Dominated Splitting and Critical Sets for Polynomials Automorphisms on $\mathbb{C}^2$

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Introduction

In the theory of dynamical systems, a well known seminal area is the study of rational maps in the Riemann sphere. For complex dynamics in several variables, polynomial automorphisms of $\mathbb{C}^2$ appear in a natural way. Their study represent the first step for a global understanding of holomorphics dynamic in higher dimension.

Moreover, this area is particularly interesting because of its connections to some fundamental questions of dynamical systems via two real dimensional dynamics and because of its connection to some powerful techniques via one dimensional complex tools.

The first results in the theory, were given by Friedland and Milnor in [FM]. They proved, that in the set of polynomial automorphism in $\mathbb{C}^2$, the only systems that exhibit rich dynamics (module, conjugation by a polynomial automorphism) are the so called generalized Hénon maps (or by simplicity, Hénon maps).

This kind of applications are characterized as the map that can be written as

$$f(x,y) = f_k \circ \cdots \circ f_1(x,y),$$

for each $j = 1, \ldots, k$

$$f_j(x,y) = (y, p_j(y) - \delta_j x)$$

where $\delta_j \neq 0$ and p_j is a monic polynomial of degree $d_j = \deg(p_j) \geq 2$. As in the one dimensional context, it can be defined the Julia set, which is the set that concentrate the interesting dynamical behavior of these type of systems. This set is compact invariant and have a series of properties similar to the ones satisfied by the one dimensional Julia set.

Several works in the study of Hénon map, has been carried out by a large number of authors including Bedford, Fornæss, Friedland, Hubbard, Lyubich, Milnor, Oberste-Vorth, Sibony, Smillie; and among other.

One of the goals of the present work is to describe the dynamics of two dimensional polynomial automorphisms under the hypotheses of *Dominated Splitting* in the Julia set. We also look to find sufficient conditions to guarantee hyperbolicity. The second main goal is to understand the dynamical obstruction for domination. This allow us to introduce a notion of critical point for polynomial automorphisms that capture many of the dynamical properties of their one-dimensional counterpart.

At the beginning of each chapter, there are extensive introductions resuming the main results. Here, we only summarize briefly the general ideas and we recommend to the reader to give a glance to that introductions.

In the first chapter, we present the generalized Hénon maps and we list a series of properties of the Julia set that give a global panorama of their dynamics. We also, formalize the notion of dominated splitting in the context of two dimensional holomorphic maps and it is given an extensive list of examples of holomorphics transformations that have invariant set with the properties of dominated splitting.

In the second chapter, we address the main part of the work about the dynamical consequences of dominated splitting. Under a topological condition in the center unstable leaves, namely expansiveness, we prove that the Julia set is a hyperbolic set (Theorem A). Another result in the second chapter gives a metrical condition that also implies hyperbolicity.

In the third chapter, we study the dynamical obstruction for domination in the two dimensional holomorphic context (not necessarily for Hénon maps), called critical points. This is a generalization in the two dimensional holomorphic context of the notion introduced in [P-RH] for surfaces maps. The main theorem of the present chapter establish, roughly speaking, that *The Julia set has dominated splitting if and only if it does not have "critical points"*. In some sense, this suggest a two dimensional counterpart for the classical one dimensional theorem about rational maps: *" If the postcritical set is disjoint from the Julia set, them the Julia set is Hyperbolic"*.

In the last chapter, we summarize a list of question that appear along this work and motivated this thesis.

Contents

Chapter 1

Preliminary Results on Polynomial Automorphisms and Dominated Splitting

Introduction

In this chapter, we study two central topics to the present work. The first of them is *generalized Hénon map*, and the second one is the notion of *dominated splitting* for two dimensional holomorphics map. We also make a list of examples of holomorphic maps in the two dimensional space, that have invariant sets with dominated splitting.

The main goal of the first section, is to present the *state of the art* in the study of Hénon map, however not all of this results are used in this Thesis works.

1.1 Background on Generalized Hénon map

In this section we study polynomial automorphisms of $\mathbb{C}^2$. Several works in the study of a special type of polynomial automorphisms called generalized Hénon map, has been carried out by a large number of authors including Bedford, Fornæss, Friedland, Hubbard, Lyubich, Milnor, Oberste-Vorth, Sibony, Smillie; and among other. We will present the most important result in this direction.

An holomorphic automorphism (or biholomorphisms) $f(x, y) = (f_1(x, y), f_2(x, y))$ of $\mathbb{C}^2$ is called a *polynomial automorphisms* if $f_1(x, y)$, $f_2(x, y)$ are polynomials in the variables x, y. The degree of f is defined by $\deg(f) = \max(\deg(f_1), \deg(f_2))$. Since f is a biholomorphism, the inverse map f^{-1} is also a polynomial automorphism of

$\mathbb{C}^2$ with the same degree of the f, and the Jacobian Determinant $J_f(x,y)$ is a non-zero constant function (see for example [MNTU]). We can describe three equivalence class of conjugation in the Group of Polynomials Automorphisms. To enunciate this result due to Friedland and Milnor in their work [FM], we first define three types of polynomial automorphism.

Definition 1.1.1. *We say that a polynomial automorphism f is:*

(a) affine map, if

$$f(x,y) = (ax + by + c, \alpha x + \beta y + \gamma), \ (a\beta - \alpha b \neq 0);$$

(b) elementary map, if

$$f(x,y) = (ax + b, sy + p(x)), \ (as \neq 0),$$

where p is a polynomial in the variable x;

(c) generalized Hénon map (or simply a Hénon map), if

$$f(x,y) = f_k \circ \cdots \circ f_1(x,y),$$

where for each $j = 1, \ldots, k$

$$f_j(x,y) = (y, p_j(y) - \delta_j x)$$

where $\delta_j \neq 0$ and p_j is a monic polynomial of degree $d_j = \deg(p_j) \geq 2$.

Theorem 1.1.1 (Friedland-Milnor). *Every polynomial automorphism of $\mathbb{C}^2$ is conjugated by a polynomial automorphism to one of the following maps:*

(1) an affine map,

(2) an elementary map,

(3) an generalized Hénon maps.

Henceforth let f be a Hénon map. It follows from the Definition 1.1.1, that the degree of f is equal to $\deg(f) = \prod_{j=1}^{k} d_j = d$ and that the Jacobian determinant is equal to $\det(Df) = \prod_{j=1}^{k} \delta_j = \delta$, we also denote $b = |\delta| = |\det(Df)|$.

We also adopt the following terminology: given f a Hénon map, let

$$K_f^{\pm} = \{z \in \mathbb{C}^2 : \{f^{\pm n}(z)\}_{n \in \mathbb{N}} \text{ is bounded } \}$$

the forward/backward filled Julia set, $J^{\pm}_f = \partial K^{\pm}_f$ the forward/backward Julia set, $K_f = K^+ \cap K^-_f$ filled Julia set, and $J_f = J^+_f \cap J^-_f = \partial K_f$ is the Julia set. Also we define the set of forward/backward escaping points by the sets $U^{\pm}_f = \mathbb{C}^2 \setminus K^{\pm}$. When the dependence is clear, we omit the subscript in the notation. We will now enumerate a series of result concerning Hénon maps.

1.1.1 Filtration

For $R > 0$ we define the sets

$$V^- = \{(x, y) \in \mathbb{C}^2 : |y| > R,\ |y| > |x|\}, \qquad V^+ = \{(x, y) \in \mathbb{C}^2 : |x| > R,\ |x| > |y|\}$$

and

$$V = \{z \in \mathbb{C}^2 \ : \ |z_i| \leq r, \ \text{ for every } \ i = 1, 2\} = \Delta(0, R)$$

the polydisc of radius R. Note that $\mathbb{C}^2 = V^- \cup V \cup V^+$. In [BS1] and [H-OV1] (see also [MNTU]), the authors show the following general properties:

Properties 1.1.1. *Given f a generalized Hénon map, there exist $R > 0$ great enough such that:*

1. $K^{\pm} \subset V \cup V^{\pm}$*;*
2. $K \subset V$ *and is closed in* V*, so* K *is a compact set;*
3. $f^{\mp}(V^{\pm}) \subsetneq V^{\pm}$ *and* $f^{\mp}(V^{\pm} \cup V) \subset V^{\pm} \cup V$*;*
4. $f^{\pm n}(V^{\pm}) \subset f^{\pm(n+1)}(V^{\pm})$ *for every* $n \geq 0$ *and* $U^{\pm} = \cup_{n \geq 0} f^{\pm n}(V^{\pm})$*;*
5. $W^{s/u}(K) = K^{\pm}$*;*
6. *If* $|\det(Df)| < 1$*, then Int*$(K^-) = \emptyset$.
7. *The chain recurrent set* $R(f) \subset K$.
8. *The sets* $K^{\pm}$ *and* $J^{\pm}$ *are connected.*
9. *If* f *is hyperbolic in* J*, then* f *has a finitely many periodic sink* $q_1, \ldots, q_k$*, whose stable manifolds* W^s *are Fatou-Bieberbach domains (open set biholomorphic to* $\mathbb{C}^2$*). Also* $\partial W^s(q_i) = J^+$ *for all* $i = 1, \ldots, k$*. Moreover* $\Omega(f) = J \cup \{q_1, \ldots, q_k\}$.

We can think, in the set K as a *"fat hyperbolic fixed point"* (see Fig.1).

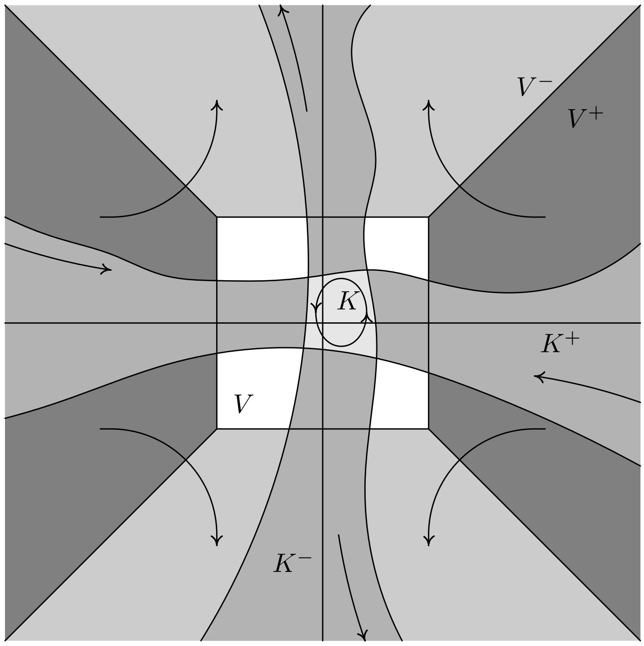

Figure 1

1.1.2 Metrical Properties of Hénon maps

In [BS1] is proved that for every Hénon map there exists an invariant measure μ that plays an important role in the studying of these applications. To complement, in the Appendix A we developed main tools to show existence of this measure.

We denote by J^* to the support of the measure μ defined above.

Properties 1.1.2. *The main properties about μ are the following:*

1. $J^* \subset J$.

2. *If J is uniformly hyperbolic, then $J = J^*$. Moreover, the invariant manifolds $W^{s/u}(x)$ of any $x \in J$, are conformally equivalent to the complex plane (see [BS1]).*

3. *If J^* is hyperbolic and f is not volume preserving, then $J = J^*$ (see [F]).*

4. *The measure μ is mixing, so ergodic (see [BS3] and [BLS1]).*

5. *The measure satisfies $h_\mu(f) = h_\mu(f|_K) = h_\mu(f|_J) = h_\mu(f|_{J^*}) = \log(d)$, where d is the degree of f (see [BS3]).*

6. *The measure μ is the unique of maximal entropy (see [BLS1]).*

Another important Property to recall, is that the periodic point are equidistributed with respect to the measure μ. This result is due to Bedford, Lyubich and Smillie (see [BLS2]).

Firstly, we introduce some notation: Fix_n denote the set of fixed point of f^n, Per_n the set of points of period exactly n. Then $\mathrm{Fix}_n = \cup \mathrm{Per}_k$, where the union is taken over all k dividing n. Also denote by SPer_n, the set of all periodic saddle points with period exactly n, hence

$$\mathrm{SPer}_n \subset \mathrm{Per}_n \subset \mathrm{Fix}_n$$

$$\#\mathrm{SPer}_n \leq \#\mathrm{Per}_n \leq \#\mathrm{Fix}_n \leq d^n,$$

where d is the degree of f.

Theorem 1.1.2. *Let P_n denotes any of the three sets above, then*

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{p \in P_n} \delta_p = \mu.$$

1.1.3 Invariant Sets, Pesin Theory

For polynomial maps of $\mathbb{C}$, Fatou and Julia used Montel's theorem to show that the expanding periodic points are dense in J. In the two dimensional context, the natural analogs of expanding point are the periodic saddle points. It is clear that periodic saddle point, are inside of J. In [BS1], the authors shows that J^* is contained in closure of periodic saddle points. Also we can relate the invariant manifolds $W^{s/u}(p)$ of a saddle point $p \in J^*$ with the set $J^{\pm}$, but the same relation can be showed for regular points in the Oseledets sense. Denote by $\mathcal{R}$ the set of regular point in J^*. The following properties are showed in [BLS1], and we can replace Per(f) by $\mathcal{R}$ and "all" by "μ-a.e." in the following results.

Properties 1.1.3. *One generalized Hénon map, has the following properties:*

1. *If p is periodic saddle point, then $p \in J^*$, in particular, the regular point are dense in J^*.*

2. *For every $p \in \mathcal{R}$ denote by $\lambda^s(p)$ and $\lambda^u(p)$ the Lyapunov exponent. Then for μ-a.e. p*

$$\lambda^u(p) = \lim_{n\to\infty} \frac{1}{n} \int \log \|Df^n(x)\| d\mu(x),$$

and

$$\lambda^s(p) = \log(b) - \lambda^u(p) \leq -\log(d) < 0 < \log(d) \leq \lambda^u(p).$$

3. *The stable and unstable manifolds $W^s(p)$ and $W^u(p)$ are conformally equivalent to the complex plane for μ-a.e. p.*

4. *For μ-a.e. p, $W^u(p)$ is dense in J^- and $W^s(p)$ is dense in J^+. In particular, $J = J^+ \cap J^- = \overline{W^s(p)} \cap \overline{W^u(q)}$ for any pair $p, q \in \mathcal{R}$.*

5. *For any pair of periodic saddle points, p and q, hold that $J^* = \overline{W^s(p) \cap W^u(q)}$.*

Remark 1. 1. *The property above, in particular implies that J^* is a homoclinic class, of any periodic point p.*

2. *The expression μ-a.e., in the previous properties, is in fact for μ-a.e. regular points.*

3. *The same previous properties are true, if we replace μ by any measure f-invariant ν, that is ergodic and hyperbolic supported in J.*

1.2 Dominated Splitting and Examples

For a long time the goal in the theory of dynamical systems is to describe the dynamics of "big set" (generic, residual, dense, etc.) in the space of all dynamical systems. It was thought in the sixties that this could be realized by the so-called hyperbolic systems: systems that have a splitting $T_{L(f)}M = E^s \oplus E^u$, where M is a manifolds and $L(f)$ is the limit set, such that vectors in E^s (respectively E^u) are uniformly forward (respectively backward) contracted by the linear cocycle (or tangent map) Df, (see [BDV] for more details, and properties).

However, hyperbolicity is far from being a generic property: it was shown that there are open sets in the space of dynamics which are non-hyperbolic. For example, in the case of surface diffeomorphisms, Newhouse show that hyperbolicity was not dense in the space of C^r-diffeomorphisms of compact surfaces for $r \geq 2$ (the case $r = 1$ is still an open problem). A similar phenomena holds for polynomial automorphisms: there are open sets of polynomial automorphisms such that their Julia set is not hyperbolic (see [Bu]).

We are interested in the study of a weaker form of hyperbolicity known as *Dominated Splitting*. The concept of dominated splitting was introduced independently by Mañé, Liao and Pliss, as a first step in the attempt to prove that structurally stable systems satisfy a hyperbolic condition on the tangent map. In fact, under the assumption of C^1-structural stability, the closure of the periodic points exhibits dominated splitting.

In this work, we are interested in the holomorphic context. More precisely, the notion of dominated splitting for a two-dimensional holomorphics systems.

In this section, we present this notion and describe equivalents definitions that only holds for polynomial automorphisms. Moreover, we present a series of examples of holomorphic systems, that have an invariant set with dominated splitting.

In what follows, U and V are open connected sets of $\mathbb{C}^2$, f is a holomorphic diffeomorphisms of U to V, and $\Lambda \subset U \cap V$ is a compact f-invariant set.

Definition 1.2.1. *A splitting $T_\Lambda\mathbb{C}^2 = E \oplus F$, with $\dim_{\mathbb{C}} E_z = \dim_{\mathbb{C}} F_z = 1$ is dominated (l-dominated), if it is invariant under the derivative Df and there exists a positive integer $l > 0$ such that*

$$||Df^l(z)|_{E(z)}|| \cdot ||Df^{-l}(f^l(z))|_{F(f^l(z))}|| < \frac{1}{2},$$

for every $z \in \Lambda$.

The following classical proposition establish properties equivalent with the dominated splitting notion.

Proposition 1.2.1. *The following statement are equivalent:*

1. *The f-invariant set Λ has dominated splitting.*

2. *There exist an splitting $T_\Lambda\mathbb{C}^2 = E \oplus F$, of Df-invariant one-dimensional complex planes, and positive real numbers $C > 0$ and $0 < \lambda < 1$ such that*

$$||Df(z)^n|_{E(z)}|| \cdot ||Df^{-n}(z)|_{F(f^n(z))}|| \leq C\lambda^n.$$

3. *There exists an splitting $T_\Lambda\mathbb{C}^2 = E \oplus F$, of one-dimensional complex planes (not necessarily Df-invariant), such that there exists $l > 0$ and cone fields $K(\alpha, E)$ and $K(\beta, F)$, namely*

$$K(\alpha, E(z)) = \{u + v \in E(z) \oplus F(z) \ : \ ||u|| \leq \alpha||v||\}$$

and

$$K(\beta, F(z)) = \{u + v \in E(z) \oplus F(z) \ : \ ||v|| \leq \beta||u||\},$$

such that

$$Df^{-l}(f^l(z))(K(\alpha, E(f^l(z)))) \subset K(\alpha, E(z))^\circ, \ \ Df^l(z)(K(\beta, F(z))) \subset K(\beta, F(f^l(z)))^\circ$$

and

$$\|Df^l(z)|_{K(\alpha,E(z))}\| \cdot \|Df^{-l}(z)|_{K(\beta,F(f^l(z)))}\| < \frac{1}{2},$$ [1]

where $K^\circ = \mathrm{int}(K) \cup \{0\}$. *We say that such of those cones are* Df^l*-invariant and have the property of l-domination.*

Remark 2. *We recall that the notion of dominated splitting, does not depend of the norm function, considered in* $\mathbb{C}^2$.

However in the context of Hénon maps, it is possible to get a stronger equivalent definitions, of dominated splitting.

Definition 1.2.2. *Let* $f : \mathbb{C}^n \to \mathbb{C}^n$ *be a biholomorphisms and* $\Lambda \subset \mathbb{C}^n$ *denote be a compact* f*-invariant set. We say that* f *is* ρ_0*-pseudo hyperbolic in* Λ*, if there exist a continuous* Df*-invariant splitting* $T_\Lambda\mathbb{C}^n = E \oplus F$*, and constants* $0 < \lambda_0 < \rho_0 < \mu_0$ *and* $C > 0$ *such that,*

1. $\|Df^n(x)|_{E(x)}\| \leq C\lambda_0^n$*, for* $n \geq 0$*,*
2. $\|Df^n(x)|_{F(x)}\| \geq C\mu_0^n$*, for* $n \geq 0$.

Remark 3. *In general, the notion of* ρ_0*-pseudo hyperbolicity is strong than dominated splitting. For example, a dominated splitting in* Λ*,* $T_\Lambda\mathbb{C}^2 = E \oplus F$ *is equivalent with the fact (maybe working whit* f^n *instead* f*, with n great enough) that for each* $x \in \Lambda$

$$\frac{\|Df(x)|E(x)\|}{\|Df(x)|F(x)\|} < \lambda < 1,$$

for some λ*. Thus there exist* τ_x *such that*

$$\|Df(x)|E(x)\| < \tau_x < \lambda\|Df(x)|F(x)\|,$$

but we not have any condition to standardize the value τ_x *over* Λ*. In other words, we not have a uniform control of the spectrum of* Df.

The following Lemma, establish that for a Hénon maps dominated splitting and ρ_0-pseudo hyperbolicity are equivalent notions, and the keys for this, is that the Jacobian determinant of a Hénon map is constant.

[1] This condition appear for real cocycle of each dimension. In our case, complex two-dimensional dominated splitting is not necessary. To see this fact, see the Proposition 3.1.5, and the proof of the Proposition 3.1.6, in the Chapter 3.

Lemma 1.1. *Let f be a generalized Hénon. Then f has dominated splitting in J with $T_J\mathbb{C}^2 = E \oplus F$ if and only if there exists ρ_0 such that f is ρ_0-pseudo hyperbolic in J. Moreover, if f is dissipative (i.e. $b < 1$), then direction E is a stable direction.*

We prove this this Lemma in the Chapter 2, in the Section 2.2. Also, in Chapter 2, we studies the dynamical consequences of dominated splitting in the context of Hénon maps.

1.2.1 Examples of Holomorphic systems with dominated splitting

Before introducing some examples, we consider the following Lemma.

Lemma 1.2. *Let $\lambda' \in \mathbb{C}$ such that $|\lambda'| > \lambda > 0$, and define the matrix*

$$A(\delta) = \begin{pmatrix} 0 & 1 \\ -\delta & \lambda' \end{pmatrix}.$$

Then there exist $\delta_0(\lambda) > 0$ and positive numbers α and β, with $\alpha\beta < 1$ such that

$$A^{-1}(\delta)(K^h_\alpha) \subset K^{h\circ}_\alpha, \;\; A(\delta)(K^v_\beta) \subset K^{v\circ}_\beta,$$

where

$$K^h_\alpha = \{(u, v) \in \mathbb{C}^2 \; : \; |u| \leq \alpha|v|\}$$

and

$$K^v_\beta = \{(u, v) \in \mathbb{C}^2 \; : \; |v| \leq \beta|u|\},$$

for every $0 < |\delta| < \delta_0(\lambda)$.

Proof. First one, note that we can find constant α, β and δ such that:

a. $1 < \alpha\lambda$,

b. β and $|\delta|$ small such that $\alpha\beta < 1$, $1 + \alpha^2|\delta| < \alpha\lambda$ and $|\delta| + \beta^2 < \beta\lambda$.

With this, if we take $(u, v) \in K^v_\alpha$, denote $I^v(u, v) = |u|/|v| \leq \alpha$; then $A(\delta)(u, v) = (v, \lambda' v - \delta u)$ so

$$I^v(A(\delta)(u, v)) = \frac{|v|}{|\lambda' v - \delta u|} \leq \frac{1}{|\lambda'| - |\delta| I^v(u, v)}.$$

On the other hand,

$$1 + \alpha|\delta| I^v(u, v) \leq 1 + \alpha^2|\delta| \leq \alpha\lambda \leq \alpha|\lambda'|,$$

hence

$$1 \leq \alpha(|\lambda'| - |\delta| I^v(u, v)),$$

it follows that $I^v(A(\delta)(u, v)) \leq \alpha$.

Working with the inverse matrix, and by a similar calculus, it follows that, if $I^h(u, v) = |v|/|u|$ and $I^h(u, v) \leq \beta$, then $I^h(A^{-1}(\delta)(u, v)) \leq \beta$. ■

Remark 4. *The idea for the previous Lemma is that the matrix $A(0)$, "super-contract" any vertical cone (collapsed into, the y-axis), and "super-expand" any horizontal cone. Since that the contraction/expansion of cones, is a robustly property, the same is true for $A(\delta)$, for δ close to* 0.

Also we remark that the constants α, β and δ only depend of λ and not of $|\lambda'|$.

The previous Lemma, allows us to "inflate", one dimensional maps into a two dimensional action.

Proposition 1.2.2. *Let $f_\delta(x, y) = (y, p(y) - \delta x)$ be a Hénon map, and denote by J_δ the Julia set of f_δ. Suppose that for $|\delta|$ small enough, we have the following property: for every $(x, y) \in J_\delta$, $|p'(y)| > 0$; then J_δ has dominated.*

Proof. Denote by pr_2 the projection of $\mathbb{C}^2$ in the second variable. The set $pr_2(J_\delta)$ is compact in the non-critical set

$$\mathcal{NC} = \{z \in \mathbb{C} : |p'(z)| > 0\},$$

and we can take an uniform constant λ such that $|p'(z)| > \lambda > 0$ for all $z \in pr_2(J_\delta)$. Since that

$$Jf(x, y) = \begin{pmatrix} 0 & 1 \\ -\delta & p'(y) \end{pmatrix},$$

we are in the hypothesis of the previous Lemma, so for $|\delta| < \delta_0(\lambda)$ there exist two cones fields Df-invariant. Hence J_δ has dominate splitting. ■

The previous proposition establish that when the projection of the Julia set J_δ is projected in the second coordinate to the set of not-critical points of p, then J_δ has dominated splitting when $|\delta|$ is small. However, we not have information over how close is this projection to J_p, the Julia set of p.

