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Generalized reduction and pure spinors.

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Chapter 1

Introduction.

The idea of "reducing" geometric structures is as old as the very notion of symmetry. The interplay between symmetries and reduction was significantly explored by the founders of classical mechanics (e.g. Poisson, Jacobi) who realized that integrals of motion of a mechanical system could be used to reduce its degrees of freedom (see e.g. [6]) and produce a "smaller" phase space. The intimate connection between symmetries and conservation laws is generally referred to as "Noether's principle" and is central in the study of differential equations and dynamics (see e.g. [37, 43]).

The modern mathematical formulation of the theory of reduction takes place in the context of symplectic geometry. The usual set-up involves an action of a Lie group on a symplectic manifold equipped with a moment map (see e.g. [34, 46]); then the Marsden-Weinstein reduction theorem [39] produces a quotient symplectic manifold (physically representing the phase space with reduced degrees of freedom). This reduction procedure has played a key role in different areas of mathematics, including the study of moduli spaces in gauge theory and their applications to mathematical physics, see e.g. [5, 21].

In recent years, mathematical physics has motivated the study of a much broader class of geometrical structures beyond symplectic geometry; these include e.g. Dirac structures [17, 45] and generalized complex structures [28, 24] and are commonly referred to as "generalized geometries". The main subject of this thesis is the study of symmetries and reduction of generalized geometries, extending symplectic reduction.

1.1 Generalized geometry.

For a smooth manifold M, its generalized tangent bundle is $\mathbb{T}M = TM \oplus T^*M$. In [17], T. Courant, following ideas of A. Weinstein (see also [18]), realized that by considering the generalized tangent bundle, one can unify different kinds of geometric structures (including e.g. pre-symplectic forms, Poisson structures and foliations). The motivation to treat Poisson and pre-symplectic geometry on equal footing lies on the work of P. Dirac on constrained mechanics [20] where both geometries arise naturally.

There is a natural symmetric non-degenerate bilinear form on $\mathbb{T}M$ given by

$$g_{\mathrm{can}}(X+\xi, Y+\eta) = i_X\eta + i_Y\xi, \quad X,\, Y \in \Gamma(TM);\ \xi,\, \eta \in \Gamma(T^*M).$$

For a 2-form $\omega \in \Omega^2(M)$ (resp. a bivector field $\pi \in \mathfrak{X}^2(M)$), there corresponds a subbundle of $\mathbb{T}M$ given by

$$L_\omega = \{(X, i_X\omega) \mid X \in TM\} \ \ (\text{resp. } L_\pi = \{(i_\xi\pi, \xi) \mid \xi \in T^*M\}).$$

Both L_ω and L_π share the property of being Lagrangian (i.e., maximal isotropic) with respect to g_{can}. Following [17], we call Lagrangian subbundles of $\mathbb{T}M$ *almost Dirac structures*. For a distribution $\Delta \subset TM$,

$$L_\Delta = \Delta \oplus \mathrm{Ann}\,(\Delta) \subset \mathbb{T}M$$

is also an almost Dirac structure. The main achievement of T. Courant [17] was the definition of a bracket - the *Courant bracket* - on the sections of $\mathbb{T}M$ which encompasses the well-known integrability conditions of Poisson and presymplectic structures as well as distributions. Its formula is

$$[\![X+\xi, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi, \tag{1.1}$$

for $X+\xi,\, Y+\eta \in \Gamma(TM \oplus T^*M)$. The Courant bracket, as written above, is not skew symmetric, hence it is not a Lie bracket. It does satisfy, however, a version of the Jacobi identity. Noticing that

$$[\![X+\xi, Y+\eta]\!] = -[\![Y+\eta, X+\xi]\!] + d\, g_{\mathrm{can}}(X+\xi, Y+\eta), \tag{1.2}$$

one may verify that $[\![\cdot,\cdot]\!]$ is a Lie bracket when restricted to isotropic subbundles of $\mathbb{T}M$.

A *Dirac structure* is an almost Dirac structure $L \subset \mathbb{T}M$ such that

$$[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L).$$

For a distribution $\Delta \subset TM$, the corresponding subbundle L_Δ is a Dirac structure if and only if Δ is involutive. Also, L_ω (resp. L_π) is a Dirac structure if and only if $d\omega = 0$ (resp. $[\pi,\pi] = 0$, where $[\cdot,\cdot]$ is the Schouten bracket on multivector fields).

It is also possible to incorporate a description of complex structures in this context, a fact which was realized by N. Hitchin [28]. Let $J : TM \to TM$ be an almost complex structure on M and consider its $-i$-eigenbundle $T_{0,1} \subset TM \otimes \mathbb{C}$. Then,

$$L_{0,1} = T_{0,1} \oplus \mathrm{Ann}\,(T_{0,1}) \subset \mathbb{T}M \otimes \mathbb{C}$$

is an almost Dirac structure relative to the $\mathbb{C}$-bilinear extension of g_{can}. By complexifying the Courant bracket, one has that $L_{0,1}$ is a Dirac structure if and only if J is integrable. This motivates the definition of *generalized complex structures* on M as being maximal isotropic subbundles L of $\mathbb{T}M \otimes \mathbb{C}$ satisfying

(i) $L \cap \bar{L} = 0$;

(ii) $[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$.

It is also possible to see a symplectic structure on M as a generalized complex structure given by

$$L_{i\,\omega} = \{(X,\, i\,\omega(X,\cdot)) \mid X \in TM \otimes \mathbb{C}\} \subset \mathbb{T}M \otimes \mathbb{C}, \tag{1.3}$$

where $\omega \in \Omega^2(M, \mathbb{R})$ is the symplectic 2-form. Generalized complex structures were intensively studied in [24]. Their importance lies on the fact that they provide a unified view of symplectic and complex geometries much desired by physicists who study mirror symmetry (for more on the relation between generalized complex structures and physics, see [24] and references therein).

Motivated also by some constructions from physics (e.g. [32]), P. Ševera and A. Weinstein [45] realized that the Courant bracket could be twisted by a closed 3-form $H \in \Omega^3(M)$, defining the *H-twisted Courant bracket* $[\![\cdot,\cdot]\!]_H$ given, for $X + \xi,\, Y + \eta \in \Gamma(\mathbb{T}M)$, by

$$[\![X + \xi, Y + \eta]\!]_H = [X, Y] + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H. \tag{1.4}$$

The bundle $\mathbb{T}M$ endowed with g_{can} and $[\![\cdot,\cdot]\!]_H$ is the main example of an *exact Courant algebroid* [36, 45]. In general, an exact Courant algebroid consists of a vector bundle E over M endowed with a bracket $[\![\cdot,\cdot]\!]$ on its sections, a non-degenerate symmetric bilinear form g and a map $p : E \to TM$, called the *anchor*, satisfying a set of axioms presented in §4.1. In the case of $\mathbb{T}M$, the anchor is pr_{TM}, the projection on TM. One particularly important property of E is the existence of an exact sequence

$$0 \longrightarrow T^*M \xrightarrow{p^*} E \xrightarrow{p} TM \longrightarrow 0, \tag{1.5}$$

where we have identified $E \cong E^*$ using g. Exact Courant algebroids are the space in which our geometric structures sit as Lagrangian subbundles.

Given an exact Courant algebroid, it is always possible to find a splitting $\nabla : TM \to E$ of (1.5) such that $\nabla(TM)$ is a Lagrangian subbundle (see the discussion on §2.1). Such splittings are called *isotropic splittings*. In this case (see Chapter 4),

$$\nabla + p^* : \mathbb{T}M \longrightarrow E$$

is an isomorphism which identifies $(E, [\![\cdot,\cdot]\!])$ with $(\mathbb{T}M, [\![\cdot,\cdot]\!]_H)$, where $H \in \Omega^3(M)$ is a closed 3-form on M defined by

$$H(X, Y, Z) = g([\![\nabla X, \nabla Y]\!], \nabla Z),\ \ X, Y, Z \in \Gamma(TM). \tag{1.6}$$

For two isotropic splittings ∇_1, ∇_2, there exists a 2-form $B \in \Omega^2(M)$ uniquely defined such that

$$\nabla_1 X - \nabla_2 X = p^*(i_X B),\ \forall\, X \in \Gamma(TM).$$

The 3-forms H_1, H_2 corresponding to ∇_1 and ∇_2, respectively, are related by $H_1 = H_2 + dB$ (see [45]). Hence, the cohomology class $[H] \in H^3(M, \mathbb{R})$ does not depend on ∇ and is called the *Ševera class* of E [45].

1.2 Generalized reduction.

For each of the key examples of generalized geometric structures, there is an appropriate "reduction" procedure when a Lie group G acts on M by symmetry (for example, Poisson reduction [38], symplectic reduction [39]). It is natural, as well as important in applications, to seek for a reduction procedure for generalized geometries extending the familiar situations. The construction of such procedure is the subject of [11] (see also [35] and [48]).

There is a strong analogy between the generalized reduction of [11] and reduction [39] in the symplectic context. In fact, from a super-geometric point of view, this analogy is not surprising because an exact Courant algebroid E is a symplectic (super)manifold [44] (see [10] for the study of the super-geometry underpinning generalized reduction). In the Marsden-Weinstein reduction, one needs a (compact, for simplicity) Lie group G acting on a symplectic manifold preserving the symplectic structure, as well as a coisotropic submanifold suitably compatible with the action. Similarly, the reduction data needed to perform the generalized reduction of [11] includes a compact Lie group acting by automorphisms on the Courant algebroid E over M (the analog of symplectic diffeomorphisms) and a pair (N, K) (which should be seen as the analogue of the coisotropic submanifold), where $N \subset M$ is an invariant submanifold and $K \subset E|_N$ is an equivariant isotropic subbundle. With the reduction data in place, one can construct [11] the reduced exact Courant algebroid E_{red} over N/G and, more importantly, there is a map

$$L \longmapsto L_{red} \tag{1.7}$$

associating to every invariant Dirac structure $L \subset E$, satisfying a "clean intersection hypothesis", a reduced Dirac structure $L_{red} \subset E_{red}$. The clean intersection hypothesis for the reduction (1.7) is that $L|_N \cap K$ has constant rank; a more restritive hypothesis is the transversality condition $L|_N \cap K = 0$ (following the analogy with (super-)symplectic geometry, similar conditions appear in the reduction of Lagrangian submanifolds of symplectic manifolds, see e.g. [50]).

As an example of how this procedure works, let us show how Marsden-Weinstein reduction fits into this general framework. Consider a symplectic manifold (M, ω) acted upon by a compact Lie group G such that there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ (with respect to the co-adjoint action) satisfying

$$i_{u_M}\omega = d\langle \mu, u\rangle, \ \forall\, u \in \mathfrak{g},$$

where u_M is the infinitesimal generator of the action. The map μ is called a *moment map* for the action. By choosing $E = \mathbb{T}M$ with the standard Courant bracket (1.1),

$$N = \mu^{-1}(0) \text{ and } K = \Delta_{\mathfrak{g}} \oplus \mathrm{Ann}\left(T\mu^{-1}(0)\right)),$$

where $\Delta_{\mathfrak{g}}$ is the distribution tangent to the G-orbits, the reduced Courant algebroid E_{red} over $M_{red} = \mu^{-1}(0)/G$ is naturally isomorphic to $\mathbb{T}M_{red}$ with the standard Courant bracket. Moreover, by taking $L = L_\omega$, the Dirac structure corresponding to ω, its reduction L_{red} is exactly the Dirac structure corresponding to the Marsden-Weinstein reduced symplectic structure ω_{red} on M_{red} [39] (see Example 4.56 for more details.)

Another simple instance of generalized reduction is the restriction of a Dirac structure $L \subset (\mathbb{T}M, [\![\cdot,\cdot]\!]_H)$, $H \in \Omega^3_{cl}(M)$, to a submanifold $N \subset M$; this restriction operation was originally considered in [17]. In this case, there is no Lie group action; only the pair (N, K), where $K = \mathrm{Ann}\,(TN) \subset \mathbb{T}M|_N$. The reduced Courant algebroid E_{red} is naturally identified with $(\mathbb{T}N, [\![\cdot,\cdot]\!]_{j^*H})$, where $j : N \to M$ is the inclusion map. The reduced Dirac structure L_{red} is given by

$$\frac{L|_N \cap (TN \oplus T^*M|_N)}{\mathrm{Ann}\,(TN)} \tag{1.8}$$

(see Example 4.55 for more details); the cleaness condition in this case is that $L|_N \cap \mathrm{Ann}\,(TN)$ has constant rank.

In [11, 12], the map (1.7) (providing the necessary complexifications) extends to complex Dirac structures $L \subset E \otimes \mathbb{C}$. This is fundamental when dealing with generalized complex structures.

1.3 The pure spinor point of view.

The approach N. Hitchin originally followed in [28] to define generalized complex structures relies on the Clifford bundle $Cl(E, g)$ associated to an exact Courant algebroid E. To any isotropic splitting $\nabla : TM \to E$ of (1.5), there corresponds a representation of $Cl(E, g)$,

$$\Pi_\nabla : Cl(E, g) \longrightarrow \mathrm{End}\,(\wedge^\bullet T^*M)\,.$$

By a well-known result of E. Cartan [14], almost Dirac structures $L \subset E$ correspond to specific line bundles $U^\nabla(L) \subset \wedge^\bullet T^*M$ called *pure spinor line bundles.* At a point $x \in M$, one has

$$U^\nabla(L)_x = \{\varphi \in \wedge^\bullet T^*_xM \mid \Pi_\nabla(e)\varphi = 0, \forall\, e \in L_x\}.$$

By complexifying the whole picture, the correspondence between almost Dirac structures and pure spinor line bundles extends to $E \otimes \mathbb{C}$, by switching $\wedge^\bullet T^*M$ with $\wedge^\bullet T^*M \otimes \mathbb{C}$ (this will be important for generalized complex structures; see Example 5.4 for more details). Let us give an example. For a distribution $\Delta \subset TM$, let $\{\xi_1, \ldots, \xi_n\}$ be a frame for $\mathrm{Ann}\,(\Delta) \subset T^*M$ over some open neighborhood $\mathcal{V} \subset M$. Then $\varphi = \xi_1 \wedge \cdots \wedge \xi_n$ is a section of $U^{\nabla_{\mathrm{can}}}(\Delta \oplus \mathrm{Ann}\,(\Delta))$ over $\mathcal{V}$ corresponding to the canonical splitting

$$\begin{array}{rccc} \nabla_{\mathrm{can}} : & TM & \longrightarrow & \mathbb{T}M \\ & X & \longmapsto & (X, 0). \end{array}$$

Hence,

$$U^{\nabla_{\mathrm{can}}}(\Delta \oplus \mathrm{Ann}\,(\Delta)) = \det(\mathrm{Ann}\,(\Delta)). \tag{1.9}$$

In particular, by considering a complex structure $J : TM \to TM$, the associated (complex) Dirac structure, $L_{0,1} = T_{0,1} \oplus \mathrm{Ann}\,(T_{0,1})$, corresponds to the canonical line bundle of (M, J),

$$U^{\nabla_{\mathrm{can}}}(L_{0,1}) = \det(\mathrm{Ann}\,(T_{0,1})) = \wedge^{n,0}\, T^*M, \tag{1.10}$$

where $n = \dim(M)$.

In [28], a class of generalized complex structures $L \subset \mathbb{T}M \otimes \mathbb{C}$ for which there exists a *closed* global section of $U^{\nabla_{\mathrm{can}}}(L)$ was studied. For (1.10), the existence of a closed global section turns (M, J) into what N. Hitchin [28] calls a Calabi-Yau manifold (although in the literature it is more common to ask, additionally, for (M, J) to be Kähler). This new class of generalized complex structures is called *generalized Calabi-Yau*. Another example of a generalized Calabi-Yau manifold is given by the generalized complex structure (1.3) associated to a symplectic manifold (M, ω). Indeed, its pure spinor line bundle $U^{\nabla_{\mathrm{can}}}(L_{i\,\omega})$ has a canonical global section given by

$$e^{-i\omega} = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!}\, \omega^k$$

for which

$$d\, e^{-i\omega} = -i\, d\omega \wedge e^{-i\omega} = 0.$$

The major role played by the pure spinor line bundle in generalized Calabi-Yau geometry shows that it is an important piece of information. The pure spinor line bundle provides an alternative, easier-to-handle in concrete examples (see e.g. [24]) framework in which, in principle, every construction in generalized geometry can be understood. As an illustration of how useful pure spinors can be, we mention the work of A. Alekseev, H. Bursztyn, E. Meinrenken [1] about G-valued moment maps [2], where pure spinors were used sucessfully to obtain a much simpler construction than [3] (which also extends to the case where G is non-compact) for the volume forms in quasi-Hamiltonian G-spaces.

Our goal in this thesis is to provide a refined description of the generalized reduction procedure constructed in [11] (see §1.2) by using pure spinors. More precisely, given a Dirac structure $L \subset E$ and an isotropic splitting $\nabla : TM \to E$ satisfying suitable (invariance and cleaness) conditions, we want to find the pure-spinor counterpart of (1.7); i.e., we would like to construct an explicit map

$$\Gamma(U^{\nabla}(L)) \longrightarrow \Gamma(U^{\nabla_{red}}(L_{red})), \quad \varphi \longmapsto \varphi_{red}, \tag{1.11}$$

relating pure spinors of L and L_{red} (here ∇_{red} is an isotropic splitting for E_{red} corresponding to ∇).

The main technical difficulty to describe (1.11) is that redution via pure spinors is more sensitive to transversality issues than (1.7). Usually, one has to distiguish between $L|_N \cap K$ with constant rank equal to zero or not. Let us illustrate this issue with the simple case of restriction as in (1.8).

The map

$$L \longmapsto L_{red} = \frac{L|_N \cap (TN \oplus T^*M|_N)}{\mathrm{Ann}\,(TN)},$$

where $N \subset M$ is a submanifold, is well-defined as long as $L|_N \cap \mathrm{Ann}\,(TN)$ has constant rank. It is well known (see e.g. [1]) that, in case $L|_N \cap \mathrm{Ann}\,(TN) = 0$, the pull-back map

$$j^* : \wedge^\bullet T^*M \to \wedge^\bullet T^*N$$

(corresponding to the inclusion map $j : N \to M$) restricted to the pure spinor line bundle $U^\nabla(L) \subset \wedge^\bullet T^*M$ relates the pure spinors of L and L_{red}. The problem is that for a section φ of $U^\nabla(L)$, if $L|_N \cap \mathrm{Ann}\,(TN)$ has constant *non-zero* rank, then

$$j^*\varphi = 0.$$

In other words, all geometric information in the pure spinor is lost upon restriction to a submanifold, even though the geometric structure itself admits a restriction.

One of our main achievements is a method (see the details in §3.2.2) which circumvents the non-transversality issues by finding a Lagrangian subbundle $L' \subset E|_N$ (that we call the *pertubation* of L) for which

(i) $L'_{red} = L_{red}$;

(ii) $L' \cap K = 0$. Moreover, $L' = L|_N$ if and only if $L|_N \cap K = 0$.

The definition of the pertubation relies on the choice of an equivariant isotropic subbundle $D \subset E|_N$ such that (see Proposition 4.32)

$$(L|_N \cap K)^\perp \oplus D = E|_N,$$

where $(\,\cdot\,)^\perp$ refers to orthogonal with respect to g. Our main result (see Theorem 5.29) says that if φ is an invariant section of $U^\nabla(L)$ (with respect to a G-action on $\wedge^\bullet T^*M$ which corresponds to an action by automorphisms on E; see §5.1.2) over an invariant neighborhood $\mathcal{V}$ of M and $\{d_1, \ldots, d_r\}$ is an invariant frame for D over $N \cap \mathcal{V}$, then the formula

$$\varphi_{red} = q_* \circ j^*(e^B \wedge \Pi_\nabla(d_1 \cdots d_r)\varphi) \tag{1.12}$$

defines a section of $U^{\nabla_{red}}(L_{red})$ over $(N \cap \mathcal{V})/G$; here $B \in \Omega^2(M)$ is an invariant 2-form satisfying

$$\nabla u_M + p^*(i_{u_M} B) \in \Gamma(K),\ \forall\, u \in \mathfrak{g}$$

(see the discussion in §4.2.2) and $q_* : \Omega(N) \longrightarrow \Omega(N/G)$ is the push-forward of differential forms (integration along the fibers) corresponding to the principal bundle $q : N \to N/G$ (see Appendix B).

1.4 Contents.

We now summarize the contents of this thesis.

In Chapter 2, we present the linear algebra involved in the reduction procedure of [11]. For this, following previous work of H. Bursztyn and O. Radko [13], we consider the split-quadratic category, an exact analogue of the linear symplectic category (see [26] and [50]). This category has as its objects vector spaces E endowed with non-degenerate symmetric bilinear forms g admiting Lagrangian subspaces. The morphisms from (E_1, g_1) to (E_2, g_2) are Lagrangian subspaces $\Lambda \subset (E_1, -g_1) \times (E_2, g_2)$. It was shown in [11] how to interpret the reduction procedure as a quotient in this category. Our main result in this part is a factorization (see Theorem 2.35) of the quotient morphism into simpler pieces. The main ingredient for this factorization is the notion of K-admissible splitting (see Definition 2.32).

In Chapter 3, we review the well-known theory of Clifford algebras and pure spinors. Our references for this part are [16, 24, 40]. In this chapter, we define pure spinors and see how the linear reduction defined in Chapter 2 operates on pures spinors at the linear-algebra level (see Theorem 3.24). At the end of this chapter (see §3.2.2), we will present our pertubative method to solve the technical problem of how to deal with $L|_N \cap K \neq 0$.

We review the work in [11, 12] in Chapter 4. We begin by defining properly the main objects (e.g. Courant algebroids, Dirac structures, etc) and by studying the group of automorphisms of a Courant algebroid and the corresponding Lie algebra of derivations. In §4.2, we then show how to obtain the reduction data which is used in §4.3 to construct the reduced Courant algebroid E_{red} over M_{red}. Instead of just quoting the results from [11, 12], we have chosen to adapt the general construction of [11] to our simpler case. The main reason for this is the need of having a working expression of the reduced bracket on $\Gamma(E_{red})$ to prove our result on the Ševera class of E_{red} (see Proposition 4.47). This result will be important to our alternative proof of the integrability of the reduced Dirac structures (see Theorem 5.41).

Chapter 5 is where the main results of this thesis are presented. We begin by constructing the Clifford bundle. Specializing to the case of Courant algebroids, we follow [29] to study the structure of $Cl(E, g)$-module on $\Gamma(\wedge^\bullet T^*M)$ induced by isotropic splittings $\nabla : TM \to E$. In §5.1.2, we show how automorphisms of E act on $\Gamma(\wedge^\bullet T^*M)$ following ideas of [29], where this action was defined (we shall clarify a few points concerning its definition). This action allows one to define a Cartan-like calculus on $\Gamma(\wedge^\bullet T^*M)$ extending the Lie derivative [29] and to give an interpretation, first obtained in [4], of the Courant bracket as a derived bracket [33]; it also allows one to relate invariance of a Dirac structure with invariance of its pure spinor line bundle (see Proposition 5.22). In §5.3, we present our main results. The first one (Theorem 5.29) proves formula (1.12). The second one (Theorem 5.41) provides an alternative proof that the reduced Dirac structure is integrable only in pure spinors terms. We finish this chapter with some applications of our results. First, we illustrate formula (1.12) in some examples in §5.4.1. We also give conditions for the reduction of generalized Calabi-Yau to be also generalized Calabi-Yau in §5.4.2; our result generalizes the main result of [41]. At last, we shed new light on previous work of G. Cavalcanti and M. Gualtieri [15] on T-duality. In their work, they use the mathematical model of T-duality proposed in [8] (see also [9]) to construct a map relating Dirac structures on T-dual spaces. Using Theorem 5.29, we are able to show that their map, when interpreted as acting on pure spinors, is exactly the isomorphism constructed in [8] relating the twisted cohomologies of the T-dual spaces (see Theorem 5.53).

In the Appendix A, we extend the results of §3.2 on how the quotient morphism acts on pure spinor to more general morphisms (in the sense of the split-quadratic category). This appendix was inspired by unpublished work of M. Gualtieri [25]. In the Appendix B, we collect some results on push-forward of differential forms $\pi_* : \Omega(P) \to \Omega(N)$, where $\pi : P \to N$ is a G-principal bundle with G compact and connected, following [7]. The proofs of some known results, included for the sake of completeness, are collected in Appendix C.

Chapter 2

Reduction on the linear algebra setting

In this chapter, we study the linear algebra related to the geometry of the extended tangent bundle $\mathbb{T}M := TM \oplus T^*M$ for a smooth manifold M. The main algebraic features of $\mathbb{T}M$ which concern us are the existence of a non-degenerate symmetric bilinear form given by

$$g_{\text{can}}(X+\xi, Y+\eta) = i_X\eta + i_Y\xi, \text{ for } X, Y \in \Gamma(TM) \text{ and } \xi, \eta \in \Gamma(T^*M)$$

and of an exact sequence

$$0 \longrightarrow T^*M \overset{\text{pr}_{TM}^*}{\longrightarrow} \mathbb{T}M \overset{\text{pr}_{TM}}{\longrightarrow} TM \longrightarrow 0$$

(where we have identified $\mathbb{T}M \cong (\mathbb{T}M)^*$ using g_{can}). These constitute the ingredients of an extension of a vector space, as first defined in [29]. In §2.1, we study a general extension of a fixed vector space and focus on its Lagrangian subspaces. In the case of $\mathbb{T}M$, Lagrangian subbundles are called (almost) Dirac structures on M and they were studied thoroughly by T. Courant [17] as a generalization of both pre-symplectic and Poisson geometries. In §2.2, we follow previous work of H.Bursztyn and O.Radko [13] to construct a category in which Lagrangian subspaces are morphisms; this category is the exact analogue of the symplectic category [26, 50]. In §2.3, we define a quotient procedure in this category following [11]. In the last section, we prove a result which gives a factorization of the quotient morphism (Proposition 2.35) which is of central importance in this thesis.

2.1 Extensions of finite-dimensional vector spaces.

Let V be a finite-dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We say that a vector space E over $\mathbb{F}$ is an **extension** of V if it has a non-degenerate symmetric bilinear form $g : E \times E \to \mathbb{F}$ and there is a linear map $p : E \to V$ such that the sequence

$$0 \longrightarrow V^* \xrightarrow{p^*} E \xrightarrow{p} V \longrightarrow 0 \tag{2.1}$$

is exact, where we have identified $E = E^*$ using the isomorphism

$$\begin{array}{rccc} g_\sharp : & E & \longrightarrow & E^* \\ & e & \longmapsto & g(e, \cdot). \end{array}$$

Remark 2.1. In the following, by vector space we mean finite-dimensional $\mathbb{F}$-vector space, where $\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{C}$. When necessary, we say a real (or complex) vector space.

Example 2.2. For any vector space V, consider $E = V \oplus V^*$ together with the bilinear form g_{can} given by

$$g_{\text{can}}(X + \xi, Y + \eta) = i_X\eta + i_Y\xi, \text{ where } X, Y \in V \text{ and } \xi, \eta \in V^*$$

and $\text{pr}_V : V \oplus V^* \to V$. The triple $(V \oplus V^*, g_{\text{can}}, \text{pr}_V)$ defines an extension of V which we call the **canonical extension of** V. We denote it by $\mathcal{D}(V)$.

A vector space E endowed with a non-degenerate symmetric bilinear form $g : E \times E \to \mathbb{F}$ is called a **quadratic vector space**.

Definition 2.3. Let (E, g) be a quadratic vector space. For a subspace $K \subset E$, define its **orthogonal** to be

$$K^\perp = \{e_1 \in E \mid g(e_1, e_2) = 0 \text{ for every } e_2 \in K\}.$$

We say that the subspace K is **isotropic** if $K \subset K^\perp$ and **Lagrangian** if $K^\perp = K$.

Let (E, g) be a quadratic vector space and $K \subset E$ a subspace. Under the isomorphism $g^\sharp : E \to E^*$, $K^\perp$ is sent to the annihilator of K and thus

$$\dim(K) + \dim(K^\perp) = \dim(E). \tag{2.2}$$

Therefore, if K is isotropic, then

$$2\dim(K) \leq \dim(E).$$

and the equality holds if and only if K is Lagrangian.

Example 2.4. Consider $\mathbb{C}^{2n}$ with its canonical bilinear form

$$g(z, w) = \sum_{j=1}^{n} z_j w_j. \tag{2.3}$$

Let $\{e_1, \ldots, e_{2n}\}$ be the canonical basis. The subspace L spanned by

$$e^1 + i\, e^{n+1}, \ldots, e^n + i\, e^{2n}$$

is Lagrangian as well as its complement L' spanned by

$$e^1 - i\, e^{n+1}, \ldots, e^n - i\, e^{2n}.$$

This decomposes $\mathbb{C}^{2n}$ into a sum of Lagrangian subspaces

$$\mathbb{C}^{2n} = L \oplus L'.$$

Example 2.5. For any extension (E, g, p) of V, the subspace $p^*(V^*)$ is Lagrangian. Indeed, for $\xi_1, \xi_2 \in V^*$,

$$g(p^*(\xi_1), p^*(\xi_2)) = \xi_1(p \circ p^*(\xi_2)) = 0.$$

which proves that $p^*(V^*)$ is an isotropic subspace of E. By (2.1),

$$2 \dim(p^*(V^*)) = \dim(E)$$

which proves that $p^*(V^*)$ is Lagrangian.

Example 2.6. Let $\omega \in \wedge^2 V^*$ be a 2-form. The induced map

$$\begin{array}{rccc} \omega_\sharp : & V & \longrightarrow & V^* \\ & X & \mapsto & \omega(X, \cdot) \end{array}$$

defines a Lagrangian subspace of $\mathcal{D}(V)$ given by

$$\mathrm{Graph}\,(\omega_\sharp) = \{X + \xi \in V \oplus V^* \mid \xi = \omega(X, \cdot)\}.$$

We claim that any Lagrangian $L \subset \mathcal{D}(V)$ such that

$$L \cap V^* = 0 \tag{2.4}$$

is the graph of 2-form ω. Indeed, (2.4) implies that

$$\mathrm{pr}_V|_L : L \to V$$

is injective and as they have the same dimension, it is an isomorphism. Now, $L = \mathrm{Graph}\,(F)$, where $F : V \to V^*$ is the composition

$$V \overset{\mathrm{pr}_V|_L^{-1}}{\longrightarrow} L \overset{\mathrm{pr}_{V^*}}{\longrightarrow} V^*.$$

A straightforward calculation shows that the fact that L is isotropic implies that $\omega : V \times V \to \mathbb{F}$ defined by

$$\omega(X, Y) = i_Y\, F(X)$$

is antisymmetric and $F = \omega_\sharp$.

Proposition 2.7 ([40]). *Let (E, g) be a quadratic vector space. Given any isotropic subspace K_0 of E, there exists another isotropic subspace K_1 such that $K_1 \oplus K_0^\perp = E$.*

Proof. Take any complement D to $K_0^\perp$. As

$$0 = (D \oplus K_0^\perp)^\perp = D^\perp \cap K_0,$$

the map $g^\sharp : K_0 \to D^*$ given by $g^\sharp(x) = g(x, \cdot)|_D$ is an isomorphism. Let $A : D \to K_0$ be the composition

$$D \xrightarrow{(x \mapsto g(x,\cdot)|_D)} D^* \xrightarrow{(g^\sharp)^{-1}} K_0.$$

It clearly satisfies $g(Ax, y) = g(x, y)$ and therefore

$$g(x - \frac{1}{2}Ax, y - \frac{1}{2}Ay) = g(x, y) - \frac{1}{2}(g(x, Ay) + g(Ax, y)) = 0,$$

so that $K_1 = \{x - \frac{1}{2}Ax \mid x \in D\}$ is an isotropic complement to $K_0^\perp$ as we wanted. □

Corollary 2.8. *Let (E, g) be a quadratic vector space. Suppose E has a Lagrangian subspace L; then there exists a Lagrangian complement L' to L.*

Proof. By Proposition 2.5, there exists an isotropic subspace $K_1 \subset E$ such that

$$E = K_1 \oplus L^\perp = K_1 \oplus L.$$

Therefore, as $2\dim(L) = \dim(E)$ (because L is Lagrangian)

$$2\dim(K_1) = 2\dim(E) - 2\dim(L) = 2\dim(E) - \dim(E) = \dim(E).$$

This proves that K_1 is also Lagrangian. □

Remark 2.9. In the case (E, g) is a $2n$-dimensional quadratic vector space over $\mathbb{R}$, the existence of a Lagrangian subspace L is equivalent to g having signature (n, n). Indeed, let L' be a Lagrangian complement given by Corollary 2.8. Now, let $\{e_1, \cdots, e_n\}$ be a basis of L. As $L \cap L' = 0$ and g is non-degenerate, there exists a basis $\{e^1, \ldots, e^n\}$ of L' such that $g(e_i, e^j) = \delta_i^j$. For the orthonormal basis $\{e_1^+, \ldots, e_n^+, e_1^- \ldots, e_n^-\}$ of E given by

$$e_i^+ = \frac{1}{\sqrt{2}}(e_i + e^i) \text{ and } e_i^- = \frac{1}{\sqrt{2}}(e_i - e^i), \text{ for } i = 1, \ldots, n,$$

one has

$$g(e_i^+, e_j^+) = -g(e_i^-, e_j^-) = \delta_{ij} \text{ and } g(e_i^+, e_j^-) = 0.$$

Hence, $\{e_1^+, \ldots, e_n^+\}$ (resp. $\{e_1^- \ldots, e_n^-\}$) spans a n-dimensional subspace E^+ (resp. E^-) of E where g is positive definite (resp. negative definite). This proves that g has split signature (n, n). Conversely, given a decomposition

$$E = E^+ \oplus E^-$$

where $g|_{E^+}$ is positive definite and $g|_{E^-}$ is negative definite with $\dim(E^+) = \dim(E^-) = n$, the graph $\mathrm{Graph}\,(A) \subset E$ of any isomorphism

$$A : (E^+, g|_{E^+}) \longrightarrow (E^-, -g|_{E^-})$$

is Lagrangian.

Remark 2.10. For a quadratic vector space (E, g) over $\mathbb{C}$, there always exists an orthonormal basis $\{e_1, \ldots, e_n\}$ with $g(e_i, e_i) = 1$ for all $i = 1, \ldots, n$ (see [40]). This gives an isomorphism of (E, g) with $\mathbb{C}^n$ with the bilinear form g given by (2.3). In particular,

$$\mathcal{D}(\mathbb{C}^n) \cong (\mathbb{C}^{2n}, g).$$

Note that for any even-dimensional quadratic vector space (E, g) over $\mathbb{C}$, there is always a decomposition of E into a sum of Lagrangian subspaces (see Example 2.4).

Definition 2.11. A split-quadratic vector space is a pair (E, g) where E is a vector space and g is a symmetric bilinear form admiting Lagrangian subspaces.

Note that whereas every even-dimensional complex vector space E with a non-degenerate symmetric bilinear form is split-quadratic, in the real case, the signature of g is an obstruction to the existence of Lagrangian subspaces (see Remark 2.9).

Let (E, g, p) be an extension of V and consider $L = p^*(V^*)$. As we saw in Example 2.5, L is Lagrangian. Hence, Proposition 2.8 guarantees the existence of a Lagrangian complement L' to L. In this case, $p|_{L'} : L' \to V$ is an isomorphism. Define

$$\nabla = p|_{L'}^{-1} : V \longrightarrow E.$$

Note that ∇ is a splitting for (2.1) such that $\nabla(V) = K_1$ is Lagrangian.

Definition 2.12. Let (E, g, p) be an extension of V. Any splitting $\nabla : V \to E$ of (2.1) such that its image is isotropic is called an **isotropic splitting**.

Fix an extension (E, g, p) of V and consider ∇ an isotropic splitting for E. For $e \in E$, one has that $p(e - \nabla p(e)) = 0$. As the sequence (2.1) is exact, there exists $s_\nabla(e) \in V^*$ such that $p^*(s_\nabla(e)) = e - \nabla p(e)$. Note that s_∇ is characterized by the equation

$$i_X s_\nabla(e) = g(e, \nabla X), \, \forall \, X \in V. \tag{2.5}$$

Lemma 2.13. *Let (E, g, p) be an extension of V. For any isotropic splitting $\nabla : V \to E$, the map*

$$\begin{array}{rccl} \Phi_\nabla : & E & \longrightarrow & V \oplus V^* \\ & e & \longmapsto & (p(e), s_\nabla(e)) \end{array} \tag{2.6}$$

is an isomorphism satisfying:

(1) $\Phi_\nabla^* \, g_{\mathrm{can}} = g$;

(2) the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi_\nabla} & V \oplus V^* \\ \big\downarrow p & & \big\downarrow \mathrm{pr}_V \\ V & \xrightarrow{id} & V \end{array}$$

is commutative

The inverse of Φ_∇ *is given by*

$$\Phi_\nabla^{-1}(X,\xi) = \nabla X + p^*\xi, \quad \textit{for } X \in V,\, \xi \in V^*. \tag{2.7}$$

Proof. For $e_1, e_2 \in E$,

$$\begin{aligned} g_{\mathrm{can}}(\Phi_\nabla(e_1), \Phi_\nabla(e_2)) &= i_{p(e_2)} s_\nabla(e_1) + i_{p(e_1)} s_\nabla(e_2) \\ &= g(e_1, \nabla p(e_2)) + g(e_2, \nabla p(e_1)) \\ &= g(e_1, e_2) - g(e_1, e_2 - \nabla p(e_2)) + g(e_2, \nabla p(e_1)) \end{aligned}$$

Now, observe that

$$\begin{aligned} g(e_1, e_2 - \nabla p(e_2)) &= g(e_1 - \nabla p(e_1), e_2 - \nabla p(e_2)) + g(e_2, \nabla p(e_1)) \\ &= g(e_2, \nabla p(e_1)) \end{aligned}$$

because $e_i - \nabla p(e_i) \in p^*(V^*)$ for $i = 1, 2$. Combining the last two equations yields

$$g_{\mathrm{can}}(\Phi_\nabla(e_1), \Phi_\nabla(e_2)) = g(e_1, e_2).$$

It is straightforward to check that (2) holds. To finish the proof, note that for $X \in V$ and $\xi \in V^*$,

$$p^* s_\nabla(\nabla X + p^*\xi) = \nabla X + p^*\xi - \nabla p(\nabla X + p^*\xi) = \nabla X + p^*\xi - \nabla X = p^*\xi$$

and therefore $s_\nabla(\nabla X + p^*\xi) = \xi$. Thus

$$\Phi_\nabla(\nabla X + p^*\xi) = (p(\nabla X + p^*\xi), s_\nabla(\nabla X + p^*\xi) = (X, \xi).$$

□

Proposition 2.14. *Let* $\nabla_i : V \to E$ *be an isotropic splitting for* $i = 1, 2$. *There is a unique 2-form* $B \in \wedge^2 V^*$ *such that*

$$\nabla_2 X = \nabla_1 X + p^*(i_X B), \ \ \forall X \in V. \tag{2.8}$$

In this case, we denote ∇_2 *by* $\nabla_1 + B$.

Proof. The subspace $\nabla_2(V) \subset E$ is Lagrangian and $\nabla_2(V) \cap p^*(V^*) = 0$. Hence, its image $\Phi_{\nabla_1}(\nabla_2(V))$ under the isomorphism given by (2.6) for ∇_1 is a Lagrangian complement to V^*. By Example 2.6, there exists a 2-form $B \in \wedge^2 V^*$ such that

$$\Phi_{\nabla_1}(\nabla_2(V)) = \mathrm{Graph}\,(B_\sharp)$$

or, equivalently,

$$\nabla_2(V) = \Phi_{\nabla_1}^{-1}(\mathrm{Graph}\,(B_\sharp)).$$

One obtains (2.8) using the expression for $\Phi_{\nabla_1}^{-1}$ given by (2.7). □

For ∇ an isotropic splitting and $B \in \wedge^2 V^*$, the map $\tau_B : V \oplus V^* \to V \oplus V^*$ given by

$$\tau_B(X + \xi) = X + i_X B + \xi \tag{2.9}$$

makes the diagram below commutative

$$\begin{array}{ccc} E & \xrightarrow{\Phi_{\nabla+B}} & V \oplus V^* \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \tau_B} \\ E & \xrightarrow{\Phi_{\nabla}} & V \oplus V^*. \end{array} \tag{2.10}$$

We call τ_B a **B-field transformation**.

Remark 2.15. Let (E, g, p) be a real extension of a real vector space V. Its complexification $E \otimes \mathbb{C}$ together with the $\mathbb{C}$-bilinear extension $g_{\mathbb{C}}$ and the map

$$p \otimes \mathrm{id} : E \otimes \mathbb{C} \longrightarrow V \otimes \mathbb{C}$$

is a complex extension of $V \otimes \mathbb{C}$. Any isotropic splitting $\nabla : V \to E$ induces an isotropic splitting for $E \otimes \mathbb{C}$ given by

$$\nabla \otimes \mathrm{id} : V \otimes \mathbb{C} \longrightarrow E \otimes \mathbb{C}.$$

One can check directly that the isomorphism $\Phi_{\nabla \otimes \mathrm{id}} : E \otimes \mathbb{C} \to (V \oplus V^*) \otimes \mathbb{C}$ given by (2.6) is equal to $\Phi_\nabla \otimes \mathrm{id}$, the $\mathbb{C}$-linear extension of $\Phi_\nabla : E \to V \oplus V^*$.

From what has been done so far, it is clear that to study Lagrangian subspaces of an arbitrary extension of a vector space V, it suffices to study Lagrangian subspaces of $\mathcal{D}(V)$ and see how they transform under τ_B, for $B \in \wedge^2 V^*$.

Example 2.16. Let $S \subset V$ be a linear subspace and let

$$\mathrm{Ann}\,(S) = \{\xi \in V^* \mid \xi|_S = 0\} \subset V^*.$$

The subspace

$$L = S \oplus \mathrm{Ann}\,(S) \subset V \oplus V^*$$

is Lagrangian. For a 2-form $B \in \wedge^2 V^*$,

$$\tau_B(L) = \{(X, \xi) \in V \oplus V^* \mid \xi|_S = i_X j^* B\}.$$

Example 2.17. Let $\pi \in \wedge^2 V$ be a bivector and consider the induced map

$$\begin{array}{cccc} \pi^\sharp : & V^* & \to & V \\ & \xi & \longmapsto & \pi(\xi, \cdot). \end{array}$$

One has that

$$\mathrm{Graph}\left(\pi^\sharp\right) = \{X + \xi \in V \oplus V^* \mid X = \pi(\xi, \cdot)\}$$

is a Lagrangian subspace of $\mathcal{D}(V)$. Arguing as in Example 2.6, one can prove that a Lagrangian subspace $L \subset \mathcal{D}(V)$ is the graph of a bivector if and only if

$$L \cap V = 0.$$

To see how L transforms under τ_B, for $B \in \wedge^2 V^*$, we need to study more of the structure of Lagrangian subspaces of $\mathcal{D}(V)$. In general, $\tau_B(L)$ will no longer be the graph of a bivector.

A Lagrangian subspace L of $\mathcal{D}(V)$ can be fully characterized by its projection $S = \mathrm{pr}_V(L)$ on V and a 2-form $\omega_S \in \wedge^2 S^*$ defined by

$$\omega_S(X,Y) = g(X+\xi, Y) = \xi(Y), \text{ where } \xi \in V^* \text{ is such that } X+\xi \in L. \tag{2.11}$$

Note that if $\eta \in V^*$ is any other 1-form such that $X+\eta \in L$, then $\xi - \eta \in L \cap V^*$ and as L is isotropic, it follows that $\xi(Y) = \eta(Y)$ for every $Y \in S$.

Remark 2.18. For a Lagrangian subspace $L \subset \mathcal{D}(V)$, one has

$$L \cap V^* = \mathrm{Ann}\left(\mathrm{pr}_V(L)\right). \tag{2.12}$$

The pair (S, ω_S) determines L by

$$L = \{X + \xi \in S \oplus V^* \mid i_X \omega_S = \xi|_S\}. \tag{2.13}$$

It can be checked by a straightforward calculation that for $B \in \wedge^2 V^*$ and a Lagrangian subspace $L \subset \mathcal{D}(V)$ associated to the pair (S, ω_S), $\tau_B(L)$ is the Lagrangian subspace associated to the pair $(S, \omega_S + j^* B)$, where $j : S \to V$ is the inclusion.

Remark 2.19. Note that by taking any $B \in \wedge^2 V^*$ such that $j^* B = \omega_S$, one has that $\tau_B(S \oplus \mathrm{Ann}(S)) = L$.

Remark 2.20. For a Lagrangian subspace L corresponding to a pair (S, ω_S), it is straightforward to see that

$$L \cap V = \{X \in S \mid i_X \omega_S = 0\}.$$

Hence, graphs of bivectors corresponds to pairs (S, ω_S) whose 2-form $\omega_S \in \wedge^2 S^*$ is non-degenerate.

Remark 2.21. A dual characterization of L is obtained by changing the roles of V and V^*. Note that

$$\mathrm{pr}_{V^*}(L) = \mathrm{Ann}\left(L \cap V\right)$$

and therefore

$$(\mathrm{pr}_{V^*}(L))^* = V/(L \cap V).$$

The analog of ω_S (2.11) in this case is a bivector $\pi_{L\cap V} \in \wedge^2(V/L\cap V)$ and

$$L = \{X + \xi \in V \oplus \mathrm{Ann}\left(L \cap V\right) \mid X|_{\mathrm{Ann}(L\cap V)} = i_\xi \pi_{L\cap V}\}. \tag{2.14}$$

where we used the isomorphism $V = (V^*)^*$.

2.2 The split-quadratic category.

Let (E_i, g_i) be a split-quadratic vector space for $i = 1, 2$. Denote the split-quadratic vector space $(E_1, -g_1)$ by $\bar{E}_1$. Following ideas from Guillemin - Sternberg [26] and Weinstein [50] in the setting of symplectic geometry, Bursztyn - Radko [13] introduced a category whose objects are $\mathcal{D}(V)$ (see Example 2.2) for a vector space V and morphisms are Lagrangian subspaces of $\overline{\mathcal{D}(V)} \times \mathcal{D}(W)$. The motivation was to find a proper setting to define pull-back and push-forward of Lagrangian subspaces of $\mathcal{D}(V)$ (see (2.22) and (2.23) below) generalizing the notion of symplectomorphisms and Poisson maps. In this section, we naturally extend their category by considering more general split-quadratic vector spaces as objects. We call split-quadratic category this enhanced category.

The starting principle for the split-quadratic category is the observation that when $F : (E_1, g_1) \to (E_2, g_2)$ is an isomorphism, its graph

$$\Lambda_F = \{(e_1, F(e_1)) \mid e_1 \in E_1\} \tag{2.15}$$

is a Lagrangian subspace of $\bar{E}_1 \times E_2$. Moreover, if L is a Lagrangian subspace of E_1, one can obtain $F(L)$ as

$$F(L) = \{e_2 \in E_2 \mid \exists e_1 \in E_1 \,\text{s.t.}\, (e_1, e_2) \in \Lambda_F\}.$$

This suggests the following: for Lagrangian subspaces $\Lambda \subset \bar{E}_1 \times E_2$ and $L \subset E_1$, define

$$\Lambda(L) = \{e_2 \in E_2 \mid \exists e_1 \in L \text{ s.t } (e_1, e_2) \in \Lambda\} \tag{2.16}$$

Proposition 2.22 (See e.g. [13]). *$\Lambda(L)$ is a Lagrangian subspace of E_2.*

We will give a well-known proof of this proposition in the next section as an instance of a reduction procedure.

If $\Lambda_1 \subset \bar{E}_1 \times E_2$, $\Lambda_2 \subset \bar{E}_2 \times E_3$ are Lagrangian subspaces, their composition is defined by

$$\Lambda_2 \circ \Lambda_1 = \{(e_1, e_3) \mid \exists e_2 \in E_2 \text{ s.t. } (e_2, e_3) \in \Lambda_2 \,\text{and}\, (e_1, e_2) \in \Lambda_1\}. \tag{2.17}$$

A direct computation shows that if $F_1 : E_1 \to E_2$ and $F_2 : E_2 \to E_3$ are isomorphisms then

$$\Lambda_{F_2} \circ \Lambda_{F_1} = \Lambda_{F_2 \circ F_1}.$$

The **split-quadratic category** is the category where the objects are split-quadratic vector spaces (E, g) and morphisms from (E_1, g_1) to (E_2, g_2) are Lagrangian subspaces of $\bar{E}_1 \times E_2$. This category has a point space $E = \{0\}$ and $g = 0$. For any split-quadratic vector space (E, g) its Lagrangian subspaces can be seen as Lagrangian subspaces of $\overline{\{0\}} \times E$, which are morphisms from the point space to E. One then immediately checks that, for Lagrangian subspaces $\Lambda \subset \bar{E}_1 \times E_2$ and $L \subset E_1$,

$$\Lambda(L) = \Lambda \circ L,$$

where the right-hand side is interpreted as composition of morphisms

$$\{0\} \xrightarrow{L} E_1 \xrightarrow{\Lambda} E_2.$$

Remark 2.23. Under this point of view, Lagrangian subspaces of $(E, -g)$ are morphisms from (E, g) to $\{0\}$. Although as a set any Lagrangian subspace L of (E, g) is also a Lagrangian subspace of $(E, -g)$, its categorical interpretation changes when one passes from g to $-g$. This difference will play an important role.

We give a proof that $\Lambda_2 \circ \Lambda_1$ is a Lagrangian subspace of $\bar{E}_1 \times E_3$ as an application of Proposition 2.22.

Proposition 2.24. *The subspace* $\Lambda_2^{big} = \{(e_1, e_2, e_1, e_3) \mid (e_2, e_3) \in \Lambda_2\}$ *is Lagrangian in* $E_1 \times \bar{E}_2 \times \bar{E}_1 \times E_3$ *and*

$$\Lambda_2^{big}(\Lambda_1) = \Lambda_2 \circ \Lambda_1$$

Proof. Note that Λ_2^{big} is the image of $\Delta \times \Lambda_2$ under the natural isometry

$$E_1 \times \bar{E}_1 \times \bar{E}_2 \times E_3 \longrightarrow E_1 \times \bar{E}_2 \times \bar{E}_1 \times E_3$$

which exchanges the second and third factors and where $\Delta = \{(e_1, e_1) \mid e_1 \in E_1\}$ is the diagonal. It is straightforward to see that $\Delta \times \Lambda_2$ is a Lagrangian subspace of $E_1 \times \bar{E}_1 \times \bar{E}_2 \times E_3$.

For the second statement (see (2.16)),

$$\begin{aligned} \Lambda_2^{big}(\Lambda_1) = &\ \{(e_1, e_3) \mid \exists (e_1', e_2) \in \Lambda_1 \text{ s.t. } (e_1', e_2, e_1, e_3) \in \Lambda_2^{big}\} \\ = &\ \{(e_1, e_3) \mid \exists e_2 \in E_2 \text{ s.t. } (e_1, e_2) \in \Lambda_1 \text{ and } (e_2, e_3) \in \Lambda_2\} \\ = &\ \Lambda_2 \circ \Lambda_1. \end{aligned}$$

□

Remark 2.25. To any extension (E, g, p) of a vector space V, there is an associated split-quadratic vector space (E, g) together with a distinguished Lagrangian subspace $p^*(V^*)$. Conversely, to any split-quadratic vector space (E, g) together with a Lagrangian subspace $L \subset E$, the adjoint $p : E \to L^*$ of the inclusion $L \hookrightarrow E$ makes (E, g, p) an extension of L^* (as usual, we use g to identify E with E^*).

For $\Lambda_1 \subset \bar{E}_1 \times E_2$ a morphism, the transpose of Λ_1 is defined as

$$\Lambda_1^t = \{(e_2, e_1) \mid (e_1, e_2) \in \Lambda\}. \tag{2.18}$$

It is clearly a Lagrangian subspace of $\bar{E}_2 \times E_1$. If $\Lambda_2 \subset \bar{E}_2 \times E_3$ is another morphism, one has that

$$(\Lambda_2 \circ \Lambda_1)^t = \Lambda_1^t \circ \Lambda_2^t. \tag{2.19}$$

Let (E_i, g_i) be split-quadratic vector spaces for $i = 1, 2$ and let $\Lambda \subset \bar{E}_1 \times E_2$ be a Lagrangian subspace. Define

$$\ker(\Lambda) = \{e_1 \in E_1 \mid (e_1, 0) \in \Lambda\} \tag{2.20}$$

and

$$\operatorname{im}(\Lambda) = \{e_2 \in E_2 \mid \exists e_1 \in E_1 \text{ s.t. } (e_1, e_2) \in \Lambda\}. \tag{2.21}$$

Lemma 2.26. *One has that*

$$\ker(\Lambda) = \mathrm{im}(\Lambda^t)^{\perp}.$$

In particular, $\ker(\Lambda)$ *is an isotropic subspace of* E_1*.*

Proof. For $e_1 \in E_1$ we have

$$e_1 \in \ker(\Lambda) \Leftrightarrow (e_1, 0) \in \Lambda$$

As Λ is Lagrangian, this is equivalent to

$$0 = g((e_1, 0), (e_1', e_2)) = -g_1(e_1, e_1'), \ \forall (e_1', e_2) \in \Lambda$$

where $g = -g_1 + g_2$ is the bilinear form on $\bar{E}_1 \times E_2$. This proves that $e_1 \in \ker(\Lambda)$ if and only if $e_1 \in \mathrm{im}(\Lambda^t)^{\perp}$. For the second statement, note that $\ker(\Lambda) \subset \mathrm{im}(\Lambda^t)$. □

In the rest of the section, we recall some results from [13]. Let V_1 and V_2 be vector spaces and let $f : V_1 \to V_2$ be a linear homomorphism. Define

$$\Lambda_f = \{(X, f^*\eta, f(X), \eta) \mid X \in V_1, \eta \in V_2^*\} \subset \overline{\mathcal{D}(V_1)} \times \mathcal{D}(V_2) \tag{2.22}$$

and

$$\Lambda_f^t = \{(f(X), \eta, X, f^*\eta) \mid X \in V_1, \eta \in V_2^*\} \subset \overline{\mathcal{D}(V_2)} \times \mathcal{D}(V_1) \tag{2.23}$$

(by abuse of notation, we denote the pair $(V_i \oplus V_i^*, g_{\mathrm{can}})$ by the same symbol of the canonical extension $\mathcal{D}(V_i)$). It is easy to check that Λ_f and Λ_f^t are Lagrangian. For a Lagrangian subspace $L_1 \subset \mathcal{D}(V_1)$

$$\Lambda_f(L_1) = \{(f(X), \eta) \in \mathcal{D}(V_2) \mid (X, f^*\eta) \in L_1\}.$$

By Proposition 2.22, it is a Lagrangian subspace of $\mathcal{D}(V_2)$ called the **push-forward** of L_1 by f. Similarly, for a Lagrangian subspace $L_2 \subset \mathcal{D}(V_2)$,

$$\Lambda_f^t(L_2) = \{(X, f^*\eta) \in \mathcal{D}(V_1) \mid (f(X), \eta) \in L_2\}$$

is a Lagrangian subspace of $\mathcal{D}(V_1)$ called the **pull-back** of L_2 by f. For the next proposition, recall the characterizations of Lagrangian subspaces of a canonical extension given by (2.13) and (2.14).

Proposition 2.27. *(Bursztyn - Radko [13].) Let* $L_1 \subset \mathcal{D}(V_1)$ *and* $L_2 \subset \mathcal{D}(V_2)$ *be Lagrangian subspaces.*

(1) If $L_2 = \Lambda_f(L_1)$*, then* $f(L_1 \cap V_1) = L_2 \cap V_2$ *and the induced transformation*

$$f : \frac{V_1}{L_1 \cap V_1} \longrightarrow \frac{V_2}{L_2 \cap V_2}$$

satisfies $f_*(\pi_{L_1 \cap V_1}) = \pi_{L_2 \cap V_2}$*.*

(2) If $L_1 = \Lambda^t_f(L_2)$*, then* $f^{-1}(\mathrm{pr}_{V_2}(L_2)) = \mathrm{pr}_{V_1}(L_1)$ *and* $f^*\omega_{\mathrm{pr}_{V_2}(L_2)} = \omega_{\mathrm{pr}_{V_1}(L_1)}$.

In particular, if $L_2 = \mathrm{Graph}\,(\omega_\sharp)$, for a 2-form $\omega \in \wedge^2 V_2^*$ (see Example 2.6), then

$$\Lambda^t_f(L_2) = \mathrm{Graph}\,(f^*\omega)\,.$$

Similarly, if $L_1 = \mathrm{Graph}\left(\pi^\sharp\right)$, for a bivector $\pi \in \wedge^2 V_1$ (see Example 2.17), then

$$\Lambda_f(L_1) = \mathrm{Graph}\left((f_*\pi)^\sharp\right)$$

In [13], Bursztyn-Radko also give the following definition: given $L_1 \subset \mathcal{D}\,(V_1)$ and $L_2 \subset \mathcal{D}\,(V_2)$ Lagrangian subspaces and a map $f : V_1 \to V_2$ we say that

(1) f is **forward Dirac** if

$$\Lambda_f(L_1) = L_2.$$

It is **strong forward Dirac** [1] if it is forward Dirac and $L_1 \cap \ker(f) = 0$;

(2) f is **backward Dirac** if

$$\Lambda^t_f(L_2) = L_1.$$

It is **strong backward Dirac** [1] if it is backward Dirac and $L_2 \cap \ker(f^*) = 0$.

2.3 A reduction procedure.

In this section, we develop a quotient procedure in the split-quadratic category. Given a split-quadratic vector space (E, g) and an isotropic subspace $K \subset E$, there is an induced symmetric bilinear form on $K^\perp/K$ which turns it into a split-quadratic vector space. Associated to it, there is a canonical Lagrangian subspace $\Lambda_K \subset \bar{E} \times K^\perp/K$ which defines a quotient map

$$\begin{array}{ccc} Lag(E) & \xrightarrow{\Lambda_K} & Lag(K^\perp/K) \\ L & \longmapsto & \Lambda_K(L). \end{array}$$

This can be seen as the linear algebra model for the reduction framework developed in [11]. As an application to this procedure, we prove Proposition 2.22 following unpublished notes of M.Gualtieri [25]. When (E, g) comes from an extension (E, g, p) of some vector space V, we prove (see Proposition 2.30) that $K^\perp/K$ has the structure of an extension of Q/R, where $Q = p(K^\perp)$ and $R = p(K)$. When studying isotropic splittings for $K^\perp/K$, the fundamental notion of K-admissible splittings is introduced in Definition 2.32.

