := ((\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\cdot\widetilde{S})$ *and* $\mathbf{d} := ((\mathfrak{d} : \widetilde{S}) : \mathfrak{d}\cdot\widetilde{S})$.

Proof. From Formula 3.4 we deduce

$$P_S(\mathfrak{a},\mathfrak{b},\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^m} \#((\mathfrak{b}:\mathfrak{a})_{\mathbf{n}}/U_{\mathfrak{b}})\mathbf{t}^{\mathbf{v}(\mathfrak{a}\cdot\widetilde{S})-\mathbf{v}(\mathfrak{b}\cdot\widetilde{S})+\mathbf{n}} \tag{3.6}$$

and

$$P_S(S,\mathfrak{b}:\mathfrak{a},\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^m} \#((\mathfrak{b}:\mathfrak{a})_{\mathbf{n}}/U_{\mathfrak{b}:\mathfrak{a}})\mathbf{t}^{-\mathbf{v}((\mathfrak{b}:\mathfrak{a})\cdot\widetilde{S})+\mathbf{n}}.$$

If we restrict the action of $U_{\mathfrak{b}:\mathfrak{a}}$ to $U_{\mathfrak{b}}$, then we have to divide by the index $[U_{\mathfrak{b}:\mathfrak{a}}:U_{\mathfrak{b}}]$. Hence

$$P_S(S,\mathfrak{b}:\mathfrak{a},\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^m} \frac{\#((\mathfrak{b}:\mathfrak{a})_{\mathbf{n}}/U_{\mathfrak{b}})}{[U_{\mathfrak{b}:\mathfrak{a}}:U_{\mathfrak{b}}]}\mathbf{t}^{-\mathbf{v}((\mathfrak{b}:\mathfrak{a})\cdot\widetilde{S})+\mathbf{n}}. \tag{3.7}$$

We now obtain the first result of this theorem by comparing the coefficients of the two series in Formula 3.6 and Formula 3.7. On the other hand, since $(\mathfrak{b}:\mathfrak{a})\cdot\mathfrak{a}\subseteq\mathfrak{b}$, it follows that $((\mathfrak{b}:\mathfrak{a})\cdot\widetilde{S})\cdot(\mathfrak{a}\cdot\widetilde{S})\subseteq\mathfrak{b}\cdot\widetilde{S}$, that is, the multi-exponent $\mathbf{b}-\mathbf{d}\geq\mathbf{0}$. Moreover,

$$(\mathfrak{b}:\mathfrak{a}):\widetilde{S} = (\mathfrak{b}:\widetilde{S}):(\mathfrak{a}\cdot\widetilde{S}).$$

Indeed, $z\in(\mathfrak{b}:\widetilde{S}):(\mathfrak{a}\cdot\widetilde{S})$ if and only if $z(\mathfrak{a}\cdot\widetilde{S})\subseteq\mathfrak{b}$, i.e. $z\widetilde{S}\subseteq\mathfrak{b}:\mathfrak{a}$, i.e. $z\in(\mathfrak{b}:\mathfrak{a}):\widetilde{S}$. Thus, $\mathbf{v}((\mathfrak{b}:\mathfrak{a}):\widetilde{S})=\mathbf{v}(\mathfrak{b}:\widetilde{S})-\mathbf{v}(\mathfrak{a}\cdot\widetilde{S})$, hence $\mathbf{b}-\mathbf{d}=\mathbf{v}((\mathfrak{b}:\mathfrak{a})\cdot\widetilde{S})+\mathbf{v}(\mathfrak{a}\cdot\widetilde{S})-\mathbf{v}(\mathfrak{b}\cdot\widetilde{S})$ and therefore $\mathbf{b}-\mathbf{d}\leq\mathbf{b}$. ■

The previous theorem justifies that, from now on, we will sometimes assume that the S-ideal $\mathfrak{a}$ is equal to the semilocal ring S, as in the local case.

Corollary 32 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be S-ideals. Then*

$$Z_S(\mathfrak{a},\mathfrak{b},t) = [U_{\mathfrak{b}:\mathfrak{a}}:U_{\mathfrak{b}}]t^{\deg_S(\mathfrak{b})-\deg_S(\mathfrak{a})-\deg_S(\mathfrak{b}:\mathfrak{a})}Z_S(S,\mathfrak{b}:\mathfrak{a},t)$$

3.3 Euler product identities

Now, we establish the link between the local and semilocal theory.

Lemma 33 ***Euler product identity***

$$\zeta_S(\mathfrak{a},z) = \prod_{j=1}^{s}\zeta_{\mathcal{O}_j}(\mathfrak{a}\mathcal{O}_j,z)$$

for each S-ideal $\mathfrak{a}$.

Proof. From Proposition 17, $\mathfrak{d} = \mathfrak{d}\mathcal{O}_1 \cap \cdots \cap \mathfrak{d}\mathcal{O}_s$ and $\dim(\mathfrak{d}/\mathfrak{a}) = \sum_{j=1}^{s} \dim(\mathfrak{d}\mathcal{O}_j/\mathfrak{a}\mathcal{O}_j)$ for each S-ideal $\mathfrak{d}$ that contains $\mathfrak{a}$. On the other hand, $\#(\mathfrak{d}/\mathfrak{a}) = q^{\dim(\mathfrak{d}/\mathfrak{a})}$ and $\#(\mathfrak{d}\mathcal{O}_j/\mathfrak{a}\mathcal{O}_j) = q^{\dim(\mathfrak{d}\mathcal{O}_j/\mathfrak{a}\mathcal{O}_j)}$ for each $j = 1, \cdots, s$. Thus $\#(\mathfrak{d}/\mathfrak{a}) = \prod_{j=1}^{s} \#(\mathfrak{d}\mathcal{O}_j/\mathfrak{a}\mathcal{O}_j)$. By Proposition 19, if $\mathfrak{d}_j$ is an $\mathcal{O}_j$-ideal, for each $j = 1, \cdots, s$, then there exists an S-ideal $\mathfrak{d}$ such that $\mathfrak{d}\mathcal{O}_j = \mathfrak{d}_j$ for each $j = 1, \cdots, s$. So the product identity follows from this. ■

Theorem 34 ***(Euler product identity of partial zeta functions)*** *Let S be a semilocal ring of a geometrically integral algebraic curve X, defined over a field $\mathbb{F}_q$ of q elements whose expression as intersection of local rings is $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$, where $\mathcal{O}_j$ is a local ring of the curve X for $j = 1, \cdots, s$. Then*

$$Z_S(\mathfrak{a}, \mathfrak{b}, t) = \prod_{j=1}^{s} Z_{\mathcal{O}_j}(\mathfrak{a}\mathcal{O}_j, \mathfrak{b}\mathcal{O}_j, t) \text{ for each pair of } S\text{-ideals } \mathfrak{a} \text{ and } \mathfrak{b}.$$

Proof. We prove the particular case $\mathfrak{a} = S$. The general case follows from the case $\mathfrak{a} = S$, by applying Corollary 18 and Corollary 21. By Formula 3.3,

$$Z_S(S, \mathfrak{b}, t) = \sum_{z \in \mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}} t^{\deg_S(z^{-1}\mathfrak{b}) - \deg_S(\mathfrak{a})}$$

where $\mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}$ denotes the quotient of $\mathfrak{b}\backslash\{0\}$ by the action of the group $U_{\mathfrak{b}}$ and z varies over a complete system of representatives of $\mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}$. Similarly, by Formula 3.3,

$$Z_{\mathcal{O}_j}(\mathcal{O}_j, \mathfrak{b}\mathcal{O}_j, t) = \sum_{z \in \mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}} t^{\deg_{\mathcal{O}_j}(z^{-1}\mathfrak{b}\mathcal{O}_j) - \deg_{\mathcal{O}_j}(\mathfrak{a}\mathcal{O}_j)} \text{ for each } j = 1, \cdots, s,$$

where $\mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}$ denotes the quotient of $\mathfrak{b}\mathcal{O}_j\backslash\{0\}$ by the action of the group $U_{\mathfrak{b}\mathcal{O}_j}$ and z varies over a complete system of representatives of $\mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}$. By Proposition 20, the assignment $zU_{\mathfrak{b}} \longmapsto (zU_{\mathfrak{b}\mathcal{O}_1}, \cdots, zU_{\mathfrak{b}\mathcal{O}_s})$ defines a bijection between the quotient $\mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}$ and the product of quotients $(\mathfrak{b}\mathcal{O}_1\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_1} \times \cdots \times (\mathfrak{b}\mathcal{O}_s\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_s}$. Now, we conclude from Corollary 18 that

$$\deg_S(z^{-1}\mathfrak{b}) - \deg_S(\mathfrak{a}) = \sum_{j=1}^{s} \deg_{\mathcal{O}_j}(z^{-1}\mathfrak{b}\mathcal{O}_j) - \deg_{\mathcal{O}_j}(\mathfrak{a}\mathcal{O}_j),$$

hence, that

$$\sum_{z \in \mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}} t^{\deg_S(z^{-1}\mathfrak{b}) - \deg_S(\mathfrak{a})} = \prod_{j=1}^{s} \sum_{z_j \in \mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}} t^{\deg_{\mathcal{O}_j}(z^{-1}\mathfrak{b}\mathcal{O}_j) - \deg_{\mathcal{O}_j}(\mathfrak{a}\mathcal{O}_j)}$$

and finally that the Euler product identity for partial zeta functions holds. ■

In the following result we prove the link between the multi-variable Poincaré series associated to a semilocal ring and those series of its several local ring components.

Theorem 35 ***(Euler product identity of Poincaré series)*** *Let S be a semilocal ring of a geometrically integral algebraic curve X, defined over a finite field $\mathbb{F}_q$ of q elements whose expression as intersection of local rings is $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$, where $\mathcal{O}_j$ is a local ring of the curve X for $j = 1, \cdots, s$. Then*

$$P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \prod_{j=1}^{s} P_{\mathcal{O}_j}(\mathfrak{a}\mathcal{O}_j, \mathfrak{b}\mathcal{O}_j, \mathbf{t}_j) \text{ for each pair of } S\text{-ideals } \mathfrak{a} \text{ and } \mathfrak{b}$$

where $\mathbf{t} := (\mathbf{t}_1, \cdots, \mathbf{t}_s)$ *and* $\mathbf{t}_j := \left(t_1, \cdots, t_{m_j}\right)$ *for* $j = 1, \cdots, s$.

Proof. We can obtain this identity from the case $\mathfrak{a} = S$, and from Corollary 21. Thus we assume that $\mathfrak{a} = S$. We have, from Formula 3.4, that

$$P_S(S, \mathfrak{b}, \mathbf{t}) = \sum_{z \in \mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}} \mathbf{t}^{-\mathbf{v}(\mathfrak{b}\cdot\widetilde{S}) + \mathbf{v}(z)}$$

and

$$P_{\mathcal{O}_j}(\mathcal{O}_j, \mathfrak{b}\mathcal{O}_j, \mathbf{t}) = \sum_{z \in \mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}} \mathbf{t}_j^{-\mathbf{v}(\mathfrak{b}\mathcal{O}_j\cdot\widetilde{\mathcal{O}_j}) + \mathbf{v}_j(z)}, \text{ for each } j = 1, \cdots, s$$

where, as in the previous theorem, $\mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}$ and $\mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}$ denote the quotient of $\mathfrak{b}\backslash\{0\}$ and $\mathfrak{b}\mathcal{O}_j\backslash\{0\}$ by the action of the group $U_{\mathfrak{b}}$ and $U_{\mathfrak{b}\mathcal{O}_j}$, respectively; and z varies over a complete system of representatives of $\mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}$ and $\mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}$, respectively. From Proposition 20, the assignment $zU_{\mathfrak{b}} \longmapsto (zU_{\mathfrak{b}\mathcal{O}_1}, \cdots, zU_{\mathfrak{b}\mathcal{O}_s})$ define a bijection between $\mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}$ and the product $(\mathfrak{b}\mathcal{O}_1\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_1} \times \cdots \times (\mathfrak{b}\mathcal{O}_s\backslash\{0\})/U_{\mathfrak{b}\mathcal{O}_s}$ such that $\mathbf{v}(z) = (\mathbf{v}_1(z), \cdots, \mathbf{v}_s(z))$ for each representative z of the class $zU_{\mathfrak{b}} \in \mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}$. Thus, due to

$$\mathbf{v}(\mathfrak{b}\cdot\widetilde{S}) = (\mathbf{v}_1(\mathfrak{b}\mathcal{O}_1\cdot\widetilde{\mathcal{O}_1}), \cdots, \mathbf{v}_s(\mathfrak{b}\mathcal{O}_s\cdot\widetilde{\mathcal{O}_s})),$$

we have

$$\sum_{z \in \mathfrak{b}\backslash\{0\}/U_{\mathfrak{b}}} \mathbf{t}^{-\mathbf{v}(\mathfrak{b}\cdot\widetilde{S}) + \mathbf{v}(z)} = \prod_{j=1}^{s} \sum_{z_j \in \mathfrak{b}\mathcal{O}_j\backslash\{0\}/U_{\mathfrak{b}\mathcal{O}_j}} \mathbf{t}_j^{-\mathbf{v}(\mathfrak{b}\mathcal{O}_j\cdot\widetilde{\mathcal{O}_j}) + \mathbf{v}_j(z_j)}.$$

So, the Euler product identity for Poincaré series holds. ■

As consequence of the previous theorem, we obtain for each S-ideal $\mathfrak{a}$ the multi-variable geometric series

$$P_S(\mathfrak{a}, \widetilde{S}, \mathbf{t}) = \frac{1}{(1-t_1)\cdots(1-t_m)}.$$

We observe that the Euler product identity of partial zeta functions can be obtained from the Euler product identity of multi-variable Poincaré series and from the relation between the partial zeta functions and multi-variable Poincaré series. Indeed, from the last one we obtain

$$Z_S\left(S,\mathfrak{b},t\right)=t^{\dim(\widetilde{S}/S)-\dim(\mathfrak{b}\cdot\widetilde{S}/\mathfrak{b})}P_S(S,\mathfrak{b},t^{r_1},\cdots,t^{r_m})$$

and

$$Z_{\mathcal{O}_j}\left(\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j,t\right)=t^{\dim(\widetilde{\mathcal{O}_j}/\mathcal{O}_j)-\dim(\mathfrak{b}\cdot\widetilde{\mathcal{O}_j}/\mathfrak{b}\mathcal{O}_j)}P_{\mathcal{O}_j}(\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j,t^{r_{j1}},\cdots,t^{r_{jm_j}})\text{ for }j=1,\cdots,s.$$

Thus, due to $\dim(\widetilde{S}/S)=\sum\limits_{j=1}^{s}\dim(\widetilde{\mathcal{O}_j}/\mathcal{O}_j)$ and $\dim(\mathfrak{b}\cdot\widetilde{S}/\mathfrak{b})=\sum\limits_{j=1}^{s}\dim(\mathfrak{b}\cdot\widetilde{\mathcal{O}_j}/\mathfrak{b}\mathcal{O}_j)$ we deduce the Euler identity for partial zeta functions from that for Poincaré series.

Similarly, we can prove the relation between the partial zeta functions and multi-variable Poincaré series from the Euler product identities of partial zeta functions and multi-variable Poincaré series. In fact,

$$\begin{aligned}
Z_S(S,\mathfrak{b},t)&=\prod_{j=1}^{s}Z_{\mathcal{O}_j}(\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j,t)\\
&=\prod_{j=1}^{s}t^{\dim(\widetilde{\mathcal{O}_j}/\mathcal{O}_j)-\dim(\mathfrak{b}\cdot\widetilde{\mathcal{O}_j}/\mathfrak{b}\mathcal{O}_j)}P_{\mathcal{O}_j}(\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j,t^{r_{j1}},\cdots,t^{r_{jm_j}})\\
&=t^{\sum_{j=1}^{s}(\dim(\widetilde{\mathcal{O}_j}/\mathcal{O}_j)-\dim(\mathfrak{b}\cdot\widetilde{\mathcal{O}_j}/\mathfrak{b}\mathcal{O}_j))}\prod_{j=1}^{s}P_{\mathcal{O}_j}(\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j,t^{r_{j1}},\cdots,t^{r_{jm_j}})\\
&=t^{\dim(\widetilde{S}/S)-\dim(\mathfrak{b}\cdot\widetilde{S}/\mathfrak{b})}P_S(S,\mathfrak{b},t^{r_1},\cdots,t^{r_m}).
\end{aligned}$$

Corollary 36 *Let $S_1,\cdots,S_n$ be semilocal subrings of the function field $K\mid\mathbb{F}_q$ such that no two of them are contained in the same valuation ring, and let $\mathfrak{b}$ be an S-ideal, where $S:=S_1\cap\cdots\cap S_n$. Then*

$$P_S(S,\mathfrak{b},\mathbf{t})=\prod_{j=1}^{n}P_{S_j}(S_j,\mathfrak{b}S_j,\mathbf{t}_j)$$

where $\mathbf{t}:=(\mathbf{t}_1,\cdots,\mathbf{t}_n)$ and $\mathbf{t}_j:=\left(t_{j1},\cdots,t_{jm_j}\right)$ for $j=1,\cdots,n$.

Proof. We obtain this corollary from Theorem 35, by induction over n. ■

The Poincaré series $P_S(\mathfrak{a},\mathfrak{b},\mathbf{t})$ is a power series in $t_1,\cdots,t_m$, say

$$P_S(\mathfrak{a},\mathfrak{b},\mathbf{t})=\sum\eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b})\mathbf{t}^{\mathbf{n}}\in\mathbb{Z}[t_1,\cdots,t_m],$$

that encodes the cardinalities of certain sets of S-ideals:

$$\eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b}) = \#\{S\text{-ideals } \mathfrak{d} \text{ satisfying } \mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b} \text{ and } \mathfrak{d}\cdot\widetilde{S} = \mathfrak{a}\cdot\mathfrak{p}^{-\mathbf{n}}\}.$$

It is clear, from Euler product and from Theorem 8, that these cardinalities are actually finite and that the radius of convergence of $P_S(\mathfrak{a},\mathfrak{b},\mathbf{t})$ is equal to one. Using the local properties of the multiple Poincaré series we can prove some properties of the multiple Poincaré series $P_S(\mathfrak{a},\mathfrak{b},\mathbf{t})$, for example its rationality. We can also investigate how it behaves if the S-ideals $\mathfrak{a}$ and $\mathfrak{b}$ are replaced by the dual S-ideals $\mathfrak{c}:\mathfrak{a}$ and $\mathfrak{c}:\mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$, respectively, which is expressed in a reciprocity formula. Furthermore, we can derive explicit formulae for the coefficients of this series.

Theorem 37 *The coefficients of the Poincaré series $P_S(\mathfrak{a},\mathfrak{b},\mathbf{t}) := \sum \eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b})\,\mathbf{t}^{\mathbf{n}}$ satisfy:*

1. $\eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b}) = \frac{\#\left((\mathfrak{b}:\mathfrak{a})_{\mathbf{j}}/U_S\right)}{[U_{\mathfrak{b}}:U_S]}$ *where* $\mathbf{j} = \mathbf{n} - \mathbf{v}(\mathfrak{a}\cdot\widetilde{S}) + \mathbf{v}(\mathfrak{b}\cdot\widetilde{S})$ *for each* $\mathbf{n} \in \mathbb{Z}^m$.
2. $\eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b}) > 0$ *if and only if* $\mathbf{n} \in S(\mathfrak{b})$.
3. $0 \le \eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b}) \le [U_{\widetilde{S}} : U_{\mathfrak{b}}]$ *for each* $\mathbf{n} \in \mathbb{Z}^m$.
4. $\mathbf{b}$ *is the smallest vector in the partial order of* $\mathbb{N}^m$ *with the following property: if* $\mathbf{n} \ge \mathbf{b}$ *then* $\eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b}) = [U_{\widetilde{S}} : U_{\mathfrak{b}}]$.

Proof. (1) From Formula 3.5 we obtain the first sentence.

Let $\mathbf{n} = (\mathbf{n}_1,\cdots,\mathbf{n}_s)$ be an integer vector in $\mathbb{Z}^{m_1}\times\cdots\times\mathbb{Z}^{m_s}$, where each m_j is the number of branches of the local ring $\mathcal{O}_j$. We have from the Euler product identity that

$$\eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b}) = \prod_{j=1}^{s} \eta_{\mathcal{O}_j,\mathbf{n}_j}(\mathfrak{a}\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j).$$

(2) So, $\eta_{S,\mathbf{n}}(\mathfrak{a},\mathfrak{b}) > 0$ if and only if $\eta_{\mathcal{O}_j,\mathbf{n}_j}(\mathfrak{a}\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j) > 0$ for each $j = 1,\cdots,s$. Hence, from Proposition 20, we obtain the second sentence by applying the local case (see [27], Theorem 3.2 (i)).

(3) By the local case (see [27], Theorem 3.2(ii)), $0 \le \eta_{\mathcal{O}_j,\mathbf{n}_j}(\mathfrak{a}\mathcal{O}_j,\mathfrak{b}\mathcal{O}_j) \le [U_{\widetilde{\mathcal{O}_j}} : U_{\mathfrak{b}\mathcal{O}_j}]$ for each $j = 1,\cdots,s$. Therefore we obtain the third claim from Corollary 21.

(4) Let $\mathbf{b}_j$ be the multi-exponent of the $\widetilde{\mathcal{O}_j}$-ideal $(\mathfrak{b}\mathcal{O}_j : \widetilde{\mathcal{O}_j}) : \mathfrak{b}\mathcal{O}_j$. We have that $\mathbf{b} = (\mathbf{b}_1,\cdots,\mathbf{b}_s)$ and, by the local case (see [27], Theorem 3.2(iii)), we have that $\mathbf{b}_j$

is the smallest vector in the partial order of $\mathbb{N}^{m_j}$ satisfying the following property: if $\mathbf{n}_j \geq \mathbf{b}_j$ then $\eta_{\mathcal{O}_j,\mathbf{n}_j}(\mathfrak{a}\mathcal{O}_j, \mathfrak{b}\mathcal{O}_j) = [U_{\widetilde{\mathcal{O}_j}} : U_{\mathfrak{b}\mathcal{O}_j}]$. Thus, the fourth claim follows from Corollary 21. ■

The semilocal ring $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$ has a finite number of maximal ideals, say $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$. Let us denote by

$$\varrho_j := \dim_k (\mathcal{O}_j / \mathfrak{m}_j \mathcal{O}_j)$$

and by

$$\delta_j := \dim_k (\widetilde{\mathcal{O}_j} / \mathcal{O}_j)$$

the degree of the residue field and the singularity degree of the local ring $\mathcal{O}_j$, respectively, for each $j = 1, \cdots, s$.

Theorem 38

$$\eta_{S,\mathbf{n}}(S, \mathfrak{b}) = \frac{q^{\sum_{j=1}^{s}(\delta_j + \varrho_j)}}{[U_{\mathfrak{b}} : U_S] \prod_{j=1}^{s} (q^{\varrho_j} - 1)} \sum_{\mathbf{i} \in \{0,1\}^m} (-1)^{|\mathbf{i}|} q^{-\dim_k \left(\mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{i}}\right)}.$$

where the sum is taken over the vectors $\mathbf{i} = (i_1, \cdots, i_m) \in \{0,1\}^m$.

Proof. Let $\mathfrak{b}_j$ denote $\mathfrak{b}\mathcal{O}_j$, for $j = 1, \cdots, s$. Let $\mathbf{n} = (\mathbf{n}_1, \cdots, \mathbf{n}_s) \in \mathbb{N}^{m_1} \times \cdots \times \mathbb{N}^{m_s}$. By Theorem 6 in [27], it follows that

$$\eta_{\mathcal{O}_j,\mathbf{n}}(\mathcal{O}_j, \mathfrak{b}_j) = \frac{q^{\varrho_j + \delta_j}}{[U_{\mathfrak{b}_j} : U_{\mathcal{O}_j}] (q^{\varrho} - 1)} \sum_{\mathbf{i}_j \in \{0,1\}^{m_j}} (-1)^{|\mathbf{i}_j|} q^{-\dim(\mathfrak{b}_j \mathfrak{p}_j^{\mathbf{n}_j} / \mathfrak{b}_j \cap \mathfrak{p}_j^{\mathbf{n}_j + \mathbf{i}_j})}$$

for each $j = 1 \cdots, s$.Then, by Euler product identity for Poincaré series, we obtain

$$\begin{aligned}
\eta_{S,\mathbf{n}}(S, \mathfrak{b}) &= \prod_{j=1}^{s} \eta_{\mathcal{O}_j,\mathbf{n}}(\mathcal{O}_j, \mathfrak{b}_j) \\
&= \prod_{j=1}^{s} \left(\frac{q^{\varrho_j + \delta_j}}{[U_{\mathfrak{b}_j} : U_{\mathcal{O}_j}] (q^{\varrho_j} - 1)} \sum_{\mathbf{i}_j \in \{0,1\}^{m_j}} (-1)^{|\mathbf{i}_j|} q^{-\dim\left(\mathfrak{b}_j \mathfrak{p}_j^{\mathbf{n}_j} / \mathfrak{b}_j \cap \mathfrak{p}_j^{\mathbf{n}_j + \mathbf{i}_j}\right)} \right).
\end{aligned}$$

Therefore,

$$\eta_{S,\mathbf{n}}(S, \mathfrak{b}) = \frac{q^{\sum_{j=1}^{s}(\delta_j + \varrho_j)}}{[U_{\mathfrak{b}} : U_S] \prod_{j=1}^{s} (q^{\varrho_j} - 1)} \sum_{\mathbf{i} \in \{0,1\}^m} (-1)^{|\mathbf{i}|} q^{-\dim_k \left(\mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{i}}\right)}.$$

The last equality being a consequence of

$$\dim_k \left(\mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{i}}\right) = \sum_{j=1}^{s} \dim_k (\mathfrak{b}_j \mathfrak{p}_j^{\mathbf{n}_j} / \mathfrak{b}_j \cap \mathfrak{b}_j \mathfrak{p}_j^{\mathbf{n}_j + \mathbf{i}_j}) \tag{3.8}$$

for each $\mathbf{i} = (\mathbf{i_1}, \cdots, \mathbf{i_s}) \in \{0,1\}^{m_1} \times \cdots \times \{0,1\}^{m_s} = \{0,1\}^m$, and

$$\prod_{j=1}^{s} [U_{\mathfrak{b}_j} : U_{\mathcal{O}_j}] = [U_{\mathfrak{b}} : U_S] \tag{3.9}$$

which are due to Lemma 22 and Corollary 21, respectively. ■

In [27], for each local ring $\mathcal{O}$ of a geometrically integral algebraic curve defined over a field $\mathbb{F}_q$, and for each set M in the Boolean algebra generated by the cosets $z + \mathfrak{a}$ ($z \in K$) of the $\mathcal{O}$-ideals was attributed a volume $\mu(M) \geq 0$, uniquely determined by the three axioms: $\mu(\mathcal{O}) = 1$, $\mu(z + M) = \mu(M)$ and $\mu(M \cup N) = \mu(M) + \mu(N)$ whenever $M \cap N = \emptyset$. Using this measure Stöhr proved the precedent theorem in the local case. Similarly, we can attribute to each $M \in \mathcal{M}_S$, where $\mathcal{M}_S$ is the Boolean algebra generated by the cosets $z + \mathfrak{a}$ ($z \in K$) of the S-ideals, a volume $\mu_S(M) \geq 0$, uniquely determined by the three axioms: $\mu_S(S) = 1$, $\mu_S(z + M) = \mu_S(M)$ and $\mu_S(M \cup N) = \mu_S(M) + \mu_S(N)$ whenever $M \cap N = \emptyset$. By using this measure and following the same ideas that were used by Stöhr to prove the local case, we can prove the above theorem without any use the Euler product identity. We can also prove Euler product identity for Poincaré series in a similar fashion.

Indeed, since the S-ideal $\mathfrak{b}$ is a disjoint finite union of cosets of the S-ideal $\mathfrak{a}$, whenever $\mathfrak{a} \subseteq \mathfrak{b}$, it follows that the volume

$$\mu_S(\mathfrak{b}) = \#(\mathfrak{b}/\mathfrak{a})\,\mu_S(\mathfrak{a}) \text{ whenever } \mathfrak{a} \subseteq \mathfrak{b} \text{ are } S\text{-ideals.}$$

Thus, for each S-ideal $\mathfrak{b}$ we have that $z\mathfrak{b} \subseteq S$ for some $z \in S$, hence we get that $\mu_S(S) = \#(S/z\mathfrak{b})\,\mu_S(z\mathfrak{b}) = q^{-\deg_S(z\mathfrak{b})}\mu_S(z\mathfrak{b})$ and, finally, we get that

$$\mu_S(\mathfrak{b}) = \#(\mathfrak{b}/z\mathfrak{b})\,\mu_S(z\mathfrak{b}) = q^{\deg_S(\mathfrak{b}) - \deg_S(z\mathfrak{b})}.$$

Then, due to the normalization $\mu_S(S) = 1$,

$$\mu_S(\mathfrak{b}) = q^{\deg_S(\mathfrak{b})} \text{ for each } S\text{- ideal } \mathfrak{b}.$$

The group U_S, of units of the semilocal ring S, is the complement of the union of the finite maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$ of S. Hence

$$\mu_S(U_S) = \mu_S(S \setminus \cup_{i=1}^{s} \mathfrak{m}_i) = 1 - \mu_S(\cup_{i=1}^{s} \mathfrak{m}_i).$$

On the other hand, since $\mu_S(\cdot)$ is additive, it follows from the inclusion-exclusion principle that

$$\begin{aligned}
\mu_S(U_S) &= 1 - \sum\nolimits_{j=1}^{s} (-1)^{j-1} \sum_{0 \le i_1 < \cdots < i_j \le s} \mu_S(\mathfrak{m}_{i_1} \cap \cdots \cap \mathfrak{m}_{i_j}) \\
&= 1 - \sum\nolimits_{j=1}^{s} (-1)^{j-1} \sum_{0 \le i_1 < \cdots < i_j \le s} q^{\deg_S(\mathfrak{m}_{i_1} \cap \cdots \cap \mathfrak{m}_{i_j})} \\
&= 1 - \sum\nolimits_{j=1}^{s} (-1)^{j-1} \sum_{0 \le i_1 < \cdots < i_j \le s} q^{\sum_{k=1}^{j} \deg_{\mathcal{O}_{i_k}}(\mathfrak{m}_{i_k}\mathcal{O}_{i_k})} \\
&= (1 - q^{\deg_{\mathcal{O}_1}(\mathfrak{m}_1\mathcal{O}_1)}) \cdots (1 - q^{\deg_{\mathcal{O}_s}(\mathfrak{m}_s\mathcal{O}_s)}).
\end{aligned}$$

Therefore, since each $\varrho_i = \dim_k(\mathcal{O}_i/\mathfrak{m}_i\mathcal{O}_i)$, we obtain

$$\mu_S(U_S) = \left(1 - q^{-\varrho_1}\right) \cdots \left(1 - q^{-\varrho_s}\right).$$

Now, the Euler product identity for multiple Poincaré series follows from 3.8 and 3.9 in the proof of Theorem 38.