1.2.2 Examples

Now we present a several list of examples.

Example 1. Hyperbolic maps: A Hénon map f, is said that is hyperbolic if the Julia set J_f is a uniform hyperbolic set. It is clear that they is an examples of systems with dominated splitting.

In the Example 2, we give an example of hyperbolic Hénon map. However this example, and in all examples that we are presented here, they are essentially a one dimensional phenomena. In the article [I], Yutaka Ishii give examples of hyperbolic Hénon map that are *non-planar*, i.e. which is not topologically conjugate on its Julia set to a small perturbation of any expanding polynomial in one variable.

In the Example 7, we present an one-parameter family f_δ, that have dominated splitting but are non-hyperbolic. In these example the Julia set J_δ is planar because is a small perturbation of a planar map. An interesting question is:
Question: There exist a Hénon map whose Julia set is non-planar, has dominated splitting and is non-hyperbolic?

Example 2. Inflate of a Hyperbolic polynomial: It is possible to inflate a hyperbolic polynomial p, in the Hénon map of the form

$$f_\delta(x, y) = (y, p(y) - \delta x),$$

such that if $|\delta|$ is small enough, the resultant Hénon map is hyperbolic. The idea behind of this fact, is the persistence of hyperbolic set, including the context of endomorphisms. We present a small sketch of the proof of the hyperbolicity of f_δ. For reference of hyperbolicity in the context of endomorphisms, see [J].

For a compact f-invariant set Λ of a Riemannian manifold M, we define the *set of histories* of Λ by

$$\widetilde{\Lambda} = \{\widetilde{x} = (x_i)_{i\leq 0} \ : \ x_i \in \Lambda \ \text{and} \ f(x_i) = x_{i+1}\}.$$

This set is a compact metrizable subspace of $\Lambda^{\mathbb{N}}$. The restriction of f to Λ lifts to a homeomorphism $\widetilde{f}$ of $\widetilde{\Lambda}$ by $\widetilde{f}((x_i)_{i\leq 0}) = (f(x_i))_{i\leq 0}$. There is a natural projection π from $\widetilde{\Lambda}$ to Λ sending $\widetilde{x}$ to x_0 and the pullback under π of the tangent bundle $T_\Lambda M$ is a bundle $T\widetilde{\Lambda}$ which we call the tangent bundle. Explicitly, a point in $T\widetilde{\Lambda}$ is of the form $\widetilde{v} = (\widetilde{x}, v)$ where $\widetilde{x} \in \widetilde{\Lambda}$ and $v \in T_{x_0}M$. The derivative Df lifts to a map $D\widetilde{f}$ of $T\widetilde{\Lambda}$ in a natural way, i.e., $D\widetilde{f}(\widetilde{v}) = (\widetilde{f}(\widetilde{x}), Df_{x_0}v)$.

Now, we say that f is hyperbolic in Λ if the linear cocycle $D\widetilde{f}$ is hyperbolic in the bundle $T\widetilde{\Lambda}$ with the natural bundle metric induced, i.e., if there exist a $D\widetilde{f}$-invariant splitting $T\widetilde{\Lambda} = E^s \oplus E^u$, and constants $C > 0$ and $\tau > 1$ such that

$$|D\widetilde{f}^n(\widetilde{v})| \geq C\tau^n|\widetilde{v}|, \;\; \widetilde{v} \in E^u,$$

$$|D\widetilde{f}^n(\widetilde{v})| \leq C^{-1}\tau^{-n}|\widetilde{v}|, \;\; \widetilde{v} \in E^s.$$

We have that following.

Proposition 1.2.3. *If f is hyperbolic on $\Lambda = \Lambda_f$ and g is C^1-close to f, then there exist a continuous map $h : \widetilde{\Lambda}_f \to M$ close to the projection $\pi(\widetilde{x}) = x_0$ such that $g \circ h = h \circ \widetilde{f}$ and that g is hyperbolic on $\Lambda_g = h(\widetilde{\Lambda}_f)$. The map h lifts to a homeomorphism $\widetilde{h} : \widetilde{\Lambda}_f \to \widetilde{\Lambda}_g$ with $\widetilde{g} \circ \widetilde{h} = \widetilde{h} \circ \widetilde{f}$, and h depends continuously on g in the C^r topology ,* $1 \leq r \leq \infty$.

Now, we present the sketch of the proof to statement: *If p is hyperbolic, then f_δ is for $|\delta|$ small enough.*
From hyperbolicity of p it follows that there exist $C_0 > 0$ and $\tau > 1$ such that for every $z \in J_p$

$$|(p^n)'(z)| \geq C\tau^n.$$

The map $f_0(x, y) = (y, p(y))$ is hyperbolic in

$$J_0 = \{(y, p(y)) \; : \; y \in J_p\}.$$

This because J_0 is f_0-invariant, and defining the splitting $T\widetilde{J_0} = E^s \oplus E^u$ where $\widetilde{v} \in E^u_{\widetilde{x}}$ if $v = \alpha(1, p'(x_0))$ with $\alpha \in \mathbb{C}$, and $\widetilde{v} \in E^s_{\widetilde{x}}$ if $v = (v_1, 0)$. Thus is clear that E^s is invariant an that for every constant $C > 0$

$$|D\widetilde{f_0}^n(\widetilde{v})| \leq C^{-1}\tau^{-n}|\widetilde{v}|, \;\; \widetilde{v} \in E^s.$$

On the other hand, is not difficult to see that E^u is also invariant and that

$$|D\widetilde{f_0}^n|E^u_{\widetilde{x}}|^2 = \frac{|Df_0^n(x_0)(1, p'(x_0))|^2}{|(1, p'(x_0))|^2} = \frac{|(p^n)'(x_0)|^2 + |(p^{n+1})'(x_0)|^2}{1 + |p'(x_0)|^2} \geq \frac{C_0^2}{1 + M(J_p)}(\tau^n)^2,$$

where $M(J_p) = \sup\{|p'(z)| \; : \; z \in J_p\}$.

Applying the previous Proposition to f_0, we conclude that for $|\delta|$ small, f_δ is an hyperbolic Hénon map, so has dominated splitting.

Example 3. Inflate of a non Hyperbolic Polynomial: To present another examples of holomorphic systems with dominated splitting, we need introduce the following notion.

Definition 1.2.3. *We said that a family of Hénon maps*

$$f_\delta(x, y) = (y, p(y) - \delta x),$$

is regular, if the function $\delta \mapsto J_\delta$ is continuous, as a function of $\mathbb{C}$ into the space of all compact set of $\mathbb{C}^2$, endowed with the Hausdorff metric, and $J_0 = \{(y, p(y)) : y \in J_p\}$.

We have the following:

Proposition 1.2.4. *Let us assume that f_δ is a regular family. If J_p is far of critical set of p*

$$C(p) = \{z \in \mathbb{C} : p'(z) = 0\}.$$

Then for all δ with $|\delta|$ small, J_δ has dominated splitting.

This it is follows directly from Proposition 1.2.2: from continuity variation of J_δ, and for $|\delta|$ small, the projection on the second variable of points of J_δ, are far of the critical set of p.

We conjecture that this condition (J_p is far of critical set of p) is sufficient to guarantee continuity of the family. For a non-hyperbolic polynomial, we do not have a criteria to establish continuity of the Julia set with respect to the parameter δ, in the Hausdorff topology, in fact, the problem is to determine if the family is continuous in 0, and if his value in 0 is in fact J_0 as above.

Remark 5. In that follows we assume the **regularity** of the family of compact invariant set, in the parameter $\delta = 0$.

Example 4. Polynomial-like maps: The same construction in the previous examples can be used for a polynomial-like map $p : U \to V$. Note that the map $f_\delta(x, y)$ defined as a Hénon map, is a diffeomorphisms between $U \times \mathbb{C}$ onto its image. For $|\delta|$ small enough, f_δ has a compact invariant set J_δ set, and the projection in the second variable is close to Julia set J_p. If J_p is the interior of the non-critical set, from Proposition 1.2.2, it follows that J_δ has dominated splitting.

Example 5. Siegel Disk and Herman Rings: Consider now a rational map R and either a Siegel disk or a Herman ring denoted by U. It is known that ∂U is contained in the post-critical set. Moreover a Siegel disc may have a critical point in the boundary.

However, since R is conjugated in U to an irrational rotation in the unitary disc, then R has invariant Jordan curves in the interior of U that are conjugated to an irrational rotation in the circle. Moreover, U contain an exhaustion of compact R-invariant set, each of them are either topological disc or topological rings, and whose boundaries are the invariant Jordan curves.

Let K some of this compact invariant set. In the Siegel disk case this compacts set are topological disks, and in the Herman ring case, are rings.

We proceed as the previous examples. The map $f_\delta(x,y) = (y, R(y) - \delta x)$ is a diffeomorphisms between $U \times \mathbb{C}$ onto its image, and since that K is far of the critical set, f_δ has an invariant set $\widetilde{K}$, such that the projection in the second variable is close to K, so $\widetilde{K}$ has dominated splitting.

Question: Is the dynamics conjugated to a one dimensional Siegel disk or Herman ring multiplied by a uniform contraction?

Example 6. Hénon maps with periodic components. Rotation Domains has dominated splitting:

We now consider f, be a dissipative Hénon map, with $|\det(Df)| = |\delta| < 1$.

Let U be a connected component of $\mathrm{Int}(K^+)$. We say that U is *periodic*, if there exist a integer n such that $f^n(U) = U$. Also, we say that U is *recurrent*, if there exist $w \in U$ and a compact set $L \subset U$ such that $f^n(x) \in L$ for infinitely many $n > 0$.

In the article [BS2] (see also [MNTU]), the author proof that any recurrent domain U in the interior of K^+, is indeed a periodic component, and is either:

1. The basin of attraction of an attracting periodic point a, $W^s(a)$,
2. The stable set of a Siegel disk, $W^s(\mathcal{D})$,
3. The stable set of a Herman ring, $W^s(\mathcal{H})$.

In the two dimensional context, we say that a one-dimensional complex set $\mathcal{S}$, is a Siegel disk (resp. a Herman ring) when, there exist κ be an irrational number and an injective holomorphic map $\varphi : \Delta \to \mathbb{C}^2$, where $\Delta = \{z \in \mathbb{C} : |z| < r\}$ for some $r > 0$ (resp. $\Delta = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ for some $0 < r_1 < r_2$), and a positive integer n such that

$$f^n(\varphi(w)) = \varphi(bw),$$

where $b = e^{i\pi\kappa}$. Then the set $\mathcal{S} = \varphi(\Delta)$ is a Siegel disk (resp. Herman ring), if this set is maximal with respect to the inclusion. In the previous conditions we say that the set $\mathcal{S}$, is a *rotation domain* of period n. Note also that the sets $\mathcal{S}_0 = \mathcal{S}$, $\mathcal{S}_1 = f(\mathcal{S})$, $\mathcal{S}_2 = f^2(\mathcal{S}), \ldots, \mathcal{S}_{n-1} = f^{n-1}(\mathcal{S})$, are all rotation domains.

We can find a "linearization" for rotation domain. As before, Δ denote both a (euclidean) disk or a ring, and denote by $\Delta^* = \Delta \setminus \{0\}$. Then there exist a neighborhood $\mathcal{N}$

of $\Delta^* \times \{0\}$ in $\Delta^* \times \mathbb{C}$ and a biholomorphisms $S : \mathcal{N} \to \mathbb{C}^2$ satisfying $S(z, 0) = \varphi(z)$ such that $F = S^{-1} \circ f^n \circ S : \mathcal{N} \to \mathbb{C}^2$ has the form

$$F(z, w) = (bz + w^2 f_1, \delta^n b^{-1} + w^2 f_2),$$

where f_1 and f_2 are holomorphic functions in $\mathcal{N}$.

The set $\Delta^* \times \{0\}$ is F-invariant, and that the Jacobian matrix in points of $\Delta^* \times \{0\}$ is

$$DF(z, 0) = \begin{pmatrix} b & 0 \\ 0 & \delta^n b^{-1} \end{pmatrix}.$$

Since that we can find n-linearizations function for each $\mathcal{S}_i$ with $i = 0, \dots, n-1$, we conclude that (each in the orbit) the rotation domain has dominated splitting. This is follows, since that conjugated system with one that has dominated splitting, has dominated splitting; and this is because, the conjugated system contract and expand families of cones.

Example 7. Inflate of a parabolic fixed point: Now we consider a polynomial p of degree greater than 2, with a parabolic point in the Julia set of the form $p(q) = q$ and $p'(q) = 1$, and such that all critical point are far to the Julia set J_p. For example, $p(z) = z^2 + z$ has 0 as a parabolic fixed point, and the critical point $z = -1/2$ goes to 0 by iterates of p, however $-1/2 \notin J_p$.

We define the two parametric family of Hénon map $f_{\mu,\delta}$ by the equation

$$f_{\mu,\delta}(x, y) = (y, p(y) + \mu - \delta x).$$

We assert that we can find a function $\delta \mapsto \mu(\delta)$ such that $f_\delta = f_{\mu(\delta),\delta}$ has a periodic fixed point (q_δ, q_δ) with an eigenvalue equal to one. With this, and taking $|\delta|$ small, we have a Hénon map with dominated splitting that is not hyperbolic. More over, J_δ is not expansive, since that J_δ^* is a homoclinic class of periodic saddle points, and they are dense them, it follows the assertion.

To end, we must construct the explicit family. Since that p' has degree greater or equal to 1, for every $\delta > 0$ we can find a continuous parametric point q_δ close to q such that $p'(q_\delta) = 1 + \delta$. Define

$$\mu(\delta) = (1 + \delta)q_\delta - p(q_\delta),$$

then μ is continuous and equal to 0 for $\delta = 0$. With this, is easy to see that f_δ has (q_δ, q_δ) as a fixed point, and that the Jacobian matrix

$$Jf_\delta(q_\delta, q_\delta) = \begin{pmatrix} 0 & 1 \\ -\delta & 1 + \delta \end{pmatrix},$$

has eigenvalues $\lambda_- = \delta$ and $\lambda_+ = 1$.

To end, and assuming that the family f_δ is regular, it follows from Proposition 1.2.4, that J_δ has dominated splitting, but this is not hyperbolic because the point (q_δ, q_δ) has an eigenvalue equal to 1.

Chapter 2

Dynamical Consequences of Dominated Splitting in the Holomorphic Context

Introduction

The main purpose of this chapter is the study generalized Hénon maps in $\mathbb{C}^2$, with the hypothesis of dominated splitting in the Julia set J (see Chapter 1 for preliminaries results on Hénon maps). The initial motivation for this study is to answer the question:
Question 1: *Under which conditions the hypothesis of dominated splitting implies hyperbolicity on J?*
This question appears in a natural way, since that the set K and also $J = \partial K$, in a topological point of view, is a *"fat hyperbolic fixed point"*. In fact we have the following conjecture:
Conjecture: *If f is a dissipative Hénon map with dominated splitting in J^*, and all periodic point in J^* are hyperbolic, then f is hyperbolic in J.*

We recall that a consequence of hyperbolicity of J, is the hyperbolicity of the non-wandering set $\Omega(f)$, which is also is equal to $J \cup \{p_1, \ldots, p_k\}$, where each p_i is a sink (see 1.1.1, item 9 on the previous Chapter).

The first approach to answer the Question 1, is to show that under the hypothesis of domination, a Hénon map is ρ-pseudo hyperbolic, that is slightly strong property than domination (see Definition 1.2.2 and Remark 3, in the previous Chapter).

This establish that the spectrum of Df over J is lies off the circle of radius ρ (Lemma

1.1). A numerical condition for hyperbolicity is the following: suppose that

$$\frac{\|Df|E(x)\|}{\|Df|F(x)\|} < \lambda < 1,$$

where $T_J\mathbb{C}^2 = E \oplus F$ is the dominated splitting and $b = |\det(Df(x))|$ the norm of the Jacobian determinant. Thus if $\lambda < \min(b, b^{-1})$, then J is hyperbolic, and in particular if f is volume preserving ($b = 1$), then f is hyperbolic in J (see Corollary 2.2.1). Thus the interesting case is when $b < 1$, that is the dissipative case.

Another way of trying to solve the conjecture in the dissipative case, is under a topological condition in some neighborhood of J, namely forward expansiveness in some neighborhood of J, or in a more weaker sense, forward expansiveness in the center unstable leaves.

For a dissipative function f with dominated splitting, the direction E is a stable direction (Lemma 1.1), and there exist stables local manifolds and center unstable local manifolds over J. If we assume that the center unstable leaves are forward expansive, then it is proved that they are dynamically defined (they are in fact the unstable local set), and this last condition in the holomorphic context, implies that the center unstable manifolds are holomorphic submanifolds of $\mathbb{C}^2$. With the above, and a simply application of the Schwarz Lemma, we conclude that the direction center unstable are in fact an unstable direction, implying hyperbolicity of J.

Another approach, also under the hypothesis of dissipativity and domination, is to give a more metrical condition to have hyperbolicity. This establish that J is hyperbolic if and only if, J only supports measures f-invariant, that are hyperbolic, i.e., that its Lyapunov exponents related with each regular point $\lambda^- \leq \lambda^+$, satisfying $\lambda^- < 0 < \lambda^+$. In this results, play an important role, the Fornæss's Theorem:

Theorem 2.0.1 (Fornæss). *Let f be a complex Hénon map which is hyperbolic in J^*. If f is not volume preserving, then $J^* = J$.*

This chapter is organized as follows: In the section 2.1 we state the main result of this chapter, in the direction of proof the following Theorem:

Theorem A. *Let f be dissipative Hénon map, with dominated splitting. If f is cu-forward expansive, then f is hyperbolic.*

In the section 2.2, we prove all results stated in the section 2.1, including the Theorem A.

In the section 2.3, we give a simply description of the set supp (J), that roughly speaking, is the set that support all invariant measures. We introduce the set J_0, how the set that support all measures that have a null Lyapunov exponent, and that play an important role in the section 2.5. We prove that $\mathrm{supp}\,(J) = J^* \cup J_0$. We do note that the set $\mathrm{supp}\,(J)$ in a more general context play an essential role in the Section 3.3.3, in the following Chapter.

In the section 2.4 and 2.5 we prove the following theorem:

Theorem B. *Let f be a dissipative complex Hénon map, with dominated splitting in J. The following statement are equivalents:*

1. *J is hyperbolic,*
2. *$J_0 = \emptyset$,*
3. *The set of periodic (saddle) point are uniformly hyperbolic.*
4. *The set of periodic (saddle) point are uniformly expanding at the period.*

As a Corollary of the previous Theorem we have:

Corollary of Theorem B. *Let f be a dissipative complex Hénon map, with dominated splitting in J. Then J is hyperbolic if and only if all measure f-invariant supported in J is hyperbolic.*

2.1 Preliminaries and Main Results

In this section we recall several classic results in the context of complex and holomorphic dynamics. Some of this results, can be enunciated in $\mathbb{C}^n$ for any $n \geq 2$.

We define the closed polydisc of center 0 and radio $r > 0$ in $\mathbb{C}^k$ as the set

$$\Delta_k(0, r) = \{z \in \mathbb{C}^k \;:\; |z_i| \leq r, \;\text{ for every }\; i = 1, \dots, k\}.$$

Denote by $Emb^1(\Delta_k(0, 1), \mathbb{C}^n)$ the set of C^1-embeddings of $\Delta_k(0, 1)$ on $\mathbb{C}^n$. Two point $x, y \in \mathbb{C}^n$ are *forward ρ-asymptotic* under f, if $d(f^n(x), f^n(y)) \leq C\rho^n$ for all $n \geq 0$ and some constant $C > 0$. Similarly, we define *backward ρ-asymptotic* as forward ρ-asymptotic for f^{-1}.

We recall that Hénon map with dominated splitting in the Julia, is in fact ρ-pseudo hyperbolic (see Definition 1.2.2), this assert the Lemma 1.1. We present his proof in the following section. Recall by [HPS] that a ρ-pseudo hyperbolic has the following property.

Theorem 2.1.1. *Let f be a biholomorphisms in $\mathbb{C}^n$, such that f is ρ-pseudo hyperbolic in Λ. Then there exist two continuous functions $\phi^{cs} : \Lambda \to Emb^1(\Delta_k(0,1), \mathbb{C}^n)$ and $\phi^{cu} : \Lambda \to Emb^1(\Delta_l(0,1), \mathbb{C}^n)$ such that, with $W^{cs}_\varepsilon(x) = \phi^{cs}(x)\Delta_k(0,\varepsilon)$ and $W^{cu}_\varepsilon(x) = \phi^{cu}(x)\Delta_l(0,\varepsilon)$, the following properties hold:*

a) $T_x W^{cs}_\varepsilon(x) = E(x)$ and $T_x W^{cu}_\varepsilon(x) = F(x)$,

b) for all $0 < \varepsilon_1 < 1$ there exist ε_2 such that

$$f(W^{cs}_{\varepsilon_2}(x)) \subset W^{cs}_{\varepsilon_1}(f(x))$$

and

$$f^{-1}(W^{cu}_{\varepsilon_2}(x)) \subset W^{cu}_{\varepsilon_1}(f^{-1}(x)).$$

If $\rho \leq 1$ (resp. $\rho \geq 1$) then $\{W^{cs}_1(x)\}_{x\in\Lambda}$ (resp. $\{W^{cu}_1(x)\}_{x\in\Lambda}$) are C^∞ submanifolds of $\mathbb{C}^n$, and $W^{cs}_1(x)$ (resp. $W^{cu}_1(x)$) is characterized as those points locally forward (resp. backward) ρ-asymptotic with x.

We name the sets $W^{cs}_\varepsilon(x)$, the center stable leaf or cs-leaf, and similarly for cu-leaf. The following Theorem is part of the folklore and we prove them in the following section. In his proof is introduced an important technique, that we use later in the proof of Proposition 2.1.5.

Theorem 2.1.2. *Let f be a ρ-pseudo hyperbolic map in Λ with splitting $T_\Lambda\mathbb{C}^n = E \oplus F$. Suppose that the direction F, is an unstable direction, then the cu-leaves are holomorphics unstable manifolds.*

Remark 6. *When Λ is hyperbolic, is known that the invariant manifolds are holomorphics and that when $n = 2$ (see [BS1] or [BS2]), this manifolds are biholomorphic to $\mathbb{C}$. In the previous Theorem, we don't have hypothesis that the direction E is stable.*

A first dynamic consequence for Hénon map with dominated splitting in the Julia set, is the following Proposition.

Proposition 2.1.1. *Let f be dissipative Hénon map, with dominated splitting in J^*. Then for every $x \in J^*$, holds that $W^{cu}_{loc}(x) \cap U^+ \neq \emptyset$.*

Remark 7. *The Proposition above say roughly speaking, that there are point (not necessarily every point) in the center unstable leaf, that for forward iterates escaping to infinity. We proof this Proposition after the proof of Theorem A.*

We recall some basic definitions. The *unstable set* of a point x for f, is the set

$$W^u(x) = \{y \in \mathbb{C}^n \,:\, d(f^{-n}(x), f^{-n}(y)) \to 0, \text{ when } n \to \infty\},$$

where d is the euclidean distance. Similarly, the *local unstable set* of size ε is the set

$$W^u_\varepsilon(x) = \{y \in W^u(x) \,:\, d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon, \text{ for every } n \geq 0\}.$$

It is know that for every $0 < \varepsilon \leq 1$ there exist $\delta > 0$ such that for every $x \in \Lambda$, $W^u_\delta(x) \subseteq W^{cu}_\varepsilon(x)$, however in general the opposite inclusion not hold if we not have good properties in the asymptotic behavior of Df.

Definition 2.1.1. *We say that the cu-leaves are dynamically defined, if for every* $0 < \varepsilon \ll 1$, $W^{cu}_\varepsilon(x) \subset W^u_{loc}(x)$ *for all* $x \in \Lambda$.

In holomorphic context, *cu*-leaves dynamically defined has an important analytic consequence.

Theorem 2.1.3. *Let f a biholomorphisms ρ-pseudo hyperbolic in $\Lambda \subset \mathbb{C}^n$. If the cu-leaves are dynamically defined, then they are holomorphic submanifolds of* $\mathbb{C}^n$.

In the following definition we introduce a local topological property, that implies that *cu*-leaves are dynamically defined. This is motivated by a small modification of the property before mentioned in the Remark 7.

Definition 2.1.2. *We say that f is expansive in the center unstable leaves or cu-expansive (resp. forward expansive), if there exist a uniform constant $c > 0$ such that for every $x \in \Lambda$, and any $y \in W^{cu}_\varepsilon(x)$, there exists $n \in \mathbb{Z}$ (resp. $n \in \mathbb{N}$), such that* $\operatorname{dist}(f^n(y), f^n(x)) > c$. *We say that the constant c is the expansiveness constant.*

Theorem 2.1.4. *If f is cu-forward expansive then the cu-leaves are dynamically defined.*

In dimension two, we also have that the condition of *cu*-forward expansive and *cu*-dynamically defined are equivalent conditions.

Theorem 2.1.5. *Let f a biholomorphisms ρ-pseudo hyperbolic in $\Lambda \subset \mathbb{C}^2$ with splitting $T_\Lambda\mathbb{C}^2 = E \oplus F$. Then the cu-leaves are dynamically defined if and only if $F = E^u$, i.e., to F is an unstable direction.*

In other words, cu-forward expansive and cu-dynamically defined are equivalent conditions.