Let (E, g) be a split-quadratic vector space and let $K \subset E$ be an isotropic subspace. On $K^\perp/K$, consider the symmetric bilinear form

$$g_K(k_1^\perp + K, k_2^\perp + K) = g(k_1^\perp, k_2^\perp), \text{ for every } k_1^\perp, k_2^\perp \in K^\perp, \tag{2.24}$$

which is easily seen to be non-degenerate. Hence, $(K^\perp/K, g_K)$ is a quadratic vector space. For a Lagrangian subspace $L \subset E$ define

$$L_K = \frac{L \cap K^\perp + K}{K} = \left\{ k^\perp + K \in \frac{K^\perp}{K} \;\middle|\; k^\perp \in L \cap K^\perp \right\}. \tag{2.25}$$

Proposition 2.28. *For any Lagrangian subspace L of E, L_K defines a Lagrangian subspace of E_K. Therefore, $(K^\perp/K,\, g_K)$ is a split-quadratic vector space.*

Proof. Let $k_1^\perp + K,\, k_2^\perp + K \in K^\perp/K$ with $k_1^\perp,\, k_2^\perp \in L \cap K^\perp$. Then,

$$g_K(k_1^\perp + K, k_2^\perp + K) = g(k_1^\perp, k_2^\perp) = 0$$

because L is isotropic. This proves that $L_K \subset K^\perp/K$ is isotropic. To prove that L_K is Lagrangian, note that

$$\dim(L_K) = \dim(L \cap K^\perp + K) - \dim(K).$$

Now, we claim that $L \cap K^\perp + K$ is Lagrangian. Indeed,

$$[L \cap K^\perp + K]^\perp = [(L + K) \cap K^\perp]^\perp = (L + K)^\perp + K = L \cap K^\perp + K. \tag{2.26}$$

Therefore,

$$\begin{aligned} 2\dim(L_K) = 2\dim(L \cap K^\perp + K) - 2\dim(K) \quad &= \dim(E) - 2\dim(K) \\ &= \dim(K^\perp) - \dim(K) \\ &= \dim(K^\perp/K) \end{aligned}$$

which proves that L_K is Lagrangian and implies that $(K^\perp/K,\, g_K)$ is a split-quadratic vector space (see Definition 2.11). ☐

Consider the canonical morphism $\Lambda_K \subset \bar{E} \times K^\perp/K$ defined by

$$\Lambda_K = \{(k^\perp, k^\perp + K) \mid k^\perp \in K^\perp\}. \tag{2.27}$$

We claim that for a Lagrangian subspace L of E,

$$L_K = \Lambda_K(L).$$

Indeed (see (2.16)),

$$\Lambda_K(L) = \quad \{k^\perp + K \in K^\perp/K \mid \exists e \in L \text{ s.t. } (e, k^\perp + K) \in \Lambda_{_K}\},$$

but as $(e, k^\perp + K) \in \Lambda_{_K}$ if and only if $e \in K^\perp$ and $e + K = k^\perp + K$, one has

$$\Lambda_K(L) = \{k^\perp + K \in K^\perp/K \mid k^\perp \in L \cap K^\perp\} = L_K.$$

We call Λ_K the **quotient morphism** from E to $K^\perp/K$. Let us prove Proposition 2.22 as an application of Proposition 2.28.

Proof of proposition 2.22. The argument is an adaptation of Weinstein's [50]. Begin by noticing that inside $E_1 \times \overline{E}_1 \times E_2$, the diagonal $K = \Delta_{E_1} \times \{0\}$ is isotropic. Also $K^\perp = \Delta_{E_1} \times E_2$. It is straightforward to check that the split-quadratic vector space $(K^\perp/K,\, g_K)$ is isomorphic to (E_2, g_2). The key point is to find $\Lambda(L)$ as a the image of some Lagrangian subspace of $E_2 \times \overline{E_1} \times E_1$ under the quotient morphism Λ_K; the result that $\Lambda(L)$ is Lagrangian will follow from Proposition 2.28. Now, $L \times \Lambda$ is clearly Lagrangian and

$$\Lambda_K(L \times \Lambda) = \frac{(L \times \Lambda) \cap K^\perp + K}{K}$$

is exactly $\Lambda(L)$. □

Let us pass to the case where (E, g) comes from an extension (E, g, p) of a vector space V. Let K be an isotropic subspace of E and call $R = p(K)$ and $Q = p(K^\perp)$. It is clear that $R \subset Q$.

Lemma 2.29. *One has that* $p^*(V^*) \cap K = p^*(\mathrm{Ann}\,(Q))$ *and* $p^*(V^*) \cap K^\perp = p^*(\mathrm{Ann}\,(R))$. *Moreover, the sequences*

$$0 \longrightarrow \mathrm{Ann}\,(Q) \xrightarrow{p^*} K \xrightarrow{p} R \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ann}\,(R) \xrightarrow{p^*} K^\perp \xrightarrow{p} Q \longrightarrow 0$$

are exact.

Proof. We have just to prove that $p^*(V^*) \cap K = p^*(\mathrm{Ann}\,(Q))$ and $p^*(V^*) \cap K^\perp = p^*(\mathrm{Ann}\,(R))$ as the exactness of both sequences follows from (2.1). We check the first as the other follows by similar arguments. For $\xi \in V^*$ and $e \in E$, one has that

$$g(p^*\xi, e) = \xi(p(e)).$$

Thus, if $p^*\xi \in K$, then $\xi(p(k^\perp)) = 0$ for every $k^\perp \in K^\perp$, which proves that $\xi \in \mathrm{Ann}\,(Q)$. On the other hand, if $\xi \in \mathrm{Ann}\,(Q)$, then $g(p^*(\xi), k^\perp) = 0$ for every $k^\perp \in K^\perp$ which proves that $p^*\xi \in (K^\perp)^\perp = K$. This completes the proof. □

The map $p : E \to V$ induces a map $p_K : K^\perp/K \to Q/R$ defined by

$$p_K(k^\perp + K) = q(p(k^\perp)) + R, \tag{2.28}$$

where $q : Q \to Q/R$ is the quotient map.

Proposition 2.30. $E_K := (K^\perp/K, g_K, p_K)$ *is an extension of* Q/R.

Proof. As g_K is non-degenerate, it remains to prove that

$$0 \longrightarrow \left(\frac{Q}{R}\right)^* \xrightarrow{p_K^*} \frac{K^\perp}{K} \xrightarrow{p_K} \frac{Q}{R} \longrightarrow 0 \tag{2.29}$$

is exact. Consider the isomorphism

$$\begin{array}{ccccc} \mathrm{Ann}\,(R)\,/\mathrm{Ann}\,(Q) & \xrightarrow{T_1} & (Q/R)^* & & \\ \xi + \mathrm{Ann}\,(Q) & \longmapsto & \hat{\xi}: \ Q/R & \to & \mathbb{R} \\ & & X + R & \mapsto & \xi(R). \end{array}$$

and the map (well-defined by Lemma 2.29)

$$\begin{array}{ccc} \mathrm{Ann}\,(R)\,/\mathrm{Ann}\,(Q) & \xrightarrow{T_2} & K^{\perp}/K \\ \xi + \mathrm{Ann}\,(Q) & \longmapsto & p^*\xi + K. \end{array}$$

We claim that

$$p_{_K}^* \circ T_1 = T_2.$$

Indeed, for $\xi \in \mathrm{Ann}\,(R)$ and $k^{\perp} + K \in K^{\perp}/K$,

$$g_{_K}(T_2(\xi + \mathrm{Ann}\,(Q)),\, k^{\perp} + K) = g_K(p^*\xi + K,\, k^{\perp} + K) = g(p^*\xi, k^{\perp}) = \xi(p(k^{\perp}))$$

and

$$\begin{aligned} g_{_K}(p_{_K}^* \circ T_1(\xi + \mathrm{Ann}\,(Q)),\, k^{\perp} + K) = \hat{\xi}(p_{_K}(k^{\perp} + K)) &= \hat{\xi}(q(p(k^{\perp}))) \\ &= \hat{\xi}(p(k^{\perp}) + R) \\ &= \xi(p(k^{\perp})). \end{aligned}$$

Hence, as g_K is non-degenerate, it follows that $p_{_K}^* \circ T_1 = T_2$. Therefore, the chain map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ann}\,(R)\,/\mathrm{Ann}\,(Q) & \xrightarrow{T_2} & K^{\perp}/K & \xrightarrow{p_K} & Q/R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & (Q/R)^* & \xrightarrow{p_K^*} & K^{\perp}/K & \xrightarrow{p_K} & Q/R & \longrightarrow & 0 \end{array}$$

is a chain isomorphism and as the first complex is exact (by Lemma. 2.29) it follows that the second is too. □

Example 2.31. Let V be a vector space and $R \subset Q \subset V$ subspaces. Define $K = R \oplus \mathrm{Ann}\,(Q) \subset E$. Then $K^{\perp} = Q \oplus \mathrm{Ann}\,(R)$ and K is an isotropic subspace. Moreover, $E_{_K} = \mathcal{D}\,(Q/R)$.

To study isotropic splittings for the extension $(K^{\perp}/K, g_K, p)$, the following notion is fundamental.

Definition 2.32. We say that an isotropic splitting $\nabla : V \to E$ is K-**admissible** if $\Phi_{\nabla}(K) = R \oplus \mathrm{Ann}\,(Q)$, where Φ_{∇} is the isomorphism (2.6).

Lemma 2.33. $\nabla : V \to E$ is K-admissible if and only if $\nabla(R) \subset K$. This is equivalent to $\nabla(Q) \subset K^{\perp}$.

Proof. If $\nabla(R) \subset K$, then $\nabla|_R$ provides a splitting for the exact sequence

$$0 \longrightarrow \mathrm{Ann}\,(Q) \xrightarrow{p^*} K \xrightarrow{p} R \longrightarrow 0$$

in which case $K = \nabla(R) \oplus \mathrm{Ann}\,(Q)$. Applying Φ_∇, one obtains that

$$\Phi_\nabla(K) = \Phi_\nabla(\nabla(R) \oplus \mathrm{Ann}\,(Q)) = R \oplus \mathrm{Ann}\,(Q)$$

proving that ∇ is K-admissible. On the other hand, if $\Phi_\nabla(K) = R \oplus \mathrm{Ann}\,(Q)$, then restricting Φ_∇^{-1} to R gives that $\nabla(R) \subset K$. For the last statement, if ∇ is K-admissible, then

$$\Phi_\nabla(K^\perp) = (\Phi_\nabla(K))^\perp = (R \oplus \mathrm{Ann}\,(Q))^\perp = Q \oplus \mathrm{Ann}\,(R)\,.$$

Hence, by restricting Φ_∇^{-1} to Q, one has that $\nabla(Q) \subset K^\perp$. Conversely, if $\nabla(Q) \subset K^\perp$, then for every $k \in K$ and $X \in Q$ (see (2.5))

$$i_X s_\nabla(k) = g(k, \nabla X) = 0$$

which proves that $s_\nabla(K) \subset \mathrm{Ann}\,(Q)$ and therefore

$$\Phi_\nabla(K) = (p(K), s_\nabla(K)) \subset R \oplus \mathrm{Ann}\,(Q)\,.$$

The equality follows from dimension count (see Lemma 2.29). □

Any K-admissible splitting ∇ induces an isotropic splitting ∇_K for E_K given by

$$\nabla_K\, q(X) = \nabla X + K, \text{ for } X \in Q. \tag{2.30}$$

For the next proposition, let $j : Q \to V$ be the inclusion map.

Proposition 2.34. *For two K-admissible splittings ∇^1 and ∇^2, the two form $B \in \wedge^2 V^*$ defined by (see Proposition 2.14)*

$$\nabla^2 - \nabla^1 = B$$

satisfies

$$i_X j^* B = 0, \text{ for every } X \in R.$$

The 2-form $B_K \in \wedge^2(Q/R)^*$ defined by $q^* B_K = j^* B$ equals $\nabla_K^2 - \nabla_K^1$.

Proof. Let $X \in R$ and $Y \in Q$. Choose $k^\perp \in K^\perp$ such that $p(k^\perp) = Y$. Then by definition,

$$B(X, Y) = g(p^* i_X B, k^\perp) = g(\nabla^2 X - \nabla^1 X, k^\perp).$$

As both ∇^1 and ∇^2 are K-admissible, by Lemma 2.33, one has that

$$\nabla^2 X - \nabla^1 X \in K.$$

Therefore $g(\nabla^2 X - \nabla^1 X, k^\perp) = 0$ which proves the first claim. As for the second claim, for $Z \in Q$ and $k^\perp \in K^\perp$

$$g_{_K}(p^*_{_K} i_{q(Y)} B_{_K}, k^\perp + K) = B_{_K}(\,q(Y), q(p(k^\perp))\,) = B(Y, p(k^\perp)), \text{ by definition.}$$

On the other hand,

$$\begin{aligned} g_{_K}(\nabla^2_{_K} q(Y) - \nabla^1_{_K} q(Y), k^\perp + K) \quad &= g_{_K}(\nabla^2 Y - \nabla^1 Y + K, k^\perp + K) \\ &= g(\nabla^2 Y - \nabla^1 Y, k^\perp) \\ &= B(Y, p(k^\perp)). \end{aligned}$$

□

We will show later (see Corollary 4.35) that K-admissible splittings exist in a more general context.

2.4 Quotient morphism as a composition of pull-back and push-forward.

In this section we prove a result about the quotient morphism (2.27) which together with Proposition 3.35 provides the linear algebra framework for the main result of this thesis.

Let (E, g, p) be an extension of V, K an isotropic subspace and $\nabla : V \to E$ be a K-admissible splitting (see (2.32)). We have proved in Proposition 2.30 that $(K^\perp/K, p_{_K}, g_{_K})$ is an extension over Q/R , where $Q = p(K^\perp)$ and $p_{_K}$ and $g_{_K}$ were defined in (2.28) and (2.24) respectively. Also ∇ induces an isotropic splitting

$$\nabla_K : \frac{Q}{R} \longrightarrow \frac{K^\perp}{K}$$

defined by (2.30).

Theorem 2.35. *Let $q : Q \to Q/R$ be the quotient map and $j : Q \to V$ be the inclusion. Under the isomorphism*

$$\Phi_\nabla \times \Phi_{\nabla_{_K}} : \bar{E} \times (K^\perp/K) \longrightarrow \overline{\mathcal{D}\,(V)} \times \mathcal{D}\,(Q/R)$$

the quotient morphism (see (2.27)) $\Lambda_{_K}$ is sent to $\Lambda_q \circ \Lambda_j^t$.

Proof. One may directly verify from the definitions that

$$(X, \xi, Y, \eta) \in \Lambda_q \circ \Lambda_j^t \Longleftrightarrow q^*\eta = j^*\xi,\ X \in Q \text{ and } q(X) = Y$$

Let $k^\perp \in K^\perp$. For $(k^\perp, k^\perp + K) \in \Lambda_{_K}$, one has

$$\Phi_\nabla \times \Phi_{\nabla_{_K}}(k^\perp, k^\perp + K) = (p(k^\perp), s_\nabla(k^\perp), p_{_K}(k^\perp + K), s_{\nabla_{_K}}(k^\perp + K))$$

where $s_{\nabla_K}(k^\perp + K) \in (Q/R)^*$ and $s_\nabla(k^\perp) \in V^*$ are defined by (2.5). First, we have that $p(k^\perp) \in Q$ and $q(p(k^\perp)) = p_K(k^\perp + K)$ by definition. We claim that

$$q^* s_{\nabla_K}(k^\perp + K) = j^* s_\nabla(k^\perp)$$

Indeed, by (2.5), for every $X \in Q$

$$\begin{aligned} i_{q(X)} s_{\nabla_K}(k^\perp + K) = \quad g_K(k^\perp + K, \nabla_K\, q(X)) &= \quad g_K(k^\perp + K, \nabla X + K) \\ &= \quad g(k^\perp, \nabla X) \\ &= \quad i_X s_\nabla(k^\perp). \end{aligned}$$

This proves that

$$\Phi_\nabla \times \Phi_{\nabla_K}(\Lambda_K) \subset \Lambda_q \circ \Lambda_j^t.$$

As both have the same dimension, the equality holds. □

Remark 2.36. In general, if we start with a general isotropic splitting $\nabla : V \to E$, there exists $B \in \wedge^2 V^*$ such that $\nabla + B$ is K-admissible. Call $\nabla_K^B := (\nabla + B)_K$ the induced isotropic splitting for $K^\perp/K$. In this case,

$$\Phi_\nabla \times \Phi_{\nabla_K^B}(\Lambda_K) = \Lambda_q \circ \Lambda_j^t \circ \Lambda_{\tau_{-B}}.$$

Corollary 2.37. *For any $L \subset E$ Lagrangian subspace,*

$$\Phi_{\nabla_K}\left(\frac{L \cap K^\perp + K}{K}\right) = \Lambda_q \circ \Lambda_j^t\,(\Phi_\nabla(L)).$$

Example 2.38. Let V be a vector space and $Q \subset V$ a subspace. Take $K = \mathrm{Ann}\,(Q) \subset \mathcal{D}\,(V)$, which is an isotropic subspace with $p(K^\perp) = Q$ and $R = p(K) = 0$ (in this case $q : Q \to Q/R$ is the identity map of Q). Any isotropic splitting ∇ for $\mathcal{D}\,(V)$ is trivially K-admissible, and the induced splitting ∇_K for the extension E_K of Q is given by

$$\nabla_K : Q \ni X \mapsto \nabla X + K \in \frac{K^\perp}{K}.$$

For any Dirac structure $L \subset \mathcal{D}\,(V)$ on V,

$$\frac{L \cap K^\perp + K}{K} = \frac{L \cap (Q \oplus V^*)}{L \cap \mathrm{Ann}\,(Q)}.$$

This is the restriction of L to Q as defined by T. Courant in [17]. In this case, Corollary 2.37 gives that

$$\Phi_{\nabla_K}\left(\frac{L \cap (Q \oplus V^*)}{L \cap \mathrm{Ann}\,(Q)}\right) = \Lambda_j^t(\Phi_\nabla(L)).$$

In the case ∇ is the canonical splitting ∇_{can} for $\mathcal{D}\,(V)$ (see Example 2.2), $\Phi_\nabla = Id$.

Remark 2.39. Let (E, g, p) be a real extension of a real vector space V. Consider the complex extension $(E \otimes \mathbb{C}, p \otimes \mathrm{id}, g_{\mathbb{C}})$ of $V \otimes \mathbb{C}$ (see Example 2.15). Any isotropic subspace K of E induces an isotropic subspace of $E \otimes \mathbb{C}$ given by $K_{\mathbb{C}} := K \otimes \mathbb{C}$, its orthogonal being $K_{\mathbb{C}}^{\perp} = K^{\perp} \otimes \mathbb{C}$. Proposition 2.30 implies that

$$\left(\frac{K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}}},\, (p \otimes \mathrm{id})_{K_{\mathbb{C}}},\, (g_{\mathbb{C}})_{K_{\mathbb{C}}} \right) \text{ is an extension of } \frac{Q \otimes \mathbb{C}}{R \otimes \mathbb{C}}, \tag{2.31}$$

where $Q = p(K^{\perp})$ and $R = p(K)$ as usual. By identifying $(K^{\perp} \otimes \mathbb{C})/(K \otimes \mathbb{C})$ and $(Q \otimes \mathbb{C})/(R \otimes \mathbb{C})$ with $(K^{\perp}/K) \otimes \mathbb{C}$ and $(Q/R) \otimes \mathbb{C}$ respectively, it is not difficult to see that (2.31) is the complexification of the real extension $(K^{\perp}/K, p_K, g_K)$, i.e.

$$(p \otimes \mathrm{id})_{K_{\mathbb{C}}} = p_K \otimes \mathrm{id} \quad \text{and} \quad (g_{\mathbb{C}})_{K_{\mathbb{C}}} = (g_K)_{\mathbb{C}}.$$

Let $\nabla : V \to E$ be a K-admissible isotropic splitting for E. Then $\nabla \otimes \mathrm{id} : V \otimes \mathbb{C} \to E \otimes \mathbb{C}$ is $K_{\mathbb{C}}$-admissible and

$$(\nabla \otimes \mathrm{id})_{K_{\mathbb{C}}} = \nabla_K \otimes \mathrm{id}$$

for the induced splitting $\nabla_K : Q/R \to K^{\perp}/K$. Hence, the isomorphism

$$\Phi_{(\nabla \otimes \mathrm{id})_{K_{\mathbb{C}}}} : \frac{K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}}} \longrightarrow \left[\left(\frac{Q}{R} \right) \oplus \left(\frac{Q}{R} \right)^* \right] \otimes \mathbb{C}$$

is just the $\mathbb{C}$-linear extension of $\Phi_{\nabla_K} : K^{\perp}/K \to \mathcal{D}\,(Q/R)$.

Let $L \subset E \otimes \mathbb{C}$ be a Lagrangian subspace and consider the Lagrangian subspace of $(K^{\perp}/K) \otimes \mathbb{C}$ given by

$$L_{red} = \frac{L \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}}{K_{\mathbb{C}}}.$$

Corollary 2.37 gives that

$$\Phi_{\nabla_K} \otimes \mathrm{id}\,(L_{red}) = \Lambda_{q \otimes \mathrm{id}} \circ \Lambda^t_{j \otimes \mathrm{id}}(\Phi_{\nabla} \otimes \mathrm{id}\,(L)),$$

where $j : Q \to V$ is the inclusion, $q : Q \to Q/R$ is the quotient and $j \otimes \mathrm{id}$ and $q \otimes \mathrm{id}$ are their respective $\mathbb{C}$-linear extensions.

Chapter 3

Spinors: Part I.

In this chapter, we give another point of view to the constructions of Chapter 1. Given an extension (E, g, p) of a vector space V there is a correspondence between its Lagrangian subspaces and some special elements of irreducible modules for the Clifford algebra $Cl(E, g)$ called **pure spinors**. This correspondence can be traced back to the work of Cartan [14] (see also [16]) and should be seen as an analogue of the correspondence between Lagrangian submanifolds of the cotangent bundle and generating functions ([27, 50]); this analogy will be developed in future work using super-geometry [22]. Given an isotropic subspace $K \subset E$ and a Lagrangian subspace $L \subset E$, we describe how pure spinors corresponding to L_K (see 2.37) can be obtained from pure spinors corresponding to L. This construction can be generalized by substituting the quotient morphism (2.27) by any morphism (in the sense of the split-quadratic category). We will develop this more general approach in the Appendix A. The sources for the classical results on spinors and Clifford algebras will be [16, 24, 40].

3.1 Clifford algebra

Let (E, g) be a quadratic vector space (see §2.1). Its **Clifford algebra** $Cl(E, g)$ is the algebra generated by the elements of E subject to the relation

$$e_1 e_2 + e_2 e_1 = g(e_1, e_2). \tag{3.1}$$

It can be alternatively defined as the quotient of the tensor algebra

$$T(E) = \bigoplus_{i \geq 0} \underbrace{E \otimes \cdots \otimes E}_{\text{i times}}$$

by the ideal $\mathfrak{I}$ generated by

$$e_1 \otimes e_2 + e_2 \otimes e_1 - g(e_1, e_2). \tag{3.2}$$

The tensor algebra has a natural $\mathbb{Z}$ grading. Although the ideal generated by (3.2) is not $\mathbb{Z}$ homogeneous, it is $\mathbb{Z}_2$ homogeneous (for the underlying $\mathbb{Z}_2$

grading). Therefore, the Clifford algebra inherits a $\mathbb{Z}_2$ grading which we denote by

$$Cl(E,g) = Cl_0(E,g) \oplus Cl_1(E,g),$$

where $Cl_0(E,g)$ denotes the even part and $Cl_1(E,g)$ the odd part. It is a $\mathbb{Z}_2$-graded algebra in the sense that

$$Cl_i(E,g)Cl_j(E,g) \subset Cl_{i+j}(E,g),$$

where the sum is modulo 2.

Remark 3.1. For a real quadratic vector space (E,g), consider its complexification $E \otimes \mathbb{C}$ together with the $\mathbb{C}$-bilinear extension $g_{\mathbb{C}}$ of g. Then $(E \otimes \mathbb{C}, g_{\mathbb{C}})$ is a complex quadratic vector space and

$$Cl(E \otimes \mathbb{C}, g_{\mathbb{C}}) = Cl(E,g) \otimes \mathbb{C}.$$

This follows from the identification $T(E \otimes \mathbb{C}) = T(E) \otimes \mathbb{C}$ together with the fact that ideal generated by

$$(e_1 + i\,e_2) \otimes (e_3 + i\,e_4) - (e_3 + i\,e_4) \otimes (e_1 + i\,e_2) - g_{\mathbb{C}}(e_1 + i\,e_2, e_3 + i\,e_4)$$

for $e_1, \ldots, e_4 \in E$ is equal to $\mathfrak{I} \otimes \mathbb{C}$, where $\mathfrak{I} \subset T(E)$ is the ideal generated by (3.2). See [16] for more details.

Besides the $\mathbb{Z}_2$ grading, the Clifford algebra $Cl(E,g)$ also inherits from the tensor algebra a universal property (for a proof see [16]): If A is an associative algebra with unit and $f : E \to A$ is a vector-space homomorphism such that

$$f(e_1) \cdot f(e_2) + f(e_2) \cdot f(e_1) = g(e_1, e_2) 1_A,$$

then f admits a unique extension to an algebra homomorphism $Cl(f)$ making the diagram below commutative:

$$\begin{array}{ccc} & & Cl(E,g) \\ & \nearrow & \big\downarrow{\scriptstyle Cl(f)} \\ E & \xrightarrow{\ f\ } & A. \end{array}$$

We shall refer to this property as the *universal property* of the Clifford algebra.

Example 3.2. Let

$$O(E,g) = \{A \in GL(E) \mid A^* g = g\}$$

be the symmetry group of (E,g). For $A \in O(E,g)$, the composition (which we continue to call A)

$$E \xrightarrow{A} E \hookrightarrow Cl(E,g)$$

satisfies for $e_1, e_2 \in E$

$$A(e_1)A(e_2) + A(e_2)A(e_1) = g(A(e_1), A(e_2)) = g(e_1, e_2).$$

Therefore, there is an extension $Cl(A) : Cl(E, g) \longrightarrow Cl(E, g)$ to an algebra homomorphism. Doing the same with A^{-1} and by uniqueness we get that $Cl(A)$ is an automorphism. Also, for $A_1, A_2 \in O(E, g)$, by uniqueness

$$Cl(A_1A_2) = Cl(A_1)Cl(A_2).$$

In this way, we get an representation

$$\rho : O(E, g) \to \mathrm{Aut}(Cl(E, g)). \tag{3.3}$$

Example 3.3. The map $\sigma : E \longrightarrow E$ defined by $\sigma(e) = -e$ belongs to $O(E, g)$. Let $Cl(\sigma) : Cl(E, g) \longrightarrow Cl(E, g)$ be its corresponding automorphism. It is easy to see that $Cl(\sigma)|_{Cl_0(E,g)} = Id$ and $Cl(\sigma)|_{Cl_1(E,g)} = -Id$. For simplicity, we denote $Cl(\sigma)(a)$ by a^σ for $a \in Cl(E, g)$. It is fairly easy to check that $(a^\sigma)^\sigma = a$.

Let $Cl^\times(E, g)$ be the the group of invertible elements of $Cl(E, g)$ and define the Clifford group

$$\Gamma = \{a \in Cl^\times(E, g) \mid a^\sigma e a^{-1} \in E \;\forall\, e \in E\}.$$

For $a \in \Gamma$

$$-a^\sigma e a^{-1} = (a^\sigma e a^{-1})^\sigma = -ae(a^\sigma)^{-1},$$

which implies that $a^\sigma \in \Gamma$ and

$$a^\sigma e a^{-1} = ae(a^\sigma)^{-1}.$$

Thus, for $e_1, e_2 \in E$

$$\begin{aligned} g(a^\sigma e_1 a^{-1}, a^\sigma e_2 a^{-1}) &= (a^\sigma e_1 a^{-1})(a^\sigma e_2 a^{-1}) + (a^\sigma e_2 a^{-1})(a^\sigma e_1 a^{-1}) \\ &= ae_1(a^\sigma)^{-1}\, a^\sigma e_2 a^{-1} + ae_2(a^\sigma)^{-1}\, a^\sigma e_1 a^{-1} \\ &= a(e_1e_2 + e_2e_1)a^{-1} \\ &= g(e_1, e_2). \end{aligned}$$

Thus $A = a^\sigma(\cdot)a^{-1} \in O(E, g)$ and moreover $Cl(A)(\cdot) = a^\sigma \cdot a^{-1}$ (if $a \in Cl_0(E, g)$, $Cl(A)$ is inner).

It can be proved (see e.g. [40]) that the Clifford group of a quadratic vector space (E, g) fits into an exact sequence of groups

$$0 \longrightarrow \mathbb{F}^* \longrightarrow \Gamma \longrightarrow O(E, g) \longrightarrow 0.$$

For $a \in Cl^\times(E, g)$, $a \in \Gamma$ if and only if a is a product $e_1 \cdots e_k$ of elements of E with $g(e_i, e_i) \neq 0$. To obtain a double cover of $O(E, g)$, one has to normalize the $e_i's$. To do this, we first have to define the main antiautomorphism of $Cl(E, g)$.

The tensor algebra $T(E)$ has a canonical anti-automorphism given in homogeneous elements by $e_1 \otimes \cdots \otimes e_r \mapsto e_r \otimes \cdots \otimes e_1$, which preserves the ideal generated by (3.2) and thus descends to $Cl(E, g)$, giving rise to an anti-automorphism which takes homogeneous elements $h = e_1 \cdots e_n \in Cl(E, g)$ to

$$h^t = e_n \cdots e_1. \tag{3.4}$$

Now,

$$Pin(E) = \{a \in \Gamma \mid a^t a = \pm 1\}$$

is a double cover of $O(E, g)$ and

$$Spin(E) = Pin(E) \cap Cl_0(E, g)$$

is a double cover $SO(E, g)$.

Example 3.4. Consider a split-quadratic vector space (E, g) (see 2.11). Let $L' \subset E$ be a Lagrangian subspace of E. The subalgebra of $Cl(E, g)$ generated by L' is isomorphic to the exterior algebra $\wedge^\bullet L'$ because of the defining relations (3.1). Let $B \in \wedge^2 L'$ and consider

$$e^B = \sum_{n=0}^{\infty} \frac{1}{n} B^n \text{ (finite sum).}$$

We shall see that $e^B \in Spin(E, g)$. Let L be a Lagrangian complement (it exists by Corollary 2.8). Using the non-degerate pairing g, every element of $x \in L$ defines a left derivation $D^g(x)$ of degree -1 on $\wedge^\bullet L'$ acting on generators $y \in L'$ by

$$D^g(x)y = g(x, y).$$

Using the relation (3.1), one has that for $x \in L$

$$xe^B = D^g(x)e^B + e^B x = (D^g(x)B)e^B + e^B x.$$

As e^B has even degree, $(D^g(x)B)e^B = e^B(D^g(x)B)$ and thus

$$e^{-B} x e^B = D^g(x)B + x.$$

Also, for $y \in L'$

$$e^{-B} y e^B = y e^{-B} e^B = y.$$

Therefore, for a general element of E, $e = x + y \in L \oplus L'$,

$$e^{-B} (x + y) e^B = x + y + D^g(x)y$$

so that $e^B \in \Gamma$. As it is even and $(e^B)^t e^B = e^{-B} e^B = 1$, $e^B \in Spin(E, g)$.

Remark 3.5. We would like to point out a particular instance of this construction. Let $E = V \oplus V^*$ for some vector space V and $g = g_{can}$ (see Example 2.2). Consider $L = V$ and $L' = V^*$; in this case, the derivation $D^g(\cdot)$ is just the interior product and the map $e^{-B}(\cdot)e^B$ is the B-field transformation τ_B given by (2.9).

3.1.1 Clifford modules for split-quadratic vector spaces.

Let (E, g) be a split-quadratic vector space.

Definition 3.6. *A* ***Clifford module*** *for (E, g) is a vector space S together with an algebra homomorphism*

$$\rho : Cl(E, g) \longrightarrow \mathrm{End}\,(S)\,.$$

It is called irreducible if there is no proper subspace $\tilde{S}$ of S such that $\rho(a)$ leaves $\tilde{S}$ invariant for every $a \in Cl(E)$. It is called faithful if $\rho(a) = 0$ implies $a = 0$.

A **polarization** of E is an ordered pair $l = (L, L')$ of Lagrangian subspaces such that $E = L \oplus L'$ (it always exists by Corollary 2.8). We study a class of irreducible Clifford modules for (E, g) parametrized by polarizations of E.

Example 3.7. For $(V \oplus V^*, g_{\mathrm{can}})$ (see Example 2.2), the pair (V, V^*) is a polarization called the **canonical polarization**. Its opposite polarization is (V^*, V).

Let $l = (L, L')$ be a polarization of E. Choose a basis $\{e_1, \cdots, e_n\}$ of L and let $\{e^1, \cdots, e^n\}$ be the basis of L' which satisfies $g(e_i, e^j) = \delta_i^j$. Recall that as L' is isotropic, the subalgebra of $Cl(E, g)$ generated by L' is isomorphic to $\wedge^\bullet L'$.

Lemma 3.8. *$Cl(E, g) = \wedge^\bullet L' \oplus \langle L \rangle$, where $\langle L \rangle$ is the left ideal generated by L.*

Proof. Any element $a \in Cl(E)$ can be written as a sum of products of $e_1, \ldots, e_n, e^1, \ldots, e^n$. Products which involve only elements $e^1, \ldots, e^n$ are in $\wedge^\bullet L'$. In any product which has e_r's as factors, one can use relations (3.1) to substitute it for a sum of products where the e_r's only appear in the right end and thus are in the ideal $\langle L \rangle$. For example,

$$e^r e_s e_t e^u = \delta_t^u e^r e_s - e^r e_s e^u e_t.$$

This completes the proof. □

Define

$$\begin{array}{rcccl} \Pi_l : & Cl(E, g) & \longrightarrow & \mathrm{End}\,(\wedge^\bullet L') & \\ & a & \longmapsto & \mathrm{pr}_{\wedge^\bullet L'}(a\,\cdot) : & \wedge^\bullet L' \;\; \to \;\; \wedge^\bullet L' \\ & & & & \beta \;\; \mapsto \;\; \mathrm{pr}_{\wedge^\bullet L'}(a\beta) \end{array} \tag{3.5}$$

where $\mathrm{pr}_{\wedge^\bullet L'}$ is the projection onto $\wedge^\bullet L'$ relative to the decomposition given by the Lemma 3.8.

Proposition 3.9 ([16])**.** *Π_l is an irreducible and faithful representation of $Cl(E, g)$.*

Proof. Let $a_1, a_2 \in Cl(E, g)$ and $\beta \in \wedge^\bullet L'$.

$$\begin{aligned} (a_1 a_2)\beta = \;\; a_1(a_2\beta) &= \;\; a_1(\mathrm{pr}_{\wedge^\bullet L'}(a_2\beta) + \mathrm{pr}_{\langle L \rangle}(a_2\beta)) \\ &= \;\; a_1\,(\Pi_l(a_2)\beta) + a_1\,\mathrm{pr}_{\langle L \rangle}(a_2\beta). \end{aligned}$$

As $\langle L\rangle$ is a left ideal, it follows that $a_1 \operatorname{pr}_{\langle L\rangle}(a_2\beta) \in \langle L\rangle$. Therefore,

$$\Pi_l(a_1a_2)\beta = \operatorname{pr}_{\wedge^\bullet L'}((a_1a_2)\beta) = \operatorname{pr}_{\wedge^\bullet L'}(a_1\,(\Pi_l(a_2)\beta)) = \Pi_l(a_1)\Pi_l(a_2)\beta.$$

It is clear that $\Pi_l(1) = Id$ and thus Π_l is a representation.

Let us prove that it is irreducible. Let $0 \neq \beta \in \wedge^\bullet L'$. Choose the biggest k such that there exists $I = \{i_1 < \cdots < i_k\} \subset \{1, \cdots, n\}$ with the coordinate of β corresponding to e^I different from zero. Then, using relations (3.1) and the fact that $g(e_i, e^j) = \delta_{ij}$, one has that

$$e_{i_k}\cdots e_{i_1}\beta = c + \hat{\beta}, \text{ where } \hat{\beta} \in \langle L\rangle \text{ and } c \in \mathbb{F}\backslash\{0\}.$$

Therefore,

$$\Pi_l\left(\frac{1}{c}e_{i_k}\cdots e_{i_1}\right)\beta = 1$$

and hence, for any $\alpha \in \wedge^\bullet L'$,

$$\Pi_l\left(\frac{1}{c}\,\alpha\, e_{i_k}\cdots e_{i_1}\right)\beta = \alpha,$$

which proves that

$$\Pi_l(Cl(E,g))\,\beta = \wedge^\bullet L'$$

for any $0 \neq \beta \in \wedge^\bullet L'$. This is clearly equivalent to irreducibility.

As for faithfulness, let $a \in Cl(E)$ and $a = a_1 + a_2 \in \wedge^\bullet L' \oplus \langle L\rangle$ be its decomposition according to Lemma 3.8. If $a_1 \neq 0$, then $\Pi_l(a)1 = a_1 \neq 0$. If $a_1 = 0$, write

$$a_2 = \sum_{\{i_1<\cdots<i_k\}\subset\{1,\cdots,n\}} a^{i_1\dots i_k} e_{i_1}\cdots e_{i_k},$$

with $a^{i_1\dots i_k} \in \wedge^\bullet L'$. Take the smallest k such that $a^{i_1\dots i_k} \neq 0$ and let $\beta = e^{i_k} \wedge \cdots \wedge e^{i_1}$. Using relations (3.1)

$$a_2\beta = a^{i_1\dots i_k} + \langle L\rangle$$

and therefore

$$\Pi_l(a_2)\beta = a^{i_1\dots i_k} \neq 0.$$

Thus $\Pi(a) \neq 0$ for every $a \in Cl(E)$.

□

A word about notation. From now on, we will denote the ordered set $\{i < i+1 < \cdots < j\}$ by the interval notation $[i,j]$; by $(i,j]$ we mean the ordered set $\{i+1, \cdots, j\}$. Also, all subsets I of intervals are supposed to be ordered from the smallest element to the largest unless otherwise stated. We denote the cardinality of I by $|I|$; its j-th element by i_j and for $j \in [1, |I|]$, I_j is $I - \{i_j\}$.

Remark 3.10. For any polarization $l = (L, L')$ of a real quadratic vector space (E, g), one has that $(L \otimes \mathbb{C}, L' \otimes \mathbb{C})$ is a polarization of $(E \otimes \mathbb{C}, g_{\mathbb{C}})$. Proposition 3.9 gives that
$$\wedge^\bullet (L' \otimes \mathbb{C}) = \wedge^\bullet L' \otimes \mathbb{C}$$
is a Clifford module for $(E \otimes \mathbb{C}, g_{\mathbb{C}})$.

Using relations (3.1), one can find explicit formulas for Π_l. Let $\alpha = e^I \in \wedge^\bullet L', I \subset [1, n]$ and $x \in L$,

$$x\alpha = \sum_{j=1}^{k} (-1)^{j+1} g(x, e^{i_j}) e^{I_j} + (-1)^k \alpha x.$$

As $\alpha x \in \langle L \rangle$, this implies that (see Example 3.4)

$$\Pi_l(x)\alpha = \sum_{j=1}^{k} (-1)^{j+1} g(x, e^{i_j}) e^{I_j} = D^g(x)\alpha. \tag{3.6}$$

Similarly, for $y \in L'$,
$$\Pi_l(y)\alpha = y \wedge \alpha. \tag{3.7}$$

It is a trivial but important observation that if $l_1 = (L, L')$ is a polarization of (E, g) then so is $l_2 = (L', L)$. Note that the resulting representation $\Pi_{l_2} : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet L)$ exchanges the roles of L and L', that is

$$\Pi_{l_2}(x) = x \wedge \cdot\,, \text{ for } x \in L$$

and
$$\Pi_{l_2}(y) = D^g(y)\,, \text{ for } y \in L'.$$

Example 3.11. Let (E, g, p) be an extension of a vector space V and consider the Clifford algebra $Cl(E, g)$. Every isotropic splitting $\nabla : V \to E$ induces a polarization $(\nabla V, V^*)$ (we have identified V^* with $p^*(V^*)$) which turns $\wedge^\bullet V^*$ into a module for $Cl(E, g)$. Let

$$\Pi_\nabla : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet V^*)$$

be the representation. For an element $e \in E$, we have

$$\Pi_\nabla(e) = i_{p(e)} + s_\nabla(e) \wedge \cdot \tag{3.8}$$

where $s_\nabla(e) \in \wedge^\bullet V^*$ is the element given by (2.5).

A particular instance of Example 3.11 is when $E = \mathcal{D}(V)$ for some vector space V. In this case, we denote the representation $\Pi_{\nabla_{\mathrm{can}}} : Cl(V \oplus V^*, g_{\mathrm{can}}) \to \wedge^\bullet V^*$ obtained from the canonical splitting simply by Π. From (3.8),

$$\Pi(X + \xi) = i_X + \xi \wedge \cdot\,, \text{ for } X + \xi \in V \oplus V^*;$$

The representation on $\wedge^\bullet V$ obtained by switching V^* with V is denoted by $\widehat{\Pi}$ and given by

$$\widehat{\Pi}(X+\xi) = X \wedge \cdot + i_\xi\,, \text{ for } X+\xi \in V \oplus V^*.$$

For E an extension over V and ∇ an isotropic splitting, we have the important relation:

$$\Pi_\nabla(e) = \Pi(p(e) + s_\nabla(e)) = \Pi(\Phi_\nabla(e)), \text{ for } e \in E, \tag{3.9}$$

where Φ_∇ is the isomorphism (2.6).

Example 3.12. Let (E, g, p) be a real extension of a real vector space V and consider its complexification $(E \otimes \mathbb{C}, p \otimes \mathrm{id}, g_\mathbb{C})$ (see Example 2.15). Let $\nabla : V \to E$ be an isotropic splitting and consider $\nabla \otimes \mathrm{id} : V \otimes \mathbb{C} \to E \otimes \mathbb{C}$. Its image is just $(\nabla V) \otimes \mathbb{C}$ which together with $p^*(V^*) \otimes \mathbb{C}$ gives a polarization of $E \otimes \mathbb{C}$. By identifying $p^*(V^*)$ with V^* one has that

$$\wedge^\bullet(V^* \otimes \mathbb{C}) = \wedge^\bullet V^* \otimes \mathbb{C}$$

is a module for $Cl(E \otimes \mathbb{C}, g_\mathbb{C}) \cong Cl(E, g) \otimes \mathbb{C}$. The representation

$$\Pi_{\nabla\otimes\mathrm{id}} : Cl(E, g) \otimes \mathbb{C} \longrightarrow \mathrm{End}\,(\wedge^\bullet V^* \otimes \mathbb{C})$$

is just the $\mathbb{C}$-linear extension of the representation $\Pi_\nabla : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet V^*)$ constructed in Example 3.11 after identifying

$$\mathrm{End}\,(\wedge^\bullet V^* \otimes \mathbb{C}) = \mathrm{End}\,(\wedge^\bullet V^*) \otimes \mathbb{C}.$$

Example 3.13. Let (E, g) be a split-quadratic vector space. It is clear that if $l = (L, L')$ is a polarization for (E, g), then l is also a polarization for $(E, -g)$. We denote the representation of $Cl(E, -g)$ associated with l by Π_l^-. For $x+y \in L \oplus L'$,

$$\Pi_l^-(x+y)\alpha = -D^g(x)\alpha + y \wedge \alpha,$$

where $\alpha \in \wedge^\bullet L'$.

Example 3.14. Let (E_i, g_i) be a split-quadratic vector space and let $l_i = (L_i, L_i')$ be a polarization of E_i for $i = 1, 2$. Then $(E = E_1 \times E_2, g = g_1 + g_2)$ is a split-quadratic vector space and $l_1 \times l_2 = (L_1 \times L_2, L_1' \times L_2')$ is a polarization of E. Using Proposition 3.9, one gets a representation

$$\Pi_{l_1\times l_2} : Cl(E) \to \mathrm{End}\,(\wedge^\bullet(L_1' \times L_2'))\,.$$

If $e_1 \in E_1$ and $e_2 \in E_2$, then $g(e_1, e_2) = g(e_2, e_1) = 0$. Therefore, using relation (3.1), $Cl(E) = Cl(E_1) \otimes Cl(E_2)$ as vector spaces. It is straightforward to prove (again using (3.1)) that for $a_1, b_1 \in Cl(E_1)$ and $a_2, b_2 \in Cl(E_2)$,

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{|a_2||b_2|} a_1 b_1 \otimes a_2 b_2. \tag{3.10}$$

Under this isomorphism, the subalgebra $\wedge^\bullet(L_1' \times L_2')$ is taken to $\wedge^\bullet L_1' \otimes \wedge^\bullet L_2'$. One can check that for $a_1 \in Cl(E_1, g_1)$, $a_2 \in Cl(E_2, g_2)$ and $\alpha\otimes\beta \in \wedge^\bullet L_1' \otimes \wedge^\bullet L_2'$,

$$\Pi_{l_1\times l_2}(a_1 \otimes a_2)\alpha \otimes \beta = (-1)^{|a_2||\alpha|}\Pi_{l_1}(a_1)\alpha \otimes \Pi_{l_2}(a_2)\beta.$$

We will see in the next paragraph that all these representations induced by polarizations of (E, g) are isomorphic.

3.1.2 Pure spinors.

Let (E, g) be a split-quadratic vector space and $l = (L, L')$ be an arbitrary polarization of E. We now proceed to study special elements of the module $\wedge^\bullet L'$ called pure spinors. For any non-zero element $\varphi \in \wedge^\bullet L'$, define

$$\mathcal{N}_l(\varphi) = \{e \in E : \Pi_l(e)\varphi = 0\}.$$

Let $e_1, e_2 \in \mathcal{N}_l(\varphi)$, from (3.1),

$$g(e_1, e_2)\,\varphi = [\Pi_l(e_1)\Pi_l(e_2) + \Pi_l(e_2)\Pi_l(e_1)]\,\varphi = 0.$$

Therefore, $\mathcal{N}_l(\varphi)$ is an isotropic subspace of E.

Definition 3.15 ([16]). *$\varphi \in \wedge^\bullet L'$ is said to be a* ***pure spinor*** *if $\mathcal{N}_l(\varphi)$ is Lagrangian.*

It is clear that if φ is a pure spinor, then any element of $\mathbb{F}\varphi$ is also a pure spinor.

Example 3.16. Let $S \subset L$. Define $L'' = L' \cap S^\perp \oplus S$, which is a Lagrangian subspace of E (compare with (2.26) and note that S is isotropic). Let $\{e_1, \cdots, e_n\}$ be a basis of L such that $\{e_{s+1}, \cdots, e_n\}$ generates S and let $\{e^1, \cdots, e^n\}$ be the basis of L' such that $g(e_i, e^j) = \delta_i^j$; $S^\perp \cap L'$ is generated by $\{e^1, \cdots, e^s\}$. Let

$$\varphi = e^1 \wedge \cdots \wedge e^s \in \wedge^\bullet L'.$$

We claim that $\mathcal{N}_l(\varphi) = L''$. Indeed, for $e \in E$, let $e = x + y \in L \oplus L'$ be its decomposition, then

$$0 = \Pi_l(e)\varphi = D^g(x)\varphi + y \wedge \varphi \Leftrightarrow \begin{cases} D^g(x)\varphi = 0 \\ y \wedge \varphi = 0, \end{cases}$$

because $D^g(x)\varphi$ has degree $s-1$ and $y\wedge\varphi$ has degree $s+1$. The second equation holds if and only if $y \in S^\perp \cap L'$ and the first holds (see formula (3.6)) if and only if $x \in L \cap (S^\perp \cap L')^\perp = S$. This proves our claim.

The next lemma is a useful tool to find pure spinors.

Lemma 3.17 ([16]). *Let $A \in O(E, g)$ be an orthogonal transformation and let $a \in \Gamma$ (the Clifford group) be such that $a^\sigma(\cdot)a^{-1} = A(\cdot)$. Then for any $\varphi \in \wedge^\bullet L'$,*

$$\mathcal{N}_l(\Pi_l(a)\varphi) = A(\mathcal{N}_l(\varphi)).$$

Proof. For $e \in E$,

$$\Pi_l(e)\Pi_l(a)\varphi = \Pi_l(a^\sigma)\Pi_l((a^{-1})^\sigma ea)\varphi = \Pi_l(a^\sigma)\Pi_l(A^{-1}(e))\varphi$$

and as $\Pi_l(a^\sigma)$ is invertible one has that $e \in \mathcal{N}_l(\Pi_l(a)\varphi)$ if and only if $A^{-1}(e) \in \mathcal{N}_l(\varphi)$ as we wanted to show. □

A particularly useful application of the Lemma 3.17 is the following: if $\varphi \in \wedge^\bullet L'$ is a pure spinor and $B \in \wedge^2 L'$, then $e^{-B} \in Spin(E)$ (see Example 3.4) and $e^{-B} \wedge \varphi$ is a pure spinor whose annihilator is $e^{-B}\mathcal{N}_l(\varphi)e^B$. In the case $(V \oplus V^*, g_{\text{can}})$ and $l = (V, V^*)$ is the canonical polarization, we saw (see Remarks 2.19 and 3.5) that any Lagrangian subspace of $V \oplus V^*$ can be obtained as

$$\tau_B(S \oplus \text{Ann}\,(S)) = e^{-B}(S \oplus \text{Ann}\,(S))e^B$$

for some $B \in \wedge^2 V^*$ and $S \subset V$ uniquely determined by the Lagrangian subspace. It is a simple matter to adapt it to a general split-quadratic vector space (E, g). For $l = (L, L')$ an arbitrary polarization of E and L'' a Lagrangian subspace, let $S \subset L$ be the projection of L'' on L and $B \in \wedge^2 L'$ be a two form extending $\omega_S \in \wedge^2 S^*$ (compare with (2.11)) defined by

$$\omega_S(x_1, x_2) = g(y_1, x_2) \text{ where } y_1 \in L' \text{ is such that } x_1 + y_1 \in L'' \tag{3.11}$$

(we implicitly identify L' with L^* via g in this construction). Then,

$$L'' = e^{-B}(S \oplus (S^\perp \cap L'))e^B. \tag{3.12}$$

Proposition 3.18 ([16])**.** *Let $l = (L, L')$ be an arbitrary polarization of E and L'' be a Lagrangian subspace of E. Let $S \subset L$ be its projection and $B \in \wedge^2 L'$ a 2-form extending ω_S (see 3.11). Then, for any $\Omega \in \det(\text{Ann}\,(S)) \subset \wedge^\bullet L'$,*

$$\varphi = e^{-B} \wedge \Omega \tag{3.13}$$

is a pure spinor such that

$$\mathcal{N}_l(\varphi) = L''.$$

Moreover,

$$U^l(L'') := \{\theta \in \wedge^\bullet L' : \Pi_l(e)\theta = 0, \forall e \in L''\} = \mathbb{F}\varphi \tag{3.14}$$

Proof. Once we have (3.12), the expression (3.13) for a pure spinor corresponding to L'' is a direct application of Example 3.16 and Lemma 3.17 . It remains to show the last assertion. Note that if $\theta \in U^l(L'')$ then, by Lemma 3.17, $e^B \wedge \theta$ is a pure spinor for $S \oplus (S^\perp \cap L')$. Now, let $\{e^1, \cdots, e^n\}$ be a basis of L' such that $\{e^1, \cdots, e^s\}$ generates $\text{Ann}\,(S)$. Then

$$e^B \wedge \theta = \sum_{I \subset \{1, \cdots, n\}} a_I e^I, \;\; a_I \in \mathbb{F},$$

and it satisfies

$$\begin{cases} i_{e_j}(e^B \wedge \theta) = 0, & \text{for } j > s \\ e^j \wedge (e^B \wedge \theta) = 0, & \text{for } j \le s \end{cases}$$

Now the first set of equations implies that $a_I = 0$ if $I \not\subset \{1, \cdots, s\}$ and the second set of equations implies that $e^B \wedge \theta = a_{[1,s]} e^{[1,s]}$ and therefore $\theta \in \mathbb{F}\varphi$. $\square$

In the following, for the canonical polarization $l_1 = (V, V^*)$ of $V \oplus V^*$ and its opposite $l_2 = (V^*, V)$ we will denote $\mathcal{N}_{l_1}(\cdot)$ (resp. $U^{l_1}(\cdot)$) and $\mathcal{N}_{l_2}(\cdot)$ (resp. $U^{l_2}(\cdot)$) simply by $\mathcal{N}(\cdot)$(resp. $U(\cdot)$) and $\hat{\mathcal{N}}(\cdot)$(resp. $\widehat{U}(\cdot)$) respectively.

Example 3.19. For $B \in \wedge^2 V^*$, Lemma 3.17 gives that

$$\mathcal{N}(e^{-B}) = e^{-B}\mathcal{N}(1)e^B = \tau_B(V) = \{(X, i_X B) \mid X \in V\}.$$

Example 3.20. Similarly, for $\pi \in \wedge^2 V$, one has

$$\hat{\mathcal{N}}(e^{-\pi}) = \mathrm{Graph}\,(\pi) = \{(\pi^\sharp(\xi), \xi) \mid \xi \in V^*\}$$

The next example fits into the more general construction of Example 3.16. It is the prototype of an important construction given in Chapter 3 (see Definition 4.9).

Example 3.21. Let V be a real vector space with $\dim(V) = 2n$ and let $J : V \to V$ be a complex structure on V (i.e. $J^2 = -id$). Consider the $+i$-eigenspace $V_{0,\,1}$ of the $\mathbb{C}$-linear extension $J : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ and the Lagrangian subspace

$$L = V_{0,\,1} \oplus \mathrm{Ann}\,(V_{0,\,1}) = V_{0,\,1} \oplus V^{1,\,0}$$

of $(V \oplus V^*) \otimes \mathbb{C}$. Using the polarization $(V \otimes \mathbb{C}, V^* \otimes \mathbb{C})$, one has that

$$\wedge^\bullet V^* \otimes \mathbb{C}$$

is a Clifford module for $((V \oplus V^*) \otimes \mathbb{C}, (g_{\mathrm{can}})_{\mathbb{C}})$. By Example 3.16,

$$U(L) = \wedge^n V^{1,\,0} = \wedge^{n,\,0} V$$

is the pure spinor line corresponding to L.

Remark 3.22. Let (E, g, p) be an extension of a vector space V and consider an isotropic splitting $\nabla : V \to E$. In Example 3.11, we have associated a representation Π_∇ of $Cl(E, g)$ on $\wedge^\bullet V^*$ to this data. Let $L \subset E$ be a Lagrangian subspace of E and $\varphi \in \wedge^\bullet V^*$ such that

$$\mathcal{N}_\nabla(\varphi) := \{e \in E \mid \Pi_\nabla(e)\varphi = 0\} = L.$$

Now, formula (3.9) implies that

$$\mathcal{N}(\varphi) = \Phi_\nabla(\mathcal{N}_\nabla(\varphi)) = \Phi_\nabla(L) \tag{3.15}$$

and therefore

$$U^\nabla(L) = \{\theta \in \wedge^\bullet V^* \mid \Pi_\nabla(e)\,\theta = 0, \forall\, e \in L\} = U(\Phi_\nabla(L)) \subset \wedge^\bullet V^*.$$

One of the applications of pure spinors in the context of Clifford modules is to relate Clifford modules arising from different polarizations. This will be extremely important for example to relate the modules arising from different isotropic splittings of an arbitrary extension. Let $l_i = (L_i, L_i')$ be an arbitrary polarization of E for $i = 1, 2$. Take $\varphi \in U^{l_2}(L_1) \subset \wedge^\bullet L_2'$ to be any pure spinor whose annihilator is L_1.

Proposition 3.23. *The map*

$$\begin{array}{rlcl} F_{l_1 l_2}: & \wedge^\bullet L_1' & \longrightarrow & \wedge^\bullet L_2' \\ & \beta & \longmapsto & \Pi_{l_2}(\beta)\varphi \end{array} \tag{3.16}$$

satisfies

$$F_{l_1 l_2} \circ \Pi_{l_1}(a) = \Pi_{l_2}(a) \circ F_{l_1 l_2}.$$

for every $a \in Cl(E)$.

Proof. Take $\varphi \in U^{l_2}(L_1)$ and define

$$\begin{array}{rlcl} F: & Cl(E) & \longrightarrow & \wedge^\bullet L_2' \\ & a & \mapsto & \Pi_{l_2}(a)\varphi. \end{array}$$

As Π_{l_2} is irreducible, the map is surjective and its kernel is the left ideal $\langle L_1 \rangle$ generated by L_1. By Lemma 3.8, it defines an isomorphism

$$F_{l_1 l_2}: \wedge^\bullet L_1' \to \wedge^\bullet L_2'.$$

For $a \in Cl(E)$ and $\beta \in \wedge^\bullet L_1'$

$$\Pi_{l_2}(a) F_{l_1 l_2}(\beta) = \Pi_{l_2}(a)\Pi_{l_2}(\beta)\varphi = \Pi_{l_2}(a\beta)\varphi = F(a\beta).$$

Finally, one has that $a\beta - \Pi_{l_1}(a)\beta \in \langle L_1 \rangle$ by definition of the representation (see Proposition 3.5). Therefore

$$F(a\beta) = F(\Pi_{l_1}(a)\beta) = F_{l_1 l_2}(\Pi_{l_1}(a)\beta)$$

This finishes the proof. □

Example 3.24. Let $E = V \oplus V^*$, $l_1 = (V, V^*)$ and $l_2 = (V^*, V)$. Any $0 \neq \nu \in \det(V)$ is a pure spinor for which $\mathcal{N}_{l_2}(\nu) = V$. In this case, we denote $F_{l_1 l_2}: \wedge^\bullet V^* \longrightarrow \wedge^\bullet V$ by the Hodge star symbol $\star$. For $\alpha \in \wedge^\bullet V^*$

$$\star\alpha = i_\alpha \nu, \tag{3.17}$$

where we have extended the interior product to $\wedge^\bullet V^*$ by the formula

$$i_{\alpha\wedge\beta} = i_\alpha \circ i_\beta.$$

Example 3.25. Let (E, g, p) be an extension of a vector space V. In Example 3.11, we saw how an isotropic splitting ∇ induces a polarization of (E, g) and gives a representation Π_∇ of $Cl(E, g)$ on $\wedge^\bullet V^*$. Let $B \in \wedge^2 V^*$ and consider the representation $\Pi_{\nabla+B}$. We shall use Proposition 3.23 to relate the Clifford modules induced from $(\nabla V, V^*)$ and $((\nabla + B)V, V^*)$. To do that, it suffices to find $\varphi \in \wedge^\bullet V^*$ such that

$$\mathcal{N}_{\nabla+B}(\varphi) = \nabla V = \Phi_\nabla^{-1}(V).$$

Equivalently, by (3.15), φ has to satisfy

$$\mathcal{N}(\varphi) = \Phi_{\nabla+B}(\mathcal{N}_{\nabla+B}(\varphi)) = \Phi_{\nabla+B}(\Phi_{\nabla}^{-1}(V)).$$

But as $\Phi_{\nabla+B} = \tau_{-B} \circ \Phi_B$ by (2.10), φ has to fulfill

$$\mathcal{N}(\varphi) = \tau_{-B}(V) = \mathrm{Graph}\,(-B)\,.$$

Now, Example 3.19 gives that

$$\varphi = e^B$$

Therefore, Proposition 3.23 guarantees that

$$\begin{array}{cccc} F_B: & \wedge^\bullet V^* & \to & \wedge^\bullet V^* \\ & \alpha & \longmapsto & \Pi_{\nabla+B}(\alpha)e^B \end{array}$$

intertwines Π_∇ with $\Pi_{\nabla+B}$, that is

$$\Pi_{\nabla+B}(a) \circ F_B = F_B \circ \Pi_\nabla(a),\, \forall\, a \in Cl(E,g). \tag{3.18}$$

To finish this example, just observe that as $V^* \subset E$ acts on $\wedge^\bullet V^*$ by exterior multiplication, one has that

$$F_B(\alpha) = \alpha \wedge e^B = e^B \wedge \alpha.$$

Example 3.26. Let $\pi \in \wedge^2 V$ be a bivector and consider its graph (see Example 2.17)

$$L = \mathrm{Graph}\left(\pi^\sharp\right) \subset V \oplus V^*.$$

Using Examples 3.20 and 3.24, one can find a generator $\varphi \in U(L)$ once a volume element $\nu \in \wedge^{\mathrm{top}} V^*$ is chosen:

$$\varphi = \star\, e^{-\pi} = \sum_{i=0}^{\infty} \frac{(-1)^n}{n!}\, i_{\pi^n}\, \nu = \nu - i_\pi\, \nu + \frac{1}{2} i_{\pi^2}\, \nu + \dots.$$

Parity. By Proposition 3.18, any pure spinor is a $\mathbb{Z}_2$ homogeneous form. It is an even or odd form depending on $\dim(S^\perp \cap L')$. This property relates to a simple topological aspect of the space of Lagrangians of E: it has two connected components (see [40]). A pure spinor $\varphi \in \wedge^\bullet L'$ has even exterior degree if and only if $\mathcal{N}_l(\varphi)$ is in the connected component of L. Note that if there is $A \in O(E,g)$ such that $A(L) = \mathcal{N}_l(\varphi)$, then by Lemma 3.17 $\varphi = \Pi_l(a)1$ for $a \in \Gamma$ (the Clifford group) with $a^\sigma(\cdot)a^{-1} = A(\cdot)$. In this case, φ is even or odd depending on whether $a \in Cl_0(E)$ or $Cl_1(E)$ respectively, or equivalently, $\det(A) = 1$ or -1 respectively. As we shall show (see Proposition 3.27), such A always exists. We denote the $\mathbb{Z}_2$ degree of φ by $|\varphi|$. Henceforth, in every formula where $|\cdot|$ appears, it is assumed that the elements involved are $\mathbb{Z}_2$ homogeneous.