In [27] (Proposition 2.6) it was proved that, if $\mathfrak{b}$ is an $\mathcal{O}$-ideal of a local ring $\mathcal{O}$, then

$$P_{\mathcal{O}}(\mathcal{O}, \mathfrak{b}, \mathbf{t}) = P_{\mathcal{O}}(\mathfrak{b} : \mathfrak{b}, \mathfrak{b}, \mathbf{t}) = P_{\mathfrak{b}:\mathfrak{b}}(\mathfrak{b} : \mathfrak{b}, \mathfrak{b}, \mathbf{t}).$$

Similarly, we can extend this property to semilocal rings.

Proposition 39 *If $\mathfrak{b}$ is an S-ideal of a semilocal ring S, then*

$$P_S(S, \mathfrak{b}, \mathbf{t}) = P_S(\mathfrak{b} : \mathfrak{b}, \mathfrak{b}, \mathbf{t}) = P_{\mathfrak{b}:\mathfrak{b}}(\mathfrak{b} : \mathfrak{b}, \mathfrak{b}, \mathbf{t}). \tag{3.10}$$

Proof. We have that $S \subseteq \mathfrak{b} : \mathfrak{b} \subseteq \widetilde{S}$, then the ring $\mathfrak{b} : \mathfrak{b}$ is a semilocal ring. According to [27] we first observe that the S-ideal $\mathfrak{b}$, and even any S-ideal equivalent to $\mathfrak{b}$, may be viewed as an $(\mathfrak{b} : \mathfrak{b})$-ideal and if $\mathfrak{d}$ is an S-ideal equivalent to $\mathfrak{b}$, then $\mathfrak{d} \supseteq S$ if and only if $\mathfrak{d} \supseteq \mathfrak{b} : \mathfrak{b}$. Furthermore, since $\mathfrak{b} : \mathfrak{b}$ is a ring, it follows that $(\mathfrak{b} : \mathfrak{b})\widetilde{S} = \widetilde{S}$. Now, the proposition follows from Definition 29. ■

Corollary 40 *If S' is any subring of $\widetilde{S}$ containing the semilocal ring S, then*

$$P_S(S, S', \mathbf{t}) = P_{S'}(S', S', \mathbf{t}).$$

In particular, if $S' := S^{\mathfrak{a}}$ is the blow-up of S with respect to an ideal $\mathfrak{a}$, then

$$P_S(S, S^{\mathfrak{a}}, \mathbf{t}) = P_{S^{\mathfrak{a}}}(S^{\mathfrak{a}}, S^{\mathfrak{a}}, \mathbf{t}) = P_S(S, \mathfrak{a}^n : \mathfrak{a}^n, \mathbf{t})$$

for all sufficiently large integer n.

Proof. We observe that S' is an S-ideal and $S' : S' = S'$. Thus the first equality follows from the previous proposition. On the other hand, it is known that $S^{\mathfrak{a}} = \mathfrak{a}^n : \mathfrak{a}^n$ for all sufficiently large integer n (cf. [20] Proposition 4.3). Hence the second equality holds. ■

Let $\mathfrak{b}$ be an $\mathcal{O}$-ideal of a local ring $\mathcal{O}$. The ring $S := \mathfrak{b} : \mathfrak{b}$ is a semilocal ring and it may be expressed as a finite intersection of local rings $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$. We denote by $\varrho_j := \dim_k (\mathcal{O}_j/\mathfrak{m}_j\mathcal{O}_j)$ and $\delta_j := \dim_k(\widetilde{\mathcal{O}}_j/\mathcal{O}_j)$ the degree of the residue field and the singularity degree of the local ring $\mathcal{O}_j$, respectively, for each $j = 1, \cdots, s$.

From Identity 3.10 and from the Euler product identity of Poincaré series we obtain the following result:

Corollary 41 $[U_{\mathfrak{b}} : U_{\mathcal{O}}] = \frac{q^{\varrho+\delta-\sum_{j=1}^{s}\left(\delta_j+\varrho_j\right)}\prod_{j=1}^{s}\left(q^{\varrho_j}-1\right)}{(q^{\varrho}-1)}$.

Proof. Let $\mathbf{n} = (\mathbf{n}_1, \cdots, \mathbf{n}_s) \in \mathbb{N}^{m_1} \times \cdots \times \mathbb{N}^{m_s}$. From Theorem 6 in [27], it follows that

$$\eta_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) = \frac{q^{\varrho+\delta}}{[U_{\mathfrak{b}} : U_{\mathcal{O}}]\,(q^{\varrho}-1)} \sum_{\mathbf{i}\in\{0,1\}^m} (-1)^{|\mathbf{i}|}\, q^{-\dim_k(\mathfrak{b}\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{i}})}. \tag{3.11}$$

On the other hand, since $U_{\mathfrak{b}} = U_{\mathfrak{b}:\mathfrak{b}}$, $\mathfrak{b} : \mathfrak{b} = S$ and, hence, $[U_{\mathfrak{b}} : U_S] = 1$; it follows from Theorem 38 that

$$\eta_{S,\mathbf{n}}(S, \mathfrak{b}) = \frac{q^{\sum_{j=1}^{s}(\delta_j+\varrho_j)}}{\prod_{j=1}^{s}(q^{\varrho_j}-1)} \sum_{\mathbf{i}\in\{0,1\}^m} (-1)^{|\mathbf{i}|}\, q^{-\dim_k(\mathfrak{b}\mathfrak{p}^{\mathbf{n}}/\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{i}})}. \tag{3.12}$$

So, by comparing 3.11 and 3.12, we obtain the result. ■

In particular, if $\mathcal{O}_1 := (\mathfrak{b} : \mathfrak{b})$ is a local ring, then $\mathfrak{m}_1 := \mathcal{O}_1 \cap (\cap_{i=1}^{m}\mathfrak{p}_i)$ is the maximal ideal of $\mathcal{O}_1$, and $[U_{\mathfrak{b}} : U_{\mathcal{O}}] = \frac{q^{\varrho+\delta-(\delta_1+\varrho_1)}(q^{\varrho_1}-1)}{(q^{\varrho}-1)}$, where $\varrho_1 := \dim_k(\mathcal{O}_1/\mathfrak{m}_1)$ and $\delta_1 := \dim_k(\widetilde{\mathcal{O}}_1/\mathcal{O}_1)$. Thus, in this case, ϱ divides ϱ_1 and the natural homomorphism $\mathcal{O}/\mathfrak{m} \hookrightarrow \mathcal{O}_1/\mathfrak{m}_1$ is an isomorphism of fields if and only if $\varrho = \varrho_1$ if and only if $[U_{\mathfrak{b}} : U_{\mathcal{O}}] = q^{\delta-\delta_1}$ if and only if $\mathcal{O}_1 = \mathcal{O}+\mathfrak{m}_1$ i.e. $\mathcal{O}_1 \subseteq \mathcal{O}+(\cap_{i=1}^{m}\mathfrak{p}_i)$. Hence, according to Stöhr the local ring $\mathcal{O}' := \mathcal{O}+(\cap_{i=1}^{m}\mathfrak{p}_i)$ is the largest local ring between $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ whose residue field is equal to the residue field of $\mathcal{O}$. It follows that $[U_{\mathcal{O}'} : U_{\mathcal{O}}] = q^{\varrho+\delta-|\mathbf{r}|}$, $[U_{\widetilde{\mathcal{O}}} : U_{\mathcal{O}'}] = \frac{\prod_{i=1}^{m}(q^{r_i}-1)}{q^{\varrho}-1}$ and $U_{\mathcal{O}'}/U_{\mathcal{O}}$ is the maximal p-subgroup of $U_{\widetilde{\mathcal{O}}}/U_{\mathcal{O}}$.

We observe that, if S is a semilocal ring of a geometrically integral algebraic curve X defined over a field $\mathbb{F}_q$ of q elements, whose expression as an intersection of local rings is $S = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$, where each $\mathcal{O}_j$ is a local ring of the curve X, and, if $\mathfrak{b}$ is an S-ideal, then $\mathfrak{b} : \mathfrak{b}$ is a semilocal ring and it may be expressed as the finite intersection of local rings $\mathfrak{b}:\mathfrak{b}=\mathcal{O}_1^{(\mathfrak{b})}\cap\cdots\cap\mathcal{O}_{s_\mathfrak{b}}^{(\mathfrak{b})}$, where $\mathcal{O}_i^{(\mathfrak{b})}$ is a local ring for each $i=1,\cdots,s_\mathfrak{b}$. Hence $(\mathfrak{b}\mathcal{O}_1:\mathfrak{b}\mathcal{O}_1)\cap\cdots\cap(\mathfrak{b}\mathcal{O}_s:\mathfrak{b}\mathcal{O}_s)=\mathcal{O}_1^{(\mathfrak{b})}\cap\cdots\cap\mathcal{O}_{s_\mathfrak{b}}^{(\mathfrak{b})}$, with $s_\mathfrak{b}\geq s$. As before, let us denote by ϱ_j and by $\varrho_i^{(\mathfrak{b})}$ the degree of the residue field of the local rings $\mathcal{O}_j$ and $\mathcal{O}_i^{(\mathfrak{b})}$, respectively. From the previous corollary we deduce that

$$[U_\mathfrak{b}:U_S]=\frac{q^{\delta_S+\varrho_S-(\delta_{\mathfrak{b}:\mathfrak{b}}+\varrho_{\mathfrak{b}:\mathfrak{b}})}\prod_{i=1}^{s_\mathfrak{b}}(q^{\varrho_i^{(\mathfrak{b})}}-1)}{\prod_{j=1}^{s}(q^{\varrho_j}-1)} \tag{3.13}$$

where $\delta_S:=\dim_k(\widetilde{S}/S)$ and $\delta_{\mathfrak{b}:\mathfrak{b}}:=\dim_k(\widetilde{S}/(\mathfrak{b}:\mathfrak{b}))$ are the singularity degree of the semilocal rings S and $\mathfrak{b}:\mathfrak{b}$, respectively; and $\varrho_S:=\dim_k(S/\mathfrak{r})$, $\varrho_{\mathfrak{b}:\mathfrak{b}}:=\dim_k\left(\mathfrak{b}:\mathfrak{b}/\mathfrak{r}^{(\mathfrak{b})}\right)$, where $\mathfrak{r}$ and $\mathfrak{r}^{(\mathfrak{b})}$ are the Jacobson radical of S and $\mathfrak{b}:\mathfrak{b}$, respectively.

3.4 Integral representation

Let $\mathcal{R}:=\prod_{i=1}^m\widehat{K}_{v_i}$ be the locally compact total ring of fractions of the completion $\widehat{S}$ of the semilocal ring S, and let $U_\mathcal{R}:=\prod_{i=1}^m\widehat{K}_{v_i}^*$ be its group of units. As in the local case (cf. [27]), the assignment $\mathfrak{a}\longmapsto\widehat{\mathfrak{a}}=\widehat{S}\cdot\mathfrak{a}$ defines a one-to-one monotone degree-preserving bijection between the S-ideals and the regular $\widehat{S}$-ideals. Its inverse mapping is given by $\widehat{\mathfrak{a}}\longmapsto\widehat{\mathfrak{a}}\cap K$. Two S-ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent if and only if the corresponding $\widehat{S}$-ideals $\widehat{\mathfrak{a}}$ and $\widehat{\mathfrak{b}}$ are equivalent, that is, there exists $z\in U_\mathcal{R}$ such that $\widehat{\mathfrak{b}}=z\widehat{\mathfrak{a}}$ (see [14], section 3). Moreover, $\widehat{\mathfrak{b}:\mathfrak{a}}=\widehat{\mathfrak{b}}:\widehat{\mathfrak{a}}$ for each pair of S-ideals $\mathfrak{a}$ and $\mathfrak{b}$ (see [27], section 5).

The homomorphism $\mathbf{v}:K^*\longrightarrow\mathbb{Z}^m$ extends naturally to the group homomorphism $\mathbf{v}:U_\mathcal{R}\longrightarrow\mathbb{Z}^m$ that maps each unity $u:=(u_1,\cdots,u_m)$ in $U_\mathcal{R}$ to the integer vector $\mathbf{v}(u)=(\widehat{v}_1(u_1),\cdots,\widehat{v}_m(u_m))$ in $\mathbb{Z}^m$.

Let $\widehat{\mu}_S$ be the Haar measure on the additive group of the locally compact $\mathbb{F}_q$-algebra $\mathcal{R}$, normalized so that $\widehat{\mu}_S(\widehat{S})=1$. The multiple Poincaré series $P_S(\mathfrak{a},\mathfrak{b},\mathbf{t})$ may be expressed as an integral in terms of this measure.

Since the $\widehat{S}$-ideal $\widehat{\mathfrak{b}}$ is a finite disjoint union of cosets of the S-ideal $\widehat{\mathfrak{a}}$, whenever $\mathfrak{a}\subseteq\mathfrak{b}$, it follows that the volume

$$\widehat{\mu}_S(\widehat{\mathfrak{b}})=\#(\widehat{\mathfrak{b}}/\widehat{\mathfrak{a}})\widehat{\mu}_S(\widehat{\mathfrak{a}})\text{ whenever }\mathfrak{a}\subseteq\mathfrak{b}\text{ are }S\text{-ideals.}$$

Thus, for each S-ideal $\mathfrak{b}$ we have that $z\widehat{\mathfrak{b}}\subseteq\widehat{S}$ for some $z\in\widehat{S}$, hence, we have that $\widehat{\mu}_S(\widehat{S})=\#(\widehat{S}/z\widehat{\mathfrak{b}})\widehat{\mu}_S(z\widehat{\mathfrak{b}})=q^{-\deg_S(z\widehat{\mathfrak{b}})}\widehat{\mu}_S(z\widehat{\mathfrak{b}})$. Finally we have that

$$\widehat{\mu}_S(\widehat{\mathfrak{b}})=\#(\widehat{\mathfrak{b}}/z\widehat{\mathfrak{b}})\widehat{\mu}_S(z\widehat{\mathfrak{b}})=q^{\deg_S(\widehat{\mathfrak{b}})-\deg_S(z\widehat{\mathfrak{b}})}.$$

Then, due to the normalization $\widehat{\mu}_S(\widehat{S}) = 1$,

$$\widehat{\mu}_S(\widehat{\mathfrak{b}}) = q^{\deg_S(\mathfrak{b})} \text{ for each } S\text{- ideal } \mathfrak{b}.$$

Because, the group $U_{\widehat{S}}$ of units of the semilocal ring $\widehat{S}$ is the complement of the union of the finite maximal ideals, say $\widehat{\mathfrak{m}}_1, \cdots, \widehat{\mathfrak{m}}_s$, of $\widehat{S}$ we conclude

$$\widehat{\mu}_S(U_{\widehat{S}}) = \widehat{\mu}_S(\widehat{S} \setminus \cup_{i=1}^s \widehat{\mathfrak{m}}_i) = 1 - \widehat{\mu}_S(\cup_{i=1}^s \widehat{\mathfrak{m}}_i).$$

On the other hand, since $\widehat{\mu}_S(\cdot)$ is additive it follows from the inclusion-exclusion principle that

$$\begin{aligned}
\widehat{\mu}_S(U_S) &= 1 - \sum\nolimits_{j=1}^s (-1)^{j-1} \sum_{0 \le i_1 < \cdots < i_j \le s} \widehat{\mu}_S(\widehat{\mathfrak{m}}_{i_1} \cap \cdots \cap \widehat{\mathfrak{m}}_{i_j}) \\
&= 1 - \sum\nolimits_{j=1}^s (-1)^{j-1} \sum_{0 \le i_1 < \cdots < i_j \le s} q^{\deg_S(\mathfrak{m}_{i_1} \cap \cdots \cap \mathfrak{m}_{i_j})} \\
&= 1 - \sum\nolimits_{j=1}^s (-1)^{j-1} \sum_{0 \le i_1 < \cdots < i_j \le s} q^{\sum_{k=1}^j \deg_{\mathcal{O}_{i_k}}(\mathfrak{m}_{i_k}\mathcal{O}_{i_k})} \\
&= (1 - q^{\deg_{\mathcal{O}_1}(\mathfrak{m}_1\mathcal{O}_1)}) \cdots (1 - q^{\deg_{\mathcal{O}_s}(\mathfrak{m}_s\mathcal{O}_s)}).
\end{aligned}$$

Thus, given that $\varrho_i = \dim_k(\mathcal{O}_i/\mathfrak{m}_i\mathcal{O}_i)$, we obtain

$$\widehat{\mu}_S\left(U_{\widehat{S}}\right) = \left(1 - q^{-\varrho_1}\right) \cdots \left(1 - q^{-\varrho_s}\right).$$

Moreover, we have $\widehat{\mu}_S(z\widehat{S}) = q^{-\mathbf{r}\cdot\mathbf{v}(z)}$ for each $z \in K^*$ and even more, for each $z \in U_{\mathcal{R}}$. Now, from the uniqueness of the normalized Haar measure we get

$$\widehat{\mu}_S(zM) = q^{-\mathbf{r}\cdot\mathbf{v}(z)} \widehat{\mu}_S(M)$$

for each $z \in K^*$ and for each measurable subset M of $\mathcal{R}$.

The multiple power series $P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ can be realized by an integral in the following form:

Theorem 42

$$P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \frac{\prod_{j=1}^s (1 - q^{-\varrho_j})}{[U_{\mathfrak{b}} : U_S]} \int_{(\widehat{\mathfrak{b}}:\widehat{\mathfrak{a}}) \cap U_{\mathcal{R}}} q^{\mathbf{r}\cdot\mathbf{v}(z)} \mathbf{t}^{\mathbf{v}(z)} d\widehat{\mu}_S(z)$$

in the unit poly-disk $|t_1| < 1, \cdots, |t_m| < 1$, *where* $\varrho_j := \dim_k(\mathcal{O}_j/\mathfrak{m}_j\mathcal{O}_j)$ *is the degree of the residue field of the local ring* $\mathcal{O}_j$ *over the constant field* k *for each* $j = 1, \cdots, s$.

Proof. Since $\widehat{\mathfrak{d}} \cap U_{\mathcal{R}}$ is the disjoint union of the sets $\widehat{\mathfrak{d}}_{\mathbf{n}} = \{z \in \widehat{\mathfrak{d}} : \mathbf{v}(z) = \mathbf{n}\}$ and since $\mathbf{v}(z)$ assumes on $\widehat{\mathfrak{d}}_{\mathbf{n}}$ the constant value $\mathbf{n}$, we have

$$\int_{\widehat{\mathfrak{d}} \cap U_{\mathcal{R}}} q^{\mathbf{r}\cdot\mathbf{v}(z)} \mathbf{t}^{\mathbf{v}(z)} d\widehat{\mu}_S(z) = \sum_{\mathbf{n}\in\mathbb{Z}^m} q^{\mathbf{r}\cdot\mathbf{n}} \mathbf{t}^{\mathbf{n}} \widehat{\mu}_S(\widehat{\mathfrak{d}}_{\mathbf{n}}) \in \mathbb{C}[[t_1, \cdots, t_m]]\mathbf{t}^{\mathbf{v}(\mathfrak{d}\cdot\widetilde{S})}$$

in the domain of the absolute convergence of the Laurent series on the right hand side series. Since $\widehat{\mathfrak{d}}_{\mathbf{n}}$ is the disjoint union of the cosets $zU_{\widehat{S}}$, where z varies over a complete system of representatives of $\mathfrak{d}_{\mathbf{n}}$ modulo U_S, and each of this cosets has the volume

$$\widehat{\mu}_S\left(zU_{\widehat{S}}\right) = q^{-\mathbf{r}\cdot\mathbf{v}(z)} \widehat{\mu}_S\left(U_{\widehat{S}}\right) = q^{-\mathbf{r}\cdot\mathbf{v}(z)} \left(1 - q^{-\varrho_1}\right) \cdots \left(1 - q^{-\varrho_s}\right)$$

we obtain

$$\widehat{\mu}_S(\widehat{\mathfrak{d}}_{\mathbf{n}}) = \#(\mathfrak{d}_{\mathbf{n}}/U_S) q^{-\mathbf{r}\cdot\mathbf{n}} \left(1 - q^{-\varrho_1}\right) \cdots \left(1 - q^{-\varrho_s}\right).$$

So,

$$\int_{\widehat{\mathfrak{d}} \cap U_{\mathcal{R}}} q^{\mathbf{r}\cdot\mathbf{v}(z)} \mathbf{t}^{\mathbf{v}(z)} d\widehat{\mu}_S(z) = \left(1 - q^{-\varrho_1}\right) \cdots \left(1 - q^{-\varrho_s}\right) \sum_{\mathbf{n}\in\mathbb{Z}^m} \#(\mathfrak{d}_{\mathbf{n}}/U_S) \mathbf{t}^{\mathbf{n}}$$

for each S-ideal $\mathfrak{d}$. Thus, by setting $\mathfrak{d} = \mathfrak{b} : \mathfrak{a}$ and by applying Formula 3.5, we obtain the integral representation of the Poincaré series. ■

3.5 Functional equation

Theorem 43 *The Poincaré series $P_S(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ is a rational function of the form*

$$P_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m) = \frac{\Lambda_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)}{(1 - t_1) \cdots (1 - t_m)}$$

where $\Lambda_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$ is a polynomial in $t_1, \cdots, t_m$ with integer coefficients of multi-degree smaller than or equal to $\mathbf{b} = (b_1, \cdots, b_m) := \mathbf{v}((\mathfrak{b}{:}\widetilde{S}) : (\mathfrak{b}{\cdot}\widetilde{S}))$, which satisfies the ***functional equation****:*

$$\Lambda_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m) = [U_{\mathfrak{b}:\mathfrak{a}} : U_{\mathfrak{b}}] q^{\dim(\mathfrak{b}:\mathfrak{a}/(\mathfrak{b}:\mathfrak{a}):\widetilde{S})} t_1^{b_1} \cdots t_m^{b_m} \Lambda_S(S, \mathfrak{a}\cdot\mathfrak{b}^*, \frac{1}{q^{r_1}t_1}, \cdots, \frac{1}{q^{r_m}t_m}). \tag{3.14}$$

In particular,

$$\Lambda_S(S, \mathfrak{b}, t_1, \cdots, t_m) = q^{\dim(\mathfrak{b}/\mathfrak{b}:\widetilde{S})} t_1^{b_1} \cdots t_m^{b_m} \Lambda_S(S, \mathfrak{b}^*, \frac{1}{q^{r_1}t_1}, \cdots, \frac{1}{q^{r_m}t_m}),$$

or equivalently,

$$\Lambda_S(S, \mathfrak{b}^*, t_1, \cdots, t_m) = q^{\dim(\mathfrak{b}\cdot\widetilde{S}/\mathfrak{b})} t_1^{b_1} \cdots t_m^{b_m} \Lambda_S(S, \mathfrak{b}, \frac{1}{q^{r_1}t_1}, \cdots, \frac{1}{q^{r_m}t_m}).$$

Proof. We put $\mathfrak{b}_j = \mathfrak{b}\mathcal{O}_j$, $\mathfrak{a}_j = \mathfrak{a}\mathcal{O}_j$ and $\mathfrak{d}_j = \mathfrak{b}_j : \mathfrak{a}_j$, for $j = 1, \cdots, s$. By the local case (see [27], theorem 7.1), we have

$$P_{\mathcal{O}_j}(\mathfrak{a}_j, \mathfrak{b}_j, t_1, \cdots, t_m) = \frac{\Lambda_{\mathcal{O}_j}(\mathfrak{a}_j, \mathfrak{b}_j, t_{j1}, \cdots, t_{jm_j})}{(1 - t_{j1}) \cdots (1 - t_{jm_j})}$$

where each $\Lambda_{\mathcal{O}_j}(\mathfrak{a}_j, \mathfrak{b}_j, t_{j1}, \cdots, t_{jm_j})$ is a polynomial in $t_{j1}, \cdots, t_{jm_j}$ with integer coefficients of multi-degree smaller than or equal to $\mathbf{b}_j = (b_{j1}, \cdots, b_{jm_j}) := \mathbf{v}((\mathfrak{b}_j{:}\widetilde{\mathcal{O}_j}) : (\mathfrak{b}_j\widetilde{\mathcal{O}_j}))$ which satisfies the functional equation equation

$$\Lambda_{\mathcal{O}_j}(\mathfrak{a}_j, \mathfrak{b}_j, \mathbf{t}_j) = [U_{\mathfrak{d}_j} : U_{\mathfrak{b}_j}] q^{\dim(\mathfrak{d}_j/\mathfrak{d}_j : \widetilde{\mathcal{O}}_j)} t_{j1}^{b_{j1}} \cdots t_{jm_j}^{b_{jm_j}} \Lambda_{\mathcal{O}_j}(\mathcal{O}_j, \mathfrak{a}_j \cdot \mathfrak{b}_j{}^*, \frac{1}{q^{r_{j1}} t_{j1}}, \cdots, \frac{1}{q^{r_{jm_j}} t_{jm_j}}) \tag{3.15}$$

where $\mathbf{t}_j := \left(t_{j1}, \cdots, t_{jm_j}\right)$ for $j = 1, \cdots, n$. Now, set

$$\Lambda_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m) := \prod_{j=1}^{s} \Lambda_{\mathcal{O}_j}(\mathfrak{a}_j, \mathfrak{b}_j, t_{j1}, \cdots, t_{jm_j}).$$

Then, $\Lambda_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$ is a polynomial in $t_1, \cdots, t_m$ with integer coefficients of multi-degree smaller than or equal to $\mathbf{b} = (b_1, \cdots, b_m) := \mathbf{v}((\mathfrak{b}{:}\widetilde{S}) : (\mathfrak{b}{\cdot}\widetilde{S}))$ and so, by the Euler product identity, $P_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m) = \frac{\Lambda_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)}{(1-t_1)\cdots(1-t_m)}$. From Proposition 19 and Corollary 21, $\mathfrak{a} \cdot \mathfrak{b}^* = \mathfrak{a}_1 \cdot \mathfrak{b}_j{}^* \cap \cdots \cap \mathfrak{a}_s \cdot \mathfrak{b}_s{}^*$ and $\prod_{j=1}^{s} [U_{\mathfrak{d}_j} : U_{\mathfrak{b}_j}] = [U_{\mathfrak{b}:\mathfrak{a}} : U_{\mathfrak{b}}]$, respectively. Thus, From 3.15 we conclude that $\Lambda_S(\mathfrak{a}, \mathfrak{b}, t_1, \cdots, t_m)$ satisfies the functional equation 3.14. ■

Chapter 4

Computation of Poincaré series and ground field extension

4.1 Computation of Poincaré series

In this section we indicate a procedure which is useful to determine the ideal classes of a local ring $\mathcal{O}$ and to compute the Poincaré series $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ for each pair of ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. Let $\mathcal{O}$ be a local ring of a geometrically irreducible algebraic curve defined over a finite field $k = \mathbb{F}_q$ with rational function field K, and let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathcal{O}$-ideals.

Let us consider the semilocal subring $\mathcal{O}_0 := k \oplus \mathfrak{f}$ of $\mathcal{O}$, where $\mathfrak{f} = \mathcal{O} : \widetilde{\mathcal{O}}$ is the conductor ideal of the local ring $\mathcal{O}$. We first give a procedure to compute the multiple Poincaré series $P_{\mathcal{O}_0}(\mathcal{O}_0, \mathcal{O}_0, \mathbf{t})$ which indicates a general procedure to compute the multi-variable Poincaré series $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$.

We observe that $\mathcal{O}_0 \subseteq \mathcal{O} \subseteq \widetilde{\mathcal{O}}$ and that the valuation rings of $K \mid k$ containing $\mathcal{O}_0$ are precisely the valuation rings of $K \mid k$ containing $\mathcal{O}$, hence $\widetilde{\mathcal{O}_0} = \widetilde{\mathcal{O}}$. Moreover, $\mathfrak{f} = \mathcal{O}_0 : \widetilde{\mathcal{O}}$, that is, $\mathfrak{f}$ is also the conductor ideal of $\mathcal{O}_0$. The completion of the ring $\mathcal{O}_0$ is the ring $\widehat{\mathcal{O}_0} = k(1, \cdots, 1) \oplus \widehat{\mathfrak{f}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathcal{O}_0$-ideals.