The previous results, applied to the case of Hénon maps, can be summarized as the following:

Theorem A. *Let f be dissipative Hénon map, with dominated splitting in J^*. If f is cu-forward expansive, then f is hyperbolic in J.*

2.2 Proofs of Main Results

We begin this section proving the equivalence between domination and ρ-pseudo hyperbolicity, in the context of Hénon map. For this, we first present a equivalent notion of domination for Hénon maps. This is proved in the Proposition 3.1.6 in the following Chapter, in a more general context.

Proposition 2.2.1. *Let f be a Hénon map with $b = |\det(Df)|$, and let $T_J\mathbb{C}^2 = E \oplus F$ be a splitting. The following statement are equivalents:*

1. *The splitting $T_J\mathbb{C}^2 = E \oplus F$ is dominated;*
2. *There exist $C > 0$ and $0 < \lambda < 1$ such that:*
   a) *For every unitary vector $v \in F$ and $n \geq 1$*
   $$\frac{b^n}{||Df^n v||^2} \leq C\lambda^n,$$
   b) *For every unitary vector $v \in E$ and $n \geq 1$*
   $$\frac{b^{-n}}{||Df^{-n} v||^2} \leq C\lambda^n.$$

Proof of Lemma 1.1. From the previous Proposition, and maybe taking an iterate of f instead f, we can assume that there exist $0 < \lambda < 1$ such that

(a) $\dfrac{b^n}{||Df_x^n u_x||^2} < \lambda^n$, for every n and $x \in J$,

(b) $\dfrac{b^{-n}}{||Df_x^{-n} v_x||^2} < \lambda^n$, for every n and $x \in J$,

where $u_x \in F(x)$ and $v_x \in E(x)$ are unitary vectors.

Replacing the previous inequality for the direction $E(x)$, it follows that

$$||Df_x^{-n} v_x||^2 > \left(\frac{1}{b\lambda}\right)^n.$$

Replacing the inverse function of Df^{-n} in the previous inequality, and taking $\lambda_0 = \sqrt{b\lambda}$, we obtain that

$$\|Df_x^n v_x\| \le \lambda_0^n \implies \|Df_x^n|_{E(x)}\| \le \lambda_0^n.$$

Similarly for the direction $F(x)$, let u_x a unitary vector in this direction we obtain

$$\|Df_x^n u_x\|^2 > \left(\frac{b}{\lambda}\right)^n,$$

and taking $\mu_0 = \sqrt{b/\lambda}$ it follows that

$$\|Df^n|_{F_x}\| \ge \mu_0^n.$$

Thus we have

$$\lambda^2 < 1 \iff b\lambda < \frac{b}{\lambda} \iff \lambda_0 < \mu_0,$$

then we can find a positive real number ρ_0 that satisfies the inequality $\lambda_0 < \rho_0 < \mu_0$, so f is ρ_0-pseudo hyperbolic in J.

On the other hand, if f is ρ_0-pseudo hyperbolic in J (see Definition 1.2.2), is not difficult to see that

$$\frac{\|Df^n|E(x)\|}{\|Df^n|F(x)\|} < \left(\frac{\lambda_0}{\mu_0}\right)^n =: \lambda^n,$$

where $\lambda < 1$; so it follows from Proposition 1.2.1 that J has dominated splitting.

To end, for the dissipative case $b < 1$, so $\lambda_0 = \sqrt{\lambda b} < 1$, then E is a stable direction.

■

Corollary 2.2.1. *The constants λ_0 and μ_0 given in the proof of previous Lemma, satisfy $\lambda_0 < 1 < \mu_0$ if and only if $\lambda < \min(b, b^{-1})$. In particular, if f has dominated splitting and is volume preserving (i.e., $b = 1$), then f is hyperbolic.*

Proof. From the previous proof, we conclude that

$$\|Df^n|F(x)\| \ge \mu_0^n \quad \text{and} \quad \|Df^n|E(x)\| \le \lambda_0^n,$$

where $\lambda_0 = \sqrt{\lambda b}$, $\mu_0 = \sqrt{b/\lambda}$ and $b = |\det(Df)|$. From this equation is easy to see that

$$\frac{\|Df^n|E(x)\|}{\|Df^n|F(x)\|} < \lambda^n.$$

Suppose that $\lambda_0 < 1 < \mu_0$ is equivalent to haves

$$\lambda^2 b < \lambda < b,$$

that is $\lambda < \min(b, b^{-1})$. ■

Now we enunciate a classical version of the Theorem of existence of center stables/center unstable manifold, knowing as Hadamard-Perron Theorem. We will use the notation and the "technique" of this Theorem, to prove many of the statement in the previous section. We use the version of the theorem stated in the book [KH].

Theorem 2.2.2 (Hadamard-Perron Theorem). *Let $\lambda < \mu$, $r \geq 1$ and for each $m \in \mathbb{Z}$ let $f_m : \mathbb{C}^n \to \mathbb{C}^n$ be a (onto) C^r diffeomorphisms such that for $(x, y) \in \mathbb{C}^l \oplus \mathbb{C}^k$,*

$$f_m(x, y) = (A_m x + \alpha_m(x, y), B_m y + \beta_m(x, y)),$$

for some linear maps $A_m : \mathbb{C}^l \to \mathbb{C}^l$ and $B_m : \mathbb{C}^k \to \mathbb{C}^k$ with $\|A_m^{-1}\| \leq \mu^{-1}$, and $\|B_m\| \leq \lambda$ and $\alpha_m(0) = 0$, $\beta_m(0) = 0$.

Then for $0 < \gamma < \min(1, \sqrt{\mu/\lambda} - 1)$ and

$$0 < \delta < \min\left(\frac{\mu - \lambda}{\gamma + 2 + \gamma^{-1}}, \frac{\mu - (1+\gamma)^2\lambda}{(1+\gamma)(\gamma^2 + 2\gamma + 2)}\right) \tag{2.1}$$

we have the following property: If $\|\alpha_m\|_{C^1} < \delta$ and $\|\beta_m\|_{C^1} < \delta$ for all $m \in \mathbb{Z}$ then there is

(1) a unique family $\{W_m^+\}_{m\in\mathbb{Z}}$ of l-dimensional C^1 manifolds

$$W_m^+ = \{(x, \varphi_m^+(x)) \ : \ x \in \mathbb{C}^l\} = \operatorname{graph} \varphi_m^+$$

and

(2) a unique family $\{W_m^-\}_{m\in\mathbb{Z}}$ of k-dimensional C^1 manifolds

$$W_m^- = \{(\varphi_m^-(y), y) \ : \ y \in \mathbb{C}^{n-l}\} = \operatorname{graph} \varphi_m^-,$$

where $\varphi_m^+ : \mathbb{C}^l \to \mathbb{C}^k$, $\varphi_m^- : \mathbb{C}^k \to \mathbb{C}^l$, $\sup_{m\in\mathbb{Z}} \|D\varphi_m^\pm\| < \gamma$, and the following properties holds:

(i) $f_m(W_m^-) = W_{m+1}^-$, $f_m(W_m^+) = W_{m+1}^+$.

(ii) The inequalities

$$\|f_m(z)\| < \lambda'\|z\| \text{ for } z \in W_m^-,$$

and

$$\|f_{m-1}^{-1}(z)\| < \mu'\|z\| \text{ for } z \in W_m^+$$

hold, where $\lambda' = (1+\gamma)(\lambda + \delta(1+\gamma)) < \frac{\mu}{1+\gamma} - \delta = \mu'$.

(iii) *Let* $\lambda' < \nu < \mu'$. *If* $\|f_{m+j-1} \circ \cdots \circ f_m(z)\| < C\nu^j$ *for all* $j \geq 0$ *and some* $C > 0$ *then* $z \in W_m^-$.
Similarly, if $\|f_{m-j}^{-1} \circ \cdots \circ f_{m-1}^{-1}(z)\| < C\nu^{-j}$ *for all* $j \geq 0$ *and some* $C > 0$ *then* $z \in W_m^+$.

Finally, in the hyperbolic case $\lambda < 1 < \mu$ *the families* $\{W_m^+\}_{m\in\mathbb{Z}}$ *and* $\{W_m^-\}_{m\in\mathbb{Z}}$ *consist of* C^r *manifolds.*

It is important also to note the following proposition, proved in [KH].

Proposition 2.2.2. *The invariant manifolds (with* C^1 *topology) obtained in the Hadamard-Perron Theorem, have continuous dependence with respect to the family* $f = \{f_m\}_{m\in\mathbb{Z}}$, *with the* C^1 *topology defined by*

$$d_1(f, g) = \sup_{m\in\mathbb{Z}} d_{C^1}(f_m, g_m).$$

Remark 8. *To apply the previous Theorem and the subsequent results, firstly we construct the family* $\{f_m\}_{m\in\mathbb{Z}}$, *that carries the asymptotic information of the map* f, *along the whole orbit of some point* $x \in \Lambda$.

For this, first note that given $\delta > 0$ *we can find* $R > 0$ *such that for every* $x_0 \in \Lambda$ *we can write*

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + R_{x_0}(x - x_0)$$

on $\mathbb{C}^n$, *and* $\|R_{x_0}(x - x_0)\|_{C^1} < \delta$ *for all* $x \in \Delta_n(x_0, R)$. *Moreover, for every* $\delta > 0$ *we can find* $R > 0$ *(uniformly in* Λ*) and a function* f_{x_0} *such that:* $f_{x_0}(h) = f(x + h) - f(x_0)$ *with* $h \in \Delta_n(0, R)$, f_{x_0} *is a diffeomorphisms in* $\mathbb{C}^n$ *and* $\|f_{x_0}(h) - Df(x_0)(h)\|_{C^1} < \delta$ *for all* $h \in \mathbb{C}^n$.

Now taking $L_{x_0} : \mathbb{C}^n = \mathbb{C}^l \oplus \mathbb{C}^k \to \mathbb{C}^n$ *a linear orthogonal complex map such that* $L_{x_0}(\mathbb{C}^l) = F(x_0)$ *and* $L_{x_0}(\mathbb{C}^k) = F(x_0)^\perp$, *and define the maps* $\widehat{f_{x_0}} = L_{f(x_0)}^{-1} \circ f_{x_0} \circ L_{x_0}$, *then* $\widehat{f_{x_0}}$ *has the form*

$$\widehat{f_{x_0}}(x, y) = (A_{x_0}x + \alpha_{x_0}(x, y), B_{x_0}y + \beta_{x_0}(x, y)).$$

To finish, we denote $x_m = f^m(x_0)$ *with* $m \in \mathbb{Z}$ *and* $f_m = \widehat{f_{x_m}}$, *then:*

1) f_m *is holomorphic in* $\Delta_n(0, R')$ *for every* $R' < R$,

2) *since that the angle between the direction* F *and* E *are uniformly away from zero, it follows that there exist* $\lambda < \mu$ *such that are satisfied the hypothesis of Hadamard-Perron Theorem.*

2.2.1 Proof of Theorem 2.1.2

To proof this Theorem, is only necessary to observe the Proposition 2.2.3. The principal ideas are summarizes right away.

The sketch of the proof is essentially the following: in the Hadamard-Perron Theorem, we find the invariant manifolds as an application of the Contraction Theorem for Lipschitz maps. We define the graph transform operator, in an appropriated space of functions $C^0_\gamma(\mathbb{C}^l)$, and it is possible to proof that the graph transform operator is a contraction in this space, that is a completed space with certain metric.

In this space of functions, we will prove that under the hypothesis of holomorphic of the functions f_m and forward expansiveness in the center unstable direction, the graph transform operator leaves invariant the subspace of functions of $C^0_\gamma(\mathbb{C}^l)$ that are holomorphic in some small neighborhood of zero.

The metric considerer in this space is the metric of uniform convergence in compact set, and with this metric, the space of holomorphic function is closed, concluding that the limit of holomorphic function is holomorphic, so the invariant manifold.

Proposition 2.2.3. *Under the hypothesis of Theorem 2.2.2, suppose that the following additional conditions hold:*

1. $\mu > 1$.
2. *There exists $R > 0$ such that, for each $m \in \mathbb{Z}$, the map f_m is holomorphic in some neighborhood of the closed polydisc $\Delta_n(0, R) \subset \mathbb{C}^n$.*

Then there exists $0 < r < R$ such that each φ^+_m is holomorphic in some neighborhood of $\Delta_l(0, r) \subset \mathbb{C}^l$, where φ^+_m is as in (1) in the Theorem 2.2.2.

Proof. In the proof of Theorem 2.2.2 (see [KH]), the functions φ^+_m are obtained as fixed point of a contractive operator in a space of Lipschitz maps. We enumerate the main fact:

1. Let C^0_γ the space of sequences as form $\varphi_* = \{\varphi_m\}_{m\in\mathbb{Z}}$ where each φ_m is in the set
$$C^0_\gamma(\mathbb{C}^l) = \{\varphi : \mathbb{C}^l \to \mathbb{C}^{n-l} \ : \ Lip(\varphi) < \gamma, \text{ and } \varphi(0) = 0\}.$$
2. The set C^0_γ is a compact metric space with the metric defined by
$$d_*(\varphi_*, \phi_*) = \sup_{m\in\mathbb{Z}} d(\varphi_m, \phi_m);$$

where

$$d(\varphi, \phi) = \sup_{x \in \mathbb{C}^l \setminus \{0\}} \frac{\|\varphi(x) - \phi(x)\|}{\|x\|}$$

is a metric in $C^0_\gamma(\mathbb{C}^l)$. Note that $(C^0_\gamma(\mathbb{C}^l), d)$ is also compact metric space.

3. The action of $f = \{f_m\}_{m\in\mathbb{Z}}$ in the space C^0_γ is the desired contraction; this action is defined as follows: denote by $(f_m)_*\varphi$ the unique Lipschitz map that satisfy the equation

$$f_m(\operatorname{graph}\varphi) = \operatorname{graph}((f_m)_*\varphi).$$

On the other hand, we have the bijection $G^m_\varphi : \mathbb{C}^l \to \mathbb{C}^l$ defined by

$$G^m_\varphi(x) = A_m x + \alpha_m(x, \varphi(x)),$$

and the map $F^m_\varphi : \mathbb{C}^l \to \mathbb{C}^l$ given by

$$F^m_\varphi(x) = B_m\varphi(x) + \beta_m(x, \varphi(x)),$$

it follows that the function $(f_m)_*\varphi$ is given by the expression

$$(f_m)_*\varphi(x) = F^m_\varphi \circ (G^m_\varphi)^{-1}(x).$$

Finally if we define $f\varphi_* = \{\psi_m\}_{m\in\mathbb{Z}}$, whit $\psi_{m+1} = (f_m)_*\varphi_m$, we have that

$$\lim_{n\to\infty} f^n\varphi_* = \varphi^+_*, \tag{2.2}$$

where $\varphi^* \in C^0_\gamma$ and $\varphi^+_* = \{\varphi^+_m\}_{m\in\mathbb{Z}}$ is the sequences of function given by the Hadamard-Perron Theorem.

Denote by $O^0_\gamma(r) \subset C^0_\gamma$, the set of sequences of functions that are holomorphic in some neighborhood of the closed polydisc $\Delta_l(0, r)$ in each level $m \in \mathbb{Z}$. To prove the Proposition, is only necessary to prove that there exists $0 < r < R$ such that:

(a) $O^0_\gamma(r)$ is a closed space in C^0_γ,

(b) $O^0_\gamma(r)$ is invariant by the action f.

If we assume that (a) and (b) holds, and since that equation (2.2) hold for every $\varphi_* \in O^0_\gamma(r)$, the limit

$$\lim_{n\to\infty} f^n\varphi_* = \varphi^+_*$$

there exists and is an element of $O^0_\gamma(r)$, so each function φ^+_m is holomorphic in some neighborhood of $\Delta_l(0, r)$.

Observe that for proof the two previous assertions, is only necessary proof that:

(a') $\mathcal{O}^0_\gamma(r, \mathbb{C}^l)$ is a closed space in $C^0_\gamma(\mathbb{C}^l)$,

(b') $\mathcal{O}^0_\gamma(r, \mathbb{C}^l)$ is invariant by the action f_m, for all $m \in \mathbb{Z}$.

where $\mathcal{O}^0_\gamma(r, \mathbb{C}^l)$ is the subset of $C^0_\gamma(\mathbb{C}^l)$, whose elements are holomorphics function in some neighborhood of the polydisc $\Delta_l(0, r)$.

The first assertion (a'), follows after observing that the metric defined in the paragraph (2.), induce the uniformly convergence on compact topology in $\mathcal{O}^0_\gamma(r, \mathbb{C}^l)$, so if $\varphi_n \in \mathcal{O}^0_\gamma(r, \mathbb{C}^l)$ and $\varphi_n \to \varphi$ for some $\varphi \in C^0_\gamma(\mathbb{C}^l)$ then, the limit map φ is an element of the set $\mathcal{O}^0_\gamma(r, \mathbb{C}^l)$.

The proposition (b'), it follows from the following: in the proof of the Theorem 2.2.2, we can see that

$$\|G^m_\varphi(x)\| \geq \mu_0 \|x\|. \tag{2.3}$$

where the constant is $\mu_0 = (\mu - \delta(1 + \gamma))$. This constant is greater than 1 if and only if, $\mu > 1$ and δ and γ are small enough. If we take $r = \mu_0^{-1} R$, the functions F^m_φ and G^m_φ are holomorphics in some neighborhood of $\Delta_l(0, r)$ when $\varphi \in \mathcal{O}^0_\gamma(r, \mathbb{C}^l)$. It follows by the equation (2.3) that $\Delta_l(0, R) \subset G^m_\varphi(\Delta_l(0, r))$, then the function $(G^m_\varphi)^{-1}$ is holomorphic in $\Delta_l(0, r)$, and also by equation (2.3), it follows that

$$(G^m_\varphi)^{-1}(\Delta_l(0, r)) \subset \Delta_l(0, \mu_0^{-1} r) \subset \Delta_l(0, r).$$

We obtain that $F^m_\varphi \circ (G^m_\varphi)^{-1}$ is holomorphic in some neighborhood of $\Delta_l(0, r)$, is as desired. ■

2.2.2 Proof of Theorem 2.1.3

To prove this Theorem, first we present a necessary and sufficient condition that to have *cu*-leaf are dynamically defined. The proof is left to the reader.

Lemma 2.1. *The cu-leaves are dynamically defined, if and only if, there exists $r \ll 1$ uniform constant such that for all $x \in \Lambda$, the following statement holds:*

1. *For any $r_1 < r$, there exist $r_0 < r_1$ such that for every $n \geq 0$ and $x \in \Lambda$, $f^{-n}(W^{cu}_{r_0}(x)) \subset W^{cu}_{r_1}(f^{-n}(x))$.*

2. *For every $r_1 < r$ and $r_0 < r_1$, there exists $N = N(r_0, r_1)$ such that for all $x \in \Lambda$ and $n \geq N$ $f^{-n}(W^{cu}_{r_1}(x)) \subset W^{cu}_{r_0}(f^{-n}(x))$.*

An important consequence of this easy observation is the following.

Lemma 2.2. *Let f such that the cu-leaves are dynamically defined, and let $0 < r \ll 1$ as in the previous Lemma. Then for every $0 < r_2 < r$ there exit $0 < r_{-1} < r_0 < r_1 < r_2$, a number $N = N(r_0, r_1)$ and closed topological balls $B^{cu}(x) \subset (W^{cu}_{r_0}(x))^\circ$ for every $x \in \Lambda$, such that the following statement holds:*

1. *For every $n \geq N$ we have the inclusion $f^{-n}(W^{cu}_{r_1}(x)) \subset (W^{cu}_{r_0}(f^{-n}(x)))^\circ$,*
2. *$W^{cu}_{r_{-1}}(f^{-N}(x)) \subset f^{-N}(W^{cu}_{r_1}(x)) \subset (B^{cu}(f^{-N}(x)))^\circ$,*
3. *For every $0 \leq k \leq N$, we have $f^k(B^{cu}(f^{-N}(x))) \subset (W^{cu}_r(f^{N-k}(x)))^\circ$.*

Proof. If let us take $r_2 < r$, then for every $x \in \Lambda$, $\text{dist}\left(\partial W^{cu}_{r_2}(x), \partial W^{cu}_r(x)\right) > 0$, where dist is the induced distance in the center unstable leaf. It follows by compactness of Λ and continuity of the *cu*-leaves, that there exist a positive number $\delta > 0$ such that $\text{dist}\left(\partial W^{cu}_{r_2}(x), \partial W^{cu}_r(x)\right) > \delta$.

Now let us take $r_1 < r_2$ as in the item 1 in the previous Lemma. Since that for every $n \geq 0$, $f^{-n}(W^{cu}_{r_1}(x)) \subset W^{cu}_{r_2}(f^{-n}(x))$ we have in particular that

$$\text{dist}\left(\partial f^{-n}(W^{cu}_{r_1}(x)), \partial W^{cu}_r(f^{-n}(x))\right) \geq \text{dist}\left(\partial W^{cu}_{r_2}(f^{-n}(x)), \partial W^{cu}_r(f^{-n}(x))\right) > \delta.$$

If we take $r_0 < r_1$, and ε small enough such that $r_0 - \varepsilon > 0$, we know from the item 2 in the previous Lemma, that there exit $N = N(r_0 - \varepsilon, r_1)$ such that

$$f^{-n}(W^{cu}_{r_1}(x)) \subset W^{cu}_{r_0-\varepsilon}(f^{-n}(x)) \subset (W^{cu}_{r_0}(f^{-n}(x)))^\circ$$

for every $n \geq N$, and this implies the first item.

On the other hand, we can define the function $\rho(x) = \text{dist}\left(f^{-N}(x), \partial f^{-N}(W^{cu}_{r_1}(x))\right) > 0$ that is continuous in Λ. Let $\rho_0 = \inf_{x\in\Lambda} \rho(x)$. Then for every x there exist a neighborhood U_x and a radius r_x such that for $y \in U_x$

$$\text{dist}\left(W^{cu}_{r_x}(f^{-N}(y)), \partial f^{-N}(W^{cu}_{r_1}(y))\right) > \frac{\rho_0}{2}.$$

So by compactness, there exist a r_{-1} such that

$$\text{dist}\left(W^{cu}_{r_{-1}}(f^{-N}(x)), \partial f^{-N}(W^{cu}_{r_1}(x))\right) > \frac{\rho_0}{2}$$

in Λ and in particular, $W^{cu}_{r_{-1}}(f^{-N}(x)) \subset f^{-N}(W^{cu}_{r_1}(x))$, that is the first inclusion of the second item.

For the second inclusion of the item 2 and the item 3, we first will construct the sets $B^{cu}(x)$. For this, let us take

$$B(x) = \{z \in W^{cu}_r(x) \, : \, \text{dist}\,(z, \partial W^{cu}_r(x)) \geq \delta/2\}.$$

Then it is clear that $f^{-n}(W^{cu}_{r_1}(x)) \subset (B(f^{-n}(x)))^{\circ}$ for all $n \geq 0$, thus we define $B^{cu}(f^{-N}(x))$ has the connected component that contain $f^{-N}(W^{cu}_{r_1}(x))$ of the intersection

$$W^{cu}_{r_0-\varepsilon}(f^{-N}(x))) \cap f^{-1}(B(f^{-(N-1)}(x))) \cap \ldots \cap f^{-N}(B(x)).$$

By construction the set $(B^{cu}(f^{-N}(x)))^{\circ}$ contain $f^{-N}(W^{cu}_{r_1}(x))$ and it follows the third item, that conclude the proof of this Lemma. ■

Remark 9. *It is important to recall that the election of the constant $r_0 < r_1$ is arbitrary, once we take $r_1 < r_2$.*

Now, we want to highlight the main difference of the Theorem 2.1.2 with the Theorem 2.1.3. In the first of them, it is assumed that F is an unstable direction. Here we only assume that the center unstable manifold are dynamically defined.

However, the states of the proof has many similarities. The goal is to show that using the graph transform operator it is possible to prove that the *cu*-leaves are limits of the graph of uniformly bounded holomorphic function, and therefore it is also holomorphic. The main difficulty is to show that only using the dynamically defined property, is arrange to recover, after some iterate, the overflowing property of the graph transform operator.

We recall that in the Hadamard-Perron Theorem (Theorem 2.2.2), we obtain the existence of invariant manifolds, that are denoted by $\varphi^{\pm}_m$. The are, by some local change of chart, the center stable/unstable manifolds given in the Theorem 2.1.1.

The proof of Theorem 2.1.3, it follows immediately from the following Proposition.

Proposition 2.2.4. *Under the hypothesis of Theorem 2.2.2, suppose that the following additional conditions hold:*

1. *There exists $R > 0$ such that, for each $m \in \mathbb{Z}$, the map f_m is holomorphic in some neighborhood of the closed polydisc $\Delta(0, R) \subset \mathbb{C}^n$,*
2. *For every $0 < r < R$, are satisfied the three items of the Lemma 2.2,*

then there exit $R' < R$ such that φ^{+}_m is holomorphic in $\Delta_l(0, R')$.