Proposition 3.27 ([40])**.** *$O(E,g)$ acts transitively on the space of Lagrangian subspaces of E.*

Proof. Let $l = (L, L')$ be a polarization of E and let L'' be any Lagrangian. We have already shown that $L'' = e^{-B}(S \oplus (S^\perp \cap L'))e^B$ for some $S \subset L$ and $B \in \wedge^2 L'$ (see the discussion before Proposition 3.18) so that it is sufficient to take $L = S \oplus (S^\perp \cap L')$ as $e^{-B}(\cdot)e^B \in O(E, g)$. Choose a basis $\{e_1, \cdots, e_n\}$ of L such that $\{e_{s+1}, \cdots, e_n\}$ spans S, where $s = \dim(\mathrm{Ann}\,(S))$. Let $\{e^1, \cdots, e^n\}$ be the basis of L' such that $g(e_i, e^j) = \delta_i^j$. Take $f_1 : L \to L$ to be the projection onto S; $\omega : L \to L'$ by

$$\omega(e_i) = \begin{cases} e^i, & \text{if } i \leq s \\ 0, & \text{if } i > s\,; \end{cases}$$

and $\pi : L' \to L$ by

$$\pi(e^i) = \begin{cases} e_i\,, & \text{if } i \leq s \\ 0\,, & \text{if } i > s\,. \end{cases}$$

Using the decomposition $E = L \oplus L'$ to write elements of $GL(E)$ in matrix form, define

$$A = \begin{pmatrix} f_1 & \pi \\ \omega & f_2 \end{pmatrix},$$

where $f_2 : L' \to L'$ is uniquely defined by $g(f_2(e^i), e_j) = g(e^i, f_1(e_j))$ (or equivalently, the projection onto the subspace generated by $\{e^{s+1}, \cdots, e^n\}$). It is straightforward to check that $A \in O(E, g)$ and takes L to $S \oplus (S^\perp \cap L')$. □

Actually, one has a stronger result:

Proposition 3.28 ([40])**.** *$O(E, g)$ acts transitively on the space of polarizations of E.*

Proof. Let $l_i = (L_i, L_i')$ be an arbitrary polarization for $i = 1, 2$. We will prove that there exists $A \in O(E, g)$ such that $l_1 = (A(L_2), A(L_2'))$. Indeed, by the Proposition 3.27, there exists $A_1 \in O(E, g)$ such that $A_1(L_2') = L_1'$. Let $L'' = A_1(L_2)$. It is clear that $L'' \cap L_1' = 0$, so that there exists a map $B : L_1 \longrightarrow L_1'$ such that $L'' = \mathrm{Graph}\,(B)$ (compare with Example 2.6). As L'' is Lagrangian,

$$0 = g(e_1 + Be_1, e_2 + Be_2) = g(Be_1, e_2) + g(e_1, Be_2);$$

so that B defines an element of $\wedge^2 L_1^*$ by

$$B(e_1, e_2) = g(Be_1, e_2) \text{ for } e_1, e_2 \in L_1.$$

Identifying L' with L' via g, consider $A_2 = e^B(\cdot)e^{-B} \in O(E, g)$ (see Example 3.4). It satisfies $A_2(L_1') = L'$ and $A_2(L'') = L_1$. Therefore $A = A_2 \circ A_1$ satisfies $l_1 = (A(L_2), A(L_2'))$ as we wanted to show. □

Remark 3.29. For any two polarizations $l_1 = (L_1, L_1')$ and $l_2 = (L_2, L_2')$, the map $F_{l_1 l_2} : \wedge^\bullet L_1' \longrightarrow \wedge^\bullet L_2'$ of Proposition 3.23 preserves the $\mathbb{Z}_2$ degree depending on the parity of the elements of the pure spinor line $U^{l_2}(L_1) \subset \wedge^\bullet L_2'$. It preserves parity if and only if $U^{l_2}(L_1)$ has a generator of even degree which corresponds to L_1 and L_2 being in the same connected component.

For an extension (E, g, p) of a vector space V, any isotropic splitting ∇ induces a representation $\Pi_\nabla : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet V^*)$. As we saw in Example 3.25, for a 2-form $B \in \wedge^2 V^*$ the representations $\Pi_{\nabla+B}$ and Π_∇ are intertwined by $F_B(\alpha) = e^B \wedge \alpha$, for $\alpha \in \wedge^\bullet V^*$. As e^B is even, F_B is parity preserving and therefore the decomposition

$$\wedge^\bullet V^* = (\wedge^\bullet V^*)_0 \oplus (\wedge^\bullet V^*)_1 = \bigoplus_{i\, even} \wedge^i V^* \oplus \bigoplus_{i\ odd} \wedge^i V^* \tag{3.19}$$

independs of the splitting ∇. This is due to the fact that the whole set

$$\{\nabla(V) \mid \nabla \text{ is an isotropic splitting } \}$$

is contained in the same component. The parity of a pure spinor $\varphi \in \wedge^\bullet V^*$ in the representation space corresponding to some ∇ is even if and only if $\mathcal{N}_\nabla(\varphi)$ is in this component.

For an endomorphism $A : \wedge^\bullet V^* \to \wedge^\bullet V^*$ we say that A is **even** if it preserves the decomposition (3.19) and **odd** if A sends the odd forms to even forms and vice-versa. The $\mathbb{Z}_2$ degree of A is denoted by $|A|$ and it is 0 if A is even and 1 if it is odd. This defines a decomposition

$$\mathrm{End}\,(\wedge^\bullet V^*) = \mathrm{End}_0(\wedge^\bullet V^*) \oplus \mathrm{End}_1(\wedge^\bullet V^*)$$

and

$$\mathrm{End}_i(\wedge^\bullet V^*)\mathrm{End}_j(\wedge^\bullet V^*) \subset \mathrm{End}_{i+j}(\wedge^\bullet V^*) \text{ for } i, j \in \mathbb{Z}_2$$

for the composition of endomorphisms. In this way, $\mathrm{End}\,(\wedge^\bullet V^*)$ is a $\mathbb{Z}_2$-algebra.

If Π_∇ is the representation of $Cl(E, g)$ corresponding to some isotropic splitting ∇, then for an $\mathbb{Z}_2$ homogeneous element $a \in Cl(E, g)$, one has that

$$\begin{cases} \Pi_\nabla(a) \text{ is even }, & \text{if } a \text{ is even;} \\ \Pi_\nabla(a) \text{ is odd }, & \text{if } a \text{ is odd.} \end{cases}$$

Therefore, $\Pi_\nabla : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet V^*)$ is an isomorphism of $\mathbb{Z}_2$-graded algebras.

The *supercommutator* of two homogeneous elements $A_1, A_2 \in \mathrm{End}\,(\wedge^\bullet V^*)$ is given by

$$[A_1, A_2] = A_1 A_2 - (-1)^{|A_1||A_2|} A_2 A_1. \tag{3.20}$$

3.2 Pull-back and push-forward of spinors.

In §3.1, we saw Lagrangian subspaces of a split-quadratic vector space correspond to pure spinors. In this section, we shall go a little further to investigate how this correspondence behave under pull-back and push-forward morphisms. We are mostly interested in seeing how the pure spinor of a Lagrangian subspace is transformed under the reduction procedure developed in §2.3 (see Theorem 3.35 below). This is part of a story that will be further developed in Appendix A.

3.2.1 Main theorem at the linear algebra level.

Let us start with a construction for isomorphisms. Let (E_1, g_1) and (E_2, g_2) bw two split-quadratic vector spaces and $F : (E_1, g_1) \to (E_2, g_2)$ be an isomorphism. By the universal property of Clifford algebras, we have an induced isomorphism of Clifford algebras:

$$Cl(F) : Cl(E_1, g_1) \longrightarrow Cl(E_2, g_2).$$

Let $l_1 = (L_1, L_1')$ be a polarization of E_1 and consider the induced polarization $l_2 = (L_2, L_2')$ with $L_2 = F(L_1)$ and $L_2' = F(L_1')$. It is immediate to check that $Cl(F)$ preserves the decompositions given by Proposition 3.8. Thus, we have an isomorphism

$$T := Cl(F)|_{\wedge^\bullet L_1'} : \wedge^\bullet L_1' \longrightarrow \wedge^\bullet L_2'$$

which sends $v_1 \wedge \cdots \wedge v_k$ to $F(v_1) \wedge \cdots \wedge F(v_k)$ for $v_i \in L_1'$, $i = 1, \ldots, k$ which satisfies:

$$T \circ \Pi_{l_1}(e_1) = \Pi_{l_2}(F(e_1)) \circ T, \text{ for } e_1 \in E_1. \tag{3.21}$$

Therefore, we have that for any pure spinor $\varphi \in \wedge^\bullet L_1'$,

$$F(\mathcal{N}_{l_1}(\varphi)) = \mathcal{N}_{l_2}(T(\varphi)).$$

What we shall do for push-forward and pull-back morphisms can be seen as an extension of this construction. In these cases, one cannot expect to have isomorphisms between Clifford modules and the problem of finding conditions to determine when the maps constructed are zero will be essential.

The case of pull-back morphisms was treated in the paper [1]. We recall their result here. Let V, W be vector spaces. For $f : V \to W$ a homomorphism, let Λ_f^t (see (2.23)) be the pull-back morphism.

Although Λ_f^t is not the graph of an isomorphism, we have an analog of formula (3.21) in this case. For any linear map $F : W \oplus W^* \to V \oplus V^*$ such that $\mathrm{Graph}\,(F) \subset \Lambda_f^t$, one has

$$\Pi(X + \xi) f^*\varphi = f^*(\Pi(F(X + \xi))\varphi),$$

for $\varphi \in \wedge^\bullet V^*$ and $X + \xi \in W \oplus W^*$. This follows from the formula

$$i_X f^*\varphi + f^*\eta \wedge f^*\varphi = f^*(i_{f(X)}\varphi + \eta \wedge \varphi). \tag{3.22}$$

This is sufficient to have a relation between pure spinors of a Lagrangian $L \subset W \oplus W^*$ and $\Lambda_f^t(L)$:

Proposition 3.30. *(A.B.M. [1]) Let $L \subset W \oplus W^*$ be a Lagrangian subspace. If $\varphi \in \wedge^\bullet W^*$ is a pure spinor such that $\mathcal{N}(\varphi) = L$, then $f^*\varphi \in \wedge^\bullet V^*$ is a pure spinor for $Cl(V \oplus V^*, g_{\mathrm{can}})$ as long as it is non-zero. In this case,*

$$\mathcal{N}(f^*\varphi) = \Lambda_f^t(L).$$

Proof. For $X+\xi \in V \oplus V^*$, by definition, $X+\xi \in \Lambda^t_f(L)$ if and only if $\xi = f^*\eta$ for some $\eta \in W^*$ and $f(X)+\eta \in L$. Assume that $f^*\varphi \neq 0$. If $X+\xi \in \Lambda^t_f(L)$, then

$$i_X f^*\varphi + \xi \wedge f^*\varphi = f^*(i_{f(X)}\varphi) + f^*(\eta \wedge \varphi) = f^*(i_{f(X)}\varphi + \eta \wedge \varphi) = 0.$$

This proves that $\Lambda^t_f(L) \subset \mathcal{N}(f^*\varphi)$. But $\Lambda^t_f(L)$ is maximal isotropic (by Proposition 2.22), so the equality holds. □

The problem now is to determine when $f^*\varphi \neq 0$ for a pure spinor φ.

Proposition 3.31. *(A.B.M. [1]) For a pure spinor $\varphi \in \wedge^\bullet V^*$, $f^*\varphi \neq 0$ if and only if $\mathcal{N}(\varphi) \cap \ker(f^*) = 0$.*

Proof. We recall the formula for φ given (3.13). By associating to $\mathcal{N}(\varphi)$ the pair (ω_S, S) given by (2.13), one has that

$$\varphi = e^{-B} \wedge e^{[1,s]},$$

where $\{e^1, \cdots, e^s\}$ is a basis of $\mathrm{Ann}\,(S) = \mathcal{N}(\varphi) \cap V^*$ and $B \in \wedge^2 V^*$ extends ω_S. Now,

$$f^*\varphi = e^{-f^*B} f^*e^1 \wedge \cdots \wedge f^*e^s = 0$$

if and only if $\{f^*e^1, \cdots, f^*e^s\}$ is a linearly dependent set. But the last condition is clearly equivalent to $\mathrm{Ann}\,(S) \cap \ker(f^*) \neq 0$ as we wanted to show. □

We now prove similar propositions for the push-forward morphism. In this case, we have to work with the polarizations (V^*, V) and (W^*, W). Recall that

$$f_* : \wedge^\bullet V \longrightarrow \wedge^\bullet W$$

is the natural extension of $f : V \to W$ as an exterior algebra homomorphism.

Proposition 3.32. *Let $L \subset V \oplus V^*$ be a Lagrangian subspace and let $\mathfrak{X} \in \wedge^\bullet V$ be a pure spinor for $Cl(V \oplus V^*, g_{\mathrm{can}})$ such that $\widehat{\mathcal{N}}(\mathfrak{X}) = L$. Then, if $f_*(\mathfrak{X}) \neq 0$, it is a pure spinor for $Cl(W \oplus W^*, g_{\mathrm{can}})$. Moreover,*

$$\widehat{\mathcal{N}}(f_*(\mathfrak{X}) = \Lambda_f(L).$$

Proof. For $X+\xi \in W \oplus W^*$, by definition $X+\xi \in \Lambda_f(L)$ if and only if $X = f(Y)$ for some $Y \in V$ and $Y + f^*\xi \in L$. Assume that $f_*(\mathfrak{X}) \neq 0$. If $X+\xi \in \Lambda^t_f(L)$, then

$$\widehat{\Pi}(X+\xi)f_*(\mathfrak{X}) = X \wedge f^*\mathfrak{X} + i_\xi f_*(\mathfrak{X}) = f(Y) \wedge \mathfrak{X} + f_*(i_{f^*\xi}\mathfrak{X}) = f_*(Y \wedge \mathfrak{X} + i_{f^*\xi}\mathfrak{X}) = 0.$$

This proves that $\Lambda_f(L) \subset \widehat{\mathcal{N}}(f_*\mathfrak{X})$. But $\Lambda_f(L)$ is maximal isotropic (by Proposition 2.22), so the equality holds. □

Proposition 3.33. *For a pure spinor $\mathfrak{X} \in \wedge^\bullet V$, $f_*(\mathfrak{X}) \neq 0$ if and only if $\widehat{\mathcal{N}}(\mathfrak{X}) \cap \ker(f) = 0$.*

Proof. We use again the representation of $\mathfrak{X}$ given by (3.13). It is

$$\mathfrak{X} = e^{-B} \wedge e_{[1,s]}$$

for some $B \in \wedge^2 V$ and $\{e_1, \ldots, e_s\}$ is a basis of $\widehat{\mathcal{N}}(\mathfrak{X}) \cap V$. Again,

$$f_*(e^{-B} e_{[1,s]}) = e^{-f_*(B)} f(e_1) \wedge \cdots \wedge f(e_s) = 0$$

if and only if $\{f(e_1), \ldots, f(e_s)\}$ is linearly dependent which is equivalent to

$$\ker(f) \cap \widehat{\mathcal{N}}(\mathfrak{X}) \cap V = \ker(f) \cap \widehat{\mathcal{N}}(\mathfrak{X}) \neq 0$$

□

Comparing Propositions 3.30 and 3.32, it is worth noting the difference between the polarizations used in the pull-back and push-forward cases. When composing pull-back with push-forward morphisms (e.g. the quotient morphism) it will be crucial to uniformize the polarizations in question. We will choose to work with canonical polarizations and their corresponding modules. Thus, it will be important to understand the push-forward transform in this setting.

Lemma 3.34. *Let $\nu_1 \in \det(V)$ and $\nu_2 \in \det(W^*)$ and consider the corresponding maps (see (3.17))*

$$\star_1 : \wedge^\bullet V^* \longrightarrow \wedge^\bullet V \text{ and } \star_2 : \wedge^\bullet W \longrightarrow \wedge^\bullet W^*.$$

If $\varphi \in \wedge^\bullet V^*$ is a pure spinor, then

$$\theta := \star_2 f_*(\star_1 \varphi) \in \wedge^\bullet W^* \tag{3.23}$$

is not zero if and only if $\mathcal{N}(\varphi) \cap \ker(f) = 0$. In this case, it is a pure spinor and moreover

$$\mathcal{N}(\theta) = \Lambda_f(\mathcal{N}(\varphi)).$$

Proof. As $\star_1$ is a Clifford module isomorphism, it follows that $\star_1\varphi \in \wedge^\bullet V$ is a pure spinor with respect to the action induced by the polarization (V^*, V). Moreover,

$$\widehat{\mathcal{N}}(\star_1\varphi) = \mathcal{N}(\varphi).$$

Therefore, Propositions 3.32 and 3.33 give that $f_*(\star_1\varphi) \neq 0$ if and only if $\mathcal{N}(\varphi) \cap \ker(f) = 0$, and in this case

$$\widehat{\mathcal{N}}(f_*(\star_1\varphi)) = \Lambda_f(\mathcal{N}(\varphi)).$$

Using once more that $\star_2$ is a Clifford module isomorphism gives that

$$\mathcal{N}(\star_2 f_*(\star_1\varphi)) = \widehat{\mathcal{N}}(f_*(\star_1\varphi)) = \Lambda_f(\mathcal{N}(\varphi)).$$

□

Let now (E, g, p) be an extension of V and $K \subset E$ be an isotropic subspace. Consider $L \subset E$ a Lagrangian subspace. We are now ready to relate pure spinors corresponding to L to those corresponding to $(L \cap K^\perp + K)/K$. First choose a K-admissible isotropic splitting ∇ for E and consider the induced isotropic splitting ∇_K of $K^\perp/K$ (see 2.30). These splittings induce representations

$$\Pi_\nabla : Cl(E, g) \longrightarrow \mathrm{End}\,(\wedge^\bullet V^*) \text{ and } \Pi_{\nabla_K} : Cl\left(\frac{K^\perp}{K}, g_K\right) \longrightarrow \mathrm{End}\left(\wedge^\bullet \left(\frac{Q}{R}\right)^*\right),$$

where $Q = p(K^\perp)$ and $R = p(K)$. Note that the problem is equivalent to finding $\varphi \in \wedge^\bullet(Q/R)^*$ such that

$$\mathcal{N}(\varphi) = \Lambda_q \circ \Lambda_j^t(\,\Phi_\nabla(L)).$$

Indeed, by Corollary 2.37 and Equation (3.15),

$$\mathcal{N}_{\nabla_K}(\varphi) = \Phi_{\nabla_K}^{-1}(\mathcal{N}(\varphi)) = \frac{L \cap K^\perp + K}{K}.$$

Theorem 3.35. *Let $j : Q \to V$ be the inclusion map and $q : Q \to Q/R$ the quotient map. Choose $\nu_1 \in \det(Q^*)$ and $\nu_2 \in \det(Q/R)$ and let $\star_1$, $\star_2$ be the corresponding maps. If $\varphi \in \wedge^\bullet V^*$ is a pure spinor, then*

$$\star_2 \circ q_* \circ \star_1(j^*\varphi) \in \wedge^\bullet \left(\frac{Q}{R}\right)^* \tag{3.24}$$

is not zero if and only if $\mathcal{N}(\varphi) \cap (R \oplus \mathrm{Ann}\,(Q)) = 0$. In this case, it is a pure spinor for $\Lambda_q \circ \Lambda_j^t(\mathcal{N}(\varphi))$.

Proof. We already know from Proposition 3.30 and 3.31 that $j^*\varphi \neq 0$ if and only if $\mathcal{N}(\varphi) \cap \mathrm{Ann}\,(Q) = 0$ and that in this case $\mathcal{N}(j^*\varphi) = \Lambda_j^t(\mathcal{N}(\varphi))$. Suppose that this is the case. For $X + \xi \in Q \oplus Q^*$,

$$\begin{aligned} X + \xi \in \mathcal{N}(j^*\varphi) \cap R \quad &\Longleftrightarrow \exists \eta \in V^* \text{ s.t. } q^*\eta = \xi = 0 \text{ and } j(X) + \eta \in \mathcal{N}(\varphi) \\ &\Longleftrightarrow \exists \eta \in V^* \text{ s.t. } j(X) + \eta \in \mathcal{N}(\varphi) \cap (R \oplus \mathrm{Ann}\,(Q)) \\ &\quad\ \text{and } q^*\eta = \xi. \end{aligned}$$

Thus

$$\mathcal{N}(\varphi) \cap (R \oplus \mathrm{Ann}\,(Q)) = 0 \Longleftrightarrow \mathcal{N}(j^*\varphi) \cap R = 0 \Longleftrightarrow \star_2 f_*(\star_1 j^*\varphi) \neq 0.$$

where the last implication follows by Lemma 3.34. Again, by Lemma 3.34, if $\star_2 f_*(\star_1 j^*\varphi) \neq 0$, then

$$\mathcal{N}(\star_2 f_*(\star_1 j^*\varphi)) = \Lambda_q(\mathcal{N}(j^*\varphi)) = \Lambda_q(\Lambda_j^t(\mathcal{N}(\varphi))) = \Lambda_q \circ \Lambda_j^t(\,\mathcal{N}(\varphi)).$$

□

Remark 3.36. Recall that, for a K-admissible splitting ∇,

$$\Phi_\nabla(K) = R \oplus \mathrm{Ann}\,(Q)\,.$$

Thus, the condition

$$\mathcal{N}(\varphi) \cap (R \oplus \mathrm{Ann}\,(Q)) = 0$$

is equivalent to

$$\mathcal{N}_\nabla(\varphi) \cap K = 0.$$

Remark 3.37. The map

$$\begin{array}{rccc} C: & \det(Q) \times \det((Q/R)^*) & \longrightarrow & \mathrm{End}\left(\wedge^\bullet Q^*, \wedge^\bullet (Q/R)^*\right) \\ & (\nu_1, \nu_2) & \longmapsto & \star_2 \circ q_* \circ \star_1. \end{array}$$

is bilinear and therefore induces a linear map

$$C : \det(Q) \otimes \det((Q/R)^*) \longrightarrow \mathrm{End}\left(\wedge^\bullet Q^*, \wedge^\bullet (Q/R)^*\right).$$

There is an isomorphism between $\det(Q) \otimes \det((Q/R)^*)$ and $\det(R)$ given by

$$\nu_1 \otimes \nu_2 \mapsto \delta := \star_1(q^* \nu_2).$$

Therefore, to realize $\star_2 \circ q_* \circ \star_1$ one only needs an element of $\delta \in \det(R)$; this is very useful in the manifold setting as we can drop orientability assumptions. Let us conclude this remark by giving the expression of $\star_2 \circ q_* \circ \star_1$ in terms of δ.

Let $\{\xi^1, \ldots, \xi^m\}$ be a basis of Q^* such that $\{\xi^1, \ldots, \xi^r\}$ generates $\mathrm{Ann}\,(R)$. Any element of $\wedge^\bullet Q^*$ is a sum of forms of the type

$$q^*\alpha \wedge \xi^I, \ I \subset (r, n].$$

We claim that $\star_2 \circ q_* \circ \star_1$ coincides with the map $C_\delta : \wedge^\bullet Q^* \to \wedge^\bullet (Q/R)^*$ defined by

$$C_\delta : q^*\alpha \wedge \xi^I \longmapsto \begin{cases} 0, & \text{if } I \neq (r, n] \\ (i_{\xi^I}\delta)\,\alpha, & \text{if } I = (r, n]\,. \end{cases} \tag{3.25}$$

To prove our claim, note that

$$\begin{aligned} \star_2 \circ q_* \circ \star_1 (q^*\alpha \wedge \xi^I) &= \star_2\, q_* \left[i_{q^*\alpha}(\star_1 \xi^I) \right] \\ &= \star_2\, i_\alpha (q_*(\star_1 \xi^I)) \\ &= \alpha \wedge \star_2\, q_*(\star_1 \xi^I), \end{aligned}$$

where we used in the first and third equality the fact that $\star_i$ intertwines interior product with exterior multiplitication for $i = 1, 2$. Now,

$$q_*(\star_1 \xi^I) \neq 0 \Leftrightarrow I = (r, n],$$

and in this case

$$\star_2\, q_*(\star_1 \xi^I) = i_{q_*(\star_1 \xi^I)} \nu_2 = i_{\star_1 \xi^I} q^* \nu_2 = i_{\xi^I} \star_1 q^* \nu_2 = i_{\xi^I} \delta.$$

Remark 3.38. Let $\alpha \in \wedge^\bullet Q^*$. For the map $C_\delta : \wedge^\bullet Q^* \to \wedge^\bullet (Q/R)^*$ defined by (3.25) corresponding to $\delta \in \det(R)$, one has

$$C_\delta(\alpha) \neq 0 \Longleftrightarrow i_{X_1 \wedge \cdots \wedge X_{m-r}}\, \alpha \neq 0,$$

where $\{X_1, \ldots, X_{m-r}\}$ is a basis of R.

Remark 3.39. Let (E, g, p) be a real extension of a real vector space V and consider its complexification $(E \otimes \mathbb{C}, g_{\mathbb{C}}, p \otimes \mathrm{id})$. Given an isotropic subspace $K \subset E$, let $K_{\mathbb{C}} = K \otimes \mathbb{C} \subset E \otimes \mathbb{C}$ be its complexification and consider the quotient extension

$$\left(\frac{K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}}} \cong \left(\frac{K^{\perp}}{K} \right) \otimes \mathbb{C},\ (g_K)_{\mathbb{C}},\ p_K \otimes \mathrm{id} \right)$$

of $(Q/R) \otimes \mathbb{C}$ where as usual $Q = p(K^{\perp})$ and $R = p(K)$. For a K-admissible splitting $\nabla : V \to E$, let $\nabla_K : Q/R \to K^{\perp}/K$ be the induced splitting (2.30). Then, by Remark 2.39 and Theorem 3.35 given a pure spinor $\varphi \in \wedge^\bullet V^* \otimes \mathbb{C}$

$$\mathcal{N}_{\nabla_K \otimes \mathrm{id}}(\star_2 \circ q_* \circ \star_1 (j^* \varphi)) = \frac{L \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}}{K_{\mathbb{C}}}$$

(in case $\mathcal{N}_{\nabla \otimes \mathrm{id}}(\varphi) \cap K_{\mathbb{C}} = 0$) where (by abuse of notation)

$$q_* : \wedge^\bullet Q \otimes \mathbb{C} \to \wedge^\bullet (Q/R) \otimes \mathbb{C} \ \text{ and } \ j^* : \wedge^\bullet V^* \otimes \mathbb{C} \to \wedge^\bullet Q^* \otimes \mathbb{C}$$

are the $\mathbb{C}$-linear extension of the respective ($\mathbb{R}$-linear) push-forward and pull-back maps and

$$\star_1 : \wedge^\bullet Q^* \otimes \mathbb{C} \to \wedge^\bullet Q \otimes \mathbb{C} \ \text{ and } \ \star_2 : \wedge^\bullet (Q/R) \otimes \mathbb{C} \to \wedge^\bullet (Q/R)^* \otimes \mathbb{C}$$

are the star maps corresponding to $\nu_1 \in \wedge^{\mathrm{top}} Q \otimes \mathbb{C}$ and $\nu_2 \in \wedge^{\mathrm{top}} R^* \otimes \mathbb{C}$ respectively.

By choosing real elements $\nu_1 \otimes 1 \in \wedge^{\mathrm{top}} Q \otimes \mathbb{C}$ and $\nu_2 \otimes 1 \in \wedge^{\mathrm{top}} (Q/R)^* \otimes \mathbb{C}$ for $\nu_1 \in \wedge^{\mathrm{top}} Q$ and $\nu_2 \in \wedge^{\mathrm{top}} (Q/R)^*$, the corresponding map

$$\star_2 \circ q_* \circ \star_1 : \wedge^\bullet Q^* \otimes \mathbb{C} \longrightarrow \wedge^\bullet (Q/R)^* \otimes \mathbb{C}$$

is just the $\mathbb{C}$-linear extension of

$$\wedge^\bullet Q^* \ni \alpha \longmapsto i_{q_*(i_\alpha \nu_1)}\, \nu_2 \in \wedge^\bullet (Q/R)^*.$$

3.2.2 Dealing with non-transversality.

When comparing the procedure to reduce Lagrangian subspaces of a given split-quadratic vector space (E, g) with an isotropic subspace $K \subset E$ given by Proposition 2.28 and its pure spinor counterpart, Theorem 3.35 (see also Remark 3.36), one notes a discrepancy: namely, the first is defined for all Lagrangian subspaces of E and the second, only for those Lagrangian which satisfies $L \cap K = 0$. To have a complete description of the reduction procedure at the spinorial level, it is necesary to understand how to construct the reduced spinor when $L \cap K \neq 0$. In this section, we deal with this problem.

Let (E, g) be a split-quadratic vector space and $K \subset E$ an isotropic subspace. Consider $L \subset E$ a Lagrangian subspace. The idea of our method is to substitute L by an other Lagrangian subspace L' which satisfies the following properties:

(i) $L = L'$ if and only if $L \cap K = 0$;

(ii) $L' \cap K = 0$;

(iii) $L' \cap K^\perp + K = L \cap K^\perp + K$ and

(iv) $L \to L'$ is computable at the spinor level.

Note that such substitution solves our problem. Indeed, property (iv) implies that once we have a pure spinor for L we can find one for L'. Now, property (ii) allows us to apply Theorem 3.35 for the pure spinor of L' to find a pure spinor corresponding to

$$\frac{L' \cap K^\perp + K}{K} = \frac{L \cap K^\perp + K}{K} \subset \frac{K^\perp}{K}$$

where the equality holds because of property (iii). Property (i) guarantees that one doesn't have to substitute L in the case where we already know how to deal with pure spinor reduction.

Let us explain the method. Take $D \subset E$ to be any isotropic subspace such that $(L \cap K)^\perp \oplus D = E$ (it exists by Proposition 2.7). Define

$$L' = L_D := L \cap D^\perp + D.$$

Proposition 3.40. *L_D satisfies (i),(ii) and (iii)*

Proof. First note that if $L \cap K = 0$, then D is necessarily 0 and therefore $L_D = L$. Conversely, if $L_D = L$, then $D \subset L$. But, by construction

$$0 = (L \cap K)^\perp \cap D = (L + K^\perp) \cap D \Rightarrow D \cap L = 0.$$

Hence, $D = 0$ and therefore $(L \cap K)^\perp = E$, or equivalently, $L \cap K = 0$. This proves that L_D satisfies property (i). To prove (ii) note that $(L \cap K)^\perp \cap D = 0$ implies $(L \cap K) \oplus D^\perp = E$; this in turn implies that

$$L = L \cap K \oplus L \cap D^\perp. \tag{3.26}$$

Thus,

$$\begin{aligned} E = D + (L \cap K)^\perp = D + L + K^\perp &\overset{(3.26)}{=} D + (L \cap D^\perp + L \cap K) + K^\perp \\ &= L_D + L \cap K + K^\perp \\ &= L_D + K^\perp. \end{aligned}$$

So $0 = (L_D + K^\perp)^\perp = L_D \cap K$ as we wanted.

To prove (iii), note that, by (3.26),

$$L + D = (L \cap D^\perp + D) + L \cap K = L_D + L \cap K,$$

which implies that $L_D + K = L + K + D$. Thus

$$L_D \cap K^\perp = (L_D + K)^\perp = (L + K + D)^\perp \subset L \cap K^\perp.$$

So

$$L_D \cap K^\perp + K \subset L \cap K^\perp + K,$$

and, as both subspaces are Lagrangians (see (2.26)), they are equal, thus proving (iii). □

To prove (iv) we specialize to the situation where (E, g) comes from an extension (E, g, p) of a vector space V. We choose an isotropic splitting $\nabla : V \to E$ so as to have a representation $\Pi_\nabla : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet V^*)$ (see Example 5.6).

Proposition 3.41. *If* $\varphi \in \wedge^\bullet V^*$ *is a pure spinor for* L*, then*

$$\varphi_D := \Pi_\nabla(d_1 \cdots d_r)\varphi \neq 0$$

and

$$\mathcal{N}_\nabla(\varphi_D) = L \cap D^\perp + D.$$

where $\{d_1, \ldots, d_r\} \subset E$ *is a basis of* D*.*

Proof. We are going to use a result which will be proved only ahead in greater generality (see Proposition 4.34) which states that there exists a Lagrangian subspace $L_0 \subset E$ such that

$$E = L \oplus L_0 \quad \text{and} \quad D \subset L_0.$$

As such, $l_1 = (L, L_0)$ defines a polarization of E which we can relate to $l_2 = (\nabla V, V^*)$ via Proposition 3.23: namely, the isomorphism

$$\begin{array}{rccc} F_{l_1 l_2}: & \wedge^\bullet L_0 & \longrightarrow & \wedge^\bullet V^* \\ & \alpha & \longmapsto & \Pi_\nabla(\alpha)\varphi \end{array}$$

intertwines Π_{l_1} with Π_∇. Consequently, if $0 \neq \theta \in \wedge^\bullet L_0$ is a pure spinor with $\mathcal{N}_{l_1}(\theta) = L \cap D^\perp + D$, then $F_{l_1 l_2}(\theta) \neq 0$ and

$$\mathcal{N}_\nabla(F_{l_1 l_2}(\theta)) = L \cap D^\perp + D.$$

Now, Proposition 3.18 says that for any basis $\{d_1, \ldots, d_r\}$ of D, the element $\theta = d_1 \wedge \cdots \wedge d_n \in \wedge^\bullet L_0$ is a pure spinor such that

$$\mathcal{N}_{l_1}(\theta) = L \cap D^\perp + D.$$

In this case

$$F_{l_1 l_2}(\theta) = \Pi_\nabla(d_1 \wedge \cdots \wedge d_n)\varphi = \varphi_D.$$

This concludes the proof. □

Example 3.42. Let $E = (V \oplus V^*, g_{\mathrm{can}}, \mathrm{pr}_V)$ for a vector space V. Let $R \subset V$ and $L \subset V \oplus V^*$ be a Lagrangian subspace. Choose $S \subset V$ such that

$$(L \cap R) \oplus S = V.$$

Then $D = \mathrm{Ann}\,(S) \subset V \oplus V^*$ satisfies

$$(L \cap R) \oplus D^{\perp} = V \oplus V^*.$$

If $\varphi \in \wedge^{\bullet} V^*$ is such that $\mathcal{N}(\varphi) = L$, then for $\Omega \in \det(\mathrm{Ann}\,(S)) \subset \wedge^{\bullet} V^*$

$$\varphi_D = \Omega \wedge \varphi$$

is a pure spinor for $L \cap D^{\perp} + D$ by Proposition 3.41.

Chapter 4

Reduction of generalized structures.

In this chapter, we recall the reduction procedure for Dirac structures from [11, 12]; its spinorial counterpart will be treated in the next chapter.

The chapter is organized as follows: in §4.1 and §4.2, we spend much of our time defining the main ingredients of the reduction procedure; they are a Lie group G acting on a Courant algebroid E over M by automorphisms (see Definition 4.1) and the choice of an invariant manifold $N \subset M$ given by the zero level set of a moment map (see §4.2.3). In §4.3, we show how to associated to these data a Courant algebroid E_{red} over $M_{red} = N/G$; the reduced Dirac structures will be subbundles of E_{red}. We focus on the construction of the bracket on the sections $\Gamma(E_{red})$ of E_{red} adapting the general construction of [11] to the more particular construction considered in [12] and used in this thesis. At last, we close this chapter with §4.4, where the reduction studied in §2.3. will be used pointwise to reduce a Dirac structure on M satisfying some natural conditions to a Dirac structure on M_{red}.

4.1 Generalized geometry.

Let M be a smooth manifold. In this section, we study a class of Courant algebroids (see Definition 4.1) called exact. In §4.1.2, we study the group of automorphism of exact Courant algebroids and its Lie algebra of derivations.

4.1.1 Courant algebroids.

Let M be a smooth manifold.

Definition 4.1 ([36]). A **Courant algebroid** over M is a real vector bundle $E \to M$ equipped with a fibrewise non-degenerate symmetric bilinear form g, a bilinear bracket $[\![,]\!]$ on the smooth sections $\Gamma(E)$, and a bundle map p :

$E \to TM$ called the **anchor**, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

(C1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,

(C2) $p([\![e_1, e_2]\!]) = [p(e_1), p(e_2)]$,

(C3) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\mathcal{L}_{p(e_1)}f)e_2$,

(C4) $\mathcal{L}_{p(e_1)}\, g(e_2, e_3) = g([\![e_1, e_2]\!], e_3) + g(e_2, [\![e_1, e_3]\!])$,

(C5) $[\![e_1, e_2]\!] = -[\![e_2, e_1]\!] + p^*(d\, g(e_1, e_2))$.

The main example of a Courant algebroid is $\mathbb{T}M := TM \oplus T^*M$ with $\mathrm{pr}_{TM} : \mathbb{T}M \to TM$ as anchor, the canonical bilinear symmetric form g_{can} given by

$$g_{\mathrm{can}}(X + \xi, Y + \eta) = i_X\eta + i_Y\xi, \text{ for } X + \xi, Y + \eta \in \Gamma(\mathbb{T}M)$$

and the bracket $[\![\cdot,\cdot]\!]$ given by

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - i_Y d\eta = [X, Y] + (\mathcal{L}_X\eta - \mathcal{L}_Y\xi) + di_Y\xi. \quad (4.1)$$

The bundle $\mathbb{T}M$ endowed with g_{can} and the bracket (4.1) was studied thoroughly by T. Courant in [17] where it was used to unify different kinds of geometry, including pre-symplectic, Poisson and the geometry of regular foliations. In the original paper [36], Courant algebroids were seen as a generalization of the double of Lie bialgebras, an important structure in Manin-Drinfeld's theory of Poisson-Lie groups. Recently, extending previous work of N.Hitchin [28], M. Gualtieri used this framework to study generalized complex structures [24] (see Definition 4.9). Here, we study a special class of Courant algebroids.

First, note that axiom (C5) implies that

$$[\![e, fe]\!] + [\![fe, e]\!] = p^*(d(f\, g(e, e))), \text{ for } e \in \Gamma(E) \text{ and } f \in C^\infty(M);$$

applying the anchor on both sides and using (C2), one has

$$0 = p \circ p^*(d(f\, g(e, e))).$$

We can always choose $e \in \Gamma(E)$ such that $g(e, e) = \pm 1$ in a neighborhood of M (see Lemma 5.1) so that

$$p \circ p^*(df) = 0, \text{ for every } f \in C^\infty(M).$$

Thus we have a complex

$$0 \longrightarrow T^*M \xrightarrow{p^*} E \xrightarrow{p} TM \longrightarrow 0. \quad (4.2)$$

Definition 4.2 ([45])**.** A Courant algebroid E is **exact** if for every $x \in M$, the triple $(E_x, g_x, p|_{E_x})$ is an extension of T_xM (see §2.1).

Example 4.3. For M a smooth manifold, the Courant algebroid $(\mathbb{T}M, g_{\text{can}}, \text{pr}_{TM}, [\![\cdot,\cdot]\!])$ with the bracket $[\![\cdot,\cdot]\!]$ given by (4.1) is exact. For every $x \in M$,

$$(T_xM \oplus T_x^*M, g_{\text{can}}, \text{pr}_{TM}) = \mathcal{D}\,(T_xM) \text{ (see Example 2.2).}$$

The theory developed in the first chapter applies pointwise to the case of exact Courant algebroids. For instance, an isotropic splitting for a Courant algebroid E is a bundle map $\nabla : TM \to E$ such that for every $x \in M$, $\nabla_x : T_xM \to E_x$ is an isotropic splitting for the extension $(E_x, g_x, p|_{E_x})$.

As before, we concentrate on subbundles L of E such that $L_x \subset E_x$ is a Lagrangian subspace; again, we will be considering T^*M as a Lagrangian subbundle of E via its image $p^*(T^*M)$.

Definition 4.4 ([17])**.** A **Dirac structure** on M is a Lagrangian subbundle L of E such that its sections $\Gamma(L)$ are involutive under the bracket $[\![\cdot,\cdot]\!]$ (in which case we call L **integrable**).

Example 4.5. For any exact Courant algebroid E, $T^*M \subset E$ defines a Dirac structure. Indeed, it is Lagrangian by Example 2.5. Also, for $\xi, \eta \in \Gamma(T^*M)$, axiom (C2) implies that

$$p([\![\xi,\eta]\!]) = [\![p(\xi), p(\eta)]\!] = 0.$$

As $T^*M = \ker(p)$, $[\![\xi,\eta]\!] \in \Gamma(T^*M)$. Actually, restricted to $\Gamma(T^*M)$, the bracket $[\![\cdot,\cdot]\!]$ is zero. Indeed, let $e \in \Gamma(E)$. By axiom (C4) and (C2) (recall the formula $g(\eta,\cdot) = \eta(p(\cdot))$),

$$\begin{aligned} g([\![\xi,\eta]\!], e) &= \mathcal{L}_{p(\xi)} g(\eta, e) - g(\eta, [\![\xi, e]\!]) \\ &= -\eta(p([\![\xi, e]\!])) \\ &= -\eta([\![p(\xi), p(e)]\!]) \\ &= 0. \end{aligned}$$

As g is non-degenerate, it follows that

$$[\![\xi,\eta]\!] = 0,\ \forall\, \xi,\ \eta \in \Gamma(T^*M). \tag{4.3}$$

Remark 4.6. The description (2.13) applies pointwise to Lagrangian subbundles of $\mathbb{T}M$. They are characterized by a singular distribution $S \subset TM$ and an element ω_S of $\Gamma(\wedge^2 S^*)$. The distribution S is integrable in the sense of Stefan and Sussman [49] and ω_S defines a 2-form on each leaf of the singular foliation. In [17], it is proved that the pair (S, ω_S) defines a Dirac subbundle if and only if $d\omega_S = 0$ over each leaf of the foliation.

Let E be an exact Courant algebroid and let $\nabla : TM \to E$ be an isotropic splitting. Its image $\nabla(TM)$ is a Lagrangian subbundle; it will be a Dirac structure if and only if

$$H(X,Y,Z) := g([\![\nabla X, \nabla Y]\!], \nabla Z) = 0, \quad \text{for every } X,\ Y,\ Z \in \Gamma(TM).$$

One can show that H is a closed 3-form on M (see [45]). Its cohomology class $[H] \in H^3(M,\mathbb{R})$ does not depend on the splitting. In fact, changing ∇ to $\nabla + B$ has the effect of changing H to $H + dB$, where $B \in \Omega^2(M)$. The 3-form H is called the **curvature** of the splitting ∇ and its cohomology class $[H] \in H^3(M,\mathbb{R})$ is called the **Ševera class** of E; as we shall see, it completely determines E.

Giving an isotropic splitting $\nabla : TM \to E$, one can use the fibrewise isomorphism given by (2.6) to construct a bundle isomorphism $\Phi_\nabla : E \to \mathbb{T}M$ given by

$$\Phi(e) = (p(e), s_\nabla(e)),$$

where $s_\nabla(e) \in T^*M$ is such that

$$p^* s_\nabla(e) = e - \nabla p(e). \tag{4.4}$$

Let us see how the bracket is transformed. By Axiom (C2) and (4.3), for $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^*M)$, one has that

$$\begin{aligned} \Phi_\nabla([\![\nabla X + \xi, \nabla Y + \eta]\!]) &= ([X,Y], s_\nabla([\![\nabla X + \xi, \nabla Y + \eta]\!])) \\ &= ([X,Y], s_\nabla([\![\nabla X, \nabla Y]\!]) + s_\nabla(\nabla X, \xi) + s_\nabla(\xi, \nabla Y). \end{aligned}$$

For $Z \in \Gamma(TM)$,

$$i_Z s_\nabla([\![\nabla X, \nabla Y]\!]) = g([\![\nabla X, \nabla Y]\!], \nabla Z) = H(X,Y,Z).$$

We claim that

$$s_\nabla([\![\nabla X, \eta]\!]) = \mathcal{L}_X \eta$$

and

$$s_\nabla([\![\xi, \nabla Y]\!]) = i_Y d\xi.$$

Indeed, from (2.5) and Axioms (C2) and (C4), one has that for $Z \in \Gamma(TM)$,

$$\begin{aligned} i_Z\, s_\nabla([\![\nabla X, \eta]\!]) = ([\![\nabla X, \eta]\!], \nabla Z) &= \mathcal{L}_X\, g(\eta, Z) - g(\eta, [\![\nabla X, \nabla Z]\!]) \\ &= \mathcal{L}_X i_Z \eta - i_{[X,Z]}\eta \\ &= i_Z \mathcal{L}_X \eta \end{aligned}$$

Also, by axiom (C5),

$$[\![\xi, \nabla Y]\!] = -[\![\nabla Y, \xi]\!] + dg(\xi, \nabla Y).$$

Therefore,

$$s_\nabla([\![\xi, \nabla Y]\!]) = -s_\nabla([\![\nabla Y, \xi]\!]) + d i_Y \xi = -\mathcal{L}_Y \xi + d i_Y \xi = -i_Y d\xi.$$

Finally,

$$\Phi_\nabla([\![\nabla X + \xi, \nabla Y + \eta]\!]) = ([X,Y], i_Y i_X H + \mathcal{L}_X \eta - i_Y d\xi). \tag{4.5}$$

Definition 4.7. For a closed 3-form $H \in \Omega^3(M)$, we call the bracket on the sections of $\mathbb{T}M$ given by

$$[\![X+\xi, Y+\eta]\!]_H = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H \tag{4.6}$$

the *H-twisted Courant bracket* [45].

Example 4.8. Let $\omega \in \Omega^2(M)$ be a 2-form on M and $H \in \Omega^3(M)$ be a closed 3-form. Let

$$L = \text{Graph}\,(\omega) = \{(X, i_X\omega) \in \mathbb{T}M \mid X \in TM)\}.$$

It is a Lagrangian subbundle of $(\mathbb{T}M, g_{\text{can}})$. Its sections $\Gamma(L)$ are involutive with respect to the H-twisted Courant bracket if and only if $d\omega = -H$.

Using Φ_∇ to transport Lagrangian subbundles of E to Lagrangian subbundles of $\mathbb{T}M$ reduces the problem of understanding the integrability condition on the definition of Dirac subbundles to that for H-twisted brackets. In this case, an argument similar to that of [17] shows that (see Example 4.6)

$$(S, \omega_S) \text{ is } [\![\cdot,\cdot]\!]_H \text{ involutive if and only if } d\omega_S = j^*H,$$

where $j : \mathcal{S} \to M$ is the immersion of the leaf of the foliation tangent to S.

Generalized complex structures. Let E be a Courant algebroid over M and consider its complexification $E_\mathbb{C} = E \otimes \mathbb{C}$. By extending $\mathbb{C}$-bilinearly both the metric and the Courant bracket, we can study the Lagrangian subbundles of $E_\mathbb{C}$ whose sections are closed under $[\![\cdot,\cdot]\!]_\mathbb{C}$. We call such Lagrangian subspaces **complex Dirac structures** on M.

Definition 4.9 (M. Gualtieri [24])**.** A generalized complex structure on M is a complex Dirac structure $L \subset E_\mathbb{C}$ such that

$$L \cap \overline{L} = 0, \tag{4.7}$$

where $\overline{L}$ is the conjugate subbundle. A general Lagrangian subbundle $L \subset E_\mathbb{C}$ such that (4.7) holds is called a **generalized almost complex structure**.

To a generalized almost complex structure $L \subset E_\mathbb{C}$ on M, one can associate (see [19, 24]) a bundle map

$$\mathcal{J} : E \longrightarrow E$$

such that $\mathcal{J}^2 = -Id$ and $g(\mathcal{J}\cdot, \mathcal{J}\cdot) = g(\cdot,\cdot)$ such that

$$L = \{e - i\mathcal{J}e \mid e \in E\}.$$

The integrability of L translates to

$$[\![\mathcal{J}e_1, \mathcal{J}e_2]\!] - [\![e_1, e_2]\!] - \mathcal{J}([\![\mathcal{J}e_1, e_2]\!] + [\![e_1, \mathcal{J}e_2]\!]) = 0, \;\; \forall\, e_1,\, e_2 \in \Gamma(E). \tag{4.8}$$

By choosing an isotropic splitting ∇, one has (in matrix notation)

$$\Phi_\nabla \circ \mathcal{J} \circ \Phi_\nabla^{-1} = \begin{pmatrix} a & \pi^\sharp \\ \omega_\sharp & -a^* \end{pmatrix}$$

where $\pi^\sharp : T^*M \to TM$ is the map induced by a bivector field π, $\omega_\sharp : TM \to T^*M$ is the map induced by a 2-form ω and $a : TM \to TM$ is a bundle map. The generalized Nijenhuis equation (4.8) implies certain compatibilities between these structure maps(see [19] for details). One has

$$\Phi_\nabla(L) = \{(X - i(a(X) + \pi^\sharp(\xi)),\, \xi - i(\omega_\sharp(X) - a^*(\xi))) \mid (X, \xi) \in \Gamma(TM \oplus T^*M)\}$$

In the next two examples we consider the Courant algebroid given by $\mathbb{T}M$ with the standard Courant bracket (4.1).

Example 4.10 ([24])**.** Let $J : TM \to TM$ be an almost complex structure on M. By defining

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix},$$

we get generalized almost complex structure on M given by

$$\begin{aligned} L &= \{(X, \xi) - i\,\mathcal{J}(X, \xi) \mid (X, \xi) \in \Gamma(\mathbb{T}M)\} \\ &= \{(X - iJX,\, \xi + iJ^*\xi) \mid (X, \xi) \in \Gamma(\mathbb{T}M)\} \end{aligned}$$

In the usual notation from complex geometry, one has

$$L = T_{1,0} \oplus \mathrm{Ann}\,(T_{1,0}) = T_{1,0} \oplus T^{0,1}.$$

By taking vector fields $e_1 = X$ and $e_2 = Y$ in (4.8), one obtains the usual condition of integrability of J given by the vanishing of the Ninjehuis tensor.

Example 4.11 ([24])**.** Given a non-degenerate 2-form ω on M, define

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega_\sharp^{-1} \\ \omega_\sharp & 0 \end{pmatrix}$$

It defines a generalized almost complex structure on M given by

$$L = \{X - i\omega_\sharp(X) \mid X \in \Gamma(TM \otimes \mathbb{C})\}.$$

L is integrable if and only if $d\omega = 0$.

4.1.2 Symmetries of the Courant bracket.

Let E be an exact Courant algebroid. We study the group $\mathrm{Aut}(E)$ of bundle automorphism preserving the underlying structures and its Lie algebra $\mathrm{Der}(E)$. We follow [11].

Definition 4.12. The automorphism group $\mathrm{Aut}(E)$ of a Courant algebroid E is the group of pairs (Ψ, ψ), where $\Psi : E \to E$ is a bundle automorphism covering $\psi \in \mathrm{Diff}(M)$ such that

(1) $\psi^* g(\Psi(\cdot), \Psi(\cdot)) = g(\cdot, \cdot)$

(2) $[\![\Psi(\cdot), \Psi(\cdot)]\!] = \Psi[\![\cdot, \cdot]\!]$;

(3) $p \circ \Psi = \psi_* \circ p$.

Remark 4.13. It can be seen that axiom (C2) and both the properties (1) and (2) above imply (3) (see [30]).

Let us give some examples of elements of $\mathrm{Aut}(E)$ for $E = (\mathbb{T}M, [\![\cdot, \cdot]\!]_H)$.

Example 4.14. For $\psi \in \mathrm{Diff}(M)$, define

$$\Psi_\psi = \psi_* + (\psi^*)^{-1},$$

which is easily seen to preserve g_{can}. For $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^*M)$, one has that

(i) $[\psi_* X, \psi_* Y] = \psi_*[X, Y]$;

(ii) $\mathcal{L}_{\psi_* X}(\psi^*)^{-1}\eta = (\psi^*)^{-1}\mathcal{L}_X \eta$;

(iii) $i_{\psi_* Y} d(\psi^*)^{-1}\xi = (\psi^*)^{-1} i_Y \xi$ and

(iv) $i_{\psi_* Y} i_{\psi_*} H = (\psi^*)^{-1} i_Y i_X\, \psi^* H$.

Thus,

$$[\![\Psi_\psi(X + \xi), \Psi_\psi(Y + \eta)]\!]_H = \Psi_\psi([\![X + \xi, Y + \eta]\!]_{\psi^* H})$$

which implies that for every $\psi \in \mathrm{Diff}(M)$ such that $\psi^* H = H$, $\Psi_\psi \in \mathrm{Aut}(E)$.

Example 4.15. Let $B \in \Omega^2(M)$ and let $\tau_B : \mathbb{T}M \to \mathbb{T}M$ be the bundle map which is fibrewise given by (2.9), that is, for $X + \xi \in \Gamma(\mathbb{T}M)$

$$\tau_B(X + \xi) = X + i_X B + \xi.$$

We saw that τ_B preserves g_{can}. For $X + \xi, Y + \eta \in \Gamma(\mathbb{T}M)$,

$$\begin{aligned} [\![\tau_B(X + \xi), \tau_B(Y + \eta)]\!]_H = &\ [X, Y] + \mathcal{L}_X(\eta + i_Y B) - i_Y d(\xi + i_X B) + i_Y i_X H \\ = &\ [X, Y] + \mathcal{L}_X \eta - i_Y d\xi + i_{[X,Y]} B - i_Y i_X (H + dB) \\ = &\ \tau_B([\![X + \xi, Y + \eta]\!]_{H + dB}). \end{aligned}$$

Therefore, for any closed 2-form B, one has that $\tau_B \in \mathrm{Aut}(E)$.

Proposition 4.16 ([11])**.** *Let E be a Courant algebroid and $\nabla : TM \to E$ an isotropic splitting. For $(\Psi, \psi) \in \mathrm{Aut}(E)$, there exists $B \in \Omega^2(M)$ such that*

$$\Phi_\nabla \circ \Psi \circ \Phi_\nabla^{-1} = \Psi_\psi \circ \tau_B. \tag{4.9}$$

Moreover, if $H \in \Omega^3(M)$ is the curvature of ∇,

$$H - \psi^* H = dB, \tag{4.10}$$

which implies that $\psi \in \mathrm{Diff}_{[H]}(M)$, the group of diffeomorphisms preserving the cohomology class of H.

Proof. First note that condition (3) in the definition of $\mathrm{Aut}(E)$ gives that Ψ preserves $T^*M = \ker(p)$. More precisely, for $\xi \in \Gamma(T^*M)$ and $X \in \Gamma(TM)$

$$g(\Psi(\xi), Y) = g(\xi, \Psi^{-1}(Y)) = \xi(p(\Psi^{-1}(Y))) = \xi((\psi^{-1})_* p(Y)) = (\psi^{-1})^*\xi(Y).$$

This proves that

$$\Psi(\xi) = (\psi^{-1})^*\xi, \ \forall \xi \in \Gamma(T^*M). \tag{4.11}$$

For $X + \xi \in \Gamma(\mathbb{T}M)$,

$$\Phi_\nabla \circ \Psi \circ \Phi_\nabla^{-1}(X+\xi) = \psi_* X + \left[(\psi^{-1})^*\xi + \Psi(\nabla X) - \nabla\psi_* X\right]$$

Now, note that ψ^*E together with $(\psi^{-1})_* \circ p : \psi^*E \to TM$ and the induced bilinear form g is a bundle of extensions of TM in the sense that $(\psi^*E)_x$ is an extension of T_x^*M for every $x \in M$. Moreover, $\Psi \circ \nabla$ and $\nabla \circ \psi_*$ are isotropic splittings for ψ^*E. Thus, by proposition 2.14, there exists $B \in \Omega^2(M)$ such that

$$\Psi(\nabla X) - \nabla \psi_* X = i_X B.$$

This proves that

$$\Phi_\nabla \circ \Psi \circ \Phi_\nabla^{-1} = \Psi_\psi \circ \tau_B.$$

To finish the proof, note that Ψ satisfies property (2) in the definition of $\mathrm{Aut}(E)$ if and only if $\Phi_\nabla \circ \Psi \circ \Phi_\nabla^{-1}$ preserves the H-twisted Courant bracket. But by Examples 4.14 and 4.15,

$$[\![\Psi_\psi \circ \tau_B(\cdot), \Psi_\psi \circ \tau_B(\cdot)]\!]_H = \Psi([\![\cdot,\cdot]\!]_{\psi^* H + dB}).$$

This finishes the proof. □

With an isotropic splitting ∇ chosen, an element of $\mathrm{Aut}(E)$ can be seen as

$$(\psi, B) \in \mathrm{Diff}_{[H]}(M) \times \Omega^2(M) \text{ such that } H - \psi^* H = dB,$$

where $H \in \Omega^3(M)$ is the curvature of the splitting. This description of elements of $\mathrm{Aut}(E)$ depends on the chosen splitting. To see how the 2-form appearing on (4.9) behaves if the splitting changes, let $B' \in \Omega^2(M)$. By formula (2.10),

$$\Phi_{\nabla+B'} \circ \Psi \circ \Phi_{\nabla+B'}^{-1} = \tau_{-B'} \circ (\Psi_\psi \circ \tau_B) \circ \tau_{B'} = \Psi_\psi \circ \tau_{B' - \psi^* B' + B}.$$

Thus, if we change ∇ to $\nabla + B'$, then

$$(\psi, B) \mapsto (\psi, B' - \psi^* B' + B). \tag{4.12}$$

Remark 4.17. It is shown in [11] that $\mathrm{Aut}(E)$ is an abelian extension of $\mathrm{Diff}(M)$.