For each $\mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m$ the coefficient $\eta_{\mathcal{O}_0,\mathbf{n}}(\mathfrak{a}, \mathfrak{b})$ of the multi-variable

Poincaré series $P_{\mathcal{O}_0}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ is

$$\begin{aligned}\eta_{\mathcal{O}_0,\mathbf{n}}(\mathfrak{a}, \mathfrak{b}) &= \#\{\mathcal{O}_0\text{-ideals } \mathfrak{d} \text{ satisfying } \mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b} \text{ and } \mathfrak{d} \cdot \widetilde{\mathcal{O}} = \mathfrak{a} \cdot \mathfrak{p}^{-\mathbf{n}}\} \\ &= \#\{\mathcal{O}_0\text{-ideals } \mathfrak{d} \text{ satisfying } \mathfrak{d} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}},\ \ \mathfrak{d} \sim \mathfrak{b}\ \text{ and } \mathfrak{p}^{\mathbf{n}}\mathfrak{a} \subseteq \mathfrak{d}\} \\ &= \#\{\widehat{\mathcal{O}_0}\text{-ideals } \widehat{\mathfrak{d}} \text{ satisfying } \widehat{\mathfrak{d}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}},\ \ \widehat{\mathfrak{d}} \sim \widehat{\mathfrak{b}}\ \text{ and } \widehat{\mathfrak{p}}^{\mathbf{n}}\widehat{\mathfrak{a}} \subseteq \widehat{\mathfrak{d}}\}.\end{aligned}$$

Given that $\widehat{\mathcal{O}_0}$ is a subring of $\widehat{\widetilde{\mathcal{O}}}$ that contains the conductor ideal $\widehat{\mathfrak{f}}$, we have $\widehat{\widetilde{\mathcal{O}}} = \widetilde{V} \oplus \widehat{\mathfrak{f}}$ for some vector space $\widetilde{V}$. Since $\mathfrak{f} \subseteq (\mathfrak{d} : \widetilde{\mathcal{O}}) : (\mathfrak{d}\widetilde{\mathcal{O}}) \subseteq \widetilde{\mathcal{O}}$ for each $\mathcal{O}_0$-ideal $\mathfrak{d}$ (cf. [27] Lemma 3.1), it follows that each $\widehat{\mathcal{O}_0}$-ideal $\widehat{\mathfrak{d}}$, which satisfies $\widehat{\mathfrak{d}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$, is of the form $\widehat{\mathfrak{d}} = D \oplus \widehat{\mathfrak{f}}$, where D is a vector subspaces of $\widetilde{V}$ satisfying $k(1, \cdots, 1) \cdot D \subseteq D \oplus \widehat{\mathfrak{f}}$ and contains for each $i = 1, \cdots, m$ a vector whose i-th entry has order 0. In particular, $\widehat{\mathfrak{a}} = A \oplus \widehat{\mathfrak{f}}$ and $\widehat{\mathfrak{b}} = B \oplus \widehat{\mathfrak{f}}$ where A and B are vector subspaces of $\widetilde{V}$ that satisfy $k(1, \cdots, 1) \cdot A \subseteq A \oplus \widehat{\mathfrak{f}}$ and $k(1, \cdots, 1) \cdot B \subseteq B \oplus \widehat{\mathfrak{f}}$ and they contain for each $i = 1, \cdots, m$ a vector whose i-th entry has order 0. Another such an ideal $\widehat{\mathfrak{d}} = D \oplus \widehat{\mathfrak{f}}$ is equivalent to $\widehat{\mathfrak{b}} = B \oplus \widehat{\mathfrak{f}}$ if and only if there is a vector $z \in \widetilde{V}$ with entries of order 0 such that $D \oplus \widehat{\mathfrak{f}} = zB + \widehat{\mathfrak{f}}$. In particular (for $\widehat{\mathfrak{a}} = \widehat{\mathcal{O}_0}$ and $\widehat{\mathfrak{b}} = \widehat{\mathcal{O}_0}$) the $\widehat{\mathcal{O}_0}$-ideals $\widehat{\mathfrak{d}}$ satisfying $\widehat{\mathfrak{d}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$, $\widehat{\mathfrak{d}} \sim \widehat{\mathcal{O}_0}$ and $\widehat{\mathfrak{p}}^{\mathbf{n}}\widehat{\mathcal{O}_0} \subseteq \widehat{\mathfrak{d}}$ correspond to the vector subspaces D of $\widetilde{V}$ satisfying: $k(1, \cdots, 1) \cdot D \subseteq D \oplus \widehat{\mathfrak{f}}$, D contains for each $i = 1, \cdots, m$ a vector whose i-th entry has order 0, there is a vector $z \in \widetilde{V}$ with entries of order 0 such that $D \oplus \widehat{\mathfrak{f}} = zk(1, \cdots, 1) + \widehat{\mathfrak{f}}$, and $\widehat{\mathfrak{p}}^{\mathbf{n}} \subseteq D \oplus \widehat{\mathfrak{f}}$. Thus, we obtain (see Theorem 44)

$$P_{\mathcal{O}_0}(\mathcal{O}_0, \mathcal{O}_0, \mathbf{t}) = \frac{(1 - t_1) \cdots (1 - t_m) + (q^{\mathbf{r} \cdot \mathbf{f} - |\mathbf{r}|} \prod_{i=1}^{m} (q^{r_i} - 1)/(q - 1))\mathbf{t}^{\mathbf{f}}}{(1 - t_1) \cdots (1 - t_m)}.$$

Similarly, the completion of the local ring $\mathcal{O}$ can be expressed in the form $\widehat{\mathcal{O}} = V \oplus \widehat{\mathfrak{f}}$ for some vector subspace V of $\widetilde{V}$ such that $k(1, \cdots, 1) \cdot V \subseteq V \oplus \widehat{\mathfrak{f}}$ and V contains for each $i = 1, \cdots, m$ a vector whose i-th entry has order 0. Moreover, each $\widehat{\mathcal{O}_0}$-ideals $\widehat{\mathfrak{d}}$ satisfying $\widehat{\mathfrak{d}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$, $\widehat{\mathfrak{d}} \sim \widehat{\mathcal{O}}$ and $\widehat{\mathfrak{p}}^{\mathbf{n}}\widehat{\mathcal{O}_0} \subseteq \widehat{\mathfrak{d}}$ corresponds to a vector subspace D of $\widetilde{V}$ satisfying: $k(1, \cdots, 1) \cdot D \subseteq D \oplus \widehat{\mathfrak{f}}$, D contains for each $i = 1, \cdots, m$ a vector whose i-th entry has order 0, there is a vector $z \in \widetilde{V}$ with entries of order 0 such that $D \oplus \widehat{\mathfrak{f}} = zV + \widehat{\mathfrak{f}}$ and $\widehat{\mathfrak{p}}^{\mathbf{n}} \subseteq D \oplus \widehat{\mathfrak{f}}$. From Proposition 39, $P_{\mathcal{O}_0}(\mathcal{O}_0, \mathcal{O}, \mathbf{t}) = P_{\mathcal{O}_0}(\mathcal{O}, \mathcal{O}, \mathbf{t}) = P_{\mathcal{O}}(\mathcal{O}, \mathcal{O}, \mathbf{t})$. Thus, in this way, we can compute the Poincaré series $P_{\mathcal{O}}(\mathcal{O}, \mathcal{O}, \mathbf{t})$.

Now, we proceed to indicate a general procedure to compute the Poincaré series $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ for each pair of ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. Since the normalization $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is a semilocal principal ideal domain, we choose generators $\pi_1, \cdots, \pi_m$ of the maximal integral $\widetilde{\mathcal{O}}$-ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$, respectively. By the weak approximation theorem we can assume that

$$v_j (\pi_i - 1) \geq f_j \text{ whenever } j \neq i.$$

Observe that each $\mathcal{O}$-ideal can be written in a unique way in the form $\pi^{-\mathbf{n}}\mathfrak{d}$ where $\pi^{\mathbf{n}} := \pi_1^{n_1} \cdots \pi_m^{n_m}$ for some $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$ and where $\mathfrak{d}$ is one of the finitely many $\mathcal{O}$-ideals satisfying $\mathfrak{d} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and that the $\mathcal{O}$-ideal $\pi^{-\mathbf{n}}\mathfrak{d}$ contains $\mathfrak{a}$ if and only if $\pi^{\mathbf{n}}\mathfrak{a} \subseteq \mathfrak{d}$, that is, $\pi^{\mathbf{n}} \in \mathfrak{d} : \mathfrak{a}$. Hence the Poincaré series $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ admits the following partition

$$P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \sum_{\mathfrak{d} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}},\ \mathfrak{d} \sim \mathfrak{b}} \left(\sum_{\pi^{\mathbf{n}} \in \mathfrak{d} : \mathfrak{a}} \mathbf{t}^{\mathbf{n} + \mathbf{v}(\mathfrak{a} \cdot \widetilde{\mathcal{O}})} \right)$$

where $\mathfrak{d}$ varies over the $\mathcal{O}$-ideals that are equivalent to $\mathfrak{b}$ and that satisfy $\mathfrak{d} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ (the number of $\mathcal{O}$-ideals satisfying these properties is $[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}}]$), and $\mathbf{n}$ varies over the integer vectors satisfying $\pi^{\mathbf{n}} \in \mathfrak{d} : \mathfrak{a}$. Therefore, the coefficient $\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b})$ of $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ satisfies

$$\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b}) = \#\{\mathcal{O}\text{-ideals } \mathfrak{d} \text{ satisfying } \mathfrak{d} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}, \mathfrak{d} \sim \mathfrak{b} \text{ and } \pi^{\mathbf{n} - \mathbf{v}(\mathfrak{a} \cdot \widetilde{\mathcal{O}})}\mathfrak{a} \subseteq \mathfrak{d}\}. \tag{4.1}$$

Theorem 44 *The Poincaré series $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ converges absolutely in the unit poly-disk $|t_1| < 1, \cdots, |t_m| < 1$ to a rational function*

$$P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \frac{\Lambda_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})}{(1 - t_1) \cdots (1 - t_m)},$$

where $\Lambda_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) \in \mathbb{Z}[t_1, \cdots, t_m]$ is a polynomial of multi-degree $\leq \mathbf{b}$. More precisely,

$$\Lambda_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{b}} \eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b})\, \mathbf{t}^{\mathbf{n}} \prod_{n_i < b_i} (1 - t_i)$$

where the index i runs through the integers $i = 1, \cdots, m$ with $n_i < b_i$ in the product.

Proof. The polynomial $\Lambda_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t})$ can be obtained from the Functional Equation. Here we give an algorithmic proof. The theorem will follow from the next assertion:

$$\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b}) = \eta_{\inf(\mathbf{n}, \mathbf{b})}(\mathfrak{a}, \mathfrak{b})$$

where $\inf(\mathbf{n}, \mathbf{b}) := (\min(n_1, b_1), \cdots, \min(n_m, b_m))$ for each $\mathbf{n}$. To prove this assertion we can assume that $\mathfrak{a} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$. Since $v_i(\pi_i) = 1, v_j(\pi_i - 1) \geq f_j$ whenever $j \neq i$, and $f_j \geq b_j$, we deduce

$$\mathbf{v}\left(\pi^{\mathbf{n}} a - \pi^{\inf(\mathbf{n}, \mathbf{b})} a\right) \geq \mathbf{b} \text{ for each } \ a \in \widetilde{\mathcal{O}}.$$

If $\mathfrak{d} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and $\mathfrak{d} \sim \mathfrak{b}$ (and therefore $\mathfrak{d} \supseteq \mathfrak{p}^{\mathbf{b}}$), then for each element $a \in \widetilde{\mathcal{O}}$ we obtain $\pi^{\mathbf{n}} a - \pi^{\inf(\mathbf{n}, \mathbf{b})} a \in \mathfrak{d}$, hence $\pi^{\mathbf{n}} a \in \mathfrak{d}$ if and only if $\pi^{\inf(\mathbf{n}, \mathbf{b})} a \in \mathfrak{d}$, and therefore $\pi^{\mathbf{n}}\mathfrak{a} \subseteq \mathfrak{d}$

if and only if $\pi^{\inf(\mathbf{n},\mathbf{b})}\mathfrak{a} \subseteq \mathfrak{d}$, this proves the above assertion. Therefore, the Poincaré series $P_{\mathcal{O}}(\mathfrak{a},\mathfrak{b},\mathbf{t})$ is a sum of $(b_1+1)\cdots(b_m+1)$ multiple geometric series

$$P_{\mathcal{O}}(\mathfrak{a},\mathfrak{b},\mathbf{t}) = \sum_{\mathbf{0}\leq\mathbf{n}\leq\mathbf{b}} \eta_{\mathbf{n}}(\mathfrak{a},\mathfrak{b})\,\mathbf{t}^{\mathbf{n}} \prod_{n_i=b_i} \frac{1}{1-t_i}$$

and so the theorem will follow. ■

By previous Theorem, we have only to compute the finitely many coefficients $\eta_{\mathbf{n}}(\mathfrak{a},\mathfrak{b})$ with $\mathbf{0} \leq \mathbf{n} \leq \mathbf{b} = (b_1,\cdots,b_m) := \mathbf{v}((\mathfrak{b}:\widetilde{\mathcal{O}}):\mathfrak{b}\cdot\widetilde{\mathcal{O}})$.

Since the coefficients $\eta_{\mathbf{n}}(\mathfrak{a},\mathfrak{b})$ only depend on the classes of the $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$, we will assume that $\mathfrak{a}\cdot\widetilde{\mathcal{O}} = \mathfrak{b}\cdot\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$. The right-hand side of the equations 4.1 remains unchanged, if $\mathcal{O}$, $\widetilde{\mathcal{O}}$, $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{d}$ are replaced by their respective completions:

$$\begin{aligned}\eta_{\mathbf{n}}(\mathfrak{a},\mathfrak{b}) &= \#\{\widehat{\mathcal{O}}\text{-ideals } \widehat{\mathfrak{d}} \text{ satisfying } \widehat{\mathfrak{d}}\cdot\widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}, \widehat{\mathfrak{d}} \sim \widehat{\mathfrak{b}} \text{ and } \pi^{\mathbf{n}-\mathbf{v}(\widehat{\mathfrak{a}}\cdot\widehat{\widetilde{\mathcal{O}}})}\widehat{\mathfrak{a}} \subseteq \widehat{\mathfrak{d}}\} \\ &= \#\{\widehat{\mathcal{O}}\text{-ideals } \widehat{\mathfrak{d}} \text{ satisfying } \widehat{\mathfrak{d}}\cdot\widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}, \widehat{\mathfrak{d}} \sim \widehat{\mathfrak{b}} \text{ and } \pi^{\mathbf{n}}\widehat{\mathfrak{a}} \subseteq \widehat{\mathfrak{d}}\}.\end{aligned} \tag{4.2}$$

By Cohen's Structure Theorem, the completion $\widehat{\widetilde{\mathcal{O}}}$ of the normalization of $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is the direct product of formal power series rings

$$\widehat{\widetilde{\mathcal{O}}} = k_1[[\pi]] \times \cdots \times k_m[[\pi]]$$

where $\pi := \pi_1\cdots\pi_m$ and $k_1 = \widetilde{\mathcal{O}}/\mathfrak{p}_1,\cdots,\ k_m = \widetilde{\mathcal{O}}/\mathfrak{p}_m$ are the residue fields of $\widetilde{\mathcal{O}}$ at $\mathfrak{p}_1,\cdots,\mathfrak{p}_m$, respectively. By passing from the constant field $k = \mathbb{F}_q$ to the residue field of $\mathcal{O}$, which coincides with the residue field of $\widehat{\mathcal{O}}$ and is contained in $\widehat{\mathcal{O}}$, and by passing from $t_1,\cdots,t_m$ to $t_1^{\rho},\cdots,t_m^{\rho}$ we could assume that k is the residue field of $\widehat{\mathcal{O}}$, that is, $\rho = 1$.

Since $\widehat{\mathcal{O}}$ is a subring of $\widehat{\widetilde{\mathcal{O}}} = \prod_{i=1}^{m} k_i[[\pi]]$ that contains the conductor ideal $\widehat{\mathfrak{f}} = \prod_{i=1}^{m} \pi^{f_i}k_i[[\pi]]$, we have $\widehat{\mathcal{O}} = V \oplus \widehat{\mathfrak{f}}$ for some vector subspace V of $\widetilde{V} := \prod_{i=1}^{m}\bigoplus_{j=0}^{f_i-1} k_i\pi^j$ that satisfies $V\cdot V \subseteq V \oplus \widehat{\mathfrak{f}}$, V contains $\mathbf{1} := (\pi^0,\cdots,\pi^0)$, and V does not contain any of the m one-dimensional vector spaces $\{0\}\times\cdots\times k_i\pi^{f_i-1}\times\cdots\times\{0\}$. Since $\mathfrak{f} \subseteq (\mathfrak{d}:\widetilde{\mathcal{O}}):(\mathfrak{d}\widetilde{\mathcal{O}}) \subseteq \widetilde{\mathcal{O}}$ for each $\mathcal{O}$-ideal $\mathfrak{d}$ (cf. [27] Lemma 3.1), it follows that any $\widehat{\mathcal{O}}$-ideal $\widehat{\mathfrak{d}}$, which satisfies $\widehat{\mathfrak{d}}\cdot\widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$, is of the form $\widehat{\mathfrak{d}} = D \oplus \widehat{\mathfrak{f}}$, where D is a vector subspace of $\widetilde{V}$ satisfying $V\cdot D \subseteq D\oplus\widehat{\mathfrak{f}}$ and D contains for each $i = 1,\cdots,m$ a vector whose i-th entry has order 0. In particular, $\widehat{\mathfrak{a}} = A\oplus\widehat{\mathfrak{f}}$ and $\widehat{\mathfrak{b}} = B\oplus\widehat{\mathfrak{f}}$, where A and B are vector subspaces of $\widetilde{V}$ satisfying $V\cdot A \subseteq A\oplus\widehat{\mathfrak{f}}$, $V\cdot B\subseteq B\oplus\widehat{\mathfrak{f}}$ and they contain for each $i = 1,\cdots,m$ a vector whose i-th entry has order 0. Another such ideal $\widehat{\mathfrak{d}} = D\oplus\widehat{\mathfrak{f}}$

is equivalent to $\widehat{\mathfrak{b}} = B \oplus \widehat{\mathfrak{f}}$ if and only if there is a vector $z \in \widetilde{V}$ with entries of order 0 such that $D \oplus \widehat{\mathfrak{f}} = zB + \widehat{\mathfrak{f}}$. The number of these vector spaces D is equal to the index $[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}}]$.

To compute the coefficients $\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b})$ by Formula 4.2 we discuss the condition $\pi^{\mathbf{n}}\widehat{\mathfrak{a}} \subseteq \widehat{\mathfrak{d}}$. By our choice of the generators π_i of the ideals $\mathfrak{p}_i$, we have $\pi^{\mathbf{n}} \equiv (\pi^{n_1}, \cdots, \pi^{n_m})$ mod $\widehat{\mathfrak{f}}$ for each $\mathbf{n} \in \mathbb{N}^m$. Alternatively, the maximal ideal $\widehat{\mathfrak{p}_i}$ is generated by vector $\widehat{\pi_i} := (1, \cdots, \pi, \cdots, 1)$ with i-th entry is equal to π and any other entry is equal to 1, and we have $\widehat{\pi_1}^{n_1} \cdots \widehat{\pi_m}^{n_m} = (\pi^{n_1}, \cdots, \pi^{n_m})$. The condition $\pi^{\mathbf{n}}\widehat{\mathfrak{a}} \subseteq \widehat{\mathfrak{d}}$ simply means that $(\pi^{n_1}, \cdots, \pi^{n_m}) A \subseteq D \oplus \widehat{\mathfrak{f}}$. Hence we have a procedure in terms of Linear Algebra to compute the coefficients $\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b})$ of the Poincaré series $P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{t})$.

Proposition 45 $\eta_{\mathbf{n}}(\mathfrak{a}, \mathfrak{b})$ *is equal to the number of vector subspaces D of $\widetilde{V}$ such that for each $i = 1, \cdots, m$, D contains a vector with i-th entry of order 0 and it satisfies $V \cdot D \subseteq D \oplus \widehat{\mathfrak{f}}$, $D \oplus \widehat{\mathfrak{f}} \sim B \oplus \widehat{\mathfrak{f}}$ and $(\pi^{n_1}, \cdots, \pi^{n_m}) A \subseteq D \oplus \widehat{\mathfrak{f}}$.*

4.2 Examples

In this section we present some examples of zeta functions and multi-variable Poincaré series of local rings of singular curves defined over a finite field. We observe that this series are determined by the semigroup of values of the local ring when it is a residually rational ring. This is no longer true when the local ring is not residually rational, as the following example shows. Some examples also suggest the behavior of these series under ground field extension.

Example 46 Let X be the plane projective cubic curve cut out by the absolutely irreducible homogeneous equation $y^2z + x^2z - x^3 = 0$ which is defined over the field $\mathbb{F}_q$ of characteristic different from 2 and such that $q - 1$ is not divisible by 4. This curve has its unique singularity at the point $P = (0 : 0 : 1)$. We denote by $\mathcal{O}$ the local ring $\mathcal{O}_{X,P}$ and by $\widehat{\mathcal{O}}$ its completion. Then $\widehat{\mathcal{O}} \simeq \mathbb{F}_q[[x, y]]/ (y^2 + x^2 - x^3)$, whose minimal primes correspond bijectively to the irreducible factors of $y^2 + x^2 - x^3$ in $\mathbb{F}_q[[x, y]]$, that is, the branches of X at the point P. Let us consider the finite field extension $\mathbb{F}_{q^2} \mid \mathbb{F}_q$, so that $\mathbb{F}_{q^2}$ is isomorphic to the field $\mathbb{F}_q[T]/ (T^2 + 1)$. Thus

$$\begin{aligned}\widehat{\mathcal{O}} &\simeq \mathbb{F}_q[[x]][y]/ \left(y^2 + x^2 - x^3\right) \\ &\subseteq \mathbb{F}_q((x))[y]/ \left(y^2 + x^2 - x^3\right) \\ &\subseteq \mathbb{F}_{q^2}((x))[y]/ \left(y^2 + x^2 - x^3\right).\end{aligned}$$

Since, in $\mathbb{F}_{q^2}((x))[y]$, $y^2+x^2-x^3 = (y-\xi x \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} x^n)(y+\xi x \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} x^n)$, where $\xi \in \mathbb{F}_{q^2}$ is a zero of T^2+1 in $\mathbb{F}_{q^2}$, then it follows by the Chinese remainder theorem that

$$\begin{aligned}\frac{\mathbb{F}_{q^2}((x))[y]}{(y^2+x^2-x^3)} &\simeq \mathbb{F}_{q^2}((x))[y]/(y-\xi x\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n)(y+\xi x\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n) \\ &\simeq \mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x)).\end{aligned}$$

Therefore the ring $\widehat{\mathcal{O}}$ is isomorphic to some subring of $\mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x))$. By Weierstrass division theorem, it follows that $\widehat{\mathcal{O}} \approx \mathbb{F}_q[[x]] \oplus \mathbb{F}_q[[x]]y$. That means, if $f \in \mathbb{F}_q[[x]][y]$, then $f = h\,(y^2+x^2-x^3)+r$, where $h, r \in \mathbb{F}_q[[x]][y]$ and $\deg r \leq 1$, so $\overline{f} = \overline{r}$ in $\mathbb{F}_q[[x]][y]/\,(y^2+x^2-x^3)$ and the image in $\mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x))$ of the class $\overline{f}$, under above isomorphism, is

$$(r(x,\xi x\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n), r(x,-\xi x\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n)) \in \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_{q^2}[[x]].$$

Thus, the ring $\widehat{\mathcal{O}}$ is isomorphic to the subring

$$\{(r(x,\xi x\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n), r(x,-\xi x\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n)) : r \in \mathbb{F}_q[[x]][y], \deg r \leq 1\}$$

of the ring $\mathbb{F}_{q^2}[[x]] \times \mathbb{F}_{q^2}[[x]]$. On the other hand, Galois group of the field extension $\mathbb{F}_{q^2} \mid \mathbb{F}_q$ is generated by Frobenius automorphism σ which acts on the field $\mathbb{F}_{q^2}$ by $\sigma(c) = c^q$. Note that ξ and $-\xi$ are the zeros of the polynomial T^2+1 in $\mathbb{F}_{q^2}$ and $\sigma(\xi) = -\xi$. Then the ring $\widehat{\mathcal{O}}$ is isomorphic to the ring

$$\{r(x,\xi x\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}x^n) \in \mathbb{F}_{q^2}[[x]] : r \in \mathbb{F}_q[[x]][y], \deg r \leq 1\},$$

that is, $\widehat{\mathcal{O}}$ is isomorphic to the ring $\mathbb{F}_q \oplus x\mathbb{F}_{q^2}[[x]]$. Thus, we have $\varrho = 1$, $m = 1$, $\mathbf{r} = 2$, $\widehat{\widetilde{\mathcal{O}}} = \mathbb{F}_{q^2}[[x]]$ and

$$\widehat{\mathcal{O}} \simeq \mathbb{F}_q \oplus \widehat{\mathfrak{f}},$$

where $\widehat{\mathfrak{f}} = x\mathbb{F}_{q^2}[[x]]$. Hence $\delta = 1$ and $\widehat{\widetilde{\mathcal{O}}}/\widehat{\mathfrak{f}} \cong \mathbb{F}_{q^2}$. The $\widehat{\mathcal{O}}$-ideals $\widehat{\mathfrak{b}}$ that satisfy $\widehat{\mathfrak{b}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$ contain the conductor ideal $\widehat{\mathfrak{f}}$ and, therefore, correspond bijectively to the vector subspaces of $\mathbb{F}_{q^2}$ that contain a vector of order 0. By writing their bases into standard forms we obtain the following list:

$$\begin{aligned}&\mathbb{F}_q\,(1+a\xi) \oplus \widehat{\mathfrak{f}}, \text{ with } a \in \mathbb{F}_q \\ &\mathbb{F}_q\,(\xi) \oplus \widehat{\mathfrak{f}} \\ &\mathbb{F}_{q^2} \oplus \widehat{\mathfrak{f}}.\end{aligned}$$

We pick up representatives

$$\widehat{\mathfrak{b}}_1 = \mathbb{F}_q \oplus \widehat{\mathfrak{f}},$$

by putting $a = 0$ in the first line, and

$$\widehat{\mathfrak{b}}_2 = \mathbb{F}_{q^2} \oplus \widehat{\mathfrak{f}}.$$

The first and second lines are in the ideal class represented by $\widehat{\mathfrak{b}}_1$. Let $\mathfrak{b}_i := \widehat{\mathfrak{b}}_i \cap K$ (i=1,2) be the corresponding representatives of the ideal classes of $\mathcal{O}$. Then $\mathfrak{b}_1 = \mathcal{O}$, and $\mathfrak{b}_2 = \widetilde{\mathcal{O}}$ and we obtain: $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathcal{O}}\right] = q + 1$,

$$\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t\right) = 1 + qt \text{ and } \Lambda_{\mathcal{O}}(\mathcal{O}, \widetilde{\mathcal{O}}, t) = 1.$$

Moreover, $Z_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t\right) = \frac{1+qt^2}{1-t^2}$, $Z_{\mathcal{O}}(\mathcal{O}, \widetilde{\mathcal{O}}, t) = \frac{t}{1-t^2}$, $Z_{\mathcal{O}}\left(\mathcal{O}, t\right) = \frac{1+t+qt^2}{1-t^2}$ and $S\left(\mathcal{O}\right) = \mathbb{N}$.

In the precedent example the local ring $\mathcal{O}$ of a singular curve defined over a finite field $\mathbb{F}_q$ is not residually rational and for each ideal class $\mathfrak{b}$ the set $S\left(\mathfrak{b}\right)$ is equal to $\mathbb{N}$. In the next example we consider the curve defined by the same equation of the precedent example but, in this case, the local ring is residually rational. We observe that, if $\widehat{\mathbb{F}_{q^2}\mathcal{O}}$ is the completion of the semilocal ring $\mathbb{F}_{q^2}\mathcal{O}$ which is the extension of the local ring $\mathcal{O}$ to the function field $\mathbb{F}_{q^2}K \mid \mathbb{F}_{q^2}$, then $\widehat{\mathbb{F}_{q^2}\mathcal{O}} \simeq \mathbb{F}_{q^2}\left(1,1\right) \oplus \widehat{\mathfrak{f}}$ where $\widehat{\mathfrak{f}} = x\mathbb{F}_{q^2}[[x]] \times x\mathbb{F}_{q^2}[[x]]$.

Example 47 We consider the completion of the local ring of a rational node: $\widehat{\mathcal{O}} = \mathbb{F}_q\left(1,1\right) \oplus \widehat{\mathfrak{f}}$ where $\widehat{\mathfrak{f}} = \pi\mathbb{F}_q[[\pi]] \times \pi\mathbb{F}_q[[\pi]]$. Then $\mathcal{O}$ is a Gorenstein ring of singularity degree $\delta = 1$, and $\widehat{\widetilde{\mathcal{O}}}/\widehat{\mathfrak{f}} \cong \mathbb{F}_q \times \mathbb{F}_q$. The $\mathcal{O}$-ideals $\mathfrak{b}$ that satisfy $\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ contain the conductor $\mathfrak{f}$ and, therefore, correspond bijectively to the vector subspaces of $\mathbb{F}_q \times \mathbb{F}_q$ that, for $i = 1, 2$, contain a vector whose i-th entry is not zero. There are only two bases in standard forms, namely

$$\begin{array}{l} \mathbb{F}_q\left(1, a\right) \\ \mathbb{F}_q\left(1, 0\right) \oplus \mathbb{F}_q\left(0, 1\right) \end{array}$$

where a varies over the multiplicative group $\mathbb{F}_q^*$. Each of them represent one ideal class, namely the class of principal ideals and the class of $\widetilde{\mathcal{O}}$-ideals. Thus, we obtain $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathcal{O}}\right] = q - 1$,

$$P_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t_1, t_2\right) = \frac{1 - t_1 - t_2 + qt_1t_2}{\left(1 - t_1\right)\left(1 - t_2\right)},$$

$Z_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t\right) = \frac{1-2t+qt^2}{\left(1-t\right)^2}$, $Z_{\mathcal{O}}\left(\mathcal{O}, t\right) = \frac{1-t+qt^2}{\left(1-t\right)^2}$ and $S(\mathcal{O}) = \{(0,0)\} \cup ((1,1) + \mathbb{N}^2)$.

In the following example we compute the zeta series and the multi-variable Poincaré series of a singularity with two brunches which is not residually rational.