Proof. We use the same notation of the proof of Proposition 2.2.3. Firstly let us take $r_2 < r < R$ with

$$2\gamma r_2 < \frac{R-r}{2}. \tag{2.4}$$

We remark that from the item 2 of the hypothesis, we have that there exist $r_{-1} < r_0 < r_1 < r_2$, an integer $N = N(r_0, r_1)$, and a family of closed topological balls U_m with

$\Delta_l(0, r_{-1}) \subset \overline{U_m} \subset \Delta_l(0, r_0)$, such that if we denote $D_m^+ = \text{graph}\,(\varphi_m^+|U_m)$ and $W_m^+(r') = \text{graph}\,(\varphi_m^+|\Delta_l(0, r'))$ are satisfied the following properties:

a) For every $n \geq N$ and $m \in \mathbb{Z}$, hold that $f_{m-n}^{-1} \circ \cdots \circ f_{m-1}^{-1}(W_m^+(r_1)) \subset (W_m^+(r_0))^\circ$,

b) $W_{m-N}^+(r_{-1}) \subset f_{m-N}^{-1} \circ \cdots \circ f_{m-1}^{-1}(W_m^+(r_1)) \subset (D_{m-N}^+)^\circ$,

c) For every $0 \leq k \leq N-1$ we have $f_{m-(N-(k-1))} \circ \cdots \circ f_{m-(N-1)} \circ f_{m-N}(D_{m-N}^+) \subset W_{m-(N-k)}^+(r)^\circ$.

We recall that from the remark 9, we can take $r_0 < r_1$ small enough such that

$$2\gamma r_0 < \frac{r_1 - r_0}{2}. \tag{2.5}$$

The proof goes through a series of claims.

Claim 1: *There exists $\lambda_0 < 1$, such that for every φ, $\phi \in C_\gamma^0(\mathbb{C}^l)$, $m \in \mathbb{Z}$ and $x \in \mathbb{C}^l$ we have the inequality*

$$\|f_m(x, \varphi(x)) - f_m(x, \phi(x))\| \leq \lambda_0 \|\varphi(x) - \phi(x)\|.$$

Proof of Claim 1. We recall that $f_m(x, \varphi(x)) = (G_\varphi^m(x), F_\varphi^m(x))$, then we have

$$\begin{aligned}
\|f_m(x, \varphi(x)) - f_m(x, \phi(x))\| &= \|(G_\varphi^m(x), F_\varphi^m(x)) - (G_\phi^m(x), F_\phi^m(x))\| \\
&\leq \|G_\varphi^m(x) - G_\phi^m(x)\| + \|F_\varphi^m(x) - F_\phi^m(x)\| \\
&\leq \|(A_m x + \alpha_m(x, \varphi(x))) - (A_m x + \alpha_m(x, \phi(x)))\| \\
&\qquad + \|(B_m\varphi(x) + \beta_m(x, \varphi(x))) - (B_m\phi(x) + \beta_m(x, \phi(x)))\| \\
&\leq \|\alpha_m(x, \varphi(x)) - \alpha_m(x, \phi(x))\| + \|B_m(\varphi(x) - \phi(x))\| \\
&\qquad + \|\beta_m(x, \varphi(x)) - \beta_m(x, \phi(x))\| \\
&\leq (\lambda + 2\delta)\|\varphi(x) - \phi(x)\|,
\end{aligned}$$

then let us take $\lambda_0 = (\lambda + 2\delta)$, and we will prove that $\lambda_0 < 1$. Firstly note that we can assume that $\mu \leq 1$, if not by Proposition 2.2.3 it follows that φ_m^+ is holomorphic in a polydisc $\Delta_l(0, R')$ for some $R' < R$, that is we want to prove. On other hand, by inequality (2.1) in the Theorem 2.2.2

$$\delta < \frac{\mu - \lambda}{\gamma + \gamma^{-1} + 2},$$

and this is less than $(1-\lambda)/2 < 1$. This end the proof of the claim. ∎

In that follows, we fix $m \in \mathbb{Z}$ and define

$$g_k = f_{(m+kN)+(N-1)} \circ f_{(m+kN)+(N-2)} \circ \cdots \circ f_{(m+kN)+1} \circ f_{(m+kN)}. \tag{2.6}$$

Then we can write g_k as the form

$$g_k(x,y) = (C_k x + c_k(x,y), D_k y + d_k(x,y)),$$

where

$$C_k = A_{(m+kN)+(N-1)} \cdot A_{(m+kN)+(N-2)} \cdot \ldots \cdot A_{(m+kN)+1} \cdot A_{(m+kN)}$$

and

$$D_k = B_{(m+kN)+(N-1)} \cdot B_{(m+kN)+(N-2)} \cdot \ldots \cdot B_{(m+kN)+1} \cdot B_{(m+kN)}.$$

We recall that graph transform operator $(f_m)_*$ of a Lipschitz function φ is defined by the equation

$$(x', (f_m)_*\varphi(x')) = f_m(x, \varphi(x)) = (A_m x + \alpha_m(x, \varphi(x)), B_m\varphi(x) + \beta_m(x, \varphi(x))).$$

It is possible to prove that the map $G_\varphi^m : \mathbb{C}^l \to \mathbb{C}^l$ given by

$$G_\varphi^m(x) = A_m x + \alpha_m(x, \varphi(x)),$$

is a bijection, and that if we define $F_\varphi^m : \mathbb{C}^l \to \mathbb{C}^l$ by

$$F_\varphi^m(x) = B_m\varphi(x) + \beta_m(x, \varphi(x)),$$

then the graph transform operator $(f_m)_*\varphi$, is given by the expression

$$(f_m)_*\varphi(x) = F_\varphi^m \circ (G_\varphi^m)^{-1}(x).$$

Similarly, we denote by

$$\widetilde{G}_\varphi^k(x) = C_k x + c_k(x, \varphi(x)),$$

and

$$\widetilde{F}_\varphi^k(x) = D_k\varphi(x) + d_k(x, \varphi(x)),$$

the coordinates maps related with g_k and φ.

For a fixed k and φ, we denote:

1. $\varphi_1 = (f_{m+kN})_*\varphi$,

2. $\varphi_{j+1} = (f_{m+kN+j})_*\varphi_j$, for every $j = 1, \ldots, N-2$,

3. $G_j = G_{\varphi_j}^{m+kN+j}$, for every $j = 1, \ldots, N-1$.

Claim 2: *We have that*

$$\widetilde{G}_\varphi^k = G_{N-1} \circ G_{N-2} \circ \ldots G_1 \circ G_\varphi^{m+kN},$$

and the graph transform operator of g_k, given by equality 2.6, is equal to

$$(g_k)_* = (f_{(m+kN)+(N-1)})_*(f_{(m+kN)+(N-2)})_* \cdots (f_{(m+kN)+1})_*(f_{(m+kN)})_*.$$

Proof. This is elementary, and the proof is left to the reader. ■

As a consequence of the previous claim, we conclude that $\widetilde{G}_\varphi^k$ is a bijection of $\mathbb{C}^l$, and that the graph transform operator related with g_k is given by the equality

$$(g_k)_*\varphi(x) = \widetilde{F}_\varphi^k \circ (\widetilde{G}_\varphi^k)^{-1}(x).$$

In that follows by simplicity, we will work with $m = k = 0$, and the function $g_0 = f_{N-1} \circ \cdots \circ f_0$, but all the following results are true for any m and k.

Claim 3: *If $\varphi \in C_\gamma^0(\mathbb{C}^l)$ is holomorphic in some neighborhood of U_0^+, then the function $\widetilde{G}_\varphi^0$ is holomorphic in some neighborhood of U_0^+.*

Proof. From the inequality (2.4), for any point x in the closed polydisc $\Delta_l(0, r_2)$, we have that

$$\|\varphi_0^+(x) - \varphi(x)\| \leq 2\gamma\|x\| \leq 2\gamma r_2 < \frac{R-r}{2}.$$

We recall that each map f_j is holomorphic in the closed polydisc $\Delta(0, R)$. From the item (c), it follows that $f_0(D_0^+) \subset (W_1^+(r))^\circ$ and we conclude that $G_{\varphi_0^+}^0(U_0) \subset \Delta_l(0, r)^\circ$. It follows from the Claim 1 that for every $x \in U_0$

$$\|G_{\varphi_0^+}^0(x) - G_\varphi^0(x)\| \leq \|f_0(x, \varphi_0^+(x)) - f_0(x, \varphi(x))\| \leq \lambda_0\|\varphi_0^+(x) - \varphi(x)\| < \lambda_0\frac{R-r}{2},$$

this implies that

$$\|G_\varphi^0(x)\| \leq \|G_{\varphi_0^+}^0(x) - G_\varphi^0(x)\| + \|G_{\varphi_0^+}^0(x)\| < \frac{R-r}{2} + r = \frac{R+r}{2} < R.$$

As before, we denote $\varphi_1 = (f_0)_*\varphi$, $\varphi_{j+1} = (f_j)_*\varphi_j$, for every $j = 1, \ldots, N-2$; and $G_j = G_{\varphi_j}^j$, for every $j = 1, \ldots, N-1$. Again by item (c), for every $1 \leq k \leq N-1$ we have $f_k \circ \cdots \circ f_0(D_0^+) \subset (W_{k+1}^+(r))^\circ$, it follows that $G_{\varphi_k^+}^k \circ \cdots \circ G_{\varphi_0^+}^0(U_0) \subset \Delta_l(0, r)^\circ$. We use the following notation:

$$x_k^+ = G_{\varphi_k^+}^k \circ \cdots \circ G_{\varphi_0^+}^0(x) \quad \text{and} \quad x_k = G_k \circ \ldots \circ G_\varphi^0(x).$$

Then as before, we conclude that for every $x \in U_0$

$$\begin{aligned}
||x_k^+ - x_k|| &\leq ||f_k(x_{k-1}^+, \varphi_k^+(x_{k-1}^+)) - f_k(x_{k-1}, \varphi_k(x_{k-1}))|| \\
&\leq \lambda_0 ||\varphi_k^+(x_{k-1}^+) - \varphi_k(x_{k-1})|| \\
&\leq \lambda_0 ||f_{k-1}(x_{k-2}^+, \varphi_{k-1}^+(x_{k-2}^+)) - f_{k-1}(x_{k-2}, \varphi_{k-1}(x_{k-2}))|| \\
&\quad \vdots \\
&\leq \lambda_0^k ||\varphi_0^+(x) - \varphi(x)|| \\
&< \lambda_0^k \frac{R-r}{2}
\end{aligned}$$

then is follows that

$$||G_k \circ \ldots \circ G_\varphi^0(x)|| \leq ||x_k^+ - x_k|| + ||x_k^+|| < \frac{R-r}{2} + r = \frac{R+r}{2} < R.$$

To end, since that φ is holomorphic in some neighborhood of U_0, f_0 is holomorphic in $\Delta_l(0,R)$ and $\overline{\mathrm{Im}\,(G_\varphi^0(U_0))} \subset \Delta_l(0,R)^\circ$ it follows that the map φ_1 is holomorphic in some neighborhood of $\mathrm{Im}\,(G_\varphi^0(U_0))$. Similarly, since that $\overline{G_1(\mathrm{Im}\,(G_\varphi^0(U_0)))} \subset \Delta_l(0,R)^\circ$, and f_1 is holomorphic in this domain, we conclude that φ_2 is holomorphic in $\mathrm{Im}\,(G_1(\mathrm{Im}\,(G_\varphi^0(U_0))))$, and so on. This implies that the map $\widetilde{G}_\varphi^0 = G_{N-1} \circ \cdots G_1 \circ G_\varphi^0$ is holomorphic in U_0 and $\overline{\mathrm{Im}\,(\widetilde{G}_\varphi^0(U_0))} \subset \Delta_l(0,R)^\circ$. ■

Claim 4: *The image of U_0 from the map $\widetilde{G}_\varphi^0$, contain the polydisc $\Delta_l(0,r_0)$.*

Proof. From the item (b), we have that $W_N^+(r_1) \subset (g_0(D_0^+))^\circ$, and we recall that $g_0(D_0^+)$ is a topological ball that contain 0. Now for a point $x \in pr_1(g_0^{-1}(W_N^+(r_1))) \subset \Delta_l(0,r_0)$ we have that $||G_{\varphi_{N-1}^+}^{N-1} \circ \cdots \circ G_{\varphi_0^+}^0(x)|| \leq r_1$ and that

$$||G_{\varphi_{N-1}^+}^{N-1} \circ \cdots \circ G_{\varphi_0^+}^0(x) - \widetilde{G}_\varphi^0(x)|| \leq ||g_0(x, \varphi_0^+(x)) - g_0(x, \varphi(x))|| < \lambda_0^N 2\gamma r_0 < \frac{r_1 - r_0}{2},$$

and this last inequality comes from the inequality (2.5). This conclude the proof of the claim. ■

From the previous claim, in particular we have that $U_N \subset \mathrm{Im}\,(g_0(D_0^+))$. Since $\widetilde{F}_\varphi^0$ be a holomorphic map in some neighborhood of U_0, and $(\widetilde{G}_\varphi^0)^{-1}$ is holomorphic in $\Delta_l(0,r_0) \supset U_N$ (and this because $\widetilde{G}_\varphi^0$ is holomorphic and injective), it follows that the map $\varphi'(x) = (g_0)_* \varphi(x) = \widetilde{F}_\varphi^0 \circ (\widetilde{G}_\varphi^0)^{-1}(x)$ is holomorphic in U_N.

We conclude that for any m, the action of the graph transform operator associated with the family $g = \{g_k\}_{k \in \mathbb{Z}}$ defined as in the equation (2.6), leaves invariant the set of sequences of Lipschitz functions that in each level is holomorphic in some neighborhood of the sets U's; and note that this set contain the linear maps. Passing to limit, we conclude that each φ_m^+ is holomorphic in the set $U_m \supset \Delta_l(0, r_{-1})$. Thus taking $R' = r_{-1}$, we completed the proof of the Proposition. ∎

2.2.3 Proof of Theorem 2.1.4

For this purpose, is only necessary to prove that are satisfied the equivalents condition in the Lemma 2.1.

Proposition 2.2.5. *Let f be a forward expansive map in the cu-leaves, with constant of expansiveness c. Then for every $r_1 < c$ there exist $r_0 < r_1$ such that for all $x \in \Lambda$ and $n \geq 0$*

$$f^{-n}(W^{cu}_{r_0}(x)) \subset W^{cu}_{r_1}(f^{-n}(x)).$$

Proof. We suppose that is not true, thus there exists r_1 such that the previous proposition not holds. Let $\rho \gtrapprox 1$ such that $\rho r_1 < c$ and let $(r_k)_k$ be a sequence of positive numbers such that $r_k \to 0$ and $r_k < r_1$. Thus there exist $x_k \in \Lambda$ and $(n_k)_k \nearrow \infty$ such that

$$f^{-n_k}(W^{cu}_{r_k}(x_k)) \not\subseteq W^{cu}_{r_1}(f^{-n_k}(x_k)) \subset W^{cu}_{\rho r_1}(f^{-n_k}(x_k)).$$

We take each n_k minimal with this property. Let us take $y_k = f^{-n_k}(x_k)$ and take z_k some point in the following intersection

$$f^{-n_k}(W^{cu}_{r_k}(x_k)) \cap \overline{W^{cu}_{\rho r_1}(y_k) \setminus W^{cu}_{r_1}(y_k)}.$$

Also we take y_0 and z_0 such that $z_k \to z_0$ and $y_k \to y_0$. By construction (and C^1 continuity of the *cu*-leaves) we have that $z_0 \in \overline{W^{cu}_{\rho r_1}(y_0) \setminus W^{cu}_{r_1}(y_0)}$.

We assert that

$$\operatorname{dist}(f^n(y_0), f^n(z_0)) \leq \rho r_1$$

for each $n \geq 1$, and since $\rho r_1 < c$ we have a contradiction with the expansiveness in the *cu*-leaves. Then to conclude the proof, is only necessary to prove the previous assertion.

By contradiction, we assume that there exist n such that $\operatorname{dist}(f^n(y_0), f^n(z_0)) = \gamma > \rho r_1$. By continuity of f^n, given $\varepsilon > 0$ we can take $k \gg 1$ such that $n_k > n$ and satisfied

$$\operatorname{dist}(f^n(y_k), f^n(y_0)) < \varepsilon \quad \text{and} \quad \operatorname{dist}(f^n(z_k), f^n(z_0)) < \varepsilon.$$

If we take ε such that $\gamma - 2\varepsilon > \rho r_1$ we conclude that $\operatorname{dist}(f^n(z_k), f^n(y_k)) > \gamma - 2\varepsilon > \rho r_1$. To end, taking $\widetilde{z}_k \in W^{cu}_{r_1}(x_k)$ such that $f^{n_k}(\widetilde{z}_k) = z_k$, the previous inequality implies that

$$\operatorname{dist}(f^{n-n_k}(\widetilde{z}_k), f^{n-n_k}(x_k)) > \rho r_1,$$

that is

$$f^{n-n_k}(W^{cu}_{r_k}(x_k)) \not\subseteq W^{cu}_{r_1}(f^{n-n_k}(x_k)),$$

that contradict the minimality of n_k. This ends the proof. ■

Proposition 2.2.6. *Let f be a forward expansive map in the cu-leaves, and $r_0 < r_1$ such that $r_0 \in I(r_1)$. Then for every $0 < \varepsilon < r_1 < c$ there exists $N = N(\varepsilon, r_0)$ such that for all $x \in \Lambda$ and $n \geq N$*

$$f^{-n}(W^{cu}_{r_0}(x)) \subset W^{cu}_{\varepsilon}(f^{-n}(x)).$$

Proof. We suppose that is not true. Thus there exist ε such that for all $k \geq 0$ there exist $x_k \in \Lambda$ and $n_k > k$ such that

$$f^{-n_k}(W^{cu}_{r_0}(x_k)) \not\subseteq W^{cu}_{\varepsilon}(f^{-n_k}(x_k)) \subset W^{cu}_{r_1}(f^{-n_k}(x_k)).$$

We take each n_k minimal with this property. Let us take $y_k = f^{-n_k}(x_k)$ and take z_k some point in the following intersection

$$f^{-n_k}(W^{cu}_{r_k}(x_k)) \cap \overline{W^{cu}_{r_1}(y_k) \setminus W^{cu}_{\varepsilon}(y_k)}.$$

Note that in particular $\operatorname{dist}(y_k, z_k) < c$.

Also we take y_0 and z_0 such that $z_k \to z_0$ and $y_k \to y_0$. By construction (and C^1 continuity of the cu-leaves) we have that $z_0 \in \overline{W^{cu}_{r_1}(y_0) \setminus W^{cu}_{\varepsilon}(y_0)}$ and $\operatorname{dist}(y_0, z_0) \leq c$.

We assert that

$$\operatorname{dist}(f^n(y_0), f^n(z_0)) \leq c$$

for each $n \geq 1$, and since c the expansiveness constant, we have a contradiction with the hypothesis of expansiveness in the cu-leaves. Then to conclude the proof, is only necessary to prove the previous assertion.

By contradiction, and arguing as in the previous proposition, if we assume that there exist n such that $\operatorname{dist}(f^n(y_0), f^n(z_0)) > c > \varepsilon$,there exist $k \gg 1$ such that $n_k > n$ and satisfies $\operatorname{dist}(f^n(z_k), f^n(y_k)) > \varepsilon$. Thus

$$f^{n-n_k}(W^{cu}_{r_k}(x_k)) \not\subseteq W^{cu}_{r_1}(f^{n-n_k}(x_k)),$$

that contradict the minimality of n_k. ■

2.2.4 Proof of Theorem 2.1.5

Proof. It is clear that if F is unstable, then the cu-leaves are unstable manifolds, so are dynamically defined.

For the reciprocal, we suppose that cu-leaves are dynamically defined. It follows from the Theorem 2.1.3, that the cu-leaves are holomorphic manifolds, then the map ϕ^{cu} given by Theorem 2.1.1, satisfy $\phi^{cu} : \Lambda \to Emb_{Hol}(\mathbb{D}, \mathbb{C}^2)$.

On the other hand, from continuity of ϕ^{cu} and compactness of Λ, there exist a constant $c > 1$ such that $c^{-1} < \|(\phi^{cu}(x))'(0)\| < c$ for all $x \in \Lambda$. Let

$$e_x = \frac{(\phi^{cu}(x))'(0)}{\|(\phi^{cu}(x))'(0)\|},$$

and write

$$(\phi^{cu}(x))'(0) = \alpha_x e_x.$$

It follows that the maps $x \mapsto E_x$ and $x \mapsto \alpha_x$ varies continuously in Λ and that $c^{-1} < |\alpha_x| < c$.

Working with f^n for n great instead of f if for necessary, we can assume that

$$f^{-1}(W^{cu}_1(x)) \subset W^{cu}_{1/2}(f^{-1}(x)). \tag{2.7}$$

For an element $v \in F(x)$ with $v = \alpha e_x$ we write

$$Df^{-1}(x)(v) = \alpha \cdot Df^{-1}(x)(e_x) = \alpha \cdot t_x \cdot e_{f^{-1}(x)}.$$

For each $x \in \Lambda$, we define the holomorphic map $f_x : \mathbb{D} \to \mathbb{D}$ given by

$$f_x(z) = (\phi^{cu}(f^{-1}(x)))^{-1} \circ f^{-1} \circ \phi^{cu}_x(x),$$

then by construction the maps $x \mapsto f_x$ is continuous.

Now, from the equation (2.7), $f_x(\mathbb{D}) \subset \{|z| < 1/2\}$ and $f_x(0) = 0$, and applying Schwarz Lemma we conclude that $|f_x'(0)| = \lambda_x < 1/2$.

Now note that

$$\lambda_x = |\alpha^{-1}_{f^{-1}(x)} \cdot t_x \cdot \alpha_x|,$$

thus for any $n \geq 1$ we have that

$$\lambda_{f^{-(n-1)}(x)} \cdot \ldots \cdot \lambda_x = |\alpha^{-1}_{f^{-1}(x)} \cdot t_{f^{-(n-1)}(x)} \cdot \ldots \cdot t_x \cdot \alpha_x| < (1/2)^n,$$

and it follows that

$$\|Df^{-n}(x)|F(x)\| = |t_{f^{-(n-1)}(x)} \cdot \ldots \cdot t_x| \leq |\alpha_{f^{-1}(x)}| \cdot |\alpha_x^{-1}|(1/2)^n < c^2(1/2)^n,$$

we conclude that the direction F is an unstable direction. ∎

2.2.5 Proof of Theorem A.

Proof. We consider the dominated splitting as $T_\Lambda\mathbb{C}^2 = E \oplus F$. From Lemma 1.1, and working with f^{n_0} instead f if necessary, there exist $1 > \rho > 0$ such that f is ρ-pseudo hyperbolic. Thus J has a foliation local stable and local center unstable.

On the other hand, since that f is cu-forward expansiveness, Theorem 2.1.4 implies that the cu-leaves are dynamically defined; and Theorem 2.1.5 implies that the direction F is an unstable direction, that is follows the hyperbolicity. ■

2.2.6 Proof of Proposition 2.1.1

Proof. The statement of the Lemma is true for saddle periodic points. In fact, for a saddle periodic point p, we have that $W^u(p)$ is a copy of $\mathbb{C}$, and is dense in J^- (see Property 1.1.3 in the previous Chapter, item 3 and 4). Also, we have that $J^- \cap U^+ \neq \emptyset$. This is follows from the Property 1.1.1 in the previous Chapter (see also Figure 1, in page 4).

Thus, to proof this Lemma, we assert that the stable manifold of p intersect any local center unstable disk. This it follows from the fact that $J^* = H(p)$ is a homoclinic class of any periodic saddle point, and that there is a uniformly contractive sub-bundle, i.e., the direction E (see Lemma 3).

Let p_k be a sequence of periodic saddle points, $p_k \to x \in J^*$. From the continuity of the splitting, it follows that for k great enough $W^s_{loc}(p_k) \cap W^{cu}_{loc}(x) \neq \emptyset$. Since that $p_k \in H(p)$, each $W^s_{loc}(p_k)$ is approximated by disc contained in $W^s(p)$. More precisely, there exist a disc $D_k \subset W^s(p)$ such that $\mathrm{dist}_1(D_k, W^s_{loc}(p_k)) < 1/k$, where dist_1 is the metric of the C^1 topology. It follows that for k great enough, $D_k \cap W^{cu}_{loc}(x) \neq \emptyset$ thus $W^s(p)$ intersect to $W^{cu}_{loc}(x)$.

Now since that $W^u(p) \cap U^+ \neq \emptyset$ and U^+ is open, backward iterates of U^+ accumulates on any compact part of $W^s(p)$, and imply that backward iterates of U^+ intersects $W^{cu}_{loc}(x)$. ■

2.3 Zero Lyapunov Exponent Measures

In this section, we assume that f is a dissipative polynomial diffeomorphisms in $\mathbb{C}^2$, with $|\det(Df)| = b < 1$.

In what follows, ν is a f-invariant measure whose support is contained in J. Also, we denote by $\mathcal{R}(\nu)$, the set of all regular point in $\mathrm{supp}\,(\nu)$. By the classical Oseledets

Theorem, we know that $\nu(\mathcal{R}(\nu)) = 1$. Let $x \in J$ be a regular point and let $\lambda^-(x) \leq \lambda^+(x)$ its Lyapunov exponents, then they are related with a splitting E_x^- and E_x^+ respectively. Since J has no attracting periodic points, from the equation $\lambda^-(x) + \lambda^+(x) = \log(b)$ it follows that $\lambda^-(x) \leq \log(b) < 0 \leq \lambda^+(x)$.