The Lie algebra of the group $\mathrm{Aut}(E)$ is $\mathrm{Der}(E)$, the infinitesimal symmetries of E. An element of $\mathrm{Der}(E)$ is a pair (A, X) where $A : \Gamma(E) \to \Gamma(E)$ and $X \in \Gamma(TM)$ such that, for every $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,

(i) $A(fe_1) = fA(e_1) + (\mathcal{L}_X f)e_1$;

(ii) $g(A(e_1), e_2) + g(e_1, A(e_2)) = \mathcal{L}_X g(e_1, e_2)$;

(iii) $A([\![e_1, e_2]\!]) = [\![A(e_1), e_2]\!] + [\![e_1, A(e_2)]\!]$

(iv) $p(A(e_1)) = [X, p(e_1)]$.

Remark 4.18. As before, (iv) follows from (i),(ii) and (iii) together with the axioms (C1), ..., (C5). Indeed, for $\xi \in \Gamma(T^*M)$ and $e \in \Gamma(E)$

$$\begin{array}{rcl} \xi(p(A(e))) = & g(A(e), p^*\xi) = & \mathcal{L}_X g(e, p^*\xi) - g(e, A(p^*\xi)) \\ . & = & (\mathcal{L}_X\xi)(p(e)) + \xi([X, p(e)]) - g(e, A(p^*e)). \end{array}$$

Now, axiom (C5) together with (iii) and (i) implies that $A(p^*\xi) = p^*\mathcal{L}_X\xi$, and thus

$$\xi(p(A(e))) = \xi([X, p(e)]),\ \forall\, \xi \in \Gamma(T^*M).$$

Example 4.19 ([11])**.** For any $e \in \Gamma(E)$, let $A : \Gamma(E) \to \Gamma(E)$ be given by $A = [\![e, \cdot]\!]$ and $X = p(e) \in \Gamma(TM)$. Properties (i), (ii), (iii) and (iv) above correspond exactly to axioms (C3), (C4) , (C1) and (C2) in Definition 4.1. Infinitesimal symmetries of this kind are called **inner symmetries**.

Let $\nabla : TM \to E$ be an isotropic splitting for E with curvature $H \in \Omega^3(M)$. By the description of $\mathrm{Aut}(TM \oplus T^*M, [\![\cdot,\cdot]\!]_H)$ given by Proposition 4.16, an infinitesimal symmetry is given by a pair $(X, B) \in \Gamma(TM) \times \Omega^2(M)$ given by differentiation of an one-parameter subgroup $(\psi_t, B_t) \in \mathrm{Aut}(E)$ at $t = 0$. Equation (4.10) gives that

$$\mathcal{L}_X H = -dB.$$

By changing the splitting ∇ via a 2-form $B' \in \Omega^2(M)$ to $\nabla + B'$, the one-parameter subgroup of $\mathrm{Aut}(E)$ has to change according to (4.12):

$$(\psi_t, B_t) \mapsto (\psi_t, B_t + B' - \psi_t^* B').$$

By differentiation, it follows that the corresponding infinitesimal symmetry changes by

$$(X, B) \mapsto (X, B - \mathcal{L}_X B'). \tag{4.13}$$

Let us see which map $A : \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ corresponds to $(X, B) \in \mathrm{Der}(E)$. For $Y + \eta \in \Gamma(\mathbb{T}M)$,

$$\begin{array}{rcl} \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} (\psi_{-t})_* & 0 \\ 0 & (\psi_t^*) \end{pmatrix} \circ \tau_{B_{-t}}(Y + \eta) & = & \frac{d}{dt}\Big|_{t=0} [\,(\psi_{-t})_*Y + (\psi_t)^*(\eta + i_Y B_{-t})\,] \\ & = & [X, Y] + \mathcal{L}_X\eta - i_Y B. \end{array}$$

Thus, the map $A : \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ corresponding to (X, B) is

$$A(Y + \eta) = [X, Y] + \mathcal{L}_X \eta - i_Y B.$$

For inner symmetries, note that by writing

$$[\![X + \xi, Y + \eta]\!]_H = [X, Y] + \mathcal{L}_X \eta - i_Y (d\xi - i_X H),$$

it is evident that $(X, d\xi - i_X H) \in \Gamma(TM) \times \Omega^2(M)$ is the pair associated to the inner symmetry given by $X + \xi$. Define

$$\begin{array}{rccl} ad: & \Gamma(\mathbb{T}M) & \longrightarrow & \mathrm{Der}(\mathbb{T}M, [\![\cdot,\cdot]\!]_H) \\ & X + \xi & \longmapsto & (X, d\xi - i_X H). \end{array} \tag{4.14}$$

For every $(X, B) \in \mathrm{Der}(E)$ (as $d(i_X H + B) = \mathcal{L}_X H + dB = 0$), there is an associated cohomology class $[i_X H + B] \in H^2(M, \mathbb{R})$. This class is exact if and only if there exists $\xi \in \Omega^1(M)$ such that $B = d\xi - i_X H$. Thus, the inner simmetries are exactly the kernel of the surjective map

$$\mathrm{Der}(E) \ni (X, B) \longrightarrow [i_X H + B] \in H^2(M, \mathbb{R}),$$

showing, in particular, that ad is not always surjective. It is not injective either; indeed if $ad(X + \xi) = 0$, then in particular

$$[\![X + \xi, Y + \eta]\!]_H = 0 \text{ for every } Y + \eta \in \Gamma(\mathbb{T}M).$$

By choosing $Y = 0$ and $\eta = df$ for any function $f \in C^\infty(M)$, one has that $df(X) = 0$ for every $f \in C^\infty(M)$. This implies that $X = 0$. By choosing $\eta = 0$, one has that $i_Y d\xi = 0$ for every $Y \in \Gamma(TM)$ which implies that $d\xi = 0$ and gives that the sequence

$$0 \longrightarrow \Omega^1_{cl}(M) \xrightarrow{p^*} \Gamma(E) \xrightarrow{ad} \mathrm{Der}(E) \xrightarrow{\chi} H^2(M, \mathbb{R}) \longrightarrow 0 \tag{4.15}$$

is exact.

4.2 Actions on Courant algebroids.

Let M be a smooth manifold and G a Lie group acting on M. In this section, given a Courant algebroid E over M, we review the constructions given in [11, 12] to lift the G action to E.

4.2.1 Extended actions.

Let M be a smooth manifold and E be a Courant algebroid over M. Consider G a connected, compact Lie group acting on a manifold M by

$$G \ni g \mapsto \psi_g \in \mathrm{Diff}(M)$$

and let

$$\begin{array}{rccc} \Sigma: & \mathfrak{g} & \longrightarrow & \mathfrak{X}(M) \\ & u & \longmapsto & u_M \end{array}$$

be the infinitesimal action. We will be interested in lifting the action of G to E by automorphisms. Infinitesimally, this amounts to finding an homomorphism $\mathfrak{g} \to \mathrm{Der}(E)$ such that

$$\begin{array}{ccccc} \mathfrak{g} & \longrightarrow & \mathrm{Der}(E) & & (X, B) \\ & \overset{\Sigma}{\searrow} & \downarrow & & \downarrow \\ & & \mathfrak{X}(M) & & X \end{array}$$

commutes. We will be mainly interested in actions by inner symmetries:

$$\mathfrak{g} \longrightarrow \Gamma(E) \xrightarrow{\mathrm{ad}} \mathrm{Der}(E).$$

Example 4.20. If $\psi_g^* H = H$ for every $g \in G$, one can trivially lift the action by (see Example 4.14)

$$g \mapsto \Psi_g = \begin{pmatrix} (\psi_g)_* & 0 \\ 0 & (\psi_{-g})^* \end{pmatrix}.$$

Infinitesimally, by considering the natural inclusion $\Gamma(TM) \subset \Gamma(\mathbb{T}M)$, one has that

$$\mathfrak{g} \xrightarrow{\Sigma} \Gamma(\mathbb{T}M) \xrightarrow{\mathrm{ad}} \mathrm{Der}(E).$$

To consider only maps $\mathfrak{g} \to \Gamma(E)$ is insufficient if one thinks of TM and T^*M on equal footing. Even conceptually, the Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and $(\Gamma(E), [\![\cdot,\cdot]\!])$ are distinct as axiom (C5) says that $[\![\cdot,\cdot]\!]$ is not antisymmetric. We shall follow [11] to define a certain structure which in a sense allows one to treat tangent and cotangent directions equally.

Definition 4.21 ([11])**.** A **Courant algebra** over a Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ endowed with a bilinear bracket $[\![\cdot,\cdot]\!] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$, a map $\rho : \mathfrak{a} \to \mathfrak{g}$ such that

$$[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!], \quad a_1, a_2, a_3 \in \mathfrak{a} \tag{4.16}$$

and $\rho([\![a_1, a_2]\!]) = [\rho(a_1), \rho(a_2)]$ for all $a_1, a_2 \in \mathfrak{a}$.

A Courant algebra is **exact** if ρ is surjective and $[\![h_1, h_2]\!] = 0$ for all $h_1, h_2 \in \ker(\rho)$.

Example 4.22. For any Courant algebroid E over M, by considering the natural extension of the map $p : E \to TM$ to sections $p : \Gamma(E) \to \Gamma(TM)$, axioms (C1) and (C2) give that $\Gamma(E)$ is a Courant algebra over $\Gamma(TM)$. If E is exact, then by definition $\ker(p) = \Gamma(T^*M)$ and as we saw (Example 4.5)

$$[\![\Gamma(T^*M), \Gamma(T^*M)]\!] = 0.$$

Thus $\Gamma(E)$ is an exact Courant algebra over $\Gamma(TM)$ if and only if E is an exact Courant algebroid.

Example 4.23 (Hemisemidirect product [11, 31]). Let $\mathfrak{h}$ be a $\mathfrak{g}$-module. We can endow $\mathfrak{a} := \mathfrak{g} \oplus \mathfrak{h}$ with an exact Courant algebra structure by taking $\rho : \mathfrak{a} \to \mathfrak{g}$ to be the natural projection and the bracket defined by

$$[\![(u_1, h_1), (u_2, h_2)]\!] = ([u_1, u_2], u_1 \cdot h_2).$$

It is straightforward to see that ρ is surjective and $[\![\ker(\rho), \ker(\rho)]\!] = 0$. As for condition (4.16), the Jacobi identity for $\mathfrak{g}$ and the module structure of $\mathfrak{h}$ imply it. This construction is called **hemisemidirect product**. It was first studied in [31], in the context of Leibniz algebras.

Now we are able to define an extended G-action and its infinitesimal counterpart, an extended $\mathfrak{g}$-action.

Definition 4.24 ([11]). Let $\mathfrak{g}$ be a Lie algebra acting on M infinitesimally by $\Sigma : \mathfrak{g} \to \Gamma(TM)$. An **extended $\mathfrak{g}$-action** on an exact Courant algebroid E over M is given by an exact Courant algebra $\rho : \mathfrak{a} \to \mathfrak{g}$ together with a bracket preserving map linear map $\chi : \mathfrak{a} \to \Gamma(E)$ which satisfies two conditions

(i) $ad \circ \chi|_{\ker(\rho)} = 0$ and

(ii) $\Sigma \circ \rho = p \circ \chi$.

The first condition together with the exact sequence (4.15) give that the image of $\chi|_{\ker(\rho)}$ is a subset of the closed 1-forms $\Omega^1_{cl}(M)$ of M. Moreover, by quotienting out $\ker(\rho)$, one has that $\mathrm{ad}\circ\chi$ descends to the quotient $\mathfrak{g} = \mathfrak{a}/\ker(\rho)$ giving a Lie algebra homomorphism

$$\widetilde{ad \circ \chi} : \mathfrak{g} \to \mathrm{Der}(E) \tag{4.17}$$

as we first wanted. Altogether, these two condition guarantees the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\rho) & \longrightarrow & \mathfrak{a} & \xrightarrow{\rho} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi|_{\ker(\rho)}} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \Sigma} & & \\ 0 & \longrightarrow & \Omega^1_{cl}(M) & \longrightarrow & \Gamma(E) & \xrightarrow{p} & \mathfrak{X}(M) & \longrightarrow & 0 \end{array} \tag{4.18}$$

An **extended G-action** is an extended $\mathfrak{g}$-action (where $\mathfrak{g} = Lie(G)$) such that the induced infinitesimal action (4.17) integrates to a group homomorphism $G \to \mathrm{Aut}(E)$.

Let us give two extreme examples to illustrate our idea of treating TM and T^*M equally.

Example 4.25. Let $F : M \to \mathbb{R}^n$, $F = (f_1, \ldots, f_n)$ be a submersion. Let $\mathfrak{g} = 0$ and $\Sigma : \mathfrak{g} \to \Gamma(TM)$ identically zero. By considering $\mathbb{R}^n$ as the trivial $\mathfrak{g}$-module, one can form the hemisemidirect product $\mathfrak{a} = \{0\} \oplus \mathbb{R}^n$(the bracket is zero). Define, for the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$

$$\begin{array}{rccc} \chi : & \mathfrak{a} & \longrightarrow & \Gamma(\mathbb{T}M) \\ & (0, e_i) & \longmapsto & df_i \end{array}$$

Fix any closed $H \in \Omega^3(M)$ and endow $E = \mathbb{T}M$ with the H-twisted Courant bracket $[\![\cdot,\cdot]\!]_H$. As $[\![df_i,\cdot]\!]_H = 0$, χ is bracket-preserving. and $ad \circ \chi = 0$. Condition (ii) is trivial. Therefore, $\chi : \mathfrak{a} \to \Gamma(E)$ is an extended $\{0\}$-action (it integrates to the identity group $G = \{e\}$ giving an extended G-action). This action is purely cotangent.

Example 4.26 ([11])**.** Let $E = \mathbb{T}M$ with the standard Courant bracket (4.1) and $\Sigma : \mathfrak{g} \to \Gamma(TM)$ an infinitesimal action of the Lie algebra $\mathfrak{g}$ on M. Let $\mathfrak{a} = \mathfrak{g}$ with $\rho : \mathfrak{g} \to \mathfrak{g}$ the identity map. We can trivially extend the action by choosing χ equal to the natural extension $\Sigma : \mathfrak{g} \to \Gamma(\mathbb{T}M)$. For $u, v \in \mathfrak{g}$,

$$\chi([u,v]) = \Sigma([u,v]) = [\Sigma(u),\Sigma(v)] = [\![\Sigma(u),\Sigma(v)]\!] \text{ see (4.1).}$$

Also, as $\ker(\rho) = 0$, the condition $ad \circ \chi|_{\ker(\rho)} = 0$ is satisfied trivially. At last,

$$p \circ \chi = \Sigma = \Sigma \circ \rho.$$

When Σ integrates to a G action ($G \ni g \mapsto \psi_g \in \mathrm{Diff}(M)$), then this trivially extended $\mathfrak{g}$-action integrates to an extended G-action given by (4.14)

$$\Psi_g = (\psi_g)_* + (\psi_g^{-1})^* \in \mathrm{Aut}(E).$$

This kind of action is purely tangent.

Example 4.27 ([11])**.** Let M be a smooth manifold and let $\omega \in \Omega^2(M)$ be a closed two form. Let $\Sigma : \mathfrak{g} \to \Gamma(TM)$ be an infinitesimal symplectic action (i.e. $\mathcal{L}_{u_M}\,\omega = 0$). Define

$$\begin{array}{rccc} \chi : & \mathfrak{g} \oplus \mathfrak{g} & \longrightarrow & \Gamma(\mathbb{T}M) \\ & (u,v) & \longmapsto & u_M + i_{v_M}\omega. \end{array}$$

We claim that χ is a $\mathfrak{g}$-extended action if we give $\mathbb{T}M$ the standard Courant bracket (4.1) and $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{g}$ the hemisemidirect structure given by the $\mathfrak{g}$-module structure on $\mathfrak{g}$ inherited by the bracket ($u \cdot v = [u,v]$, for $u, v \in \mathfrak{g}$).

$$\begin{array}{rcl} [\![\chi(u_1,v_1),\chi(u_2,v_2)]\!] = & [(u_1)_M,(u_2)_M] + \mathcal{L}_{(u_1)_M}\, i_{(v_2)_M}\,\omega - i_{(u_2)_M} d i_{(u_1)_M}\omega \\ = & [u_1,u_2]_M + i_{[u_1,v_2]_M}\omega, \\ = & \chi([(u_1,v_1),(u_2,v_2)]) \end{array}$$

where we used that

$$\mathcal{L}_{(u_1)_M}\, i_{(v_2)_M}\,\omega = i_{(v_2)_M}\,\mathcal{L}_{(u_1)_M}\,\omega + i_{[u_1,v_2]_M}\omega = i_{[u_1,v_2]_M}\omega.$$

and

$$i_{(u_2)_M} d i_{(u_1)_M}\omega = i_{(u_2)_M}\mathcal{L}_{(u_1)_M}\omega - i_{(u_2)_M} d\omega = 0$$

by hypothesis. Also, as $\{(0,v) \in \mathfrak{a} \mid v \in \mathfrak{g}\} \subset \mathfrak{a}$ is the kernel of the projection on the first factor and

$$\chi(0,v) = i_{v_M}\omega$$

is a closed 1-form, the map χ satisfies condition (i) on the definition of extended action. As for the second condition, it is trivially satisfied. Therefore, χ is an $\mathfrak{g}$-extended action as we wanted.

4.2.2 Lifted actions.

In this subsection, we focus on a particularly important kind of extended action called isotropic lifted actions. Associated to an isotropic lifted action, there is an isotropic subbundle $K_{\mathfrak{g}}$ of E. We study isotropic splittings $\nabla : TM \to E$ adapted to such actions. One of the properties of these splitting is $K_{\mathfrak{g}}$-admissibility (see Definition 2.32). Our main result is the existence of such splittings (see Corollary 4.35).

Following [11, 12], we shall call a $\mathfrak{g}$-extended actions $\chi : \mathfrak{a} \to \Gamma(E)$ such that $\mathfrak{a} = \mathfrak{g}$ and $\rho : \mathfrak{g} \to \mathfrak{g}$ is the identity map **lifted $\mathfrak{g}$-actions**. The definition is summarized by the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{id} & \mathfrak{g} \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \Sigma} \\ \Gamma(E) & \xrightarrow{p} & \Gamma(TM) \end{array}$$

(if χ integrates to a G-action, we shall call it **lifted G-action**).

Let E be a Courant algebroid over M and let $\mathfrak{g}$ be a Lie algebra acting infinitesimally on M by $\Sigma : \mathfrak{g} \to \Gamma(TM)$ and let $\chi : \mathfrak{g} \to \Gamma(E)$ be a lifted $\mathfrak{g}$-action. Define $K_{\mathfrak{g}} \subset E$ as the image of the induced map

$$\mathfrak{g} \times M \ni (u, x) \mapsto \chi(u)(x).$$

Remark 4.28. Note that $K_{\mathfrak{g}} \subset E$ is a subbundle of E if and only if $\{u_M(x) \mid u \in \mathfrak{g}\}$, $x \in M$ has constant dimension.

Henceforth, we suppose that the infinitesimal action $\Sigma : \mathfrak{g} \to \Gamma(TM)$ integrates to a free action of a connected, compact Lie group G on M. In particular, $\{u_M(x) \mid u \in \mathfrak{g}\}$ has dimension $\dim(\mathfrak{g})$ for every $x \in M$ and we denote the corresponding subbundle of TM by $\Delta_{\mathfrak{g}}$. Note that in this case

$$K_{\mathfrak{g}} \cap T^*M = 0. \tag{4.19}$$

Definition 4.29 ([11])**.** Given a lifted G-action $\chi : \mathfrak{g} \to \Gamma(E)$, we say that an isotropic splitting $\nabla : TM \to E$ is **invariant** if

$$\nabla \circ (\psi_g)_* = \Psi_g \circ \nabla, \text{ for every } g \in G. \tag{4.20}$$

Infinitesimally, condition (4.20) reads:

$$\nabla[u_M, X] = [\![\chi(u), \nabla X]\!], \forall u \in \mathfrak{g} \text{ and } X \in \Gamma(TM).$$

Fix an isotropic splitting ∇ with curvature $H \in \Omega^3(M)$ and let $\xi_u := \mathrm{pr}_{T^*M}(\Phi_\nabla \circ \chi(u)) \in \Gamma(T^*M)$. One has

$$\Phi_\nabla \circ \chi(u) = u_M + \xi_u, \text{ for } u \in \mathfrak{g}.$$

Proposition 4.30 ([11])**.** *∇ is invariant if and only if*

$$ad(u_M + \xi_u) = (u_M, 0) \ (i.e.\ d\xi_u - i_{u_M} H = 0).$$

for every $u \in \mathfrak{g}$. In this case,

$$\mathcal{L}_{u_M} H = 0.$$

Proof. By identifying E with $\mathbb{T}M$ via Φ_∇, one has to prove that

$$[\![u_M + \xi_u, X]\!]_H = [u_M, X] \Longleftrightarrow d\xi_u - i_{u_M} H = 0.$$

for every $u \in \mathfrak{g}$ and $X \in \Gamma(TM)$. This follows from

$$[\![u_M + \xi_u, X]\!]_H = [u_M, X] + i_X(i_{u_M} H - d\xi_u).$$

In this case, by Cartan formula,

$$\mathcal{L}_{u_M} H = di_{u_M} H + i_{u_M} dH = dd\xi_u = 0.$$

□

Corollary 4.31. *Let $B \in \Omega^2(M)$. If ∇ is invariant, then $\nabla + B$ is invariant if and only if $\mathcal{L}_{u_M} B = 0$*

Proof. If we change the splitting by the 2-form B, then by (4.13), the inner symmetries generated by $\chi(u)$, for $u \in \mathfrak{g}$, changes as

$$(u_M, d\xi_u - i_{u_M} H) \mapsto (u_M, (d\xi_u - i_{u_M} H) - \mathcal{L}_{u_M} B).$$

As ∇ is invariant, $d\xi_u - i_{u_M} H = 0$ and therefore, by Proposition 4.30, $\nabla + B$ is invariant if and only if $\mathcal{L}_{u_M} B = 0$. □

We now prove that invariant splittings for lifted G-actions always exist under the hypothesis that G is compact.

Proposition 4.32. *Given any equivariant isotropic subbundle K_0 of E, there exists an equivariant* ***isotropic*** *subbundle K_1 such that $K_1 \oplus K_0^\perp = E$.*

Proof. First we prove that $K_0^\perp$ admits an equivariant complement. As $K_0^\perp$ is equivariant (G preserves the bilinear form), every automorphism Ψ_g descends to the quotient bundle $E/K_0^\perp$ (we continue to call the action in the quotient by Ψ_g). Let $s : E/K_0^\perp \to E$ be any bundle map splitting the exact sequence

$$0 \longrightarrow K_0^\perp \longrightarrow E \longrightarrow E/K_0^\perp \longrightarrow 0$$

and define $s_g := \Psi_g \circ s \circ \Psi_{g^{-1}}$. Now, let $s_{inv} : E/K_0^\perp \to E$ be defined as

$$s_{inv}(\cdot) = \int_G s_g(\cdot)\, d\mu(g)$$

where μ is the left Haar measure of G. It clearly satisfies $\Psi_g \circ s_{inv} = s_{inv}$ for every $g \in G$ and therefore its image D is an equivariant complement to $K_0^\perp$.

Recall the map $A : D \to K_0$ constructed in the proof of Proposition 2.7: it is caracterized by

$$g(Ae_1, e_2) = g(e_1, e_2)$$

for every $e_1, e_2 \in D$. We will prove that A is an equivariant bundle map. Indeed,

$$\begin{aligned} g(A(\Psi_g(e_1)), e_2) &= g(\Psi_g(e_1), e_2) \\ &= g(e_1, \Psi_{g^{-1}}(e_2)) \\ &= g(A(e_1), \Psi_{g^{-1}}(e_2)) \\ &= g(\Psi_g(A(e_1)), e_2). \end{aligned}$$

By the non-degeneracy of the form, it follows that $A \circ \Psi_g = \Psi_g \circ A$. Therefore, the subbundle

$$K_1 = \left\{ e - \frac{1}{2} A(e) \mid e \in D \right\}$$

is an equivariant isotropic complement to $K_0^{\perp}$. □

Corollary 4.33 ([11])**.** *Let G be a compact Lie group. For any lifted G action on E, there exists an invariant isotropic splitting.*

Proof. We have already proved that $T^*M \subset E$ is an equivariant Lagrangian subbundle of E (see (4.11)), so that there exists an equivariant isotropic subbundle $K_1 \subset E$ (by Proposition 4.32) such that

$$T^*M \oplus K_1^{\perp} = E$$

It is easy to see that K_1 is Lagrangian (by dimension count) and defines an invariant splitting by taking the inverse ∇ of $p|_{K_1} : K_1 \to TM$. Note that it satisfies

$$\nabla \circ (\psi_g)_* = \Psi_g \circ \nabla.$$

because $p \circ \Psi_g = (\psi_g)_* \circ p$. □

We are particularly interested in lifted G-actions for which the subbundle $K_{\mathfrak{g}}$ is isotropic. We call such action **isotropic G-lifted actions**(see [11, 12]). Recall from §2.3 that a $K_{\mathfrak{g}}$-admissible splitting $\nabla : TM \to E$ is an isotropic splitting such that

$$\Phi_{\nabla}(K_{\mathfrak{g}}) = \Delta_{\mathfrak{g}}.$$

For a $K_{\mathfrak{g}}$-admissible splitting ∇,

$$\xi_u = \mathrm{pr}_{T^*M}(\Phi_{\nabla}(\chi(u))) = 0$$

and thus, if ∇ is also invariant, its curvature H is a basic form 3-form (i.e. $\mathcal{L}_{u_M} H = 0$ and $i_{u_M} H = 0$ for any $u \in \mathfrak{g}$). We now prove that $K_{\mathfrak{g}}$-admissible splittings exists as corollary of a general result.

Proposition 4.34. *Let L be an equivariant Lagrangian subbundle of E and K be an equivariant isotropic subbundle such that $K \cap L = 0$. Then there exists another equivariant Lagragian subbundle L' such that $K \subset L'$ and $L \oplus L' = E$.*

Proof. Let $L^{(1)} = L \cap K^{\perp} + K$. It is an equivariant Lagrangian subbundle of E (we use that $L \cap K = 0$ to guarantee that $L \cap K^{\perp}$ has constant dimension). Let $L^{(2)}$ be any equivariant Lagrangian complement to $L^{(1)}$ (which exists by Proposition 4.32). We claim that

$$L' = L^{(2)} \cap K^{\perp} + K$$

is an equivariant complement to L such that $K \subset L'$. Indeed, L' is clearly equivariant and contains K. To see that it is a complement to L, observe that

$$K^{\perp} = L^{(2)} \cap K^{\perp} \oplus L^{(1)} = L' + L \cap K^{\perp}.$$

Thus, as $L \cap K = 0$,

$$E = K^{\perp} + L = (L' + L \cap K^{\perp}) + L = L' + L.$$

As $\dim(L') = \dim(L) = \frac{1}{2}\dim(E)$ the result follows. □

Corollary 4.35. *Let χ be an isotropic lifted G action on E. Then there always exists a $K_{\mathfrak{g}}$-admissible invariant splitting.*

Proof. We can take $L = T^*M$ and $K = K_{\mathfrak{g}}$ in Proposition 4.34 as equation (4.19) guarantees that $T^*M \cap K_{\mathfrak{g}} = 0$ and the fact that χ preserves bracket and G is connected implies that $K_{\mathfrak{g}}$ is equivariant. Thus, there exists an equivariant Lagrangian subbundle $L' \subset E$ such that $K_{\mathfrak{g}} \subset L'$ and $L' \oplus T^*M = E$. Therefore,

$$p|_{L'} : L' \to TM$$

is an isomorphism and $\nabla = (p|_{L'})^{-1}$ is the isotropic splitting wanted. It is invariant as L' is invariant and it is $K_{\mathfrak{g}}$-admissible as $\nabla(\Delta_{\mathfrak{g}}) \subset K_{\mathfrak{g}}$ (see Lemma 2.33). □

Let E be a Courant algebroid over M and χ be an isotropic lifted action. Consider ∇_1 an invariant splitting with curvature $H \in \Omega^3(M)$ and let

$$\Phi_{\nabla_1}(\chi(u)) = u_M + \xi_u, \text{ for } u \in \mathfrak{g}.$$

Every $K_{\mathfrak{g}}$-admissible invariant splitting ∇_2 is given as $\nabla_1 + B$ for some invariant 2-form B (see 4.31). Now, as ∇_2 is $K_{\mathfrak{g}}$-admissible,

$$u_M = \Phi_{\nabla_2}(\chi(u)) = \tau_{-B} \circ \Phi_{\nabla_1}(\chi(u)) = u_M + (\xi_u - i_{u_M}B)$$

which proves that

$$i_{u_M}B = \xi_u, \text{ for every } u \in \mathfrak{g}. \tag{4.21}$$

It is easy to see that if $B \in \Omega^2(M)$ is any invariant 2-form which satisfies (4.21), then $\nabla_1 + B$ is invariant and $K_{\mathfrak{g}}$-admissible.

Example 4.36 ([11])**.** Let $\pi : P \to N$ be a principal S^1 bundle and let $M = P \times S^1$. Let $\xi \in \Omega^1(S^1)$ be a volume form and let $\alpha \in \Omega(P)$ be a connection 1-form (i.e. $i_{u_M}\alpha = 1$ and $\mathcal{L}_{u_M}\alpha = 0$ for the infinitesimal generator u_M corresponding to $1 \in \mathbb{R} = Lie(S^1)$). We can define an isotropic lifted action on the standard Courant algebroid $\mathbb{T}M$ given by

$$\chi(1) = u_M + \xi.$$

The canonical splitting is invariant as $d\xi = 0$. Now, the 2-form defined by

$$B = \alpha \wedge \xi$$

is clearly invariant and satisfies (4.21) as

$$i_{u_M}B = i_{u_M}\alpha \wedge \xi - \alpha \wedge i_{u_M}\xi = \xi.$$

Therefore $\nabla_{\text{can}} + B$ is an invariant $K_{\mathfrak{g}}$-admissible splitting. The curvature of $\nabla_{\text{can}} + B$ is

$$dB = d\alpha \wedge \xi - \alpha \wedge d\xi = \pi^*\mathcal{F} \wedge \xi$$

where $\mathcal{F} \in \Omega^2(N)$ is the curvature form of P.

4.2.3 Moment maps.

Let $\rho : \mathfrak{a} \to \mathfrak{g}$ be an exact Courant algebra over $\mathfrak{g}$ and let $\mathfrak{h} = \ker(\rho)$. Consider an extended $\mathfrak{g}$-action $\chi : \mathfrak{a} \to \Gamma(E)$ on an exact Courant algebroid E over M. As we saw before (see (4.18)), $\chi|_{\mathfrak{h}} : \mathfrak{h} \to \Omega^1_{cl}(M)$. Following [11], we say that the $\mathfrak{g}$-extended action has moment map if we can find primitives for these closed forms in an equivariant fashion. More precisely, there is a map $\mu : M \to \mathfrak{h}^*$ such that, for each $h \in \mathfrak{h}$,

(i) $\chi|_{\mathfrak{h}}(h) = d\mu^h$, where $\mu^h \in C^\infty(M)$ is given by $\mu^h(x) = \mu(x)(h)$ for $x \in M$;

(ii) $\mathcal{L}_{u_M}\mu^h = \mu^{[u,h]}$.

We call μ the **moment map** [11] for the action.

The next two examples show the level of generality of the definition.

Example 4.37. For lifted actions $\chi : \mathfrak{g} \to \Gamma(E)$, the zero map $\mu : M \to \{0\}$ is a moment map.

Example 4.38. Let $F : M \to \mathbb{R}^n$ be a submersion and let $F = (f_1, \ldots, f_n)$, $f_i \in C^\infty(M)$ for $i = 1, \ldots, n$. We saw (Example 4.25) that $\chi : \{0\} \times \mathbb{R}^n \to \Gamma(\mathbb{T}M)$ defined by

$$\chi(0, e_i) = df_i$$

is a $\{0\}$-extended action. In this case, $\mathfrak{h} = \mathbb{R}^n$ and, by identifying $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ via the canonical inner product, a moment map $\mu : M \to \mathbb{R}^n$ is given by $\mu = F$.

The next example shows how the usual moment map from the theory of Hamiltonian actions appears in this context.

Example 4.39 ([11])**.** Let M be a symplectic manifold with 2-form ω and let $\Sigma : \mathfrak{g} \to \mathfrak{X}(M)$ be an infinitesimal symplectic action. In Example 4.27, we saw that

$$\chi(u,v) = \chi(u) + i_{v_M}\,\omega,\ \ u,\, v \in \mathfrak{g}.$$

defines a $\mathfrak{g}$-extended action of the hemisemidirect product $\mathfrak{g} \oplus \mathfrak{g}$ on the standard Courant algebroid. A moment map for this action is a map $\mu : M \to \mathfrak{g}^*$ such that

(i) $i_{v_M}\,\omega = d\mu^v$, for $v \in \mathfrak{g}$ and

(ii) $\mathcal{L}_{u_M}\mu^v = \mu^{[u,v]}$.

This is exactly the definition of a moment map from the theory of infinitesimal Hamiltonian actions of symplectic geometry [34, 46]

Following [12], we now present a construction which is fundamental for what we will do. Suppose one has an isotropic lifted action $\chi : \mathfrak{g} \to \Gamma(E)$ on the Courant algebroid E over M. Let $\mathfrak{h}$ be a $\mathfrak{g}$-module and $\mu : M \to \mathfrak{h}^*$ an equivariant map (i.e. $\mathcal{L}_{u_M}\mu^h = \mu^{u\cdot h}$). Define, for $(u,h) \in \mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$,

$$\chi_{\mathfrak{h}}(u,h) = \chi(u) + d\mu^h\ \in \Gamma(E).$$

For $u_1, u_2 \in \mathfrak{g}$ and $h_1, h_2 \in \mathfrak{h}$, one has

$$\begin{aligned}[\![\chi_{\mathfrak{h}}(u_1,h_1), \chi_{\mathfrak{h}}(u_2,h_2)]\!] &= [\![\chi(u_1),\chi(u_2)]\!] + [\![\chi(u_1), d\mu^{h_2}]\!] \\ &= \chi([u_1,u_2]) + \mathcal{L}_{(u_1)_M}\, d\mu^{h_2} \\ &= \chi([u_1,u_2]) + d\mu^{u_1\cdot h_2} \\ &= \chi_{\mathfrak{h}}([u_1,u_2], u_1\cdot h_2).\end{aligned} \tag{4.22}$$

Recall that

$$[(u_1,h_1),(u_2,h_2)] = ([u_1,u_2], u_1\cdot h_2)$$

together with the projection $\rho : \mathfrak{g}\oplus\mathfrak{h} \to \mathfrak{g}$ on the first factor gives $\mathfrak{a} = \mathfrak{g}\oplus\mathfrak{h}$ the structure of a Courant algebra (Example 4.23) over $\mathfrak{g}$. The calculation (4.22) shows that $\chi_{\mathfrak{h}} : \mathfrak{a} \to \Gamma(E)$ is a $\mathfrak{g}$-extended action with moment map $\mu : M \to \mathfrak{h}^*$.

Definition 4.40 ([12])**.** We call the triple $(\chi, \mathfrak{h}, \mu)$ **reduction data** on E if 0 is a regular value for μ and the $\mathfrak{g}$-extended action $\chi_{\mathfrak{h}}$ integrates to a G-extended action by automorphisms of E with its corresponding G-action on $\mu^{-1}(0)$ being free.

4.3 The reduced Courant algebroid.

In this section, given the reduction data $(\chi, \mathfrak{h}, \mu)$ on a Courant algebroid E over M, we adapt the general construction of [11] to obtain an exact Courant algebroid E_{red} over $\mu^{-1}(0)/G$. This is the Courant algebroid in which we will find the reduced Dirac subbundles.

Let $(\chi, \mathfrak{h}, \mu)$ be the reduction data on an exact Courant algebroid E. Let $K \subset E|_{\mu^{-1}(0)}$ be defined by

$$K = K_{\mathfrak{g}} \oplus \mathrm{Ann}\left(T\mu^{-1}(0)\right), \tag{4.23}$$

where $K_{\mathfrak{g}}$ is the isotropic subbundle corresponding to χ (see §4.2.2).

Lemma 4.41 ([11]). *K is an isotropic equivariant subbundle of $E|_{\mu^{-1}(0)}$.*

Proof. First, note that as 0 is a regular value for μ, it follows that K is a subbundle of $E|_{\mu^{-1}(0)}$. Let $u_1, u_2 \in \mathfrak{g}$ and $h_1, h_2 \in \mathfrak{h}$. One has for any $x \in \mu^{-1}(0)$

$$\begin{aligned} g(\chi_{\mathfrak{h}}(u_1, h_1), \chi_{\mathfrak{h}}(u_2, h_1))_x = & \; i_{(u_1)_M}\, d_x\mu^{h_2} + i_{(u_2)_M}\, d_x\mu^{h_1} \\ = & \; (\mathcal{L}_{(u_1)_M}\mu^{h_2})(x) + (\mathcal{L}_{(u_2)_M}\mu^{h_1})(x) \\ = & \; \mu^{u_1\cdot h_2}(x) + \mu^{u_2\cdot h_1}(x) \;=\; 0 \end{aligned}$$

This proves that K is isotropic. The equivariance of K follows directly from the fact that $\chi_{\mathfrak{h}}$ is bracket preserving and G is connected. □

As the G action on E preserves the bracket, $K^\perp$ is also an equivariant G-bundle and thus the quotient bundle $K^\perp/K$ over $\mu^{-1}(0)$ inherits a G-action by bundle maps. We keep denoting the bundle map of $K^\perp/K$ corresponding to g by Ψ_g; it covers the restriction of ψ_g to $\mu^{-1}(0)$ which we denote by $\psi_g|_{\mu^{-1}(0)}$.

We can apply pointwise the analysis done in Section 1.1 in this case. Define $R := p(K) \subset TM$ and $Q := p(K^\perp) \subset TM$. They will be subbundles of $TM|_{\mu^{-1}(0)}$ as long $K \cap T^*M = \mathrm{Ann}\,(Q)$ and $K^\perp \cap T^*M = \mathrm{Ann}\,(R)$ have constant dimension (see Lemma 2.29). Fix arbitrary $u \in \mathfrak{g}$ and $h \in \mathfrak{h}$ and $x \in M$. As

$$p(\chi(u)(x) + d_x\mu^h) = u_M(x),$$

it follows that $R_x = T_x\,(G \cdot x) = (\Delta_{\mathfrak{g}})_x$, the tangent space at x of its G-orbit. It has the same dimension as G, as we assumed the G-action to be free. Also,

$$\chi(u)(x) + d_x\mu^h \in K_x \cap T_x^*M \Leftrightarrow \chi(u)(x) \in T_x^*M \Leftrightarrow p \circ \chi(u)(x) = u_M(x) = 0.$$

Again, as G acts freely, it follows that u must be zero. Therefore

$$\mathrm{Ann}\,(Q_x) = K_x \cap T_x^*M = \{d_x\mu^h \mid h \in \mathfrak{h}\} = \mathrm{Ann}\left(T_x\mu^{-1}(0)\right)$$

and thus $Q_x = T_x\mu^{-1}(0)$.

The bundle $K^\perp/K$ over $\mu^{-1}(0)$ has a fibrewise defined non-degenerate symmetric bilinear form g_K given by (2.24) and a bundle map

$$p_K : \frac{K^\perp}{K} \longrightarrow \frac{T\mu^{-1}(0)}{\Delta_{\mathfrak{g}}}$$

given by (2.28).

Lemma 4.42. *g_K is preserved by the G-action on $K^\perp/K$ and p_K is equivariant if we endow $T\mu^{-1}(0)/\Delta_{\mathfrak{g}}$ with its quotient action.*

Proof. For $g \in G$, let $\Psi_g \in \mathrm{Aut}(E)$ be the corresponding automorphism of E. It covers $\psi_g \in \mathrm{Diff}(M)$ and $g \mapsto \psi_g$ is the G action on M. As G preserves $\mu^{-1}(0)$,

$$(\psi_g)_*(T\mu^{-1}(0)) = T\mu^{-1}(0).$$

Also, it is a general fact that $(\psi_g)_*(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{g}}$. Therefore, it induces an action of G by bundle maps $(\psi_g)_* : T\mu^{-1}(0)/\Delta_{\mathfrak{g}} \to T\mu^{-1}(0)/\Delta_{\mathfrak{g}}$ covering $\psi_g|_{\mu^{-1}(0)}$. Now, as condition (3) on the definition of an automorphism of E says that $p \circ \Psi_g = (\psi_g)_* \circ p$, one has for $k^\perp \in K^\perp$,

$$\begin{aligned} p_K \circ \Psi_g(k^\perp + K) = &\ p_K(\Psi_g(k^\perp) + K) \\ = &\ p\,(\Psi_g(k^\perp)) + \Delta_{\mathfrak{g}} \\ = &\ (\psi_g)_* p(k^\perp) + \Delta_{\mathfrak{g}} \\ = &\ (\psi_g)_* p_K(k^\perp + K) \end{aligned}$$

which proves the assertion about p_K. The assertion about g_K is a direct consequence of the fact that Ψ_g preserves the metric g. □

Lemma 4.42 implies that on the vector bundle

$$E_{red} = \frac{K^\perp}{K} \Big/ G. \tag{4.24}$$

over $M_{red} = \mu^{-1}(0)/G$ we have a non-degenerate symmetric bilinear form g_{red} and a bundle map

$$p_{red} : E_{red} \to \frac{T\mu^{-1}(0)}{\Delta_{\mathfrak{g}}} \Big/ G.$$

Remark 4.43. The bundle $(T\mu^{-1}(0)/\Delta_{\mathfrak{g}})/G$ over M_{red} is naturally isomorphic to TM_{red} via the differential of $q : \mu^{-1}(0) \to M_{red}$, the quotient map. By abuse of notation, we continue to call p_{red} the map $E_{red} \to TM_{red}$ obtained by composition with dq.

To define a bracket on $\Gamma(E_{red})$, we need a lemma first.

Lemma 4.44. *Let $e_1, e_2 \in \Gamma(E|_{\mu^{-1}}(0))$ and suppose $e_i \in \Gamma(K^\perp)$ for $i = 1, 2$. If $e_i^{(j)} \in \Gamma(E)$ satisfies $e_i^{(j)}|_{\mu^{-1}(0)} = e_i$ for $i, j = 1, 2$, then*

$$\left(\llbracket e_1^{(1)}, e_2^{(1)} \rrbracket - \llbracket e_1^{(2)}, e_2^{(2)} \rrbracket\right)\Big|_{T\mu^{-1}(0)} \in \Gamma(K)$$

Proof. By axiom (C2) on the definition of a Courant algebroid, it follows that for any $e_1^{(3)}$, $e_2^{(3)} \in \Gamma(E)$ and $x \in M$

$$\llbracket e_1^{(3)}, e_2^{(3)} \rrbracket(x) \in E_x$$

depends only on $e_1^{(3)}|_U$ and $e_2^{(3)}|_U$ where U is a neighborhood of x.

So let $x \in \mu^{-1}(0)$ and let U be an open neighborhood such that there exists a local frame $\{e_1^{(3)}, \ldots, e_{2n}^{(3)}\} \in \Gamma(E|_U)$ with $\{e_1^{(3)}, \ldots, e_k^{(3)}\}$ a local frame for $\Gamma(K^{\perp}|_{\mu^{-1}(0)\cap U})$. Write for $i = 1, 2$,

$$e_i^{(1)} - e_i^{(2)} = \sum_{j=1}^{k} f_i^j\, e_j^{(3)},\ \ f_i^j \in C^{\infty}(U)$$

By hypothesis, $f_i^j|_{\mu^{-1}(0)\cap U} \equiv 0$. Now,

$$\begin{aligned} [\![e_1^{(1)} - e_1^{(2)}, e_2^{(1)}]\!] &= \textstyle\sum_{i=j}^{k} [\![f_1^j\, e_j^{(3)}, e_2^{(1)}]\!] \\ &= \textstyle\sum_{j=1}^{k} f_1^j [\![e_j^{(3)}, e_2^{(1)}]\!] - \mathcal{L}_{p(e_2^{(1)})} f_1^j\, e_j^{(3)} + g(e_j^{(3)}, e_2^{(1)})\, df_1^j \end{aligned}$$

and

$$\begin{aligned} [\![e_1^{(2)}, e_2^{(1)} - e_2^{(2)}]\!] &= \textstyle\sum_{j=1}^{k} [\![e_1^{(2)},\, f_2^j\, e_j^{(3)}]\!] \\ &= \textstyle\sum_{j=1}^{k} f_2^j [\![e_1^{(2)}, e_j^{(3)}]\!] + \mathcal{L}_{p(e_1^{(2)})} f_2^j\, e_j^{(3)}. \end{aligned}$$

Note that as $e_1, e_2 \in \Gamma(K^{\perp})$, one has that $p(e_1), p(e_2) \in T\mu^{-1}(0)$ and therefore restricting to $\mu^{-1}(0) \cap U$,

$$\mathcal{L}_{p(e_1)} f_2^j = \mathcal{L}_{p(e_2)} f_1^j = 0 \ \text{ for every } j = 1, \ldots, k.$$

Therefore,

$$[\![e_1^{(2)}, e_2^{(2)}]\!]|_{\mu^{-1}(0)\cap U} = [\![e_1^{(2)}, e_2^{(1)}]\!]|_{\mu^{-1}(0)\cap U}$$

and

$$\begin{aligned} \Big([\![e_1^{(1)}, e_2^{(1)}]\!] - [\![e_1^{(2)}, e_2^{(2)}]\!]\Big)\Big|_{\mu^{-1}(0)\cap U} &= [\![e_1^{(1)} - e_1^{(2)}, e_2^{(1)}]\!]\Big|_{\mu^{-1}(0)\cap U} \\ &= \textstyle\sum_{j=1}^{k} g(e_j^{(3)}, e_2^{(1)})\, df_1^j\,|_{\mu^{-1}(0)\cap U}. \end{aligned}$$

To finish, just note that for any $x \in \mu^{-1}(0) \cap U$, $df_1^j(x) \in \mathrm{Ann}\left(T_x\mu^{-1}(0)\right) = K_x \cap T_x^*M$. □

We say that a section e_1 of $\Gamma(K^{\perp})$ is invariant if for any section e of K

$$[\![\tilde{e}, \tilde{e}_1]\!]\,|_{\mu^{-1}(0)} = [\![\tilde{e}_1, \tilde{e}]\!]\,|_{\mu^{-1}(0)} \in \Gamma(K), \tag{4.25}$$

where $\tilde{e}$, $\tilde{e}_1 \in \Gamma(E)$ are any extension of e, e_1 respectively. Note that (4.25) independs on the extension by Lemma 4.44. We denote the space of invariant section by $\Gamma_{inv}(K^{\perp})$. Note that $\Gamma(K) \subset \Gamma_{inv}(K^{\perp})$.

Theorem 4.45 (Bursztyn - Cavalcanti-Gualtieri [11])**.** *E_{red} is an exact Courant algebroid over M_{red}.*

Sketch of the proof. We just construct the bracket on the sections of E_{red} following [11] and point to this paper to the proof that with this bracket, g_{red} and p_{red}, E_{red} is an exact Courant algebroid. Consider the map

$$\begin{array}{ccc} \Gamma_{inv}(K^{\perp}) & \longrightarrow & \Gamma(E_{red}) \\ e & \longmapsto & [e + K], \end{array}$$

where $[e+K](q(x)) = [e(x)+K]$ for $x \in \mu^{-1}(0)$ and $[\cdot]$ denotes the G-orbit on $K^\perp/K$. It is well-defined and surjective. The fact that e is an invariant section of $K^\perp$ and G is connected implies that $[e+K]$ is well-define. To prove surjectivity, it suffices to find an equivariant split of the exact sequence

$$0 \longrightarrow K \longrightarrow K^\perp \longrightarrow \frac{K^\perp}{K} \longrightarrow 0$$

(it exists by compactness of G). If $e_1, e_2 \in \Gamma_{inv}(K^\perp)$ and $e_3 \in \Gamma(K)$, denote by $\tilde{e}_i \in \Gamma(E)$ an extension for $i = 1, 2, 3$. One has

$$g([\![\tilde{e}_1, \tilde{e}_2]\!], \tilde{e}_3) = -g(\tilde{e}_2, [\![\tilde{e}_1, \tilde{e_3}]\!]) + \mathcal{L}_{p(\tilde{e}_1)} g(\tilde{e_2}, \tilde{e}_3).$$

By restricting to $\mu^{-1}(0)$, one proves that the right side is zero and therefore $[\![\tilde{e}_1, \tilde{e}_2]\!]|_{\mu^{-1}(0)} \in \Gamma(K^\perp)$. By Lemma 4.44, this last property does not depend on the extensions, as $\Gamma(K) \subset \Gamma(K^\perp)$. More is true; it holds that $[\![\tilde{e}_1, \tilde{e}_2]\!]|_{\mu^{-1}(0)} \in \Gamma_{inv}(K^\perp)$.

Thus, for any sections $(e_1)_{red}, (e_2)_{red}$ of $\Gamma(E_{red})$, let e_1, e_2 be any section of $\Gamma_{inv}(K^\perp)$ such that $(e_i)_{red} = [e_i + K]$ for $i = 1, 2$. Define

$$[\![(e_1)_{red}, (e_2)_{red}]\!] = \left[\, [\![\tilde{e}_1, \tilde{e}_2]\!]|_{\mu^{-1}(0)} + K \,\right]. \tag{4.26}$$

Note that the expression on the right side is extension-invariant by Lemma 4.44 and does not depend on the element of $\Gamma_{inv}(K^\perp)$ one chooses to represent $(e_i)_{red}$ as any two of them differ by an element of $\Gamma(K)$ and $[\![\Gamma(K), \Gamma_{inv}(K^\perp)]\!] \subset \Gamma(K)$. □

We will focus now on how one can find an isotropic splitting for E_{red}. We saw in Proposition4.35 how to find invariant $K_{\mathfrak{g}}$-admissible splittings.

Lemma 4.46. *Any $K_{\mathfrak{g}}$-admissible splitting ∇ is also K-admissible.*

Proof. By Lemma 2.33, ∇ is K-admissible if $\nabla|_{T\mu^{-1}(0)} : TM|_{\mu^{-1}(0)} \to E|_{\mu^{-1}(0)}$ satisfies $\nabla(p(K)) \subset K$. But, as ∇ is $K_{\mathfrak{g}}$-admissible and $p(K) = p(K_{\mathfrak{g}}) = \Delta_{\mathfrak{g}}$, one has that

$$\nabla(\Delta_{\mathfrak{g}}) \subset K_{\mathfrak{g}} \subset K.$$

Thus, ∇ is K-admissible. □

Let ∇ be an invariant $K_{\mathfrak{g}}$-admissible splitting. Using Lemma 4.46 together with Lemma 2.33, one hat that $\nabla|_{T\mu^{-1}(0)} : TM|_{\mu^{-1}(0)} \to E|_{\mu^{-1}(0)}$ takes $T\mu^{-1}(0)$ to $K^\perp$ and thus defines a map

$$\nabla_{K} : \frac{T\mu^{-1}(0)}{\Delta_{\mathfrak{g}}} \longrightarrow \frac{K^\perp}{K} \tag{4.27}$$

which is G-invariant and satisfies

$$g_{K}(\nabla_{K}(\cdot), \nabla_{K}(\cdot)) = 0;$$

Therefore, it descends to the quotient by the G-action giving an isotropic splitting

$$\nabla_{red} : TM_{red} \longrightarrow E_{red}.$$

For a point $x \in \mu^{-1}(0)$,

$$\begin{array}{rccl} \nabla_{red}: & (TM_{red})_{q(x)} & \longrightarrow & (E_{red})_{q(x)} \\ & d_x q(X) & \longmapsto & [\nabla X + K_x]. \end{array} \tag{4.28}$$

Recall that the curvature H of ∇ is a basic 3-form.

Proposition 4.47. *The curvature $H_{red} \in \Omega^3(M_{red})$ of ∇_{red} is the 3-form such that $q^* H_{red} = j^* H$, where $j : \mu^{-1}(0) \to M$ is the inclusion map.*

Proof. We choose a connection $F \oplus \Delta_{\mathfrak{g}} = T\mu^{-1}(0)$; as we already know that $j^* H$ is basic, it suffices to show that

$$H_{red}(dq_x\, w_1, dq_x\, w_2, dq_x\, w_3) = H(w_1, w_2, w_3)$$

for any $x \in \mu^{-1}(0)$ and $w_i \in F_x$ for $i = 1, 2, 3$. Let $X_1, X_2, X_3 \in \Gamma(F)$ be invariant sections of F such that $X_i(x) = w_i$ and extend them to vector fields $\hat{X}_i$ defined all over M for $i = 1, 2, 3$. As ∇ is invariant, it follows that for $u \in \mathfrak{g}$ and $h \in \mathfrak{h}$ that

$$[\![\chi(u) + d\mu^h, \nabla \hat{X}_i]\!]|_{\mu^{-1}(0)} = \nabla[u_M, \hat{X}_i]|_{\mu^{-1}(0)} = \nabla[u_M, X] = 0, \text{ for } i = 1, 2, 3.$$

and therefore $\hat{X}_i \in \Gamma_{inv}(K^\perp)$. Now, as ∇ is also $K_{\mathfrak{g}}$-admissible, one has

$$\nabla_{red}\, q_*(X_i) = [\nabla \hat{X}_i|_{\mu^{-1}(0)} + K], \text{ for } i = 1, 2, 3.$$

By the construction of the reduced bracket on Theorem 4.45,

$$[\![\nabla_{red}\, q_*(X_1),\, \nabla_{red}\, q_*(X_2)]\!] = \Big[\, [\![\nabla \hat{X}_1, \nabla \hat{X}_2]\!]|_{\mu^{-1}(0)} + K \Big]$$

and therefore

$$\begin{aligned} H_{red}(dq_x\, w_1, dq_x\, w_2, dq_x\, w_3) &= g_{red}([\![\nabla_{red}\, q_*(X_1),\, \nabla_{red}\, q_*(X_2)]\!], \nabla_{red}\, q_*(X_3))_{q(x)} \\ &= g([\![\nabla \hat{X}_1, \nabla \hat{X}_2]\!], \nabla \hat{X}_3)_x \\ &= H(\hat{X}_1, \hat{X}_2, \hat{X}_3)_x \\ &= H(w_1, w_2, w_3). \end{aligned}$$

This concludes the proof. □

Remark 4.48. In [11], the problem of giving a description of the Ševera class of E_{red} was treated only for the case where $\mu^{-1}(0) = M$. Also, the techiques used there were different from the ones used here.

Example 4.49 ([11])**.** Let $\pi : P \to N$ be a S^1 principal bundle and consider $M = P \times S^1$ as in Example 4.36. We saw that by choosing a connection form α and a volume form $\xi \in \Omega^1(S^1)$ then

$$\nabla = \nabla_{\text{can}} + \alpha \wedge \xi$$

is an invariant $K_{\mathfrak{g}}$-admissible splitting for the isotropic lifted action

$$\chi(1) = u_M + \xi$$

on the standard Courant algebroid. In this case, we can take $\mathfrak{h} = 0$ and the constant map $\mu : M \to \{0\}$ to have reduction data $(\chi, \mathfrak{h}, \mu)$. Therefore,

$$M_{red} = \frac{\mu^{-1}(0)}{G} = N \times S^1$$

and the splitting ∇_{red} identifies E_{red} with $(E_{\text{can}}(M_{red}), [\![\cdot,\cdot]\!]_{H_{red}})$ where

$$H_{red} = \mathcal{F} \wedge \xi, \;\; \mathcal{F} \in \Omega^2(N) \text{ the curvature form of } P.$$

So if $0 \neq [\mathcal{F}] \in H^2(N)$ (e.g., the Hopf bundle $S^3 \to S^2$), the reduced Courant algebroid has non-zero Ševera class $[H_{red}]$ although we started with one (the standard Courant algebroid) with zero Ševera class.

Remark 4.50. Let E be a Courant algebroid over M and consider reduction data on E given by $(\chi, \mathfrak{h}, \mu)$. Let $K \subset E$ be the isotropic subbundle (4.23). Consider the complexification $E_{\mathbb{C}} = E \otimes \mathbb{C}$ together with $g_{\mathbb{C}}$ and $[\![\cdot,\cdot]\!]_{\mathbb{C}}$. The Lie group G acts on $E_{\mathbb{C}}$ by extending its action $G \ni g \mapsto \Psi_g \in \text{Aut}(E)$ to the complexification

$$\Psi_g \otimes \text{id} : E_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}.$$

The subbundles $K \otimes \mathbb{C}$ and $K^\perp \otimes \mathbb{C}$ of $E_{\mathbb{C}}|_{\mu^{-1}(0)}$ are easily seen to be invariant and one can form the quotient

$$(E_{\mathbb{C}})_{red} = \frac{K^\perp \otimes \mathbb{C}}{K \otimes \mathbb{C}} \Big/ G$$

which is naturally identified with $(E_{red})_{\mathbb{C}} = E_{red} \otimes \mathbb{C}$. This will be important when reducing generalized complex structures.

4.4 Reduction of Dirac structures.

Let M be a manifold, E a Courant algebroid over M and consider reduction data $(\chi, \mathfrak{h}, \mu)$ on E. Let $K \subset E$ be the isotropic subbundle defined by (4.23).