Example 48 Let X be the plane projective quartic curve cut out by the absolutely irreducible homogeneous equation $y^3z - (x^3z + x^4) = 0$ with base field $\mathbb{F}_q$. Assume that in the field $\mathbb{F}_q$ the polynomial $T^2 + T + 1 \in \mathbb{F}_q[T]$ is irreducible. This curve has its unique singularity at the point $P = (0 : 0 : 1)$. We denote by $\mathcal{O}$ the local ring $\mathcal{O}_{X,P}$ and by $\widehat{\mathcal{O}}$ its completion. Then $\widehat{\mathcal{O}} \simeq \mathbb{F}_q[[x, y]]/\,(y^3 - (x^3 + x^4))$, whose minimal primes correspond bijectively to the irreducible factors of $y^3 - (x^3 + x^4)$ in $\mathbb{F}_q[[x, y]]$, that is, the branches of X at the point P. Let us consider the finite field extension $\mathbb{F}_{q^2} \mid \mathbb{F}_q$, so that $\mathbb{F}_{q^2}$ is isomorphic to the field $\mathbb{F}_q[T]/\,(T^2 + T + 1)$. Thus

$$\begin{aligned}\widehat{\mathcal{O}} &\simeq \mathbb{F}_q[[x]][y]/\left(y^3 - \left(x^3 + x^4\right)\right)\\ &\subseteq \mathbb{F}_q((x))[y]/\left(y^3 - \left(x^3 + x^4\right)\right)\\ &\subseteq \mathbb{F}_{q^2}((x))[y]/\left(y^3 - \left(x^3 + x^4\right)\right).\end{aligned}$$

Since, in $\mathbb{F}_{q^2}((x))[y]$, $y^3 - (x^3 + x^4) = \prod_{j=1}^{3}(y - \xi^j x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n)$, where $\xi \in \mathbb{F}_{q^2}$ is a zero of $T^2 + T + 1$ in $\mathbb{F}_{q^2}$, it follows by the Chinese remainder theorem that

$$\begin{aligned}\mathbb{F}_{q^2}((x))[y]/\left(y^3 - \left(x^3 + x^4\right)\right) &\simeq \prod_{j=1}^{3} \mathbb{F}_{q^2}((x))[y]/(y - \xi^j x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n)\\ &\simeq \mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x)).\end{aligned}$$

Therefore the ring $\widehat{\mathcal{O}}$ is isomorphic to some subring of $\mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x))$. By Weierstrass division theorem it follows that $\widehat{\mathcal{O}} \approx \mathbb{F}_q[[x]] \oplus \mathbb{F}_q[[x]]y \oplus \mathbb{F}_q[[x]]y^2$. That means, if $f \in \mathbb{F}_q[[x]][y]$ then $f = q\,(y^3 - (x^3 + x^4)) + r$, where $q, r \in \mathbb{F}_q[[x]][y]$ and $\deg r \leq 2$, so that $\overline{f} = \overline{r}$ in $\mathbb{F}_q[[x]][y]/\,(y^3 - (x^3 + x^4))$ and the image in $\mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x)) \times \mathbb{F}_{q^2}((x))$ of the class $\overline{f}$, under above isomorphism, is

$$(r(x, \xi^j x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n))_{j=1,2,3} \in \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_q[[x]].$$

Thus, the ring $\widehat{\mathcal{O}}$ is isomorphic to the subring

$$\{(r(x, \xi^j x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n))_{j=1,2,3} \in \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_q[[x]] : r \in \mathbb{F}_q[[x]][y], \deg r \leq 2\}$$

of the ring $\mathbb{F}_{q^2}[[x]] \times \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_q[[x]]$. On the other hand, Galois group of the field extension $\mathbb{F}_{q^2} \mid \mathbb{F}_q$ is generated by Frobenius automorphism σ which acts on $\mathbb{F}_{q^2}$

by $\sigma(c) = c^q$. Note that ξ, and ξ^2 are the zeros of the polynomial T^2+T+1 in $\mathbb{F}_{q^2}$ and $\sigma(\xi) = \xi^2$ and $\sigma(\xi^3) = \xi$. Then the ring $\widehat{\mathcal{O}}$ is isomorphic to the ring $\{(r(x, \xi x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n), r(x, x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n)) \in \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_q[[x]] : r \in \mathbb{F}_q[[x]][y], \deg r \leq 2\}$, that is, $\widehat{\mathcal{O}}$ is isomorphic to the ring $\mathbb{F}_q(1,1) \oplus \mathbb{F}_q(x,x) \oplus \mathbb{F}_q(\xi x, x) \oplus x^2\mathbb{F}_{q^2}[[x]] \times x^2\mathbb{F}_q[[x]]$. Therefore, we have $\varrho = 1$, $m = 2$, $\mathbf{r} = (2,1)$, $\widehat{\widetilde{\mathcal{O}}} = \mathbb{F}_{q^2}[[x]] \times \mathbb{F}_q[[x]]$ and

$$\widehat{\mathcal{O}} \simeq \mathbb{F}_q(1,1) \oplus \mathbb{F}_q(x,x) \oplus \mathbb{F}_q(\xi x, x) \oplus \widehat{\mathfrak{f}},$$

where $\widehat{\mathfrak{f}} = x^2\mathbb{F}_{q^2}[[x]] \times x^2\mathbb{F}_q[[x]]$. Hence $\delta = 3$ and $\widehat{\widetilde{\mathcal{O}}}/\widehat{\mathfrak{f}} \cong (\mathbb{F}_{q^2} \oplus \mathbb{F}_{q^2}x) \times (\mathbb{F}_q \oplus \mathbb{F}_q x)$. The $\widehat{\mathcal{O}}$-ideals $\widehat{\mathfrak{b}}$ that satisfy $\widehat{\mathfrak{b}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$ contain the conductor ideal $\widehat{\mathfrak{f}}$ and, therefore, correspond bijectively to the vector subspaces of $(\mathbb{F}_{q^2} \oplus \mathbb{F}_{q^2}x) \times (\mathbb{F}_q \oplus \mathbb{F}_q x)$ that contain, for $i = 1,2$, a vector whose i-th entry has order 0. By writing their bases into standard forms we obtain the following list:

$\mathbb{F}_q(a_1 - a_1\xi + a_2 x, 1) \oplus \mathbb{F}_q(a_1 x, x) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$, with $a_1 \neq 0$
$\mathbb{F}_q(a_1 + a_2\xi + a_3\xi x, 1) \oplus \mathbb{F}_q(a_4\xi x, x) \oplus \mathbb{F}_q(x + a_5\xi x, 0) \oplus \widehat{\mathfrak{f}}$, with $a_1 + a_2\xi \neq 0$,
$a_2 = a_4 + a_1 a_5$, and $a_1 - a_2 = a_4 - a_2 a_5$
$\mathbb{F}_q(a_1 + a_2\xi, 1) \oplus \mathbb{F}_q(0, x) \oplus \mathbb{F}_q(x, 0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$, with $a_1 + a_2\xi \neq 0$
$\mathbb{F}_q(a_1, 1) \oplus \mathbb{F}_q(\xi, 0) \oplus \mathbb{F}_q(0, x) \oplus \mathbb{F}_q(x, 0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$,
$\mathbb{F}_q(a_1\xi, 1) \oplus \mathbb{F}_q(1 + a_2\xi, 0) \oplus \mathbb{F}_q(0, x) \oplus \mathbb{F}_q(x, 0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$,
$\mathbb{F}_q(0, 1) \oplus \mathbb{F}_q(1, 0) \oplus \mathbb{F}_q(\xi, 0) \oplus \mathbb{F}_q(0, x) \oplus \mathbb{F}_q(x, 0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$

where $a_i \in \mathbb{F}_q$, for each $i = 1,2,3,4,5$ (in each line). We pick up representatives

$$\widehat{\mathfrak{b}}_1 = \mathbb{F}_q(1,1) \oplus \mathbb{F}_q(\xi x, x) \oplus \mathbb{F}_q(x - \xi x, 0) \oplus \widehat{\mathfrak{f}}$$

by putting $a_1 = 1$ and $a_2 = 0$ in the second line,

$$\widehat{\mathfrak{b}}_2 = \mathbb{F}_q(1,1) \oplus \mathbb{F}_q(0,x) \oplus \mathbb{F}_q(x,0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$$

by putting $a_1 = 1$ and $a_2 = 0$ in the third line,

$$\widehat{\mathfrak{b}}_3 = \mathbb{F}_q(1,1) \oplus \mathbb{F}_q(\xi,0) \oplus \mathbb{F}_q(0,x) \oplus \mathbb{F}_q(x,0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$$

by putting $a_1 = 1$ in the fourth line,

$$\widehat{\mathfrak{b}}_4 = \mathbb{F}_q(0,1) \oplus \mathbb{F}_q(\xi,0) \oplus \mathbb{F}_q(0,x) \oplus \mathbb{F}_q(x,0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$$

by putting $a_1 = 0$ in the fourth line, and

$$\widehat{\mathfrak{b}}_5 = \mathbb{F}_q(0,1) \oplus \mathbb{F}_q(1,0) \oplus \mathbb{F}_q(\xi,0) \oplus \mathbb{F}_q(0,x) \oplus \mathbb{F}_q(x,0) \oplus \mathbb{F}_q(\xi x, 0) \oplus \widehat{\mathfrak{f}}$$

of each ideal class. The first and second lines are in the ideal class represented by $\widehat{\mathfrak{b}}_1$, the third line is in the ideal class represented by $\widehat{\mathfrak{b}}_2$. If $a_1 \neq 0$ in the fourth and fifth lines, then they are in the ideal class represented by $\widehat{\mathfrak{b}}_3$, otherwise they are in the ideal class represented by $\widehat{\mathfrak{b}}_4$. Let $\mathfrak{b}_i := \widehat{\mathfrak{b}}_i \cap K$ (i=1,2,3,4,5) be the corresponding representatives of the ideal classes of $\mathcal{O}$. Then $\mathfrak{b}_1 = \mathcal{O}$, $\mathfrak{b}_5 = \widetilde{\mathcal{O}}$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathcal{O}}\right] = q\left(q^2-1\right)$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_2}\right] = q^2-1$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_3}\right] = q^2-1$, and $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_4}\right] = q+1$. Moreover, we obtain:

$$\begin{aligned}
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t_1, t_2\right) &= 1 - t_1 - t_2 + \left(q^2+1\right) t_1 t_2 - q^2 t_1 t_2^2 - q t_1^2 t_2 + q^3 t_1^2 t_2^2 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_2, t_1, t_2\right) &= 1 - t_1 - t_2 + q^2 t_1 t_2 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_3, t_1, t_2\right) &= q - t_1 - q t_2 + q^2 t_1 t_2 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_4, t_1, t_2\right) &= 1 + q t_2 \\
\Lambda_{\mathcal{O}}(\mathcal{O}, \widetilde{\mathcal{O}}, t_1, t_2) &= 1,
\end{aligned}$$

and

$$\begin{aligned}
Z_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t\right) &= \frac{1 - t - t^2 + \left(q^2+1\right) t^3 - q t^4 - q^2 t^5 + q^3 t^6}{(1-t)\left(1-t^2\right)} \\
Z_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_2, t\right) &= \frac{t - t^2 - t^3 + q^2 t^4}{(1-t)\left(1-t^2\right)} \\
Z_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_3, t\right) &= \frac{q t^2 - t^3 - q t^4 + q^2 t^5}{(1-t)\left(1-t^2\right)} \\
Z_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_4, t\right) &= \frac{t^2 + q t^4}{(1-t)\left(1-t^2\right)} \\
Z_{\mathcal{O}}(\mathcal{O}, \widetilde{\mathcal{O}}, t) &= \frac{t^3}{(1-t)\left(1-t^2\right)}.
\end{aligned}$$

Moreover, $Z_{\mathcal{O}}\left(\mathcal{O}, t\right) = \frac{1+(q-1)t^2+q^2t^3+\left(q^2-q\right)t^4+q^3t^6}{(1-t)(1-t^2)}$

$$S\left(\mathcal{O}\right) = \{(0,0), (1,1)\} \cup ((2,1) + \mathbb{N} \times \{0\}) \cup ((2,2) + \mathbb{N} \times \mathbb{N})$$

$$S\left(\mathfrak{b}_2\right) = \{(0,0)\} \cup ((1,1) + \mathbb{N} \times \mathbb{N})$$

$$S\left(\mathfrak{b}_2\right) = \{(0,0)\} \cup ((1,0) + \mathbb{N} \times \{0\}) \cup ((1,1) + \mathbb{N} \times \mathbb{N})$$

$$S\left(\mathfrak{b}_3\right) = ((0,0) + \mathbb{N} \times \{0\}) \cup ((0,1) + \mathbb{N} \times \mathbb{N}).$$

Example 49 Let X be the plane projective quartic curve cut out by the absolutely irreducible homogeneous equation $y^3 z - a\left(x^3 z + x^4\right) = 0$ with base field $\mathbb{F}_q$, where $a \in \mathbb{F}_q$ and $a \notin \mathbb{F}_q^3$ i. e. the polynomial $T^3 - a \in \mathbb{F}_q[T]$ is irreducible over $\mathbb{F}_q$. This curve has its unique singularity at the point $P = (0:0:1)$. We denote by $\mathcal{O}$ the local ring $\mathcal{O}_{X,P}$ and by $\widehat{\mathcal{O}}$ its completion. Then $\widehat{\mathcal{O}} \simeq \mathbb{F}_q[[x,y]]/\left(y^3 - a\left(x^3 + x^4\right)\right)$, whose minimal primes correspond bijectively to the irreducible factors of $y^3 - a\left(x^3 + x^4\right)$ in

$\mathbb{F}_q[[x,y]]$, that is, the branches of X in the point P. Since $a \in \mathbb{F}_q$ and $a \notin \mathbb{F}_q^3$ i.e. $q \equiv 1$ mod 3, the Weierstrass polynomial $y^3 - a\left(x^3 + x^4\right)$ is irreducible in $\mathbb{F}_q[[x]][y]$, hence in $\mathbb{F}_q[[x,y]]$, and it follows that $(a^{\frac{q-1}{3}})^j \in \mathbb{F}_q$ for each non negative integer j, $(a^{\frac{q-1}{3}})^3 = 1$ and $1 + a^{\frac{q-1}{3}} + (a^{\frac{q-1}{3}})^2 = 0$. Let us consider the finite field extension $\mathbb{F}_{q^3} \mid \mathbb{F}_q$, so that $a \in \mathbb{F}_{q^3}^3$, that is, $\mathbb{F}_{q^3}$ is isomorphic to the field $\mathbb{F}_q[T]/\left(T^3 - a\right)$. Thus

$$\begin{aligned}\widehat{\mathcal{O}} &\simeq \mathbb{F}_q[[x]][y]/\left(y^3 - a\left(x^3 + x^4\right)\right)\\ &\subseteq \mathbb{F}_q((x))[y]/\left(y^3 - a\left(x^3 + x^4\right)\right)\\ &\subseteq \mathbb{F}_{q^3}((x))[y]/\left(y^3 - a\left(x^3 + x^4\right)\right).\end{aligned}$$

Since, in $\mathbb{F}_{q^3}((x))[y]$, $y^3 - a\left(x^3 + x^4\right) = \prod\limits_{j=1}^{3}(y - (a^{\frac{q-1}{3}})^j \alpha x \sum\limits_{n=0}^{\infty} \binom{1/3}{n} x^n)$, where $\alpha \in \mathbb{F}_{q^3}$ is a zero of $T^3 - a$ in $\mathbb{F}_{q^3}$, it follows by the Chinese remainder theorem that

$$\begin{aligned}\mathbb{F}_{q^3}((x))[y]/\left(y^3 - a\left(x^3 + x^4\right)\right) &\simeq \prod_{j=1}^{3} \mathbb{F}_{q^3}((x))[y]/(y - (a^{\frac{q-1}{3}})^j \alpha x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n)\\ &\simeq \mathbb{F}_{q^3}((x)) \times \mathbb{F}_{q^3}((x)) \times \mathbb{F}_{q^3}((x)).\end{aligned}$$

Therefore the ring $\widehat{\mathcal{O}}$ is isomorphic to some subring of $\mathbb{F}_{q^3}((x)) \times \mathbb{F}_{q^3}((x)) \times \mathbb{F}_{q^3}((x))$. By Weierstrass division theorem it follows that $\widehat{\mathcal{O}} \approx \mathbb{F}_q[[x]] \oplus \mathbb{F}_q[[x]]y \oplus \mathbb{F}_q[[x]]y^2$. That means, if $f \in \mathbb{F}_q[[x]][y]$ then $f = q\left(y^3 - a\left(x^3 + x^4\right)\right) + r$, where $q, r \in \mathbb{F}_q[[x]][y]$ and $\deg r \leq 2$, so that $\overline{f} = \overline{r}$ in $\mathbb{F}_q[[x]][y]/\left(y^3 - a\left(x^3 + x^4\right)\right)$ and the image in $\mathbb{F}_{q^3}((x)) \times \mathbb{F}_{q^3}((x)) \times \mathbb{F}_{q^3}((x))$ of the class $\overline{f}$, under above isomorphism, is

$$(r(x, (a^{\frac{q-1}{3}})^j \alpha x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n))_{j=1,2,3} \in \mathbb{F}_{q^3}[[x]] \times \mathbb{F}_{q^3}[[x]] \times \mathbb{F}_{q^3}[[x]].$$

Thus, the ring $\widehat{\mathcal{O}}$ is isomorphic to the subring

$$\{(r(x, (a^{\frac{q-1}{3}})^j \alpha x \sum_{n=0}^{\infty} \binom{1/3}{n} x^n))_{j=1,2,3} \in \left(\mathbb{F}_{q^3}[[x]]\right)^{\oplus 3} : r \in \mathbb{F}_q[[x]][y], \deg r \leq 2\}$$

of the ring $\mathbb{F}_{q^3}[[x]] \times \mathbb{F}_{q^3}[[x]] \times \mathbb{F}_{q^3}[[x]]$. On the other hand, Galois group of the field extension $\mathbb{F}_{q^3} \mid \mathbb{F}_q$ is generated by Frobenius automorphism σ which acts on $\mathbb{F}_{q^3}$ by $\sigma(c) = c^q$. Note that $a^{\frac{q-1}{3}}\alpha$, $(a^{\frac{q-1}{3}})^2\alpha$, and $(a^{\frac{q-1}{3}})^3\alpha$ are the zeros of the polynomial $T^3 - a$ in $\mathbb{F}_{q^3}$ and $\sigma(\alpha) = a^{\frac{q-1}{3}}\alpha$. Then the ring $\widehat{\mathcal{O}}$ is isomorphic to the ring $\{r(x, \alpha x \sum\limits_{n=0}^{\infty} \binom{1/3}{n} x^n) \in \mathbb{F}_{q^3}[[x]] : r \in \mathbb{F}_q[[x]][y], \deg r \leq 2\}$, that is, $\widehat{\mathcal{O}}$ is isomorphic to the ring
$\{A + \alpha x(\sum\limits_{n=0}^{\infty} \binom{1/3}{n} x^n)B + \alpha^2 x^2(\sum\limits_{n=0}^{\infty} \binom{1/3}{n} x^n)^2 C \in \mathbb{F}_{q^3}[[x]] : A, B, C \in \mathbb{F}_q[[x]]\}$ i.e.

$$\widehat{\mathcal{O}} \simeq \mathbb{F}_q \oplus x\mathbb{F}_q \oplus \alpha x\mathbb{F}_q \oplus x^2\mathbb{F}_{q^3}[[x]].$$

Thus, we have $\varrho = 1$, $m = 1$, $\mathbf{r} = 3$, $\widehat{\widetilde{\mathcal{O}}} = \mathbb{F}_{q^3}[[x]]$ and $\widehat{\mathcal{O}} \simeq \mathbb{F}_q \oplus x\mathbb{F}_q \oplus \alpha x\mathbb{F}_q \oplus \widehat{\mathfrak{f}}$ where $\widehat{\mathfrak{f}} = x^2\mathbb{F}_{q^3}[[x]]$. Hence $\delta = 3$ and $\widehat{\widetilde{\mathcal{O}}}/\widehat{\mathfrak{f}} \cong \mathbb{F}_{q^3} \oplus \mathbb{F}_{q^3}x$. The $\widehat{\mathcal{O}}$-ideals $\widehat{\mathfrak{b}}$ that satisfy $\widehat{\mathfrak{b}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$ contain the conductor ideal $\widehat{\mathfrak{f}}$ and, therefore, correspond bijectively to the vector subspaces of $\mathbb{F}_{q^3} \oplus \mathbb{F}_{q^3}x$ that contain a vector of order 0. By writing their bases into standard forms we obtain the following list:

$$
\begin{aligned}
&\mathbb{F}_q\left(1 + a_1\alpha + a_2\alpha^2 + a_3\alpha^2 x\right) \oplus \mathbb{F}_q\left(x + a_4\alpha^2 x\right) \oplus \mathbb{F}_q\left(\alpha x + a_5\alpha^2 x\right) \oplus \widehat{\mathfrak{f}},\\
&\text{with } a_1 = aa_2a_4 + a_5 \text{ and } a_2 = a_4 + a_1a_5\\
&\mathbb{F}_q\left(1 + a_1\alpha + \tfrac{1}{aa_1}\alpha^2 + a_3\alpha x\right) \oplus \mathbb{F}_q\left(x + a_1\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}, \text{ with } \ a_1 \neq 0\\
&\mathbb{F}_q\left(\alpha + a_1\alpha^2 + a_2\alpha^2 x\right) \oplus \mathbb{F}_q\left(x + \tfrac{1}{aa_1}\alpha x\right) \oplus \mathbb{F}_q\left(\alpha x + a_1\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}, \text{ with } a_1 \neq 0\\
&\mathbb{F}_q\left(\alpha + a_1 x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}},\\
&\mathbb{F}_q\left(\alpha^2 + a_1\alpha x\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}},\\
&\mathbb{F}_q\left(1 + a_1\alpha + a_2\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}\\
&\mathbb{F}_q\left(\alpha + a_1\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}\\
&\mathbb{F}_q\left(\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}\\
&\mathbb{F}_q\left(1 + a_1\alpha^2\right) \oplus \mathbb{F}_q\left(\alpha + a_2\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}\\
&\mathbb{F}_q\left(1 + a_1\alpha\right) \oplus \mathbb{F}_q\left(\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}\\
&\mathbb{F}_q\left(\alpha\right) \oplus \mathbb{F}_q\left(\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}\\
&\mathbb{F}_q\left(1\right) \oplus \mathbb{F}_q\left(\alpha\right) \oplus \mathbb{F}_q\left(\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}
\end{aligned}
$$

where $a_i \in \mathbb{F}_q$, for each $i = 1, 2, 3, 4, 5$ (in each line). We pick up representatives of each ideal class by putting $a_1 = 0$ and $a_2 = 0$ in the first, sixth, and ninth line, respectively, so

$$
\begin{aligned}
\widehat{\mathfrak{b}}_1 &= \mathbb{F}_q\left(1\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \widehat{\mathfrak{f}},\\
\widehat{\mathfrak{b}}_2 &= \mathbb{F}_q\left(1\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}},\\
\widehat{\mathfrak{b}}_3 &= \mathbb{F}_q\left(1\right) \oplus \mathbb{F}_q\left(\alpha\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}} \text{ and}\\
\widehat{\mathfrak{b}}_4 &= \mathbb{F}_q\left(1\right) \oplus \mathbb{F}_q\left(\alpha\right) \oplus \mathbb{F}_q\left(\alpha^2\right) \oplus \mathbb{F}_q\left(x\right) \oplus \mathbb{F}_q\left(\alpha x\right) \oplus \mathbb{F}_q\left(\alpha^2 x\right) \oplus \widehat{\mathfrak{f}}.
\end{aligned}
$$

The ideals in the lines 1,2,3,4, and 5 are in the ideal class represented by $\widehat{\mathfrak{b}}_1$; the ideals in the lines 6,7, and 8 are in the ideal class represented by $\widehat{\mathfrak{b}}_2$; the ideals in the lines 9,10, and 11 are in the ideal class represented by $\widehat{\mathfrak{b}}_3$. Let $\mathfrak{b}_i := \widehat{\mathfrak{b}}_i \cap K$ (i=1,2,3,4) be the corresponding representatives of the ideal classes of $\mathcal{O}$. Then $\mathfrak{b}_1 = \mathcal{O}$, $\mathfrak{b}_4 = \widetilde{\mathcal{O}}$, $[U_{\widetilde{\mathcal{O}}} : U_{\mathcal{O}}] = q^3 + q^2 + q$, $[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_2}] = q^2 + q + 1$, $[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_3}] = q^2 + q + 1$. Moreover, we obtain:

$$
\begin{aligned}
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t\right) &= 1 + \left(q^2 + q - 1\right)t + q^3t^2\\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_2, t\right) &= 1 + \left(q^2 + q\right)t\\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_3, t\right) &= (q + 1) + q^2t\\
\Lambda_{\mathcal{O}}(\mathcal{O}, \widetilde{\mathcal{O}}, t) &= 1,
\end{aligned}
$$

and

$$\begin{aligned} Z_{\mathcal{O}}(\mathcal{O},\mathcal{O},t) &= \frac{1+(q^2+q-1)\,t^3+q^3t^6}{1-t^3} \\ Z_{\mathcal{O}}(\mathcal{O},\mathfrak{b}_2,t) &= \frac{t+(q^2+q)\,t^4}{1-t^3} \\ Z_{\mathcal{O}}(\mathcal{O},\mathfrak{b}_3,t) &= \frac{(q+1)\,t^2+q^2t^5}{1-t^3} \\ Z_{\mathcal{O}}(\mathcal{O},\widetilde{\mathcal{O}},t) &= \frac{t^3}{1-t^3}. \end{aligned}$$

Furthermore, $Z_{\mathcal{O}}(\mathcal{O},t) = \frac{1+t+(q+1)t^2+\left(q^2+q\right)t^3+\left(q^2+q\right)t^4+q^2t^5+q^3t^6}{1-t^3}$ and $S(\mathfrak{b}_i) = \mathbb{N}$ for $i=1,2,3,4$.

In the earlier example the local ring $\mathcal{O}$ of a singular curve defined over a finite field $\mathbb{F}_q$ is not residually rational. As in the first example, we have that for each ideal class $\mathfrak{b}$ the set $S(\mathfrak{b})$ is equal to $\mathbb{N}$.

Now, we consider the curve defined by the same equation of the precedent example but, in this case, the local ring is residually rational. We can obtain it by doing extension of the ground field.

Example 50 Let X be the plane projective quartic curve cut out by the absolutely irreducible homogeneous equation $y^3z - a\,(x^3z+x^4) = 0$, where $a \in \mathbb{F}_q$ and there exists $\alpha \in \mathbb{F}_q$ such that $\alpha^3 = a$. This curve has its unique singularity at the point $P = (0:0:1)$. We denote by $\mathcal{O}$ the local ring $\mathcal{O}_{X,P}$ and by $\widehat{\mathcal{O}}$ its completion. Then $\widehat{\mathcal{O}} \simeq \mathbb{F}_q[[x,y]]/\left(y^3 - a\,(x^3+x^4)\right)$, whose minimal primes correspond bijectively to the irreducible factors of $y^3 - a\,(x^3+x^4)$ in $\mathbb{F}_q[[x,y]]$, that is, the branches of X at the point P. Thus

$$\begin{aligned} \widehat{\mathcal{O}} &\simeq \mathbb{F}_q[[x]][y]/\left(y^3 - a\left(x^3+x^4\right)\right) \\ &\subseteq \mathbb{F}_q((x))[y]/\left(y^3 - a\left(x^3+x^4\right)\right). \end{aligned}$$

If, in $\mathbb{F}_q((x))[y]$, $y^3 - a\,(x^3+x^4) = \prod_{\theta^3=1}(y - \theta\alpha x\sum_{n=0}^{\infty}\binom{1/3}{n}x^n)$, then, by the Chinese remainder theorem, it follows that

$$\begin{aligned} \mathbb{F}_q((x))[y]/\left(y^3 - a\left(x^3+x^4\right)\right) &\simeq \prod_{\theta^3=1}\mathbb{F}_q((x))[y]/(y-\theta\alpha x\sum_{n=0}^{\infty}\tbinom{1/3}{n}x^n) \\ &\simeq \mathbb{F}_q((x)) \times \mathbb{F}_q((x)) \times \mathbb{F}_q((x)). \end{aligned}$$

Therefore, the ring $\widehat{\mathcal{O}}$ is isomorphic to some subring of $\mathbb{F}_q((x)) \times \mathbb{F}_q((x)) \times \mathbb{F}_q((x))$. By Weierstrass division theorem it follows that $\widehat{\mathcal{O}} \approx \mathbb{F}_q[[x]] \oplus \mathbb{F}_q[[x]]y \oplus \mathbb{F}_q[[x]]y^2$. That means, if $f \in \mathbb{F}_q[[x]][y]$ then $f = q\left(y^3 - a\left(x^3 + x^4\right)\right) + r$, where $q, r \in \mathbb{F}_q[[x]][y]$ and $\deg r \leq 2$, so that $\overline{f} = \overline{r}$ in $\mathbb{F}_q[[x]][y]/\left(y^3 - a\left(x^3 + x^4\right)\right)$ and the image in $\mathbb{F}_q((x)) \times \mathbb{F}_q((x)) \times \mathbb{F}_q((x))$ of the class $\overline{f}$, under above isomorphism, is

$$(r(x, \xi^j \alpha x \sum_{n=0}^{\infty} \tbinom{1/3}{n} x^n))_{j=1,2,3} \in \mathbb{F}_q[[x]] \times \mathbb{F}_q[[x]] \times \mathbb{F}_q[[x]],$$

where $\xi^2 + \xi + 1 = 0$ and $\xi \neq 1$. Thus, the ring $\widehat{\mathcal{O}}$ is isomorphic to the subring

$$\{(r(x, \xi^j \alpha x \sum_{n=0}^{\infty} \tbinom{1/3}{n} x^n))_{j=1,2,3} \in (\mathbb{F}_q[[x]])^{\oplus 3} : r \in \mathbb{F}_q[[x]][y], \deg r \leq 2\}$$

of the ring $\mathbb{F}_q[[x]] \times \mathbb{F}_q[[x]] \times \mathbb{F}_q[[x]]$. Therefore, the ring $\widehat{\mathcal{O}}$ is isomorphic to the ring $\mathbb{F}_q(1,1,1) \oplus \mathbb{F}_q(x,x,x) \oplus \mathbb{F}_q(\xi\alpha x, \xi^2\alpha x, \alpha x) \oplus \widehat{\mathfrak{f}}$, that is,

$$\widehat{\mathcal{O}} \simeq \mathbb{F}_q(1,1,1) \oplus \mathbb{F}_q(x,x,x) \oplus \mathbb{F}_q\left(\xi\alpha x, \xi^2\alpha x, \alpha x\right) \oplus \widehat{\mathfrak{f}},$$

where $\widehat{\mathfrak{f}} = \left(x^2\mathbb{F}_q[[x]]\right)^{\oplus 3}$. Thus, $\varrho = 1$, $m = 3$, $\mathbf{r} = (1,1,1)$ and $\widehat{\widetilde{\mathcal{O}}} = \mathbb{F}_q[[x]]^{\oplus 3}$. Hence $\delta = 3$ and $\widehat{\widetilde{\mathcal{O}}}/\widehat{\mathfrak{f}} \cong (\mathbb{F}_q \oplus \mathbb{F}_q x) \times (\mathbb{F}_q \oplus \mathbb{F}_q x) \times (\mathbb{F}_q \oplus \mathbb{F}_q x)$. The $\widehat{\mathcal{O}}$-ideals $\widehat{\mathfrak{b}}$ that satisfy $\widehat{\mathfrak{b}} \cdot \widehat{\widetilde{\mathcal{O}}} = \widehat{\widetilde{\mathcal{O}}}$ contain the conductor ideal $\widehat{\mathfrak{f}}$ and, therefore, correspond bijectively to the vector subspaces of $(\mathbb{F}_q \oplus \mathbb{F}_q x) \times (\mathbb{F}_q \oplus \mathbb{F}_q x) \times (\mathbb{F}_q \oplus \mathbb{F}_q x)$ that contain, for $i = 1,2,3$, a vector whose i-th entry has order 0. By writing their bases into standard forms we obtain the following list:

$\mathbb{F}_q(1, a_1, a_2 + a_3x) \oplus \mathbb{F}_q(x, 0, -\xi a_2 x) \oplus \mathbb{F}_q\left(0, x, -\xi^2 \frac{a_2}{a_1} x\right) \oplus \widehat{\mathfrak{f}}$, with $a_1 \neq 0$ and $a_2 \neq 0$,
$\mathbb{F}_q(1, a_1, a_2) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}}$, with $a_1 \neq 0$ and $a_2 \neq 0$
$\mathbb{F}_q(1,0,a_1) \oplus \mathbb{F}_q(0,1,a_2) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}}$, with $a_1 \neq 0$ or $a_2 \neq 0$
$\mathbb{F}_q(1,a_1,0) \oplus \mathbb{F}_q(0,0,1) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}}$, with $a_1 \neq 0$
$\mathbb{F}_q(1,0,0) \oplus \mathbb{F}_q(0,1,0) \oplus \mathbb{F}_q(0,0,1) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}}$

where $a_i \in \mathbb{F}_q$, for each $i = 1,2,3$ (in each line). We pick up representatives:

$$\widehat{\mathfrak{b}}_1 = \mathbb{F}_q(1,1,1) \oplus \mathbb{F}_q(x,0,-\xi x) \oplus \mathbb{F}_q\left(0,x,-\xi^2 x\right) \oplus \widehat{\mathfrak{f}}$$

by putting $a_1 = 1$, $a_2 = 1$ and $a_3 = 0$ in the first line;

$$\widehat{\mathfrak{b}}_2 = \mathbb{F}_q(1,1,1) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}}$$

by putting $a_1 = 1$ and $a_2 = 1$ in the second line;

$$\widehat{\mathfrak{b}}_3 = \mathbb{F}_q(1,0,0) \oplus \mathbb{F}_q(0,1,1) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}},$$

by putting $a_1 = 0$ and $a_2 = 1$ in the third line;

$$\widehat{\mathfrak{b}}_4 = \mathbb{F}_q(1,0,1) \oplus \mathbb{F}_q(0,1,0) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}},$$

by putting $a_1 = 1$ and $a_2 = 0$ in the third line;

$$\widehat{\mathfrak{b}}_5 = \mathbb{F}_q(1,0,1) \oplus \mathbb{F}_q(0,1,1) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}},$$

by putting $a_1 = 1$ and $a_2 = 1$ in the third line;

$$\widehat{\mathfrak{b}}_6 = \mathbb{F}_q(1,1,0) \oplus \mathbb{F}_q(0,0,1) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}},$$

by putting $a_1 = 1$ in the fourth line, and

$$\widehat{\mathfrak{b}}_7 = \mathbb{F}_q(1,0,0) \oplus \mathbb{F}_q(0,1,0) \oplus \mathbb{F}_q(0,0,1) \oplus \mathbb{F}_q(x,0,0) \oplus \mathbb{F}_q(0,x,0) \oplus \mathbb{F}_q(0,0,x) \oplus \widehat{\mathfrak{f}}.$$