Definition 2.3.1. *We say that a measure ν:*

1. *is hyperbolic, if $\lambda^+(x) > 0$ for ν-a.e.,*
2. *has a zero exponent, if $\lambda^+(x) = 0$ for ν-a.e.,*

Give ν a measure, we denote by $\mathcal{R}^+(\nu)$ (resp. $\mathcal{R}^0(\nu)$), the set of all regular points, that has the maximal exponent positive (resp. null). It is clear that $\mathcal{R}(\nu) = \mathcal{R}^+(\nu) \sqcup \mathcal{R}^0(\nu)$, where $\sqcup$ is a disjoint union. It is easy to see from the definition that ν is hyperbolic (resp. be a zero exponent) if and only if $\nu(\mathcal{R}^+(\nu)) = 1$ (resp. $\nu(\mathcal{R}^0(\nu)) = 1$). A measure, is not of the above types if and only if $\nu(\mathcal{R}^+(\nu))$, $\nu(\mathcal{R}^0(\nu)) > 0$. We recall that $\operatorname{supp}(\nu) = \overline{\mathcal{R}(\nu)}(\text{mod}0) = \mathcal{R}(\nu)(\text{mod}0)$.

We can write every measure ν, as a direct sum of the form $\nu = \nu^+ \oplus \nu^0$, where $\nu^+ = \nu|_{\mathcal{R}^+(\nu)}$ is hyperbolic and $\nu^0 = \nu|_{\mathcal{R}^0(\nu)}$ is has a zero exponent. Naturally $\nu^0 \equiv 0$ when ν is hyperbolic, and $\nu^+ \equiv 0$ when ν has a zero exponent.

Remark 10. *It is important to recall that, for a measure that is neither hyperbolic nor has zero exponent, the supports* $\operatorname{supp}(\nu^0) = \mathcal{R}^0(\nu)(\text{mod}0)$ *and* $\operatorname{supp}(\nu^+) = \mathcal{R}^+(\nu)(\text{mod}0)$ *can intersect, but this intersection has measure zero both for ν^0 and for ν^+.*

We define the set support of J, as the set

$$\operatorname{supp}(J) = \overline{\cup\{\operatorname{supp}(\nu) \ : \ \nu \ \text{is} \ f\text{-invariant}\}}.$$

This set can by defined in more general context, namely, linear cocycles (see Chapter 3 for details), and play an important role in the proof of Theorem D in this chapter.

In the paper [BLS1], the authors proof that the set $J^* = \operatorname{supp}(\mu)$, where μ is the unique measure of maximal entropy $\log(\deg(f))$, and that any hyperbolic measure has support contained in J^*. Then we have that

$$J^* = \overline{\cup\{\operatorname{supp}(\nu) \ : \ \nu \ \text{is hyperbolic}\}}.$$

Also we define the set

$$J_0 = \overline{\cup\{\operatorname{supp}(\nu) \ : \ \nu \ \text{is has a zero exponent}\}}.$$

Note that by definition, J_0 is a compact f-invariant set.

Proposition 2.3.1. *The equality* $\operatorname{supp}(J) = J^* \cup J_0$ *holds.*

Proof. It is clear that $J^* \cup J_0 \subset \operatorname{supp}(J)$. On the other hand, Let $x_n \to x \in \operatorname{supp}(J)$ with $x_n \in \operatorname{supp}(\nu_n)$. Writing $\nu_n = \nu_n^+ \oplus \nu_n^0$, we have that there is an infinity times n such that either $x_n \in \operatorname{supp}(\nu_n^+)$ or $x_n \in \operatorname{supp}(\nu_n^0)$, and we can take a subsequence converging to x. This conclude the proof. ∎

2.4 Equivalence to Hyperbolicity in the Dominated Case

The main goal of this section is to proof the following Theorem.

Theorem B. *Let f be a dissipative complex Hénon map, with dominated splitting in J. The following statement are equivalents:*

1. *J is hyperbolic,*
2. *$J_0 = \emptyset$,*
3. *The set of periodic (saddle) point are uniformly hyperbolic.*
4. *The set of periodic (saddle) point are uniformly expanding at the period.*

An immediate Corollary from the Theorem above is the following.

Corollary of Theorem B. *Let f be a dissipative complex Hénon map, with dominated splitting in J. Then J is hyperbolic if and only if all measure f-invariant supported in J is hyperbolic.*

In the next subsection, we shall prove this result. The proof will be supported essentially in the Fornæss Theorem (see [F]), and the Theorem 2.4.2.

Theorem 2.4.1 (Fornæss). *Let f be a complex Hénon map which is hyperbolic in J^*. If f is not volume preserving, then $J^* = J$.*

This implies that is sufficient to see hyperbolicity of the J^*. This allows enunciate the following result.

Theorem 2.4.2. *Let f be a complex Hénon map, dissipative with dominated splitting in J^*. Then we have the following dichotomy:*

i. *The set J^* is hyperbolic.*

ii. $J^* \cap J_0 \neq \emptyset$.

In the next section, we shall prove this result, as a corollary of the Theorem 2.1 of the celebrated work of R. Mañé "A proof of the C^1 Stability Conjecture". This Theorem can be also proved independently of the Mañé work, and we present this proof in the Chapter 3, subsection 3.5. With this another proof, we can conclude that in the context of Hénon map, Mañé Theorem and Theorem 2.4.2 are equivalent.

As a corollary of the previous Theorem, we have.

Corollary 2.4.3. *The set* $J_0 \neq \emptyset$ *if and only if* $J_0 \cap J^* \neq \emptyset$.

Proof. If $J_0 \cap J^* = \emptyset$, then J^* is hyperbolic. Thus from Fornæss Theorem, J is hyperbolic and $J_0 = \emptyset$. ■

We also have the following question.

Question 2. *Let f be a dissipative Hénon map with dominated splitting in J. If f is also expansive in J, is the set J_0 empty?* An affirmative answer for this question, allows to enunciate the following statement: *If f is expansive in J, the any measure invariant supported them, is hyperbolic.*

Question 3. *Let f be a dissipative Hénon map with dominated splitting in J. If the set J_0 is not empty, there exist some condition under which $J_0 \setminus J^* \neq \emptyset$?* An affirmative answer, give examples for which $J^* \subsetneq J$.

2.4.1 Proof of Theorem B

First we define some basically notions. Let $\mathcal{P}er$ the set of all periodic point contained in J. From [BS1] any periodic saddle point p of f is on $\mathcal{P}er$, and $J^* = \overline{\mathcal{P}er}$.

We recall that from Lemma 1.1, the dominated direction E in each periodic point is a stable direction. This justify the following definition.

Definition 2.4.1. 1. *We say that $\mathcal{P}er$ is uniformly hyperbolic if there exist a $C \geq 1$ and $0 < \lambda_1 < 1$ such that for every $n \geq 1$*

$$\|Df^{-n}|_{F(p)}\| \leq C\lambda_1^n,$$

for every $p \in \mathcal{P}er$.

2. *We say that $\mathcal{P}er$ is uniformly expanding at the period, if there exist a $C \geq 1$ and $0 < \lambda_1 < 1$ such that*

$$\|Df^{-\pi(p)}|_{F(p)}\| \leq C\lambda_1^{\pi(p)},$$

for every $p \in \mathcal{P}er$, where $\pi(p)$ is the period of p.

Proof of (1) $\Leftrightarrow$ (2) . From the Theorem 2.4.2, it follows that if J is hyperbolic, then $J_0 \cap J^* = \emptyset$. Thus, from Corollary 2.4.3 it follows that $J_0 = \emptyset$.

The reciprocal direction, is essentially the same: Corollary 2.4.3 say that if $J_0 = \emptyset$, then $J_0 \cap J^* = \emptyset$. The hyperbolicity of J, it follows from the Theorem 2.4.2, and the Fornæss Theorem. ∎

It is clear that **(1)** $\Rightarrow$ **(3)** $\Rightarrow$ **(4)**. Then is only necessary to proof that **(4)** $\Rightarrow$ **(2)** and we conclude the proof of Theorem B. This it follows directly from the following fact.

Proposition 2.4.1. *Let f be a complex Hénon map, dissipative with dominated splitting in J^*. Then we have the following dichotomy:*

i. The set $\mathcal{P}er$ is uniformly expanding at the period.

ii. $J^* \cap J_0 \neq \emptyset$.

Proof. We assume that $J^* \cap J_0 = \emptyset$, and that the set $\mathcal{P}er$ is not uniformly expanding at the period. In this case we can assume that for every $n \geq 1$, there exist a periodic point p_n such that

$$\|Df^{-k\pi(p_n)}|_{F(p_n)}\| < \left(\frac{n-1}{n}\right)^{k\pi(p_n)},$$

for every $k \geq 1$. Thus we have

$$\log\left(\frac{n}{n-1}\right) > \frac{1}{k\pi(p_n)} \log\left(\|Df^{k\pi(p_n)}|_{F(p_n)}\|\right). \tag{2.8}$$

Since that $\lambda^+(p_n) > 0$, we can find k_n great enough such that

$$\frac{1}{k_n\pi(p_n)} \log\left(\|Df^{k_n\pi(p_n)}|_{F(p_n)}\|\right) > \frac{1}{n}. \tag{2.9}$$

Now we define

$$\nu_n = \frac{1}{k_n\pi(p_n)} \sum_{j=1}^{k_n\pi(p_n)} \delta_{f^j(p_n)},$$

be a sequence of f-invariant measures that, taking a subsequence if necessary, we can assume that $\nu_n \to \nu$. It follows from the inequalities (2.8) and (2.9) that

$$\int \log \|Df|_F\| d\nu = \lim_{n\to\infty} \int \log \|Df|_F\| d\nu_n = 0,$$

that is a contradiction with $J_0 = \emptyset$. ∎

Recently Christian Bonatti, Shaobo Gan and Dawei Yang, have proven an more general case of the previous proposition and that contain this (see [BGY] for instance). In the work of Bonatti *Et al.*, an important hypothesis in the proof is that his compact invariant set is a homoclinic class, and these is the case of J^*; but we don't use this fact in the previous proof, however homoclinic class is a hypothesis used in the proof of Fornæss Theorem. We conclude this section with the statement of Theorem of Bonatti Et al..

Theorem 2.4.4 (**Bonatti-Gan-Yang).** *Let p be a hyperbolic periodic point of a diffeomorphism f on a compact manifold M. Assume that its homoclinic class $H(p)$ admits a (homogeneous) dominated splitting $T_{H(p)}M = E \oplus F$ with E contracting and* $\dim(E) = ind(p)$.

If f is uniformly F-expanding at the period on the set of periodic points q homoclinically related to p, then F is uniformly expanding on $H(p)$.

2.5 Proof of Theorem 2.4.2

First one, we present the Theorem 2.1 due to Mañé in [Ma1]. Let f be a diffeomorphisms of C^1 class in a Riemannian manifold M of any dimension, and Λ be a compact invariant by f. A dominated splitting $T\Lambda = E \oplus F$ is say *homogeneous* if the dimension of the subspace $E(x)$ is constant for every $x \in \Lambda$. We say that a compact neighborhood U of Λ is *admissible* if the set $M(f, U) = \cap_{n\in\mathbb{Z}} f^n(U)$ has one and exactly one homogeneous dominated splitting $TM(f, U) = \widehat{E} \oplus \widehat{F}$ extending the splitting $T\Lambda = E \oplus F$. It is known, that if $T\Lambda$ has a homogeneous dominated splitting, then Λ has an admissible neighborhood U (see [HPS] for instance).

Theorem 2.5.1. *Let Λ be a compact invariant set of $f \in Diff^1(M)$ such that $\Omega(f|_\Lambda) = \Lambda$, let $T_\Lambda M = E \oplus F$ be a homogeneous dominated splitting such that E is contracting and suppose $c > 0$ is such that the inequality*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|(Df)^{-1}|_{F(f^j(x))}\| < -c \tag{2.10}$$

holds for a dense set of points $x \in \Lambda$. Then either F is expanding (and therefore Λ is hyperbolic) or for every admissible neighborhood V of Λ and every $0 < \gamma < 1$ there exists a periodic point $p \in M(f, V)$ with arbitrarily large period N and satisfying

$$\gamma^N \leq \prod_{j=1}^{N} \|(Df)^{-1}|_{\widehat{F}(f^j(p))}\| < 1$$

where $\widehat{F}$ *is given be the unique homogeneous dominated splitting* $TM(f,V) = \widehat{E} \oplus \widehat{F}$ *that extend* $T\Lambda = E \oplus F$.

In terms of the hypothesis of the Mañé Theorem, is clear that are satisfied: f of C^1 class and homogeneous dominated splitting. From item (2) above, is satisfied the inequality (2.10) with $c = \log(d)$ where d is the degree of the map f (see Properties 1.1.2 in the previous chapter). Also we remark that any periodic point in $M(f,V)$ for some V a admissible neighborhood of J^*, is in fact an element of $\mathcal{P}er$.

Proof of Theorem 2.4.2. By the Mañé Theorem, if J^* is not hyperbolic then, in particular, for every $n > 0$ there exist a periodic point p_n of period $N(n) \geq n$ such that

$$\log\left(\frac{n-1}{n}\right) \leq \frac{1}{N(n)} \log \|(Df)^{-N(n)}|_{F(p_n)}\| < 0.$$

To end the proof, proceed in the same way as in the Proposition 2.4.1. ■

Chapter 3

Critical Points for Projective Cocycle

Introduction

The main purpose of this chapter is to study the dynamical obstruction to dominated splitting for two dimensional holomorphic systems.

In the real and complex one-dimensional context, this phenomena is already known: for one-dimensional endomorphism, the presence of critical points (points with zero derivative) in the Limit set is an obstruction to hyperbolicity. In the one-dimensional real case, Mañé showed that smooth and generic (Kupka-Smale) one-dimensional endomorphisms without critical points are either hyperbolic or conjugate to an irrational rotation (see [Ma2]). So we could say that, for generic smooth one-dimensional endomorphisms, any compact invariant set is hyperbolic if, and only if, it does not contain critical points. In the one-dimensional complex case (more precisely for rational maps), the Julia set J is hyperbolic if, and only if, J is disjoint of the post critical set (see for example, [MNTU]).

On the other hand, E. Pujals and F. Rodriguez Hertz in their work [P-RH], introduce a notion of critical point for surfaces maps and they show that under the hypothesis of "dissipation":

Main Theorem. [Pujals-Rodriguez Hertz] *For a generic C^2 system in a compact surface, a systems has dominated splitting, if and only if the set of critical point is empty.*

From Theorem B of Pujals-Sambarino in [P-S], the authors of [P-RH] conclude that (generically) a invariant set (under certain hypothesis) is either a hyperbolic set or a invariant closed curve that is normally hyperbolic and on which the dynamics is conjugate to an irrational rotation if and only if the set of critical points is empty. Also we remark that in [P-RH], the authors perform the proof of their main result, using the

Theorem B on [P-S].

Later, Silvan Crovisier in [Cr], give a proof of the Main Theorem on [P-RH], independent of the Pujals-Sambarino's Theorem.

In this Chapter, we generalize the notion of critical point for two dimensional holomorphics maps. In few words, given a dissipative compact invariant set without attracting periodic points, the main result (Theorem D) establish that: *One systems of this nature have dominated splitting if and only if there are not critical points*.

Roughly speaking, it is said that a point z is a critical point, if this has a direction, a one dimensional complex space, that in the projective action, is expansive for backward/backward iterates. The notion of critical point that we define in this work, is slightly different to the original given in [P-RH]. In general, critical point are uniquely defined along the orbit, and the set of all critical points (critical set) is compact and far of dominated set and hyperbolic blocks. With the new definition, the critical set are also invariant by metric and are persistent by conjugation close to the identity. To justify the notion of critical points, we refer to section 3.4 where a set of properties of the critical points are listed and proved.

Notice that, in the two dimensional holomorphics context, in principle, there are not a Pujals-Sambarino Theorem. Thus to prove the Theorem D, we adapt the ideas of Crovisier for the two dimensional holomorphics maps.

Another important remark is the following: however that in particular a holomorphic map is a C^2 map, this is not an important point in the proof of theorem, it is only necessary have continuous variation in the derivative of the map. It is not even necessary to be holomorphic.

To show the independence of the class of differentiability, we prove this result in a more general context: namely complex linear cocycles acting in a complex fiber bundle of dimension two. This is possible because domination and the notion of critical point (introduce in the section 1.7) defined for a differentiable map f, are intrinsic properties of the projective dynamics induced by the cocycle (f, Df); in particular, we can define them in general cocycles acting in a vectorial bundle.

This chapter is organized as follows:

In the section 3.1-3.5 we introduce the basic definition and preliminaries to define critical point. In the section 3.6-3.8 we introduce the notion of critical point and demonstrate the Theorem D.

More precisely:

In the subsections 3.1.1 and 3.1.2, we present the notions of multiplier (and the norm of the multiplier) for a holomorphic function on the Riemann sphere, and we calculate this for the case of Möbius transformation.

In the subsection 3.1.3, we recall the notions of linear and projective bundle and present the natural system acting in there, namely, linear and projective cocycles.

In the subsection 3.1.4, relate the linear and projective bundle with the Riemannian and Spherical metric defined there.

In the subsection 3.1.5, present the notion of dominated splitting for a linear cocycle.

In the subsections 3.1.6 to 3.1.8, is shown that linear cocycles has a related projective cocycle. Moreover, we introduce the notion of *"hyperbolicity"* for this projective cocycle and it is proved that: *A linear cocycle has dominated splitting if and only if its related projective cocycle is hyperbolic.* Also we prove that hyperbolicity is equivalent with some more weak condition in the projective cocycle.

In the section 3.2, we recall some technical considerations, in particular related with the projective cocycles and the norm of the multiplier.

In the section 3.3, we define the notions of critical point and it is defined the sets called *"block of domination"*, which exhibits some kind of asymptotic dominations.

In the subsection 3.3.1 it is proved the Theorem C, that establish that this blocks of domination always exists. This it follows from a easy observation of Oseledets theorem.

In the subsection 3.3.2, we describe two criteria for obtaining domination.

In the subsection 3.3.3, based in the ideas of Sylvain Crovisier (see [Cr]) for the proof of the Theorem B in [P-RH], we prove the Theorem D.

In the section 3.4, we describe a series of properties of critical points.

3.1 Dominated Splitting and Hyperbolic Projective Cocycle

This section is devote to show the main elements used to introduce the notion of critical point for linear cocycles.

3.1.1 Multiplier

In the studies of rational function in the Riemann sphere, an important tool to describe the dynamics near fixed (or periodic) points is the notion of multiplier. By theBöcher's Theorem, the dynamics in a neighborhood of a fixed point is, via conjugation, given by the dynamics of the map $w \mapsto \lambda w$, where λ is called the multiplier. Based on this approaches, the main goal of this subsection is to introduce the notion of the multiplier for any point (not necessarily for fixed or periodic points), that in the particular case of Möbius transformation determine the behavior of the dynamics. First we remember the definition of the multiplier for fixed points of rational map.

Definition 3.1.1. *Let R be a rational function in the Riemann sphere $\overline{\mathbb{C}}$, and let $z \in \overline{\mathbb{C}}$ be a fixed point for R.*

i) If $z \in \mathbb{C}$ we define the multiplier of R in the point z by $R'(z)$, and is denoted by $\lambda(z, R)$.

ii) If $z = \infty$ we choose a Möbius map f such that $f(\infty) \in \mathbb{C}$, and it is defined $\lambda(\infty, R) = \lambda(f(\infty), f \circ R \circ f^{-1})$.

Note that in the previous Definition, the value of $\lambda(z, R)$ when $z \in \mathbb{C}$ is invariant under conjugation. It follows that $\lambda(\infty, R)$ is a well-defined.

Now we may define the norm of the multiplier for a rational map R in any $z \in \overline{\mathbb{C}}$. In what follows, $\mathrm{Isom}(\overline{\mathbb{C}})$ denote the set of all isomorphisms in the Riemann Sphere with the spherical metric.

Definition 3.1.2. *Let $U \subset \overline{\mathbb{C}}$ be an open set and $R : U \to \overline{\mathbb{C}}$ be an holomorphic map. Let $z \in U$ and $f_z, h_z \in \mathrm{Isom}(\overline{\mathbb{C}})$ such that $f_z(z) = h_z(R(z)) = 0$. Then the "norm of the multiplier" of R in a point z is defined as*

$$g(z, R) = |\lambda(0, h_z \circ R \circ f_z^{-1})| = |(h_z \circ R \circ f_z^{-1})'(0)|. \tag{3.1}$$

Observe that the Definition above, is invariant under conjugation by isomorphisms.

Proposition 3.1.1. *Let $D \subset \overline{\mathbb{C}}$ be a topological disc and $R : D \to \overline{\mathbb{C}}$ be an holomorphic map. Let $z \in D \mapsto f_{i,z}, h_{i,z} \in \mathrm{Isom}(\overline{\mathbb{C}})$, with $i = 1, 2$, be continuous correspondence such that $f_{i,z}(z) = h_{i,z}(R(z)) = 0$. If we define $F_{1,z}(w) = h_{1,z} \circ R \circ f_{1,z}^{-1}(w)$ and $F_{2,z}(w) = h_{2,z} \circ R \circ f_{2,z}^{-1}(w)$ in some neighborhood of zero, then there exists a unique continuous function $\xi : D \to \mathbb{S}^1$ such that*

$$F'_{1,z}(0) = \xi(z) F'_{2,z}(0). \tag{3.2}$$

Proof. If we write $f_{i,z} = T_{a_i(z), b_i(z)}$ and $h_{i,z} = T_{c_i(z), d_i(z)}$, then $f_{1,z} \circ f_{2,z}^{-1}(w) = \xi_1(z) \cdot w$ and $h_{2,z} \circ h_{1,z}^{-1}(w) = \xi_2(z) \cdot w$ where $\xi_1(z) = \zeta_1(z) \cdot (\overline{\zeta}_1(z))^{-1}$, $\xi_2(z) = \zeta_2(z) \cdot (\overline{\zeta}_2(z))^{-1}$, $\zeta_1(z) = a_1(z)\overline{a}_2(z) + \overline{b}_1(z) b_2(z)$ and $\zeta_2(z) = c_2(z)\overline{c}_1(z) + \overline{d}_2(z) d_1(z)$. Hence taking $\xi(z) = (\xi_1(z))^{-1} \cdot \xi_2(z)$ is proved the thesis. ∎

The previous Proposition assert that the definition of g given in the Definition 3.1.2, is independent of the isometries f_z and h_z considered to calculate it, that is, g is well defined.

We would like give three remark to justify Definition 3.1.2. The reader can skip this part.

1. *The derivative in the point does not give good information for the behavior of the dynamics.* For example the map $M(w) = 2w$ has infinity as attracting fixed point with multiplier $1/2$, but points near of infinity has derivative equal two. So this is not a well definition because in the context of surfaces, we expect that the "derivative" be a continuous function.

2. *The multiplier provide good information in the case of fixed points,* so we need transform any point in a fixed point by Möbius transformations; but we can not work with all transformations in this process, because we can lose information about the local behavior in this point. For example, for the map $M(z) = z$, we can take $f(z) = z - 1$ and $h(z) = 3z - 3$ for calculate a multiplier in the point $z = 1$; then the map $F(z) = h \circ M \circ f^{-1}$ has zero as fixed point and his multiplier is equal to $F'(0) = 3$. However, M not expand any neighborhood of $z = 1$. To avoid this inconvenience, we only work with spherical isometries.

3. *Bad definition:* Restricting the conjugations to spherical isometries, it is natural to defines the multiplier as value $F'_{1,z}(0)$ given in the equation (3.2). But precisely this equation say that this value depends of the conjugation considered. For this reason, we take the norm of this value.

Remember that a Möbius transformation T is an isometry in the Riemann sphere whit the spherical metric, if and only if T can be written in the form

$$T(w) = T_{a,b}(w) = \frac{\overline{b}w + a}{-\overline{a}w + b},$$

with a and b complex number and $|a|^2 + |b|^2 = 1$. Observe that if we write $z = a/b$ (and $z = \infty$ if $b = 0$) then $T(0) = z$ and since T is an isometry, $T(\infty) = z^* = -1/\overline{z}$, that is the antipodal point of z in the Riemann sphere.

We recall that the norm of the multiplier g, is given by $g(z, R) = |(h_z \circ R \circ f_z^{-1})'(0)|$, as in the Definition 3.1.2.

Proposition 3.1.2. *The definition of g is half of the norm of the derivative of M in the spherical metric.*

Proof. The spherical norm of a vector $v \in \mathbb{C}$ at a finite point z is given by

$$\|v\|_z = \frac{2}{1+|z|^2} \cdot |v|,$$

then if R is a rational function and z is such that $R(z)$ is finite, we can take f_z and h_z isometries as in the Proposition 3.1.1 and obtain

$$\|R'(z)\|_{R(z)} = \|(h_z \circ R \circ f_z^{-1})'(0)\|_0 = 2g(z,R).$$

■

3.1.2 Calculating the function g

To calculate the function g for a rational map R we can the choice isometries as the form:

$$\text{if } \; w \in \mathbb{C} \mapsto \; f_w(z) = \frac{z+w}{-\overline{w}z+1}, \; \text{ if } \; w \in \overline{\mathbb{C}} \setminus \{0\} \mapsto \; f_w(z) = \frac{-\overline{w}^{-1}z+1}{z+w^{-1}}.$$

This allows define $g(z) := g(z,R)$ for any $z \in \overline{\mathbb{C}}$ as a continuous function.