For any point of $x \in \mu^{-1}(0)$, consider the Lagrangian subspace $(\Lambda_K)_x$ of $\bar{E}_x \times (K^\perp/K)_x$ given by (compare with (2.27))

$$\left\{ (k^\perp, k^\perp + K_x) \in \bar{E}_x \times \left(\frac{K^\perp}{K}\right)_x \;\middle|\; k^\perp \in K_x^\perp \right\}.$$

Given $L \subset E$ a Lagrangian subbundle, we can apply the reduction procedure of §2.3 to get a Lagrangian subspace of $(K^{\perp}/K)_x$ given by (see (2.16) and Lemma 2.28)

$$(L_K)_x = (\Lambda_K)_x(L_x) = \frac{L_x \cap K_x^{\perp} + K_x}{K_x}.$$

Now, by taking the G-orbit $[\cdot]$ on $K^{\perp}/K$, we have a Lagrangian subspace of $(E_{red})_{q(x)}$:

$$(L_{red})_x := \{ [k^{\perp} + K] \mid k^{\perp} \in (L \cap K^{\perp})_x \}. \tag{4.29}$$

We wish to proceed in this way to define a Lagrangian subbundle of E_{red}; the problem with this pointwise definition is that for any other point y in the G-orbit of x (or, equivalently, $y \in q^{-1}(q(x))$), one does not have in general

$$(L_{red})_x = (L_{red})_y.$$

Even if this happens, one has to guarantee that the result is a subbundle of E_{red}. In this section, we shall indicate how to deal with this.

Remark 4.51. These listed problems should be seen as analogues of the difficulties one has to deal with in order to extend the linear symplectic category to symplectic manifolds (see [50], Lecture III for a detailed discussion). In this sense, the kind of hypothesis we are looking for are similar to clean intersection hypothesis one has to do in the symplectic case.

Definition 4.52. A subbundle $L \subset E$ is $\mathfrak{g}$-invariant if

$$[\![\chi(u), \Gamma(L)]\!] \subset \Gamma(L),$$

for every $u \in \mathfrak{g}$. In particular, for $g \in G$ its corresponding automorphism $\Psi_g : E \to E$ preserves L (i.e. $\Psi_g(L_x) = L_{\psi_g(x)}$).

Lemma 4.53. *If L is a $\mathfrak{g}$-invariant Lagrangian subbundle, then for every $x \in \mu^{-1}(0)$ and $g \in G$*

$$(L_{red})_x = (L_{red})_{\psi_g(x)}.$$

Proof. For any $x \in \mu^{-1}(0)$, to prove that

$$(L_{red})_x = (L_{red})_{\psi_g(x)}$$

it suffices to show that

$$\Psi_g\left(\frac{L_x \cap K_x^{\perp} + K_x}{K_x}\right) = \frac{L_{\psi_g(x)} \cap K_{\psi_g(x)}^{\perp} + K_{\psi_g(x)}}{K_{\psi_g(x)}},$$

where $\Psi_g : K^{\perp}/K \to K^{\perp}/K$ is the G-action induced by the G-action on E. This follows immediately from the equivariance of L, $K^{\perp}$ and K. □

Theorem 4.54 (Bursztyn-Cavalcanti-Gualtieri [11])**.** *If L is a $\mathfrak{g}$-invariant Lagrangian subbundle of E and $L|_{\mu^{-1}(0)} \cap K$ (or equivalently $L|_{\mu^{-1}(0)} \cap K^\perp$) has constant rank, then*

$$L_{red} = \frac{L|_{\mu^{-1}(0)} \cap K^\perp + K}{K} \Big/ G \subset E_{red}$$

is a Dirac structure on M_{red}.

Proof. First note that for $x \in \mu^{-1}(0)$,

$$(L_x \cap K_x)^\perp = L_x + K_x^\perp.$$

This implies that $L|_{\mu^{-1}(0)} \cap K$ has constant rank if and only if $L|_{\mu^{-1}(0)} + K^\perp$ also has. This last condition is equivalent to $L|_{\mu^{-1}(0)} \cap K^\perp$ having constant rank.

Suppose $L|_{\mu^1(0)} \cap K^\perp$ has constant rank. Then as $(L|_{\mu^{-1}(0)} \cap K^\perp + K)_x$ is a Lagrangian subspace of E_x, it has rank equal to $\frac{1}{2}\dim(E) = \dim(M)$ for every $x \in \mu^{-1}(0)$. Therefore $L|_{\mu^{-1}(0)} \cap K^\perp + K$ is a subbundle of $K^\perp \subset E|_{\mu^{-1}(0)}$ and we can form the quotient bundle over $\mu^{-1}(0)$:

$$L_K = \frac{L|_{\mu^{-1}(0)} \cap K^\perp + K}{K}.$$

It is an equivariant subbundle of $K^\perp/K$ because K, $K^\perp$ and $L|_{\mu^{-1}(0)}$ are all equivariant subbundles of $E|_{\mu^{-1}(0)}$. Consider its quotient by the G action: L_K/G. It is a subbundle of E_{red}. But as we saw in the proof of Lemma 4.53, for $x \in \mu^{-1}(0)$

$$(L_{red})_x = (L_K/G)_{q(x)}$$

which proves that L_{red} is a Lagrangian subbundle of E_{red}. As for integrability, we point to [11] for the original proof (also we will give an alternative proof using pure spinor techniques in Theorem 5.41). □

Let us give some examples.

Example 4.55. Let $F : M \to \mathbb{R}^n$ be a submersion and let $N = F^{-1}(0)$. Fix a closed 3-form $H \in \Omega^3(M)$. As we saw in Example 4.38, there is a corresponding reduction data $(\chi, \mathbb{R}^n, F)$ on the standard Courant algebroid $(\mathbb{T}M, [\![\cdot,\cdot]\!]_H)$, where $\chi : \{0\} \to \Gamma(\mathbb{T}M)$ is the zero map. The associated isotropic subbundle $K \subset (\mathbb{T}M)|_N$ is

$$K = \mathrm{Ann}\,(TN) = \{\xi \in T^*M|_N \mid j^*\xi = 0\}, \text{ where } j : N \to M \text{ is the inclusion.}$$

The reduced Courant algebroid over N is

$$E_{red} = \frac{K^\perp}{K} = \frac{TN \oplus T^*M}{\mathrm{Ann}\,(TN)} = TN \oplus \frac{T^*M}{\mathrm{Ann}\,(TN)}.$$

The canonical splitting ∇ is invariant and K-admissible and induces an isomorphism

$$\begin{array}{rccl} \Phi_{\nabla_{red}}: & E_{red} & \longrightarrow & TN \oplus T^*N \\ & (X, \xi + \mathrm{Ann}\,(TN)) & \longmapsto & (X, j^*\xi). \end{array}$$

The curvature of ∇_{red} is j^*H.

As $G = \{e\}$ in this case, any Dirac structure $L \subset \mathbb{T}M$ is invariant and if $L \cap \mathrm{Ann}\,(TN)$ has constant rank, then (compare with Example 2.38)

$$L_{red} = \frac{L \cap (TN \oplus T^*M) + \mathrm{Ann}\,(TN)}{\mathrm{Ann}\,(TN)} = \frac{L \cap (TN \oplus T^*M)}{L \cap \mathrm{Ann}\,(TN)}.$$

We call L_{red} the *restriction of L to N* and denote it by $\Lambda^t_j(L)$. Note that for $x \in N$,

$$\Phi_{\nabla_{red}}(\Lambda^t_j(L)_x) = \{(X, dj^*_x\xi) \in T_xN \oplus T^*_xN \mid dj_x(X) + \xi \in L_x\} = \Lambda^t_{dj_x}(L_x),$$

where $\Lambda^t_{dj_x}$ is the pull-back morphism (2.23).

The next example shows how to fit Marsden-Weinstein reduction [39] in this setting.

Example 4.56 ([11])**.** Let (M, ω) be a symplectic manifold and let G be a connected, compact Lie group acting on M in a Hamiltonian fashion, that is, there exists a moment map $\mu : M \to \mathfrak{g}^*$ for the action (see Example 4.39). On the H-twisted Courant algebroid $\mathbb{T}M$ with $[\![\cdot,\cdot]\!]$ given by (4.1), there is a corresponding reduction data $(\chi, \mathfrak{g}, \mu)$, where

$$\begin{array}{rccc} \chi: & \mathfrak{g} & \longrightarrow & \mathbb{T}M \\ & u & \longmapsto & u_M \end{array}$$

which integrates to

$$\begin{array}{ccl} G & \longrightarrow & \mathrm{Aut}(\mathbb{T}M) \\ g & \longmapsto & \Psi_g = \begin{pmatrix} (\psi_g)_* & 0 \\ 0 & (\psi_{-g})^* \end{pmatrix}. \end{array}$$

The corresponding isotropic subbundle $K \subset (\mathbb{T}M)|_{\mu^{-1}(0)}$ is given by

$$K = \Delta_{\mathfrak{g}} \oplus \mathrm{Ann}\left(T\mu^{-1}(0)\right)$$

and

$$\frac{K^\perp}{K} = \frac{T\mu^{-1}(0) \oplus \mathrm{Ann}\,(\Delta_{\mathfrak{g}})}{\Delta_{\mathfrak{g}} \oplus \mathrm{Ann}\,(T\mu^{-1}(0))} = \frac{T\mu^{-1}(0)}{\Delta_{\mathfrak{g}}} \oplus \frac{\mathrm{Ann}\,(\Delta_{\mathfrak{g}})}{\mathrm{Ann}\,(T\mu^{-1}(0))}.$$

Note that the induced G action on $K^\perp/K$ preserves this decomposition and thus induces a decomposition on E_{red},

$$E_{red} = \left(\frac{T\mu^{-1}(0)}{\Delta_{\mathfrak{g}}}\right)\Big/ G \oplus \left(\frac{\mathrm{Ann}\,(\Delta_{\mathfrak{g}})}{\mathrm{Ann}\,(T\mu^{-1}(0))}\right)\Big/ G,$$

in a sum of Lagrangian subbundles; the second summand is $\ker(p_{red})$ and as we saw (see Remark 4.43) the first summand is isomorphic to TM_{red}. The canonical splitting ∇ for $\mathbb{T}M$ is invariant and K-admissible and the isomorphism corresponding $\Phi_{\nabla_{red}} : E_{red} \to TM_{red} \oplus T^*M_{red}$ to the induced splitting ∇_{red} is given for $(X, \xi) \in (T\mu^{-1}(0) \oplus \mathrm{Ann}\,(\Delta_{\mathfrak{g}}))\big|_{x\in\mu^{-1}(0)}$ by

$$E_{red}\big|_{q(x)} \ni [(X,\xi) + K_x] \mapsto (dq_x(X), \xi_{red}) \in (TM_{red} \oplus T^*M_{red})\big|_{q(x)},$$

where $dq_x^*\xi_{red} = dj_x^*\xi$, $q : \mu^{-1}(0) \to M_{red}$ is the quotient map and $j : \mu^{-1}(0) \to M$ is the inclusion as usual.

Let now $L \subset \mathbb{T}M$ be the graph of $\omega \in \Omega^2(M)$,

$$L = \{(X, i_X\omega) \mid X \in TM\}.$$

It is fairly easy to check that L is invariant if and only if ω is invariant (i.e. $\psi_g^*\omega = \omega$). Also, as

$$i_{u_M}\omega = d\mu^u, \text{ for } u \in \mathfrak{g} \text{ (by definition of } \mu),$$

one has that

$$L|_{\mu^{-1}(0)} \cap K = \{(u_M(x), d\mu_x^u) \mid u \in \mathfrak{g},\, x \in \mu^{-1}(0)\},$$

which has constant rank equal to $\dim(\mathfrak{g})$ as we assumed that G acts freely on M. Therefore, by Theorem 4.54,

$$L_{red} = \left.\frac{L|_{\mu^{-1}(0)} \cap K^\perp + K}{K}\right/ G$$

is a subbundle of E_{red} and for $x \in \mu^{-1}(0)$,

$$\Phi_{\nabla_{red}}(L_{red})\big|_{q(x)} = \{(dq_x(X), \xi_{red}) \in (TM_{red} \oplus T^*M_{red})\big|_{q(x)} \mid (X,\xi) \in (L \cap K^\perp)_x\}.$$

To finish, just note that as ω is invariant and

$$i_X\omega(u_M) = -d\mu^u(X) = 0, \text{ for } X \in T\mu^{-1}(0) \text{ and } u \in \mathfrak{g}$$

one has

$$j^*\omega = q^*\omega_{red}$$

for some $\omega_{red} \in \Omega^2(M_{red})$. Thus,

$$L|_{\mu^{-1}(0)} \cap K^\perp = \{(X, i_X\omega) \mid X \in T\mu^{-1}(0)\}$$

and therefore

$$\begin{aligned}\Phi_{\nabla_{red}}(L_{red}) &= \left\{(dq_x(X), \xi_{red}) \,\middle|\, \begin{array}{l} x \in \mu^{-1}(0),\ X \in T_x\mu^{-1}(0) \text{ and} \\ dq_x^*\xi_{red} = i_X j^*\omega = i_{dq_x(X)}\omega_{red} \end{array}\right\} \\ &= \{(Y, i_Y\,\omega_{red}) \mid Y \in TM_{red}\}.\end{aligned}$$

4.4.1 Reduction of generalized complex structures.

Let E be a Courant algebroid over M and consider reduction data $(\chi, \mathfrak{h}, \mu)$ on E. The proof of Theorem 4.54 adapts easily to the complexified case. That is, let L be a Lagrangian subbundle of $E_{\mathbb{C}} = E \otimes \mathbb{C}$ such that

$$L|_{\mu^{-1}(0)} \cap (K_{\mathbb{C}}) \text{ has constant rank}$$

(where K is the subbundle given by (4.23) and $K_{\mathbb{C}} = K \otimes \mathbb{C}$). Suppose L is G invariant (see Remark 4.50), then [11]

$$L_{red} = \frac{L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}}{K_{\mathbb{C}}} \Big/ G \subset (E_{red})_{\mathbb{C}}$$

is a Lagrangian subbundle.

In the case $L \subset E_{\mathbb{C}}$ is a (almost) generalized complex structure (see Definition 4.9), i.e.

$$L \cap \overline{L} = 0,$$

one would like to have conditions on L in such a way that L_{red} is also a (almost) generalized complex structure. In [11], it is proved that if

$$\left(L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}\right) \cap \left(\overline{L}\,|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}\right) \subset K_{\mathbb{C}}, \tag{4.30}$$

then L_{red} defines a (almost) generalized complex structure on M_{red}. A simple condition ensuring (4.30) is

$$\mathcal{J}(K) = K, \tag{4.31}$$

where $\mathcal{J} : E \to E$ is the bundle map whose $+i$-eingenbundle is L.

Chapter 5

Spinors: Part II

This is main chapter of this thesis. In §5.1, given an exact Courant algebroid over M, we construct the associated Clifford bundle $Cl(E, g)$. Associated to an isotropic splitting $\nabla : TM \to E$, there is a representation

$$\Pi_\nabla : Cl(E, g) \longrightarrow \mathrm{End}\,(\wedge^\bullet T^*M)$$

constructed as in Chapter 3. Following [4, 24], we explain how the deRham differential $d : \Gamma(\wedge^\bullet T^*M) \to \Gamma(\wedge^\bullet T^*M)$ is connected to the Courant bracket $[\![\cdot, \cdot]\!]$ in E, providing, in particular, an alternative approach to integrability of Dirac structures (see Corollary 5.18). In §5.3, we state and prove the main results. Theorem 5.29 gives the reduced pure spinor corresponding to the reduction of Dirac structures explained in §4.4 and Theorem 5.41 provides an alternative proof of the integrability of the reduced Dirac structure. Finally, in §5.4, we give some examples. In order to make the thesis self-contained, we choose to provide the proofs of some known results. We collect them in Appendix C.

5.1 Preliminaries

5.1.1 Clifford bundle

A split-quadratic vector bundle over M is vector bundle E endowed with a fiberwise symmetric bilinear form g such that (E_x, g_x) is a split-quadratic vector space for every $x \in M$ (see Definition 2.11). As usual, E can be a real or a complex vector bundle. In any case, E has always even rank $2n$.

Let $U \subset M$ be an open set and $\mathcal{B} = \{e_1, \cdots, e_{2n}\}$ a frame of E over U. We say that $\mathcal{B}$ is a **polarized** frame if

$$g(e_i, e_j) = \delta_{i+n,j}.$$

Lemma 5.1. *Given any point x of M, there exists a neighbourhood U over which there is a polarized frame.*

In the following, we denote by $O(n,n,\mathbb{F})$ the group of orthogonal transformations of $(\mathbb{F}^n \oplus (\mathbb{F}^n)^*, g_{\text{can}})$ (see Example 2.2) for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 5.2. For a split-quadratic vector bundle (E, g) over M its **polarized frame bundle** is the $O(n,n,\mathbb{F})$-principal bundle $Fr(E)$ over M which in the point $x \in M$ has as its fiber $Fr(E)_x$ the space of polarized basis of E_x.

Let
$$\rho : O(n,n,\mathbb{F}) \to \operatorname{Aut}(Cl(\mathbb{F}^n \oplus (\mathbb{F}^n)^*,\, g_{\text{can}}))$$
be the the representation constructed in Example 3.2 and define the Clifford bundle as the associated bundle

$$Cl(E,g) = Fr(E) \times_\rho Cl(\mathbb{F}^n \oplus (\mathbb{F}^n)^*, g_{\text{can}}).$$

As ρ represents $O(n,n,\mathbb{F})$ as automorphisms, every fiber has a Clifford algebra structure. More precisely,

$$Cl(E,g)_x = Cl(E_x, g_x).$$

For every $A \in O(n,n,\mathbb{F})$, $\rho(A)$ restricted to $\mathbb{F}^n \oplus (\mathbb{F}^n)^* \subset Cl(\mathbb{F}^n \oplus (\mathbb{F}^n)^*, g_{\text{can}})$ (the set of generators) is just A itself. This defines the inclusion $E \hookrightarrow Cl(E,g)$ as the subbundle $Fr(E) \times_\rho (\mathbb{F}^n \oplus (\mathbb{F}^n)^*)$.

Suppose we have a global polarization $l = (L, L')$ (i.e. Lagrangian subbundles L and L' such that $E = L \oplus L'$). This defines a reduction (in the sense of reduction of principal bundles) of $Fr(E)$ to a $GL(n,\mathbb{F})$-principal bundle P, where $GL(n,\mathbb{F})$ sits diagonally inside $O(n,n,\mathbb{F})$:

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}.$$

In this case,
$$Cl(E,g) = P \times_{\rho|_{GL(n,\mathbb{F})}} Cl(\mathbb{F}^n \oplus (\mathbb{F}^n)^*, g_{\text{can}}).$$

The representation of $GL(n,\mathbb{F})$ as automorphisms of $Cl(\mathbb{F}^n \oplus (\mathbb{F}^n)^*, g_{\text{can}})$ preserves the decomposition given by Proposition 3.8 corresponding to the canonical polarization $(\mathbb{F}^n, (\mathbb{F}^n)^*)$ and therefore defines a decomposition of $Cl(E,g)$ into two subbundles:
$$Cl(E,g) = \langle L \rangle \oplus \wedge L'$$

where $\langle L \rangle$ is the bundle of left ideals generated by L and $\wedge L'$ is the bundle of subalgebras generated by L'. As in Proposition 3.9, define a bundle map

$$\begin{array}{rccll} \Pi_l : & Cl(E,g) & \longrightarrow & \operatorname{End}(\wedge^\bullet L') & \\ & a & \longmapsto & \operatorname{pr}_{\wedge^\bullet L'}(a\,\cdot) : & \wedge^\bullet L' \;\to\; \wedge^\bullet L' \\ & & & & \beta \;\mapsto\; \operatorname{pr}_{\wedge^\bullet L'}(a\beta) \end{array}$$

For every $x \in M$, Π_l restricted to $Cl(E,g)_x$ gives a map $Cl(E_x, g_x) \to \operatorname{End}(\wedge^\bullet L'_x)$ which is exactly the representation constructed in §3.1.

Example 5.3. Let E be an exact Courant algebroid over M. Every isotropic splitting $\nabla : TM \to E$ defines a polarization $(\nabla TM, T^*M)$ and thus a representation

$$\Pi_\nabla : Cl(E,g) \longrightarrow \mathrm{End}\,(\wedge^\bullet T^*M)\,.$$

For $e \in E$,

$$\Pi_\nabla(e) = i_{p(e)} + (e - \nabla p(e)) \wedge \cdot\,.$$

If $B \in \Omega^2(M)$ is a 2-form, the map

$$\begin{array}{rccc} F_B: & \wedge^\bullet T^*M & \longrightarrow & \wedge^\bullet T^*M \\ & \alpha & \longmapsto & e^B \wedge \alpha \end{array}$$

satisfies (see Example 3.25)

$$\Pi_{\nabla+B}(a) \circ F_B = F_B \circ \Pi_\nabla(a),\ \forall\, a \in Cl(E,g).$$

Example 5.4. Let E be a Courant algebroid over M and consider the split-quadratic (complex) vector bundle $(E \otimes \mathbb{C}, g_\mathbb{C})$. Any isotropic splitting $\nabla : TM \to E$ for E induces a polarization for $E\otimes\mathbb{C}$ given by $(\nabla(TM)\otimes\mathbb{C},\, T^*M\otimes\mathbb{C})$ (see Example 3.12). This gives

$$\Pi_{\nabla\otimes\mathrm{id}} : Cl(E\otimes\mathbb{C},\, g_\mathbb{C}) \longrightarrow \mathrm{End}\,(\wedge^\bullet T^*M\otimes\mathbb{C})$$

which after the proper identifications is just the $\mathbb{C}$-linear extension of $\Pi_\nabla : Cl(E,g) \to \mathrm{End}\,(\wedge^\bullet T^*M)$.

Example 5.5. Let E be a Courant algebroid over M and $N \subset M$ be a submanifold. Consider the bundle $E|_N$ over N. It inherits a non-degenerate bilinear form g in each fiber and thus one has the Clifford bundle $Cl(E|_N, g)$ over N. It is not difficult to see that

$$Cl(E|_N, g) = Cl(E,g)|_N.$$

Every isotropic splitting $\nabla : TM \to E$ induces a map

$$\nabla|_N : TM|_N \to E|_N$$

which gives a decomposition of $E|_N$ into two Lagrangian subbundles and thus gives a representation $\Pi_{\nabla|_N} : Cl(E|_N, g) \to \mathrm{End}\,(\wedge^\bullet T^*M|_N)$ which is just the restriction of Π_∇ to $Cl(E,g)|_N$.

Example 5.6. Let E be a Courant algebroid over M and G a compact, connected Lie group acting on M. Consider reduction data $(\chi, \mathfrak{h}, \mu)$ on E and the corresponding isotropic subbundle K (4.23). Every K-admissible splitting ∇ induces an isotropic splitting for the exact sequence

$$0 \longrightarrow \left(\frac{T\mu^{-1}(0)}{\Delta_\mathfrak{g}}\right)^* \xrightarrow{p_K^*} \frac{K^\perp}{K} \xrightarrow{p_K} \frac{T\mu^{-1}(0)}{\Delta_\mathfrak{g}} \longrightarrow 0.$$

It thus defines a polarization for $(K^\perp/K, g_K)$ which in turn gives a representation

$$\Pi_{\nabla_K} : Cl\left(\frac{K^\perp}{K}, g_K\right) \longrightarrow \mathrm{End}\left(\wedge^\bullet\left(\frac{T\mu^{-1}(0)}{\Delta_{\mathfrak{g}}}\right)^*\right).$$

For the quotient map $q : \mu^{-1}(0) \to M_{red}$, the induced bundle map

$$dq : \frac{T\mu^{-1}(0)}{\Delta_{\mathfrak{g}}} \longrightarrow q^*TM_{red}$$

is an isomorphism and the dual map identifies $(T\mu^{-1}(0)/\Delta_{\mathfrak{g}})^*$ with $q^*T^*M_{red}$. We will henceforth consider Π_{∇_K} as a representation of $Cl(K^\perp/K, g_K)$ on $\wedge^\bullet q^*T^*M_{red}$. For $x \in \mu^{-1}(0)$, $k^\perp + K \in (K^\perp/K)_x$ and $\alpha \in T^*_{q(x)}M_{red}$

$$\Pi_{\nabla_K}(k^\perp + K)\alpha = i_{dq_x(p(k^\perp))}\alpha + s_{\nabla_K}(k^\perp + K) \wedge \alpha, \tag{5.1}$$

where $s_{\nabla_K}(k^\perp + K) \in T^*_{q(x)}M_{red}$ is such that

$$(dq_x \circ p)^* s_{\nabla_K}(k^\perp + K) = k^\perp - \nabla k^\perp.$$

5.1.2 Cartan calculus on Clifford modules.

Let E be a Courant algebroid over M and let $\nabla : TM \to E$ be an isotropic splitting with curvature $H \in \Omega^3(M)$. Consider the induced representation $\Pi_\nabla : Cl(E, g) \to \mathrm{End}(\wedge^\bullet T^*M)$. The group $\mathrm{Aut}(E)$ of automorphisms of E acts on $\Gamma(\wedge^\bullet T^*M)$ as follows: for $(\Psi, \psi) \in \mathrm{Aut}(E)$, let $(\psi, B) \in \mathrm{Diff}_{[H]}(M) \times \Omega^2(M)$ be its representation given by Proposition 4.16 and define

$$(\psi, B) \cdot \alpha = (\psi^{-1})^*(e^{-B} \wedge \alpha), \text{ for } \alpha \in \wedge^\bullet T^*M. \tag{5.2}$$

Remark 5.7. The action (5.2) was first considered in [29] although it is not clear in their paper from where this action appear.

We shall spend some time explaining how one gets (5.2). From the isotropic splitting ∇, we get polarizations $(l_1)_x = (\nabla T_xM, T^*_xM)$ of (E_x, g_x) for every $x \in M$. The element $(\Psi, \psi) \in \mathrm{Aut}(E)$ defines a second polarization

$$(l_2)_x = (\Psi(\nabla T_{\psi^{-1}(x)}M), T^*_xM),$$

(recall that Ψ preserves T^*M). As $\Psi_x : (E_x, g_x) \to (E_{\psi(x)}, g_{\psi(x)})$ is an isomorphism, it induces an isomorphism of Clifford algebras

$$Cl(\Psi_x) : Cl(E_x, g_x) \longrightarrow Cl(E_{\psi(x)}, g_{\psi(x)})$$

that when restricted to $\wedge^\bullet T^*_xM$ gives an isomorphism of the Clifford modules $\wedge^\bullet T^*_xM$ and $\wedge^\bullet T^*_{\psi(x)}M$ corresponding to $(l_1)_x$ and $(l_2)_{\psi(x)}$ respectively (see the beginning of §3.2). Now, by (4.11) it follows that this isomorphism is exactly

$$\wedge^\bullet T^*_xM \ni \alpha \longmapsto (d\psi^{-1}_{\psi(x)})^*\alpha \in \wedge^\bullet T^*_{\psi(x)}M.$$

To get back to the polarization l_1, we have to use Proposition 3.23 to obtain an isomorphism of $Cl(E_{\psi(x)}, g_{\psi(x)})$ modules:

$$F_{l_2 l_1} : \wedge^\bullet T^*_{\psi(x)}M \longrightarrow \wedge^\bullet T^*_{\psi(x)}M$$

by choosing a pure spinor $\varphi \in \wedge^\bullet T^*_{\psi(x)}M$ such that

$$\mathcal{N}_\nabla(\varphi) = \Psi(\nabla T_x M) = \Phi_\nabla^{-1}(\Psi_\psi \circ \tau_B(T_x M))$$

where the last equality follows from Proposition 4.16. Now, note that

$$\Psi_\psi \circ \tau_B(T_x M) = \tau_{(\psi^{-1})^* B}(T_{\psi(x)}M).$$

By (3.15) and Example 3.19, this implies that

$$\mathcal{N}_\nabla(e^{-(d\psi^{-1}_{\psi(x)})^* B_x}) = \Psi(\nabla T_x M)$$

and therefore

$$F_{l_2 l_1}(\beta) = e^{-(d\psi^{-1}_{\psi(x)})^* B_x} \wedge \beta.$$

By composing with $Cl(\Psi_x)|_{\wedge^\bullet T^*_x M}$, we get:

$$\begin{array}{ccc} \wedge^\bullet T_x M & \longrightarrow & \wedge^\bullet T^*_{\psi(x)}M \\ \alpha & \longmapsto & e^{-(d\psi^{-1}_{\psi(x)})^* B_x} \wedge (d\psi^{-1}_{\psi(x)})^* \alpha = (d\psi^{-1}_{\psi(x)})^*(e^{-B_x} \wedge \alpha) \end{array}$$

which is exactly the action given by (5.2).

Remark 5.8. This construction extends naturally to the complexified setting. By that we mean the following: consider the extension of $(\Psi, \psi) \in \mathrm{Aut}(E)$ to

$$\begin{array}{ccc} E \otimes \mathbb{C} & \xrightarrow{\Psi \otimes \mathrm{id}} & E \otimes \mathbb{C} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\psi} & M. \end{array}$$

Given an isotropic splitting $\nabla : TM \to E$, we have a polarization of $E \otimes \mathbb{C}$ given by $l_1 = (\nabla(TM) \otimes \mathbb{C}, T^*M \otimes \mathbb{C})$ which gives $\Gamma(\wedge^\bullet T^*M \otimes \mathbb{C})$ the structure of a Clifford module (see Example 5.4). Using the intermediary polarization $l_2 = (\Psi(\nabla(TM)) \otimes \mathbb{C}, T^*M \otimes \mathbb{C})$, we get an action of (Ψ, ψ) on $\Gamma(\wedge^\bullet T^*M \otimes \mathbb{C})$ given by

$$(\Psi, \psi) \cdot (\alpha + i\beta) = \psi^*(e^{-B} \wedge \alpha) + i\, \psi^*(e^{-B} \wedge \beta) = \psi^*(e^{-B} \wedge (\alpha + i\,\beta))$$

where $\alpha, \beta \in \Gamma(\wedge^\bullet T^*M)$ and $B \in \Omega^2(M)$ is such that (4.9) holds.

Let us see how the action (5.2) depends on the splitting. If we change the splitting ∇ by a 2-form $B' \in \Omega^2(M)$, then we have a corresponding representation $\Pi_{\nabla + B'} : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet T^*M)$. The isomorphism

$$\begin{array}{cccc} F_{B'} : & \wedge^\bullet T^*M & \to & \wedge^\bullet T^*M \\ & \alpha & \longmapsto & e^{B'} \wedge \alpha \end{array}$$

intertwines Π_∇ and $\Pi_{\nabla+B'}$ (see Example 3.25). The action of $\mathrm{Aut}(E)$ relative to $\nabla+B'$ takes place on the space of the representation $\Pi_{\nabla+B'}$. If the pair $(\psi, B) \in \mathrm{Diff}_{[H]}(M) \times \Omega^2(M)$ corresponds to (Ψ, ψ) via ∇, then the pair obtained via $\nabla + B'$ is given by (see (4.12))

$$(\psi, B' - \psi^* B' + B)$$

and therefore the action is given, for $\alpha \in \Gamma(\wedge^\bullet T^*M)$, by

$$\alpha \mapsto (\psi^{-1})^*(e^{-B'-\psi^* B'+B} \wedge \alpha).$$

We shall prove that $F_{B'}$ intertwines the actions corresponding to the different splitting, that is

$$F_{B'}((\psi^{-1})^*(e^{-B} \wedge \alpha)) = (\psi^{-1})^*(e^{-B'-\psi^* B'+B} \wedge F_{B'}(\alpha))$$

Indeed,

$$\begin{aligned}(\psi^{-1})^* \left[e^{-B'+\psi^* B'-B} \wedge (e^{B'} \wedge \alpha)\right] &= (\psi^{-1})^*(e^{\psi^* B'-B} \wedge \alpha) \\ &= e^{B'} \wedge (\psi^{-1})^*(e^{-B} \wedge \alpha)\end{aligned}$$

as we wanted.

Still with the isotropic splitting ∇, given an one-parameter subgroup $(\psi_t, B_t) \in \mathrm{Aut}(E)$ generated by the element $(X, B) \in \mathrm{Der}(E)$, one has for $\alpha \in \Gamma(\wedge^\bullet T^*M)$ the infinitesimal action:

$$(X, B) \cdot \alpha := \left.\frac{d}{dt}\right|_{t=0} \psi_t^*(e^{B_t} \wedge \alpha) = \mathcal{L}_X \alpha + B \wedge \alpha, \text{ for } \alpha \in \Gamma(\wedge^\bullet T^*M).$$

For $e \in \Gamma(TM)$, define the Lie derivative of $\alpha \in \wedge^\bullet T^*M$ along e as

$$\mathcal{L}_e^\nabla \alpha := ad(\Phi_\nabla(e)) \cdot \alpha = \mathcal{L}_X \alpha + (d\xi - i_X H) \wedge \alpha, \tag{5.3}$$

where H is the curvature of ∇ and $\Phi_\nabla(e) = X + \xi$.

In the following we shall use the $\mathbb{Z}_2$-grading of $\Gamma(\wedge^\bullet T^*M)$ in even and odd forms and the corresponding theory of super-commutator for its endomorphisms (see the discussion in end of §3.1). Just as one obtains the Lie bracket of vector fields from the interior product and the Lie derivative as

$$i_{[X,Y]} = [\mathcal{L}_X, i_Y], \text{ for } X, Y \in \Gamma(TM),$$

one can derive the Courant bracket from the action of E on $\wedge^\bullet T^*M$ (induced from $Cl(E, g)$) and the Lie derivative (5.3) (note that it is an even endomorphism). More precisely, we have:

Proposition 5.9 (Alekseev-Xu [4], Hu-Uribe[29]). *Let $e_1, e_2 \in \Gamma(E)$ and ∇ be an isotropic splitting for E. Then*

$$\Pi_\nabla([\![e_1, e_2]\!]) = [\mathcal{L}_{e_1}^\nabla, \Pi_\nabla(e_2)]$$

and

$$\mathcal{L}_{[\![e_1,e_2]\!]}^\nabla = [\mathcal{L}_{e_1}^\nabla, \mathcal{L}_{e_2}^\nabla].$$

In this context, there is also a Cartan formula for the Lie derivative as we now explain. Every representation space $\wedge^\bullet T^*M$ for $Cl(E,g)$ associated to an isotropic splitting ∇ with curvature H comes with an odd differential given by

$$d^\nabla = d - H \wedge \cdot\,.$$

These differentials are intertwined via the isomorphisms F_B given in Example 5.3. More precisely, if $B \in \Omega^2(M)$ is a 2-form, then

$$F_B \circ d^\nabla = d^{\nabla+B} \circ F_B. \tag{5.4}$$

Indeed, one has that

$$d^{\nabla+B} = d - (H + dB) \wedge \cdot.$$

and for $\alpha \in \Gamma(\wedge^\bullet T^*M)$

$$\begin{aligned} F_B(d^\nabla \alpha) = &\ e^B \wedge (d\alpha - H \wedge \alpha) \\ = &\ d(e^B \wedge \alpha) - dB \wedge e^B \wedge \alpha - H \wedge e^B \wedge \alpha \\ = &\ (d - (H + dB) \wedge \cdot\,) e^B \wedge \alpha \\ = &\ d^{\nabla+B} F_B(\alpha). \end{aligned}$$

Proposition 5.10 (Alekseev-Xu [4], Gualtieri[24])**.** *For $e \in \Gamma(E)$,*

$$\mathcal{L}_e^\nabla = [d^\nabla,\, \Pi_\nabla(e)]$$

and

$$[d^\nabla, \mathcal{L}_e^\nabla] = 0.$$

Remark 5.11. The Cartan-like identities of Propositions 5.9 and 5.10 extend to the complexified picture. For the isotropic splitting ∇, consider the Clifford module $\Gamma(\wedge^\bullet T^*M \otimes \mathbb{C})$ corresponding to $\nabla \otimes \mathrm{id}$ (see Example 5.4). Consider the $\mathbb{C}$-linear extension of the differential d^∇

$$d^\nabla : \Gamma(\wedge^\bullet T^*M \otimes \mathbb{C}) \longrightarrow \Gamma(\wedge^\bullet T^*M \otimes \mathbb{C}).$$

Then, for $e_1, \ldots, e_4 \in \Gamma(E)$, one has

$$[d^\nabla, \Pi_\nabla(e_1 + i\, e_2)] = \mathcal{L}_{e_1}^\nabla + i\, \mathcal{L}_{e_2}^\nabla$$

and

$$[\,[d^\nabla, \Pi_\nabla(e_1 + i\, e_2)], \Pi_\nabla(e_3 + i\, e_4)] = \Pi_\nabla\left([\![e_1 + i\, e_2, e_3 + i\, e_4]\!] \right).$$

Let G be a compact, connected Lie group acting freely on M and $\chi : \mathfrak{a} \to \Gamma(E)$ be an extended G-action of an exact Courant algebra $\rho : \mathfrak{a} \to \mathfrak{g}$. Consider ∇ an isotropic splitting for E and let $\alpha \in \Gamma(\wedge^\bullet T^*M)$ and $a \in \mathfrak{a}$. One has that

$$\mathcal{L}_{\chi(a)}^\nabla \alpha = (ad \circ \chi(a)) \cdot \alpha.$$

As $ad \circ \chi(a) = 0$ for $a \in \mathfrak{h} = \ker(\rho)$, we can pass to the quotient $\mathfrak{g} = \mathfrak{a}/\mathfrak{h}$ to define an infinitesimal action of $\mathfrak{g}$ on $\wedge^\bullet T^*M$

$$\begin{array}{rccl} \Sigma: & \mathfrak{g} & \longrightarrow & \operatorname{End}(\wedge^\bullet T^*M) \\ & u & \longmapsto & \mathcal{L}^\nabla_{\chi(u)}. \end{array} \tag{5.5}$$

As χ is bracket-preserving and by Proposition 5.9, one has

$$\Sigma([u,v]) = \mathcal{L}^\nabla_{\chi([u,v])} = \mathcal{L}^\nabla_{[\![\chi(u),\chi(v)]\!]} = [\mathcal{L}^\nabla_{\chi(u)}, \mathcal{L}^\nabla_{\chi(v)}] = [\Sigma(u), \Sigma(v)].$$

This infinitesimal action integrates to a G-action which is the composition of $G \to \operatorname{Aut}(E)$ with (5.2).

Remark 5.12. If χ is a lifted G-action and ∇ is invariant, then for $u \in \mathfrak{g}$,

$$\mathcal{L}^\nabla_{\chi(u)} = \mathcal{L}_{u_M}$$

and the action of G on the ∇ representation $\wedge^\bullet T^*M$ is just

$$G \ni g \mapsto (\psi_{g^{-1}})^*.$$

5.2 Pure spinors.

Let (E, g) be a split-quadratic vector bundle over M and let $l = (L, L')$ be a polarization of (E, g). Consider the representation

$$\Pi_l : Cl(E, g) \longrightarrow \operatorname{End}(\wedge^\bullet L')$$

and let L'' be a Lagrangian subbundle of E. For every $x \in M$, Proposition 3.18 gives that there exists a line $U^l(L''_x) \subset \wedge^\bullet L'_x$ such that for every $\varphi \in U^l(L''_x)$,

$$\mathcal{N}_l(\varphi) = \{e \in E_x \mid \Pi_l(e)\varphi = 0\} = L''.$$

We claim that $U^l(L'')$ defines a line bundle over M. Indeed, for each open neighborhood $\mathcal{W}$ over which L'' has a frame $\{e_1, \ldots, e_n\}$, $U^l(L'')$ can be seen as the kernel of the bundle map

$$\Theta : \wedge^\bullet L'|_{\mathcal{W}} \longrightarrow \underbrace{\wedge^\bullet L'|_{\mathcal{W}} \times \cdots \times \wedge^\bullet L'|_{\mathcal{W}}}_{\text{n times}} \tag{5.6}$$

given by $\Theta(\alpha) = (\Pi_l(e_1)\alpha, \ldots, \Pi_l(e_n)\alpha)$ (this follows from (3.14)). We call $U^l(L'')$ the **pure spinor line bundle** corresponding to L''.

Example 5.13. Let ∇ be an isotropic splitting for E and consider the representation $\Pi_\nabla : Cl(E, g) \to \operatorname{End}(\wedge^\bullet T^*M)$. For every Lagrangian subbundle L of E, one has its pure spinor line bundle

$$U^\nabla(L).$$

If we change the splitting by a 2-form B, then (see Example 5.3)

$$U^{\nabla+B}(L) = F_B(U^\nabla(L)) = \{e^B \wedge \varphi \mid \varphi \in U^\nabla(L)\}.$$

Example 5.14. Consider $(\mathbb{T}M, g_{\mathrm{can}})$ and the canonical polarization (TM, T^*M). For $L = T^*M$, the pure spinor line bundle is given by $\wedge^{\mathrm{top}} T^*M$. It has a global section if and only if M is orientable.

Example 5.15. Let $J : TM \to TM$ be a complex structure on M (see Definition 4.9) and consider its $-i$-eigenbundle $T_{0,1} \subset TM \otimes \mathbb{C}$. As shown in Example 3.21, the pure spinor line bundle corresponding to $L = T_{0,1} \oplus T^{1,0} \subset \mathbb{T}M \otimes \mathbb{C}$ is

$$U(L) = \wedge^{n,0} T^*M \subset \wedge^\bullet T^*M \otimes \mathbb{C}.$$

Examples 5.14 and 5.15 show that pure spinor line bundles are not always trivial. In general, one can guarantee only the existence of *local sections* φ of $U^l(L'')$ (by local section we mean that there exists a open neighborhood $\mathcal{V}$ such that, for every $x \in \mathcal{V}$, one has that $0 \neq \varphi_x \in U^l(L''_x)$).

Example 5.16. Let E be a Courant algebroid and let $N \subset M$ be a submanifold. As we saw in Example 5.5, every isotropic splitting ∇ induces a representation

$$\Pi_{\nabla|_N} : Cl(E|_N, g) \longrightarrow \wedge^\bullet T^*M|_N.$$

To every Lagrangian subbundle $L \subset E|_N$ there corresponds its pure spinor line bundle $U^{\nabla|_N}(L)$. Note that if $L = \tilde{L}|_N$ for some Lagrangian subbundle $\tilde{L} \subset E$, then

$$U^{\nabla|_N}(L) = U^{\nabla}(\tilde{L})|_N.$$

We now give a result from [4, 24] which translates the integrability of Lagrangian subbundles of E into properties of its pure spinor line bundle. It should be seen as an analogue of Frobenius theorem in the setting of differential ideals. Thereafter, we study the integrability of invariant Lagrangian subbundles and give some examples.

Let $L \subset E$ be a Lagrangian subbundle and φ be a local section of $U^{\nabla}(L)$ over an open neighborhood $\mathcal{W}$. Choose L' to be any Lagrangian subbundle of E such that $l = (L, L')$ is a polarization (which exists by Proposition 4.32) and consider the corresponding spinor bundle $\wedge^\bullet L'$. Consider the bundle isomorphism

$$\begin{array}{rccc} F_l: & \wedge^\bullet L'|_{\mathcal{W}} & \longrightarrow & \wedge^\bullet T^*M|_{\mathcal{W}} \\ & \alpha & \longmapsto & \Pi_{\nabla}(\alpha)\varphi \end{array}$$

which is pointwise the isomorphism of Proposition 3.23.

Proposition 5.17 ([4, 24])**.** *One has that*

$$\Upsilon = F_l^{-1}(d^{\nabla}\alpha) \in \bigoplus_{i \leq 3} \wedge^i L'|_{\mathcal{W}}. \tag{5.7}$$

Moreover, its degree 2 component is zero and, for any sections $e_1, e_2, e_3 \in \Gamma(L)$,

$$\Pi_l(e_1 \wedge e_2 \wedge e_3)\Upsilon = g(e_1, [\![e_2, e_3]\!])\ . \tag{5.8}$$

Corollary 5.18 ([4, 24])**.** *L is integrable if and only if*

$$d^{\nabla}(U^{\nabla}(L)) \subset \Pi_{\nabla}(L')\, U^{\nabla}(L).$$

Example 5.19. Let E be $\mathbb{T}M$ with the standard Courant bracket (4.1) and consider a 2-form $\omega \in \Omega^2(M)$. As we saw, its graph

$$\text{Graph}\,(\omega) = \{(X, i_X\omega) \mid X \in TM\}$$

is a Lagrangian subbundle of E. It has T^*M as a Lagrangian complement; thus, the obstruction to the integrability of Graph$\,(\omega)$ is a section Υ of T^*M with components only in degree 1 and 3. The canonical splitting ∇_{can} has zero curvature and therefore

$$d^{\nabla_{\text{can}}} = d.$$

Now, $\varphi = e^{-\omega}$ is a section of $U(\text{Graph}\,(\omega))$ and

$$d\, e^{-\omega} = -d\omega \wedge e^{-\omega}.$$

Therefore, the degree 1 component $\Upsilon \in \Gamma(T^*M)$ is zero and the degree 3 component is $-d\omega$.

Example 5.20. Let $\Delta \subset TM$ be a subbundle and consider

$$L = \Delta \oplus \text{Ann}\,(\Delta) \subset \mathbb{T}M.$$

It is a Lagrangian subbundle and its pure spinor line bundle $U(L)$ relative to the canonical polarization is given by $\det(\text{Ann}\,(\Delta)) \subset T^*M$. Let $\mathcal{V}$ be an open neighborhood of M such that there is a frame $\{X_1, \cdots, X_n\}$ of TM with $\{X_1, \cdots, X_r\}$ a frame for Δ. Consider the dual frame $\{\xi^1, \ldots, \xi^n\} \subset T^*M$ over $\mathcal{V}$. The subbundle L' of $\mathbb{T}M|_{\mathcal{V}}$ generated by $\{X_{r+1}, \ldots, X_n, \xi^1, \ldots, \xi^r\}$ is a Lagrangian complement to $L|\mathcal{V}$ and

$$\Omega = \xi^{r+1} \wedge \cdots \wedge \xi^n \in \Gamma(U(L)|_{\mathcal{V}}).$$

Thus, one has

$$d\Omega = \Pi(\Upsilon)\, \Omega, \text{ for some } \Upsilon \in \Gamma(\wedge^{\bullet} L').$$

It is straightforward to check that Υ has degree 1 component given by

$$\Upsilon_{(1)} = \sum_{1 \leq j \leq r < k \leq n} \xi^k([X_k, X_j])\, \xi^j$$

and the degree 3 component is given by

$$\Upsilon_{(3)} = \sum_{1 \leq j,\, k \leq r < i \leq n} \xi^i([X_k, X_j])\, X_i \wedge \xi^j \wedge \xi^k.$$

Example 5.21. Let M be a m-dimensional smooth manifold and let $\pi \in \Gamma(\wedge^2 TM)$ be a bivector field with its corresponding bundle map

$$\begin{array}{rccc} \pi^\sharp : & T^*M & \longrightarrow & TM \\ & \xi & \longmapsto & i_\xi \, \pi. \end{array}$$

Consider the Lagrangian subbundle of $\mathbb{T}M$ given by

$$L_\pi = \mathrm{Graph}\left(\pi^\sharp\right) = \{(\pi^\sharp(\xi), \xi) \mid \xi \in T^*M\}.$$

In [17], it is proved that π is Poisson (i.e. $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket on multivector fields) if and only if L_π is involutive under the standard Courant bracket $[\![\cdot, \cdot]\!]$ given by (4.1). Over a neighborhood $\mathcal{U}$ over which there is a section ν of $\wedge^m T^*M$,

$$\varphi = i_{e^{-\pi}} \, \nu = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} i_{\pi^k} \, \nu$$

is a local section of the pure spinor line bundle $U(L) \subset \wedge^\bullet T^*M$ by Example 3.26. The fact that $L_\pi \cap TM = 0$ implies that $TM \oplus L_\pi = \mathbb{T}M$. Hence, by Proposition 5.17, there exists $\Upsilon_{(1)} \in \Gamma(TM|_{\mathcal{U}})$ and $\Upsilon_{(3)} \in \Gamma(\wedge^3 TM|_{\mathcal{U}})$ such that

$$d\varphi = \Pi(\Upsilon_{(1)} + \Upsilon_{(3)}) \, \varphi. \tag{5.9}$$

Looking the degree $m-1$ part of (5.9), one has that

$$d \, i_\pi \nu = -i_{\Upsilon_{(1)}} \nu.$$

This is the defining equation for X_π, the *modular vector field* of π with respect to ν [51]. Also, it can be proven (see [23]) that

$$\Upsilon_{(3)} = -\frac{1}{2}[\pi, \pi]$$

so that

$$d\varphi = \Pi\left(X_\pi - \frac{1}{2}[\pi, \pi]\right) \varphi.$$

In the rest of this section, we fix an isotropic splitting ∇ for E with curvature $H \in \Omega^3(M)$ and consider the representation space $\wedge^\bullet T^*M$ for $Cl(E, g)$ associated to ∇ (see Example 5.3). Let G be a compact, connected Lie group acting freely on M and consider an isotropic lifted G-action $\chi : \mathfrak{g} \to \Gamma(E)$. Let $L \subset E$ be a Lagrangian subbundle and consider its pure spinor line bundle $U^\nabla(L) \subset \wedge^\bullet T^*M$.

Proposition 5.22. *L is an invariant subbundle of E if and only if for every open neighborhood $\mathcal{W}$ of M and every section $\varphi \in \Gamma(U^\nabla(L)|_{\mathcal{W}})$*

$$\left(\mathcal{L}^\nabla_{\chi(u)} \varphi\right)_x \in U^\nabla(L_x), \ \forall \, x \in \mathcal{W} \text{ and } \forall \, u \in \mathfrak{g}.$$

In this case, $U^\nabla(L)$ is a G-equivariant line bundle over M.

Proof. Let $\mathcal{W}$ be an open neighborhood where $U^\nabla(L)$ has a local section φ and let $e \in \Gamma(L)$. For every $u \in \mathfrak{g}$, we have from Proposition 5.9

$$\Pi_\nabla(e)\,\mathcal{L}^\nabla_{\chi(u)}\,\varphi = -[\mathcal{L}^\nabla_{\chi(u)}, \Pi_\nabla(e)]\,\varphi + \mathcal{L}^\nabla_{\chi(u)}\,\Pi_\nabla(e)\,\varphi = -\Pi_\nabla\left(\llbracket\chi(u), e\rrbracket\right)\,\varphi,$$

as $\Pi_\nabla(e)\varphi = 0$. Now, if L is invariant, then (see definition 4.52)

$$\llbracket\chi(u), e\rrbracket \in \Gamma(L)$$

and therefore

$$\Pi_\nabla(e)\,\mathcal{L}^\nabla_{\chi(u)}\,\varphi = 0.$$

Choosing a frame $\{e_1, \ldots, e_n\}$ for L over $\mathcal{W}$ (or maybe a smaller neighborhood) and considering the map Θ from (5.6) one has that $\Theta(\mathcal{L}^\nabla_{\chi(u)}\varphi) = 0$ and therefore $\mathcal{L}^\nabla_{\chi(u)}\varphi \in \Gamma(U^\nabla(L)\big|_{\mathcal{W}})$. Conversely, if $\mathcal{L}^\nabla_{\chi(u)}\varphi$ is a section of $U^\nabla(L)$ over $\mathcal{W}$, then

$$\Pi_\nabla\left(\llbracket\chi(u), e\rrbracket\right)\,\varphi = 0$$

which implies that $\llbracket\chi(u), e\rrbracket|_{\mathcal{W}} \in \Gamma(L|_{\mathcal{W}})$. By covering M with open neighborhood in which $U^\nabla(L)$ has local sections, one proves that

$$\llbracket\chi(u), e\rrbracket \in \Gamma(L).$$

As $e \in \Gamma(L)$ and $u \in \mathfrak{g}$ are arbitrary, it follows that L is invariant.

As for the last statement, recall (see the end of §5.1) that the infinitesimal action

$$\mathfrak{g} \ni u \longmapsto \mathcal{L}^\nabla_{\chi(u)} \in \mathrm{End}\left(\wedge^\bullet T^*M\right)$$

integrates for a G-action. Due to the connectedness of G, the infinitesimal invariance of $U^\nabla(L)$ is sufficient to guarantee that it is globally invariant. $\square$

Remark 5.23. The proof of Proposition 5.22 extends to the complexification $E \otimes \mathbb{C}$. One has to extend the action of G to $E \otimes \mathbb{C}$ (see Remark 4.50) and to consider the $\mathbb{C}$-linear extension of $\mathcal{L}^\nabla_{\chi(u)}$ to $\Gamma(\wedge^\bullet T^*M \otimes \mathbb{C})$. Then, as observed in Remark 5.11, Propositions 5.9 and 5.10 extend to this case, thus giving that a complex subbundle $L \subset E \otimes \mathbb{C}$ is invariant if and only if its pure spinor line $U^{\nabla\otimes\mathrm{id}}(L) \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ is invariant (see Remark 5.8).

Remark 5.24. We shall need a general result from G-equivariant fiber bundles over M. The important property here is that G acts freely on M; this implies that all these bundles have invariant sections over some invariant open neighborhoods. The idea is that for a G-equivariant fiber bundle $P \to M$, one can form the quotient bundle P/G over M/G and sections for P/G over a neighborhood $\mathcal{V}$ such that

$$q^{-1}(\mathcal{V}) \cong \mathcal{V} \times G, \text{ where } q : M \to M/G \text{ is the quotient map}$$

induces naturally invariant sections for P over $q^{-1}(\mathcal{V})$.

We will specialize Proposition 5.17 to invariant polarization of E; this will be important to prove Theorem 5.41. So let $l = (L, L')$ be an invariant polarization of E. Extend the action of G on L' to its exterior algebra by the natural extensions

$$\wedge\Psi_g : \wedge^\bullet L' \to \wedge^\bullet L', \text{ for } g \in G$$

and consider the corresponding infinitesimal action given for $u \in \mathfrak{g}$ by

$$\mathcal{L}^l_{\chi(u)}(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^{k} e_1 \wedge \cdots \wedge [\![\chi(u), e_i]\!] \wedge \cdots \wedge e_k, \text{ for } e_1, \ldots, e_k \in \Gamma(L').$$

and

$$\mathcal{L}^l_{\chi(u)} f = \mathcal{L}_{u_M} f, \text{ for } f \in \Gamma(\wedge^0 L') = C^\infty(M).$$

Observe that $\mathcal{L}^l_{\chi(u)}$ is an even derivation of $\wedge^\bullet L'$ for every $u \in \mathfrak{g}$.

Remark 5.25. If ∇ is invariant, consider the invariant polarization $l = (\nabla TM, T^*M)$. In this case, for $u \in \mathfrak{g}$

$$\mathcal{L}^l_{\chi(u)} = \mathcal{L}^\nabla_{\chi(u)} = \mathcal{L}_{u_M}.$$

Lemma 5.26. *Let $l = (L, L')$ be an invariant polarization. For $\alpha \in \Gamma(\wedge^\bullet L')$ and $u \in \mathfrak{g}$,*

$$[\mathcal{L}^\nabla_{\chi(u)}, \Pi_\nabla(\alpha)] = \Pi_\nabla(\mathcal{L}^l_{\chi(u)}\alpha).$$

Proof. We shall prove it by induction on the degree of α. For degree 0, both sides equal $\mathcal{L}_{u_M}\alpha \wedge \cdot$. For degree 1, the equality follows from Proposition 5.9. Now, suppose it holds for degree k and let α has degree $k+1$. Locally, α can be written as

$$\alpha = \alpha_{(k)} \wedge \alpha_{(1)},$$

where $\alpha_{(k)}$ has degree k and $\alpha_{(1)}$ has degree 1. Thus,

$$\begin{aligned}
[\mathcal{L}^\nabla_{\chi(u)}, \Pi_\nabla(\alpha)] = &\ \mathcal{L}^\nabla_{\chi(u)}\Pi_\nabla(\alpha_{(k)} \wedge \alpha_{(1)}) - \Pi_\nabla(\alpha_{(k)})\mathcal{L}^\nabla_{\chi(u)}\Pi_\nabla(\alpha_{(1)}) \\
&+ \Pi_\nabla(\alpha_{(k)})\mathcal{L}^\nabla_{\chi(u)}\Pi_\nabla(\alpha_{(1)}) - \Pi_\nabla(\alpha_{(k)} \wedge \alpha_{(1)})\mathcal{L}^\nabla_{\chi(u)} \\
= &\ [\mathcal{L}^\nabla_{\chi(u)}, \Pi_\nabla(\alpha_{(k)})]\, \Pi_\nabla(\alpha_{(1)}) + \Pi_\nabla(\alpha_{(k)})\, [\mathcal{L}^\nabla_{\chi(u)}, \Pi_\nabla(\alpha_{(1)})] \\
= &\ \Pi_\nabla(\mathcal{L}^l_{\chi(u)}\alpha_{(k)})\Pi_\nabla(\alpha_{(1)}) + \Pi_\nabla(\alpha_{(k)})\Pi_\nabla(\mathcal{L}^l_{\chi(u)}\alpha_{(1)}) \\
= &\ \Pi_\nabla(\mathcal{L}^l_{\chi(u)}\alpha).
\end{aligned}$$

□

Proposition 5.27. *Let $l = (L, L')$ be an invariant polarization of E. If $\varphi \in \Gamma(U^\nabla(L)|_\mathcal{V})$ is an invariant section of the pure spinor line bundle of L over $\mathcal{V}$, then the section $\Upsilon \in \Gamma(\wedge^\bullet L'|_\mathcal{V})$ defined by (5.7) is invariant, i.e.*

$$\mathcal{L}^l_{\chi(u)}\Upsilon = 0, \text{ for every } u \in \mathfrak{g}.$$

Proof. Let $\mathcal{V}$ be an invariant open neighborhood of M over which there is an invariant section φ of $U^\nabla(L)$ and let $\Upsilon \in \Gamma(\wedge^\bullet L'|_{\mathcal{V}})$ be given by (5.7). By Proposition 5.10 and the fact that φ is invariant, one has for $u \in \mathfrak{g}$

$$\mathcal{L}^\nabla_{\chi(u)} d^\nabla \varphi = d^\nabla \mathcal{L}^\nabla_{\chi(u)}\varphi + [\mathcal{L}^\nabla_{\chi(u)}, d^\nabla]\varphi = 0.$$

Thus,

$$\mathcal{L}^\nabla_{\chi(u)} \Pi_\nabla(\Upsilon)\varphi = 0.$$

Now, by Lemma 5.26 and again by the invariance of φ,

$$0 = \mathcal{L}^\nabla_{\chi(u)}\Pi_\nabla(\Upsilon)\varphi = \Pi_\nabla(\Upsilon)\mathcal{L}^\nabla_{\chi(u)}\varphi + \Pi_\nabla(\mathcal{L}^l_{\chi(u)}\Upsilon)\varphi = \Pi_\nabla(\mathcal{L}^l_{\chi(u)}\Upsilon)\varphi. \quad (5.10)$$

As

$$F_l : \wedge^\bullet L' \ni \alpha \mapsto \Pi_\nabla(\alpha)\varphi \in \wedge^\bullet T^*M$$

is an isomorphism, equation 5.10 implies that

$$\mathcal{L}^l_{\chi(u)}\Upsilon = 0$$

as we wanted to proof. □

Remark 5.28. Let $u \in \mathfrak{g}$; as $\mathcal{L}^l_{\chi(u)}$ preserves the $\mathbb{Z}$-degree of $\wedge^\bullet L'$, it follows that both the components of degree 1 and 3 of Υ are also invariant.