Let $\mathfrak{b}_i := \widehat{\mathfrak{b}}_i \cap K$ (i=1,2,3,4,5,6,7) be the corresponding representatives of the ideal classes of $\mathcal{O}$. Then $\mathfrak{b}_1 = \mathcal{O}$, $\mathfrak{b}_7 = \widetilde{\mathcal{O}}$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathcal{O}}\right] = q\,(q-1)^2$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_2}\right] = (q-1)^2$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_3}\right] = q-1$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_4}\right] = q-1$, $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_5}\right] = (q-1)^2$ and $\left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}_5}\right] = q-1$. Moreover, we obtain:

$$\begin{aligned}
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathcal{O}, t_1, t_2, t_3\right) &= q^3t_1^2t_2^2t_3^2 - q^2t_1^2t_2^2t_3 - q^2t_1^2t_2t_3^2 - q^2t_1t_2^2t_3^2 + qt_1^2t_2t_3 + qt_1t_2^2t_3 \\
&\quad + qt_1t_2t_3^2 + \left(q^2 - 2q - 1\right)t_1t_2t_3 + t_1t_2 + t_1t_3 + t_2t_3 - t_1 - t_2 - t_3 + 1 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_2, t_1, t_2, t_3\right) &= \left(q^2 - 2q\right)t_1t_2t_3 + t_1t_2 + t_1t_3 + t_2t_3 - t_1 - t_2 - t_3 + 1 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_3, t_1, t_2, t_3\right) &= qt_2t_3 - t_3 - t_2 + 1 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_4, t_1, t_2, t_3\right) &= qt_1t_3 - t_3 - t_1 + 1 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_5, t_1, t_2, t_3\right) &= q^2t_1t_2t_3 - qt_1t_2 - qt_1t_3 - qt_2t_3 + t_1 + t_2 + t_3 + q - 2 \\
\Lambda_{\mathcal{O}}\left(\mathcal{O}, \mathfrak{b}_6, t_1, t_2, t_3\right) &= qt_1t_2 - t_2 - t_1 + 1 \\
\Lambda_{\mathcal{O}}(\mathcal{O}, \widetilde{\mathcal{O}}, t_1, t_2, t_3) &= 1,
\end{aligned}$$

and

$$Z_{\mathcal{O}}(\mathcal{O},\mathcal{O},t)=\frac{1-3t+3t^2+(q^2-2q-1)\,t^3+3qt^4-3q^2t^5+q^3t^6}{(1-t)^3}$$
$$Z_{\mathcal{O}}(\mathcal{O},\mathfrak{b}_2,t)=\frac{t-3t^2+3t^3+(q^2-2q)\,t^4}{(1-t)^3}$$
$$Z_{\mathcal{O}}(\mathcal{O},\mathfrak{b}_3,t)=\frac{t^2-2t^3+qt^4}{(1-t)^3}$$
$$Z_{\mathcal{O}}(\mathcal{O},\mathfrak{b}_4,t)=\frac{t^2-2t^3+qt^4}{(1-t)^3}$$
$$Z_{\mathcal{O}}(\mathcal{O},\mathfrak{b}_5,t)=\frac{(q-2)\,t^2+3t^3-3qt^4+q^2t^5}{(1-t)^3}$$
$$Z_{\mathcal{O}}(\mathcal{O},\mathfrak{b}_6,t)=\frac{t^2-2t^3+qt^4}{(1-t)^3}$$
$$Z_{\mathcal{O}}(\mathcal{O},\widetilde{\mathcal{O}},t)=\frac{t^3}{(1-t)^3}.$$

Moreover, $Z_{\mathcal{O}}(\mathcal{O},t)=\frac{q^3t^6-2q^2t^5+\left(q^2+q\right)t^4+\left(q^2-2q\right)t^3+(q+1)t^2-2t+1}{(1-t)^3}$. and

$$\begin{aligned}
S(\mathcal{O}) &= \{(0,0,0),(1,1,1)\}\cup((1,2,1)+\{0\}\times\mathbb{N}\times\{0\})\cup((2,1,1)+\mathbb{N}\times\{0\}\times\{0\})\\
&\quad\cup((2,2,2)+\mathbb{N}^3)\\
S(\mathfrak{b}_2) &= \{(0,0,0)\}\cup\left((1,1,1)+\mathbb{N}^3\right)\\
S(\mathfrak{b}_3) &= \{(0,0,0)\}\cup((1,0,0)+\mathbb{N}\times\{0\}\times\{0\})\cup((0,1,1)+\{0\}\times\mathbb{N}\times\mathbb{N})\\
&\quad\cup\left((1,1,1)+\mathbb{N}^3\right)\\
S(\mathfrak{b}_4) &= \{(0,0,0)\}\cup((0,1,0)+\{0\}\times\mathbb{N}\times\{0\})\cup((1,0,1)+\mathbb{N}\times\{0\}\times\mathbb{N})\\
&\quad\cup\left((1,1,1)+\mathbb{N}^3\right)\\
S(\mathfrak{b}_5) &= \{(0,0,0)\}\cup((0,1,0)+\{0\}\times\mathbb{N}\times\{0\})\cup((1,0,0)+\mathbb{N}\times\{0\}\times\{0\})\\
&\quad\cup\left((1,1,1)+\mathbb{N}^3\right)\\
S(\mathfrak{b}_6) &= \{(0,0,0)\}\cup((0,0,1)+\{0\}\times\{0\}\times\mathbb{N})\cup((1,1,0)+\mathbb{N}\times\mathbb{N}\times\{0\})\\
&\quad\cup\left((1,1,1)+\mathbb{N}^3\right)\\
S(\mathfrak{b}_7) &= \mathbb{N}^3.
\end{aligned}$$

4.3 Ground Field Extensions

In this section we discuss the behavior of the multi-variable Poincaré series under ground field extensions. Before we do this we give some well-known definitions and results on the behavior of a function field in one variable under the ground field extension.

Let $K \mid k$ be a function field of one variable, let $l \mid k$ be an algebraic field extension and let S be a semilocal subring of $K \mid k$. It is known that, in this case, the tensor product $l \otimes_k K$ is a field. Thus we can identify the compositum $l \cdot K$ with $l \otimes_k K$, the field l is algebraically closed in the field $l \cdot K$ and hence $l \cdot K \mid l$ is a function field. The ring $l \otimes_k S$ can be identify with the semilocal subring $l \cdot S$, of the function field $l \cdot K \mid l$, which consists merely of all linear combinations of elements of S with coefficients in l, and hence $l \cdot S \cap K = S$ (see [28] and [22] section 3).

Let v be a valuation of the function field $K \mid k$. Recall that if w is a valuation of $l \cdot K \mid l$ that lies over v, then the *ramification index* $e_{w|v}$ of w over v is defined as the group index $[w(l \cdot K \backslash \{0\}) : w(K \backslash \{0\})]$ and the *inertia index* $f_{w|v}$ of w over v is defined as the degree of field extensions $[k_w : k_v]$. Thus, it is associated to v the divisor

$$\mathrm{Con}_{l \cdot K|K}(v) := \sum\nolimits_{w|v} e_{w|v} \cdot w$$

where the sum run over all the valuation w of $l \cdot K \mid l$ lying over v. It is called *the Conorm* of v. The following Theorem is well-known (cf. [28] Theorem III.6.3).

Theorem 51 *If $l \mid k$ is a separable algebraic extension. Then*

1. $l \cdot \mathcal{O}_v = \widetilde{\mathcal{O}}_v$, *where* $\widetilde{\mathcal{O}}_v$ *is the integral closure of the valuation ring* $\mathcal{O}_v$ *in* $l \cdot K$
2. v *is unramified in* $l \cdot K$ *i.e.* $e_{w|v} = 1$ *for each valuation* w *of* $l \cdot K \mid l$ *lying over* v.

For each valuation w of $l \cdot K \mid l$ lying over v, we may consider both the residue field k_v and the field l as subfield of the residue field l_w of w. Thus the compositum $l \cdot k_v$ of the field k_v and the field l is well defined.

Corollary 52 *If $l \mid k$ is a separable algebraic field extension. Then*

1. $l \cdot k_v = l_w$ *for each valuation* w *of* $l \cdot K \mid l$ *lying over* v.
2. $\mathrm{Con}_{l \cdot K|K}(v) = \sum_{w|v} w$.
3. *If* $l \mid k$ *is a finite field extension, then* $\deg(\mathrm{Con}_{l \cdot K|K}(v)) = \deg(v)$.

In particular, if $l = k_v$ then $\deg(w) = 1$ for each valuation w of $k_v \cdot K \mid k_v$ lying over v.

As we are mainly interested in local or semilocal subrings of a function field $K \mid k$ whose constant field is the finite field $k = \mathbb{F}_q$ with q elements, we assume from now on that $k = \mathbb{F}_q$ and that $l \mid k$ is a finite field extension. Thus, $l \mid k$ is a separable algebraic field extension. In the following, we fix an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. Then, for any positive integer n, there exists exactly one extension $\mathbb{F}_{q^n} \mid \mathbb{F}_q$ of degree n with $\mathbb{F}_{q^n} \subseteq \overline{\mathbb{F}_q}$. Thus, $l = \mathbb{F}_{q^n}$ for some positive integer n.

Let v be a valuation of $K \mid k$ of degree r, that is $r = [k_v : k]$, hence $k_v = \mathbb{F}_{q^r}$. By Theorem 51, the valuation v is unramified in $l \cdot K \mid l$. By Corollary 52, we have that for each valuation w of $l \cdot K \mid l$ lying over v the residue field of w is the compositum of $\mathbb{F}_{q^n}$ with the residue field $k_v = \mathbb{F}_{q^r}$ of v, that is, $\mathbb{F}_{q^n} \cdot \mathbb{F}_{q^r} = \mathbb{F}_{q^l}$, where $l := \operatorname{lcm}(r, n)$. Therefore,

$$\deg(w) = [\mathbb{F}_{q^l} : \mathbb{F}_{q^n}] = r / \gcd(r, n)$$

for each valuation w of $l \cdot K \mid l$ lying over v. Since $\deg(\mathrm{Con}_{l \cdot K|K}(v)) = \deg(v) = r$, we conclude the there exists exactly $d := \gcd(r, n)$ valuations w of $l \cdot K \mid l$ lying over v, each of them having degree r/d. Summarizing, we have the following proposition (cf. [28] Lemma V.1.9).

Proposition 53 *Let $K \mid k$ be a function field of one variable, whose constant field is the finite field $k = \mathbb{F}_q$ with q elements, let $l \mid k$ be a finite field extension of degree n. If v is a degree r valuation of $K \mid k$, then there exist exactly $d := \gcd(r, n)$ valuations w of $l \cdot K \mid l$ lying over v, each of them having ramification index $e_{w|v} = 1$, degree $\deg(w) = r/d$ and residue field $l_w = \mathbb{F}_{q^n} \cdot \mathbb{F}_{q^r} = \mathbb{F}_{q^l}$, where $l = \operatorname{lcm}(r, n)$.*

The ground field of a function field $K \mid k$ may be extended to the algebraic closure $\overline{k}$ of k. Thus, it is defined $\overline{K} =: \overline{k} \otimes_k K$. Since $l \otimes_k K$ is a field for every finite field extension $l \mid k$, any embedding $l \longrightarrow \overline{k}$ extends to an embedding $l \otimes_k K \longrightarrow \overline{K}$. Indeed, $\overline{K}$ is just the set-theoric union of the images of such embeddings. In particular every element of $\overline{K}$ lies in some subfield, and so $\overline{K}$ is a field. Thus $\overline{K} \mid \overline{k}$ is a function field.

Let v be a valuation of the function field $K \mid k$. There exists a finite field extension $k' \mid k$ such that $\deg(v') = 1$ for each valuation v' of $k' \cdot K \mid k'$ lying over v. Moreover, we observe that there exists a one to one bijection between valuations w of $\overline{K} \mid \overline{k}$ lying over v and valuations v' of $k' \cdot K \mid k'$ lying over v such that $e_{w|v} = e_{v'|v}$. Indeed, it is clear that given a valuation v' of $k' \cdot K \mid k'$ lying over v, there exists a valuation w of $\overline{K} \mid \overline{k}$ lying over v' and hence lying over v. Remain to prove that there exists exactly one. If, by way of contradiction, there were more than one valuations of $\overline{K} \mid \overline{k}$ lying over v', they would differ on some element in $\overline{K}$, say $u = \sum_{i=1}^{j} \alpha_i \otimes x_i \in \overline{K}$ with $\alpha_i \in \overline{k}$ and $x_i \in K$ $(i = 1, \cdots, j)$, then the field $k'' := k'(\alpha_1, \cdots, \alpha_j)$ is a finite extension of k', $u \in k'' \otimes_k K$ and there were more than one valuations v'' of $k''K \mid k''$ lying over v'. However, since $f_{v''|v'} = [k'' : k']$ and $[k'' : k'] = [k''(k'K) : k'K]$, by fundamental equality, there exists a unique valuations v'' of $k''K \mid k''$ lying over v'. With a similar argument we can prove that a local parameter t of w lies in some finite field extension $k'' \otimes_k K$, where we may assume that $k'' \supseteq k'$. By fundamental equality, $e_{w|v''} = 1$ and $e_{v''|v'} = 1$, where v'' is the valuation restriction of w to $k''K = k''(k'K)$. Thus, $e_{w|v} = e_{w|v''}e_{v''|v'}e_{v'|v} = e_{v'|v}$. Therefore, for any valuation v of the function field $K \mid k$, the Conorm $\mathrm{Con}_{\overline{K}|K}(v)$ defined by

$$\mathrm{Con}_{\overline{K}|K}(v) := \sum\nolimits_{w|v} e_{w|v} \cdot w$$

where the sum run over all the valuation w of $\overline{K} \mid \overline{k}$ lying over v, is well-defined. Let $l \mid k$ be a finite field extension and let w be a valuation w of $\overline{K} \mid \overline{k}$. The valuation w is said to be *defined over* l if there exists a valuation v of $l \cdot K \mid l$ such that $\mathrm{Con}_{\overline{K}|l \cdot K}(v) = w$. The valuations of $l \cdot K \mid l$ can be identified with the valuations of $\overline{K} \mid \overline{k}$ defined over l.

It is well-known that the Galois group of $\mathbb{F}_{q^n} \mid \mathbb{F}_q$ is cyclic Galois group of order n generated by the Frobenius map that acts on $\mathbb{F}_{q^n}$ by $\alpha \mapsto \alpha^q$. Let

$$\sigma : \mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} K \longrightarrow \mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} K$$

be the function defined by $\alpha \otimes x \mapsto \alpha^q \otimes x$. Then, by identifying the compositum $\mathbb{F}_{q^n} \cdot K$ with $\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} K$, σ is an isomorphism of $\mathbb{F}_{q^n} \cdot K$ into itself. Note that σ is the identity in K. Consequently, it is deduced the next lemma (cf. [28] Lemma V.1.9).

Lemma 54 *Let $K \mid k$ be a function field of one variable, whose constant field is the finite field $k = \mathbb{F}_q$ with q elements and let $l \mid k$ be a finite field extension of degree n. Then $l \cdot K \mid K$ is a Galois extension with cyclic Galois group* $\mathrm{Gal}(l \cdot K \mid K)$ *of order n generated by the Frobenius automorphism σ, which acts on l by $\sigma(c) = c^q$.*

Proof. Since the Galois group of $l \mid k$ is a cyclic Galois group of order n generated by the Frobenius automorphism $\alpha \mapsto \alpha^q$ and $[l \cdot K : K] = [l : k] = n$, it follows that the group of automorphisms of $l \cdot K \mid K$ is generated by the automorphism σ and it has order n. ■

It is clear that for each automorphism $\psi \in \mathrm{Gal}(l \cdot K \mid K)$ and for each valuation w of $l \cdot K \mid l$ the composition $w \circ \psi$ is another valuation of $l \cdot K \mid l$ with valuation ring $\psi^{-1}(\mathcal{O}_w)$ and maximal ideal $\mathfrak{m}_{w \circ \psi} := \psi^{-1}(\mathfrak{m}_w)$.

Proposition 55 *Let $K \mid k$ be a function field of one variable, whose constant field is the finite field $k = \mathbb{F}_q$ with q elements, let $l \mid k$ be a finite field extension of degree n and let w_1 and w_2 be valuations of $l \cdot K \mid l$ lying over the valuation v of $K \mid k$. Then there exists $\psi \in \mathrm{Gal}\,(l \cdot K \mid K)$ such that $w_2 = w_1 \circ \psi$.*

Proof. Suppose that $w_2 \neq w_1 \circ \psi$ for each $\psi \in \mathrm{Gal}\,(l \cdot K \mid K)$. By the approximation lemma, there exists $x \in l \cdot K$ such that $w_2\,(x) = 1$ and $w_1 \circ \psi\,(x) = 0$ for each $\psi \in \mathrm{Gal}\,(l \cdot K \mid K)$. If $y := N_{l \cdot K|K}\,(x)$, one has $y \in K$, and $y = \prod_{\psi \in \mathrm{Gal}(l \cdot K|K)} \psi\,(x)$, whence $w_2\,(y) > 0$ and $w_1\,(y) = 0$, which contradicts w_1 and w_2 are valuations of $l \cdot K \mid l$ lying over the valuation v of $K \mid k$. ■

This proposition permit us to give a relationship between the set $S(\mathfrak{b})$ and $S(l \cdot \mathfrak{b})$ for each S-ideal $\mathfrak{b}$ of a semilocal subring S. Let S be a semilocal subring of the function field $K \mid k$, whose constant field is the finite field $k = \mathbb{F}_q$ with q elements. Let $\mathfrak{b}$ be an S-ideal and let $l \mid k$ be a finite field extension of degree n. Since the set $S(\mathfrak{b})$ only depends on the S-ideal class of $\mathfrak{b}$, we can assume that $\mathfrak{b}\widetilde{S} = \widetilde{S}$. The valuations $w_1, \cdots, w_{m^{(l)}}$ of $l \cdot K \mid l$ that contain the semilocal ring $l \cdot S$ are precisely the extensions to $l \cdot K \mid l$ of the valuations $v_1, \cdots, v_m$ of $K \mid k$ that contain the semilocal ring S (cf. [22] Section 3). Moreover, if $\mathbf{v}(z) \in S(\mathfrak{b})$, with $z \in \mathfrak{b} \backslash \{0\}$, then $\mathbf{w}(z) \in S(l \cdot \mathfrak{b})$, where $\mathbf{v}(z)$ and $\mathbf{w}(z)$ are the multi-exponents of z. Thus, we may view the set $S(\mathfrak{b})$ as a subset of the set $S(l \cdot \mathfrak{b})$. The Galois group $G := \mathrm{Gal}\,(l \cdot K \mid K)$ acts on the set $S(l \cdot \mathfrak{b})$. Indeed, for any $\psi \in G$ and any $\mathbf{w}(z) \in S(l \cdot \mathfrak{b})$, with $z \in l \cdot \mathfrak{b} \backslash \{0\}$, we have $\mathbf{w}(\psi(z)) \in S(l \cdot \mathfrak{b})$. Furthermore, $S(\mathfrak{b})$ injects in the set of fixed points $S(l \cdot \mathfrak{b})^G$.

In the proof of the next theorem we need the following polynomial identity.

Lemma 56 *Let n and r be positive integers. Then*

$$(1 - X^{nr/\gcd(n,r)})^{\gcd(n,r)} = \prod\nolimits_{\theta^n = 1} (1 - (\theta X)^r)$$

where θ runs over over all n-th roots of the unity in the complex number $\mathbb{C}$.

Proof. We observe, if θ is a n-th root of the unity, then $\zeta := \theta^r$ is a n/d-th root of the unity, where $d := \gcd(n, r)$. Thus the result follows from the basic polynomial identity: for any integer k,

$$1 - X^k = \prod\nolimits_{\zeta^k = 1} (1 - \zeta X),$$

where ζ runs over over all k-th roots of the unity in the complex number $\mathbb{C}$. Indeed,

$$\prod\nolimits_{\theta^n = 1} (1 - (\theta X)^r) = (\prod\nolimits_{\zeta^{n/d} = 1} (1 - \zeta X^r))^d = (1 - X^{nr/d})^d$$

$\blacksquare$

Let $\mathcal{O}$ be a local ring of at a singular point of a geometrically integral algebraic curve defined over a finite field $k = \mathbb{F}_q$ and let $\mathfrak{b}$ be an $\mathcal{O}$-ideal. We denote by

$$\eta(\mathcal{O}, \mathfrak{b}, q) := \frac{q^{\rho+\delta-|\mathbf{r}|}}{[U_{\mathfrak{b}} : U_{\mathcal{O}}](q^{\rho}-1)} \prod_{i=1}^{m} (q^{r_i}-1),$$

that is, $\eta(\mathcal{O}, \mathfrak{b}, q) = \left[U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}}\right]$, where $[U_{\mathfrak{b}} : U_{\mathcal{O}}] = \frac{q^{\delta+\varrho-(\delta_{\mathfrak{b}:\mathfrak{b}}+\varrho_{\mathfrak{b}:\mathfrak{b}})}\prod_{i=1}^{s_{\mathfrak{b}}}(q^{\varrho_i^{(\mathfrak{b})}}-1)}{q^{\varrho}-1}$ depends on q too (for the notation see 3.13). The examples that we show in the precedent section indicate the following result.

Theorem 57 *Let $\mathcal{O}$ be a local ring of at a singular point of a geometrically integral algebraic curve defined over a finite field $k = \mathbb{F}_q$ and let $l = \mathbb{F}_{q^n}$ be a finite field extension of k. If $\mathcal{O}$ is the local ring at rational point, then for each $\mathcal{O}$-ideal $\mathfrak{b}$,*

$$\eta(l \cdot \mathcal{O}, l \cdot \mathfrak{b}, q) = (-1)^{m^{(l)}+mn-n-1} \prod_{\theta^n=1} \eta(\mathcal{O}, \mathfrak{b}, \theta q),$$

where $m^{(l)} := \sum_{i=1}^{m} \gcd(r_i, n)$ is equal to the number of valuations of the function field $l \cdot K \mid l$ lying over the ring $l \cdot \mathcal{O}$.

Proof. Since $\mathcal{O}$ is the local ring at rational point, it follows that $\rho^{(l)} = \rho = 1$ and $l \cdot \mathcal{O}$ is also a local ring, hence

$$\eta(l \cdot \mathcal{O}, l \cdot \mathcal{O}, q) = \frac{(q^n)^{\rho^{(l)}+\delta^{(l)}-\left|\mathbf{r}^{(l)}\right|}}{(q^n)^{\rho^{(l)}}-1} \prod_{j=1}^{m^{(l)}} ((q^n)^{r_j^{(l)}}-1).$$

Where $\delta^{(l)} := \dim_l \left(\widetilde{l \cdot \mathcal{O}}/l \cdot \mathcal{O}\right)$, $\mathbf{r}^{(l)} := (r_1^{(l)}, \cdots, r_{m^{(l)}}^{(l)})$ is the integer vector whose coordinates are the degrees of the branches centered at the singularity and $\rho^{(l)}$ is the degree of the residue field of $l \cdot \mathcal{O}$ over the constant field l.

On the other hand, $\delta^{(l)} = \dim_l \left(l \otimes_k \widetilde{\mathcal{O}}/l \otimes_k \mathcal{O}\right) = \dim_l \left(l \otimes_k \widetilde{\mathcal{O}}/\mathcal{O}\right) = \delta$. From Lemma 53, $\left|\mathbf{r}^{(l)}\right| = |\mathbf{r}|$ and

$$\begin{aligned}\prod_{j=1}^{m^{(l)}} ((q^n)^{r_j^{(l)}}-1) &= \prod_{i=1}^{m} \prod_{w|v_i} (q^{nr_i/\gcd(r_i,n)}-1) \\ &= \prod_{i=1}^{m} (-1)^{\gcd(r_i,n)} \prod_{w|v_i} (1-q^{nr_i/\gcd(r_i,n)}) \\ &= \prod_{i=1}^{m} (-1)^{\gcd(r_i,n)} (1-q^{nr_i/\gcd(r_i,n)})^{\gcd(r_i,n)}.\end{aligned}$$

By Proposition 53 $m^{(l)} := \sum_{i=1}^{m} \gcd(r_i, n)$ is the number of valuations of $l \cdot K \mid l$ lying over the ring $l \cdot \mathcal{O}$. Now, from Lemma 56,

$$\begin{aligned}
\prod_{j=1}^{m^{(l)}} ((q^n)^{r_j^{(l)}} - 1) &= \prod_{i=1}^{m} (-1)^{\gcd(r_i,n)} \prod_{\theta^n=1} (1-(\theta q)^{r_i}) \\
&= \prod_{i=1}^{m} (-1)^{\gcd(r_i,n)+n} \prod_{\theta^n=1} ((\theta q)^{r_i} - 1) \\
&= (-1)^{mn+m^{(l)}} \prod_{\theta^n=1} \prod_{i=1}^{m} ((\theta q)^{r_i} - 1) .
\end{aligned}$$

Thus,

$$\begin{aligned}
\eta(l \cdot \mathcal{O}, l \cdot \mathcal{O}, q) &= \frac{(q^n)^{\rho^{(l)}+\delta^{(l)}-\left|\mathbf{r}^{(l)}\right|}}{(q^n)^{\rho^{(l)}} - 1} \prod_{j=1}^{m^{(l)}} ((q^n)^{r_j^{(l)}} - 1) \\
&= \frac{(q^n)^{1+\delta-|\mathbf{r}|}}{q^n - 1} (-1)^{mn+m^{(l)}} \prod_{\theta^n=1} \prod_{i=1}^{m} ((\theta q)^{r_i} - 1) \\
&= (-1)^{mn-n-1+m^{(l)}} \prod_{\theta^n=1} \frac{(\theta q)^{1+\delta-|\mathbf{r}|}}{\theta q - 1} \prod_{i=1}^{m} ((\theta q)^{r_i} - 1) \\
&= (-1)^{mn-n-1+m^{(l)}} \prod_{\theta^n=1} \eta(\mathcal{O}, \mathcal{O}, \theta q) .
\end{aligned}$$

■

We observe that, if the local ring is residually rational, then

$$\eta(l \cdot \mathcal{O}, l \cdot \mathcal{O}, q) = \eta(\mathcal{O}, \mathcal{O}, q^n) .$$

Let $\mathcal{O}$ be a local ring at a singular point of a geometrically integral algebraic curve defined over a finite field $k = \mathbb{F}_q$. For each finite field extension $l \mid k$ we consider the polynomial

$$\Delta(\mathcal{O}, l \mid k, t) := L(l \cdot \mathcal{O}, t^n) - \prod_{\theta^n=1} L(\mathcal{O}, \theta t)$$

and the rational function

$$Q(\mathcal{O}, l \mid k, t) := \frac{L(l \cdot \mathcal{O}, t^n)}{\prod_{\theta^n=1} L(\mathcal{O}, \theta t)}$$

where $n := [l : k]$.

We observe that, if $\mathcal{O}$ is the local ring of the curve at non-singular point, then $\Delta(\mathcal{O}, l \mid k, t) = 0$ and $Q(\mathcal{O}, l \mid k, t) = 1$.

Proposition 58 *The polynomial $\Delta(\mathcal{O}, l \mid k, t)$ and the rational function $Q(\mathcal{O}, l \mid k, t)$ satisfy the following functional equations:*

$$(t)^{-n\delta} \Delta(\mathcal{O}, l \mid k, t) = (\frac{1}{qt})^{-n\delta} \Delta(\mathcal{O}, l \mid k, \frac{1}{qt})$$

and
$$Q(\mathcal{O},l\mid k,t)=Q(\mathcal{O},l\mid k,\frac{1}{qt})$$
respectively.

Proof. Since
$$(\frac{1}{q^nt^n})^{-\delta}L(l\cdot\mathcal{O},\frac{1}{q^nt^n})=(t^n)^{-\delta}L(l\cdot\mathcal{O},t^n)$$
and
$$(\frac{1}{q(\theta^{-1}t)})^{-\delta}L(\mathcal{O},\frac{1}{q(\theta^{-1}t)})=(\theta^{-1}t)^{-\delta}L(\mathcal{O},\theta^{-1}t)$$
for each n-th root of the unity θ, it follows that
$$\begin{aligned}(\frac{1}{qt})^{-n\delta}\Delta(\mathcal{O},l\mid k,\frac{1}{qt})&=(\frac{1}{qt})^{-n\delta}\left(L(l\cdot\mathcal{O},\frac{1}{q^nt^n})-\prod\nolimits_{\theta^n=1}L(\mathcal{O},\frac{1}{q(\theta^{-1}t)})\right)\\&=(t^n)^{-\delta}L(l\cdot\mathcal{O},t^n)-\prod\nolimits_{\theta^n=1}(\theta^{-1}t)^{-\delta}L(\mathcal{O},\theta^{-1}t)\\&=(t^n)^{-\delta}\left(L(l\cdot\mathcal{O},t^n)-\prod\nolimits_{\theta^n=1}L(\mathcal{O},\theta^{-1}t)\right)\\&=(t^n)^{-\delta}\Delta(\mathcal{O},l\mid k,t).\end{aligned}$$
Similarly, we prove the second part. ∎

Let X be a geometrically integral algebraic curve defined over a finite field $k=\mathbb{F}_q$. For each finite field extension $l\mid k$ we consider the polynomial
$$\Delta(\mathcal{O}_X,l\mid k,t):=L(\mathcal{O}_{X_l},t^n)-\prod\nolimits_{\theta^n=1}L(\mathcal{O}_X,\theta t)$$
and the rational function
$$Q(\mathcal{O}_X,l\mid k,t):=\frac{L(\mathcal{O}_{X_l},t^n)}{\prod_{\theta^n=1}L(\mathcal{O}_X,\theta t)},$$
where $n:=[l:k]$ and X_l stands for the curve $X\times_{Spec(k)}Spec\,(l)$.