Lemma 3.1. *Let R be a rational map in the Riemann sphere. Then the norm of the multiplier is given by*

$$g(z) = g(z,R) = |R'(z)| \cdot \frac{1+|z|^2}{1+|R(z)|^2}. \tag{3.3}$$

In particular, for a Möbius transformations

$$M(\xi) = \frac{a\xi+b}{c\xi+d},$$

we have

$$g(z) = g(z,R) = \frac{|\delta|}{\|Av_z\|^2}, \tag{3.4}$$

where A is the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$\delta = \det(A)$ *and v_z is a unitary vector in $\mathbb{C}^2$ such that* $[v_z] = z$.

Proof. Let $z \notin \{\infty, \text{pole}\}$ let $\xi = R(z)$. Define $F_z(w) = f_\xi^{-1} \circ R \circ f_z(w)$, and we will calculate $|F_z'(0)| = |(f_\xi^{-1})'(\xi)| \cdot |R'(z)| \cdot |f_z'(0)|$. Notice that

$$f_z(w) = \frac{w+z}{-\bar{z}w+1}, \;\; \Rightarrow f_z'(w) = \frac{1+|z|^2}{(-\bar{z}w+1)^2}$$

and

$$f_{\xi}^{-1}(w) = \frac{w-\xi}{\bar{\xi}w+1}, \quad \Rightarrow (f_{\xi}^{-1})'(w) = \frac{1+|\xi|^2}{(\bar{\xi}w+1)^2}.$$

hence, an easy calculus implies that

$$|F_z'(0)| = |(f_{\xi}^{-1})'(\xi)| \cdot |R'(z)| \cdot |f_z'(0)| = |R'(z)| \cdot \frac{1+|z|^2}{1+|R(z)|^2}.$$

Now, for a Möbious transformation, we have

$$|F_z'(0)| = |M'(z)| \cdot \frac{1+|z|^2}{1+|M(z)|^2} = \frac{|\delta|}{|cz+d|^2} \frac{1+|z|^2}{1+|M(z)|^2},$$

where $\delta = ad - bc$. To end, denote as A the linear map in $\mathbb{C}^2$ that define the Möbius transformation M and δ its determinant, also take $v_z = (v_z^1, v_z^2) \in \mathbb{C}^2$ unitary vector such that $z = v_z^1/v_z^2$. Then we have that

$$\begin{aligned}
g(z, M) &= |F_z'(0)| \\
&= |\delta| \cdot \frac{1+|z|^2}{|az+b|^2 + |cz+d|^2} \\
&= |\delta| \cdot \frac{1+\left|\frac{v_z^1}{v_z^2}\right|^2}{\left|a\frac{v_z^1}{v_z^2}+b\right|^2 + \left|c\frac{v_z^1}{v_z^2}+d\right|^2} \\
&= |\delta| \cdot \frac{|v_z^1|^2 + |v_z^2|^2}{|av_z^1 + bv_z^2|^2 + |cv_z^1 + dv_z^2|^2} \\
&= \frac{|\delta|}{||Av_z||^2}.
\end{aligned}$$

■

Remark 11. *For convenience, we will call the value* $F_z'(0)$ *the multiplier of* M *in the point* z *even knowing that this depends on the conjugation involved in its definition.*

3.1.3 Bundles and Natural Cocycles

This subsection is devoted to introduce the notion of vector and projective bundles. We consider linear cocycles acting in the two dimensional complex linear bundle and we relate it with a projective cocycle that describe the asymptotic behavior of the dynamics in the complex lines (or complex directions). Also we define a spherical metric in the projective bundle.

In this subsection, the set X denote a compact Hausdorff space.

Definition 3.1.3. *A 3-tuple* (TX, p, X) *is a vector bundle of dimension k if:*

1. TX *is a topological space,*

2. $p : TX \to X$ *is a continuous map,*

3. *for every* $z \in X$ *the set* $p^{-1}(z)$ *(that we named this as fiber and denoted by* T_z*) has structure of vector space over a field F and dimension k,*

4. *for every z there exist a pair* (U, φ) *where U is an open subset of X and the map* $\varphi : p^{-1}(U) \to U \times F^k$ *is an homomorphism and an isomorphism of vector space restricted in each fiber.*

Moreover, the set $\{(U, \varphi)\}$ *has the properties of compatibility on the change of chart, that is, if* (U_1, φ_1) *and* (U_2, φ_2) *are such that* $U_1 \cap U_2 \neq \emptyset$ *then the change of chart* $\varphi_2 \circ \varphi_1^{-1} : (U_1 \cap U_2) \times F^k \to (U_1 \cap U_2) \times F^k$ *is an homomorphism and an isomorphisms in each trivial fiber.*

The set $\{(U, \varphi)\}$ is called an atlas of trivialization maps. Since that X is a compact set, we have a good defined hermitian metric in TX (see [H]), that is, if $TX \odot TX$ denote the subset of $TX \times TX$ of pairs (u, v) such that u and v are in the same fiber, then there exist a continuous function $(\cdot|\cdot) : TX \odot TX \to \mathbb{C}$ such that $(\cdot|\cdot)|T_z \times T_z = (\cdot|\cdot)_z$ is an hermitian product in T_z.

We will only work with vector bundles of dimension two. Now we define the projective bundle space.

Definition 3.1.4. *A 3-tuple* $(\mathbb{P}, pr, X)$ *is a projective bundle if:*

1. $\mathbb{P}$ *is a topological space,*

2. $pr : \mathbb{P} \to X$ *is a continuous map,*

3. *for every* $z \in X$ *the set* $pr^{-1}(z)$ *(that we named this as projective fiber and denoted by* $\overline{\mathbb{C}}_z$*) has structure of Riemann surface biholomorphic with the sphere,*

4. *for every z there exist a pair* (U, ϕ) *where U is an open subset of X and the map* $\phi : pr^{-1}(U) \to U \times \overline{\mathbb{C}}$ *is an homomorphism and a biholomorphism restricted in each projective fiber.*

Moreover, the set $\{(U, \phi)\}$ *has the properties of compatibility on the change of chart, that is, if* (U_1, ϕ_1) *and* (U_2, ϕ_2) *are such that* $U_1 \cap U_2 \neq \emptyset$ *then the change of chart* $\phi_2 \circ \phi_1^{-1} : (U_1 \cap U_2) \times \overline{\mathbb{C}} \to (U_1 \cap U_2) \times \overline{\mathbb{C}}$ *is an homomorphism and a biholomorphism in each projective trivial fiber.*

Given a vector bundle we have a natural projective bundle defined. In fact, if $\{(U,\varphi)\}$ is an atlas of trivialization maps of TX we may construct an atlas of trivialization maps of $\mathbb{P}(X) = \cup_{z\in X}\{z\} \times \mathbb{CP}^1(T_z)$ with fiber $\overline{\mathbb{C}}$. In fact, let $\phi : pr^{-1}(U) \to U \times \overline{\mathbb{C}}$ defined by

$$\phi(w,[v]) = (w,[\pi_2(\varphi(v))]),$$

where $[\cdot]$ denote the equivalence class in the respective projective space, pr is the projection in the first coordinate and $\pi_2 : U_z \times \mathbb{C}^2 \to \mathbb{C}^2$ is the projection in the second coordinate.

3.1.4 Spherical Metric

Definition 3.1.5. *The spherical metric considered in the fiber projective bundle $\mathbb{P}(X)$, is induced by considering in each projective fiber $\overline{\mathbb{C}}_z$, the natural spherical metric defined in them. This varies continuously in the projective bundle.*

To explain with more details this definition, first we observe some basic fact of the spherical metric defined in $\overline{\mathbb{C}}$.

Consider the Riemann sphere as the Projective space of all 1-dimensional space in $\mathbb{C}^2$, and write this in homogeneous coordinates as

$$\overline{\mathbb{C}} = \{[z_1 : z_2] \,:\, (z_1,z_2) \in \mathbb{C}^2\}.$$

Also identify $z \in \mathbb{C}$ with $[z_1 : z_2] = [z : 1]$ and the point in the infinity with $[z_1 : 0] = [1 : 0]$. With this coordinates the standard spherical metric

$$d\rho = 2\frac{|dz|}{1+|z|^2}$$

has constant Gaussian curvature $+1$. The chordal metric is defined by

$$d(z,w) = \frac{2|z-w|}{[(1+|z|^2)(1+|w|^2)]^{1/2}}, \quad \text{when } z,w \in \mathbb{C},$$

and

$$d(z,\infty) = \frac{2}{(1+|z|^2)^{1/2}}, \quad \text{when } z \in \mathbb{C},$$

and we have the formula

$$d(z,w) = 2\sin\left(\frac{\rho(z,w)}{2}\right) \tag{3.5}$$

which relate the two metrics.

On the other hand, we have a natural homeomorphism h between $\overline{\mathbb{C}}$ and the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ given by

$$[z_1 : z_2] \mapsto (2\mathrm{Re}\,(z_2\bar{z}_1), 2\mathrm{Im}\,(z_2\bar{z}_1), |z_2|^2 - |z_1|^2)/(|z_1|^2 + |z_2|^2). \tag{3.6}$$

Then the chordal metric is precisely

$$d(z, w) = \|h([z_1 : z_2]) - h([w_1 : w_2])\|, \tag{3.7}$$

where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^3$.

All the previous observations are done considering homogeneous coordinates in the canonical base of $\mathbb{C}^2$. So, a natural question arises: What happens with other basin?

If we make the previous constructions with homogeneous coordinates in another orthonormal base, we obtain the same chordal metric and accordingly the same spherical metric. This is follows from the following fact.

Let A the change of basin matrix between the new orthogonal base and the canonical base and M the Möbius transformation related with the matrix A. Then A is an isometry with the hermitian metric in $\mathbb{C}^2$, and M is and isometry with the spherical metric defined in the canonical way. This last observation says that the spherical metric is only related with the Hermitian structure considered in the vectorial space, and not with the coordinate system considered in it.

Now we explain how this fact well define a spherical metric in the projective cocycle. Given $U \subset X$ an open set, we denote by $\Gamma(U, TX)$ the set of all continuous maps υ of U in TX such that $z = pr \circ \upsilon(z)$ and we call this function of *section*. Now let us take $\{U_1, \ldots, U_k\}$ a finite cover of X such that there exists sections $\upsilon_i, \upsilon_i^* \in \Gamma(U_i, TX)$ such that $(\upsilon_i(z)|\upsilon_i^*(z))_z = 0$ and $\|\upsilon_i(z)\|_z = \|\upsilon_i^*(z)\|_z = 1$, then for each $z \in U_i$ we can relate the projective space $\overline{\mathbb{C}}_z$ with the set of homogeneous coordinates in the base $\{\upsilon_i(z), \upsilon_i^*(z)\}$, that is

$$\mathbb{P}(T_z) = \{[z_1 : z_2] \,:\, z_1\upsilon_i(z) + z_2\upsilon_i^*(z) \in T_z\},$$

and we define the chordal metric in $\mathbb{P}(T_z)$ as in the equation (3.7). For points $z \in U_i \cap U_j$ with $i \neq j$ and $\xi \in \mathbb{P}(T_z)$, the homogeneous representation of ξ in the different bases $\{\sigma_i(z), \sigma_i^*(z)\}$ and $\{\sigma_j(z), \sigma_j^*(z)\}$ is not the same, however the metric induced is the same. It is clear that the chordal metric varies continuously with this local representation. This implies that we have a good defined chordal and spherical metric in the projective bundle.

3.1.5 Linear and Projective Cocycle

Now consider $f : X \to X$ an homomorphism. We denote by $\mathbb{GL}(TX, f, \mathbb{C})$ the space of all applications $A : TX \to TX$ such that for every $z \in X$ the map $A_z = A|T_z : T_z \to T_{f(z)}$ is a complex isomorphisms, that is, $A_z \in \mathbb{GL}(T_x, T_{f(z)})$. We also define

$$\|A_z\| = \sup\{\|A_z v\|_{f(z)} \ : \ v \in T_z \ , \|v\|_z = 1\}$$

and the norm of A as

$$\|A\| = \max\{|A|, |A^{-1}|\} \ \text{ where } \ |A| = \sup_{z\in X} \|A_z\|.$$

Definition 3.1.6. *A complex linear cocycle with base f, is a 4-tuple (TX, f, X, A) where X is a topological space, $f \in Hom(X)$, TX is a vector bundle with base X and $A \in \mathbb{GL}(TX, f, \mathbb{C})$ is a continuous map with $\|A\| < \infty$.*

In what follows, we refer as cocycle to the map A and its base f, omitting the other element of the tuple, since this inherent as sets of definitions of this functions.

Remark 12. *Observe that under a change of trivialization chart, the map $z \mapsto A_z$ is a continuous map $z \mapsto U_z$, where $U_z \in \mathbb{GL}(2, \mathbb{C})$.*

Now, we introduce the notion of dominated splitting.

Definition 3.1.7. *We say that a cocycle $A : TX \to TX$ has dominated splitting if there exists an splitting $TX = E \oplus F$, of A-invariant one-dimensional complex planes, and a positive integer l such that for each pair of unitary vectors $u \in E_z$ and $v \in F_{f^l(z)}$ we have*

$$\|A_z^l(u)\| \cdot \|A_{f^l(z)}^{-l}(v)\| < \frac{1}{2},$$

where $A_z^l = A_{f^{l-1}(z)} \circ \cdots \circ A_{f(z)} \circ A_z$ and $A_z^{-l} = A_{f^{-l}(z)}^{-1} \circ \cdots \circ A_{f^{-2}(z)}^{-1} \circ A_{f^{-1}(z)}^{-1}$.

The following classical proposition establishes properties equivalent with the dominated splitting notion.

Proposition 3.1.3. *The following statement are equivalent:*

1. *The cocycle $A : TX \to TX$ has dominated splitting.*

2. *There exist an splitting $TX = E \oplus F$, of A-invariant one-dimensional complex planes, and positive constants $C > 0$ and $0 < \lambda < 1$ such that*

   $$\|A_z^n|_{E_z}\| \cdot \|A_z^{-n}|_{F_{f^n(z)}}\| \leq C\lambda^n,$$

   for any $n > 0$.

3. *There exists an splitting $TX = E \oplus F$, of one-dimensional complex planes and not necessarily A-invariant, such that there exists $l > 0$ and cone fields $K(\alpha, E)$ and $K(\beta, F)$, namely*

$$K(\alpha, E_z) = \{u + v \in E_z \oplus F_z \ : \ \|u\| \leq \alpha \|v\|\}$$

and

$$K(\beta, F_z) = \{u + v \in E_z \oplus F_z \ : \ \|v\| \leq \beta \|u\|\},$$

such that

$$A^{-l}_{f^l(z)}(K(\alpha, E_{f^l(z)})) \subset K(\alpha, E_z)^\circ, \ \ A^l_z(K(\beta, F_z)) \subset K(\beta, F_{f^l(z)})^\circ,$$

where $K^\circ = \mathrm{int}(K) \cup \{0\}$. We say that such of those cones are A-invariant and have the property of l-domination.

Notice that in the condition *(3.)* in the previous Proposition, is possible to get the invariant splitting given in *(1.)* and *(2.)* by the expressions

$$\widetilde{E}_z = \bigcap_{n \geq 0} A^{-n}_{f^n(z)}(K(\alpha, E_{f^n(z)})) \quad \text{and} \quad \widetilde{F}_z = \bigcap_{n \geq 0} A^n_{f^{-n}(z)}(K(\beta, F_{f^{-n}(z)})). \tag{3.8}$$

As in a vector bundle the natural function acting in this space are linear cocycles; the natural function acting in a projective bundle are cocycles that are biholomorphisms in the fibers. More precisely we have the following definition.

Definition 3.1.8. *A projective cocycle with base f, is a continuous map $M : \mathbb{P}(X) \to \mathbb{P}(X)$ with the form $M = (f, M_*)$ where f is an homomorphism in X, and $M_z : \overline{\mathbb{C}}_z \to \overline{\mathbb{C}}_{f(z)}$ is a biholomorphism.*

Remark 13. *Note that given a linear cocycle A we can associate to it a projective cocycle M in a natural fashion as $M_z([v]) = [A_z v]$.*

For every $\xi \in \overline{\mathbb{C}}_z$ we denote by $\lambda(\xi)$ the multiplier of M_z at the point ξ (see Remark 11), and $\lambda^n(\xi)$ the multiplier of $M^n_z = M_{f^{n-1}(z)} \circ \cdots \circ M_{f(z)} \circ M_z$ in the respective point. Also we write $g(\xi) = |\lambda(\xi)|$ and $g^n(\xi) = |\lambda^n(\xi)|$.

Remember that the norm of the multiplier (see Lemma 3.1), is given by the equation (equation (3.4))

$$g(\xi) = \frac{|\det(A_z)|}{\|A_z v_\xi\|^2_{f(z)}},$$

and therefore

$$g^n(\xi) = \frac{|\det(A^n_z)|}{\|A^n_z v_\xi\|^2_{f^n(z)}} \tag{3.9}$$

where v_ξ is choose unitary and such that $[v_\xi] = \xi$.

To end, denote by $TX^* = TX \setminus \{\text{the zero section}\}$ and observe that we have defined the canonical projection $p : TX^* \to \mathbb{P}(X)$. So given A a linear cocycle with base id, p_A denotes the map from TX^* to $\mathbb{P}(X)$ given by $p_A|_{T_z^*} = p \circ A_z$, and we denote by $p_I = p$ the projection defined considering $A_z = Id$ in T_z, for every $z \in X$.

3.1.6 Conjugation of Cocycles

A well known fact about holomorphics maps, is that topological (metrical) contraction of small disc around some point implies that the norm of its derivative is smaller than one and therefore also its multiplier is smaller than one. Since that projective cocycle is holomorphic in each fiber, to determinate if the norm of the multiplier is less to one in some point, it is sufficient to determinate if this contract disc around this point. For that, it is natural to look for more simples cocycles which are conjugated to the initial one, and check if the new cocycle shrinks discs. The formal notion of conjugation is the following definition.

Definition 3.1.9. *Let $M, N : \mathbb{P}(X) \to \mathbb{P}(X)$ be two projective cocycles with $M = (f, M_*)$ and $N = (g, N_*)$. We say that M is conjugate to N if there exists a projective cocycle $H = (h, H_*) : \mathbb{P}(X) \to \mathbb{P}(X)$, where $h : X \to X$ is an homeomorphism such that $H \circ M = N \circ H$.*

The definition above says that we have simultaneously the conjugations $hf(z) = gh(z)$ and $H_{f(z)}M_z(\xi) = N_{h(z)}H_z(\xi)$.

Definition 3.1.10. *We say that a section $\sigma \in \Gamma(X, \mathbb{P}(X))$ is invariant by the projective cocycle M (or M-invariant) if $M(\sigma(z)) = \sigma(f(z))$.*

In that follows, we will work with projective cocycles with an invariant section. Under the hypothesis of the existence of a global section, the following proposition gives global coordinates in the projective space that is very useful further on.

Proposition 3.1.4. *If $\mathbb{P}(X)$ has a global section, then $\mathbb{P}(X)$ is isometrically equivalent to the trivial projective bundle $X \times \overline{\mathbb{C}}$.*

Proof. Let $\sigma \in \Gamma(X, \mathbb{P}(X))$ a global section and E a splitting in TX associated with this direction. Let us take σ^* the global section associated with the direction $E^\perp$, then $\sigma^*(z)$ is the antipodal point of $\sigma(z)$ in the sphere $\overline{\mathbb{C}}_z$.

Claim: *For every $z \in X$ there exists a biholomorphisms $H_z : \overline{\mathbb{C}}_z \to \overline{\mathbb{C}}$ such that is an* *isometry, $H_z(\sigma(z)) = 0$ and $H_z(\sigma^*(z)) = \infty$.*

Proof of Claim. Let $\{(U_i, \varphi_i) : i = 1, \dots, n\}$ a family of trivialization charts such that $X = \cup_i U_i$ and let $v_i \in \Gamma(U_i, TX)$ local sections with $\|v_i\| = 1$ and $v_i(z) \in E$. Let $L_z : T_z \to \mathbb{C}^2$ be the unique linear map such that L_z is an isometry, $T_z(v_i(z)) = (1, 0)$ and $\det(L_z) = 1$. The map L_z is unique because the only element of the group $SU(2, \mathbb{C})$ that fix the vector $(1, 0)$ is the identity map. By uniqueness and by continuity of the section v_i, the map $z \mapsto L_z$ is a continuous correspondence.

Let $v_i^*(z) = L_z^{-1}((0, 1))$, then $v_i^* \in \Gamma(U_i, TX)$ is a local section such that $\|v_i^*\| = 1$ and $v_i^*(z) \in E^\perp$. Define the splitting

$$F_z = \{v \in T_z : (v|v_i - v_i^*)_z = 0\}.$$

It is easy to see that F_z is independent of the choice of the initials section considered. Moreover, F is a continuous splitting. In the projective bundle F defines a global section $\tau \in \Gamma(X, \mathbb{P}(X))$, this is, for any $u \in F_z$ we have $\tau(z) = [u]$.

To end, we define H_z as the unique biholomorphisms such that

- $H_z(\sigma(z)) = 0$,
- $H_z(\sigma^*(z)) = \infty$,
- and $H_z(\tau(z)) = 1$,

or equivalently $H_z([v]) = [L_z(v)]$. This finishes the proof of the claim.

Continuing with the proof of the Proposition, if (U, ϕ) is a local trivialization of $\mathbb{P}(X)$, by continuity of the sections, the local expression in U of H_z is a continuous function. More precisely, there exist a continuous family $\widetilde{H} : U \times \overline{\mathbb{C}} \to U \times \overline{\mathbb{C}}$ with

$$\widetilde{H}_z(w) = \frac{a_z w + b_z}{c_z w + d_z}$$

where the maps $z \mapsto a_z, \dots, z \mapsto d_z$ are continuous function and we have the equality $H_z = \widetilde{H}_z \circ \phi$ and $\widetilde{H}_z \circ \widetilde{\sigma}(z) = 0$, $\widetilde{H}_z \circ \widetilde{\sigma^*}(z) = \infty$, where $\widetilde{\sigma} = \sigma \circ \phi$ and $\widetilde{\sigma^*} = \sigma^* \circ \phi$. It follows that the function $H = (id, H_*)$ is an homeomorphism and an isometry in each fiber. ∎

Remark 14. *After previous proposition we can assume that the bundle $\mathbb{P}(X)$ is in fact the trivial bundle $X \times \overline{\mathbb{C}}$. Moreover, given a section $\sigma \in \Gamma(X, \mathbb{P}(X))$ we can lift this section to the trivial bundle $X \times \mathbb{C}^2$ as a global section $v \in \Gamma(X, X \times \mathbb{C}^2)$ such that $\|v\| = 1$ and if we write $v = (v_1, v_2)$ then $\sigma(z) = (z, [v_1(z) : v_2(z)])$; this helps us to find global expressions of the section in the projective bundle. We call v the unitary lift of σ.*

3.1.7 Projective Hyperbolicity

In this subsection we introduce the notion of hyperbolic projective cocycle, that is, cocycles with two invariant sections that contract/expand asymptotically small disc around them. We prove that a linear cocycle A have dominated splitting if and only if the natural projective cocycle related with them is hyperbolic. Also we prove that the notion of hyperbolic is equivalent with the presence the only one invariant section that have an hyperbolic asymptotic behavior for the past or the future.

Recall that the norm of the multiplier (see Definition 3.1.2) is given by the equation (3.4).

Definition 3.1.11. *We say that a section σ is a contraction for M (or is contractive), if it is M-invariant and there exist constants $C > 0$ and $0 < \lambda < 1$ such that $g^n(\sigma(z)) \leq C\lambda^n$, for any $n \geq 0$ and $z \in X$. In the same way, we say that a section is an expansion (or is expansive) if this is a contraction for the cocycle M^{-1}.*

Definition 3.1.12. *We say that a cocycle M is hyperbolic if there exist two projective sections τ and σ in $\Gamma(X, \mathbb{P}(X))$, that are disjoint (i.e., $\tau(z) \neq \sigma(z)$ for every $z \in X$), and τ is an expansion and σ is a contraction.*

We denote the unit disc in $\mathbb{C}$ by $\mathbb{D}$. For any $\xi \in \overline{\mathbb{C}}$ an any r we denote the ρ-ball with center at ξ and radius r by $B(\xi, r)$; these sets will be called balls. For any $r > 0$ and any Möbius transformation that is an isometry in the spherical metric L with $L(0) = \xi \in \overline{\mathbb{C}}$, it follows that the set $L(r\overline{\mathbb{D}})$ does not depend on L; we will denote this set by $D_r(\xi)$, and it will be called the disc of radius r centered at ξ. Note that fixed r the disc $D_r(\xi)$ is equal to $B(\xi, \varepsilon)$, where ε satisfies the equation

$$\frac{r}{\sqrt{1 + r^2}} = \sin\left(\frac{\varepsilon}{2}\right).$$

The last equation follows by the equation (3.5).