5.3 Reduction of pure spinors.

In this section we prove the main theorem describing the spinor of the reduced Dirac structure of Theorem 4.54.

5.3.1 Main theorem.

Let us recall the setting. Let M be a smooth manifold and G a compact, connected Lie group acting freely on M. Over M, there is a Courant algebroid E on which we have reduction data $(\chi, \mu, \mathfrak{h})$ and consider its associated isotropic subbundle $K \subset E|_{\mu^{-1}(0)}$ (see (4.23)). Fix an invariant K-admissible splitting ∇ for E and let ∇_K be the induced splitting for $K^\perp/K$ (see (4.27)). Consider the induced representations

$$\Pi_\nabla : Cl(E, g) \to \mathrm{End}\,(\wedge^\bullet T^*M)$$

and

$$\Pi_{\nabla_K} : Cl\left(\frac{K^\perp}{K}, g_K\right) \longrightarrow \mathrm{End}\,(\wedge^\bullet q^*TM_{red})$$

constructed in Examples 5.3 and 5.6 respectively.

Let $L \subset E$ be an invariant Lagrangian subbundle such that $L|_{\mu^{-1}(0)} \cap K$ has constant rank and let $L_{red} \subset E_{red}$ be the reduced Lagrangian subbundle of Theorem 4.54. As $L|_{\mu^{-1}(0)} \cap K$ is an equivariant isotropic subbundle of

$E|_{\mu^{-1}(0)}$, there exists (by Proposition 4.32) an equivariant isotropic subbundle $D \subset E|_{\mu^{-1}(0)}$ such that

$$L|_{\mu^{-1}(0)} \cap K \oplus D^{\perp} = E|_{\mu^{-1}(0)}.$$

Theorem 5.29. *Let $\mathcal{W} \subset \mu^{-1}(0)$ be an invariant open neighborhood over which there is an invariant local section φ of $U^{\nabla}(L)|_{\mu^{-1}(0)}$ and an invariant frame $\{d_1, \ldots, d_r\}$ of D. Then*

$$\varphi_{red} = q_*(j^*\Pi_{\nabla}(d_1 \ldots d_r)\varphi) \tag{5.11}$$

is a local section of $U^{\nabla_{red}}(L_{red})$ over $\mathcal{W}/G$.

Remark 5.30. If $\nabla : TM \to E$ is a general invariant isotropic splitting, then any invariant K-admissible isotropic splitting ∇' is given by

$$\nabla' = \nabla + B,$$

for a uniquely defined invariant 2-form B satisfying (4.21). In this case, given an invariant section φ of $U^{\nabla}(L)$ over an invariant neighborhood $\mathcal{W} \subset \mu^{-1}(0)$, $e^B \wedge \varphi$ is an invariant section of $U^{\nabla+B}(L)$ (see Example 3.25). By (3.18) and Theorem 5.29,

$$\varphi_{red} = q_* \circ j^*(\Pi_{\nabla+B}(d_1 \cdots d_r)(e^B \wedge \varphi)) = q_* \circ j^*(e^B \wedge \Pi_{\nabla}(d_1 \cdots d_r)\varphi)$$

is a section of $U^{\nabla^B_{red}}(L_{red})$ over $\mathcal{W}/G$, where $\nabla^B_{red} = (\nabla + B)_{red}$. This is the general formula (1.12) in the introduction.

Remark 5.31. If $U^{\nabla}(L)$ has an invariant global section φ, then a sufficient condition for $U^{\nabla_{red}}(L_{red})$ to have a global section is the existence of a global frame for D. If this happens, then formula (5.11) defines a global section for $U^{\nabla_{red}}(L_{red})$. Observe that D has a global frame if and only if $L|_{\mu^{-1}(0)} \cap K$ also has.

Before proving Theorem 5.29, let us make some observations. First note that

$$\varphi_D = \Pi_{\nabla}(d_1 \ldots d_r)\varphi$$

is a section of $U^{\nabla}(L|_{\mu^{-1}(0)} \cap D^{\perp} + D)$ over $\mathcal{W}$ (this follows from Proposition 3.41).

Lemma 5.32. *φ_D is invariant.*

Proof. Let $\tilde{d}_i \in \Gamma(E)$ be extensions of d_i, for $i = 1, \ldots, r$. For $x \in \mathcal{W}$, by Proposition 5.9,

$$\begin{aligned}(\mathcal{L}^{\nabla}_{\chi(u)}\varphi_D)_x = & \sum_{i=1}^r \Pi_{\nabla}\Big(d_1(x)\ldots [\![\chi(u), \tilde{d}_i]\!](x) \ldots d_r(x)\Big)\varphi_x \\ & + \Pi_{\nabla}(d_1(x)\ldots d_r(x))(\mathcal{L}^{\nabla}_{\chi(u)}\varphi)_x.\end{aligned}$$

As φ is invariant, $\mathcal{L}^{\nabla}_{\chi(u)}\varphi = 0$ and as d_i is invariant,

$$[\![\chi(u), \tilde{d}_i]\!](x) = 0, \text{ for } i = 1, \dots, r.$$

□

As $L_D = L|_{\mu^{-1}(0)} \cap D^{\perp} + D$ is an invariant Lagrangian subbundle of $E|_{\mu^{-1}(0)}$ satisfying

$$L_D \cap K = 0 \quad \text{and} \quad (L_D)_{red} = L_{red}$$

(see Proposition 3.40), we can suppose, without loss of generality that $L|_{\mu^{-1}(0)} \cap K = 0$ by changing $L|_{\mu^{-1}(0)}$ with L_D and φ with φ_D respectively.

Let $x \in \mu^{-1}(0)$. As $\nabla : T_xM \to E_x$ is K_x-admissible, Corollary 2.37 gives that

$$\Phi_{\nabla_K}\left(\frac{L_x \cap K_x^{\perp} + K_x}{K_x}\right) = \Lambda_{dq_x} \circ \Lambda^t_{dj_x}(\Phi_{\nabla}(L_x)) \subset \left(\frac{T_x\mu^{-1}(0)}{\Delta_{\mathfrak{g},x}}\right) \oplus \left(\frac{T_x\mu^{-1}(0)}{\Delta_{\mathfrak{g},x}}\right)^*,$$

where as usual $q : \mu^{-1}(0) \to M_{red}$ and $j : \mu^{-1}(0) \to M$. As $L_x \cap K_x = 0$, Proposition 3.35 together with Remark 3.37 gives that

$$\theta_x = C_{\delta_x}(dj_x^*\varphi_x) \in \wedge^{\bullet} T^*_{q(x)} M_{red} \tag{5.12}$$

is non-zero and it is a pure spinor such that

$$\mathcal{N}_{\nabla_K}(\theta_x) = \frac{L_x \cap K_x^{\perp} + K_x}{K_x},$$

where $C_{\delta_x} : \wedge^{\bullet} T_x^* \mu^{-1}(0) \to \wedge^{\bullet} T^*_{q(x)} M_{red}$ is the map (3.25) corresponding to some element $\delta_x \in \det(\Delta_{\mathfrak{g},x})$.

Lemma 5.33. *Let ∇_{red} be the splitting for E_{red} induced from ∇ (see (4.28)) and consider the corresponding representation:*

$$\Pi_{\nabla_{red}} : Cl(E_{red}, g_{red}) \longrightarrow \operatorname{End}\left(\wedge^{\bullet} T^* M_{red}\right).$$

If $L|_{\mu^{-1}(0)} \cap K = 0$, then $0 \neq \theta_x \in \wedge^{\bullet} T^*_{q(x)} M_{red}$ given by (5.12) is a pure spinor for $Cl(E_{red}, g_{red})_{q(x)}$ and (see (4.29))

$$\mathcal{N}_{\nabla_{red}}(\theta_x) = (L_{red})_x. \tag{5.13}$$

We postpone the proof of Lemma 5.33 to the end of the section.

As ∇ is invariant, the action of G on the ∇ representation $\wedge^{\bullet} T^*M$ is given by $(\psi_{g^{-1}})^*$, for $g \in G$ (see Remark 5.12). As φ is invariant, we have that

$$\psi^*_{g^{-1}}\varphi = \varphi.$$

Let $y = \psi_g(x) \in \mu^{-1}(0)$ and define

$$\delta_y = ((d\psi_g)_x)_* \, \delta_x \in \wedge^n T_y \mu^{-1}(0) \tag{5.14}$$

(by abuse of notation, we keep calling $\psi_g|_{\mu^{-1}(0)} : \mu^{-1}(0) \to \mu^{-1}(0)$ by ψ_g).

Lemma 5.34. *One has that*

$$\delta_y \in \det(\Delta_{\mathfrak{g},\,y})$$

and

$$\theta_y := C_{\delta_y}(dj_y^*\,\varphi_y) = \theta_x,$$

where θ_x was defined by (5.12)*.*

We also postpone the proof of Lemma 5.34 to the end of the section

If we have a section $\delta_{\mu^{-1}(0)}$ of the bundle $\det(\Delta_{\mathfrak{g}}) \subset \wedge^\bullet T\mu^{-1}(0)$ such that

$$(\psi_g)_*\delta_{\mu^{-1}(0)} = \delta_{\mu^{-1}(0)},\ \forall\, g \in G,$$

then we can define a bundle map

$$C_{\delta_{\mu^{-1}(0)}} : \wedge^\bullet\, T^*\mu^{-1}(0)\big|_{\mathcal{W}} \longrightarrow q^* \wedge^\bullet\, T^*M_{red}|_{\mathcal{W}}$$

and a section θ of $q^* \wedge^\bullet T^*M_{red}$ over $\mathcal{W}$ defined by

$$\theta_x = C_{\delta_{\mu^{-1}(0)}}(dj_x^*\,\varphi_x) \in \wedge^\bullet T^*_{q(x)}M_{red}$$

for $x \in \mathcal{W}$ such that

(1) $\theta_x \neq 0$;

(2) $\mathcal{N}_{\nabla_{red}}(\theta_x) = (L_{red})_x$ (by Lemma 5.33);

(3) θ is constant along the G-orbits (by Lemma 5.34).

In this case, Theorem 5.29 will be proved if we can prove that, for every $x \in \mathcal{W}$, there exists $\lambda_x \in \mathbb{R}\backslash\{0\}$ such that

$$\varphi_{red,\,q(x)} = \lambda_x\,\theta_x. \tag{5.15}$$

Proof of Theorem 5.29. Let us first show how to define an invariant section $\delta_{\mu^{-1}(0)}$ of $\det(\Delta_{\mathfrak{g}})$. To this end, choose $\delta \in \det(\mathfrak{g})$ and define

$$\delta_{\mu^{-1}(0)} = \wedge^n\Sigma\,(\delta)$$

where $\Sigma : \mathfrak{g} \to \Gamma(T\mu^{-1}(0))$ is the infinitesimal action, $\wedge^n\Sigma$ is the natural extension to $\wedge^n\mathfrak{g}$ and $n = \dim(G)$. By the well-known relation

$$(\psi_g)_*\Sigma = \Sigma \circ Ad_{g^{-1}},\ \text{for } g \in G,$$

one has that

$$(\psi_g)_*\delta_{\mu^{-1}(0)} = \det(Ad_{g^{-1}})\,\delta_{\mu^{-1}(0)}$$

and as G is connected and compact,

$$\det(Ad_{g^{-1}}) = 1,\ \forall g \in G.$$

This shows that $\delta_{\mu^{-1}(0)}$ is invariant.

Now, relation (5.15) between θ and φ_{red} is given by a integration map (see (B.2))

$$\Gamma(q^* \wedge^\bullet T^*M_{red}) \longrightarrow \Gamma(\wedge^\bullet T^*M_{red})$$

which composed with $C_{\delta_{\mu^{-1}(0)}} : \Gamma(\wedge^\bullet T^*\mu^{-1}(0)) \to \Gamma(q^* \wedge^\bullet T^*M_{red})$ is exactly the operation of push-forward of differential forms (see the Appendix B and specially Remark B.2):

$$q_* : \Gamma(\wedge^\bullet T^*\mu^{-1}(0)) \longrightarrow \Gamma(\wedge^\bullet T^*M_{red}).$$

More precisely, for $x \in \mu^{-1}(0)$,

$$\varphi_{red,\, q(x)} = \left(\int_G \nu^L\right)\theta_x$$

where ν^L is the left invariant volume form on G generated by $\nu \in \det(\mathfrak{g}^*)$ dual to $\delta \in \det(\mathfrak{g})$. □

Remark 5.35. Theorem 5.29 extends naturally to the complexification $E \otimes \mathbb{C}$ taking the appropriate $\mathbb{C}$-linear extensions. As we saw in §4.4.1, given an invariant complex Dirac structure $L \subset E \otimes \mathbb{C}$ on M, its reduction is

$$L_{red} = \frac{L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}^\perp + K_{\mathbb{C}}}{K_{\mathbb{C}}},$$

where $K_{\mathbb{C}} = K \otimes \mathbb{C}$ and $K \subset E$ is the isotropic subbundle (4.23) corresponding to the reduction data $(\chi, \mathfrak{h}, \mu)$ on E. After choosing an invariant isotropic subbundle $D \subset E \otimes \mathbb{C}$ such that

$$L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}} \oplus D = E|_{\mu^{-1}(0)},$$

and an invariant local section of $U^{\nabla\otimes\mathrm{id}}(L) \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ (see Example 5.4), where ∇ is an invariant K-admissible splitting, one has that

$$\varphi_{red} = q_*(j^*\,\Pi_{\nabla\otimes\mathrm{id}}(d_1 \cdots d_r)\varphi)$$

is a local section of $U^{\nabla_{red}\otimes\mathrm{id}}(L_{red}) \subset \wedge^\bullet T^*M_{red} \otimes \mathbb{C}$, where $\{d_1, \dots, d_r\}$ is a local frame for D and q_*, j^* are the $\mathbb{C}$-linear extension of the respective real maps (see the discussion on Remark 3.39).

Let us now close this section with the proofs of Lemmas 5.33 and 5.34.

Proof of Lemma 5.33. The fact that $\theta_x \neq 0$ follows from Theorem 3.35. It remains to prove (5.13), i.e.

$$\Pi_{\nabla_{red}}([k^\perp + K])\theta_x = 0$$

for every $[k^\perp + K] \in (L_{red})_x$. Fix $k_1^\perp \in (L \cap K^\perp)_x$; using that

$$p_{red}([k_1^\perp + K]) = dq_x(p(k_1^\perp)) \in T_{q(x)}M_{red},$$

we have

$$\begin{aligned}\Pi_{\nabla_{red}}([k_1^\perp + K])\theta_x &= i_{p_{red}([k_1^\perp+K])}\theta_x + s_{\nabla_{red}}([k_1^\perp + K]) \wedge \theta_x \\ &= i_{dq_x(k_1^\perp)}\theta_x + s_{\nabla_{red}}([k_1^\perp + K]) \wedge \theta_x,\end{aligned} \tag{5.16}$$

where $s_{\nabla_{red}}([k_1^\perp + K]) \in T^*_{q(x)}M_{red}$ is such that

$$p^*_{red} s_{\nabla_{red}}(k_1^\perp + K) = [k_1^\perp + K] - \nabla_{red}\, p_{red}([k_1^\perp + K]) = [k_1^\perp - \nabla p(k_1^\perp) + K].$$

By comparing (5.16) with (5.1),

$$\Pi_{\nabla_K}(k_1^\perp + K)\theta_x = i_{dq_x(k_1^\perp)}\theta_x + s_{\nabla_K}(k_1^\perp + K) \wedge \theta_x,$$

where $s_{\nabla_K}(k_1^\perp + K) \in T^*_{q(x)}M_{red}$ is such that

$$(dq_x \circ p)^* s_{\nabla_K}(k_1^\perp + K) = k^\perp - \nabla p(k^\perp),$$

the result will follow if we prove that

$$s_{\nabla_{red}}([k_1^\perp + K]) = s_{\nabla_K}(k_1^\perp + K). \tag{5.17}$$

Indeed, by Theorem 3.35, we already now that

$$\Pi_{\nabla_K}(k_1^\perp + K)\theta_x = 0$$

because $k_1^\perp + K \in (L_x \cap K_x^\perp + K_x)/K_x$. Now, to prove (5.17), let $k_2^\perp \in K_x^\perp$. Then

$$\begin{aligned}&g_{red}(\, p^*_{red}\, s_{\nabla_K}(k_1^\perp + K),\, [k_2^\perp + K]\,) \\ &\qquad = i_{dq_x(p(k_2^\perp))} s_{\nabla_K}(k_1^\perp + K) \\ &\qquad = i_{p(k_2^\perp)} dq_x^*\, s_{\nabla_K}(k_1^\perp + K) \\ &\qquad = g((dq_x \circ p)^* s_{\nabla_K}(k_1^\perp + K),\, k_2^\perp) \\ &\qquad = g(k_1^\perp - \nabla p(k_1^\perp), k_2^\perp) \\ &\qquad = g_{red}([k_1^\perp - \nabla p(k_1^\perp) + K], [k_2^\perp + K])\end{aligned}$$

and as g_{red} is non-degenerate and $p^*_{red} : T^*_{q(x)}M_{red} \to (E_{red})_{q(x)}$ is injective, the result follows. □

Proof of Lemma 5.34. Let δ_y be defined by (5.14). As $(d\psi_g)_x$ sends $\Delta_{\mathfrak{g},x}$ isomorphically to $\Delta_{\mathfrak{g},y}$, its action on multivectors $((d\psi_g)_x)_*$ sends $\det(\Delta_{\mathfrak{g},x})$ to $\det(\Delta_{\mathfrak{g},y})$. Therefore, $\delta_y \in \det(\Delta_{\mathfrak{g},y})$.

Now, for $\nu_1 \otimes \nu_2 \in \det(T_x\mu^{-1}(0)) \otimes \det(T^*_{q(x)}M_{red})$ such that

$$\delta_x = \star_1 dq_x^* \nu_2 = i_{dq_x^*\nu_2}\, \nu_1$$

one has that

$$C_{\delta_x} = \star_2 \circ (dq_x)_* \circ \star_1$$

(see Remark 3.37). We claim that, for $\nu_1' = ((d\psi_g)_x)_* \nu_1 \in \det(T_y\mu^{-1}(0))$,

$$\delta_y = i_{dq_y^*\nu_2}\, \nu_1'.$$

Indeed,

$$i_{dq_y^*\nu_2}\, \nu_1' = ((d\psi_g)_x)_*\, i_{(d\psi_g)_x^*\, dq_y^*\nu_2}\, \nu_1 = ((d\psi_g)_x)_* i_{dq_x^*\nu_2}\, \nu_1 = ((d\psi_g)_x)_*\, \delta_x = \delta_y.$$

The star map $\star_1' : T_y^*\mu^{-1}(0) \to T_y\mu^{-1}(0)$ associated to ν_1' sends $dj_y^*\varphi_y$ to

$$\begin{aligned} \star_1'(dj_y^*\varphi_y) \quad &= i_{dj_y^*\varphi_y}\, ((d\psi_g)_x)_*\, \nu_1 &&= ((d\psi_g)_x)_*\, i_{(d\psi_g)_x^* dj_y^*\varphi_y}\, \nu_1 \\ & &&= ((d\psi_g)_x)_*\, i_{dj_x^*\varphi_x}\nu_1 \\ & &&= ((d\psi_g)_x)_*\, \star_1\, (dj_x^*\varphi_x). \end{aligned}$$

and therefore

$$\begin{aligned} \theta_y \quad &= C_{\delta_y}(dj_y^*\varphi_y) = \quad \star_2 \circ (dq_y)_* \circ \star_1'(dj_y^*\varphi_y) &&= \star_2 \circ (dq_y \circ (d\psi_g)_x)_* \circ \star_1(dj_x^*\varphi_x) \\ & &&= \star_2 \circ (dq_x)_* \circ \star_1(dj_x^*\varphi_x) \\ & &&= C_{\delta_x}(dj_x^*\varphi_x) \\ & &&= \theta_x. \end{aligned}$$

□

5.3.2 Integrability

We now focus on the integrability of L_{red}. We follow a different, more involved approach than the original one [11]. We choose to give this prove as it shows the flexibility of the pertubative method (see §3.2.3) in tackling the integrability problem just in terms of pure spinors. We also hope that this method allows future applications as the obstruction to integrability as a section of some exterior algebra bundle (see Proposition 5.17) carries geometric information and it could be useful to see how it reduces.

Let $D \subset E|_{\mu^{-1}(0)}$ be an invariant isotropic subbundle such that

$$L|_{\mu^{-1}(0)} \cap K \oplus D = E|_{\mu^{-1}(0)}. \tag{5.18}$$

To study the integrability of L_{red} via pure spinors, it is necessary to have a Lagrangian complement to it. We will find it by reducing a suitable complement to $L_D = L|_{\mu^{-1}(0)} \cap D^\perp + D$. To calculate the obstruction to integrability, it will be necessary to truncate the reduction at a first level: the restriction of Dirac structures as seen in Example 4.55 (this corresponds to the first factor in the factorization of the quotient morphism given by Proposition 2.35). We analyze the obstruction to integrability at this level and conclude the proof by using well-known properties of the push-forward of differential forms.

Let us first choose a complement to L_D. Begin by noticing that

$$K_D = K \cap D^\perp \oplus D \tag{5.19}$$

is an invariant isotropic subbundle of $E|_{\mu^{-1}(0)}$ such that $L \cap K_D = 0$. By Proposition 4.34, there exists an invariant Lagrangian complement $L' \subset E|_{\mu^{-1}(0)}$ to $L|_{\mu^{-1}(0)}$ such that $K_D \subset L'$.

Proposition 5.36. *The invariant subbundle of $E|_{\mu^{-1}(0)}$,*

$$L_1 = L' \cap (L|_{\mu^{-1}(0)} \cap K)^{\perp} \oplus L|_{\mu^{-1}(0)} \cap K \tag{5.20}$$

is an invariant Lagrangian complement to L_D such that $K \subset L_1$. Moreover,

$$(L_1)_{red} \oplus L_{red} = E_{red}.$$

Proof. First note that by (5.18), one has

$$L|_{\mu^{-1}(0)} = (L|_{\mu^{-1}(0)} \cap K) \oplus (L|_{\mu^{-1}(0)} \cap D^{\perp}).$$

Similarly, from $E|_{\mu^{-1}(0)} = (L|_{\mu^{-1}(0)} \cap K)^{\perp} \oplus D$ and the fact that $D \subset L'$ from construction, one obtains

$$L' = L' \cap (L|_{\mu^{-1}(0)} \cap K)^{\perp} \oplus D.$$

Hence,

$$\begin{aligned} E|_{\mu^{-1}(0)} = \ & L \oplus L' \\ = \ & \left[(L|_{\mu^{-1}(0)} \cap K) \oplus (L|_{\mu^{-1}(0)} \cap D^{\perp})\right] \oplus \left[L' \cap (L|_{\mu^{-1}(0)} \cap K)^{\perp} \oplus D\right] \\ = \ & \left[L|_{\mu^{-1}(0)} \cap D^{\perp} + D\right] + \left[L' \cap (L|_{\mu^{-1}(0)} \cap K)^{\perp} + L|_{\mu^{-1}(0)} \cap K\right] \\ = \ & L_D + L_1 \end{aligned}$$

which proves that L_1 is a Lagrangian complement to L_D. It is invariant because L', L and K are all invariant. To prove that $K \subset L_1$, note that by (5.18)

$$K = (L|_{\mu^{-1}(0)} \cap K) \oplus K \cap D^{\perp}$$

and then it suffices to prove that $K \cap D^{\perp} \subset L_1$. By construction, we know that $K \cap D^{\perp} \subset L'$; also,

$$K \cap D^{\perp} \subset K^{\perp} \subset L|_{\mu^{-1}(0)} + K^{\perp} = (L|_{\mu^{-1}(0)} \cap K)^{\perp}$$

and hence $K \cap D^{\perp} \subset L' \cap (L|_{\mu^{-1}(0)} \cap K)^{\perp} \subset L_1$.

Consider the two Lagrangian subbundles of E_{red}: L_{red} and $(L_1)_{red}$ (observe that we can reduce L_1 as it is invariant and $L_1 \cap K = K$ has constant rank). To prove that E_{red} is their direct sum it suffices to prove that $L_{red} \cap (L_1)_{red} = 0$. First note that

$$(L_D \cap K^{\perp} + K) \cap (L_1 \cap K^{\perp} + K) = (L|_{\mu^{-1}(0)} \cap K^{\perp} + K) \cap L_1 = K$$

where we used that $L_D \cap K^{\perp} + K = L|_{\mu^{-1}(0)} \cap K^{\perp} + K$ (see Proposition 3.40) and that $K \subset L_1$. Thus,

$$\left(\frac{L_D \cap K^{\perp} + K}{K}\right) \cap \left(\frac{L_1 \cap K^{\perp} + K}{K}\right) = 0$$

which is sufficient to prove $L_{red} \cap (L_1)_{red} = 0$. □

From now on, fix an invariant open set $\mathcal{W}$ of $\mu^{-1}(0)$ over which there is an invariant local section φ of $U^{\nabla}(L)|_{\mathcal{W}}$ and an invariant frame $\{d_1, \ldots, d_r\}$ of D. As we saw in Theorem 5.29,

$$\varphi_{red} = q_*(j^*\varphi_D) \in \Gamma(U^{\nabla_{red}}(L_{red})\big|_{\mathcal{W}/G}),$$

where $\varphi_D = \Pi_{\nabla|_{\mu^{-1}(0)}}(d_1 \cdots d_r)\,\varphi$. Recall from Proposition 4.47 that the curvature of the reduced splitting ∇_{red} is $H_{red} \in \Omega^3(M)$ such that $q^*H_{red} = j^*H$. Now (see Proposition B.3 and Proposition B.4),

$$d_{H_{red}}\varphi_{red} = q_*(d_{q^*H_{red}}\, j^*\varphi_D) = q_*(d_{j^*H}\, j^*\varphi_D). \tag{5.21}$$

By Proposition 5.17 and Lemma 5.36, there exists a section Υ_{red} of $(L_1)_{red} \oplus \wedge^3(L_1)_{red}$ over $\mathcal{W}/G$ such that

$$d_{H_{red}}\varphi_{red} = \Pi_{\nabla_{red}}(\Upsilon_{red})\varphi_{red}.$$

We shall prove that L_{red} is integrable by proving that the degree 3 component of Υ_{red} is zero if L is integrable.

First note that on the j^*H-twisted Courant algebroid $T\mu^{-1}(0) \oplus T^*\mu^{-1}(0)$, there is an isotropic lifted G-action given by

$$G \ni g \longmapsto \Psi_g|_{\mu^{-1}(0)} = \begin{pmatrix} \left(\psi_g|_{\mu^{-1}(0)}\right)_* & 0 \\ 0 & \left(\psi_{-g}|_{\mu^{-1}(0)}\right)^* \end{pmatrix}$$

(note that $\psi_g|^*_{\mu^{-1}(0)}\, j^*H = j^*H$ because ∇ is invariant).

Before we proceed, let us make it clear our strategy. The Lagrangian subbundles $L_D, L_1 \subset E|_{\mu^{-1}(0)}$ will induce an invariant polarization of $T\mu^{-1}(0) \oplus T^*\mu^{-1}(0)$ which we shall use to analyse $d_{j^*H}\, j^*\varphi_D$ via Proposition 5.17. The careful choice of L_1 will allow us to relate the integrability of L to that of L_{red} after a series of lemmata.

Consider $\Phi_\nabla(L_D)$, $\Phi_\nabla(L_1) \subset (TM \oplus T^*M)|_{\mu^{-1}(0)}$; one has that

$$\Phi_\nabla(L_D) \cap \mathrm{Ann}\left(T\mu^{-1}(0)\right) = 0 \text{ and } \mathrm{Ann}\left(T\mu^{-1}(0)\right) \subset \Phi_\nabla(L_1)$$

and therefore they can be restricted to $\mu^{-1}(0)$ (see Example 4.55) to define Lagrangian subbundles of $T\mu^{-1}(0) \oplus T^*\mu^{-1}(0)$:

$$L_D^{\mu^{-1}(0)} := \Lambda_j^t(\Phi_\nabla(L_D)) \text{ and } L_1^{\mu^{-1}(0)} := \Lambda_j^t(\Phi_\nabla(L_1)).$$

Lemma 5.37. *$l_1 = (L_1^{\mu^{-1}(0)}, L_D^{\mu^{-1}(0)})$ is an invariant polarization of $T\mu^{-1}(0) \oplus T^*\mu^{-1}(0)$. Moreover, there exists a bundle map over $\mu^{-1}(0)$*

$$\varrho : L_1 \to L_1^{\mu^{-1}(0)} \tag{5.22}$$

such that for $x \in \mu^{-1}(0)$, $e \in (L_1)_x$ and $\alpha \in \wedge^\bullet T_x^*M$,

$$dj_x^*\, \Pi_\nabla(e)\alpha = \Pi(\varrho(e))\, dj_x^*\, \alpha. \tag{5.23}$$

Proof. We first prove that l is a polarization; for this, it suffices to show $L_1^{\mu^{-1}(0)} \cap L_D^{\mu^{-1}(0)} = 0$. Let $x \in \mu^{-1}(0)$, $X \in T_x\mu^{-1}(0)$ and $\xi \in T_x^*\mu^{-1}(0)$ such that

$$X + \xi \in (L_1^{\mu^{-1}(0)} \cap L_D^{\mu^{-1}(0)})_x.$$

This is equivalent to the existence of $\eta_1, \eta_2 \in T_x^*M$ such that $dj_x^*\eta_1 = dj_x^*\eta_2 = \xi$ and

$$dj_x(X) + \eta_1 \in \Phi_\nabla(L_1) \text{ and } dj_x(X) + \eta_2 \in \Phi_\nabla(L_D).$$

Now, $\eta_2 - \eta_1 \in \mathrm{Ann}\left(T_x\mu^{-1}(0)\right) \subset \Phi_\nabla(L_1)$ (because $K \subset L_1$) and therefore

$$dj_x(X) + \eta_2 = dj_x(X) + \eta_1 + (\eta_2 - \eta_1) \in \Phi_\nabla(L_1) \cap \Phi_\nabla(L_D) = 0$$

which implies that $X = \xi = 0$.

To prove invariance, first observe that as ∇ is an invariant splitting, $\Phi_\nabla(L_D)$ and $\Phi_\nabla(L_1)$ are both invariant under the G-action on $(\mathbb{T}M)|_{\mu^{-1}(0)}$ given by

$$G \ni g \longmapsto \begin{pmatrix} (\psi_g)_* & 0 \\ 0 & (\psi_{-g})^* \end{pmatrix}.$$

Let $x \in \mu^{-1}(0)$ and $X + \xi \in (L_1^{\mu^{-1}(0)})_x$. Consider

$$\Psi_g|_{\mu^{-1}(0)}(X + \xi) = (d\psi_g|_{\mu^{-1}(0)})_x X + (d\psi_{g^{-1}}|_{\mu^{-1}(0)})^*_{\psi_g|_{\mu^{-1}(0)}(x)}\,\xi.$$

One has to prove that it belongs to $\Lambda^t_{dj_{\psi_g(x)}}(\Phi_\nabla(L_1)_{\psi_g(x)})$. By assumption, there exists $\eta \in T_x^*M$ such that $dj_x^*\eta = \xi$ and

$$dj_x(X) + \eta \in \Phi_\nabla(L_1)_x.$$

By the invariance of L_1, one has that

$$(d\psi_g)_x dj_x(X) + (d\psi_{g^{-1}})^*_{\psi_g(x)}\eta \in \Phi_\nabla(L_1)_{\psi_g(x)}.$$

But, by definition

$$(d\psi_g)_x dj_x(X) = dj_{\psi_g(x)}(d\psi_g|_{\mu^{-1}(0)})_x(X)$$

and consequently,

$$dj^*_{\psi_g(x)}(d\psi_{g^{-1}})^*_{\psi_g(x)}\eta = (d\psi_{g^{-1}}|_{\mu^{-1}(0)})^*_{\psi_g|_{\mu^{-1}(0)}(x)}\,dj_x^*\eta = (d\psi_{g^{-1}}|_{\mu^{-1}(0)})^*_{\psi_g|_{\mu^{-1}(0)}(x)}\,\xi.$$

This concludes the proof of the invariance of $L_1^{\mu^{-1}(0)}$. The same argument applies to $L_D^{\mu^{-1}(0)}$.

Consider the isotropic subbundle $\mathrm{Ann}\left(T\mu^{-1}(0)\right) \subset (\mathbb{T}M)|_{\mu^{-1}(0)}$ and its orthogonal

$$(\mathrm{Ann}\left(T\mu^{-1}(0)\right))^\perp = T\mu^{-1}(0) \oplus T^*M|_{\mu^{-1}(0)}.$$

Define the bundle map over $\mu^{-1}(0)$

$$\begin{array}{rcl} T\mu^{-1}(0)\oplus T^*M|_{\mu^{-1}(0)} & \longrightarrow & T\mu^{-1}(0)\oplus T^*\mu^{-1}(0) \\ dj_x(X)+\xi & \longmapsto & X+dj_x^*\xi. \end{array} \tag{5.24}$$

Now, ϱ is just the composition of Φ_∇ with the restriction of (5.24) to $\Phi_\nabla(L_1)$ (note that $\mathrm{Ann}\left(T\mu^{-1}(0)\right)\subset\Phi_\nabla(L_1)\subset(\mathrm{Ann}\left(T\mu^{-1}(0)\right))^\perp$). Equation (5.23) is an easy consequence of formula (3.22). □

Remark 5.38. The fact that $K\subset L_1$ implies that $\Delta_{\mathfrak{g}}\oplus\mathrm{Ann}\left(T\mu^{-1}(0)\right)\subset\Phi_\nabla(L_1)$ (using that ∇ is K-admissible). Applying ϱ to $\Phi_\nabla(K)$ shows that

$$\Delta_{\mathfrak{g}}\subset L_1^{\mu^{-1}(0)}$$

(where by abuse of notation $\Delta_{\mathfrak{g}}\subset T\mu^{-1}(0)$). Consequently,

$$L_1^{\mu^{-1}(0)}\subset T\mu^{-1}(0)\oplus\mathrm{Ann}\left(\Delta_g\right).$$

The section $j^*\varphi_D$ of $\wedge^\bullet T^*\mu^{-1}(0)|_{\mathcal{W}}$ is a pure spinor for $Cl(T\mu^{-1}(0)\oplus T^*\mu^{-1}(0),\, g_{\mathrm{can}})$ such that

$$U(\varphi_D)=L_D^{\mu^{-1}(0)}.$$

It is straightforward to check that it is invariant. Using the invariant polarization $l_1=(L_1^{\mu^{-1}(0)},\, L_D^{\mu^{-1}(0)})$, one can find an invariant section $\Upsilon^{\mu^{-1}(0)}$ of $\wedge^\bullet L_1^{\mu^{-1}(0)}$ over $\mathcal{W}$ such that (see Props. 5.17 and 5.27)

$$d_{j^*H}\, j^*\varphi_D=\Pi\left(\Upsilon^{\mu^{-1}(0)}\right)j^*\varphi_D.$$

To relate the integrability of L to properties of $\Upsilon^{\mu^{-1}(0)}$ we shall need an extension lemma.

Lemma 5.39. *For every $x\in\mu^{-1}(0)$, there is an open neighborhood $\mathcal{V}$ of x inside M over which we can extend D to an isotropic subbundle $\tilde{D}\subset E|_{\mathcal{V}}$ such that*

$$L|_{\mathcal{V}}\cap\tilde{D}=0.$$

Proof. Let $\mathcal{U}\subset M$ be an open neighborhood of x on which there are coordinates $x_1,\cdots,x_n$ such that $\mu^{-1}(0)\cap\mathcal{U}$ is given by

$$x_{s+1}=\cdots=x_n=0, s\leq n.$$

We can also suppose that there is a polarized frame $\{e_1,\ldots,e_{2n}\}$ for E defined over $\mathcal{U}$ (see §5.1) such that $\{e_1,\ldots,e_n\}$ spans $L|_{\mathcal{U}}$ and that D has a frame $\{d_1',\ldots,d_r'\}$ over $\mathcal{U}\cap\mu^{-1}(0)$. Call L'' the Lagrangian subbundle of $E|_{\mathcal{U}}$ spanned by $\{e_{n+1},\ldots,e_{2n}\}$. Write

$$d_i'(x_1,\ldots,x_s)=\sum_{k=1}^{2n}a_i^k(x_1,\ldots,x_s)e_k(x_1,\ldots,x_s,0),\ \text{for } a_i^k\in C^\infty(\mathbb{R}^s).$$

The fact that D is isotropic translates to

$$\sum_{k=1}^{n} a_{i_1}^k a_{i_2}^{n+k} + a_{i_1}^{k+n} a_{i_2}^k = 0 \tag{5.25}$$

for every $1 \leq i_1, i_2 \leq r$. Also, $D \cap L|_{\mu^{-1}(0)} = 0$ translates to the fact that the $r \times n$ matrix

$$(a_i^k; 1 \leq i \leq r, n \leq k \leq 2n)$$

has rank r. Therefore, if we define $\widetilde{d'}_i \in \Gamma(E|_{\mathcal{U}})$ by

$$\widetilde{d'}_i(x_1, \ldots, x_n) = \sum_{k=1}^{2n} a_i^k(x_1, \ldots, x_s) e_k(x_1, \ldots, x_n)$$

then they span a r-dimensional subbundle $\tilde{D}$ of $E|_{\mathcal{U}}$ which satisfies

$$\tilde{D} \cap L|_{\mathcal{U}} = 0 \text{ and } \tilde{D}|_{\mathcal{U} \cap \mu^{-1}(0)} = D.$$

Also, it is isotropic, because

$$g(\widetilde{d'}_{i_1}, \widetilde{d'}_{i_2})\Big|_{(x_1,\ldots,x_n)} = \left(\sum_{k=1}^{n} a_{i_1}^k a_{i_2}^{n+k} + a_{i_1}^{k+n} a_{i_2}^k\right)\Bigg|_{(x_1,\ldots,x_s)} = 0.$$

□

Let $y \in \mathcal{W}$ and consider the open neighborhood $\mathcal{V} \subset M$ of y given in Lemma 5.39. We can suppose that $\mathcal{V} \cap \mu^{-1}(0) \subset \mathcal{W}$ and that there is a frame $\{\tilde{d}_1, \ldots, \tilde{d}_r\}$ of $\tilde{D}$ over $\mathcal{V}$ extending the frame $\{d_1, \ldots, d_r\}$ of D (by possibly shrinking $\mathcal{V}$). So define

$$\tilde{\varphi} = \Pi_\nabla(\tilde{d}_1 \cdots \tilde{d}_r)\varphi \in \Gamma(\wedge^\bullet T^*M|_{\mathcal{V}}).$$

It satisfies

$$\tilde{\varphi}|_{\mathcal{V}\cap\mu^{-1}(0)} = \varphi_D|_{\mathcal{V}\cap\mu^{-1}(0)} \text{ and } \mathcal{N}_\nabla(\tilde{\varphi}) = L|_{\mathcal{V}} \cap \tilde{D}^\perp + \tilde{D}.$$

Now, by going through the proof of Proposition 5.17, one can check that there exists a section Υ of $L_1 \oplus \wedge^3 L_1$ over $\mathcal{V} \cap \mu^{-1}(0)$ such that

$$(d_H\tilde{\varphi})|_{\mathcal{V}\cap\mu^{-1}(0)} = \Pi_\nabla(\Upsilon)\,\varphi_D.$$

Moreover, by considering the representation of $\Pi_{l_2} : Cl(E|_{\mu^{-1}(0)}, g) \to \text{End}\,(\wedge^\bullet L_1)$ corresponding to the polarization $l_2 = (L_D, L_1)$, one has that for sections e_1, e_2, e_3 of L_D over $\mathcal{V} \cap \mu^{-1}(0)$,

$$\Pi_{l_2}(e_1 \wedge e_2 \wedge e_3)\Upsilon_{(3)} = g([\![\tilde{e}_1, \tilde{e}_2]\!], \tilde{e}_3),$$

where $\tilde{e}_i \in \Gamma(L|_{\mathcal{V}} \cap \tilde{D}^\perp + \tilde{D})$ is an extension of e_i for $i = 1, 2, 3$. Over $\mathcal{V} \cap \mu^{-1}(0)$, one has

$$\begin{aligned} d_{j^*H} j^*\varphi_D = \quad j^* d_H \tilde{\varphi} \quad &= j^*\Pi_\nabla(\Upsilon)\varphi_D \\ &= \Pi(\wedge\varrho\,(\Upsilon)) j^*\varphi_D, \text{ by Equation 5.23,} \end{aligned}$$

where $\wedge\varrho : \wedge^\bullet L_1 \to \wedge^\bullet L_1^{\mu^{-1}(0)}$ is the natural extension of ϱ (see (5.22)). Therefore, over $\mathcal{V} \cap \mu^{-1}(0)$,

$$\Upsilon^{\mu^{-1}(0)} = \wedge\varrho\,(\Upsilon). \tag{5.26}$$

Consider the decomposition

$$\wedge^3 L_1 = \bigoplus_{0 \le i \le 3} \wedge^i \left(L' \cap (L|_{\mu^{-1}(0)} \cap K)^\perp\right) \otimes \wedge^{3-i}(L|_{\mu^{-1}(0)} \cap K)$$

Lemma 5.40. *If L is integrable, then $\Upsilon_{(3)}$ has no component on $\wedge^3 \left(L' \cap (L|_{\mu^{-1}(0)} \cap K)^\perp\right)$.*

Proof. First observe that under the identification of L_1 with L_D^* via the bilinear form g, one has

$$L' \cap (L|_{\mu^{-1}(0)} \cap K)^\perp \cong (L|_{\mu^{-1}(0)} \cap D^\perp)^*$$

and

$$L|_{\mu^{-1}(0)} \cap K \cong D^*.$$

Indeed, from the fact that $L_D \oplus L_1 = E|_{\mu^{-1}(0)}$ one has that

$$E|_{\mu^{-1}(0)} = L' \cap (L|_{\mu^{-1}(0)} \cap K)^\perp \oplus (L|_{\mu^{-1}(0)} \oplus D).$$

By observing that $L|_{\mu^{-1}(0)} \oplus D = (L|_{\mu^{-1}(0)} \cap D^\perp)^\perp$, one obtains the first identification. The second identity follows from $(L|_{\mu^{-1}(0)} \cap K) \oplus D^\perp = E|_{\mu^{-1}(0)}$. Therefore, to prove the Lemma, one has to prove that

$$\Pi_l(e_1 \wedge e_2 \wedge e_3)\Upsilon_{(3)} = 0,$$

for e_1, e_2, $e_3 \in L|_{\mu^{-1}(0)} \cap D^\perp$. So choose extensions $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3 \in L|_{\mathcal{V}} \cap \tilde{D}^\perp$ and then

$$\Pi_l(e_1 \wedge e_2 \wedge e_3)\Upsilon_{(3)} = g([\![\tilde{e}_1, \tilde{e}_2]\!], \tilde{e}_3) = 0$$

because L is integrable. ☐

Theorem 5.41. *If L is integrable, then L_{red} is integrable.*

Proof. We first show that $q_*(\Pi(\Upsilon_{(3)}^{\mu^{-1}(0)})j^*\varphi_D) = 0$. Indeed, let $x \in \mathcal{W}$ and $\delta \in \det(\mathfrak{g})$; it follows from Remark B.2 that if for every $y \in G \cdot x$,

$$C_{\delta_{\mu^{-1}(0)}}(\Pi(\Upsilon_{(3)}^{\mu^{-1}(0)})j^*\varphi_D)\Big|_y = 0, \tag{5.27}$$

then

$$q_*(\Pi(\Upsilon_{(3)}^{\mu^{-1}(0)})j^*\varphi_D)\Big|_{q(x)} = 0.$$

Let $\{u_1, \ldots, u_n\} \in \mathfrak{g}$ and $l_1 = (L_D^{\mu^{-1}(0)}, L_1^{\mu^{-1}(0)})$; by Proposition 3.23,

$$i_{(u_1)_M \wedge \cdots \wedge (u_n)_M} \Pi(\Upsilon_{(3)}^{\mu^{-1}(0)}) j^* \varphi_D = \Pi\left(\Pi_{l_1}\left((u_1)_M \wedge \cdots \wedge (u_n)_M\right) \Upsilon_{(3)}^{\mu^{-1}(0)} \right) j^* \varphi_D$$

(and hence by Remark 5.38)

$$= \Pi\left((u_1)_M \wedge \cdots \wedge (u_n)_M \wedge \Upsilon_{(3)}^{\mu^{-1}(0)} \right) j^* \varphi_D.$$

Now, for every $y \in G \cdot x$, there is a neighborhood $\mathcal{V} \subset M$ of y such that over $\mathcal{V} \cap \mu^{-1}(0) \subset \mathcal{W}$ there is a section Υ of $\wedge^\bullet L_1$ such that (5.26) holds. Now, by Lemma 5.40, if L is integrable, then

$$\Upsilon \text{ is section of } \bigoplus_{0 \leq i < 3} \wedge^i \left(L' \cap (L|_{\mu^{-1}(0)} \cap K)^\perp \right) \otimes \wedge^{3-i} (L|_{\mu^{-1}(0)} \cap K)$$

over $\mathcal{V} \cap \mu^{-1}(0)$; as $\varrho(\, L|_{\mu^{-1}(0)} \cap K\,) \subset \Delta_{\mathfrak{g}}$ (see Remark 5.38), it follows that that

$$(u_1)_M \wedge \cdots \wedge (u_n)_M \wedge \Upsilon_{(3)}^{\mu^{-1}(0)} = 0$$

over $\mathcal{V} \cap \mu^{-1}(0)$ and hence, by Remark 3.38, equation (5.27) holds.

Now, let $X \in \Gamma(T\mu^{-1}|_{\mathcal{W}})$ and $\xi \in \Gamma(T^*\mu^{-1}(0)|_{\mathcal{W}})$ such that

$$X + \xi = \Upsilon_{(1)}^{\mu^{-1}(0)} \in \Gamma(L_1^{\mu^{-1}(0)}|_{\mathcal{W}}).$$

As $\Upsilon^{\mu^{-1}(0)}$ is invariant, it follows that for every $u \in \mathfrak{g}$,

$$\begin{aligned} 0 = \quad [\![u_M, X + \xi]\!]_{j^*H} &= [u_M, X] + \mathcal{L}_{u_M} \xi + i_X i_{u_M} j^* H \\ &= [u_M, X] + \mathcal{L}_{u_M} \xi + i_X i_{u_M} q^* H_{red} \\ &= [u_M, X] + \mathcal{L}_{u_M} \xi. \end{aligned} \tag{5.28}$$

Therefore $[u_M, X] = 0$ for every $u_M \in \mathfrak{g}$ and hence there is a well-defined vector field $X_{red} \in \Gamma(TM_{red}|_{\mathcal{W}/G})$ which is q-related to X. Also, as (see Remark 5.38)

$$L_1^{\mu^{-1}(0)} \subset T\mu^{-1}(0) \oplus \mathrm{Ann}\,(\Delta_{\mathfrak{g}})$$

it follows that $i_{u_M}\xi = 0$ and from (5.28) $\mathcal{L}_{u_M}\xi = 0$ for $u \in \mathfrak{g}$. Therefore, there exists $\xi_{red} \in \Gamma(T^*M_{red}|_{\mathcal{W}/G})$ such that $q^*\xi_{red} = \xi$. Summing up, we have

$$\begin{aligned} d_{H_{red}} \varphi_{red} = \quad q_*(d_{j^*H} j^* \varphi_D) &= q_*(\Pi(\Upsilon^{\mu^{-1}(0)}) j^* \varphi_D) \\ &= q_*(\Pi(\Upsilon_{(1)}^{\mu^{-1}(0)}) j^* \varphi_D) \\ &= q_*(i_X j^* \varphi_D + \xi \wedge j^* \varphi_D) \\ &= i_{X_{red}} \varphi_{red} + \xi_{red} \wedge \varphi_{red} \end{aligned} \tag{5.29}$$

where in the last equality we used Props. B.4 and B.5. To finish the proof, note that for every $x \in \mathcal{W}$

$$(X_{red} + \xi_{red})|_{q(x)} \in \Lambda_{dq_x}(L_1^{\mu^{-1}(0)}) = \Lambda_{dq_x}(\Lambda^t_{dj_x}(L_1)) = \Phi_{\nabla_{red}}((L_1)_{red})$$

and therefore

$$d^{\nabla_{red}}\varphi_{red} = \Pi_{\nabla_{red}}(e_{red})\,\varphi_{red},$$

for $e_{red} = \Phi^{-1}_{\nabla_{red}}(X_{red} + \xi_{red}) \in \Gamma((L_1)_{red}|_{\mathcal{W}/G})$, thus proving integrability of L_{red}. □

5.4 Applications.

5.4.1 Examples.

Poisson reduction. Let M be a m-dimensional smooth manifold and let $\pi \in \Gamma(\wedge^2 TM)$ be a bivector field such that $[\pi, \pi] = 0$ (where $[\cdot,\cdot]$ is the Schouten bracket). We say that (M, π) is a **Poisson manifold**. There is an induced Lie bracket on $C^\infty(M)$ given by

$$\{f_1, f_2\} = i_{\pi^\sharp(df_1)}\, df_2, \text{ for } f_1, f_2 \in C^\infty(M),$$

where

$$\begin{array}{rccc} \pi^\sharp : & T^*M & \longrightarrow & TM \\ & \xi & \longmapsto & i_\xi\, \pi. \end{array}$$

The map

$$\{f_1, \cdot\} : C^\infty(M) \to C^\infty(M)$$

is a derivation of the algebra of functions given by the vector field $X_{f_1} := \pi^\sharp(df_1)$.

Let G be a compact, connected Lie group acting on M by Poisson diffeomorphisms, i.e. for any $f_1,\ f_2 \in C^\infty(M)$,

$$\{\psi_g^* f_1, \psi_g^* f_2\} = \psi_g^*\{f_1, f_2\}, \text{ for every } g \in G. \tag{5.30}$$

Define the isotropic lifted G-action given by

$$\chi : \mathfrak{g} \ni u \mapsto u_M \in \Gamma(\mathbb{T}M)$$

which integrates to

$$G \ni g \longmapsto \Psi_g = \begin{pmatrix} (\psi_g)_* & 0 \\ 0 & (\psi_g^{-1})^* \end{pmatrix} \in \mathrm{Aut}(\mathbb{T}M).$$

Also, take $\mu : M \to \{0\}$ to be the zero map (see Example 4.37). $(\chi, \{0\}, \mu)$ defines reduction data on $(\mathbb{T}M, [\![\cdot,\cdot]\!])$; the corresponding isotropic subbundle $K \subset \mathbb{T}M$ (4.23) is the distribution tangent to the G-orbits

$$K = \Delta_{\mathfrak{g}}.$$

Therefore, the reduced Courant algebroid (see Theorem 4.45) is

$$E_{red} = \frac{K^\perp}{K}\Big/ G = \frac{TM \oplus \mathrm{Ann}\,(\Delta_{\mathfrak{g}})}{\Delta_{\mathfrak{g}}}\Big/ G$$

defined over

$$M_{red} = M/G \text{ with } q: M \to M/G \text{ the quotient map.}$$

The canonical splitting $\nabla : TM \ni X \mapsto (X,0) \in \mathbb{T}M$ is invariant and K-admissible. Hence, by Proposition 4.47, the induced splitting $\nabla_{red} : TM_{red} \to E_{red}$ identifies E_{red} with $\mathbb{T}M_{red}$ with the standard Courant bracket (4.1).

Corresponding to the Poisson structure on M, we have a Lagrangian subbundle of $\mathbb{T}M$ (see Example 5.21)

$$L = \{(\pi^\sharp(\xi), \xi) \mid \xi \in T^*M\}$$

which we wish to reduce using Theorem 4.54. For this, it is necessary that $L \cap K$ has constant rank and that L is $\mathfrak{g}$-invariant. First note that

$$L \cap K = 0.$$

Second, for $u \in \mathfrak{g}$ and $\xi \in \Gamma(T^*M)$,

$$[\![\chi(u), \pi^\sharp(\xi) + \xi]\!] = [u_M, \pi^\sharp(\xi)] + \mathcal{L}_{u_M}\xi \in \Gamma(L) \Leftrightarrow [u_M, \pi^\sharp(\xi)] = \pi^\sharp(\mathcal{L}_{u_M}\xi).$$

Therefore, L is invariant if and only if

$$[u_M, \pi^\sharp(\xi)] = \pi^\sharp(\mathcal{L}_{u_M}\xi), \forall\, u \in \mathfrak{g}. \tag{5.31}$$

Equation (5.31) holds if and only if for $f_1, f_2 \in C^\infty(M)$

$$\begin{aligned} i_{\pi^\sharp(\mathcal{L}_{u_M} df_1)}\, df_2 &= \left(\mathcal{L}_{u_M} i_{\pi^\sharp(df_1)} - i_{\pi^\sharp(df_1)}\mathcal{L}_{u_M}\right) df_2 \\ &= \mathcal{L}_{u_M}\{f_1, f_2\} - \{f_1, \mathcal{L}_{u_M} f_2\}, \forall\, u \in \mathfrak{g} \end{aligned}$$

which is equivalent to

$$\mathcal{L}_{u_M}\{f_1, f_2\} = \{\mathcal{L}_{u_M} f_1, f_2\} + \{f_1, \mathcal{L}_{u_M} f_2\}, \forall\, u \in \mathfrak{g}. \tag{5.32}$$

Now, (5.32) is just the infinitesimal counterpart of (5.30). Hence, L is $\mathfrak{g}$-invariant. Thus, we can reduce L to find

$$L_{red} = \frac{L \cap K^\perp + K}{K} \Big/ G \subset E_{red}$$

which is a Dirac structure on M/G by Theorem 4.54.

To find a local section of $U^{\nabla_{red}}(L_{red})$ using Theorem 5.29, first note that as $L \cap K = 0$, any isotropic subbundle $D \subset \mathbb{T}M$ with $(L \cap K)^\perp \oplus D = \mathbb{T}M$ is equal to zero.

Second, to find an invariant local section of $U(L) \subset \wedge^\bullet T^*M$ using Example 5.21, it suffices to find an invariant local section of $\Gamma(\wedge^m T^*M)$. For this, let $\mathcal{V}$ be a coordinate neighborhood of M_{red} with coordinates $(x_1, \cdots, x_{m-n})$ such that $q^{-1}(\mathcal{V}) \cong \mathcal{V} \times G$, where $n = \dim(G)$. Let

$$\nu = dx_1 \wedge \cdots \wedge dx_{m-n} \wedge \nu_G$$

where μ_G is a right-invariant volume form in G such that $\int_G \mu_G = 1$. Then (see Example 5.21),

$$\varphi = \nu - i_\pi \nu + \frac{1}{2} i_{\pi^2} \nu + \dots$$

is an invariant section of the pure spinor line bundle $U(L)$ over $q^{-1}(\mathcal{V})$. Hence, by Theorem 5.29,

$$\varphi_{red} = q_*(\varphi)$$

is a local section of $U^{\nabla_{red}}(L_{red})$ over $\mathcal{V}$ (note that as $\mu^{-1}(0) = M$, the inclusion map $j : \mu^{-1}(0) \to M$ is the identity).

Let us calculate explicitly φ_{red}. Over $q^{-1}(\mathcal{V})$, $\pi = \pi_1 + \pi_2$, where

$$\pi_1 = \sum_{1 \le i < j \le n} f_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \ f_{ij} \in C^\infty(\mathcal{V} \times G)$$

and π_2 is a sum of terms which involve derivatives in the directions of G. Hence,

$$\pi^k = \sum_{j=0}^{k} \binom{k}{j} \pi_1^j \pi_2^{k-j}.$$

For $0 \le j < k$, one has that (see the Appendix B)

$$q_*(i_{\pi_1^j \pi_2^{k-j}} \nu) = 0,$$

because the derivatives in the direction of G on π_2 turn $i_{\pi_1^j \pi_2^{k-j}} \nu$ into a type (II) form. Therefore

$$q_*(i_{\pi^k} \nu) = q_*(i_{\pi_1^k} \nu) \neq 0 \text{ only if } 2k \le m - n.$$

Now, observe that

$$f_{ij} = \pi(dx_i, dx_j) = dx_j(\pi^\sharp(dx_i))$$

satisfies

$$\mathcal{L}_{u_M} f_{ij} = \mathcal{L}_{[u_M, \pi^\sharp(dx_j)]} dx_i + i_{\pi^\sharp(dx_j)} \mathcal{L}_{u_M} dx_i, \ \forall\, u \in \mathfrak{g}.$$

By (5.31) and the fact that $\mathcal{L}_{u_M} dx_k = 0$ for every $k = 1, \cdots, n$,

$$\mathcal{L}_{u_M} f_{ij} = \mathcal{L}_{\pi^\sharp(\mathcal{L}_{u_M} dx_j)} dx_i = 0, \ \forall\, u \in \mathfrak{g},$$

which proves that there exists $\hat{f}_{ij} \in C^\infty(\mathcal{V})$ such that

$$f_{ij} = \hat{f}_{ij} \circ q.$$

Finally, for $2k \le m - n$,

$$\begin{aligned} i_{\pi_1^k} \nu = &\quad (i_{\pi_1^k}\, dx_1 \wedge \dots \wedge dx_{m-n}) \wedge \nu_G \\ = &\quad q^*(i_{\pi_{red}^k} dx_1 \wedge \dots \wedge dx_{m-n}) \wedge \nu_G, \end{aligned}$$

where

$$\pi_{red} = \sum_{1 \leq i < j \leq n} \hat{f}_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{5.33}$$

Therefore, by Proposition B.4,

$$q_*(i_{\pi_1^k}\nu) = \left(i_{\pi_{red}^k} dx_1 \wedge \cdots \wedge dx_{m-n}\right) \int_G \nu_G = i_{\pi_{red}^k} dx_1 \wedge \cdots \wedge dx_{m-n}$$

and thus

$$\begin{aligned} \varphi_{red} = q_*(\varphi) = q_* \left(\sum_{k=0}^\infty \tfrac{(-1)^k}{k!}\, i_{\pi^k}\, \nu\right) &= \sum_{k=0}^\infty \tfrac{(-1)^k}{k!}\, q_*(i_{\pi_1^k}\, \nu) \\ &= \sum_{k=0}^\infty \tfrac{(-1)^k}{k!}\, i_{\pi_{red}^k}\, dx_1 \wedge \cdots \wedge dx_{m-n} \end{aligned}$$

which proves that $\Phi_{\nabla_{red}}(L_{red})$ is the Dirac structure on M_{red} corresponding to a Poisson structure π_{red} given locally on $\mathcal{V}$ by (5.33).