We observe that, if X is a non-singular curve then $\Delta(\mathcal{O}_X,l\mid k,t)=0$ and $Q(\mathcal{O}_X,l\mid k,t)=1$. We may prove the following property. Its proof is similar to the proof of the previous proposition.

Proposition 59 *The functions $\Delta(\mathcal{O}_X,l\mid k,t)$ and $Q(\mathcal{O}_X,l\mid k,t)$ satisfy the following functional equations:*
$$(t)^{-ng}\,\Delta(\mathcal{O}_X,l\mid k,t)=(\frac{1}{qt})^{-ng}\Delta(\mathcal{O}_X,l\mid k,\frac{1}{qt})$$
and
$$Q(\mathcal{O}_X,l\mid k,t)=Q(\mathcal{O}_X,l\mid k,\frac{1}{qt}),$$
respectively.

4.4 Multi-variable Poincaré series of residually rational rings

Zuñiga proved that the partial zeta function $Z(\mathcal{O},\mathcal{O},t)$ is determined by the semigroup $S(\mathcal{O})$, if $\mathcal{O}$ is a residually rational local ring (cf. [35], Theorem 5.5). We will show that this result can be extended to multi-variable Poincaré series of residually rational semilocal rings.

From Example 49, we can see that the set $S(\mathfrak{b})$ does not always determine the multi-variable Poincaré series $P_S(S,\mathfrak{b},\mathbf{t})$, where $\mathfrak{b}$ is an S-ideal of a semilocal subring of the function field $K \mid k$ with $k=\mathbb{F}_q$. On the other hand, we had observed that, if $l=k_v$, then $\deg(w)=1$ for each valuation w of $k_v\cdot K \mid k_v$ lying over a valuation v of the function field $K \mid k$. Thus there exists a finite field extension $l \mid k$ such that the semilocal ring $l\cdot S$ is residually rational.

Assume that $S=\mathcal{O}_1\cap\cdots\cap\mathcal{O}_s$ is a residually rational semilocal subring of the function field $K \mid k$. In this situation the Poincaré series has the expansion

$$P_S(S,\mathfrak{b},\mathbf{t})=\sum\nolimits_{\mathbf{n}\in S(\mathfrak{b})}\eta_{\mathbf{n}}(S,\mathfrak{b})\,\mathbf{t}^{\mathbf{n}}$$

where

$$\begin{aligned}\eta_{\mathbf{n}}(S,\mathfrak{b}) &= \frac{q^s}{[U_{\mathfrak{b}}:U_S](q-1)^s}\sum_{\mathbf{i}\in\{0,1\}^m}(-1)^{|\mathbf{i}|}\,q^{\mathbf{1}\cdot(\mathbf{n}+\mathbf{v}(\mathfrak{b}\widetilde{S}))+\deg_S(\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{v}(\mathfrak{b}\widetilde{S})+\mathbf{i}})}\\ &= \frac{q^{s+\deg_S(\mathfrak{b})}}{[U_{\mathfrak{b}}:U_S](q-1)^s}\sum_{\mathbf{i}\in\{0,1\}^m}(-1)^{|\mathbf{i}|}\,q^{\mathbf{1}\cdot(\mathbf{n}+\mathbf{v}(\mathfrak{b}\widetilde{S}))-\dim_k(\mathfrak{b}/\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{v}(\mathfrak{b}\widetilde{S})+\mathbf{i}})}\\ &= \frac{q^{s+\deg_S(\mathfrak{b})}}{[U_{\mathfrak{b}}:U_S](q-1)^s}\sum_{\mathbf{i}\in\{0,1\}^m}(-1)^{|\mathbf{i}|}\,q^{\mathbf{1}\cdot\left(\mathbf{n}+\mathbf{v}\left(\mathfrak{b}\widetilde{S}\right)\right)-l\left(\mathfrak{b},\mathbf{n}+\mathbf{v}(\mathfrak{b}\widetilde{S})+\mathbf{i}\right)}\end{aligned}\tag{4.3}$$

where $l(\mathfrak{b},\mathbf{n}):=\dim_k(\mathfrak{b}/\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}})$ for each $\mathbf{n}\in\mathbb{N}^m$. Since the semilocal ring S is residually rational, the integers $l(\mathfrak{b},\mathbf{n})$ may be expressed by induction in terms of the set $S(\mathfrak{b})$. In particular, the coefficients

$$\eta_{\mathbf{n}}(S,S)=\frac{q^s}{(q-1)^s}\sum_{\mathbf{i}\in\{0,1\}^m}(-1)^{|\mathbf{i}|}\,q^{\mathbf{1}\cdot\mathbf{n}-l(S,\mathbf{n}+\mathbf{i})}$$

may be expressed in terms of the semigroup $S(S)$.

4.4.1 The one-branch case

In this subsection we assume that S is an unibranch residually rational semilocal ring S, that is S is a residually rational semilocal ring and $m=1$. In this especial situation S is a local ring $\mathcal{O}$ and its semigroup $S(\mathcal{O})\subseteq\mathbb{N}$ is a numerical semigroup, whose conductor is equal to the exponent $f:=v(\mathcal{O}:\widetilde{\mathcal{O}})$ and whose genus $\#(\mathbb{N}\backslash S(\mathcal{O}))$ is equal to the singularity degree δ. The δ positive integers that do not belong to $S(\mathcal{O})$ are called the *gaps* of $S(\mathcal{O})$ or more generally for each $\mathcal{O}$-ideal $\mathfrak{b}$ the positive integers that do not belong to the set $S(\mathfrak{b})$ are called the *gaps* of $S(\mathfrak{b})$.

Let $\mathfrak{b}$ be a $\mathcal{O}$-ideal such that $\mathfrak{b}\widetilde{\mathcal{O}}=\widetilde{\mathcal{O}}$. We have that

$$P(\mathcal{O},\mathfrak{b},t)=\frac{q^{1+\deg_{\mathcal{O}}(\mathfrak{b})}}{[U_{\mathfrak{b}}:U_{\mathcal{O}}](q-1)}\sum_{n\in S(\mathfrak{b})}\left(q^{n-l(\mathfrak{b},n)}-q^{n-l(\mathfrak{b},n+1)}\right)t^n$$

and since $n\in S(\mathfrak{b})$ if and only if $l(\mathfrak{b},n+1)=l(\mathfrak{b},n)+1$,

$$P(\mathcal{O},\mathfrak{b},t)=\frac{q^{\deg_{\mathcal{O}}(\mathfrak{b})}}{[U_{\mathfrak{b}}:U_{\mathcal{O}}]}\sum_{n\in S(\mathfrak{b})}q^{n-l(\mathfrak{b},n)}t^n.$$

According to Stöhr [26] and Firouzian [13] $n-l(\mathfrak{b},n)$ is equal to the number of gaps of $S(\mathfrak{b})$ smaller than n. Thus

$$P(\mathcal{O},\mathfrak{b},t)=\frac{q^{\deg_{\mathcal{O}}(\mathfrak{b})}}{[U_{\mathfrak{b}}:U_{\mathcal{O}}]}\sum_{n\in S(\mathfrak{b})}q^{\#\{gaps\ of\ S(\mathfrak{b})<n\}}t^n.$$

From 41, $[U_{\mathfrak{b}}:U_{\mathcal{O}}]=q^{\delta-\dim\left(\widetilde{\mathcal{O}}/\mathfrak{b}:\mathfrak{b}\right)}$. Then

$$P(\mathcal{O},\mathfrak{b},t)=q^{-\dim(\mathfrak{b}:\mathfrak{b}/\mathfrak{b})}\sum_{n\in S(\mathfrak{b})}q^{\#\{gaps\ of\ S(\mathfrak{b})<n\}}t^n.$$

By multiplying the series with $1-t$ it follows the following result.

Proposition 60 *Let $\mathcal{O}$ be the local ring of a rational unibranch point; then*

$$\Lambda(\mathcal{O},\mathfrak{b},t)=\sum_{i=0}^{b}\lambda_i(\mathcal{O},\mathfrak{b})t^i$$

where

$$\lambda_i(\mathcal{O},\mathfrak{b})=\begin{cases}q^{-\dim(\mathfrak{b}:\mathfrak{b}/\mathfrak{b})+\#\{gaps\ of\ S(\mathfrak{b})<i\}} & \textit{if } i\in S(\mathfrak{b}) \textit{ and } i-1\notin S(\mathfrak{b})\\ -q^{-\dim(\mathfrak{b}:\mathfrak{b}/\mathfrak{b})+\#\{gaps\ of\ S(\mathfrak{b})<i\}} & \textit{if } i\notin S(\mathfrak{b}) \textit{ and } i-1\in S(\mathfrak{b})\\ 0 & \textit{otherwise}\end{cases}$$

and $b=v(\mathfrak{b}:\widetilde{\mathcal{O}})$.

Let $m := \min\left(S\left(\mathcal{O}\right)\backslash\{0\}\right)$. The *Apery sequence* of $S\left(\mathcal{O}\right)$ is defined by picking up in each residue class of $\mathbb{Z}$ module m the smallest element belonging to $S\left(\mathcal{O}\right)$, say $\alpha_0, \cdots, \alpha_{m-1}$ with $\alpha_i \equiv i \mod m$ for each $i = 0, \cdots, m-1$ and, by writing up these elements in their natural order, say $a_0 < a_1 < \cdots < a_{m-1}$. Thus

$$S\left(\mathcal{O}\right) = \bigcup\nolimits_{i=0}^{m-1}\left(\alpha_i + m\mathbb{N}\right) = \bigcup\nolimits_{i=0}^{m-1}\left(a_i + m\mathbb{N}\right)$$

The Poincaré series of a local ring of a rational unibranch point

Proposition 61 *Let $\mathcal{O}$ be the local ring of a rational unibranch point. Then*

$$P(\mathcal{O}, \mathcal{O}, t) = \sum_{0 \leq i \leq j \leq m-1} q^{\delta - \sum_{k=j+1}^{m-1}\lfloor (a_k - a_i)/m \rfloor} t^{a_i} \sum\nolimits_{j_i = \lfloor (a_j - a_i)/m \rfloor}^{\lfloor (a_{j+1} - a_i)/m \rfloor - 1} \left(q^{(m-j-1)} t^m\right)^{j_i}$$

where $\lfloor x \rfloor$ denotes the smallest integer not less than x, that is,

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$$

for each real number x and $a_m := \infty$.

Proof. We observe that $n \in S\left(\mathcal{O}\right)$ if and only if there exists only one $i = 0, \cdots, m-1$ and only one positive integer j such that $n = a_i + mj$. Then

$$P(\mathcal{O}, \mathcal{O}, t) = \sum_{i=0}^{m-1}\sum_{j=0}^{\infty} q^{\#\{gaps\ of\ S(\mathcal{O}) < a_i + jm\}} t^{a_i + jm}.$$

Let $g_k\left(\mathcal{O}, n\right)$ be the number of gaps of $S\left(\mathcal{O}\right)$ smaller than n which are congruent with a_k module m. For each $k = 1, \cdots, m-1$, we have

$$g_k\left(\mathcal{O}, a_i + jm\right) = \begin{cases} \lfloor a_k/m \rfloor, & \text{if } k \leq i \text{ and } j \geq 0 \text{ or } k > i \text{ and } j > \lfloor \left(a_k - a_i\right)/m \rfloor \\ \lfloor a_k/m \rfloor - \lfloor \left(a_k - a_i\right)/m \rfloor + j, & \text{if } k > i \text{ and } 0 \leq j \leq \lfloor \left(a_k - a_i\right)/m \rfloor \end{cases}$$

and

$$\sum_{k=1}^{m-1} g_k\left(\mathcal{O}, a_i + jm\right) = \#\{gaps\ of\ S\left(\mathcal{O}\right) < a_i + jm\}.$$

Observing now that $\sum\limits_{k=1}^{m-1}\lfloor a_k/m \rfloor$ is the number of gaps of $S\left(\mathcal{O}\right)$ i.e

$$\sum_{k=1}^{m-1}\lfloor a_k/m \rfloor = \delta,$$

we shown the proposition. ∎

4.4.2 The Two-branch case

In this subsection we assume that S is a two-branch residually rational semilocal ring S, that is S is a residually rational semilocal ring and $m=2$. We assume also that the constant field k has more than 2 elements. In this case, its semigroup $S(S) \subseteq \mathbb{N} \times \mathbb{N}$ satisfies the two properties (cf. [4]):

i. If $(n_1, n_2) \in S(S)$ and $(m_1, m_2) \in S(S)$ then

$$(\min\{n_1, m_1\}, \min\{n_2, m_2\}) \in S(S).$$

ii. Let (m_1, m_2) be a point of $S(S)$. Then m_1 is the largest abscissa of the points in $S(S)$ with ordinate m_2 if and only if m_2 is the largest ordinate of the points of $S(S)$ with abscissa m_1.

The points of item ii. are called the maximal points of $S(S)$. The set of maximal points of $S(S)$ is denoted by $M(S)$. By projecting $S(S)$ on the two coordinates axes, we obtain the semigroup of the two branches, say $S_1(S)$ and $S_2(S)$. Let $\mathfrak{b}$ be an S-ideal. From Lemma 22, Proposition 28 and remark after it, the set $S(\mathfrak{b})$ satisfies similar properties to i. and ii. Therefore, in a similar way we can define $M(\mathfrak{b})$, $S_1(\mathfrak{b})$ and $S_2(\mathfrak{b})$.

Proposition 62 *$P_S(S, \mathfrak{b}, t_1, t_2) = \sum_{(n_1,n_2)\in S(\mathfrak{b})} \eta_{(n_1,n_2)}(S, \mathfrak{b})\, t_1^{n_1} t_2^{n_2}$ where*

$$\eta_{(n_1,n_2)}(S,\mathfrak{b}) = \begin{cases} \dfrac{q^{s-1+\deg_S(\mathfrak{b})}}{[U_{\mathfrak{b}} : U_S]\,(q-1)^{s-1}} q^{s_1(\mathfrak{b},n_1)+s_2(\mathfrak{b},n_2)+m(\mathfrak{b},(n_1,n_2))}, & \textit{if } (n_1,n_2) \in M(\mathfrak{b}) \\ \dfrac{q^{s-2+\deg_S(\mathfrak{b})}}{[U_{\mathfrak{b}} : U_S]\,(q-1)^{s-2}} q^{s_1(\mathfrak{b},n_1)+s_2(\mathfrak{b},n_2)+m(\mathfrak{b},(n_1,n_2))}, & \textit{if } (n_1,n_2) \in S(\mathfrak{b})\backslash M(\mathfrak{b}) \end{cases}$$

where $s_i(\mathfrak{b}, n_i)$ stands for the number of gaps of $S_i(\mathfrak{b})$ smaller than n_i (i=1,2) and $m(\mathfrak{b}, n_1, n_2)$ stands for the number of maximal points of $S(\mathfrak{b})$ whose abscissa and coordinate are smaller than n_1 and n_2 respectively.

Proof. We can assume that $\mathfrak{b}\widetilde{S} = \widetilde{S}$. From 4.3, we have

$$\eta_{\mathbf{n}}(S,\mathfrak{b}) = \frac{q^{s+\deg_S(\mathfrak{b})+\mathbf{1}\cdot\mathbf{n}}}{[U_{\mathfrak{b}} : U_S]\,(q-1)^{s}} (q^{-l(\mathfrak{b},\mathbf{n})} - q^{-l(\mathfrak{b},\mathbf{n}+\mathbf{e}_1)} - q^{-l(\mathfrak{b},\mathbf{n}+\mathbf{e}_2))} + q^{-l(\mathfrak{b},\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)}).$$

For each point $\mathbf{n} := (n_1, n_2) \in S(\mathfrak{b})$ we have

$$l(\mathfrak{b}, (n_1+1, n_2)) = l(\mathfrak{b}, (n_1, n_2+1)) = l(\mathfrak{b}, (n_1, n_2)) + 1$$

(see Lemma 22, Proposition 28 and remark after it). Moreover,

$$l\left(\mathfrak{b},(n_1+1,n_2+1)\right)=\begin{cases} l\left(\mathfrak{b},(n_1,n_2)\right)+1, \text{ if } (n_1,n_2)\in M(\mathfrak{b}) \\ l\left(\mathfrak{b},(n_1,n_2)\right)+2, \text{ if } (n_1,n_2)\in S(\mathfrak{b})\backslash M(\mathfrak{b}). \end{cases}$$

It follows that

$$\eta_{(n_1,n_2)}\left(S,\mathfrak{b}\right)=\begin{cases} \dfrac{q^{s-1+\deg_S(\mathfrak{b})}}{\left[U_{\mathfrak{b}}:U_S\right](q-1)^{s-1}}q^{n_1+n_2-l(\mathfrak{b},(n_1,n_2))}, \text{ if } (n_1,n_2)\in M(\mathfrak{b}) \\ \dfrac{q^{s-2+\deg_S(\mathfrak{b})}}{\left[U_{\mathfrak{b}}:U_S\right](q-1)^{s-2}}q^{n_1+n_2-l(\mathfrak{b},(n_1,n_2))}, \text{ if } (n_1,n_2)\in S(\mathfrak{b})\backslash M(\mathfrak{b}). \end{cases}$$

By induction we deduce that $l(\mathfrak{b},(n_1,0)) = s_1(\mathfrak{b},n_1)$, $l(\mathfrak{b},(0,n_2)) = s_2(\mathfrak{b},n_2)$ and $l\left(\mathfrak{b},(n_1,n_2)\right)=l\left(\mathfrak{b},(n_1,0)\right)+l\left(\mathfrak{b},(0,n_2)\right)+m(\mathfrak{b},(n_1,n_2))$. Therefore,

$$n_1+n_2-l\left(\mathfrak{b},(n_1,n_2)\right)=s_1(\mathfrak{b},n_1)+s_2(\mathfrak{b},n_2)+m(\mathfrak{b},(n_1,n_2)).$$

■

Chapter 5

Multi-variable Poincaré series of plane algebroid curves

In this chapter, we study the multi-variable Poincaré series of a class of plane algebroid curves totally defined over a finite field. We show that its multi-variable Poincaré series is a complete invariant of its equisingularity class, in the sense of Zariski. We can associate the rational function $P(\mathcal{O}, \mathcal{O}, T_1, \cdots, T_m) \mod (q-1)\,\mathbb{Z}[[T_1, \cdots, T_m]]$ to an algebroid plane curve. In the first section we observe that the mentioned definitions of zeta function, partial zeta function and multi-variable Poincaré series associated to non-zero fractional ideals of a local ring $\mathcal{O}$ of the irreducible algebraic curve X can also be defined for regular fractional ideals of a reduced local ring $\mathcal{O}$ of a possibly singular, complete, reduced algebraic curve X define over a finite field $k = \mathbb{F}_q$.

5.1 Multi-variable Poincaré Series of reduced curves over finite field

The preceding theory about zeta function and multi-variable Poincaré series can be extended to reduced curves. We now indicate the necessary modifications in order to apply the results of previous sections. Before obtaining this extension we give some preliminary known definitions and results needed in the sequel.

Let X be a complete reduced curve over the field k, an let

$$X = X_1 \cup \cdots \cup X_r$$

be its decomposition into irreducible components. For each $P \in X$ let $\mathcal{O}_P = \mathcal{O}_{X,P}$ be the local ring of X at P and $\mathfrak{m}_P = \mathfrak{m}_{X,P}$ its maximal ideal. The other prime ideals of $\mathcal{O}_{X,P}$ are minimal and correspond bijectively to the irreducible components of X passing through P. If $P \in X_j$ and if $\mathfrak{p}_{X,X_j,P}$ is the corresponding minimal prime ideal of $\mathcal{O}_{X,P}$ then

$$\mathcal{O}_{X_j,P} = \mathcal{O}_{X,P}/\mathfrak{p}_{X,X_j,P}.$$

Since the local ring $\mathcal{O}_{X,P}$ is reduced, the intersection of its minimal primes is zero, and so we can identify

$$\mathcal{O}_{X,P} \subseteq \bigoplus_{X_j \ni P} \mathcal{O}_{X_j,P}$$

where X_j varies over the irreducible components of X passing through P. The codimension of $\mathcal{O}_{X,P}$ in $\bigoplus_{X_j \ni P} \mathcal{O}_{X_j,P}$ is denoted by $I_{X,P}$. The number $I_{X,P}$ is finite and it can be interpreted in terms of the intersection multiplicities.

The total ring of fractions of $\mathcal{O}_{X,P}$ is equal to the direct product

$$Frac(\mathcal{O}_{X,P}) = \bigoplus_{X_j \ni P} k(X_j)$$

of the function fields $k(X_j)$ of the irreducible components of X passing through P. Observe that, if we put $X_P := \bigcup_{X_j \ni P} X_j$, then $Frac(\mathcal{O}_{X,P}) = k\,(X_P)$ the k-algebra of rational functions on X. The integral closure $\widetilde{\mathcal{O}}_{X,P}$ of $\mathcal{O}_{X,P}$ is the direct product

$$\widetilde{\mathcal{O}}_{X,P} = \bigoplus_{X_j \ni P} \widetilde{\mathcal{O}}_{X_j,P}.$$

Thus, denoting by

$$\delta_{X,P} := \dim_k(\widetilde{\mathcal{O}}_{X,P}/\mathcal{O}_{X,P})$$

the singularity degree of X at P, it is obtained the identity

$$\delta_{X,P} = I_{X,P} + \sum_{X_j \ni P} \delta_{X_j,P} < \infty.$$

By considering the disjoint union

$$X' = X_1 \sqcup \cdots \sqcup X_r$$

of the irreducible components of X, the non-singular model $\widetilde{X}$ of X may be expressed as the disjoint union

$$\widetilde{X} = \widetilde{X}_1 \sqcup \cdots \sqcup \widetilde{X}_r$$

of the non-singular models of the irreducible components of X. Furthermore, the elements in the inverse image $\pi^{-1}(P)$ of the morphism $\pi : \widetilde{X} \longrightarrow X' \longrightarrow X$ correspond bijectively to the branches of centred at P. Thus the branches of X are simply the branches of the irreducible components of X.

By a *regular fractional* $\mathcal{O}_{X,P}$-ideal we mean a finitely generated $\mathcal{O}_{X,P}$-submodule of the total ring of fractions $Frac(\mathcal{O}_{X,P})$ of $\mathcal{O}_{X,P}$ not contained in the set of zero divisors of $\mathcal{O}_{X,P}$. Recall that an element of $\mathcal{O}_{X,P}$ is a zero divisor if and only if it is contained in some minimal prime ideal or equivalently it is a zero divisor of the ring $\bigoplus_{X_j \ni P} \mathcal{O}_{X_j,P}$, that is, a non-unity of $\bigoplus_{X_j \ni P} k(X_j)$ i.e. it is identically zero on some irreducible component passing through P. Moreover, an $\mathcal{O}_{X,P}$-submodule $\mathfrak{a}_P$ of the total ring of fractions $Frac(\mathcal{O}_{X,P})$ of $\mathcal{O}_{X,P}$ is finitely generated if and only if there exists a non-zero divisor s of $\mathcal{O}_{X,P}$ such that $s\mathfrak{a}_P \subseteq \mathcal{O}_{X,P}$. If $\mathfrak{a}_P$ and $\mathfrak{b}_P$ are regular fractional $\mathcal{O}_{X,P}$-ideal such that $\mathfrak{a}_P \supseteq \mathfrak{b}_P$ then $\dim_k(\mathfrak{a}_P/\mathfrak{b}_P) < \infty$. Thus, the *local degree* $\deg_P(\mathfrak{a}_P)$ is defined by similar properties to that used in the irreducible case, namely:

i. $\deg_P(\mathcal{O}_P) = 0$

ii. $\dim_k(\mathfrak{a}_P/\mathfrak{b}_P) = \deg_P(\mathfrak{a}_P) - (\mathfrak{b}_P)$ whenever $\mathfrak{a}_P \supseteq \mathfrak{b}_P$.

Since $\widetilde{\mathcal{O}}_{X,P} = \bigoplus\limits_{X_j \ni P} \widetilde{\mathcal{O}}_{X_j,P}$, each $\widetilde{\mathcal{O}}_{X,P}$-ideal $\mathfrak{a}_P$ is a direct sum $\bigoplus_{X_j \ni P} \mathfrak{a}_j$ where each $\mathfrak{a}_j$ is an $\widetilde{\mathcal{O}}_{X_j,P}$-ideal. A such $\widetilde{\mathcal{O}}_{X,P}$-ideal is a maximal ideal of $\widetilde{\mathcal{O}}_{X,P}$ if and only if $\mathfrak{a}_j$ is maximal ideal of $\widetilde{\mathcal{O}}_{X_j,P}$ for some j and $\mathfrak{a}_i = \widetilde{\mathcal{O}}_{X_i,P}$ for each other i. The integral closure $\widetilde{\mathcal{O}}_{X_j,P}$ of $\mathcal{O}_{X_j,P}$ is the principal ideal domain

$$\widetilde{\mathcal{O}}_{X_j,P} = \bigcap_{Q \in X_j,\ Q|P} \widetilde{\mathcal{O}}_{X_j,Q}$$

with quotient field $k(X_j)$, whose maximal ideals correspond bijectively to the branches of X_j centered in P. Moreover, Q runs over all branches in X_j centered at P. Set $Q_1, \cdots, Q_m$ be the branches of X centered at P and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ be the corresponding maximal ideals of $\widetilde{\mathcal{O}}_{X,P}$. The $\widetilde{\mathcal{O}}_{X,P}$-ideals are exactly of the form

$$\mathfrak{p}^{\mathbf{n}} := \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}$$

where $n_1, \cdots, n_m$ are integers. If $n_1 \geq 0, \cdots, n_m \geq 0$ then

$$\begin{aligned}\widetilde{\mathcal{O}}_{X,P}/\mathfrak{p}^{\mathbf{n}} &= \widetilde{\mathcal{O}}_{X,P}/\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m} \\ &\simeq \bigoplus_{i=1}^{m} \widetilde{\mathcal{O}}_{X,P}/\mathfrak{p}_i^{n_i}\end{aligned}$$

As follows by applying the Chinese remainder theorem. By passing to the projective limit one obtains

$$\widehat{\widetilde{\mathcal{O}}}_{X,P} = \widehat{\mathcal{O}}_{\tilde{X},Q_1} \times \cdots \times \widehat{\mathcal{O}}_{\tilde{X},Q_m}$$

Since $\delta_P := \delta_{X,P} < \infty$ the topology of $\mathcal{O}_P$ is induced by the topology of $\widetilde{\mathcal{O}}_P$, and so the completion $\widehat{\mathcal{O}}_P$ is the closed subring of $\widehat{\widetilde{\mathcal{O}}}_{X,P} = \widehat{\mathcal{O}}_{\widetilde{X},Q_1} \times \cdots \times \widehat{\mathcal{O}}_{\widetilde{X},Q_m}$ of codimension δ_P.

Now we indicate a natural way to extend the mentioned definitions of zeta functions and multiple Poincaré series to local rings of a complete reduced curve defined over a finite field. Let $\mathcal{O}$ be a local ring at a singular point P of a complete reduced curve $X = X_1 \cup \cdots \cup X_r$ defined over a finite field $k = \mathbb{F}_q$, where $X_1, \cdots, X_r$ are the irreducible components of X. Without loss of generality we can assume that each irreducible component passes through P, in the otherwise we take $\mathcal{O}$ to be a local ring at P of the complete reduced curve $X_P := \bigcup\limits_{X_j \ni P} X_j$. Let $\mathcal{O}_1, \cdots, \mathcal{O}_r$ be the local rings of $X_1, \cdots, X_r$ at P, respectively. Thus,

$$\mathcal{O}_j = \mathcal{O}/\mathfrak{P}_j,$$

where $\mathfrak{P}_j$ is the minimal primes of $\mathcal{O}$ corresponding to the irreducible component X_j. We can identify

$$\mathcal{O} \subseteq \bigoplus_{j=1}^{r} \mathcal{O}_j.$$

Moreover, the total ring of fractions of $\mathcal{O}$ is equal to the direct product

$$K := Frac(\mathcal{O}) = \bigoplus_{j=1}^{r} K_j$$

of the function fields $K_j := k(X_j)$ of the irreducible components of X. Hence, $K = k(X)$ is the k-algebra of rational functions on X. The integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is the direct product

$$\widetilde{\mathcal{O}} = \bigoplus_{j=1}^{r} \widetilde{\mathcal{O}}_j.$$

The branches of X centered at P correspond to the maximal ideals of $\widetilde{\mathcal{O}}$, say $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$, and the $\widetilde{\mathcal{O}}$-ideals are exactly of the form

$$\mathfrak{p}^{\mathbf{n}} := \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}$$

where $n_1, \cdots, n_m$ are integers. We define its *multi-exponent* by $\mathbf{v}(\mathfrak{p}^{\mathbf{n}}) := \mathbf{n}$.

On the other hand, for each $j = 1, \cdots, r$ the maximal ideals, say $\mathfrak{p}_{1j}, \cdots, \mathfrak{p}_{m_j j}$, of the integral closure $\widetilde{\mathcal{O}_j}$ of $\mathcal{O}_j$ in K_j correspond bijectively to the valuations $v_{1j} = ord_{\mathfrak{p}_{1j}}, \cdots, v_{m_j j} = ord_{\mathfrak{p}_{m_j j}}$ in the function field $K_j \mid k$. Each such valuation v_{ij} in the function field $K_j \mid k$ may be extended to a map v_{ij} of K onto the set

$\mathbb{Z} \cup \{\infty\}$ that vanishes on k and has the formal properties of a valuation, by defining $v_{ij}(x_1, \cdots, x_r) = v_{ij}(x_j)$. Conversely, every map of K onto the set $\mathbb{Z} \cup \{\infty\}$ that vanishes on k and has the formal properties of a valuation is of this form. Thus, for each non-zero rational function $z \in K \backslash \{0\}$ is abbreviated

$$\mathbf{v}(z) := \mathbf{v}(z\widetilde{\mathcal{O}}) = (v_1(z), \cdots, v_m(z)) \in \mathbb{Z}^m.$$

We have in this case the ingredients needed to define in similar way the multivariable Poincaré series associated to a pair of $\mathcal{O}$-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. It is defined to be the multi-variable power series

$$P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) := \sum \eta_{\mathcal{O},\mathbf{n}}(\mathfrak{a}, \mathfrak{b})\, \mathbf{t}^{\mathbf{n}} \in \mathbb{Z}[[t_1, \ldots, t_m]]$$

whose coefficients are the cardinalities

$$\eta_{\mathcal{O},\mathbf{n}}(\mathfrak{a}, \mathfrak{b}) := \#\{\mathcal{O}\text{-ideals } \mathfrak{d} \text{ satisfying } \mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b} \text{ and } \mathfrak{d} \cdot \widetilde{\mathcal{O}} = \mathfrak{a} \cdot \mathfrak{p}^{-\mathbf{n}}\}$$

where $\mathbf{t}^{\mathbf{n}} := t_1^{n_1} \cdots t_m^{n_m}$ for each $\mathbf{n} := (n_1, \cdots, n_m) \in \mathbb{Z}^m$.