Proposition 3.1.5. *Let $\sigma \in \Gamma(X, \mathbb{P}(X))$ be a M-invariant section. Then the following statement are equivalents:*

i. The section σ is a contraction.

ii. There exist $0 < \eta < 1$ and $k > 0$ such that $g^k(\sigma(z)) < \eta$ for all z in X.

iii. There exist $k > 0$ and $r > 0$ such that $M_z^k(\overline{D}_r(\sigma(z))) \subset D_r(\sigma(f^k(z)))$.

iv. There exist $k > 0$ and $R > 0$ such that for all $0 < r \leq R$, $M_z^k(\overline{D}_r(\sigma(z))) \subset D_r(\sigma(f^k(z)))$.

Proof. It is clear that (*i*) implies (*ii*). To see that (*ii*) implies (*i*), consider for any $j = 0, \dots, k-1$ let

$$C_j = \sup\{g^j(\sigma(z)) \ : \ z \in X\}.$$

Then for any $s \geq 0$

$$g^{sk+j}(\sigma(z)) = g^{sk}(\sigma(f^j(z)))g^j(\sigma(z)) \leq C_j\eta^s = C_j\eta^{-j/k}[\eta^{1/k}]^{sk+j} \leq C\lambda^{sk+j},$$

where $C = \sup\{C_j\eta^{-j/k} \ : \ j = 0, \dots, k-1\}$ and $\lambda = \eta^{1/k} < 1$.

Also it is clear that (*ii*) is equivalent with (*iii*) and that (*iv*) implies (*ii*) and (*iii*).

To prove that (*ii*) implies (*iv*), we consider $v = (v_1, v_2)$ the unitary lift of σ then $v^* = (\overline{v}_2, -\overline{v}_1)$ is an unitary lift of σ^* where σ^* is the antipodal point of σ. Consider H_z the Möbius transformation associated to the matrix

$$B_z = \begin{pmatrix} \overline{v}_2(z) & v_1(z) \\ -\overline{v}_1(z) & v_2(z) \end{pmatrix}. \tag{3.10}$$

It is easy to see that H_z defined as above is an isometry of the Riemann sphere. To end, define $N_z = H^{-1}_{f(z)} \circ M_z \circ H_z$ and the cocycles N and H by $N = (f, N_*)$ and $H = (id, H_*)$. Notice that $H \circ N = M \circ H$ and that the section null $\xi_0 \equiv 0$ is N-invariant, so each N_z has the form

$$N_z(\xi) = \frac{\xi}{\beta_z\xi + \alpha_z}.$$

Notice that

$$g(\xi_0) = \left|\frac{1}{\alpha_z}\right|,$$

so the hypothesis imply that there exist k and η such that $g^k(\xi_0(z)) = |\alpha_z^k|^{-1} < \eta$ for all z in X.

We can take $R > 0$ uniformly in z, such that $0 < r \leq R$ then $g^k(\xi) \leq \eta$ for every ξ in $D_r(\xi_0(z))$; it follows that $g^n(\xi) \leq C\lambda^n$ for some C and $0 < \lambda < 1$ and every ξ in $D_r(\xi_0(z))$ and $n \in \mathbb{N}$. The previous observation implies that $|N^n(\xi)| < C\lambda^n r$ for all ξ in $D_r(\xi_0)$ and positive n, and it follows the proposition. ∎

Corollary 3.1.1. *Let σ and R satisfying the item (iv) of the Proposition 3.1.5, then $\rho(M_z^n(\xi), M_z^n(\sigma(z)))$ goes to zero when n goes to infinity, for every ξ in $D_r(\sigma(z))$ and $0 < r \leq R$, where ρ is the spherical metric.*

Proposition 3.1.6. *A linear cocycle A has dominated splitting if and only if M is hyperbolic where $M_z = [A_z]$.*

Proof. Let us take (U, φ) a local trivialization of TX and take sections u in $\Gamma(U, E)$ and v in $\Gamma(U, F)$ such that $\|u(z)\|_z = \|v(z)\|_z = 1$ for all $z \in U$. So we can define the sections

$$\tau(z) = (z, [u(z)]) \text{ and } \sigma(z) = (z, [v(z)]).$$

By compactness X is covered by finitely many local trivialization, it follows that this section are globally defined and that are M-invariant.

On the other hand, let us take $K(\beta, F)$ an invariant cone field uniformly contracted by A^l. Denoting $p_I(K(\beta, F_z))$ by D_z, we remark that D_z is a closed conformal disc satisfying that $\sigma(z) \in D_z$ and we have that $M_z^l(D_z) \subset \text{int}(D_{f^l(z)})$. It is possible to find a radius $r > 0$ such that the closed disc centered in $\sigma(z)$ of radius r is contained in D_z, and by equation (3.8) we can find an integer $k > 0$ such that $M^k_{f^{-k}(z)}(D_{f^{-k}(z)}) = \cap_{n=0}^k M^n_{f^{-n}(z)}(D_{f^{-n}(z)}) \subset \text{int}(D(\sigma(z), r))$. By continuity of the splitting and so of the cone field, and using the compactness of X, we can choose this integer independent of the point $z \in X$, therefore it follows that $M_z^k(D(\sigma(z), r)) \subset \text{int}(D(\sigma(f^k(z)), r))$.

Repeating this argument for τ, follows that M is hyperbolic.

We shall now show the converse. First define $E_z = p_I^{-1}(\tau(z)) \cup \{0\}$ and $F_z = p_I^{-1}(\sigma(z)) \cup \{0\}$ and it is clear that this define a A-invariant splitting. Let us take sections u an unitary lift of τ and v an unitary lift of σ. Remember that the unitary lift are element of the section of the trivial vector bundle $X \times \mathbb{C}^2$.

We shall now construct a hyperbolic cocycle N which is conjugated with M. If we write $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we define H_z as the Möbius transformation associated to the matrix

$$B_z = \begin{pmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{pmatrix}, \tag{3.11}$$

and define $N_z = H^{-1}_{f(z)} \circ M_z \circ H_z$. Now the cocycles N and H are defined by $N = (f, N_*)$ and $H = (id, H_*)$ and clearly $H \circ N = M \circ H$.

It remains to prove that N is hyperbolic. First note that by construction, the sections ξ_∞ (resp. ξ_0) that associates at each fiber the point at infinity (resp. the zero point), are N-invariant sections and so we have an explicit expression for N, that is, $N_z(\xi) = \lambda_z \xi$.

To see that N is hyperbolic, it remain only to prove that N satisfy some of the statement of the Proposition 3.1.5.

Since M is hyperbolic, then M^k shrinks small closed disc around σ and expands small disc around τ for some $k \geq 0$ (see Proposition 3.1.5), so the same is true for N^k, and this imply that $|\lambda_z^k|$ is less than one for every $z \in X$, so by compactness, this is true uniformly in z, that is, there exist $0 < \eta < 1$ such that $|\lambda_z^k| < \eta$. Note that $g^k(\xi_0(z)) = |\lambda_z^k| < \eta$. Repeating the argument for ξ_∞, it follows the hyperbolicity of N.

To end, note that we have the following algebraic equality. First let us take $\{(U_i, \varphi_i)\}$ a finite atlas of trivialization maps of TX, local sections $\widetilde{u}_i$ and $\widetilde{v}_i$ in $\Gamma(U_i, TX^*)$ such that $\widetilde{u}_i \in E_z, \widetilde{v}_i \in F_z$ and both are unitary, then we have the equality

$$p_I(\widetilde{u}_i) = \tau = (id, [u]) \quad \text{and} \quad p_I(\widetilde{v}_i) = \sigma = (id, [v]).$$

Take $z \in U_i$ with $f^n(z) \in U_j$ and write $A_z^n \widetilde{u}_i(z) = k_z^n \widetilde{u}_j(f^n(z))$, $A_z^n \widetilde{v}_i(z) = l_z^n \widetilde{v}_j(f^n(z))$ and $\widetilde{A}_z^n = (\varphi_j)_z^{-1} \circ A_z^n \circ (\varphi_i)_z$ then

$$B_{f^n(z)}^{-1} \widetilde{A}_z^n B_z = \begin{pmatrix} k_z^n & 0 \\ 0 & l_z^n \end{pmatrix}$$

and it is follows that $\lambda_z^n = k_z^n / l_z^n$. Now take $u' = su_i(z)$ in E_z and $v' = tv_j(f^n)$ in $F_{f^n(z)}$ with $|s| = |t| = 1$, then

$$\|A_z^n u'\| \cdot \|A_{f^n(z)}^{-n} v'\| = \|A_z^n u_i(z)\| \cdot \|A_{f^n(z)}^{-n} v_j(f^n(z))\| = |k_z^n| \cdot |l_z^n|^{-1} = |\lambda_z^n| \leq C\lambda^n$$

for $n \geq 1$, so A has dominated splitting. ■

3.1.8 Local manifold for invariant sections and Module

This subsection is devote to prove the following proposition.

Proposition 3.1.7. *A projective cocycle M is hyperbolic if and only if at least one of the following equivalent conditions hold:*

1. *There exist a section that is a contraction.*
2. *There exist a section that is an expansion.*

To proof this Proposition, we need establish the notion of stable and unstable set of a point ξ and denoted by $W^{s/u}(\xi)$. For a contractive section σ, we assert that for every $z \in X$, $W^s(\sigma(z))$ is biholomorphic to $\mathbb{C}$. The tool to prove this last assertion is known as module. After we give the poof of Proposition 3.1.7.

Stable and Unstable Sets

Given a cocycle M, we define the stable set of a element $\xi \in \overline{\mathbb{C}}_z$ by the set

$$W^s(\xi) = \left\{ w \in \overline{\mathbb{C}}_z \ : \ \lim_{n \to \infty} \rho(M_z^n(w), M_z^n(\xi)) = 0 \right\},$$

where ρ is the spherical metric in $\overline{\mathbb{C}}_z$. Also, we define the local stable set of size $\varepsilon > 0$ by the set

$$W^s_\varepsilon(\xi) = \left\{ w \in W^s(\xi) \ : \ \rho(M^n_z(w), M^n_z(\xi)) < \varepsilon, \quad \text{for all} \ \ n \in \mathbb{N} \right\}.$$

The unstable set is defined in the same way, but with the inverse cocycle M^{-1}. More precisely

$$W^u(\xi) = \left\{ w \in \overline{\mathbb{C}}_z \ : \ \lim_{n\to\infty} \rho(M^{-n}_z(w), M^{-n}_z(\xi)) = 0 \right\},$$

and the set

$$W^u_\varepsilon(\xi) = \left\{ w \in W^u(\xi) \ : \ \rho(M^{-n}_z(w), M^{-n}_z(\xi)) < \varepsilon, \quad \text{for all} \ \ n \in \mathbb{N} \right\}$$

is the local unstable set. Also we can write the stable set (resp. unstable set) in terms of backward (resp. forward) iteration of the local stable (resp. unstable) sets, that is, given $\varepsilon > 0$ we have

$$W^s(\xi) = \bigcup_{n=0}^{\infty} M^{-n}_z \left(W^s_\varepsilon(M^n(\xi)) \right),$$

and

$$W^u(\xi) = \bigcup_{n=0}^{\infty} M^{n}_z \left(W^u_\varepsilon(M^{-n}(\xi)) \right).$$

For the proof of the Proposition 3.1.7, we need the following lemma.

Lemma 3.2. *Let σ be a contractive section for M, then there exist constant k and $r > 0$ such that*

$$W^s(\sigma(z)) = \bigcup_{t\geq 0} M^{-tk}_z \left(D_r(\sigma(f^{tk}(z))) \right).$$

Proof. It follows from the Corollary 3.1.1 that each $D_r(\sigma(f^{tk}(z)))$ is a local stable set, and there are uniformly contractive by the cocycle. ■

Module

See [M] and [LV] for details. A *double connected domain* in $\overline{\mathbb{C}}$ is a open connected set such that its complement has two connected component. The definition of the module of a double connected domain is based in the following mapping theorem: *Every double connected domain U is biholomorphic to a ring domain of the form*

$$A(r_1, r_2) = \{ z \in \mathbb{C} \ : \ 0 \leq r_1 < |z| < r_2 \leq \infty \}.$$

If $r_1 > 0$ and $r_2 < \infty$ for one canonical image of U, then the ratio of the radii r_2/r_1 is the same for all canonical images of U. Then the number

$$\mathrm{mod}(U) = \log \frac{r_2}{r_1}$$

which then determines the conformal equivalence class of U, is called the module of U. Otherwise, we define $\mathrm{mod}(U) = \infty$ and this happens if and only if at least one boundary component of U consists of a single point.

The following classical lemma provides the module super-additivity property.

Lemma 3.3. *Let U, U_1, U_2, ... be double connected domains such that the U_i are pairwise disjoint sub-domains of U. If every U_n separates the two connected component of the complement of U, then*

$$\sum_n \mathrm{mod}(U_n) \leq \mathrm{mod}(U).$$

A immediate corollary of the previous lemma is the following.

Corollary 3.1.2. *Let D_1, D_2, D_3, ... be conformal discs in $\overline{\mathbb{C}}$ such that for every $i \geq 1$ we have $\overline{D_i} \subset D_{i+1}$. If there exist a constant $\kappa > 0$ such that* $\mathrm{mod}(\mathrm{int}(D_{i+1} \setminus D_i)) \geq \kappa$, *then the set $D = \cup_n D_n$ is biholomorphic with $\mathbb{C}$.*

Proof of Proposition 3.1.7

Proof. It is enough to prove that (2) imply (1), and the other direction follows using the inverse cocycle.

We will show that $W^s(\sigma(z))$ is biholomorphic to $\mathbb{C}$. Take k and r as in the Lemma 3.1.7 and define $D_t = M_z^{-tk}\left(D_r(\sigma(f^{tk}(z)))\right)$. It is clear that $D_{t-1} \subsetneq D_t$ and the function M_z^{tk} maps biholomorphically $D_t \setminus D_{t-1}$ on

$$A_t = D_r(\sigma(f^{tk}(z))) \setminus M^k_{f^{(t-1)k}(z)}(D_r(\sigma(f^{(t-1)k}(z)))),$$

so the module $\mathrm{mod}(D_t \setminus D_{t-1})$ and $\mathrm{mod}(A_t)$ are equal. By Corollary 3.1.2, it is enough to prove that the module of the annulus are bounded above, but if we take η uniformly in X such that $g^k(\sigma) \leq \eta$ it follows that $\mathrm{mod}(A_t) \geq \log(1/\eta)$; this proves the claim.

To end, we have that $\overline{\mathbb{C}}_z \setminus W^s(\sigma(z)) = \{\tau(z)\}$. Since W^s varies continuously and is M-invariant, it follows that τ is a M-invariant section. By the definition of τ, it follows that small disc around of τ are contracted uniformly by M^{-1}, and it follows that τ is an expansion. ∎

3.2 Some Preliminary Results about Cocycles

As before we assume that X is a metric compact space, the function f is an homomorphism of X, the linear cocycle $A : TX \to TX$ is a cocycle with base f, and M is the projective cocycle associated with A, i.e., $M_z([v]) = [A_z v]$.

3.2.1 Oseledets Theorem

A point $z \in X$ is a *regular point of A* if the fiber T_z admits a splitting $T_z = E_z \oplus F_z$ of one dimensional complex spaces, and number $\lambda^-(x) \leq \lambda^+(z)$ satisfying

$$\lim_{n\to\pm\infty} \frac{1}{n} \log(\|A_z^n u\|) = \pm\lambda^-(z) \quad \text{and} \quad \lim_{n\to\pm\infty} \frac{1}{n} \log(\|A_z^n v\|) = \pm\lambda^+(z),$$

where $u \in E_z \setminus \{0\}$ and $v \in F_z \setminus \{0\}$. Remember that we say that a set $S \subset X$ is of *total probability in* X, if for every measure μ f-invariant, $\mu(S^c) = 0$.

The following Theorem is a version of Oseledets Theorem in the cocycles context (see for references [V]).

Theorem 3.2.1. (Oseledets) *The set of regular points of A has total probability. Moreover, $z \mapsto E_z$ and $z \mapsto F_z$ are measurable subbundles and the functions $z \mapsto \lambda^{\pm}(z)$ are measurable.*

Remark 15. *We denote by $\mathcal{R}(A)$ the set of regular points for a cocycle A . The Oseledets Theorem asserts that given a measure f-invariant μ, the set of regular point in the support of μ has total measure. Indeed, we have that $\mu(\mathcal{R}(A) \cap \operatorname{supp}(\mu)) = 1$. We denote this full measure set by $\mathcal{R}(A, \mu)$.*

Definition 3.2.1. *We say that a measure f-invariant μ is partially hyperbolic, if for any $x \in \mathcal{R}(A, \mu)$ the following inequality occurs:*

$$\lambda^-(x) < 0 \leq \lambda^+(x).$$

3.2.2 Pliss's Lemma

The next Lemma is due to Pliss, which is frequently used in this work.

Lemma 3.4. (Pliss's Lemma) *Given $0 < \gamma_1 < \gamma_0$ and $a > 0$, there exist $N_0 = N_0(\gamma_0, \gamma_1, a)$ and $\delta_0 = \delta_0(\gamma_0, \gamma_1, a) > 0$ such that for any sequences of numbers $(a_l)_{l=0}^{n-1}$ with $n > N_0$, $a^{-1} < a_l < a$ and $\prod_{l=0}^{n-1} a_l \geq \gamma_0^n$ we have that*

$$\#\left\{0 \leq k < n \;:\; \forall\, k < s < n, \text{ we have that } \prod_{l=k+1}^{s} a_l \geq \gamma_1^{s-k}\right\} \geq n \cdot \delta_0. \tag{3.12}$$

We reformulate an application of the Pliss Lemma, in terms of projective cocycle and the norm of the multiplier of the cocycle. We recall that the function g is defined in Definition 3.1.2, and that by equation (3.9) we have that

$$g^n(\xi) = \frac{\det(A_z^n)}{||A_z^n v_\xi||_{f(z)}^2},$$

where $\xi \in \overline{\mathbb{C}}_z$ and v_ξ is a unitary vector satisfying $[v_\xi] = \xi$.

Corollary 3.2.2. *Given $0 < \gamma_1 < \gamma_0$, there exist two positive constants N_0 and δ_0 such that:*

If for $z \in X$ there exist $\xi \in \overline{\mathbb{C}}_z$ such that $g^n(\xi) \geq \gamma_0^n$ (resp. $g^{-n}(\xi) \geq \gamma_0^n$) for $n \geq N_0$, then there exists $0 \leq j < n$ such that $n - j > n\delta_0 - 1$ and

$$g^i(M^j(\xi)) \geq \gamma_1^i \quad \text{for every} \quad 0 < i \leq n - j,$$

(resp. $g^{-i}(M^{-j}(\xi)) \geq \gamma_1^i \quad$ for every $\quad 0 < i \leq n - j$).

Proof. It is obtained applying directly the Pliss Lemma to the sequence $(g(M^l(\xi)))_{l=0}^{n-1}$. We have that $g^n(\xi) = \prod_{l=0}^{n-1} g(M^l(\xi)) \geq \gamma_0^n$, and if let us take k_0 the smallest of k's that be in the set defined in the inequality (3.12), we have that $n - k_0 \geq n\delta_0$, and for every $k_0 < s < n$

$$\gamma_1^{s-k_0} \leq \prod_{l=k_0+1}^{s} g(M^l(\xi)) = g^{s-k_0}(M^{k_0+1}(\xi)).$$

Hence it is enough to take $j = k_0 + 1$, and we have the corollary. ■

3.2.3 Some Formulas about the map g

Lemma 3.5. *If ξ_i with $i = 1, 2$ are two different directions in $\overline{\mathbb{C}}_z$ and u_i is an unitary vector that generates the direction ξ_i for $i = 1, 2$, then*

$$g^n(\xi_1)g^n(\xi_2) = \left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2,$$

for any $n \in \mathbb{Z}$.

Proof. First observe that given two vector x and y in $\mathbb{C}^2$, we denote by $\phi(x, y)$ the area of the polygon formed by the vertices 0, x, $x + y$ and y. Then it is clear that

$$\phi(x, y) = |x| \cdot |y| \cdot \sin(\measuredangle(x, y)) = \sqrt{\det([x\ y]^* \cdot [x\ y])} = |\det([x\ y])|, \tag{3.13}$$

where $[x\ y]$ is a column matrix and $[x\ y]^*$ denote its transposed conjugate. Then, it is easy to see that $\phi(Ax, Ay) = |\det(A)|\phi(x, y)$ for any linear map A in $\mathbb{C}^2$.

So by the equation 3.13 we have that

$$\left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2 = \frac{\phi(A_z^n u_1, A_z^n u_2)^2/|A_z^n u_1|^2 \cdot |A_z^n u_2|^2}{\phi(u_1, u_2)^2/|u_1|^2 \cdot |u_2|^2} = \frac{|\det(A_z^n)|^2}{|A_z^n u_1|^2 \cdot |A_z^n u_2|^2}.$$

According to the equation (3.9) and the previous equality, it follows that

$$g^n(\xi_1)g^n(\xi_2) = \frac{|\det(A_z^n)|^2}{|A_z^n u_1|^2 \cdot |A_z^n u_2|^2} = \left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2.$$

■

Lemma 3.6. *Let $\lambda > 1$, $C > 0$ and let $z \in X$. Assume that there exists a direction ξ such that $g^n(\xi) \geq C\lambda^n$ (or $g^{-n}(\xi) \geq C\lambda^n$) for every $n \geq 1$, then this direction is unique.*

Proof. By the previous lemma, if $z \in X$ and if ξ_1 and ξ_2 are two different directions that are expanded for the future, then $\measuredangle(u_1, u_2) > 0$, and hence

$$C^2\lambda^{2n} \leq g^n(\xi_1)g^n(\xi_2) = \left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2 < \frac{1}{\sin(\measuredangle(u_1, u_2))}$$

which is a contradiction. ■

For the case that we have expansion for the past, the same proof holds.

3.3 Critical Points

In this section we can enunciate formally the main Theorem of this Chapter, the Theorem D. First, we establish the notion of critical points, and for this purpose, is important to recall that a cocycle A has domination along the orbit of a point z, if there exist a direction $F \in \overline{\mathbb{C}}_z$ that is (uniformly) expanded to the past, and (uniformly) contracted to the future. Roughly speaking it is said that a point z is a critical point if it has a direction which is projective expansive for backward iterate and it is also projective expansive for the forward iterate.

To be more precise, z is a critical point $z \in X$ is critical if there exist one direction $\xi \in \overline{\mathbb{C}}_z$ such that $g^{-n}(\xi) \geq C\gamma^n$ for every $n \geq 1$ where $\gamma > 1$ and $0 < C < 1$, but $g^n(\xi) \geq C^{-1}(1/C\gamma)^n$ (meaning that any forward iterate is bounded) and there exists $n_k \to \infty$ such that $g^{n_k}(\xi) > C\gamma^{n_k}$, i.e., also has projective expansive for the future.

Observe that in this case if there are critical points, then there is not dominated splitting (see the proof of Theorem D, in page 81), the most important part is that if there are not dominated splitting, then there are critical points.

We want to highlight certain properties of the notion of critical point. The set of all critical point (the critical set) is compact, in the orbit of a critical point only one is a critical point (uniquely defined), also a critical point are invariant by a change of metric, invariant by small (close to the identity) conjugation, remain far from hyperbolic set and non-hyperbolic set, tangencies are critical points, but not the whole orbit is a critical point. This will be discussed deeply in the subsection **??**. Moreover, in subsection **??** we show certain properties of critical point, that only holds for holomorphic dynamics.

Definition 3.3.1. *Given $0 < b < 1$, we say that X is b-dissipative (for a cocycle A) if there exists a positive constant $C > 0$ such that for every $z \in X$, $|\det(A_z^n)| \leq Cb^n$ for every $n \geq 0$.*

Definition 3.3.2. (Blocks of domination). *Given β and δ positive numbers, we define the sets*

$$\beta H^{\pm}(\delta) = \{z \in X : \exists\ \zeta \in \overline{\mathbb{C}}_z \text{ such that } g^{\pm n}(\zeta) \geq \beta(1+\delta)^n,\ \forall\, n > 0\},$$

and the sets

$$\beta \mathring{H}^{\pm}(\delta) = \{z \in X : \exists\ \zeta \in \overline{\mathbb{C}}_z \text{ such that } g^{\pm n}(\zeta) > \beta(1+\delta)^n,\ \forall\, n > 0\}.$$

Remark 16. *a) We denote by $H^{\pm}(\delta)$ and also $\mathring{H}^{\pm}(\delta)$, when in the previous definitions we take $\beta = 1$.*

b) It is easy to see, that for $0 < \delta' < \delta$ the following inclusions holds $\beta H^{\pm}(\delta) \subset \beta H^{\pm}(\delta')$ and $\beta \mathring{H}^{\pm}(\delta) \subset \beta \mathring{H}^{\pm}(\delta')$.

<u>**Notation:**</u> We denote by x, the points of $\beta H^{-}(\delta)$ and $\beta \mathring{H}^{-}(\delta)$; we denote by ξ, the direction $\zeta \in \overline{\mathbb{C}}_x$ that appears in the previous definition and call this by *critical direction of* x. Similarly, we denote by y the points in $\beta H^{+}(\delta)$ (or $\beta \mathring{H}^{-}(\delta)$), and its critical direction, by ϖ.