As for the integrability of L_{red} , we follow the recipe of §5.3.2. Begin by noticing that as $D = 0$, K_D defined by (5.19) equals $K = \Delta_{\mathfrak{g}}$ and $L' = TM$ is an invariant Lagrangian subbundle of $\mathbb{T}M$ such that $K \subset TM$. Moreover,

$$TM \oplus L = \mathbb{T}M.$$

Also, as $L \cap K = 0$, one has that $L_1 \subset \mathbb{T}M$ defined by (5.20) equals TM. Then, Propositin 5.36 asserts that

$$(TM)_{red} \oplus L_{red} = E_{red}.$$

By definition (see (4.28))

$$(TM)_{red} = \frac{TM}{\Delta_{\mathfrak{g}}} \Big/ G = \nabla_{red}(TM_{red}).$$

As we saw in Example 5.21, the section Υ of $\wedge^\bullet TM$ over $q^{-1}(\mathcal{V})$ is given by

$$\Upsilon = X_\pi - \frac{1}{2}[\pi, \pi],$$

where $X_\pi \in \Gamma(TM|_{q^{-1}(\mathcal{V})})$ is the modular vector field of π with respect to ν. It follows from Proposition 5.27 and Remark 5.28 applied to the invariant polarization (TM, L) of $\mathbb{T}M$ that

$$(\psi_g)_* X_\pi = X_\pi \quad \text{and} \quad (\psi_g)_*[\pi, \pi] = [\pi, \pi].$$

By (5.29),

$$d\varphi_{red} = i_Y \varphi_{red},$$

where $Y \in \Gamma(TM_{red}|_{\mathcal{V}})$ is such that $q_*(X_\pi) = Y$. Finally, by Example 5.21, $Y = X_{\pi_{red}}$, the modular vector field of π_{red} with respect to $dx_1 \wedge \cdots \wedge dx_{m-n}$.

Distributions. Let $\Delta \subset TM$ be a distribution and suppose a compact, connected Lie group G acts freely on M preserving Δ, i.e.,

$$(\psi_g)_*\Delta = \Delta, \forall\, g \in G. \tag{5.34}$$

Consider the isotropic lifted G-action on $\mathbb{T}M$ with the standard Courant bracket (4.1) given by

$$\begin{array}{rccc} \chi: & \mathfrak{g} & \longrightarrow & \Gamma(\mathbb{T}M) \\ & u & \longmapsto & u_M \end{array}$$

which integrates to

$$G \ni g \longmapsto \Psi_g = \begin{pmatrix} (\psi_g)_* & 0 \\ 0 & (\psi_g^{-1})^* \end{pmatrix} \in \mathrm{Aut}(\mathbb{T}M, [\![\cdot,\cdot]\!]).$$

Take $\mu : M \to \{0\}$ to be the zero map (see Example 4.37. Then $(\chi, \{0\}, \mu)$ is reduction data on $\mathbb{T}M$. The reduced Courant algebroid is naturally identified with $\mathbb{T}M_{red}$ with the standard Courant bracket. Let

$$L_\Delta = \Delta \oplus \mathrm{Ann}\,(\Delta)$$

be the almost Dirac structure corresponding to Δ. It is invariant because of (5.34). If $L_\Delta \cap \Delta_{\mathfrak{g}}$ (or, equivalently, $\Delta \cap \Delta_{\mathfrak{g}}$) has constant rank, then

$$(L_\Delta)_{red} = \Delta_{red} \oplus \mathrm{Ann}\,(\Delta_{red})\,,$$

where

$$\Delta_{red} = \{q_*(X) \mid X \in \Delta\} \subset TM_{red}.$$

Let $D_1 \subset TM$ be an invariant distribution such that

$$(\Delta \cap \Delta_{\mathfrak{g}}) \oplus D_1 = TM$$

(it exists by compactness of G) and define

$$D = \mathrm{Ann}\,(D_1)\,.$$

As

$$(L_\Delta \cap \Delta_{\mathfrak{g}})^\perp = TM \oplus \mathrm{Ann}\,(\Delta \cap \Delta_{\mathfrak{g}})\,,$$

it follows that

$$(L_\Delta \cap \Delta_{\mathfrak{g}})^\perp \oplus D = \mathbb{T}M.$$

In this case, we have

$$L_D = L_\Delta \cap D^\perp + D = (\Delta \cap D_1) \oplus \mathrm{Ann}\,(\Delta \cap D_1)\,.$$

Note that

$$\Delta = \Delta \cap \Delta_{\mathfrak{g}} \oplus \Delta \cap D_1,$$

and, as q_* sends $\Delta \cap \Delta_{\mathfrak{g}}$ to zero, it follows that

$$q_*(\Delta \cap D_1) = q_*(\Delta),$$

in accordance with the general discussion of the pertubative method in §3.2.3, which guarantees that $(L_\Delta)_{red} = (L_D)_{red}$.

Let $\mathcal{W}$ be an invariant open neighborhood of M over which there is an invariant section of φ for $U(L_\Delta) = \det(\mathrm{Ann}\,(\Delta))$ and an invariant frame $\{\xi_1, \dots, \xi_r\}$ of $\mathrm{Ann}\,(D_1) \subset T^*M$. Then, from Theorem 5.29, it follows that

$$\varphi_{red} = q_*(\xi_1 \wedge \cdots \wedge \xi_r \wedge \varphi)$$

is a section of $U(L_{red}) = \det(\mathrm{Ann}\,(\Delta_{red}))$. Also, by Theorem 5.41, it follows that if Δ is the distribution tangent to a foliation, then Δ_{red} is also integrable. The leafs of Δ_{red} are the image under $q : M \to M_{red}$ of the leafs tangent to Δ.

Remark 5.42. In case L_Δ is integrable, $U(L_\Delta)$ has a global section if and only if the foliation tangent to Δ is transversely orientable. In this case, the foliation tangent to Δ_{red} is also transversely orientable if $\Delta \cap \Delta_{\mathfrak{g}}$ has a global frame (see Remark 5.31).

5.4.2 Reduction of generalized Calabi-Yau structures.

Let M be a smooth manifold and E be an exact Courant algebroid over M. Consider a generalized almost complex structure $\mathcal{J} : E \to E$ on M (see Definition 4.9) together with its $+i$-eigenbundle $L \subset E \otimes \mathbb{C}$.

Definition 5.43 ([24, 28])**.** $(M, \mathcal{J})$ is said to be generalized Calabi-Yau if the pure spinor line bundle $U^\nabla(L) \subset \wedge^\bullet T^*M \otimes \mathbb{C}$ corresponding to an isotropic splitting $\nabla : TM \to E$ has an nowhere zero d^∇-closed global section.

Remark 5.44. Using equation (5.4), it is straighforward to check that Definition 5.43 does not depend on ∇.

Note that (M, J) being generalized Calabi-Yau implies, by Proposition 5.17, that L is integrable.

Let us give some examples

Example 5.45 ([28])**.** Let $E = \mathbb{T}M$ with the standard Courant bracket (4.1) and

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix},$$

for a complex structure $J : TM \to TM$. In this case, taking the canonical splitting ∇_{can}, one has that $d^{\nabla_{\mathrm{can}}} = d$ and, by Example 5.15, $(M, \mathcal{J})$ is generalized Calabi-Yau if and only if $\wedge^{n,0}T^*M$ has a nowhere zero closed global section (or, equivalently, a nowhere zero holomorphic section).

Example 5.46 ([28])**.** Let (M, ω) be a symplectic manifold. The $+i$-eigenbundle of the generalized complex structure (see Example 4.11)

$$\mathcal{J} = \begin{pmatrix} 0 & \omega_\sharp^{-1} \\ -\omega_\sharp & 0 \end{pmatrix}$$

is given by

$$L = \{X + i\,\omega_\sharp(X) \mid X \in \Gamma(TM) \otimes \mathbb{C}\}.$$

$(M, \mathcal{J})$ is generalized Calabi-Yau because $e^{-i\omega} \in \Gamma(\wedge^\bullet T^*M) \otimes \mathbb{C}$ is a global section of $U(L)$.

Let G be a compact, connected Lie group acting on M and consider reduction data $(\chi, \mu, \mathfrak{h})$ on E. Let $\mathcal{J} : E \to E$ be a generalized almost complex structure such that $(M, \mathcal{J})$ is generalized Calabi-Yau and its $+i$-eigenbundle L is invariant. Let $K \subset E|_{\mu^{-1}(0)}$ be the isotropic subbundle (4.23) corresponding to $(\chi, \mu, \mathfrak{h})$ and choose an invariant K-admissible isotropic splitting $\nabla : TM \to E$. Consider the nowhere zero closed global section φ of $U^\nabla(L)$. Recall the inclusion map $j : \mu^{-1}(0) \to M$ and the quotient map $q : \mu^{-1}(0) \to M_{red}$.

Theorem 5.47. *Suppose φ is invariant (i.e. $\mathcal{L}_{u_M}\varphi = 0$ for every $u \in \mathfrak{g}$). If*

(i) $p(L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}) = \Delta_{\mathfrak{g}} \otimes \mathbb{C}$;

(ii) $L|_{\mu^{-1}(0)} \cap (\mathrm{Ann}\left(T\mu^{-1}\right) \otimes \mathbb{C}) = 0$;

then there exists a nowhere zero $d^{\nabla_{red}}$-closed section φ_0 of $U^{\nabla_{red}}(L_{red})$ such that

$$q^*\varphi_0 = j^*\varphi. \tag{5.35}$$

Moreover, if

(iii) $(L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}^\perp + K_{\mathbb{C}}) \cap (\bar{L}|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}^\perp + K_{\mathbb{C}}) \subset K_{\mathbb{C}}$,

then L_{red} is the $+i$-eigenbundle of a generalized complex structure $\mathcal{J}_{red} : E_{red} \to E_{red}$ such that $(M_{red}, \mathcal{J}_{red})$ is generalized Calabi-Yau.

In order to prove Theorem 5.47, we need a couple of lemmas.

Lemma 5.48. *Conditions (i), (ii) and (iii) implies that the reduced Dirac structure L_{red} given by Theorem 4.54 is a generalized complex structure.*

Proof. First note that (i) and (ii) guarantee that $L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}$ has constant rank. Indeed, the exactness of the sequence

$$0 \longrightarrow L|_{\mu^{-1}(0)} \cap (\mathrm{Ann}\left(T\mu^{-1}(0)\right) \otimes \mathbb{C}) \xrightarrow{p^*} L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}} \xrightarrow{p} \Delta_{\mathfrak{g}} \otimes \mathbb{C} \longrightarrow 0 \tag{5.36}$$

together with (ii) implies that $L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}$ is isomorphic to $\Delta_{\mathfrak{g}} \otimes \mathbb{C}$. Second, the invariance of φ implies that L is invariant (see Proposition 5.22). Hence, we can reduce L, using Theorem 4.54, to obtain $L_{red} \subset E_{red}$. The fact that L_{red} is a generalized complex structure now follows from (iii) (see §4.4.1 and section 5 of [11]). □

Let $\theta \in \Omega(\mu^{-1}(0), \mathfrak{g})$ be a connection 1-form for the principal G-bundle $q : \mu^{-1}(0) \to M_{red}$. For any basis $\{u^1, \ldots, u^r\}$ of $\mathfrak{g}$, write

$$\theta = \sum_{i=1}^{r} \theta_i\, u^r$$

where $\theta_i \in \Omega^1(\mu^{-1}(0))$.

Lemma 5.49. *Let $D_1 \subset TM|_{\mu^{-1}(0)}$ be an equivariant complement to $T\mu^{-1}(0)$. For $i = 1, \ldots, r$, there is a uniquely defined $\hat{\theta}_i \in \Gamma(\mathrm{Ann}\,(D_1))$ such that*

$$j^*\hat{\theta}_i = \theta_i.$$

Moreover,

$$\varphi_D = \hat{\theta}_1 \wedge \cdots \wedge \hat{\theta}_r \wedge \varphi \tag{5.37}$$

is a nowhere zero invariant section over $\mu^{-1}(0)$ of the pure spinor line bundle of the pertubation $L_D = L|_{\mu^{-1}(0)} \cap D^\perp + D$ corresponding to

$$D = \mathrm{Ann}\,(D_1 \oplus D_2) \otimes \mathbb{C}, \tag{5.38}$$

where

$$D_2 = \{X \in T\mu^{-1}(0) \mid i_X\theta = 0\}.$$

Proof. First note that D_1 exists by compactness of G. Let D be given by (5.38). Then property (i) implies that

$$(L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}})^\perp \oplus D = E|_{\mu^{-1}(0)}.$$

The choice of D_1 induces an (equivariant) isomorphism

$$j^*|_{\mathrm{Ann}(D_1)} : \mathrm{Ann}\,(D_1) \longrightarrow T^*\mu^{-1}(0), \tag{5.39}$$

so that the elements $\hat{\theta}_i \in \Gamma(\mathrm{Ann}\,(D_1))$ satisfying $j^*\hat{\theta}_i = \theta_i$ for $i = 1, \ldots, r$ are unique. It is easy to see that $\{\hat{\theta}_1, \ldots, \hat{\theta}_r\}$ is a global frame for $\mathrm{Ann}\,(D)$. Hence, by Proposition 3.41,

$$\varphi_D = \Pi_\nabla(\hat{\theta}_1 \wedge \cdots \wedge \hat{\theta}_r)\,\varphi = \hat{\theta}_1 \wedge \cdots \wedge \hat{\theta}_r \wedge \varphi \;\in\; \Gamma(\wedge^\bullet T^*M \otimes \mathbb{C}\,|_{\mu^{-1}(0)})$$

is a nowhere zero global section of $U(L_D)$ for $L_D = L|_{\mu^{-1}(0)} \cap D^\perp + D$. It remains to prove that φ_D is invariant. This follows from the well-know invariance property of the connection form θ,

$$\psi_g^*\,\theta = Ad_{g^{-1}} \circ \theta = \sum_{i=1}^r \theta_i Ad_{g^{-1}}(u^i), \forall\, g \in G,$$

which implies that

$$\psi_g^*\theta_i = \sum_{i=1}^r a_i^j(g)\,\theta^j, \text{ where } Ad_{g^{-1}}(u^j) = \sum_{i=1}^r a_i^j(g)\,u^i.$$

Hence,

$$\psi_g^*(\theta_1 \wedge \cdots \wedge \theta_r) = \det(Ad_{g^{-1}})\theta_1 \wedge \cdots \wedge \theta_r = \theta_1 \wedge \cdots \wedge \theta_r$$

because G is compact (and hence unimodular). Now, just use that (5.39) is equivariant and that $\psi_g^*\,\varphi = \varphi$ by hypothesis. □

Proof of Thm. 5.47. The existence of $\varphi_0 \in \Omega(M_{red}, \mathbb{C})$ satisfying (5.35) is equivalent to

$$\mathcal{L}_{u_M} j^*\varphi = 0 \quad \text{and} \quad i_{u_M} j^*\varphi = 0, \ \forall\, u \in \mathfrak{g}.$$

As φ is already invariant, it remains to prove the second equation. Choosing a right split for the sequence (5.36), one has that for every $u \in \mathfrak{g}$, there exists a element $k_u \in \Gamma(L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}})$ such that $p(k_u) = u_M$ (it is actually unique by (ii)). Now,

$$0 = \Pi_\nabla(k_u)\varphi = i_{u_M}\varphi + s_\nabla(k_u) \wedge \varphi,$$

where $s_\nabla : E \to T^*M$ is defined via (2.5). As ∇ is K-admissible, it follows that $s_\nabla(k_u)$ is a section of $\mathrm{Ann}\big(T\mu^{-1}(0)\big) \otimes \mathbb{C}$. Hence,

$$0 = j^*\Pi_\nabla(k_u)\varphi = i_{u_M} j^*\varphi.$$

This proves the existence of φ_0 satisfying (5.35). To prove that φ_0 is a section of $U^{\nabla_{red}}(L_{red})$, it suffices to prove that it is colinear to the reduced section of the pure spinor line bundle of L_{red} given by Theorem 5.29,

$$\varphi_{red} = q_* j^* \varphi_D = q_*(\theta_1 \wedge \cdots \wedge \theta_r \wedge j^*\varphi)$$

(observe that $j^*\varphi$ is nowhere zero by condition (ii); see Proposition 3.31). Using (5.35) together with Propositions B.4 and B.6,

$$\varphi_{red} = q_*(\theta_1 \wedge \cdots \theta_r \wedge q^*\varphi_0) = \left(\int_G \nu\right) \varphi_0,$$

where $\nu \in \Omega^r(G)$ is the left-invariant volume form on G induced by the basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathfrak{g}^*$ dual to $\{u^1, \ldots, u^r\}$. To finish the proof, it remains to prove that φ_0 is $d^{\nabla_{red}}$-closed, where

$$d^{\nabla_{red}} = d - H_{red}$$

and $H_{red} \in \Omega^3(M_{red})$ is the curvature of ∇_{red}. By Proposition 4.47, we have that $q^*H_{red} = j^*H$, where $H \in \Omega^3(M)$ is the curvature of ∇ and, using (5.35),

$$\begin{aligned} q^*(d\varphi_0 - H_{red} \wedge \varphi_{red}) &= d\, q^*\varphi_0 - j^*H \wedge q^*\varphi_0 \\ &= j^*(d\varphi - H \wedge \varphi) \\ &= 0. \end{aligned}$$

As $q^* : \Omega(M_{red}) \to \Omega(\mu^{-1}(0))$ is injective, this concludes the proof.

□

Example 5.50 (Nitta's reduction [41].)**.** Let G be a compact, connected Lie group acting on M. Let $H \in \Omega^3_{cl}(M)$ be a basic form ($\mathcal{L}_{u_M} H = 0$ and $i_{u_M} H = 0$, for every $u \in \mathfrak{g}$). Consider the corresponding isotropic lifted G-action

$$\begin{array}{rccc} \chi : & \mathfrak{g} & \longrightarrow & \Gamma(\mathbb{T}M) \\ & u & \longmapsto & u_M \end{array}$$

which integrates to

$$G \ni g \longmapsto \Psi_g = \begin{pmatrix} (\psi_g)_* & 0 \\ 0 & (\psi_g^{-1})^* \end{pmatrix} \in \mathrm{Aut}(\mathbb{T}M,\, [\![\cdot,\cdot]\!]_H).$$

Let $\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$ be an invariant generalized almost complex structure such that $(M, \mathcal{J})$ is generalized Calabi-Yau and such that $U(L)$ has an nowhere zero invariant d_H-closed section φ. Suppose that there exists an equivariant map

$$\mu : M \to \mathfrak{g}^*$$

(with respect to the co-adjoint action) such that

$$\mathcal{J} u_M = d\mu^u, \; \forall\, u \in \mathfrak{g}. \tag{5.40}$$

Form the corresponding reduction data $(\chi, \mu, \mathfrak{g}^*)$ with associated isotropic subbundle $K \subset \mathbb{T}M|_{\mu^{-1}(0)}$ (4.23) given by

$$K = \Delta_{\mathfrak{g}} \oplus \mathrm{Ann}\left(T\mu^{-1}(0)\right).$$

We claim that the $+i$-eigenbundle of $\mathcal{J}$ satisfies conditions (i), (ii) and (iii) of Theorem 5.47. Indeed, (5.40) implies that

$$L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}} = \{(u_M + i\, v_M,\, d\mu^v - i\, d\mu^u) \mid u,\, v \in \mathfrak{g}\} \tag{5.41}$$

which, by its turn, implies (i). As for (ii), note that

$$L|_{\mu^{-1}(0)} \cap (\mathrm{Ann}\left(T\mu^{-1}(0)\right) \otimes \mathbb{C}) = (L|_{\mu^{-1}(0)} \cap K_{\mathbb{C}}) \cap (\mathrm{Ann}\left(T\mu^{-1}(0)\right) \otimes \mathbb{C}),$$

which is clearly 0 by (5.41) and the fact that G acts freely on $\mu^{-1}(0)$. Finally, (iii) follows from (5.40) (note that $\mathcal{J}K = K$). With these data, Theorem 5.47 implies that L_{red} is a generalized complex structure for which $U(L_{red})$ has a nowhere zero $d_{H_{red}}$-closed global section φ_0 given by

$$q^*\varphi_0 = j^*\varphi,$$

where $H_{red} \in \Omega^3(M_{red})$ satisfies

$$q^* H_{red} = j^* H.$$

This is the content of the main result of [41].

Example 5.51. Let (M, ω) be a symplectic manifold and consider the corresponding generalized Calabi-Yau structure $(M, \mathcal{J})$ constructed in Example 5.46. Suppose a compact, connected Lie group G acts on M in a Hamiltonian fashion with moment map given by

$$\mu : M \to \mathfrak{g}^*.$$

In this case, for any $u \in \mathfrak{g}$

$$\mathcal{J} u_M = \omega_\sharp(u_M) = i_{u_M}\omega = d\mu^u$$

so that (5.40) is satisfied. Also, let $\varphi = e^{-i\omega}$ be a pure spinor corresponding to the $+i$-eigenbundle $L \subset \mathbb{T}M \otimes \mathbb{C}$ of $\mathcal{J}$. As $\mathcal{L}_{u_M}\omega = 0$, $\forall\, u \in \mathfrak{g}$, one has

$$d\varphi = 0 \quad \text{and} \quad \mathcal{L}_{u_M}\varphi = 0,\, \forall\, u \in \mathfrak{g}.$$

Thus, Theorem 5.47 together with Example 5.50 gives that $M_{red} = \mu^{-1}(0)/G$ inherits a generalized complex structure $\mathcal{J}_{red}$ for which $\varphi_0 \in \Gamma(\wedge^\bullet T^* M_{red} \otimes \mathbb{C})$ satisfying (5.35) is a pure spinor for its $+i$-eigenbundle $L_{red} \subset \mathbb{T}M_{red} \otimes \mathbb{C}$. In this case, a straightforward calculation shows that

$$\varphi_0 = e^{-i\omega_{red}},$$

where $\omega_{red} \in \Omega^2(M_{red})$ satisfies

$$q^*\omega_{red} = j^*\omega.$$

Hence, $\mathcal{J}_{red}$ is the generalized complex structure corresponding to the Marsden-Weinstein [39] reduction of ω.

5.4.3 The T-duality map.

T-duality is a relation between two types of string theory. In [8, 9], a mathematical version of T-duality was introduced in the context of S^1-principal bundles (also principal torus bundles). We recall their construction here following [15].

Let $\pi_1 : P_1 \to N$ be a principal circle bundle with an invariant closed integral 3-form $H_1 \in \Omega^3(P_1)$ and a connection $\theta_1 \in \Omega^1(P_1)$ (where we have identified $\mathfrak{s}^1 \cong \mathbb{R}$ in such a way that $(\pi_1)_*\theta_1 = 1$ (see Proposition B.6)). Define

$$c_2 = (\pi_1)_* H \in \Omega^2(N)$$

and let $c_1 \in \Omega^2(N)$ be the curvature of P_1,

$$\pi^* c_1 = d\theta_1.$$

There exists $h \in \Omega^3(N)$ such that

$$H_1 = \pi_1^* c_2 \wedge \theta_1 + \pi_1^* h. \tag{5.42}$$

As $(p_1)_*$ commutes with d, it follows that c_2 is closed. It is also integral as

$$\int_S c_2 = \int_{\pi_1^{-1}(S)} H,$$

for every surface $S \subset N$. Therefore, there exists a principal circle bundle $\pi_2 : P_2 \to N$ with a connection $\theta_2 \in \Omega^2(P_2)$ such that

$$d\theta_2 = \pi_2^* c_2.$$

Define

$$H_2 = \pi_2^* c_1 \wedge \theta_2 + \pi_2^* h \in \Omega^3(P_2).$$

We call (P_2, H_2) the **T-dual space** corresponding to (P_1, H_1).

Example 5.52. Consider the Hopf fibration $\pi_1 : S^3 \to S^2$. The curvature of this bundle is a volume form $c_1 \in \Omega^2(S^2)$. By taking $H_1 = 0$, one has that $c_2 = 0 = h$. Therefore, the T-dual space is $P_2 = S^2 \times S^1$ with the connection form $\theta_2 = \mathrm{pr}^*_{S^1}\xi$, where $\xi \in \Omega^1(S^1)$ is an invariant volume form and

$$H_2 = \mathrm{pr}^*_{S^2}\, c_1 \wedge \theta_2.$$

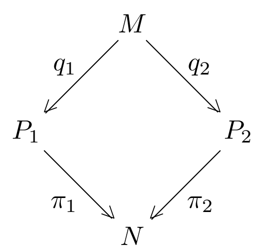
Given T-dual spaces (P_1, H_1) and (P_2, H_2), define the **correspondence space** to be the fiber product $M = P_1 \times_N P_2$ of P_1 and P_2. More precisely, M is a submanifold of $P_1 \times P_2$ defined by

$$M = \{(x, y) \in P_1 \times P_2 \mid \pi_1(x) = \pi_2(y)\}.$$

For a point $m = (x, y) \in M$, we have

$$T_m M = \{(X_1, X_2) \in T_x P_1 \times T_y P_2 \mid d\pi_1(X_1) = d\pi_2(X_2)\}.$$

We have the natural projections $q_1 : M \to P_1$ and $q_2 : M \to P_2$ which make the diagram below commutative:



(5.43)

The maps q_1 and q_2 give M the structure of a S^1 principal bundle over P_1 and P_2 respectively. Moreover, $q_1^*\theta_1$ and $q_2^*\theta_2$ are connection 1-forms on P_1 and P_2 respectively.

Using the commutativity of the diagram (5.43), one has

$$\begin{aligned} q_2^*H_2 - q_1^*H_1 &= q_2^*(\pi_2^*c_1 \wedge \theta_2) - q_1^*(\pi_1^*c_2 \wedge \theta_1) \\ &= q_1^*\pi_1^*c_1 \wedge q_2^*\theta_2 - q_2^*\pi_2^*c_2 \wedge q_1^*\theta_1 \\ &= q_1^*d\theta_1 \wedge q_2^*\theta_2 - q_2^*d\theta_2 \wedge q_1^*\theta_1 \\ &= d(q_1^*\theta_1 \wedge q_2^*\theta_2). \end{aligned}$$

Define

$$B = q_1^*\theta_1 \wedge q_2^*\theta_2 \in \Omega^2(M) \tag{5.44}$$

and let $\Omega_{S^1}(P_i)$ be the space of invariant differential forms of P_i for $i = 1, 2$. As H_1, H_2 are invariant by construction, their corresponding twisted differentials d_{H_1} and d_{H_2} turn $\Omega_{S^1}(P_1)$ and $\Omega_{S^1}(P_2)$ respectively into differential complexes. One of the main result of [8] is that this differential complexes are isomorphic. We describe the isomorphism here. Define

$$\tau : \Omega_{S^1}(P_1) \longrightarrow \Omega_{S^1}(P_2)$$

by

$$\tau(\alpha) = (q_2)_*\,(e^B \wedge q_1^*\alpha).$$

Theorem 5.53 (Bouwknegt-Evslin-Mathai[8]). *The map*

$$\tau : (\Omega_{S^1}(P_1),\, d_{H_1}) \longrightarrow (\Omega_{S^1}(P_2),\, d_{H_2})$$

is an isomorphism of differential complexes.

What we shall do here is to give an interpretation of the action of τ on pure spinors built upon previous work of G. Cavalcanti and M. Gualtieri [15]. Consider the Courant algebroid E over M which is $\mathbb{T}M$ with the $q_1^* H_1$-twisted Courant bracket. Corresponding to the S^1-principal bundle structure given on M by $q_i : M \to P_i$ (for $i = 1, 2$), we have an S^1-action on M

$$\psi^{\,i} : S^1 \longrightarrow \mathrm{Diff}(M)$$

and the corresponding infinitesimal action

$$\Sigma^i : \mathfrak{s}^1 \longrightarrow \Gamma(TM).$$

We denote the image of an element $u \in \mathfrak{s}^1$ under Σ^i by u^i_M. For $m = (x, y) \in M$,

$$\psi^1_g(m) = (x,\, g\cdot y) \text{ and } \psi^2_g(m) = (g\cdot x,\, y). \tag{5.45}$$

Lemma 5.54. *The map*

$$\begin{array}{cccc} \chi : & \mathfrak{s}^1 & \longrightarrow & \Gamma(E) \\ & u & \longmapsto & u^2_M + \xi_u, \end{array}$$

where $\xi_u = (i_{u_{P_1}}\theta_1)\, q_2^*\,\theta_2 \in \Gamma(T^*M)$ defines an isotropic lifted G-actions on E.

Proof. First we have to check that χ is bracket preserving. Let $u, v \in \mathfrak{s}^1$. The fact that

$$(q_2)_* u^2_M = (q_2)_* v^2_M = 0$$

implies

$$\mathcal{L}_{u^2_M}\xi_v = (i_{v_{P_1}}\theta_1)\, \mathcal{L}_{u^2_M} q_2^*\theta_2 = 0$$

and

$$i_{v^2_M} d\xi_u = (i_{u_{P_1}}\theta_1)\, i_{v^2_M} q_2^*\theta_2 = 0$$

(where we have used that $i_{u_{P_1}}\theta_1$ and $i_{v_{P_1}}\theta_1$ are constant functions on M). Now, using that $\mathfrak{s}^1$ is abelian together with equation (5.45),

$$\begin{aligned} [\![\chi(u), \chi(v)]\!]_{q_1^* H_1} &= [u^2_M, v^2_M] + \mathcal{L}_{u^2_M}\xi_v - i_{v^2_M}\xi_u + i_{v^2_M}\, i_{u^2_M} q_1^* H_1 \\ &= [u, v]^2_M + q_1^*\,(i_{v_{P_1}}\, i_{u_{P_1}} H_1) \\ &= q_1^*\,(i_{v_{P_1}}\, i_{u_{P_1}} H_1). \end{aligned}$$

By (5.42),

$$i_{u_{P_1}} H_1 = (i_{u_{P_1}}\theta_1)\, \pi_1^* c_2 \tag{5.46}$$

which implies

$$i_{v_{P_1}}\, i_{u_{P_1}} H_1 = 0.$$

Thus,

$$[\![\chi(u), \chi(v)]\!]_{q_1^* H_1} = 0 = \chi([u, v]).$$

Also,

$$g_{\mathrm{can}}(\chi(u), \chi(v)) = (i_{u_{P_1}} \theta_1)(i_{v^2_M} q_2^* \theta_2) + (i_{v_{P_1}} \theta_1)(i_{u^2_M} q_2^* \theta_2) = 0,$$

which proves that χ is an isotropic lifted $\mathfrak{g}$-action.

To prove that χ integrates to a G action by automorphisms of E it suffices to show that the canonical splitting ∇_{can} is invariant. This will imply that

$$g \mapsto \begin{pmatrix} \psi_g^2 & 0 \\ 0 & (\psi^2_{g^{-1}})^* \end{pmatrix}$$

integrates χ. By Proposition 4.30, we have to show that

$$d\xi_u - i_{u^2_M} q_1^* H_1 = 0, \text{ for every } u \in \mathfrak{s}^1.$$

On the one hand, we have

$$\begin{aligned} d\xi_u = (i_{u_{P_1}} \theta_1)\, dq_2^* \theta_2 &= (i_{u_{P_1}} \theta_1) q_2^* \pi_2^* c_2 \\ &= (i_{u_{P_1}} \theta_1)\, q_1^* \pi_1^* c_2 \end{aligned}$$

using the commutativity of diagram (5.43). On the other hand, by (5.46),

$$i_{u^2_M} q_1^* H_1 = q_1^*(i_{u_{P_1}} H_1) = (i_{u_{P_1}} \theta_1) q_1^*(\pi_1^* c_2).$$

This concludes the proof. □

By considering the zero moment map $\mu : M \to \{0\}$ (see Example 4.37), we have reduction data on E given by $(\chi, \{0\}, \mu)$. The corresponding isotropic subbundle of E is $K_{\mathfrak{g}}$ (see §4.2.2). To study the reduced Courant algebroid E_{red} over $M_{red} = M/S^1 = P_2$ properly it is necessary to find an invariant $K_{\mathfrak{g}}$-admissible splitting. For this, note that for $B \in \Omega^2(M)$ defined by (5.44), one has

$$\begin{aligned} i_{u^2_M} B = i_{u^2_M}(q_1^* \theta_1 \wedge q_2^* \theta_2) &= (i_{u^2_M} q_1^* \theta_1)\, q_2^* \theta_2 \\ &= (i_{u_{P_1}} \theta_1)\, q_2^* \theta_2 \\ &= \xi_u \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{u^2_M} B = d i_{u^2_M} B + i_{u^2_M} dB &= d\xi_u + i_{u^2_M}(q_2^* H_2 - q_1^* H_1) \\ &= (d\xi_u - i_{u^2_M} q_1^* H_1) + i_{u^2_M} q_2^* H_2 \\ &= 0 \end{aligned}$$

for $u \in \mathfrak{s}^1$. Thus $\nabla = \nabla_{\mathrm{can}} + B$ is an invariant $K_{\mathfrak{g}}$-admissible splitting for E. Its curvature is

$$H = q_1^* H_1 + dB = q_2^* H_2.$$

Hence (by Proposition 4.47), ∇ induces an isotropic splitting for E_{red} which identifies it with $TP_2 \oplus T^* P_2$ endowed with the H_2-twisted Courant bracket. We are now able to pass to pure spinors.

Let $\varphi \in \Omega_{S^1}(P_1)$ be a pure spinor such that

$$\mathcal{N}(\varphi) = L_1 \subset TP_1 \oplus T^*P_1.$$

Consider $q_1^*\varphi \in \Gamma(T^*M)$ viewed as a Clifford module for $Cl(E, g)$ corresponding to the canonical splitting ∇_{can}. As we saw in Proposition 3.30, $q_1^*\varphi$ is a pure spinor such that for a point $m = (x, y) \in M$,

$$\mathcal{N}(q_1^*\varphi)_m = \Lambda^t_{(dq_1)_m}(L_1|_x) \subset T_mM \oplus T^*_mM. \tag{5.47}$$

Call $L = \mathcal{N}(q_1^*\varphi)$. We claim that L is invariant under the isotropic lifted G-action χ. Indeed, as ∇_{can} is invariant,

$$\mathcal{L}^{\nabla_{\text{can}}}_{\chi(u)} q_1^*\varphi = \mathcal{L}_{u^2_M} q_1^*\varphi = q_1^*\mathcal{L}_{u_{P_1}}\varphi = 0 \ \ \forall u \in \mathfrak{s}^1$$

because $\varphi \in \Omega_{S^1}(P_1)$. Therefore, by Proposition 5.22, it follows that L is invariant. Also,

$$L \cap K_{\mathfrak{g}} = 0.$$

Indeed, let $m = (x, y) \in M$ and $(X, \xi) \in (L \cap K_{\mathfrak{g}})_m$. By (5.47) and the definition of $K_{\mathfrak{g}}$, there is some $u \in \mathfrak{s}^1$ and $\eta \in T^*_xP_1$ such that

$$X = u^2_M(m) \text{ and } (dq_1)^*_m\eta = \xi = (i_{u_{P_1}}\theta_1)\ q_2^*\,\theta_2|_m\,.$$

Now, for every $0 \neq v \in \mathfrak{s}^1$,

$$(dq_1)^*_m\eta(v^1_M) = 0 \ \text{ and } i_{v^1_M} q_2^*\theta_2|_m = \theta_2(v_{P_2}) \neq 0$$

which implies that $\xi = 0$ or, equivalently, $i_{u_{P_1}}\theta_1 = 0$. But this implies $u = 0$ and hence $X = 0$ as we wanted to show.

By Theorem 4.54, the Lagrangian subbundle $L \subset E$ can be reduced to a Lagrangian subbundle L_{red} of E_{red}. To find a pure spinor corresponding to L_{red} using Theorem 5.29, we have to pass from the ∇_{can} induced module to the $\nabla = \nabla_{\text{can}} + B$ induced module so as to have a $K_{\mathfrak{g}}$-admissible splitting. By Example 5.3,

$$\mathcal{N}_\nabla(e^B \wedge q_1^*\varphi) = L$$

and, as B is invariant, $e^B \wedge q_1^*\varphi$ is an invariant local section of $U^\nabla(L)$. Therefore, by Theorem 5.29,

$$\tau(\varphi) = (q_2)_*(e^B \wedge q_1^*\varphi) \in U^{\nabla_{red}}(L_{red}) \subset \Omega(P_2).$$

Define $L_2 = \Phi_{\nabla_{red}}(L_{red}) \subset TP_2 \oplus T^*P_2$. By construction and by Proposition 2.37, for any point $m = (x, y)$

$$\begin{aligned}(L_2)_y = \Lambda_{(dq_2)_m}(\Phi_{\nabla_{\text{can}}+B}(L)) \ &= \Lambda_{(dq_2)_m}(\tau_{-B}(L)) \\ &= \Lambda_{(dq_2)_m} \circ \Lambda_{\tau_{-B}} \circ \Lambda^t_{(dq_1)_m}(\,(L_1)_x\,).\end{aligned}$$

Let $\Lambda_{(x,y)} \subset \overline{\mathcal{D}\,(T_xP)} \times \mathcal{D}\,(T_yP_2)$ be the morphism given by

$$\Lambda = \Lambda_{(dq_2)_m} \circ \Lambda_{\tau_{-B}} \circ \Lambda^t_{(dq_1)_m}.$$

Proposition 5.55. *Let $m = (x, y) \in M$. There exists an isomorphism*

$$F : T_xP_1 \oplus T_x^*P_1 \longrightarrow T_yP_2 \oplus T_y^*P_2 \tag{5.48}$$

such that $\Lambda_{(x,y)} = \Lambda_F$.

Proof. If we prove that $\ker\left(\Lambda_{(x,y)}\right) = 0$ (see (2.20)), then by the symmetry of the situation

$$\ker\left(\Lambda^t_{(x,y)}\right) = \ker\left(\Lambda_{(dq_1)_m} \circ \Lambda_{\tau_B} \circ \Lambda^t_{(dq_2)_m}\right) = 0$$

also. In this case, (see (2.21))

$$\mathrm{im}(\Lambda_{(x,y)}) = \ker\left(\Lambda^t_{(x,y)}\right)^{\perp} = T_yP_2 \oplus T_y^*P_2$$

and this will prove the proposition. Now, it is straightforward to check that for $X + \xi \in T_xP_1 \oplus T_x^*P_1$ and $Z + \zeta \in T_yP_2 \oplus T_y^*P_2$,

$$(X{+}\xi, Z{+}\zeta) \in \Lambda_{(x,y)} \Leftrightarrow \exists (Y, \eta) \in \mathcal{D}\,(T_mM) \text{ s.t. } \begin{cases} dq_1(Y) = X; \\ dq_2(Y) = Z; \\ (dq_1)^*_x\,\xi = (dq_2)^*_y\,\zeta + i_YB. \end{cases}$$

If $(X{+}\xi, 0) \in \Lambda_{(x,y)}$, then $X = dq_1(Y)$ where $Y \in T_mM$ is such that $dq_2(Y) = 0$. In this case, there exists $u \in \mathfrak{s}^1$ such that $Y = u^2_M(m)$ and thus $X = u_{P_1}(x)$. Also,

$$(dq_1)^*_x\xi = i_YB = i_{u^2_M(m)}(q_1^*\theta_1 \wedge q_2^*\theta_2) = (i_{u_{P_1}}\theta_1)\, q_2^*\theta|_m.$$

Again, the fact that

$$i_{v^1_M(m)}(dq_1)^*_x\xi = 0 \;\text{ and }\; i_{v^1_M(m)}q_2^*\theta_2|_m = i_{v_{P_2}(x)}\theta_2 \neq 0.$$

implies that $u = 0$ and hence, both $\xi = 0$ and $X = 0$. This concludes the proof. □

Let us give the value of F for some elements of $T_xP_1 \oplus T_x^*P_1$. First observe that

$$T_xP_1 = \ker(\pi_1) \oplus \ker(\theta_{1,\,x}) = \Delta_{\mathfrak{g},\,x} \oplus \ker(\theta_{1,\,x}),$$

where $\ker(\theta_{1,\,x}) = \{X \in T_xP_1 \mid i_X\theta_{1,\,x} = 0\}$. Dualizing,

$$T_x^*P_1 = \mathrm{Ann}\,(\Delta_{\mathfrak{g},\,x}) \oplus \mathbb{R}\,\theta_{1,\,x}.$$

It is straightforward to check that

$$\begin{cases} F(\theta_{1,\,x}) = -u_{P_2}(y), \text{ where } u \in \mathfrak{s}^1 \text{ is such that } i_{u_{P_2}}\theta_2 = 1 \text{ and} \\ F(\,v_{P_1}(x)\,) = -(i_{v_{P_1}}\theta_1)\,\theta_{2,\,y} \text{ for any } v \in \mathfrak{s}^1. \end{cases}$$

Also, for $X \in T_xP_1$ such that $i_X\theta_1 = 0$, there exists a unique $Z \in T_yP_2$ such that $i_Z\theta_2 = 0$ and $d\pi_1(X) = d\pi_2(Z)$. Then

$$F(X) = Z.$$

To finish, let $\xi = (d\pi_1)^*_{\pi_1(x)}\xi_{red} \in T^*_x P_1$ for some $\xi_{red} \in T^*_{\pi_1(x)}N$; then

$$F(\xi) = (d\pi_2)^*_{\pi_2(y)}\xi_{red}.$$

Remark 5.56. The idea of G.Cavalcanti and M. Gualtieri [15] was to think of T-duality as a map from invariant Lagrangian subbundles of $(TP_1 \oplus T^*P_1, H_1)$ to invariant Lagrangian subbundles of $(TP_2 \oplus T^*P_2, H_2)$. The map they constructed is exactly the extension of the isomorphism (5.48) to invariant sections:

$$F : \Gamma_{S^1}(TP_1 \oplus T^*P_1) \to \Gamma_{S^1}(TP_2 \oplus T^*P_2).$$

Our construction completes their work by putting their map in the context of pure spinors.

Appendix A

More on pure spinors and the split-quadratic category.

In this appendix we develop further the idea of associating a transform between spinor spaces to a morphism (in the sense of the split-quadratic category) $\Lambda \subset \overline{E_1} \times E_2$, where E_1, E_2 are split vector spaces (see Definition 2.11). This should be seen as an (odd) analogue of the quantization procedure that Guillemin-Sternberg proposed in [26]. This part of this thesis grew out of the suggestion of H. Bursztyn to use the Chevalley pairing (A.1) to reduce spinors (see Theorem 5.29) and it is heavily influenced by unpublished work of M.Gualtieri [25].

A.1 The transform.

Let (E, g) be a split-quadratic vector space and let $l = (L, L')$ be a polarization of E. Consider the representation

$$\Pi_l : Cl(E, g) \longrightarrow \mathrm{End}\,(\wedge^\bullet L')$$

(see §3.1). There is a non-degenerate bilinear pairing defined in $\wedge^\bullet L'$ first introduced by E.Cartan [14] and further studied by C.Chevalley [16]. It is defined by

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & \wedge^\bullet L' \times \wedge^\bullet L' & \longrightarrow & \det(L') \\ & (\alpha, \beta) & \longmapsto & [\alpha^t \wedge \beta]^{\mathrm{top}}\,, \end{array} \tag{A.1}$$

where $\cdot^{\,t}$ is defined by (3.4) and $[\,\cdot\,]^{\mathrm{top}}$ is the projection in the top degree component of $\wedge^\bullet L'$.

Lemma A.1. *For $a \in Cl(E, g)$ and $\alpha, \beta \in \wedge^\bullet L'$*

$$\left[(\Pi_l(a)\,\alpha)^t \wedge \beta\right]^{\mathrm{top}} = \left[\alpha^t \wedge \Pi_l(a^t)\,\beta\right]^{\mathrm{top}}.$$

Proof. It suffices to prove it for $a \in E$ and use that $\Pi_l(a_1a_2) = \Pi_l(a_1) \circ \Pi_l(a_2)$. Let $a = x + y$, $x \in L$ and $y \in L'$. Recall from (3.6) and (3.7)

$$\Pi_l(a)\alpha = D^g(x)\alpha + y \wedge \alpha.$$

Thus,

$$(\Pi_l(a)\alpha)^t = (D^g(x)\alpha)^t + \alpha^t \wedge y.$$

It is straightforward to see that

$$\alpha^t = (-1)^{\frac{k(k-1)}{2}}\alpha, \text{ where } k \text{ is the degree of } \alpha$$

and therefore

$$(D^g(x)\alpha^t)^t = (-1)^{\frac{k(k-1)}{2}+\frac{(k-1)(k-2)}{2}} D^g(x)\alpha = (-1)^{k-1}D^g(x)\alpha. \tag{A.2}$$

Hence,

$$(\Pi_l(a)\alpha)^t = (-1)^{k-1}D^g(x)\alpha^t + \alpha^t \wedge y.$$

Using that $D^g(x)$ is a derivation of degree -1 and that $[D^g(x)(\alpha^t \wedge \beta)]^{\text{top}} = 0$, it is now straightforward to check the result. □

Let (E_1, g_1) and (E_2, g_2) be split-quadratic vector spaces and let $l_1 = (L_1, L_1')$ and $l_2 = (L_2, L_2')$ be polarizations for E_1 and E_2 respectively. From Example 3.14, $\wedge^\bullet L_1' \otimes \wedge^\bullet L_2'$ is a module for $Cl(E_1 \times E_2, -g_1 + g_2) = Cl(E_1, -g_1) \otimes Cl(E_2, g_2)$. The representation corresponding to the polarization $l_1 \times l_2 = (L_1 \times L_2, L_1' \times L_2')$ is given by

$$\Pi_{l_1 \times l_2}(a_1 \otimes a_2)\alpha \otimes \beta = (-1)^{|a_2||\alpha|}\Pi^-_{l_1}(a_1)\alpha \otimes \Pi_{l_2}(a_2)y, \tag{A.3}$$

for $a_1 \in Cl(E_1, -g)$, $a_2 \in Cl(E_2, g_2)$ and $\alpha \otimes \beta \in \wedge^\bullet L_1 \otimes \wedge^\bullet L_2$.

For every $\alpha \otimes \beta \in \wedge^\bullet L_1' \otimes \wedge^\bullet L_2'$ define

$$\begin{array}{rccl} \widehat{\alpha \otimes \beta}: & \wedge^\bullet L_1' \otimes \det(L_1) & \longrightarrow & \wedge^\bullet L_2' \\ & \gamma \otimes \nu & \longmapsto & \nu(\langle \alpha^t, \gamma \rangle)\beta, \end{array}$$

where $\det(L_1)$ is identified with $\det(L_1')^*$ via g_1. By linear continuation, one defines a map

$$\begin{array}{ccc} \wedge^\bullet L_1' \otimes \wedge^\bullet L_2' & \longrightarrow & \mathrm{Hom}(\wedge^\bullet L_1' \otimes \det(L_1),\ \wedge^\bullet L_2') \\ \theta & \longmapsto & \widehat{\theta}. \end{array} \tag{A.4}$$

In this way, any Lagrangian subspace $\Lambda \subset E_2 \times \bar{E}_1$ defines an one-dimensional subspace of $\mathrm{Hom}(\wedge^\bullet L_1' \otimes \det(L_1),\ \wedge^\bullet L_2')$ given by

$$\{\widehat{\theta} \mid \theta \in U^{l_1 \times l_2}(\Lambda) \subset \wedge^\bullet L_1' \otimes \wedge^\bullet L_2'\},$$

where $U^{l_1 \times l_2}(\Lambda)$ is the pure spinor line corresponding to Λ.

Theorem A.2. *Let L be a Lagrangian subspace of E_1, $\Lambda \subset \bar{E}_1 \times E_2$ a morphism and θ be a non-zero generator of $U^{l_1 \times l_2}(\Lambda)$. Let $\varphi \in U^{l_1}(L) \subset \wedge^\bullet L_1'$ and $\nu \in \det(L_1)$. If*

$$\widehat{\theta}(\varphi \otimes \nu) \in \wedge^\bullet L_2' \text{ is non-zero,} \tag{A.5}$$

then

$$\mathcal{N}_{l_2}(\widehat{\theta}(\varphi \otimes \nu)) = \Lambda(L).$$

To prove the theorem, we shall need a simple lemma.

Lemma A.3. *For $e_1 \in E_1$ and $\alpha \in \wedge^\bullet L_1'$, one has*

$$(\Pi_l^-(e_1)\alpha)^t = (-1)^{|\alpha|}\Pi_l(e_1)\alpha^t.$$

Proof. Write e_1 as $e_1 = x + y \in L_1 \oplus L_1'$. Then (see Example 3.13),

$$(\Pi_l^-(e_1)\alpha)^t = (-D^g(x)\alpha + y \wedge \alpha)^t = (-D^g(x)\alpha)^t + \alpha^t \wedge y.$$

By (A.2), $(-D^g(x)\alpha)^t = (-1)^{|\alpha|}D^g(x)\alpha^t$ and therefore

$$(\Pi_l^-(e_1)\alpha)^t = (-1)^{|\alpha|}(D^g(x)\alpha^t + y \wedge \alpha^t) = \Pi_l(e_1)\alpha^t.$$

□

Proof of Theorem A.2. Let $\{\alpha_1, \cdots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ be basis of $\wedge^\bullet L_1'$ and $\wedge^\bullet L_2'$ respectively. Write

$$\theta = \sum_{i=1}^{n}\sum_{j=1}^{m} a^{ij}\alpha_i \otimes \beta_j.$$

For $e_1 + e_2 \in \Lambda$, using formula (A.3) and the fact that $\theta \in U^{l_1 \times l_2}(\Lambda)$, one has that

$$0 = \Pi_{l_1 \times l_2}(e_1 + e_2)\theta = \sum_{i=1}^{n}\sum_{j=1}^{m} a^{ij}\left[\Pi_{l_1}^-(e_1)\alpha_i \otimes \beta_j + (-1)^{|\alpha_i|}\alpha_i \otimes \Pi_{l_2}(e_2)y_j\right].$$

which implies that for every $\nu \in \det(L_1)$

$$\begin{aligned} 0 = & \sum_{i,j} a^{ij}\ \left[\nu(\langle(\Pi_{l_1}^-(e_1)\alpha_i)^t,\, \varphi\rangle)\,\beta_j + (-1)^{|\alpha_i|}\nu(\langle\alpha_i^t, \varphi\rangle)\,\Pi_{l_2}(e_2)\beta_j\right] \\ = & \sum_{i,j}(-1)^{|\alpha_i|}a^{ij}\ \left[\nu(\langle\alpha_i^t, \Pi_{l_1}(e_1)\varphi\rangle)\,\beta_j + \nu(\langle\alpha_i^t, \varphi\rangle)\,\Pi_{l_2}(e_2)\beta_j\right], \end{aligned} \tag{A.6}$$

where we use Lemma A.1 and Lemma A.3 in the last equality. Note that as $\varphi \in \wedge^\bullet L_1'$ is a pure spinor, it has a definite $\mathbb{Z}_2$ parity (see the end of §3.1) and therefore (by definition (A.1))

$$\begin{cases} \langle\alpha_i^t, \varphi\rangle \neq 0, & \text{only if } (-1)^{|\alpha_i|+|\varphi|} = (-1)^{\dim(L_1')} \\ \langle\alpha_i^t, \Pi_{l_1}(e_1)\varphi\rangle \neq 0, & \text{only if } (-1)^{|\alpha_i|+|\varphi|-1} = (-1)^{\dim(L_1')}. \end{cases}$$

So, excluding the zero terms in the equation (A.6), one obtains

$$\sum_{i,j} a^{ij}\,\nu(\langle\alpha_i^t, \Pi_{l_1}(e_1)\varphi\rangle)\,y_j = \sum_{i,j} a^{ij}\,\nu(\langle\alpha_i^t,\varphi\rangle)\,\Pi_{l_2}(e_2)y_j$$

which is equivalent to

$$\widehat{\theta}(\Pi_{l_1}(e_1)\varphi\otimes\nu) = \Pi_{l_2}(e_2)\,\widehat{\theta}(\varphi\otimes\nu),\ \text{ for every } e_1+e_2\in\Lambda. \tag{A.7}$$

To finish the prove, suppose $\widehat{\theta}(\varphi\otimes\nu)\neq 0$. By definition, for every $e_2\in\Lambda(L)$ there exists $e_1\in L\subset E_1$ such that $e_1+e_2\in\Lambda$. As $e_1\in L$ and φ is a pure spinor for L, $\Pi_{l_1}(e_1)\varphi=0$; thus, by formula (A.7), $\Pi_{l_2}(e_2)\widehat{\theta}(\varphi\otimes\nu)=0$. This implies that

$$\Lambda(L)\subset K_{l_2}(\widehat{\theta}(\varphi\otimes\nu)).$$

But as $\Lambda(L)$ is Lagrangian and $K_{l_2}(\widehat{\theta}(\varphi\otimes\nu))$ is isotropic, the result follows. □

The following example shows the necessity of condition (A.5).

Example A.4. Let V_1, V_2 be vector spaces and consider $E_i = (V_i\oplus V_i^*, g_{\text{can}})$ (see Example 2.2) for $i=1,2$. Consider $\Lambda = V_1\times V_2$ as a Lagrangian subspace of $\overline{E}_1\times E_2$. Let L be any Lagrangian subspace of E_1 and consider $\varphi\in U(L)\subset\wedge^\bullet V_1^*$. The pure spinor line $U^{l_1\times l_2}(\Lambda)\subset\wedge^\bullet L_1'\otimes\wedge^\bullet L_2'$ is the line generated by $1\otimes 1$. For $\theta = 1\otimes 1$ one has that

$$\widehat{\theta}(\varphi\otimes\nu) = \nu(\langle 1,\varphi\rangle)\,1,\,\forall\,\nu\in\det(L_1).$$

This is non-zero if and only if $\langle 1,\varphi\rangle = [\varphi]^{\text{top}}\neq 0$. In this case, $\nu([\varphi]^{\text{top}})\,1$ is a pure spinor for $V_2=\Lambda(L)$.

Remark A.5. We shall need a simple extension of Lemma A.3 ahead in the proof of Proposition A.11. So let (E,g) be a split-quadratic vector space and $l=(L,L')$ be a polarization with the corresponding representations

$$\Pi_l : Cl(E,g)\longrightarrow \text{End}\,(\wedge^\bullet L')$$

and

$$\Pi_l^- : Cl(E,-g)\longrightarrow \text{End}\,(\wedge^\bullet L')\,.$$

Let $L''\subset E$ be a Lagrangian subspace and consider a basis $\{e_1,\dots,e_n\}$ of L''. Then for any $I=\{i_1<\cdots<i_k\}\subset[1,n]$ and $\alpha\in\wedge^\bullet L'$, using inductively Lemma A.3, one has

$$(\Pi_l^-(e_I)\alpha)^t = (-1)^{k|\alpha|+1+\cdots+k-1}\Pi_l(e_I)\alpha^t = (-1)^{k|\alpha|+\frac{k(k-1)}{2}}\Pi_l(e_I)\alpha^t$$

(note that in the left-hand side of the equation $e_I\in\wedge^\bullet L'\subset Cl(E,-g)$ and in the right-hand side of the equation $e_I\in\wedge^\bullet L'\subset Cl(E,g)$). To finish, recall that $e_I^t = (-1)^{\frac{k(k-1)}{2}}e_I$ and therefore

$$(\Pi_l^-(e_I)\alpha)^t = (-1)^{|I||\alpha|}\Pi_l(e_I^t)\alpha^t. \tag{A.8}$$

A.1.1 Pull-back and push-forward morphisms.

In this subsection, we shall reobtain the results of Propositions 3.30 and 3.32 using Theorem A.2. Let V_1, V_2 be vector spaces and $f : V_1 \to V_2$ be a linear homomorphism. Consider the push-forward morphism $\Lambda_f \subset \overline{\mathcal{D}(V_1)} \times \mathcal{D}(V_2)$ (2.22). Let us first find the pure spinor line $U^{l_1\times l_2}(\Lambda_f) \subset \wedge^\bullet V_1 \otimes \wedge^\bullet V_2$ corresponding to the polarizations $l_1 = (V_1^*, V_1)$ and $l_2 = (V_2^*, V_2)$ using Proposition 3.18.

The projection S of Λ_f on $V_1^* \times V_2^*$ is $\mathrm{Graph}\,(f^*) = \{(f^*\xi, \xi) \mid \xi \in V_2^*\}$. The 2-form associated with Λ_f is defined (see (3.11)) for $(f^*\xi_i, \xi)$ $(i = 1, 2)$ by

$$\omega_S((f^*\xi_1, \xi_1)), (f^*\xi_2, \xi_2))) = \quad (-g^1_{\mathrm{can}} + g^2_{\mathrm{can}})(X, (f^*\xi_2, \xi_2))$$

(where $X = (X_1, X_2) \in V_1 \times V_2$ is such that $X + (f^*\xi_1, \xi_1) \in \Lambda_f$. For any $X_1 \in V_1$, $X = (X_1, f(X_1)$ works)

$$\begin{aligned} &= \quad -f^*\xi_2(X_1) + \xi_2(f(X_1)) \\ &= \quad 0. \end{aligned}$$

Therefore, by Proposition 3.18, $U^{\hat{v}_1\times\hat{v}_2}(\Lambda_f) = \wedge^{\mathrm{top}}\left[S^\perp \cap (V_1 \times V_2)\right] \subset \wedge^\bullet V_1 \otimes \wedge^\bullet V_2$. For every $X = (X_1, X_2) \in V_1 \times V_2$, $X \in S^\perp$ if and only if $X_2 = f(X_1)$. Thus,

$$\theta = (e_1 \otimes 1 + 1 \otimes f(e_1)) \wedge \cdots \wedge (e_n \otimes 1 + 1 \otimes f(e_n))$$

is a generator of $U^{\hat{v}_1\times\hat{v}_2}(\Lambda_f)$ where $\{e_1, \cdots, e_n\}$ is a basis of V_1. Expanding the last expression, one obtain

$$\theta_{\mathrm{push}} = \sum_{r=0}^{n} \sum_{\sigma\in S(r,n)} (-1)^{\mathrm{sgn}(\sigma)}\, e_{\sigma(1)} \wedge\cdots\wedge e_{\sigma(r)} \otimes f(e_{\sigma(r+1)}) \wedge\cdots\wedge f(e_{\sigma(n)})\,, \quad \text{(A.9)}$$

where $S(r, n)$ is the subset of permutations of $\{1, \cdots, n\}$ such that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r+1) < \cdots < \sigma(n)$ and $\mathrm{sgn}(\sigma)$ is the sign of the permutation.

Remark A.6. Consider the pull-back morphism $\Lambda_f^t \subset \overline{\mathcal{D}(V_2)} \times \mathcal{D}(V_1)$ (see (2.23)). Choose $l_1 = (V_1, V_1^*)$ and $l_2 = (V_2, V_2^*)$ the canonical splittings. A similar calculation as for the push-forward morphism shows that

$$\theta_{\mathrm{pull}} = \sum_{r=0}^{n} \sum_{\sigma\in S(r,n)} (-1)^{\mathrm{sgn}(\sigma)}\, e^{\sigma(1)} \wedge\cdots\wedge e^{\sigma(r)} \otimes f^*e^{\sigma(r+1)} \wedge\cdots\wedge f^*e^{\sigma(n)}\,, \quad \text{(A.10)}$$

where $\{e^1, \ldots, e^n\}$ is a basis of V_2^*, is a generator for $U^{l_2\times l_1}(\Lambda_f^t) \subset \wedge^\bullet V_2^* \otimes \wedge^\bullet V_1^*$.