From the definition, this series only depends on the $\mathcal{O}$-ideal classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$ and it can be expressed in the form

$$P_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, \mathbf{t}) = \sum_{\mathfrak{d} \supset \mathfrak{a}, \mathfrak{d} \sim \mathfrak{b}} \mathbf{t}^{\mathbf{v}(\mathfrak{a} \cdot \widetilde{\mathcal{O}}) - \mathbf{v}(\mathfrak{d} \cdot \widetilde{\mathcal{O}})} \in \mathbb{Z}[[t_1, \ldots, t_m]]$$

where the sum is taken over all $\mathcal{O}$-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$. Similarly, we can associate to each $\mathcal{O}$-ideal $\mathfrak{a}$ the Stöhr Dirichlet series

$$\zeta_{\mathcal{O}}(\mathfrak{a}, s) := \sum_{\mathfrak{d} \supseteq \mathfrak{a}} \#(\mathfrak{d}/\mathfrak{a})^{-s},\ \Re(s) > 0$$

where the sum is taken over the $\mathcal{O}$-ideals $\mathfrak{d}$ that contain $\mathfrak{a}$, with

$$Z_{\mathcal{O}}(\mathfrak{a}, t) := \sum_{\mathfrak{d} \supseteq \mathfrak{a}} t^{\dim_k(\mathfrak{d}/\mathfrak{a})}.$$

where $t = q^{-s}$. Moreover, we can associate to each pair of $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ the partial zeta-function

$$\zeta_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, s) := \sum_{\mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b}} \#(\mathfrak{d}/\mathfrak{a})^{-s},\ \Re(s) > 0$$

where the sum is taken over $\mathcal{O}$-ideals that contain $\mathfrak{a}$ and are equivalent to $\mathfrak{b}$, and it may be written as a power series in $t = q^{-s}$ with integer coefficients:

$$Z_{\mathcal{O}}(\mathfrak{a}, \mathfrak{b}, t) := \sum_{\mathfrak{d} \supseteq \mathfrak{a},\ \mathfrak{d} \sim \mathfrak{b}} t^{\dim_k(\mathfrak{d}/\mathfrak{a})},\ |t| < 1$$

and it only depends on the ideals classes $[\mathfrak{a}]$ and $[\mathfrak{b}]$. They also satisfy $Z(\mathfrak{a},t) = \sum_{[\mathfrak{b}]} Z(\mathfrak{a},\mathfrak{b},t)$, where $\mathfrak{b}$ ranges in a complete system of representatives of $\mathcal{O}$-ideal class. Moreover, $Z_{\mathcal{O}}(\mathfrak{a},\mathfrak{b},t) = t^{\dim_k(\mathfrak{a}\cdot\widetilde{\mathcal{O}}/\mathfrak{a})-\dim_k(\mathfrak{b}\cdot\widetilde{\mathcal{O}}/\mathfrak{b})} P_{\mathcal{O}}(\mathfrak{a},\mathfrak{b},t^{r_1},\ldots,t^{r_m})$, where $r_1 := \dim_k(\widetilde{\mathcal{O}}/\mathfrak{p}_1), \cdots, r_m := \dim_k(\widetilde{\mathcal{O}}/\mathfrak{p}_m)$.

In similar way, many of the results in [26] and [27] about zeta function and multi-variable Poincaré series for local rings of a complete irreducible curve X may be extended to complete reduced curves. Finally, we observe that $P_{\widehat{\mathcal{O}}}(\widehat{\mathfrak{a}},\widehat{\mathfrak{b}},\mathfrak{t}) = P_{\mathcal{O}}(\mathfrak{a},\mathfrak{b},\mathfrak{t})$, $Z_{\widehat{\mathcal{O}}}\left(\widehat{\mathfrak{a}},\widehat{\mathfrak{b}},t\right) = Z_{\mathcal{O}}(\mathfrak{a},\mathfrak{b},t)$ and $Z_{\widehat{\mathcal{O}}}(\widehat{\mathfrak{a}},t) = Z_{\mathcal{O}}(\mathfrak{a},t)$, when $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathcal{O}$-ideal of a local ring $\mathcal{O}$ of a geometrically irreducible algebraic curve defined over a finite field $k = \mathbb{F}_q$.

5.2 Multi-variable Poincaré series of plane algebroid curves and equisingularity

Let k be a field non necessarily algebraically closed and let $f \in k[[X,Y]]$ be a series satisfying $f(0,0) = 0$, square free and such that each irreducible factor of it is absolutely irreducible. The reduced local ring $\mathcal{O} := k[[X,Y]]/(f)$ is called a *plane algebroid curve totally defined over* k (see [36]). It can be proved that the local ring $\mathcal{O}$ is residually rational i.e. the localization at maximal ideals of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ have the same residue field, where $\widetilde{\mathcal{O}}$ denote the integral closure of the ring $\mathcal{O}$ in its total ring of fractions. Let $f = \prod_{i=1}^{m} f_i$ be the decomposition of f into irreducible factors in $k[[X,Y]]$ and let $\mathcal{O}_i := k[[X,Y]]/(f_i)$ be the local ring, called the *irreducible component* of $\mathcal{O}$, for each $i = 1,\cdots,m$. It can be proved that $\widehat{\mathcal{O}\otimes_k \overline{k}} = \overline{k}[[X,Y]]/(f)$ and its irreducible components are of the form $\widehat{\mathcal{O}_i\otimes_k \overline{k}} = \overline{k}[[X,Y]]/(f_i)$, $i = 1,\cdots,m$. Then, there exists a finite field extension l of k such that $S(\mathcal{O}\otimes_k l) = S\left(\widehat{\mathcal{O}\otimes_k \overline{k}}\right)$ (see [36], Proposition 1.5 page. 40 and Proposition 4.7 page. 35). Thus, we may assume that $S(\mathcal{O}) = S(\mathcal{O}\otimes_k l)$, where l is any finite extension of k.

Now, let k be an algebraically closed field and let $f \in k[[X,Y]]$ be a series satisfying $f(0,0) = 0$ and square free. We denote by (f) and

$$\mathcal{O} := k[[X,Y]]/(f)$$

the plane algebroid curve defined by f over the field k and the local ring associated to it, respectively.

Let $f = \prod_{i=1}^{m} f_i$ be the corresponding decomposition of the series f into irreducible factors. The semigroup of a plane algebroid curve (f) is defined by

$$S(f) := \{(I(f_1, h), \cdots, I(f_m, h)) : h \in k[[X,Y]]\} \cap \mathbb{N}^m,$$

where $I(f_i, h)$ denotes the intersection multiplicity of f_i and h at the point $(0,0)$. It is well known that there exists a bijection between the minimal primes of the local ring $\mathcal{O}$ and the branches of (f) as well as the valuation lying over the ring $\mathcal{O}$, say $v_1, \cdots, v_m$. Moreover, if $h \in k[[X,Y]] \backslash f_i k[[X,Y]]$, then $v_i(\overline{h}) = I(f_i, h)$, where $\overline{h} := h(x,y)$ with $x := X + f_i k[[X,Y]]$ and $y := Y + f_i k[[X,Y]]$. Thus, by using the classical construction of a valuation associated to a branch, we can see that the semigroup $S(f)$ agrees with the semigroup $S(\mathcal{O})$ of the local ring $\mathcal{O}$.

The equisingularity class of a plane algebroid curves can be defined in several equivalent forms (cf. [30], [33]), for our purposes it is enough to know that two plane algebroid curves defined over an algebraically closed field are equisingular if and only if there exists a preserving intersection multiplicity bijection between their components i.e. their branches (cf. [34], Lemma 7.1). The equisingularity class was characterized by Zariski (in the case $m = 1$) and by Waldi (in the general case for any m) in terms of the semigroup, that is, the semigroup determines exactly the equisingularity class of the plane algebroid curve (cf. [30]). This notion is important because, when $k = \mathbb{C}$, the notion of equisingularity class agrees with the notion of topological class. It is known that the semigroup $S_{\mathcal{C}}$ and the Alexander polynomial $\Delta^{\mathcal{C}}(t_1, ..., t_m)$ characterize completely the topological class of a germ $(\mathcal{C}, 0) \subseteq (\mathbb{C}^2, 0)$ (cf. [32]).

A general problem in studying singularities of analytic sets is to express their topological invariants in terms of the analytic ones. The papers [7] and [8] achieve this nicely in the case of (germs of) reduced plane curves.

Let $(\mathcal{C}, 0) \subseteq (\mathbb{C}^2, 0)$ be a germ of a reduced plane curve given by $f = 0$ for $f \in \mathcal{O}_{(\mathbb{C}^2,0)}$, and let $\mathcal{C} = \bigcup_{i=1}^{m} \mathcal{C}_i$, with $r > 1$ be its decomposition into irreducible components corresponding to $f = \prod_{i=1}^{m} f_i$. Let $\Delta^{\mathcal{C}}(t_1, ..., t_m)$ be the Alexander polynomial of the link $\mathcal{C} \cap S^3_\epsilon \subseteq S^3_\epsilon$ for sufficiently small $\epsilon > 0$ (cf. [12]). The multi-variable Alexander polynomial is a complete topological invariant of the singularity $(\mathcal{C}, 0)$. A formula for $\Delta^{\mathcal{C}}(t_1, ..., t_m)$ in terms of the data of an embedded resolution of $\mathcal{C}$ was given by Eisenbud and Neumann in [12].

Let $\pi : (\mathcal{X}, D) \longrightarrow (\mathbb{C}^2, 0)$ be an embedded resolution with the exceptional divisor $D = \bigcup_{\sigma \in \Gamma} E_\sigma$ the union of irreducible components $E_\sigma \simeq \mathbb{CP}^1$, and $E^0_\sigma \subseteq E_\sigma$ be the complement in E_σ of the intersection with all other components of the total transform $(f \circ \pi)^{-1}(0)$ of the curve $\mathcal{C}$. For $\sigma \in \Gamma$ and $g \in \mathcal{O}_{(\mathbb{C}^2,0)}$, with $g \neq 0$, let $m^\sigma(g)$ be the multiplicity along E_σ of the lifting of g to $\mathcal{X}$ and let $\mathbf{m}^\sigma = (m^\sigma(f_1), ..., m^\sigma(f_m)) \in \mathbb{Z}^m_{\geq 0}$. If $S^k X$ denotes the k-th symmetric power of a topological space X, then Eisenbud and Neumann in [12] showed, by considering the space

$$Y = \prod\nolimits_\sigma \left(\bigcup\nolimits_{k=0}^{\infty} S^k E^0_\sigma \right) = \bigcup\nolimits_{k_\sigma} \left(\prod\nolimits_\sigma S^{k_\sigma} E^0_\sigma \right),$$

that the Alexander polynomial may be computed as

$$\Delta^{\mathcal{C}}(t_1, ..., t_m) = \prod\nolimits_\sigma (1 - t_1^{m^\sigma(f_1)} \cdots t_m^{m^\sigma(f_m)})^{-\chi(E^0_\sigma)},$$

where $\chi(X)$ stands for the Euler characteristic of the topological space X.

Next, let $\varphi_i : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, 0)$ be uniformizations of the branches $\mathcal{C}_i$ of $\mathcal{C}$ $(1 \leq i \leq m)$, so that, for a germ $g \in \mathcal{O}_{(\mathbb{C}^2,0)}$, with $g \neq 0$, one may denote by $v_i = v_i(g)$ and by $a_i = a_i(g)$ the exponent and the coefficient, respectively, of the leading monomial in the expansion of the germ $g \circ \varphi_i$ as a power series. Let $\mathcal{L} = \mathbb{Z}[[t_1, ..., t_m; t_1^{-1}, ..., t_m^{-1}]]$ be the set of formal Laurent series in the variables t_i ($\mathcal{L}$ is not a ring). For any $\mathbf{n} = (n_1, ..., n_m) \in \mathbb{Z}^m$ consider the ideal $J(\mathbf{n}) = \{g \in \mathcal{O}_{(\mathbb{C}^2,0)} : v_i(g) \geq n_i \; i = 1, ..., m\}$ and the set

$$P_{\mathcal{C}}(t_1, ..., t_m) = \frac{(t_1 - 1) \cdots (t_m - 1)}{t_1 \cdots t_m - 1} \sum\nolimits_{\mathbf{n}} \dim_{\mathbb{C}}(J(\mathbf{n})/J(\mathbf{n} + \mathbf{1}))\mathbf{t}^{\mathbf{n}}$$

where $\mathbf{1} = (1, ..., 1)$. Campillo, Delgado and Gusein-Zade showed in [7] that $P_{\mathcal{C}}(t_1, ..., t_m)$ is a polynomial and called it *generalized Poincaré polynomial of the multi-indexed filtration induced by the valuation-tuple* $\mathbf{v}$. In [7] they further showed that this is none other than the Alexander polynomial, that is,

$$P_{\mathcal{C}}(t_1, ..., t_m) = \prod\nolimits_\sigma (1 - t_1^{m^\sigma(f_1)} \cdots t_m^{m^\sigma(f_m)})^{-\chi(E^0_\sigma)}. \tag{5.1}$$

In the case of algebroid plane curves defined over an algebraically closed field, we want to prove a similar identity to 5.1 (proved by Campillo, Delgado and Gusein-Zade). So that, we require some terms similar to that in 5.1. First, according to the definition of Euler characteristic given in [7] (cf. [7] page 133), we observe that $\chi(E^0_\sigma) = 2 - \#\left(E_\sigma \backslash E^0_\sigma\right)$, for each $\sigma \in \Gamma$.

Let $\mathcal{O}$ be the local ring at a singular point of a geometrically integral algebraic curve defined over a finite field k and let $\mathfrak{b}$ be an $\mathcal{O}$-ideal. It was proved by Stöhr (see [27] Theorem 6.3) that

$$P(\mathcal{O}, \mathfrak{b}, \mathbf{t}) = \frac{q^{\delta - \deg(\mathfrak{b}\widetilde{\mathcal{O}})}}{[U_{\mathfrak{b}} : U_{\mathcal{O}}]\,(1 - q^{-\varrho})} \frac{\prod_{i=1}^{m} (q^{r_i} t_i - 1)}{q^{|\mathbf{r}|} t_1 \cdots t_m - 1} \sum_{\mathbf{n} \in \mathbb{Z}^m} q^{\mathbf{r} \cdot \mathbf{n}} (q^{\deg(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}})} - q^{\deg(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{1}})}) \mathbf{t}^{\mathbf{n}}$$

where $\mathbf{1} := (1, \cdots, 1)$. Let

$$c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) := \frac{q^{\delta - \deg(\mathfrak{b}\widetilde{\mathcal{O}}) + \mathbf{r}\cdot\mathbf{n}}}{[U_{\mathfrak{b}} : U_{\mathcal{O}}](1 - q^{-\varrho})} \sum_{\mathbf{i}\in\{0,1\}^m} (-1)^{m-|\mathbf{i}|} \left(q^{\deg(\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}})} - q^{\deg(\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}})}\right)$$

for each $\mathbf{n} \in \mathbb{Z}^m$. By noting that

$$\prod_{i=1}^{m} (q^{r_i} t_i - 1) = \sum_{\mathbf{i}\in\{0,1\}^m} (-1)^{m-|\mathbf{i}|} q^{\mathbf{r}\cdot\mathbf{i}} \mathbf{t}^{\mathbf{i}},$$

it follows that

$$P(\mathcal{O}, \mathfrak{b}, \mathbf{t}) = \frac{1}{q^{|\mathbf{r}|} t_1 \cdots t_m - 1} \sum_{\mathbf{n}\in\mathbb{Z}^m} c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) \mathbf{t}^{\mathbf{n}}.$$

We observe that:

Lemma 63 *Let* $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$.

1. *If* $n_j < 0$ *for some* $j = 1, \cdots, m$, *then* $c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) = 0$.
2. *If* $n_j \geq b_j + 1$ *then* $c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) = c_{\mathbf{n}+\mathbf{e}_j}(\mathcal{O}, \mathfrak{b})$.

   This means that $c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) = c_{\inf(\mathbf{n},\mathbf{b}+\mathbf{1})}(\mathcal{O}, \mathfrak{b})$ *for each* $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{N}^m$, *where* $\mathbf{b} := \mathbf{v}((\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S}) = (b_1, \cdots, b_m) \in \mathbb{Z}^m$ *is the multi-exponent of the* $\widetilde{\mathcal{O}}$*-ideal* $(\mathfrak{b} : \widetilde{S}) : \mathfrak{b}\widetilde{S}$.

Proof. Since

$$\mathfrak{b} \cap \mathfrak{b} \cdot \mathfrak{p}^{\mathbf{s}} = \{z \in \mathfrak{b} : \mathbf{v}(z) \geq s + \mathbf{v}(\mathfrak{b}\widetilde{\mathcal{O}})\} \text{ for each } \mathbf{s} \in \mathbb{Z}^m,$$

It follows that $\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{s}} = \mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{s}-\mathbf{e}_j}$ and $\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{s}+\mathbf{1}} = \mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{s}+\mathbf{1}-\mathbf{e}_j}$ for any $\mathbf{s} = (s_1, \cdots, s_m) \in \mathbb{Z}^m$ such that $s_j < 0$ for some $j = 1, \cdots, m$; and so the first sentence will follow.

To prove the second sentence let $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{N}^m$ such that $n_j \geq b_j + 1$ and let $\mathbf{i} \in \{0,1\}^m$. Let us consider a strictly increasing sequence $\left(\mathbf{n}^{(k)}\right)_{0\leq k\leq m}$, where $\mathbf{n}^{(0)} = \mathbf{n} - \mathbf{i}$, $\mathbf{n}^{(1)} = \mathbf{n} - \mathbf{i} + \mathbf{e}_j$, $\mathbf{n}^{(m)} = \mathbf{n} - \mathbf{i} + \mathbf{1}$. Then for each $k = 1, \cdots, m$ there exists $i(k) \in \{1, \cdots, m\}$ such that $\mathbf{n}^{(k)} = \mathbf{n}^{(k)} + \mathbf{e}_{i(k)}$. Hence,

$$\dim_k \left(\frac{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}}{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}}\right) = \dim_k \left(\frac{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}}{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{e}_j}}\right) + \sum_{k=1}^{m-1} \dim_k \left(\frac{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}^{(k)}}}{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}^{(k)}+\mathbf{e}_{i(k)}}}\right)$$

and

$$\dim_k \left(\frac{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{e}_j}}{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{e}_j+\mathbf{1}}}\right) = \sum_{k=1}^{m-1} \dim_k \left(\frac{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}^{(k)}}}{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}^{(k)}+\mathbf{e}_{i(k)}}}\right) + \dim_k \left(\frac{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}}{\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}+\mathbf{e}_j}}\right).$$

Since $n_j \geq b_j + 1$, it follows from Lemma 22 (3) that

$$\dim_k \left(\frac{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}}{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}} \right) = \dim_k \left(\frac{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{e}_j}}{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{e}_j+\mathbf{1}}} \right)$$

and $\deg\left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}\right) = r_j + \deg\left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_j-\mathbf{i}+\mathbf{1}}\right)$ i.e.

$$\mathbf{r} \cdot \mathbf{n} + \deg\left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}\right) = \mathbf{r} \cdot (\mathbf{n} + \mathbf{e}_j) + \deg\left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{e}_j-\mathbf{i}+\mathbf{1}}\right);$$

and so the second sentence will follow. ■

From the precedent Lemma, it follows that

$$\begin{aligned} P(\mathcal{O}, \mathfrak{b}, \mathbf{t}) &= \frac{1}{q^{|\mathbf{r}|} t_1 \cdots t_m - 1} \sum_{\mathbf{n} \in \mathbb{N}^m} c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) \, \mathbf{t}^{\mathbf{n}} \\ &= \frac{1}{q^{|\mathbf{r}|} t_1 \cdots t_m - 1} \sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{b}+\mathbf{1}} c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) \, \mathbf{t}^{\mathbf{n}} \prod_{n_j = b_j + 1} \frac{1}{1 - t_j} \end{aligned}$$

where in the product the index j runs through the integers $j = 1, \cdots, m$ with $n_j = b_j + 1$. Thus,

$$P(\mathcal{O}, \mathfrak{b}, \mathbf{t}) = \frac{\sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{b}+\mathbf{1}} c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) \, \mathbf{t}^{\mathbf{n}} \prod_{n_j < b_j + 1} (1 - t_j)}{(q^{|\mathbf{r}|} t_1 \cdots t_m - 1)(1 - t_1) \cdots (1 - t_m)}$$

where in the product the index j runs through the integers $j = 1, \cdots, m$ with $n_j < b_j$.

This is another proof of the rationality of the Poincaré series $P(\mathcal{O}, \mathfrak{b}, \mathbf{t})$.

On the other hand, we have that

$$P(\mathcal{O}, \mathfrak{b}, \mathbf{t}) = P_{\mathfrak{b}:\mathfrak{b}}(\mathfrak{b} : \mathfrak{b}, \mathfrak{b}, \mathbf{t}) \text{ and, hence, } [U_{\mathfrak{b}} : U_{\mathcal{O}}] \left(1 - q^{-\varrho}\right) = q^{\delta - \delta_{\mathfrak{b}:\mathfrak{b}}} \prod_{i=1}^{s} \left(1 - q^{-\varrho_i}\right)$$

where $\mathfrak{b} : \mathfrak{b}$ is a semilocal ring and $\mathfrak{b} : \mathfrak{b} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_s$ is its decomposition as a finite intersection of local rings, moreover, $\varrho_i := \dim_k(\mathcal{O}_i/\mathfrak{m}_i)$ and $\delta_i := \dim_k(\widetilde{\mathcal{O}_i}/\mathcal{O}_i)$ are the degree of the residue field and the singularity degree of the local ring $\mathcal{O}_j$, respectively, for each $i = 1, \cdots, s$. Thus

$$c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) = \frac{q^{\delta_{\mathfrak{b}:\mathfrak{b}} - \deg(\mathfrak{b}\widetilde{\mathcal{O}}) + \mathbf{r} \cdot \mathbf{n}}}{\prod_{i=1}^{s} (1 - q^{-\varrho_i})} \sum_{\mathbf{i} \in \{0,1\}^m} (-1)^{m - |\mathbf{i}|} \left(q^{\deg(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}})} - q^{\deg(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}})}\right).$$

Hence, from Lemma 22,

$$c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) = \frac{q^{\delta_{\mathfrak{b}:\mathfrak{b}} - \deg(\mathfrak{b}\widetilde{\mathcal{O}}) + \mathbf{r} \cdot \mathbf{n}} (q^{r_j} - 1)}{\prod_{i=1}^{s} (1 - q^{-\varrho_i})} \sum_{\mathbf{i} \in \{0,1\}^m, i_j = 0} (-1)^{m - |\mathbf{i}| - 1} \left(q^{\deg\left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}\right)} - q^{\deg(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}})}\right)$$

for each $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{N}^m$ with $n_j = b_j + 1$, where the sum is taken over the vectors $\mathbf{i} \in \{0,1\}^m$ whose j-th entry is equal to 0 i.e. $i_j = 0$. In particular, if $\mathbf{n} \geq \mathbf{b} + \mathbf{1}$ then $c_{\mathbf{n}}(\mathcal{O}, \mathfrak{b}) = \frac{q^{\delta_{\mathfrak{b}:\mathfrak{b}} - |\mathbf{r}|}}{\prod_{i=1}^{s}(1 - q^{-\varrho_i})} \left(q^{|\mathbf{r}|} - 1\right) \prod_{j=1}^{m} (q^{r_j} - 1)$.

It is well known that

$$\dim_k(\mathfrak{d}/\mathfrak{a}) = \varrho \cdot \mathrm{length}_{\mathcal{O}}(\mathfrak{d}/\mathfrak{a}) \text{ whenever } \mathfrak{d} \supseteq \mathfrak{a}.$$

Thus, we obtain the following proposition:

Proposition 64 *Let $\mathcal{O}$ be the local ring at a singular point of a geometrically integral algebraic curve defined over a finite field k and let $\mathfrak{b}$ be an $\mathcal{O}$-ideal such that the ring $\mathfrak{b} : \mathfrak{b}$ is a local ring. Then $P(\mathcal{O}, \mathfrak{b}, \mathbf{t})$ is congruent modulo $(q-1)\mathbb{Z}[[t_1, \cdots, t_m]]$ to the series $\frac{(t_1-1)\cdots(t_m-1)}{t_1\cdots t_m - 1} \sum_{\mathbf{n}\in\mathbb{Z}^m} \mathrm{length}_{\mathfrak{b}:\mathfrak{b}} \left(\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}} / \mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{1}}\right) \mathbf{t}^{\mathbf{n}}$, which is a polynomial when $m \geq 2$. In particular,*

$$P(\mathcal{O}, \mathcal{O}, \mathbf{t}) \equiv \frac{(t_1-1)\cdots(t_m-1)}{t_1\cdots t_m - 1} \sum_{\mathbf{n}\in\mathbb{Z}^m} \mathrm{length}_{\mathcal{O}} \left(\frac{\mathcal{O} \cap \mathfrak{p}^{\mathbf{n}}}{\mathcal{O} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{1}}}\right) \mathbf{t}^{\mathbf{n}}$$

$\mod (q-1)\mathbb{Z}[[t_1, \cdots, t_m]]$. *If the local ring $\mathcal{O}$ corresponds to a rational point, that is, if $\varrho = 1$, then*

$$P(\mathcal{O}, \mathcal{O}, \mathbf{t}) \equiv \frac{(t_1-1)\cdots(t_m-1)}{t_1\cdots t_m - 1} \sum_{\mathbf{n}\in\mathbb{Z}^m} \dim_k \left(\mathcal{O} \cap \mathfrak{p}^{\mathbf{n}} / \mathcal{O} \cap \mathfrak{p}^{\mathbf{n}+\mathbf{1}}\right) \mathbf{t}^{\mathbf{n}}$$

$\mod (q-1)\mathbb{Z}[[t_1, \cdots, t_m]]$

Proof. Let $\varrho_1 := \dim_k(\mathcal{O}_1/\mathfrak{m}_1)$ and $\delta_1 := \dim_k(\widetilde{\mathcal{O}_1}/\mathcal{O}_1)$ the degree of the residue field and the singularity degree of the local ring $\mathcal{O}_1 := \mathfrak{b} : \mathfrak{b}$, respectively. In this case,

$$P(\mathcal{O}, \mathfrak{b}, \mathbf{t}) = \frac{1}{q^{|\mathbf{r}|} t_1 \cdots t_m - 1} \sum_{\mathbf{n}\in\mathbb{N}^m} c_{\mathbf{n}}(\mathcal{O}_1, \mathfrak{b}) \mathbf{t}^{\mathbf{n}}$$

We observe that

$$\left(q^{\dim_k\left(\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}/\mathfrak{b}\cap\mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}\right)} - 1\right) / \left(q^{\varrho_1} - 1\right) \equiv \mathrm{length}_{\mathcal{O}_1} \left(\frac{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}}{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}}\right) \mod (q-1)$$

for each $\mathbf{n} \in \mathbb{N}^m$ and for each $\mathbf{i} \in \{0,1\}^m$. Then

$$c_{\mathbf{n}}(\mathcal{O}_1, \mathfrak{b}) \equiv \sum_{\mathbf{i}\in\{0,1\}^m} (-1)^{m-|\mathbf{i}|} \mathrm{length}_{\mathcal{O}_1} \left(\frac{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}}{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}}\right) \mod (q-1), \ \mathbf{0} \leq \mathbf{n} \leq \mathbf{b}$$

and

$$c_{\mathbf{n}}\left(\mathcal{O}_1, \mathfrak{b}\right) \equiv 0 \mod (q-1)$$

for each $\mathbf{n} = (n_1, \cdots, n_m) \in \mathbb{N}^m$ such that $n_j = b_j + 1$ for some $j = 1, \cdots, m$. Thus,

$$\begin{aligned} P(\mathcal{O}, \mathfrak{b}, \mathbf{t}) &\equiv \frac{1}{t_1 \cdots t_m - 1} \sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{b}} \left(\sum_{\mathbf{i} \in \{0,1\}^m} (-1)^{m-|\mathbf{i}|} \operatorname{length}_{\mathcal{O}_1} \left(\frac{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}}}{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}-\mathbf{i}+\mathbf{1}}} \right) \right) \mathbf{t}^{\mathbf{n}} \\ &= \frac{(t_1 - 1) \cdots (t_m - 1)}{t_1 \cdots t_m - 1} \sum_{\mathbf{n} \in \mathbb{Z}^m} \operatorname{length}_{\mathcal{O}_1} \left(\frac{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}}}{\mathfrak{b} \cap \mathfrak{b}\mathfrak{p}^{\mathbf{n}+\mathbf{1}}} \right) \mathbf{t}^{\mathbf{n}}. \end{aligned}$$

where the congruence in the first line is $\mod (q-1)\,\mathbb{Z}[[t_1, \cdots, t_m]]$ ■

We observe that the previous proposition is not always true for each $\mathcal{O}$-ideal . For example, if $\mathcal{O}$ and $\mathfrak{b}$ are the local ring and the $\mathcal{O}$-ideal $\mathfrak{b}_3$, respectively, of Example in [27], then $P(\mathcal{O}, \mathfrak{b}_3, \mathbf{t})$ is not congruent with $\frac{(t_1-1)(t_2-1)(t_3-1)}{t_1 t_2 t_3 - 1} \sum_{\mathbf{n} \in \mathbb{Z}^3} \operatorname{length}_{\mathcal{O}} \left(\frac{\mathfrak{b}_3 \cap \mathfrak{p}^{\mathbf{n}}}{\mathfrak{b}_3 \cap \mathfrak{p}^{\mathbf{n}+\mathbf{1}}} \right) \mathbf{t}^{\mathbf{n}}$ module $(q-1)\,\mathbb{Z}[[t_1, t_2, t_2]]$. Since the $\mathcal{O}$-ideal $\mathfrak{b}_3$ is a semilocal ring and

$$P(\mathcal{O}, \mathfrak{b}_3, \mathbf{t}) = P_{\mathfrak{b}_3 : \mathfrak{b}_3}(\mathfrak{b}_3 : \mathfrak{b}_3, \mathfrak{b}_3, \mathbf{t}) = P_{\mathfrak{b}_3 : \mathfrak{b}_3}(\mathfrak{b}_3 : \mathfrak{b}_3, \mathfrak{b}_3 : \mathfrak{b}_3, \mathbf{t})$$

the above proposition is not always true for semilocal subring of a function field $K \mid k$.

In [36] (cf. Proposition 4.7, page 35) Zuñiga proved that, if $\mathcal{O}$ is a residually rational local ring, then there exists a unique finite field extension $k_0 \mid k$ such that

$$S\left(\mathcal{O} \otimes_k k_0\right) = S\left(\mathcal{O} \otimes_k \overline{k}\right).$$

Moreover, $S\left(\mathcal{O} \otimes_k k_0\right) = S\left(\mathcal{O} \otimes_k l\right)$ for each finite field extension of k_0. By virtue of this we may assume that $\mathcal{O}$ is residually rational ring and $S\left(\mathcal{O}\right) = S\left(\mathcal{O} \otimes_k \overline{k}\right)$. Therefore, we can associate to $\mathcal{O} \otimes_k \overline{k}$ the multi-variable rational function

$$\overline{P}(\mathcal{O} \otimes_k \overline{k}, \mathcal{O} \otimes_k \overline{k}, T_1, \cdots, T_m) := P(\mathcal{O}, \mathcal{O}, T_1, \cdots, T_m) \qquad \mod (q-1)\,\mathbb{Z}[[T_1, \cdots, T_m]]$$

where $T_1, \cdots, T_m$ are indeterminates. This series is a polynomial when $m \geq 2$. Moreover, it depends only on $S\left(\mathcal{O}\right)$.

Now we give some well known definitions and results about embedded resolution of curves on surfaces that we use in this section. Let Y be a non-singular irreducible projective surface over an algebraically closed field k. Let X be a curve on a surface Y, this means any effective divisor on the surface Y. In particular it may be singular, reducible or even have multiple components. A point will be mean closed point, unless otherwise specified. If $\sigma : W \longrightarrow Y$ is the blow-up centered at the point P and X is a curve on the surface Y passing through P, then the inverse image $\sigma^{-1}(X)$ consist of two components: the exceptional curve E and the curve X' that can be defined as the closure in Y of $\sigma^{-1}(X\backslash P)$. It is denoted by $X' = \sigma'(X)$. By considering X as an effective divisor on the surface Y, it follows that $\sigma^*(X) = \sigma'(X) + rE$ where r is the multiplicity of P on X (cf. [18] Proposition 3.6 page 389). Moreover, we have the following properties (cf. [24] Theorem 2 page 252 and [18] Theorem 3.9 page 391):

Theorem 65 *Let $\pi : Y' \longrightarrow Y$ be a birational regular map between two non-singular projective surfaces over an algebraically closed field k. Then*

1. *If D_1 and D_2 are divisors on Y then*

$$\pi^*(D_1)\pi^*(D_2) = D_1D_2.$$

2. *If $\overline{D}$ is a divisor on Y' all of whose components are exceptional curves of f and D is any divisor on Y then*

$$\pi^*(D)\overline{D} = 0.$$

In particular, if X is a plane algebroid curve defined by $f = \prod_{i=1}^{m} f_i \in k[[X,Y]]$ and $h \in k[[X,Y]]$, then

$$v_i(\overline{h}) = I(f_i, h) = I(\pi^*(f_i), \pi^*(h)') \text{ for } i = 1, \cdots, m$$

where $\pi^*(h)'$ denotes the divisor $\pi^*(h)$ without the points in the exceptional divisor of π.

Theorem 66 ***(Embedded resolution of curves in surfaces)***

Let X be a curve on the surface Y. Then there exist a finite sequence of monoidal transformations (i.e. operations of blow-up at suitable points)

$$Y' = Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 = Y$$

such that if $\pi : Y' \longrightarrow Y$ is their composition, then the total inverse image $\pi^{-1}(X)$ is a divisor with normal crossings (this means that each irreducible component of $\pi^{-1}(X)$ is non-singular, and whenever r irreducible components $X_1, \cdots, X_r$ of $\pi^{-1}(X)$ meet at the point Q, then the local equations $g_1, \cdots, g_r$ of $X_1, \cdots, X_r$ at Q, respectively, form part of a system regular of parameters of $\mathcal{O}_{Y',Q}$).

Let $\pi : (Y', D) \longrightarrow (Y, P)$ be an embedded resolution with the exceptional divisor $D = \bigcup_{\sigma\in\Gamma} E_\sigma$ the union of irreducible components E_σ (isomorphic to the projective line $\mathbb{P}^1$), and let $E_\sigma^o \subseteq E_\sigma$ be the complement in E_σ of the intersection with all other components of the total transform $\pi^{-1}(X)$ of the curve X. It is associated to this embedded resolution a dual graph whose vertices α correspond to the irreducible components E_α (isomorphic to the projective lines) of D and two vertices are connected by an edge if the corresponding components intersect. For $\sigma \in \Gamma$ and $g \in \mathcal{O}_{Y,P}$, with $g \neq 0$, let $m^\sigma(g)$ be the multiplicity along E_σ of the lifting of g to Y' and let $\mathbf{m}^\sigma := (m^\sigma(f_1), ..., m^\sigma(f_m)) \in \mathbb{Z}^m_{\geq 0}$, where $f_1, ..., f_m$ are local equations of the irreducible components of X at the point P.

Thus, we want to study the relation between the series

$$\overline{P}(\mathcal{O}, \mathcal{O}, t_1, \cdots, t_m) = \frac{(t_1 - 1)\cdots(t_m - 1)}{t_1 \cdots t_m - 1} \sum_{\mathbf{n}\in\mathbb{Z}^m} \dim_k \left(\mathcal{O} \cap \mathfrak{p}^{\mathbf{n}} / \mathcal{O} \cap \mathfrak{p}^{\mathbf{n+1}}\right) \mathbf{t}^{\mathbf{n}}$$

and the series $\prod_\sigma (1 - t_1^{m^\sigma(f_1)} \cdots t_m^{m^\sigma(f_m)})^{\#(E_\sigma \setminus E_\sigma^0) - 2}$, whenever $\mathcal{O}$ is the local ring of an algebroid plane curve defined over an algebraically closed field k.

5.2.1 Unibranch case

Let $\mathcal{O}$ be the local ring of an irreducible plane algebroid curve totally defined over a field k defined by the irreducible power series $f \in k[[X, Y]]$ with semigroup $S(\mathcal{O})$ (where $S(\mathcal{O}) = S(\mathcal{O} \otimes_k l)$ for each finite extension l of k), whose Apery sequence respect to its multiplicity m is $a_0 < a_1 < \cdots < a_{m-1}$ and, hence, $S(\mathcal{O}) = \bigcup_{i=0}^{m-1} (a_i + m\mathbb{N})$. The blow-up of (f) is the algebroid curve $\left(f^{(1)}\right)$ given by the irreducible series $f^{(1)} = X^{-1} f(X, XY)$ where it is assumed that the tangent of (f) is the horizontal line (Y). In [3], Azevedo shows the following two important results:

(i) The semigroup $S(\mathcal{O})$ is strongly increasing, that is, $a_i + a_j \leq a_{i+j}$ whenever $0 \leq i, j, i + j \leq m$.

(ii) $S\left(\mathcal{O}^{(1)}\right) = \cup_{i=0}^{m-1} (a_i - im + m\mathbb{N})$, where $\mathcal{O}^{(1)}$ is the local ring of the algebroid curve $\left(f^{(1)}\right)$, that is, $\mathcal{O}^{(1)} = k[[X, Y]]/\left(f^{(1)}\right)$.

Thus, the semigroup $S\left(\mathcal{O}^{(1)}\right)$ is determined by the semigroup $S(\mathcal{O})$ and conversely, $S(\mathcal{O})$ is determined by the multiplicity m and the semigroup $S\left(\mathcal{O}^{(1)}\right)$.

We have proved that $P(\mathcal{O}, \mathcal{O}, t) \equiv \sum_{n\in\mathbb{Z}} \dim_k (\mathcal{O} \cap \mathfrak{p}^n / \mathcal{O} \cap \mathfrak{p}^{n+1})\, t^n \mod (q - 1)\,\mathbb{Z}[[t]]$.
Since $P(\mathcal{O}, \mathcal{O}, t)$ is determined by $S(\mathcal{O})$ and $S(\mathcal{O}) = S\left(\mathcal{O} \otimes_k \overline{k}\right)$, we may assume that

the constant field k is algebraically closed. Now, because $S(\mathcal{O}) = \bigcup_{i=0}^{m-1} (a_i + m\mathbb{N})$ and $S\left(\mathcal{O}^{(1)}\right) = \cup_{i=0}^{m-1} (a_i - im + m\mathbb{N})$, it follows that

$$\overline{P}(\mathcal{O}, \mathcal{O}, t) = \frac{1 + t^{a_1} + \cdots + t^{a_{m-1}}}{1 - t^m}$$

and

$$\overline{P}(\mathcal{O}^{(1)}, \mathcal{O}^{(1)}, t) = \frac{1 + t^{a_1 - m} + \cdots + t^{a_{m-1} - (m-1)m}}{1 - t^m}$$

Thus, we can prove by induction the following proposition.

Proposition 67 *Let $0 < v_0 < v_1 < \cdots < v_r$ be the minimal system of generators of the semigroup $S(\mathcal{O})$, let $n_0 := 1$ and $n_j := \gcd(v_0, \cdots, v_{j-1}) / \gcd(v_0, \cdots, v_j)$ for $j = 1, \cdots, r$. Then*

$$\overline{P}(\mathcal{O}, \mathcal{O}, t) = \frac{\prod_{j=1}^{r} (1 - t^{n_j v_j})}{\prod_{j=0}^{r} (1 - t^{v_j})}$$

Proof. We will prove the result by induction over the number of blowing-ups. We have that $v_0 = m$. Let $w_j := v_j - n_0 \cdots n_{j-1} v_0$ for $j = 1, \cdots, r$. In [16] was proved that the minimal system of generators of the semigroup $S\left(\mathcal{O}^{(1)}\right)$ has one of the three forms:

1. $m < w_1 < \cdots < w_r$ if $m < w_1$.
2. $w_1 < m < w_2 < \cdots < v_r$ if $w_1 < m$ and $w_1 \nmid m$.
3. $w_1 < \cdots < w_r$ if $w_1 \mid m$.

In the first case, by induction hypothesis,

$$\overline{P}(\mathcal{O}^{(1)}, \mathcal{O}^{(1)}, t) = \frac{\prod_{j=1}^{r} (1 - t^{m_j w_j})}{(1 - t^m) \prod_{j=1}^{r} (1 - t^{w_j})}$$

where $m_0 := 1, m_1 := \frac{m}{\gcd(m, w_1)}$ and $m_j := \frac{\gcd(m, w_1, \cdots, w_{j-1})}{\gcd(m, w_1, \cdots, w_j)}$ for $j = 2, \cdots, r$. Since $w_j := v_j - n_0 \cdots n_{j-1} v_0$; it follows that $m_j = n_j$ for $j = 0, \cdots, r$. Thus,

$$1 + t^{a_1 - m} + \cdots + t^{a_{m-1} - (m-1)m} = \frac{\prod_{j=1}^{r} (1 - t^{m_j w_j})}{\prod_{j=1}^{r} (1 - t^{w_j})}.$$

Hence, each $a_i - im = s_1 v_1 + \cdots + s_r v_r - (s_1 + s_2 n_1 + \cdots + s_r n_1 \cdots n_{r-1}) m$, for each $i = 0, \cdots, m-1$, is uniquely determined by the integers $s_1, \cdots, s_r$ such that

$0 \leq s_j \leq n_j - 1$ for $j = 1, \cdots, r$. On the other hand, since $v_0 < v_1 < \cdots < v_r$ is the minimal system of generators of the semigroup $S(\mathcal{O})$, each $a_i = i_1 v_1 + \cdots + i_r v_r$, for each $i = 0, \cdots, m-1$, where $i_1, \cdots, i_r$ are integers such that $0 \leq i_j \leq n_j - 1$, $j = 1, \cdots, r$ and they are uniquely determined by the expansion $i = i_1 + i_2 n_1 + \cdots + i_r n_1 \cdots n_{r-1}$.Then,

$$\begin{aligned}\overline{P}(\mathcal{O}, \mathcal{O}, t) &= \frac{1 + t^{a_1} + \cdots + t^{a_{m-1}}}{1 - t^m} \\ &= \frac{\prod_{j=1}^{r} \sum_{i_j=0}^{n_j-1} t^{i_j v_j}}{1 - t^m} \\ &= \frac{\prod_{j=1}^{r} (1 - t^{n_j v_j})}{(1 - t^m) \prod_{j=1}^{r} (1 - t^{v_j})}.\end{aligned}$$

In the second case, by induction hypothesis,

$$\overline{P}(\mathcal{O}^{(1)}, \mathcal{O}^{(1)}, t) = \frac{(1 - t^{m_1 m}) \prod_{j=2}^{r} (1 - t^{m_j w_j})}{(1 - t^{w_1})(1 - t^m) \prod_{j=2}^{r} (1 - t^{w_j})}$$

where $m_0 := 1$, $m_1 = \frac{(v_1 - m)}{\gcd(v_1 - m, m)}$, $m_2 := \frac{\gcd(v_1 - m, m)}{\gcd(v_1 - m, m, w_2)}$ and $m_j := \frac{\gcd(v_1 - m, m, w_2, \cdots, w_{j-1})}{\gcd(v_1 - m, m, w_2, \cdots, w_j)}$ for $j = 3, \cdots, r$. Since, $w_1 = v_1 - m$ and, hence,

$$m_1 m = (v_1 - m)\, m / \gcd(v_1 - m, m) = n_1 (v_1 - m) = n_1 w_1$$

and $w_j := v_j - n_0 \cdots n_{j-1} v_0$ $(j = 2, \cdots, r)$; it follows that $m_j = n_j$ for $j = 2, \cdots, r$. Thus,

$$1 + t^{a_1 - m} + \cdots + t^{a_{m-1} - (m-1)m} = \frac{\prod_{j=1}^{r} (1 - t^{m_j w_j})}{\prod_{j=1}^{r} (1 - t^{w_j})}.$$

Now, we proceed as in the first case.

In the third case, by induction hypothesis,

$$\overline{P}(\mathcal{O}^{(1)}, \mathcal{O}^{(1)}, t) = \frac{\prod_{j=2}^{r} (1 - t^{m_j w_j})}{\prod_{j=1}^{r} (1 - t^{w_j})}$$

where $m_1 := 1$, $m_2 = w_1 / \gcd(w_1, w_2)$ and $m_j := \gcd(w_1, \cdots, w_{j-1}) / \gcd(w_1, \cdots, w_j)$ for $j = 3, \cdots, r$. In this case, we have that $\gcd(v_0, v_1) = v_1 - m$ and, hence, $m = n_1 (v_1 - m)$. Moreover, Since $w_j := v_j - n_0 \cdots n_{j-1} v_0$ $(j = 1, \cdots, r)$ it follows that $m_j = n_j$ for $j = 2, \cdots, r$. Thus,

$$\frac{1 + t^{a_1 - m} + \cdots + t^{a_{m-1} - (m-1)m}}{1 - t^m} = \frac{\prod_{j=2}^{r} (1 - t^{m_j w_j})}{(1 - t^{v_1 - m}) \prod_{j=2}^{r} (1 - t^{w_j})}$$

i.e.

$$1 + t^{a_1 - m} + \cdots + t^{a_{m-1} - (m-1)m} = \frac{\left(1 - t^{n_1 (v_1 - m)}\right) \prod_{j=2}^{r} (1 - t^{m_j w_j})}{(1 - t^{v_1 - m}) \prod_{j=2}^{r} (1 - t^{w_j})} = \frac{\prod_{j=1}^{r} (1 - t^{m_j w_j})}{\prod_{j=1}^{r} (1 - t^{w_j})}.$$

Now, we proceed as in the first case. ■

We can prove precedent proposition in another way. In fact, since

$$\dim_k\left(\mathcal{O}\cap\mathfrak{p}^n/\mathcal{O}\cap\mathfrak{p}^{n+1}\right)=1 \text{ if and only if } n\in S\left(\mathcal{O}\right),$$

it follows that

$$\overline{P}(\mathcal{O},\mathcal{O},t)=\sum_{n\in S(\mathcal{O})}\dim_k\left(\mathcal{O}\cap\mathfrak{p}^n/\mathcal{O}\cap\mathfrak{p}^{n+1}\right)t^n.$$

On the other hand, each n in the semigroup $S\left(\mathcal{O}\right)$ can uniquely be represented in the form $n=s_0v_0+s_1v_1+\cdots+s_rv_r$ with $s_0\geq 0$ and $0\leq s_j<n_j$ for $j=1,\cdots,r$ (cf. [4]). Therefore,

$$\begin{aligned}\overline{P}(\mathcal{O},\mathcal{O},t)&=(\sum\nolimits_{s_0=0}^{\infty}t^{s_0})\prod\nolimits_{j=1}^{r}(\sum\nolimits_{s_j=0}^{n_j-1}t^{s_jv_j})\\&=\frac{1}{1-t^{v_0}}\prod\nolimits_{j=1}^{r}\frac{1-t^{n_jv_j}}{1-t^{v_j}}.\end{aligned}$$

Thus, $\overline{P}(\mathcal{O},\mathcal{O},t)=\frac{\prod_{j=1}^{r}(1-t^{n_jv_j})}{\prod_{j=0}^{r}(1-t^{v_j})}$.

Let $\pi:(Y',D)\longrightarrow(Y,0)$ be a minimal embedded resolution of the algebroid curve, with the exceptional divisor $D=\bigcup_{\sigma\in\Gamma}E_\sigma$. We observe that $\#(E_\sigma\backslash E_\sigma^0)=1$, $\#(E_\sigma\backslash E_\sigma^0)=2$ or $\#(E_\sigma\backslash E_\sigma^0)=3$. Moreover, the set of integers $m^\alpha(f)$ such that $\#(E_\alpha\backslash E_\alpha^0)=1$ is precisely the minimal system of generators of the semigroup $S(\mathcal{O})$, that is, there exist a bijective function j between the set of $\alpha\in\Gamma$ such that $\#(E_\alpha\backslash E_\alpha^0)=1$ and the set of integer numbers $\{v_0,\cdots,v_r\}$. Besides, for each $\beta\in\Gamma$, if $\#(E_\beta\backslash E_\beta^0)=3$ then $m^\beta(f)=n_{j(\alpha)}m^\alpha(f)$ for only one one $\alpha\in\Gamma$ such that $\#(E_\alpha\backslash E_\alpha^0)=1$ (cf. [5]). Therefore,

$$\overline{P}(\mathcal{O},\mathcal{O},t)=\prod\nolimits_{\sigma}(1-t^{m^\sigma(f)})^{\#(E_\sigma\backslash E_\sigma^0)-2}.$$

Garcia and Stöhr showed that if $S\left(\mathcal{O}'\right)$ is a semigroup associated to an irreducible algebroid plane curve of multiplicity m', then for each positive integer m, there is an irreducible algebroid curve of multiplicity m whose blow-up has $S\left(\mathcal{O}'\right)$ as its semigroup if and only if $m\in S\left(\mathcal{O}'\right)$ and $m\leq\min\left(S\left(\mathcal{O}'\right)\backslash m'\mathbb{N}\right)$. Based in this result they showed that each strongly increasing semigroup is associated to an irreducible algebroid curve (see [16]), which was first proved by Angermuller (see [1]), in characteristic zero. The Garcia and Stöhr's Theorem also allows to classify semigroups of irreducible singularities which may be resolved by a prescribed number of blowing-ups. For example, the list of semigroups with multiplicity m of irreducible singularities that can be resolved by 1, 2 or 3 blowing-ups is:

1. One blow-up, if $S\left(\mathcal{O}^{(1)}\right)=\mathbb{N}$ then $S\left(\mathcal{O}\right)=m\mathbb{N}+\left(m+1\right)\mathbb{N}$
2. Two blowing-ups, if $S\left(\mathcal{O}^{(2)}\right)=\mathbb{N}$ and $S\left(\mathcal{O}^{(1)}\right)\neq\mathbb{N}$, then $S\left(\mathcal{O}\right)=m\mathbb{N}+\left(2m+1\right)\mathbb{N}$ or $S\left(\mathcal{O}\right)=m\mathbb{N}+\left(2m-1\right)\mathbb{N}$
3. Three blowing-ups, if $S\left(\mathcal{O}^{(3)}\right)=\mathbb{N}$ and $S\left(\mathcal{O}^{(2)}\right)\neq\mathbb{N}$, then $S\left(\mathcal{O}\right)$ is equal to one of the following semigroups:
   (a) $m\mathbb{N}+\left(3m+1\right)\mathbb{N}$
   (b) $m\mathbb{N}+\left(\frac{3m}{2}\right)\mathbb{N}+\left(3m+1\right)\mathbb{N}$, where m is even
   (c) $m\mathbb{N}+\left(\frac{3m-1}{2}\right)\mathbb{N}$, where m is odd
   (d) $m\mathbb{N}+\left(3m-1\right)\mathbb{N}$
   (e) $m\mathbb{N}+\left(\frac{3m+1}{2}\right)\mathbb{N}$, where m is odd

Using 61, we can calculate the Poincaré series $P(\mathcal{O},\mathcal{O},t)$ of this semigroups, for instance:

1. If $S\left(\mathcal{O}\right)=m\mathbb{N}+\left(m+1\right)\mathbb{N}$ then
$$P(\mathcal{O},\mathcal{O},t)=\frac{\sum\limits_{i=0}^{m-2}q^{\frac{i(i+1)}{2}+i(m-i-1)}t^{im}\left(1-t^{i+1}\right)+q^{\frac{m(m-1)}{2}}t^{m(m-1)}}{1-t}$$
2. If $S\left(\mathcal{O}\right)=m\mathbb{N}+\left(2m+1\right)\mathbb{N}$ then
$$P(\mathcal{O},\mathcal{O},t)=\frac{\sum\limits_{i=0}^{m-2}q^{i(i+1)+2i(m-i-1)}t^{2im}\left(q^{m-i-1}t^{m}+1\right)\left(1-t^{i+1}\right)+q^{m(m-1)}t^{2m(m-1)}}{1-t}$$

In fact, the Apery sequence of the semigroup $S\left(\mathcal{O}\right)=m\mathbb{N}+\left(m+1\right)\mathbb{N}$ is $a_i=i\left(m+1\right)$ for $i=0,\cdots,m-1$, $a_m:=\infty$ and hence $\lfloor\left(a_k-a_i\right)/m\rfloor=k-i$ whenever $i\leq k\leq m-1$ and $\lfloor\left(a_m-a_i\right)/m\rfloor=\infty$. Thus (1) follows from this. The Apery sequence of the semigroup $S\left(\mathcal{O}\right)=m\mathbb{N}+\left(2m+1\right)\mathbb{N}$ is $a_i=i\left(2m+1\right)$ for $i=0,\cdots,m-1$, $a_m:=\infty$ and hence $\lfloor\left(a_k-a_i\right)/m\rfloor=2\left(k-i\right)$ whenever $i\leq k\leq m-1$ and $\lfloor\left(a_m-a_i\right)/m\rfloor=\infty$. Thus (2) follows from this.

5.2.2 Two-branch case

Let $\mathcal{O}$ be the local ring of an irreducible plane algebroid curve totally defined over a field k defined by the irreducible power series $f\in k[[X,Y]]$. Let $f=f_1f_2$ be the decomposition of f into irreducible factors in $k[[X,Y]]$ and let $\mathcal{O}_i := k[[X,Y]]/(f_i)$ $(i=1,2)$ be the local components of $\mathcal{O}$. We can assume that k is algebraically closed.

Let $0 < v_0^{(i)} < v_1^{(i)} < \cdots < v_{r_i}^{(i)}$ be the minimal system of generators of the semigroup $S(\mathcal{O}_i)$ $(i=1,2)$, let $\varrho := \max\{n \in \mathbb{N} : \frac{v_j^{(1)}}{v_j^{(2)}} = \frac{v_0^{(1)}}{v_0^{(2)}}$, for each $j \le n \le r_1, r_2\}$ and let $g_j^{(i)} := \gcd(v_0^{(i)}, \cdots, v_j^{(i)})$, $j = 0, \cdots, r_i$; $g_{-1}^{(i)} := 0$ $(i=1,2)$. Bayer obtained explicitly the maximal elements of the semigroup $S(\mathcal{O})$ in terms of the minimal systems of generators of the semigroups $S(\mathcal{O}_i)$ $(i=1,2)$ and the intersection multiplicity of the two components $I := I(f_1, f_2)$. He also proved that the intersection multiplicity I may be written in one of the following forms:

i. $I = \infty$ if $S(\mathcal{O}_1) = S(\mathcal{O}_2)$

ii. $I = g_{j-1}^{(1)} v_j^{(1)} + n g_j^{(1)} g_j^{(2)} = g_{j-1}^{(2)} v_j^{(2)} + n g_j^{(1)} g_j^{(2)}$ for some positive integers $1 \le j < \varrho$ and $1 \le n < \frac{v_{j+1}^{(1)}}{g_j^{(1)}} - \frac{g_{j-1}^{(1)}}{g_j^{(1)} g_j^{(1)}} v_j^{(1)}$.

iii. $I = g_{\varrho-1}^{(1)} v_\varrho^{(1)} + n g_\varrho^{(1)} g_\varrho^{(2)} = g_{\varrho-1}^{(2)} v_\varrho^{(2)} + n g_\varrho^{(1)} g_\varrho^{(2)}$ for some positive integer $1 \le n < \min\{\frac{v_{\varrho+1}^{(1)}}{g_\varrho^{(1)}} - \frac{g_{\varrho-1}^{(1)}}{g_\varrho^{(1)} g_\varrho^{(1)}} v_\varrho^{(1)}, \frac{v_{\varrho+1}^{(2)}}{g_\varrho^{(2)}} - \frac{g_{\varrho-1}^{(2)}}{g_\varrho^{(2)} g_\varrho^{(2)}} v_\varrho^{(2)}\}$.

iv. $I = g_\varrho^{(1)} v_{\varrho+1}^{(2)}$

v. $I = g_\varrho^{(2)} v_{\varrho+1}^{(1)}$

Moreover, he proved the following theorem (cf. [4] Theorems 3.10 and 3.12):

Theorem 68 *If $S(\mathcal{O}_1)$ is not equal to $S(\mathcal{O}_2)$ then*

1. *If $I = g_{j-1}^{(1)} v_j^{(1)} + n g_j^{(1)} g_j^{(2)}$ for some positive integers n and j, then the maximal elements of $S(\mathcal{O})$ are precisely of the form*

$$\left(g_j^{(1)} a, g_j^{(2)} a\right) + \sum_{k=j+1}^{r_1} i_k \left(v_k^{(1)}, \frac{I}{g_{k-1}^{(1)}}\right) + \sum_{i=j+1}^{r_2} j_i \left(\frac{I}{g_{i-1}^{(2)}}, v_i^{(2)}\right)$$

where a varies over the elements in the Apery sequence of the strongly increasing semigroup $\frac{v_0^{(1)}}{g_j^{(1)}}\mathbb{N} + \cdots + \frac{v_j^{(1)}}{g_j^{(1)}}\mathbb{N}$, with respect to the positive integer $\frac{I}{g_j^{(1)} g_j^{(2)}}$; and $0 \le i_k < \frac{g_{k-1}^{(1)}}{g_k^{(1)}}$, $0 \le j_i < \frac{g_{i-1}^{(2)}}{g_i^{(2)}}$.

2. *If $I = g_{\varrho}^{(1)} v_{\varrho+1}^{(2)}$, then then the maximal elements of $S(\mathcal{O})$ are precisely of the form*

$$\left(g_{\varrho}^{(1)}a, g_{\varrho}^{(2)}a\right) + \sum_{k=\varrho+1}^{r_1} i_k \left(v_k^{(1)}, \frac{I}{g_{k-1}^{(1)}}\right) + \sum_{i=\varrho+2}^{r_2} j_i \left(\frac{I}{g_{i-1}^{(2)}}, v_i^{(2)}\right)$$

where a varies over the elements in the Apery sequence of the strongly increasing semigroup $\frac{v_0^{(1)}}{g_{\varrho}^{(1)}}\mathbb{N} + \cdots + \frac{v_{\varrho}^{(1)}}{g_{\varrho}^{(1)}}\mathbb{N}$, with respect to the positive integer $\frac{I}{g_{\varrho}^{(1)} g_{\varrho+1}^{(2)}}$; and $0 \leq i_k < \frac{g_{k-1}^{(1)}}{g_k^{(1)}}$, $0 \leq j_i < \frac{g_{i-1}^{(2)}}{g_i^{(2)}}$.

We obtain the following proposition:

Proposition 69 *If we put $n_k^{(n)} := \frac{g_{k-1}^{(n)}}{g_k^{(n)}}$, with $0 \leq k \leq r_n$ and $1 \leq n \leq 2$, then*

1.

$$\overline{P}(\mathcal{O}, \mathcal{O}, t_1, t_2) = \prod_{k=j+1}^{r_1} \frac{1 - t_1^{n_k^{(1)} v_k^{(1)}} t_2^{n_k^{(1)} I / g_{k-1}^{(1)}}}{1 - t_1^{v_k^{(1)}} t_2^{I / g_{k-1}^{(1)}}} \prod_{i=j+1}^{r_2} \frac{1 - t_1^{n_i^{(2)} I / g_{i-1}^{(2)}} t_2^{n_i^{(2)} v_i^{(2)}}}{1 - t_1^{v_i^{(1)}} t_2^{v_i^{(2)}}} \sum_a t_1^{a g_j^{(1)}} t_2^{a g_j^{(2)}}$$

whenever $I = g_{j-1}^{(1)} v_j^{(1)} + n g_j^{(1)} g_j^{(2)}$ for some positive integers n and j, where a varies over the elements in the Apery sequence of the strongly increasing semigroup $\frac{v_0^{(1)}}{g_j^{(1)}}\mathbb{N} + \cdots + \frac{v_j^{(1)}}{g_j^{(1)}}\mathbb{N}$, with respect to the positive integer $\frac{I}{g_j^{(1)} g_j^{(2)}}$.

2.

$$\overline{P}(\mathcal{O}, \mathcal{O}, t_1, t_2) = \prod_{k=\varrho+1}^{r_1} \frac{1 - t_1^{n_k^{(1)} v_k^{(1)}} t_2^{n_k^{(1)} I / g_{k-1}^{(1)}}}{1 - t_1^{v_k^{(1)}} t_2^{I / g_{k-1}^{(1)}}} \prod_{i=\varrho+2}^{r_2} \frac{1 - t_1^{n_i^{(2)} I / g_{i-1}^{(2)}} t_2^{n_i^{(2)} v_i^{(2)}}}{1 - t_1^{I / g_{i-1}^{(2)}} t_2^{v_i^{(2)}}} \sum_a t_1^{a g_{\varrho}^{(1)}} t_2^{a g_{\varrho}^{(2)}}$$

whenever $I = g_{\varrho}^{(1)} v_{\varrho+1}^{(2)}$, where a varies over the elements in the Apery sequence of the strongly increasing semigroup $\frac{v_0^{(1)}}{g_{\varrho}^{(1)}}\mathbb{N} + \cdots + \frac{v_{\varrho}^{(1)}}{g_{\varrho}^{(1)}}\mathbb{N}$, with respect to the positive integer $\frac{I}{g_{\varrho}^{(1)} g_{\varrho+1}^{(2)}}$.

Proof. From Proposition 62, we have that

$$P_{\mathcal{O}}(\mathcal{O}, \mathcal{O}, t_1, t_2) \equiv \sum\nolimits_{(n_1, n_2) \in M(\mathcal{O})} t_1^{n_1} t_2^{n_2} \mod (q-1)\, \mathbb{Z}[[t_1, t_2]].$$

Then, by the precedent theorem, we obtain the result. ∎

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