Given the generator θ_{push} (A.9) for $U^{l_1\times l_2}(\Lambda_f)$, let us calculate the corresponding map

$$\widehat{\theta}_{\mathrm{push}} : \wedge^\bullet V_1 \otimes \det(V_1^*) \longrightarrow \wedge^\bullet V_2.$$

For $I \subset \{1, \dots, n\}$ and $\nu \in \det(V_1^*)$

$$\widehat{\theta}_{\text{push}}(e_I \otimes \nu) = \sum_{r=0}^{n} \sum_{\sigma \in S(r,n)} (-1)^{sgn(\sigma)} \, \nu\left([e_{\sigma([1,r])} \wedge e_I]^{\text{top}}\right) \; f(e_{\sigma(r+1)}) \wedge \cdots \wedge f(e_{\sigma(n)}).$$

It is straightforward to check that

$$\widehat{\theta}_{\text{push}}(e_I \otimes \nu) \neq 0 \Longleftrightarrow I = \{\sigma(r+1) < \cdots < \sigma(n)\}$$

and that in this case

$$\widehat{\theta}_{\text{push}}(e_I \otimes \nu) = \nu(e_1 \wedge \cdots \wedge e_n) \, f_*(e_I).$$

For any element $\varphi \in \wedge^{\bullet} V_1$, write

$$\varphi = \sum_{I \subset \{1,\dots,n\}} a^I \, e_I.$$

Then, by linearity,

$$\frac{1}{\nu(e_{[1,n]})} \, \widehat{\theta}_{\text{push}}(\varphi \otimes \nu) = \sum_{I \subset \{1,\dots,n\}} a_I \, f_*(e_I) = f_*(\varphi).$$

Remark A.7. Using the generator θ_{pull} (A.10) for $U^{l_2 \times l_1}(\Lambda_f^t)$, one obtains

$$\widehat{\theta}_{\text{pull}}(\varphi \otimes \nu) = \nu(e^1 \wedge \cdots \wedge e^n) \, f^* \varphi$$

for any $\nu \in \det(V_2)$.

In the case of the push-forward and pull-back morphisms, formula (A.7) reduces to the well-known formulas

$$f_*(i_{f^*\xi} \, \mathfrak{X} + X \wedge \mathfrak{X}) = i_\xi \, f_*(\mathfrak{X}) + f(X) \wedge f_*(\mathfrak{X}), \text{ for } X \in V_1, \, \xi \in V_2^* \text{ and } \mathfrak{X} \in \wedge^{\bullet} V_1.$$

and

$$f^*(i_{f(X)} \varphi + \xi \wedge \varphi) = i_X f^* \varphi + f^* \xi \wedge f^* \varphi, \text{ for } X \in V_1, \, \xi \in V_2^* \text{ and } \varphi \in \wedge^{\bullet} V_2^*$$

respectively.

A.2 Zero set and pure spinors.

In this section, we investigate thoroughly the problem of when a pure spinor is taken to zero by the transform (A.4) associated to a morphism $\Lambda \subset \overline{E}_1 \times E_2$, where (E_1, g_1) and $(E_2, g_2$ are split vector spaces. Before we define properly the zero set of Λ, we shall spend some time on the problem of how the transform (A.4) depends on the polarizations chosen.

A.2.1 Polarization dependence.

Let $l_1 = (L_1, L_1')$ and $l = (L, L')$ be two polarizations of (E_1, g_1) and $\varphi \in U^l(L_1) \subset \wedge^\bullet L'$.

Lemma A.8. *The isomorphism*

$$\begin{array}{rccc} F^-_{l_1 l}: & \wedge^\bullet L_1 & \longrightarrow & \wedge^\bullet L \\ & \alpha & \longmapsto & \Pi^-_l(\alpha)\varphi^t \end{array}$$

intertwines $\Pi^-_{l_1}$ *with* Π^-_l

Proof. Lemma A.3 implies that

$$\mathcal{N}^-_l(\varphi^t) = \{e_1 \in E_1 \mid \Pi^-_l(e_1)\varphi^t = 0\} = L_1$$

and the result follows from Proposition 3.23. □

Let (E_2, g_2) be another split vector space and choose $l_2 = (L_2, L_2')$ a polarization of E_2.

Lemma A.9. *The map*

$$\begin{array}{rccc} F^-_{l_1 l}\hat{\otimes}\,\mathrm{id}: & \wedge^\bullet L_1 \otimes \wedge^\bullet L_2 & \longrightarrow & \wedge^\bullet L \otimes \wedge^\bullet L_2 \\ & \alpha \otimes \beta & \longmapsto & (-1)^{|\varphi||\beta|} F^-_{l_1 l}(\alpha) \otimes \beta \end{array}$$

is a $Cl(E_1 \times E_2, -g_1 + g_2)$ *module isomorphism.*

Proof. Let $a_1 \in Cl(E_1, -g_1)$ and $a_2 \in Cl(E_2, g_2)$. For $\alpha \otimes \beta \in \wedge^\bullet L_1 \otimes \wedge^\bullet L_2$, one has (see Example 3.14)

$$\Pi_{l \times l_2}(a_1 \otimes a_2) \circ F^-_{l_1 l}\hat{\otimes}\,\mathrm{id}\,(\alpha \otimes \beta) = (-1)^c\, \Pi^-_l(a_1) F^-_{l_1 l}(\alpha) \otimes \Pi_{l_2}(a_2)\beta$$

where $c = |\varphi||\beta| + (|\alpha| + |\varphi|)|a_2|$. Now, using Lemma A.8, the definition of both $F^-_{l_1 l}\hat{\otimes}\,\mathrm{id}$ and $\Pi_{l_1 \times l_2}$, one has

$$\begin{aligned} &= (-1)^c F^-_{l_1 l}(\Pi^-_{l_1}(a_1)\alpha) \otimes \Pi_{l_2}(a_2)\beta \\ &= (-1)^{c + |\varphi|(|a_2| + |\beta|)}\, (F^-_{l_1 l}\hat{\otimes}\,\mathrm{id})\, (\Pi^-_{l_1}(a_1)\alpha \otimes \Pi_{l_2}(a_2)\beta) \\ &= (-1)^{c + |\varphi|(|a_2| + |\beta|) + |\alpha||a_2|}\, F^-_{l_1 l}\hat{\otimes}\,\mathrm{id} \circ \Pi_{l_1 \times l_2}(a_1 \otimes a_2)\, (\alpha \otimes \beta). \end{aligned}$$

A simple calculation shows that $c + |\varphi|(|a_2| + |\beta|) + |\alpha||a_2|$ is even and therefore

$$\Pi_{l \times l_2}(a_1 \otimes a_2) \circ F^-_{l_1 l}\hat{\otimes}\, id\, (\alpha \otimes \beta) = F^-_{l_1 l}\hat{\otimes}\, id \circ \Pi_{l_1 \times l_2}(a_1 \otimes a_2)\, (\alpha \otimes \beta)$$

proving that $F^-_{l_1 l}\hat{\otimes}\, id$ is a module isomorphism □

The idea now is the following: given $\Lambda \subset \bar{E}_1 \times E_2$ a Lagrangian subspace, by Lemma A.9

$$F^-_{l_1 l}\hat{\otimes}\,\mathrm{id}\,(U^{l_1 \times l_2}(\Lambda)) = U^{l \times l_2}(\Lambda).$$

We wish to find a map $\wedge^\bullet L_1' \otimes \det(L_1) \to \wedge^\bullet L' \otimes \det(L)$ that makes the diagram

$$\begin{array}{ccc} \wedge^\bullet L_1' \otimes \det(L_1) & \xrightarrow{\hat{\theta}} & \wedge^\bullet L_2' \\ \downarrow & & \downarrow \text{id} \\ \wedge^\bullet L' \otimes \det(L) & \xrightarrow{\widehat{F_{l_1 l}^{-}\hat{\otimes}\, id\,(\theta)}} & \wedge^\bullet L_2 \end{array} \tag{A.11}$$

commutative for every $\theta \in U^{l_1 \times l_2}(\Lambda)$. This is fundamental if we want to have any notion of zero set for Λ which independs of polarization.

Let $F_{l_1 l} : \wedge^\bullet L_1' \to \wedge^\bullet L$ be the $Cl(E_1, g_1)$ module isomorphism given by $F_{l_1 l}(\alpha) = \Pi_l(\alpha)\varphi$ (see Proposition 3.23).

Lemma A.10. *For $\alpha, \beta \in \wedge^\bullet L_1'$,*

$$\langle \alpha, \beta \rangle = 0 \textit{ if and only if } \langle F_{l_1 l}(\alpha), F_{l_1 l}(\beta) \rangle = 0$$

Proof. By Lemma A.1,

$$\begin{aligned} \langle F_{l_1 l}(\alpha), F_{l_1 l}(\beta) \rangle &= [(\Pi_l(\alpha)\varphi)^t \wedge \Pi_l(\beta)\varphi]^{\text{top}} \\ &= [\varphi^t \wedge \Pi_l(\alpha^t \wedge \beta)\varphi]^{\text{top}} \\ &= [\varphi^t \wedge \Pi_l(\gamma + \langle \alpha, \beta \rangle)\varphi]^{\text{top}} \end{aligned}$$

where $\gamma \in \wedge^\bullet L_1'$ satisfies $[\gamma]^{\text{top}} = 0$. We claim that $[\varphi^t \wedge \Pi_l(\gamma)\varphi]^{\text{top}} = 0$. Indeed, by Proposition 3.28, there exists $A \in O(E, g)$ such that $l = (A(L_1), A(L_1'))$. Let $a \in Pin(E, g)$ such that $a^\sigma(\cdot)a^{-1} = A(\cdot)$. One has

$$\Pi_l(\gamma)\varphi = \Pi_l((a^\sigma)^{-1})\Pi_l(a^\sigma \gamma a^{-1})\Pi_l(a)\varphi.$$

Observe that

(i) $a^\sigma \gamma a^{-1} \in \wedge^\bullet L_2'$ (it is just the image of γ by the natural extension $A|_{L_1'} : \wedge^\bullet L_1' \longrightarrow \wedge^\bullet L_2'$);

(ii) by Lemma 3.17,
$$\mathcal{N}_l(\Pi_l(a)\varphi) = A(L_1) = L_2$$
and thus $\Pi_l(a)\varphi \in U^l(L_2) = \wedge^0 L_2' = \mathbb{F}$;

(iii) as $a^\sigma \in Pin(E)$, $(a^\sigma)^{-1} = \pm a^t$.

Call $0 \neq \lambda = \Pi_l(a)\varphi \in \mathbb{F}$. Using (i), (ii) and (iii), one has

$$\begin{aligned} [\varphi \wedge \Pi_{l_1}(\gamma)\varphi]^{\text{top}} &= \pm[(\Pi_{l_1}(a)\varphi)^t \wedge \Pi_{l_1}(a^\sigma \gamma a^{-1})\Pi_{l_1}(a)\varphi]^{\text{top}} \\ &= \pm\lambda^2 [a^\sigma \gamma a^{-1}]^{\text{top}} \end{aligned} \tag{A.12}$$

and as $a^\sigma \gamma a^{-1}$ has the same exterior degree as γ, the last term is zero as we claimed. Thus,

$$\langle F_{l_1 l}(\alpha), F_{l_1 l}(\beta) \rangle \quad = [\varphi^t \wedge \Pi_l(\langle \alpha, \beta \rangle)\varphi]^{\text{top}}$$

and repeating the steps in (A.12), one obtains

$$= \pm\lambda^2 \left[a^\sigma \langle\alpha,\beta\rangle a^{-1}\right]^{\mathrm{top}}.$$

Using once again that $a^\sigma(\cdot)a^{-1} : \wedge^\bullet L_1' \longrightarrow \wedge^\bullet L_2'$ preserves exterior degree, the result follows. $\square$

Lemma A.10 makes it possible to define a map

$$\begin{array}{ccc} \det(L_1) & \longrightarrow & \det(L) \\ \nu & \longmapsto & \nu_\varphi \end{array}$$

implicitly by the relation (see Lemma A.10)

$$\nu_\varphi(\langle F_{l_1l}(\alpha), F_{l_1l}(\beta)\rangle) = \nu(\langle\alpha,\beta\rangle), \text{ for } \alpha,\beta \in \wedge^\bullet L_1'. \tag{A.13}$$

Proposition A.11. *The map*

$$\begin{array}{rccc} T: & \wedge^\bullet L_1' \otimes \det(L_1) & \longrightarrow & \wedge^\bullet L \otimes \det(L) \\ & \alpha \otimes \nu & \longmapsto & (-1)^{|\theta||\varphi|} F_{l_1l}(\alpha) \otimes \nu_\varphi, \end{array}$$

where $|\theta|$ is the $\mathbb{Z}_2$ degree of any element $\theta \in U^{l_1\times l_2}(\Lambda)$, makes the diagram (A.11) *commutative.*

Proof. Consider $\theta' = \gamma_1 \otimes \gamma_2 \in \wedge^\bullet L_1' \otimes \wedge^\bullet L_2'$ and call

$$\theta'_{new} = F_{l_1l}^{-}\hat{\otimes}\mathrm{id}\,(\theta) = (-1)^{|\varphi||\gamma_2|}\Pi_l^-(\gamma_1)\varphi^t \otimes \gamma_2 \in \wedge^\bullet L' \otimes \wedge^\bullet L_2.$$

For every $\nu \in \det(L_1)$ and $\alpha \in \wedge^\bullet L'$

$$\widehat{\theta'}_{new}(F_{l_1l}(\alpha)\otimes\nu_\varphi) = (-1)^{|\gamma_2||\varphi|}\nu_\varphi(\langle(\Pi_l^-(\gamma_1)\varphi^t)^t, F_{l_1l}(\alpha)\rangle)\gamma_2$$

By (A.8) and the definition of ν_φ,

$$\begin{aligned} &= (-1)^{|\gamma_2||\varphi|}(-1)^{|\gamma_1||\varphi|}\nu_\varphi(\langle\Pi_l(\gamma_1^t)\varphi, F_{l_1l}(\alpha)\rangle) \\ &= (-1)^{|\varphi|(|\gamma_1|+|\gamma_2|)}\,\nu_\varphi(\langle F_{l_1l}(\gamma_1^t), F_{l_1l}(\alpha)\rangle) \\ &= (-1)^{|\varphi||\theta'|}\nu(\langle\gamma_1^t,\alpha\rangle) \\ &= (-1)^{|\varphi||\theta'|}\widehat{\theta'}(\alpha\otimes\nu). \end{aligned}$$

The result now extends by linearity and the fact that any $\theta \in U^{l_1\times l_2}(\Lambda)$ has well-defined $\mathbb{Z}_2$ degree. $\square$

For the dependence on the polarization l_2 of (E_2,g_2) the analysis is simpler as it doesn't involve $\mathbb{Z}_2$ degree issues. Fix polarizations $l_1 = (L_1, L_1')$ and $l_2 = (L_2, L_2')$ of (E_1,g_1) and (E_2,g_2) respectively and consider $l = (L, L')$ another arbitrary polarization of (E_2,g_2). Let $F_{l_2l} : \wedge^\bullet L_2 \to \wedge^\bullet L'$ be a $Cl(E_2,g_2)$ module isomorphism (given for example by the choice of a generator of $U^l(L_2) \subset \wedge^\bullet L'$ and Proposition 3.23).

Proposition A.12. *The map* $\mathrm{id} \otimes F_{l_2 l} : \wedge^\bullet L_1' \otimes \wedge^\bullet L_2' \to \wedge^\bullet L_1' \otimes \wedge^\bullet L'$ *is a* $Cl(E_1 \otimes E_2, -g_1 + g_2)$ *module isomorphism. Moreover, the diagram*

$$\begin{array}{ccc} \wedge^\bullet L_1' \otimes \det(L_1) & \xrightarrow{\widehat{\theta}} & \wedge^\bullet L_2' \\ \Big\downarrow \mathrm{id} & & \Big\downarrow F_{l_2 l} \\ \wedge^\bullet L_1' \otimes \det(L_1) & \xrightarrow{\mathrm{id} \otimes \widehat{F_{l_2 l}}(\theta)} & \wedge^\bullet L' \end{array} \tag{A.14}$$

commutes for every $\theta \in U^{l_1 \times l_2}(\Lambda)$.

Proof. Let $\alpha \otimes \beta \in \bar{\mathcal{S}}^{l_1}(E_1) \otimes \mathcal{S}^{l_2}(E_2)$ and $a_1 \otimes a_2 \in Cl(E_1, -g_1) \otimes Cl(E_2, g_2)$. By Example 3.14 and the fact that $F_{l_2 l}$ is a $Cl(E_2, g_2)$-module isomorphism, it follows that

$$\begin{aligned} \Pi_{l_1 \times l}(a_1 \otimes a_2) \circ id \otimes F_{l_2 l}(\alpha \otimes \beta) &= (-1)^{|a_2||\alpha|} \Pi_l^-(a_1)\alpha \otimes \Pi_l F_{l_2 l}(\beta) \\ &= (-1)^{|a_2||\alpha|} \Pi_l^-(a_1)\alpha \otimes F_{l_2 l}(\Pi_{l_2}(a_2)\beta) \\ &= (-1)^{|a_2||\alpha|} id \otimes F_{l_2 l} \, (\Pi_{l_1}^-(a_1)\alpha \otimes \Pi_{l_2}(a_2)\beta) \\ &= id \otimes F_{l_2 l} \circ \Pi_{l_1 \times l_2}(a_1 \otimes a_2) \, (\alpha \otimes \beta). \end{aligned}$$

The result about θ_{new} is now just a simple calculation that we omit. □

A.2.2 Zero set.

Let (E_1, g_1) and (E_2, g_2) be split vector spaces and consider a Lagrangian subspace $\Lambda \subset \bar{E}_1 \times E_2$. Fix a polarization $l_1 = (L_1, L_1')$ of E_1.

Definition A.13. An element $\alpha \in \wedge^\bullet L_1'$ belongs to the zero set $\mathcal{Z}^{l_1}(\Lambda)$ of Λ if there exists a polarization $l_2 = (L_2, L_2')$ of (E_2, g_2) such that for every $\theta \in U^{l_1 \times l_2}(\Lambda)$ and every $\nu \in \det(L_1)$

$$\widehat{\theta}(\alpha \otimes \nu) = 0. \tag{A.15}$$

Remark A.14. Note that as formula (A.15) depends linearly on θ and ν, it is sufficient for $\alpha \in \wedge^\bullet L_1'$ to belong to $\mathcal{Z}^{l_1}(\Lambda)$ that there exists a non-zero generator θ of $U^{l_1 \times l_2}(\Lambda)$ and a non-zero volume element $\nu \in \det(L_1)$ such that (A.15) holds.

We now study how $\mathcal{Z}^{l_1}(\Lambda)$ depends on polarizations.

Proposition A.15. *Let* $\alpha \in \wedge^\bullet L_1$. *If* $\alpha \in \mathcal{Z}^{l_1}(\Lambda)$, *then for every polarization* $l = (L, L')$ *of* (E_2, g_2) *and* $\theta \in U^{l_1 \times l}(\Lambda)$

$$\widehat{\theta}(\alpha \otimes \nu) = 0, \, \forall \, \nu \in \det(L_1).$$

Proof. This is a simple application of Proposition A.12. Indeed, by definition, there exists some polarization l_2 of (E_2, g_2) for which (A.15) for every $\theta \in U^{l_1 \times l_2}(\Lambda)$. Now, if l is any other polarization of (E_2, g_2), then Proposition A.12

gives that there exists a non-zero generator $\theta_{new} \in U^{l_1 \times l}(\Lambda)$ and a $Cl(E_2, g_2)$ isomorphism $F_{l_2 l} : \mathcal{S}^{l_2}(E_2) \to \mathcal{S}^{l}(E_2)$ such that

$$\widehat{\theta}_{new}(\alpha \otimes \nu) = F_{l_2 l}(\widehat{\theta}(\alpha \otimes \nu)) = 0.$$

This proves the result. □

Proposition A.16. *Let $l = (L, L')$ be an arbitrary polarization of (E_1, g_1) and let $\varphi \in U^l(L_1)$. Consider the $Cl(E_1, g_1)$ module isomorphism $F_{l_1 l} : \wedge^\bullet L_1' \to \wedge^\bullet L'$ given by $F_{l_1 l}(\alpha) = \Pi_l(\alpha)\varphi$. Then*

$$F_{l_1 l}(\mathcal{Z}^{l_1}(\Lambda)) = \mathcal{Z}^{l}(\Lambda)$$

Proof. Let $\alpha \in \mathcal{Z}^{l_1}(\Lambda)$ and let l_2 be any polarization of (E_2, g_2). By Proposition A.15, for $0 \neq \theta \in U^{l_1 \times l_2}(\Lambda)$ and $\nu \in \det(L_1)$

$$\widehat{\theta}(\alpha \otimes \nu) = 0.$$

Let $\nu_\varphi \in \det(L)$ be determined by (A.13) (it is non-zero if ν is non-zero) and $0 \neq \theta_{new} = F_{l_1 l}^{-} \hat{\otimes}\, id\,(\theta) \in U^{l \times l_2}(\Lambda)$ (see Lemma A.9). Then, by Proposition A.11,

$$\widehat{\theta}_{new}(F_{l_1 l}(\alpha) \otimes \nu_\varphi) = (-1)^{|\theta||\varphi|}\widehat{\theta}(\alpha \otimes \nu) = 0.$$

Then, by Remark A.14, it follows that $F_{l_1 l}(\alpha) \in \mathcal{Z}^{l}(\Lambda)$. Conversely, let $\beta \in \mathcal{Z}^{l}(\Lambda)$. Then,

$$\widehat{\theta}(F_{l_1 l}^{-1}(\beta) \otimes \nu) = (-1)^{|\theta||\varphi|}\widehat{\theta}_{new}(\beta \otimes \nu_\varphi) = 0,\ \forall\, \theta \in U^{l_1 \times l_2}(\Lambda) \text{ and } \forall\, \nu \in \det(L_1)$$

which proves that $F_{l_1 l}^{-1}(\beta) \in \mathcal{Z}^{l_1}(\Lambda)$ and therefore $\beta \in F_{l_1 l}(\mathcal{Z}^{l_1}(\Lambda))$. □

We are now interested in characterizing the pure spinors which belongs to the zero set of Λ. For this, recall the definition of $\ker(\Lambda)$ (2.20).

Theorem A.17. *Let $l_1 = (L_1, L_1')$ be a polarization of (E_1, g_1). A pure spinor $\varphi \in \wedge^\bullet L_1'$ belongs to $\mathcal{Z}^{l_1}(\Lambda)$ if and only if $\mathcal{N}_l(x) \cap \ker(\Lambda) \neq 0$.*

The proof of Theorem A.17 relies on a characterization of push-forward morphisms presented by Sternberg in the Leonard M. Blumenthal Lectures in Geometry [47].

Sternberg result. Let V_1, V_2 be vector spaces and $f : V_1 \to V_2$ be a linear homomorphism. Consider the push-forward morphism $\Lambda_f \subset \overline{\mathcal{D}(V_1)} \times \mathcal{D}(V_2)$ (2.22). Note that

(i) $\Lambda_f(V_1^*) = V_2^*$;

(ii) $\ker(\Lambda_f) = \ker(f) \subset \mathcal{D}(V_1)$ doesn't intersect V_1^*.

If $F : (V_1 \oplus V_1^*, g_{can}) \to (V_1 \oplus V_1^*, g_{can})$ is any isomorphism such that $F(V_1^*) = V_1^*$, then $\Lambda_f \circ \Lambda_F$ also satisfies (i) and (ii), where Λ_F is the morphism given by (2.15). Now, any isomorphism F which leaves V_1^* invariant is given in matrix notation by

$$\begin{pmatrix} \tilde{f} & 0 \\ B_\sharp & (\tilde{f}^{-1})^* \end{pmatrix} = \begin{pmatrix} \tilde{f} & 0 \\ 0 & (\tilde{f}^{-1})^* \end{pmatrix} \circ \tau_{\tilde{f}^* B}$$

where $\tilde{f} : V_1 \to V_1$ is an isomorphism and $B_\sharp : V_1 \to V_1^*$ is the map associated to a 2-form $B \in \wedge^2 V_1^*$ and $\tau_{\tilde{f}^* B}$ is the B-field transformation defined in (2.9). It is easy to see that

$$\Lambda_f \circ \Lambda_F = \Lambda_{f \circ \tilde{f}} \circ \Lambda_{\tau_{\tilde{f}^* B}}.$$

Proposition A.18. *[47] If $\Lambda \subset \overline{\mathcal{D}(V_1)} \times \mathcal{D}(V_2)$ is any morphism such that*

(i) $\Lambda(V_1^*) = V_2^*$ *and*

(ii) $\ker(\Lambda) \cap V_1^* = 0$*,*

then there exists a map $f : V_1 \to V_2$ and a 2-form $B \in \wedge^2 V_1^*$ such that

$$\Lambda = \Lambda_f \circ \Lambda_{\tau_{-B}} = \{(X, f^*\eta + i_X B,\ f(X), \eta) \mid X \in V_1,\ \eta \in V_2^*\}.$$

Proof. First note that (i) implies that for every $\xi \in V_2^*$ there exists $\eta \in V_1^*$ such that $(\xi, \eta) \in \Lambda$. We claim that it is unique: indeed, if there is $\eta_1, \eta_2 \in V_1^*$ such that $(\xi, \eta_i) \in \Lambda$ for $i = 1, 2$, then

$$\eta_1 - \eta_2 \in V_1^* \cap \ker(\Lambda) = 0.$$

Define $g : V_2^* \to V_1^*$ by $g(\xi) = \eta$. It is clearly linear. Let $f := g^* : V_1 \to V_2$. We claim that the projection of Λ on $V_1 \times V_2$ along $V_1^* \times V_2^*$ is $\mathrm{Graph}(f)$. Indeed, if $(X, \xi, Y, \eta) \in \Lambda$, then (as Λ is isotropic)

$$\begin{aligned} 0 = & \ (-g^1_{can} + g^2_{can})((X, \xi, Y, \eta), (0, f^*\eta, 0, \eta)) \\ = & \ -g^1_{\mathrm{can}}((X, \xi), (0, f^*\eta)) + g^2_{\mathrm{can}}((Y, \eta), (0, \eta) \\ = & \ \eta(f(X)) - \eta(Y) \end{aligned}$$

This implies that $\eta(Y) = \eta(f(X))$ for every $\eta \in V_2^*$ which proves that $Y = f(X)$.

Now, we claim that there exists $B \in \wedge^2 V_1^*$ such that

$$\Lambda^t(V_2) = \tau_B(V_1),$$

where $\Lambda^t \subset \overline{\mathcal{D}(V_2)} \times \mathcal{D}(V_1)$ is the transpose (2.18) of Λ. Indeed, suppose that $\eta \in \Lambda^t(V_2) \cap V_1^*$. Then there exists $X \in V_2$ such that $(0, \eta, X, 0) \in \Lambda$, but as we saw $X = f(0) = 0$. Therefore $\eta \in V_1^* \cap \ker(\Lambda) = 0$. Thus, $\Lambda^t(V_2) \cap V_1^* = 0$. As $\Lambda^t(V_2)$ is a Lagrangian subspace of $V_1 \oplus V_1^*$, Example 2.6 says that there exists $B \in \wedge^2 V_1^*$ such that $\Lambda^t(V_2) = \mathrm{Graph}(B_\sharp) = \tau_B(V_1)$.

Finally, if $(X,\xi,f(X),\eta)\in\Lambda$, then by subtracting $(0,f^*\eta,0,\eta)$ which is also in Λ, one has that $(X,\xi-f^*\eta,f(X),0)\in\Lambda$. By definition, this implies that $(X,\xi-f^*\eta)\in\Lambda^t(V_2)=\mathrm{Graph}\,(B)$. Therefore,

$$\xi=f^*\eta+i_XB$$

as we wanted to show. □

Before proving Theorem A.17 we need a simple lemma.

Lemma A.19. *Let (E,g) be a split vector space and $K\subset E$ an isotropic subspace. There exists a Lagrangian subspace L of E such that $K\subset L$.*

Proof. By choosing a polarization $v=(V,V')$ of E and identifying V' with V^* via g one can suppose that $(E,g)=(V\oplus V^*,g_{\mathrm{can}})$. Now, let $S=\mathrm{pr}_V(K)$ and define (see (2.11)) $\omega_S\in\wedge^2S^*$ by

$$\omega(X,Y)=\xi(Y),\ \text{for}\ X,Y\in S,$$

where $\xi\in V^*$ is such that $X+\xi\in K$. A pair such (S,ω_S) as we saw (2.13) always define a Lagrangian subspace by

$$L=\{(Y,\eta)\mid Y\in S\ \text{and}\ \eta|_S=i_Y\omega_S\}.$$

It is clear that $K\subset L$. □

Proof. (Thm. A.17). To prove the Theorem, the idea is that by changing l_1 to a suitable polarization, it is possible to reduce the problem to push-forward morphisms using Proposition A.18.

First, by Lemma A.19, as $\ker(\Lambda)$ is isotropic, there exists a Lagrangian subspace $L^{^{(1)}}\subset E_1$ such that $\ker(\Lambda)\subset L^{^{(1)}}$. Let $L'^{^{(1)}}$ be any Lagrangian complement to $L^{^{(1)}}$ (it exists by Corollary 2.8) and consider the polarization $l^{^{(1)}}=(L^{^{(1)}},L'^{^{(1)}})$ of E_1. Using g_1 to identify $L'^{^{(1)}}$ with $(L^{^{(1)}})^*$, one has an identification

$$(E_1,g_1)=\left(L^{^{(1)}}\oplus(L^{^{(1)}})^*,g_{\mathrm{can}}\right)$$

Let $L_2'\subset E_2$ now be the Lagrangian subspace $\Lambda(L'^{^{(1)}})$ and let L_2 be any Lagrangian complement to L_2 so that $l_2=(L_2,L_2')$ is a polarization of E_2. By identifying L_2' with L_2^* via g_2, one has that

$$(E_2,g_2)=(L_2\oplus L_2^*,g_{\mathrm{can}})$$

and that Λ is a morphism from $\mathcal{D}\left(L^{^{(1)}}\right)$ to $\mathcal{D}\,(L_2)$. The choice of $l^{^{(1)}}$ and l_2 guarantees that

(i) $\Lambda((L^{^{(1)}})^*)=L_2^*$ and

(ii) $\ker(\Lambda)\cap(L^{^{(1)}})^*=0$.

Therefore by Proposition (A.18), there exists a map $f : L^{(1)} \to L_2$ and a 2-form $B \in \wedge^2(L^{(1)})^*$ such that

$$\Lambda = \Lambda_f \circ \Lambda_{\tau_{-B}}.$$

To finish our choices of polarizations, let $l^{(2)} = (L^{(2)}, L'^{(2)})$ be a polarization of $E_1 \cong \mathcal{D}\left(L^{(1)}\right)$ where

$$L^{(2)} = \left\{(X, i_X B) \in \mathcal{D}\left(L^{(1)}\right) \mid X \in L^{(1)}\right\}$$

and

$$L'^{(2)} = (L^{(1)})^*.$$

Again, by identifying $L'^{(2)}$ with $(L^{(2)})^*$ via g_{can}, one obtains that

$$\Lambda = \Lambda_{\tilde{f}},$$

where $\tilde{f} : L^{(2)} \to L_2$ is defined by $\tilde{f}(X, i_X B) = f(X)$, for $X \in L^{(1)}$.

By Proposition 3.33 and §A.1.1, one knows that for a pure spinor $\alpha \in \wedge^\bullet(L^{(2)})^*$

$$\alpha \in \mathcal{Z}^{l^{(2)}}(\Lambda) \text{ if and only if } \mathcal{N}_{l^{(2)}}(\alpha) \cap \ker(f) (= \ker(\Lambda)) \neq 0.$$

Now, for any $Cl(E_1, g_1)$-module isomorphism $F_{l_1 \hat{l}^{(2)}} : \wedge^\bullet L_1' \to \wedge^\bullet(L^{(2)})^*$, one has that

$$\mathcal{N}_{l_1}(\varphi) = \mathcal{N}_{l^{(2)}}(F_{l_1 l^{(2)}}(\varphi))$$

and

$$F_{l_1 l^{(2)}}(\mathcal{Z}^{l_1}(\Lambda)) = \mathcal{Z}^{l^{(2)}}(\Lambda), \quad \text{by Proposition A.16.}$$

Therefore, for any pure spinor $\varphi \in \wedge^\bullet L_1'$,

$$\begin{aligned} \varphi \in \mathcal{Z}^{l_1}(\Lambda) \Leftrightarrow &\quad F_{l_1 l^{(2)}}(\varphi) \in \mathcal{Z}^{l^{(2)}}(\Lambda) \\ \Leftrightarrow &\quad \mathcal{N}_{l^{(2)}}(F_{l_1 l^{(2)}}(\varphi)) \cap \ker(\Lambda) \neq 0 \\ \Leftrightarrow &\quad \mathcal{N}_{l_1}(\varphi) \cap \ker(\Lambda) \neq 0 \end{aligned}$$

as we wanted to prove. □

Example A.20. Let (E, g) be a split vector space and let $l_1 = (L_1, L_1')$ be a polarization. We want to re-interpret a classical result of E.Cartan (see [16]) about the the pairing $\langle \cdot, \cdot \rangle$ (A.1) which says that for two pure spinors $\varphi_1, \varphi_2 \in \wedge^\bullet L_1'$

$$\langle \varphi_1, \varphi_2 \rangle \neq 0 \Longleftrightarrow \mathcal{N}_{l_1}(\varphi_1) \cap \mathcal{N}_{l_1}(\varphi_2) = 0.$$

Let L be any Lagrangian subspace of $(E, -g)$. Thought as a morphism from E to the point space $\{0\}$ (a Lagrangian subspace of $\bar{E} \times \{0\}$) it defines a constant map

$$Lag(E) \longrightarrow Lag(\{0\}).$$

Corresponding to the unique (tautological) polarization of $\{0\}$, one can associate the Clifford module $\wedge^\bullet 0 = \mathbb{R}$. The transform associated to L as a morphism is given by a map

$$\widehat{\theta} : \wedge^\bullet L_1' \times \det(L_1) \to \mathbb{R}$$

where $\theta \in U^{l_1 \times \{0\}}(L) \subset \wedge^\bullet L_1' \otimes \mathbb{R}$. To choose θ is sufficient to choose $\alpha \in \wedge^\bullet L_1'$ such that

$$\mathcal{N}_{l_1}^-(\alpha) = \{e \in E \mid \Pi_l^-(e)\alpha = 0\} = L$$

as $\theta = \alpha \otimes 1 \in U^{l_1 \times \{0\}}(L)$. One such choice is $\alpha = \varphi_1^t$, where

$$\mathcal{N}_{l_1}(\varphi_1) = L$$

(see Lemma A.8). Thus, the transform is (see (A.4))

$$\widehat{\theta}(\beta \otimes \nu) = \nu(\langle \alpha^t, \beta \rangle)1 = \nu(\langle \varphi_1, \beta \rangle).$$

Now, note that $\ker(L) = L$ as a morphism. Hence, Theorem A.17 asserts that for a pure spinor $\varphi_2 \in \wedge^\bullet L_1'$,

$$0 = \widehat{\theta}(\varphi_2 \otimes \nu) = \nu(\langle \varphi_1, \varphi_2 \rangle) \Longleftrightarrow \mathcal{N}_{l_1}(\varphi_2) \cap \ker(L) = \mathcal{N}_{l_1}(\varphi_2) \cap \mathcal{N}_{l_1}(\varphi_1) \neq 0$$

which is exactly the result of E. Cartan.

Appendix B

Push-forward on principal bundles.

We follow [7] to define push-forward in the setting of principal bundles. Start with the local model. Let N be a manifold and G a connected, compact Lie group of dimension r. Consider $P = N \times G$ with the projections $\mathrm{pr}_1 : P \to N$ and $\mathrm{pr}_2 : P \to G$. Differential forms in P can be written as sums of two types:

$$f(x,g)\,\mathrm{pr}_1^*\alpha \wedge \mathrm{pr}_2^*\nu, \text{ where } \alpha \in \Omega(N) \text{ and } \begin{cases} \nu \in \Omega^r(G), & \text{type (I);} \\ \nu \in \Omega^k(G),\ k < r, & \text{type (II)} \end{cases}$$

with $f \in C^\infty(P)$. The **push-forward** $\mathrm{pr}_{1*} : \Omega(P) \to \Omega(N)$ is defined as the linear map which send forms of type (I) to

$$\left(\int_G f(\cdot, g)\,\nu \right) \alpha$$

and forms of type (II) to zero.

To define it globally in a general principal bundle P over N, we have to glue these local definitions. So let $\pi : P \to N$ be a G principal bundle and $\{\mathcal{U}_i\}$ an open covering of N such that there exists trivialization $\phi_i : \mathcal{U}_i \times G \to \pi^{-1}(\mathcal{U}_i)$ with $\phi_{ij} = \phi_i^{-1} \circ \phi_j : (\mathcal{U}_i \cap \mathcal{U}_j) \times G \to (\mathcal{U}_i \cap \mathcal{U}_j) \times G$ given by

$$\phi_{ij}(x,g) = (x, g_{ij}(x)g)$$

where $g_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to G$ is a smooth map.

Remark B.1. In the local picture, the Lie group G acts on P via

$$h \cdot (x,g) = (x, gh).$$

This local actions glue together to define a global action of G on P. Note that the infinitesimal vector field u_P corresponding to $u \in \mathfrak{g}$ locally satisfies

$$(pr_2)_* u_P = u^L,$$

where $u^L \in \Gamma(TG)$ is the left-invariant vector field generated by u.

Locally, any differential form on P is a sum of forms of type (I) and (II). For forms of type (II), we define π_* to be zero. For a form ω of type (I), let $\omega_i = \phi_i^* \, \omega|_{\pi^{-1}(\mathcal{U}_i)}$ (similarly for ω_j). One has

$$\omega_i = f_i(x,g)\, pr_1^* \alpha_i \wedge \mathrm{pr}_2^* \nu_i.$$

and

$$\omega_j = f_j(x,g)\, pr_1^* \alpha_j \wedge \mathrm{pr}_2^* \nu_j$$

where both ν_i and ν_j are volume forms in G, $\alpha_i \in \Omega(\mathcal{U}_i)$ and $\alpha_j \in \Omega(\mathcal{U}_j)$. One has to check that

$$\left(\int_G f_i(x,g)\nu_i\right)\alpha_i = \left(\int_G f_j(x,g)\nu_j\right)\alpha_j$$

over $\mathcal{U}_i \cap \mathcal{U}_j$. Now, as $\phi_{ij}^* \omega_i = \omega_j$, one has that for $x \in \mathcal{U}_i \cap \mathcal{U}_j$

$$f_j(x,g) = f_i(x, g_{ij}(x)g),\ \alpha_i = \alpha_j\ ,\ \text{and } \nu_j = L^*_{g_{ij}(x)} \nu_i, \tag{B.1}$$

where $L_g : G \to G$ is left multiplication by $g \in G$. As G is connected, $L_g : G \to G$ preserves orientation for every g and therefore, for every $x \in \mathcal{U}_i \cap \mathcal{U}_j$,

$$\int_G f_j(x,g)\nu_j = \int_G f_i(x, g_{ij}(x)g) L^*_{g_{ij}(x)} \nu_i = \int_G L^*_{g_{ij}(x)}(f_i(x,g)\nu_i) = \int_G f_i(x,g)\nu_i$$

which proves that the local constructions glue.

Remark B.2. The push-forward can be seen as a two-step process. The first step is a bundle map

$$\wedge^\bullet T^* P \longrightarrow \pi^* \wedge^\bullet T^* N,$$

where $\pi^* \wedge^\bullet T^* N$ is the pull-back bundle over P (whose fiber in the point $x \in P$ is $\wedge^\bullet T_{\pi(x)} N$) which we now describe. For $\delta \in \wedge^n \mathfrak{g}$, let $\delta_P \in \Gamma(\wedge^n TP)$ be the image of δ under the natural extension

$$\wedge^r \Sigma : \wedge^r \mathfrak{g} \longrightarrow \Gamma(\wedge^r TP)$$

of the infinitesimal action $\Sigma : \mathfrak{g} \to \Gamma(TP)$. For $x \in P$, let $\{\xi^1, \ldots, \xi^n\} \subset T_x^* P$ be a basis such that $\{\xi^1, \ldots, \xi^{n-r}\}$ generates $\mathrm{Ann}\,(T_x(G \cdot x))$. Any element of $\wedge^\bullet T_x^* P$ can be written as a sum of

$$d\pi_x^* \alpha \wedge \xi^I,\ I \subset (n-r, n] \text{ and } \alpha \in \wedge^\bullet T^*_{\pi(x)} N$$

and the bundle map restricted to the fiber $\wedge^\bullet T_x^* P$ is just $C_{\delta_P(x)}$ as defined in (3.25):

$$C_{\delta_P(x)} : \pi^* \alpha \wedge \xi^I \longmapsto \begin{cases} 0, & \text{if } I \neq (n-r, n] \\ (i_{\xi^I}\, \delta_x)\, \alpha, & \text{if } I = (n-r, n] \end{cases}$$

As shown in remark 3.37,

$$C_{\delta_P(x)} = \star_2 \circ (d\pi_x)_* \circ \star_1,$$

where $\star_1 : \wedge^\bullet T_x^* P \to \wedge^\bullet T_x P$ and $\star_2 : \wedge^\bullet T_{\pi(x)} N \to \wedge^\bullet T_{\pi(x)}^* N$ are the star maps (see 3.24) corresponding to $\nu_1 \in \det(T_x P)$ and $\nu_2 \in \det(T_{q(x)}^* N)$ such that

$$\star_1 d\pi_x^* \nu_2 = \delta_x.$$

The second step is integration. Let $\nu \in \wedge^r \mathfrak{g}^*$ be such that $i_\delta \nu = 1$ and consider the left invariant volume form ν^L on G. For any coordinate neighborhood $\mathcal{U}$ of N with coordinates $(x_1, \ldots, x_{n-r})$ and such that $q^{-1}(\mathcal{U}) \cong \mathcal{U} \times G$, define

$$\begin{array}{ccc} \Gamma(\pi^* \wedge^\bullet T^* N|_{\pi^{-1}(\mathcal{U})}) & \longrightarrow & \Gamma(\wedge^\bullet T^* N|_{\mathcal{U}}) \\ f(x,g) dx_I & \longmapsto & \left(\int_G f(x,\cdot) d\nu^L\right) dx_I. \end{array} \tag{B.2}$$

These local constructions glue together to give a map

$$\Gamma(\pi^* \wedge^\bullet T^* N) \longrightarrow \Gamma(\wedge^\bullet T^* N)$$

that when composed to $C_\delta : \Gamma(\wedge^\bullet T^* P) \to \Gamma(\wedge^\bullet \pi^* \wedge^\bullet T^* N)$ is exactly the push-forward map.

We now proceed to collect the main properties of the push-forward map.

Proposition B.3. *The push-forward map $\pi_* : \Omega(P) \to \Omega(N)$ commutes with the deRham differential.*

Proof. It is a local result, so it suffices to prove it for $P = \mathbb{R}^{n-r} \times G$. Let $(x_1, \cdots, x_{n-r})$ be global coordinates in $\mathbb{R}^{n-r}$ and $\omega = f(x,g)\,\mathrm{pr}_1^* \alpha \wedge \mathrm{pr}_2^* \nu$ be a form of type I. One has

$$d\omega = \sum_{i=1}^{n-r} \frac{\partial f}{\partial x_i}(x,g)\,\mathrm{pr}_1^*(dx_i \wedge \alpha) \wedge \mathrm{pr}_2^* \nu + f(x,g)\mathrm{pr}_1^* d\alpha \wedge \mathrm{pr}_2^* \nu.$$

Therefore

$$\pi_* d\omega = \left(\int_G \sum_{i=1}^{n-r} \frac{\partial f}{\partial x_i}(x,g)\nu\right) dx_i \wedge \alpha + \left(\int_G f(x,g)\nu\right) d\alpha.$$

On the other hand,

$$d\pi_*\omega = d\left[\left(\int_G f(x,g)\nu\right)\alpha\right] = \sum_{i=1}^{n-r} \frac{\partial}{\partial x_i}\left(\int_G f(x,g)\nu\right) dx_i \wedge \alpha + \left(\int_G f(x,g)\nu\right) d\alpha.$$

By taking derivatives under the integral sign,

$$= \left(\int_G \sum_{i=1}^{n-r} \frac{\partial f}{\partial x_i}(x,g)\nu\right) dx_i \wedge \alpha + \left(\int_G f(x,g)\nu\right) d\alpha = \pi_* d\omega.$$

For forms $\omega = f(x,g)\mathrm{pr}_1^* \alpha \wedge \mathrm{pr}_2^* \nu$ of type II, where $\nu \in \Omega^k(G)$ $(k < r)$ one has that

$$d\pi_*\omega = 0.$$

For $x \in \mathbb{R}^{n-r}$, let $j_x : G \to \mathbb{R}^{n-r} \times G$ be the inclusion $g \mapsto (x, g)$. Then

$$(\pi_* d\omega)|_x = \begin{cases} 0\,, & \text{if}\, k < r-1; \\ \left(\int_G d(j_x^* f \wedge \nu)\right) \alpha \quad , & \text{if}\, k = r-1. \end{cases}$$

As G is compact, Stokes theorem guarantees that $\int_G d(j_x^* f \wedge \nu) = 0$. □

Proposition B.4. *For any $\alpha \in \Omega(N)$ and $\omega \in \Omega(P)$, one has*

$$\pi_*(\pi^*\alpha \wedge \omega) = \alpha \wedge \pi_*\omega.$$

Proof. Note that multiplication by $\pi^*\alpha$ doesn't change the type of ω. If ω is of type II, both sides are zero. If $\omega = f(x,g)\mathrm{pr}_1^*\beta \wedge \mathrm{pr}_2^*\nu$ is of type I, then by definition

$$\pi_*(\pi^*\alpha \wedge \omega) = \left(\int_G f(x,g)\nu\right) \alpha \wedge \beta = \alpha \wedge \left(\int_G f(x,g)\nu\right) \beta = \alpha \wedge \pi_*\omega.$$

□

Proposition B.5. *If a vector field X in P is π-related with a vector field Y in N, then*

$$\pi_* \circ i_X = i_Y \circ \pi_*$$

Proof. As a local problem, suppose that $N = \mathbb{R}^{n-r}$ with global coordinates $(x_1, \cdots, x_{n-r})$ and $P = N \times G$. For a basis $\{u_1, \cdots, u_r\}$ of $\mathfrak{g}$, one can write

$$X = \sum_{i=1}^{n-r} f_i(x,g) \frac{\partial}{\partial x_i} + \sum_{i=1}^{r} h_i(x,g) u_i^L$$

where u_i^L is the left invariant vector-field generated by u_i. The hypothesis of π-relation implies that $f_i(x,g) = \hat{f}_i(x)$, for $\hat{f}_i \in C^\infty(\mathbb{R}^{n-r})$ and

$$Y = \sum_{i=1}^{k} \hat{f}_i(x) \frac{\partial}{\partial x_i}.$$

Now, if $\omega = f(x,g)\mathrm{pr}_1^*\alpha \wedge \mathrm{pr}_2^*\mu$ is of type (I), then

$$i_X\omega = f(x,g)\, \mathrm{pr}_1^*\, i_Y\alpha \wedge \mathrm{pr}_2^*\mu + \text{ forms of type}(II).$$

Therefore,

$$\pi_*(i_X\omega) = \left(\int_G f(x,g)\mu\right) i_Y\alpha = i_Y\pi_*\omega.$$

It is fairly easy to check that i_X takes forms of type II to a sum of forms of the same type II and thus, in this case, both sides are zero. □

Now, we wish to prove a useful proposition. Suppose that $\theta \in \Omega^1(P, \mathfrak{g})$ is a connection form in P. Let $\{u^1, \cdots, u^r\}$ be a basis of $\mathfrak{g}$ and

$$\theta = \sum_{i=1}^{r} \theta_i u^i,$$

for $\theta_i \in \Omega^1(P)$. For $i = 1, \cdots, r$, consider $\xi_i \in \mathfrak{g}^*$ such that $\xi_i(u^j) = \delta_{ij}$ and $\nu \in \Omega^n(G)$ the left-invariant volume form generated by $\xi_1 \wedge \cdots \wedge \xi_n \in \wedge^r \mathfrak{g}^*$. By definition, for any $u \in \mathfrak{g}$,

$$\theta_i(u_P) = \xi_i(u). \tag{B.3}$$

Proposition B.6.

$$\pi_*(\theta_1 \wedge \cdots \theta_n) = \int_G \nu$$

Proof. By considering a neighborhood $\mathcal{U}$ of N with coordinates $(x_1, \cdots, x_{n-r})$ and such that $\pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times G$, one can assume that $P = \mathcal{U} \times G$. In this case, consider the connection form $pr_2^* \theta_{MC}$, where $\theta_{MC} \in \Omega^1(G, \mathfrak{g})$ is the left Maurer-Cartan 1-form given by

$$(\theta_{MC})_g (v) = (dL_{g^{-1}})_g(v), \text{ for } v \in T_g G$$

It is straightforward to see that

$$\theta_{MC} = \sum_{i=1}^{n} \xi_i^L u^i.$$

By equation (B.3)

$$\alpha_i := \theta_i - \mathrm{pr}_2^* \xi_i^L \in \Omega^1(P)$$

satisfies

$$i_{u_P} \alpha_i = \xi_i(u) - \xi_i^L((pr_2)_* u_P) = \xi_i(u) - \xi_i^L(u^L) = 0.$$

for every $u \in \mathfrak{g}$ and $i = 1, \cdots, n$. Thus, by expanding

$$\theta_1 \wedge \cdots \wedge \theta_n = \left(\alpha_1 + \mathrm{pr}_2^* \xi_1^L\right) \wedge \cdots \wedge \left(\alpha_n + \mathrm{pr}_2^* \xi_n^L\right),$$

we see that

$$\theta_1 \wedge \cdots \wedge \theta_n = \mathrm{pr}_2^* \nu + \text{forms of type}\, (II)$$

and thus the result follows. $\square$

Appendix C

Some proofs.

Proof of Lemma 5.1. First we implement Lagrange method to diagonalize bilinear forms. Let $\mathcal{B} = \{e_1, \cdots, e_{2n}\}$ be a frame over a neighbourhood U of x. Let $b_{ij} = g(e_i, e_j)$. One can suppose $b_{ii} \neq 0$ for some $i = 1, \ldots, 2n$ (by reordering the $e_i's$, choose $i = 1$). Indeed, if all diagonal terms were 0, one can choose $r, s \in \{1, \ldots, 2n\}$ such that $b_{rs} \neq 0$ (g is non-degenerate) over a possible smaller neighbourhood. For any permutation σ of $\{1, \ldots, 2n\}$ such that $\sigma(1) = r$ and $\sigma(2) = s$, define

$$e_i^{(1)} = \begin{cases} e_r + e_s, & \text{if}\, \sigma(i) = r; \\ e_r - e_s, & \text{if}\, \sigma(i) = s; \\ e_i, & \text{otherwise.} \end{cases}$$

Then $c_{11} \neq 0$, where $c_{ij} = g(e_i^{(1)}, e_j^{(1)})$. Let $c_j = c_{1j}/c_{11}$. Then

$$e_j^{(2)} = \begin{cases} \frac{1}{\sqrt{|c_{11}|}} e_1^{(1)}, & \text{if}\, i = r; \\ e_j^{(1)} - c_j e_1^{(1)}, & \text{otherwise.} \end{cases}$$

One has $g(e_1^{(3)}, e_j^{(3)}) = 0$ for all $j \neq 1$. Now repeat the argument for the set $\{e_2^{(3)}, \ldots, e_{2n}^{(3)}\}$ and by induction one obtains an orthonormal frame $\{e_1^{(3)}, \ldots, e_{2n}^{(3)}\}$. As the bilinear form is split, after a permutation of the basis one can assume

$$g(e_i^{(3)}, e_i^{(3)}) = 1\, (\text{resp.} -1) \text{ for } i \leq (\text{resp.} >)\, n.$$

Finally

$$e_i^{(3)} = \begin{cases} \frac{1}{\sqrt{2}}(e_i^{(3)} + e_{i+n}^{(3)}), & \text{if}\, i \leq n; \\ \frac{1}{\sqrt{2}}(e_{i-n}^{(3)} - e_i^{(3)}), & \text{if}\, i > n \end{cases}$$

define a polarized frame over a neighbourhood of x. □

Proof of Proposition 5.9. Let $H \in \Omega^3(M)$ be the curvature of ∇ and $\Phi_\nabla(e_1) = X + \xi$, $\Phi_\nabla(e_2) = Y + \eta$, for $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^*M)$. The first equation follows from

$$\begin{cases} i_{[X,Y]} + \mathcal{L}_X\eta\wedge\cdot\ = [\mathcal{L}_X,\,(i_Y+\eta\wedge\cdot)]; \\ i_Y(i_XH - d\xi)\wedge\cdot\ = [(d\xi - i_XH)\wedge\cdot\,,\,i_Y] \ \text{ and} \\ [(d\xi - i_XH)\wedge\cdot\,,\,\eta\wedge\cdot] = 0. \end{cases}$$

As for the second, observe that

$$\begin{aligned} d(\mathcal{L}_X\eta - i_Yd\xi + i_Yi_XH) - i_{[X,Y]}H \\ &= \mathcal{L}_Xd\eta - \mathcal{L}_Y(d\xi - i_XH) - i_Ydi_XH - i_{[X,Y]}H \\ &= \mathcal{L}_Xd\eta - \mathcal{L}_Y(d\xi - i_XH) + i_Yi_XdH - i_Y\mathcal{L}_XH - i_{[X,Y]}H \\ &= \mathcal{L}_X(d\eta - i_YH) - \mathcal{L}_Y(d\xi - i_XH) \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{L}_{[X,Y]} + (d(\mathcal{L}_X\eta - i_Yd\xi + i_Yi_XH) - i_{[X,Y]}H)\wedge\cdot \\ &= [\mathcal{L}_X,\mathcal{L}_Y] + (\mathcal{L}_X(d\eta - i_YH) - \mathcal{L}_Y(d\xi - di_XH))\wedge\cdot \\ &= [\mathcal{L}_X + (d\xi - di_XH)\wedge\cdot\,,\,\mathcal{L}_Y + (d\eta - i_YH)\wedge\cdot] \end{aligned}$$

which proves the second equality. □

Proof of Proposition 5.10. Let $\Phi_\nabla(e) = X+\xi \in \Gamma(\mathbb{T}M)$. One has to prove that

$$[d_H,\,(i_X+\xi\wedge\cdot)] = \mathcal{L}_X + (d\xi - i_XH)\wedge\cdot.$$

But this is an straightforward consequence of the identities

$$\begin{aligned} [d, i_X] &= \mathcal{L}_X\,, & [d,\xi\wedge\cdot] &= d\xi\wedge\cdot \\ [H\wedge\cdot,\,i_X] &= i_XH\wedge\cdot\,, & [H\wedge\cdot,\,\xi\wedge\cdot] &= 0. \end{aligned}$$

To prove the second equation, recall that d comutes with $\mathcal{L}_X$ and that

$$[H\wedge\cdot\,,\,(d\xi - i_XH)\wedge\cdot] = 0.$$

Therefore,

$$\begin{aligned} [d^\nabla,\mathcal{L}_e^\nabla] = \quad [d,(d\xi - i_XH)\wedge\cdot] - [H\wedge\cdot\,,\,\mathcal{L}_X] \quad &= -di_XH\wedge\cdot\ + \mathcal{L}_XH\wedge\cdot \\ &= i_XdH\wedge\cdot \\ &= 0. \end{aligned}$$

□

Proof of Proposition 5.17. First note that φ and $d^\nabla\varphi$ have opposite parity, Υ has to be an odd element of $\Gamma(\wedge^\bullet L'|_{\mathcal{W}})$. Moreover, as

$$\Pi_l(e_1\wedge e_2\wedge e_3)\wedge^i L' \subset \wedge^{i-3}L', \text{ for } e_3,e_2,e_1\in L \text{ and } i\geq 0,$$

it suffices to show (5.8) to prove the Proposition. Now, by Proposition 3.23, F_l intertwines Π_l with Π_∇ and therefore

$$\Pi_l(e_1\wedge e_2\wedge e_3)\Upsilon = F_l^{-1}(\Pi_\nabla(e_1\wedge e_2\wedge e_3)\,d^\nabla\varphi) = F_l^{-1}(\Pi_\nabla(e_1)\Pi_\nabla(e_2)\Pi_\nabla(e_3)\,d^\nabla\varphi).$$

Using that $\Pi_\nabla(e_i)\varphi = 0$ as $e_i \in \Gamma(L)$ for $i = 1, 2, 3$ and the Cartan-like formulas from Proposition 5.9 and 5.10, we obtain

$$\Pi_l(e_1 \wedge e_2 \wedge e_3)\Upsilon = F_l^{-1}(\Pi_\nabla(e_1)\Pi_\nabla([\![e_2, e_3]\!])\varphi).$$

To finish, use that

$$\Pi_\nabla(e_1)\Pi_\nabla([\![e_2, e_3]\!]) + \Pi_\nabla([\![e_2, e_3]\!])\Pi_\nabla(e_1) = g(e_1, [\![e_2, e_3]\!])\, Id$$

and $\Pi_\nabla(e_1)\varphi = 0$ to conclude

$$\Pi_l(e_1 \wedge e_2 \wedge e_3)\Upsilon = F_l^{-1}(g(e_1, [\![e_2, e_3]\!])\,\varphi) = g(e_1, [\![e_2, e_3]\!]).$$

□

Proof of Corollary 5.18. If L is integrable, then $g(e_1, [\![e_2, e_3]\!]) = 0$ for $e_1, e_2, e_3 \in \Gamma(L)$. Therefore for any local section φ of $U^\nabla(L)$, one has $\Gamma_{(3)} = 0$ in Proposition 5.17 and therefore

$$d^\nabla\varphi = \Pi_\nabla(\Upsilon^{(1)})\,\varphi,$$

where $\Upsilon^{(1)}$ is the component of Υ in $\wedge^1 L' = L'$. Conversely, if

$$d^\nabla(U^\nabla(L)) \subset \Pi_\nabla(L')\, U^\nabla(L),$$

then for any local section φ, one has that

$$\Upsilon = F_l^{-1}(d^\nabla\varphi) \in \wedge^1 L'$$

and therefore for any $e_1, e_2, e_3 \in \Gamma(L)$

$$g(e_1, [\![e_2, e_3]\!]) = \Pi_l(e_1 \wedge e_2 \wedge e_3)\Upsilon = 0.$$

This proves that $[\![e_2, e_3]\!] \in \Gamma(L^\perp) = \Gamma(L)$ and therefore L is integrable. □

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