Definition 3.3.3. *Let $\alpha \geq 1$ and $\delta > 0$. We say that a point $(x, \xi) \in TX$ is a (α, δ)-critical point, for the linear cocycle A, if:*

1. *$x \in \alpha^{-1} H^{-}(\delta)$,*

2. *ξ is the critical direction of x,*

3. $f^n(x) \notin \alpha \mathring{H}^-(\delta)$ *for every* $n \geq 0$,

4. x *is maximal in the orbit* $O(x)$ *with the three properties above: if* (x', ξ') *satisfies the previous three properties then there exist* $n_0 > 0$ *such that* $x' = f^{-n_0}(x)$ *and* $\xi' = M^{-n_0}(\xi)$

A point $(y, \varpi) \in TX$ *is a* (α, δ)*-critical value, if* (y, ϖ) *is* (α, δ)*-critical point for* A^{-1}.

We denote by $C(\alpha, \delta)$ *the set of all* (α, δ)*-critical point, and by* $V(\alpha, \delta)$ *the set of all* (α, δ)*-critical value.*

In the Remark 19, we show that in the orbit of a a point x such that have a direction ξ satisfying the three first properties in the previous definition, there exist a maximal element, and this justify the previous definition.

Remark 17. *We remark that since* $0 < b < 1$ *then* $1 - b < b^{-1} - 1$. *We also introduce the following notation. When* $0 < \delta < b^{-1} - 1$ *we denote by* $\alpha_0(\delta) = b^{-1}/(1 + \delta)$. *If* $1 \leq \alpha < \alpha_0(\delta)$ *we can find* δ' *such that* $\delta < \delta' < b^{-1} - 1$ *and* $\alpha(1 + \delta) \leq 1 + \delta'$. *Similarly when* $0 < \delta < 1 - b$ *we denote by* $\alpha_1(\delta) = (2 - b)/(1 + \delta)$, *then* $1 < \alpha_1(\delta) < \alpha_0(\delta)$. *If* $1 \leq \alpha < \alpha_1(\delta)$ *we can find* δ' *such that* $\delta < \delta' < 1 - b$ *and* $\alpha(1 + \delta) \leq 1 + \delta'$

Now we will devote to prove the following two theorems:

Theorem C. *Let* A *be a linear cocycle such that* X *is* b*-dissipative, such that every* f*-invariant measure is partially hyperbolic. Then for any* $0 < \delta < b^{-1} - 1$ *and any* $\alpha_0(\delta) > \alpha \geq 1$, *the blocks of dominations* $\alpha H^+(\delta)$ *and* $\alpha^{-1} H^-(\delta)$ *are not empty compact sets. Moreover, the sets*

$$X_0^+ = \cup_{n \in \mathbb{Z}} f^n(\alpha H^+(\delta)) \text{ and } X_0^- = \cup_{n \in \mathbb{Z}} f^n(\alpha^{-1} H^-(\delta))$$

have total measure for any invariant measure ν *with support in* X.

Also we have:

Theorem D. *Let* A *be a linear cocycle such that* X *is* b*-dissipative, such that every* f*-invariant measure is partially hyperbolic. Then* A *has dominated splitting if and only if* $C(\alpha, \delta) = \emptyset$ *for every* $0 < \delta < 1 - b$ *and* $1 \leq \alpha < \alpha_1(\delta)$.

To prove the theorems, first we describe some basics fact with respect to critical directions.

It follows directly from the Lemma 3.6 the following Corollary.

Corollary 3.3.1. *The critical direction is unique.*

Proposition 3.3.1. *For $i = 1, 2$, let β_i and δ_i positive numbers. Assume that $x \in \beta_1 H^-(\delta_1)$ with critical direction ξ_x and let $l > 0$. Then the following statements occur:*

i) If $f^l(x) \in \beta_2 H^-(\delta_2)$ (or in $\beta_2 \mathring{H}^-(\delta_2)$), then $\xi_{f^l(x)} = M^l(F_x)$ (i.e. $M^l(F_x)$ is the critical direction for $f^l(x)$).

ii) If $\beta_2 \geq 1$ and $g^l(\xi_x) > \beta_2(1+\delta_2)^l$ then $f^l(x) \notin \beta_2 \mathring{H}^-(\delta_2)$.

iii) If $g^{-l}(M^l(\xi_x)) \leq \beta_2(1+\delta_2)^l$ then $f^l(x) \notin \beta_2 \mathring{H}^-(\delta_2)$.

Proof. Take v_x (resp. $v_{f^l(x)}$) a unitary vector such that $[v_x] = \xi_x$ (resp. $[v_{f^l(x)}] = \xi_{f^l(x)}$). For the first statement, note that if $\measuredangle(v_{f^l(x)}, A_x^l v_x) > 0$ then for every $n > l$ we have that

$$g^{-n}(M^l(\xi_x)) = g^{-(n-l)}(\xi_x) g^{-l}(M^l(\xi_x)) \geq \beta_1(1+\delta_1)^{n-l} g^{-l}(M^l(\xi_x)).$$

Defining $\beta = \min(\beta_1, \beta_2)$ and $\delta = \min(\delta_1, \delta_2)$ we have that for any $n > 0$ holds

$$\begin{aligned} g^{-l}(M^l(\xi_x))\beta^2(1+\delta)^{2n-l} &\leq g^{-l}(M^l(\xi_x))\beta_1(1+\delta_1)^{n-l}\beta_2(1+\delta_2)^n \\ &\leq g^{-n}(M^l(\xi_x)) g^{-n}(\xi_{f^l(x)}) \\ &= \left(\frac{\sin(\measuredangle(A_{f^l(x)}^{-n} v_{f^l(x)}, A_{f^l(x)}^{-n+l} v_x))}{\sin(\measuredangle(v_{f^l(x)}, A_x^l v_x))} \right)^2 \\ &< \frac{1}{\sin(\measuredangle(v_{f^l(x)}, A_x^l v_x))}, \end{aligned}$$

that is a contradiction.

For the second, if we suppose that $f^l(x) \in \beta_2 \mathring{H}^-(\delta_2)$ it follows from the first part that $g^{-n}(M^l(\xi_x)) > \beta_2(1+\delta_2)^n$ for every $n > 0$; in particular $g^{-l}(M^l(\xi_x)) > \beta_2(1+\delta_2)^l$, hence

$$\beta_2(1+\delta_2)^l < g^{-l}(M^l(\xi_x)) = \frac{1}{g^l(\xi_x)} < \frac{1}{\beta_2(1+\delta_2)^l}.$$

A contradiction.

The third part it follows easily because, if we suppose that $f^l(x) \in \beta_2 \mathring{H}^-(\delta_2)$, we have that

$$\beta_2(1+\delta_2)^l < g^{-l}(M^l(\xi_x)) \leq \beta_2(1+\delta_2)^l$$

that is a contradiction. ■

Corollary 3.3.2. *Let $\alpha \geq 1$ and $x \in \alpha^{-1} H^-(\delta)$ with critical direction ξ. If for every $n \geq 1$ we have that $g^{-n}(M^n(\xi)) \leq \alpha(1+\delta)^n$, then $(x, \xi) \in C(\alpha, \delta)$.*

Proposition 3.3.2. *Let $(x,\xi) \in TX$. Then $(x,\xi) \in C(\alpha,\delta)$ then for every $n \geq 1$, $g^{-n}(\xi) \geq \alpha^{-1}(1+\delta)^n$ and $g^{-n}(M^n(\xi)) \leq \alpha^{n+1}(1+\delta)^n$.*

Proof. Since that $x \in \alpha^{-1}H^-(\delta)$ the first inequality is clear. For the second inequality we proceed by induction.

First one, we assume that $g^{-1}(M(\xi)) > \alpha^2(1+\delta) > \alpha(1+\delta)$. In particular, for every $n \geq 2$

$$g^{-n}(M(\xi)) = g^{-(n-1)}(\xi)g^{-1}(M(\xi)) > \alpha^{-1}(1+\delta)^{(n-1)}\alpha^2(1+\delta) = \alpha(1+\delta)^n,$$

that implies that $f(x) \in \alpha\mathring{H}^-(\delta)$ that is a contradiction.

Now, we assume that the second inequality is true for every $0 < n < m$ and that $g^{-m}(M^m(\xi)) > \alpha^{m+1}(1+\delta)^m$. For every $0 < s < m$ we have

$$\begin{aligned}\alpha^{m+1}(1+\delta)^m < g^{-m}(M^m(\xi)) &= g^{-s}(M^m(\xi)) \cdot g^{-(m-s)}(M^{m-s}(\xi)) \\ &\leq \alpha^{(m-s)+1}(1+\delta)^{m-s}g^{-s}(M^m(\xi)),\end{aligned}$$

hence we conclude that

$$g^{-s}(M^m(\xi)) > \alpha^s(1+\delta)^s > \alpha(1+\delta)^s.$$

Then for every $0 < n \leq m$ we have that $g^{-n}(M^m(\xi)) > \alpha(1+\delta)^n$. To finish, for $n > m$ we have

$$g^{-n}(M^m(\xi)) = g^{-(n-m)}(\xi) \cdot g^{-m}(M^m(\xi)) > \alpha^{-1}(1+\delta)^{n-m}\alpha^{m+1}(1+\delta)^m > \alpha(1+\delta)^n,$$

that contradicts the fact that $f^m(x) \notin \alpha\mathring{H}^-(\delta)$. ■

3.3.1 Proof of Theorem C

Proof of Theorem C. First, we fix $1 \leq \alpha < \alpha_0(\delta)$ and take $\delta' < b^{-1} - 1$ as in Remark 17, it is $\delta < \delta' < b^{-1} - 1$ and $\alpha(1+\delta) \leq 1 + \delta'$.

On the other hand, let ν a f-invariant measure and let $x \in \mathcal{R}(A,\nu)$, then

$$\lim_{n\to+\infty} \frac{1}{n}\log(g^n(E_x)) = \lambda^+(x) - \lambda^-(x) \geq \lambda^-(x) \geq -\log(b).$$

If we choose c satisfying

$$1 + \delta' < c < b^{-1}$$

it follows that for m great enough, we have

$$g^m(E_x) \geq c^m,$$

and by Pliss's Lemma, there exist a sequence $(m_k)_k \nearrow \infty$ such that

$$g^n(M^{m_k}(E_x)) \geq (1+\delta')^n > \alpha^n(1+\delta)^n > \alpha(1+\delta)^n, \text{ for every } n \geq 1.$$

It follows that $\alpha H^+(\delta)$ is not empty. Moreover, $\alpha H^+(\delta)$ contains all accumulation points of the set $(f^{m_k}(x))_k$ with critical directions gives by an accumulation point of $(M^{m_k}(E_x))_k$.

Arguing in the same way, for $x \in \mathcal{R}(A, \nu)$ we have that

$$\lim_{n\to+\infty} \frac{1}{n} \log(g^{-n}(F_x)) = \lambda^+(x) - \lambda^-(x) \geq \lambda^-(x) \geq -\log(b),$$

and we can find a sequence $(n_k)_k \nearrow \infty$ such that

$$g^{-n}(M^{-n_k}(F_x)) \geq (1+\delta)^n > \alpha^{-1}(1+\delta)^n, \text{ for every } n \geq 1,$$

and conclude that $\alpha^{-1}H^-(\delta)$ is not empty.

The compactness follows because given a sequence $(y_n)_n \subset \alpha H^+(\delta)$ with critical direction $(\varpi_n)_n$, any accumulation point y of $(y_n)_n$, has a direction ϖ_y accumulated by the directions $(\varpi_n)_n$ that satisfy $g^n(\varpi_x) \geq \alpha(1+\delta)^n$, then $y \in \alpha H^+(\delta)$.

To end note that the set $X_0^+ = \cup_{n\in\mathbb{Z}} f^n(\alpha H^+(\delta))$ and for any regular point $x \in \mathcal{R}(A, \nu)$ there exist an element of its orbit that is in $\alpha H^+(\delta)$, then $\mathcal{R}(A, \nu) \subset X_0^+$, and X_0^+ have total measure. ■

3.3.2 Criteria of Domination

Now we present two criteria for the existence of dominated splitting that are essential to prove the Theorem D.

Proposition 3.3.3. (Criteria of Domination I) *Let us assume that X is b-dissipative, every f-invariant measure is partially hyperbolic and let us take* $0 < \delta < 1 - b$.

If there exist $k, m_0 > 0$ such that, for all $z \in X$ there exist one direction $\xi_z \in \overline{\mathbb{C}}_z$ such that

$$g^k(M^m(\xi_z)) \leq (1+\delta)^k, \quad \textit{for every} \quad m > m_0; \tag{3.14}$$

then X has dominated splitting.

Proposition 3.3.4. (Criteria of Domination II) *Let us assume that there exist positive integers k, m_0 and $\gamma < 1$ such that: for any $z \in X$ there exists one direction $\xi_z \in \overline{\mathbb{C}}_z$ such that*

$$g^k(M^m(\xi_z)) < \gamma, \quad \textit{for every} \quad m > m_0;$$

then X has dominated splitting.

Proof. Fix $z_0 \in X$ and denote by $\xi_0 = M^{m_0}(\xi_{z_0})$ and $\xi_t = M^t(\xi_0)$, then we have that $g^k(\xi_t) < \gamma$ for every $t \geq 0$. Let us take, for $j = 0, \ldots, k-1$

$$C_j = \sup\{g^j(w) \,:\, w \in \overline{\mathbb{C}}_z\,,\; z \in X\},$$

it follows that

$$g^{nk+j}(\xi_0) = g^l(\xi_0)g^{nk}(\xi_j) \leq C_j\gamma^n \leq C\lambda_0^{nk+j},$$

where $\lambda_0 = \gamma^{1/k} < 1$ and $C_0 = \sup\{C_j\gamma^{-j/k} \,:\, j = 1, \ldots, k-1\}$.

To end, for every $z \in X$ let us take $z_0 = f^{-m_0}(z)$ and $\sigma_z = M^{m_0}(\xi_{z_0})$, it follows that $g^{-n}(\sigma_z) \geq C\lambda^n$, where $C = C_0^{-1}$ and $\lambda = \lambda_0^{-1}$. The domination in X, is immediate after the following lemma (Lemma 3.7). ■

Lemma 3.7. *Let $C > 0$ and $\lambda > 1$ be two constant. Suppose that for every $z \in X$ there exists one direction $\tau_z \in \overline{\mathbb{C}}_z$ (resp. $\sigma_z \in \overline{\mathbb{C}}_z$) such that $g^n(\tau_z) \geq C\lambda^n$ (resp. $g^{-n}(\sigma_z) \geq C\lambda^n$) for every $n > 0$, then the function $\tau(z) = \tau_z$ is a continuous section M-invariant that is an expansion (resp. contraction).*

Proof. Denote by u_z some unitary vector that define the direction τ_z and suppose that $M(\tau_{f^{-1}(z)}) \neq \tau_z$, then it follows from the lemma 3.5 that $C^2\lambda^{2n} \leq 1/\sin(\measuredangle(v_z, A_z v_{f^{-1}(z)}))$ for every $n > 0$, that is a contradictions; so τ_z is M-invariant.

On the other hand, let $z_n \to z$ in X, then $\tau_{z_n} \to \tau_z$. In fact, by compactness there exists some adherence point for the sequence $(\tau_{z_n})_n$, named $\tau' \in \overline{\mathbb{C}}_z$ that is expansive for the future. From the uniqueness of the expansive direction for the future, it follows that τ' is equal to τ_z. So we prove continuity. Also is clear from the hypothesis that τ is an expansion. ■

A fundamental tool to prove the Criteria of Domination I, is the following lemma. This establish that if there exists one direction that is neither contracted nor expanded for the future, then the largest Lyapunov exponent in the omega limit of this point is negative.

Lemma 3.8. (Criteria of Negative Exponent) *Let $0 < \delta' < 1 - b < b^{-1} - 1$, x be a point in X and ξ_x a direction in $\overline{\mathbb{C}}_x$ verifying that there exist constants $n_0, m_0 \in \mathbb{N}$ such that*

i) *$\omega(x)$ is b-dissipative,*

ii) *$(1-\delta')^n \leq g^n(\xi_x)$ for every $n \geq n_0$.*

iii) *$g^n(M^m\xi_x) \leq (1+\delta')^n$ for every $m > m_0$ and $n \geq n_0$.*

Then $\omega(x)$ supports a measure that is not partially hyperbolic.

Proof. We may assume that $\omega(x)$, only support partially hyperbolic measures, This implies that the biggest exponent is positive. Let us take $n_k \to \infty$ such that the following limit exists,

$$\mu = \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{M^i(\xi_x)}.$$

We have that $\widehat{K} = \operatorname{supp}(\mu)$ is a compact set of TX, and that the projection $K = pr(\widehat{K}) \subset \omega(x)$ is the support in X, of the measure μ' that is the projection of μ in the first coordinate; that is

$$\mu' = \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i(x)}.$$

Since μ' be a f-invariant measure, we have that for any $z_0 \in \mathcal{R}(A, \mu')$ (the set of regular point in the support of μ') and $w \in \overline{\mathbb{C}}_{z_0}$, the limit

$$\lim_{k\to\infty} \frac{1}{n_k} \log(g^{n_k}(w)) = \lim_{k\to\infty} \frac{1}{n_k} (\log(|\det A_{z_0}^{n_k}|) - 2\log(|A_{z_0}^{n_k} w|)) = \lambda^+(z_0) + \lambda^-(z_0) - 2\lambda(w),$$

where $\lambda(w)$ is the Lyapunov exponent associated with the direction w. Hence this limit, denoted by $I(z_0, w)$, takes the values $\lambda^+(z_0) - \lambda^-(z_0)$ or $\lambda^-(z_0) - \lambda^+(z_0)$.

Since that $z_0 \in \omega(x)$, and assume that $f^{m_k}(x) \to z_0$; taking a subsequence if necessary there exists $w_0 \in \overline{\mathbb{C}}_z$ such that $M^{m_k}(\xi_x) \to w_0$ and (z_0, w_0) is a point in $\widehat{K}$. By condition (iii) we have that $I(z_0, w_0) \leq \log(1 + \delta')$. Moreover, this inequality is true for every $(z, w) \in \widehat{K}$, with $z \in \mathcal{R}(A, \mu')$.

On the other hand, we remark that $\lambda^-(z_0) \leq \log b < 0 \leq \lambda^+(z_0)$ and $\lambda^-(z_0) - \lambda^+(z_0) \leq \lambda^-(z_0) + \lambda^+(z_0) = \log b$. Hence either $I(z_0, w_0) \leq \log b$ or $I(z_0, w_0) > -\log b$; but if the second inequality holds, then $I(z_0, w_0) > \log(b^{-1}) \geq \log(1 + \delta')$ that is a contradiction. We conclude that for every $(z, w) \in \widehat{K}$ with z a regular point in the Oseledets sense, the limit $I(z, w)$ is equal to

$$\lim_{k\to\infty} \frac{1}{n_k} \log(g^{n_k}(w)) = \lambda^-(z) - \lambda^+(z) \leq \log(b).$$

Claim. $\mu(\widehat{K} \cap pr^{-1}(\mathcal{R}(A, \mu')) = 1$.

Proof of the Claim. The Ergodic Decomposition Theorem assert that: There exists a set Σ of full probability in $\mathbb{P}(X)$ such that for all $(z, w) \in \Sigma$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{M^j(z,w)} = \mu_{(z,w)}$$

is a ergodic measure, and for all $h \in \mathcal{L}^1(\mathbb{P}(X), \mu)$ we have

$$\int \left(\int h d\mu_{(z,w)} \right) d\mu = \int h d\mu.$$

In particular, the projection

$$\mu'_z = pr \circ \mu_{(z,w)} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)}$$

is ergodic and the previous integral equality occurs for the measures μ' and μ'_z. To end, note that the claim is true for ergodic measures. The Lyapunov exponent are invariant function which are constant in the support of the measures μ'_z. Now $R(z) = pr^{-1}(\mathcal{R}(A, \mu'_z))$ is invariant by the projective cocycle, so has $\mu_{(z,w)}$-measure 0 or 1, but

$$\mu_{(z,w)}(\log(g)) = \mu_{(y,t)}(\log(g)) = \lambda^-(y) - \lambda^+(y)$$

for $(y, t) \in R(z)$ $\mu_{(z,w)}$-a.e. this implies that $\mu_{(z,w)}(R(z)) \neq 0$ so it is equal to 1. This end the proof of the Claim.

Continuing with the proof of the Lemma, applying Birkhoff Ergodic Theorem to the function $\phi = \log(g)$, μ-a.e. $(z, w) \in \widehat{K}$, there exist the limit

$$\widetilde{\phi}(z, w) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ M^j(w) = \lambda^-(z) - \lambda^+(z),$$

hence it follows that

$$\log(b) \geq \int \widetilde{\phi} d\mu(z, w) = \int \phi d\mu(z, w) = \lim_{k\to\infty} \frac{1}{n_k} \log(g^{n_k}(\xi_x)) \geq \log(1 - \delta'),$$

therefore, there are measure with the two negative Lyapunov exponent, that is a contradiction. ■

Proof of Criteria of Domination I.

From the Criteria of Domination II, it is enough to prove that there exist some positive integers k and m_0, such that for every z there exists one direction ξ_z such that $g^k(M^m(\xi_z)) \leq (1 - \delta)^k$ for all $m > m_0$.

If not, it follows that for every pair k, $m_0 \in \mathbb{N}$, there exists $z \in X$ such that for every $\xi \in \overline{\mathbb{C}}_z$ we have that $g^k(M^m(\xi)) \geq (1 - \delta)^k$ for some $m > m_0$. In particular, for every k there exists $z_k \in X$ and $m_k > k$ such that $g^k(M^{m_k}(\xi_k)) \geq (1 - \delta)^k$, where ξ_k satisfies the equation (3.14). Let us take $\delta' > \delta$; applying the Corollary 3.2.2, it follows that there exists a sequence $(r_k)_{k\geq 1}$ with $k - r_k \to \infty$ such that

$$g^s(M^{r_k}(\xi_k)) \geq (1 - \delta')^s, \quad \text{for every } \ 0 < s \leq k - r_k.$$

Taking z and ϖ as an accumulations point of $(f^{r_k}(z_k))_k$ and $(M^{r_k}(\xi_k))_k$, respectively, it follows that

$$(1-\delta')^n \leq g^n(\varpi), \quad \text{for every } \; n > 0.$$

On the other hand, note that there exist a constant $C > 0$ such that for every $k > 0$, $n \geq 0$ and $m \geq m_0$

$$g^n(M^m(\xi_k)) \leq C(1+\delta)^n \leq (1+\delta')^n,$$

hence passing to the limit it follows that

$$g^n(M^m(\varpi)) \leq (1+\delta')^n, \quad \text{for every} \quad n \geq 0.$$

Therefore, by the Criteria of Negative Exponent, we conclude that there exists a invariant measure that is not partially hyperbolic supported in X, which is a contradiction. ■

3.3.3 Proof of Theorem D

This chapter is based in the ideas of Sylvain Crovisier in [Cr], for the proof of the same theorem in the context of C^2 generic diffeomorphisms in compact manifolds. Our exhibition presents significant changes compared with that of Silvan, among others, we have a different definition of critical point. Now we present a notion that allows to prove the Theorem D.

Definition 3.3.4. *Given $0 < \delta$, and $\alpha \geq 1$, we say that a projective cocycle M satisfies the property $P(\alpha, \delta)$ if there exist $k_0 > 0$, such that for every $k > k_0$ there exit $x_k \in X$, $\xi_k \in \overline{\mathbb{C}}_{x_k}$ and $m_k \geq 0$ such that:*

1. $g^{-n}(\xi_k) \geq \alpha^{-1}(1+\delta)^n$*, for every* $1 \leq n \leq k$*,*
2. $g^k(M^{m_k}(\xi_k)) \geq 1$.

Proposition 3.3.5. *If the projective cocycle M satisfies the property $P(\alpha, \delta)$, then $C(\alpha, \delta) \neq \emptyset$.*

Proof. We will apply the Corollary 3.2.2. Let $k > 0$, $\gamma_0 = 1$, $\gamma_1 = \alpha^{-1/k}(1+\delta)^{-1}$. Also take n_0 and $\delta_0 > 0$ be the numbers given by this Corollary. If we choose $s > n_0$ such that $s\delta_0 - 1 > k$, since that $g^s(M^{m_s}(\xi_s)) \geq 1$, then there exist $0 \leq j < s$ such that $s - j > s\delta_0 - 1 > k$ and

$$g^i(M^{m_s+j}(\xi_s)) \geq \gamma_1^i, \quad \text{for every} \quad 0 < i \leq s-j.$$

Then we can take $y_k = x_s$, $\upsilon_k = \xi_s$ and $n_k = m_s + j$ and obtain that for every $k > 0$, there exist $y_k \in X$, $\upsilon_k \in \overline{\mathbb{C}}_{x_k}$ and $n_k \geq 0$ such that, for every $0 < n \leq k$,

$$g^{-n}(\upsilon_k) \geq \alpha^{-1}(1+\delta)^n \text{ and } g^n(M^{n_k}(\upsilon_k)) \geq \alpha^{-n/k}(1+\delta)^{-n} > \alpha^{-1}(1+\delta)^{-n}.$$

For every k, let us take $0 \leq l_k \leq n_k$ maximal such that for every $0 < n \leq k + l_k$ we have that $g^{-n}(M^{l_k}(\upsilon_k)) \geq \alpha^{-1}(1+\delta)^n$.

Claim. For every $0 < n \leq n_k - l_k$ we have that $g^{-n}(M^{l_k+n}\upsilon_k) < (1+\delta)^n$.

Proof of the Claim. It is clear that this is true for $n = 1$, and we assume that the property hold for every $0 < n < m$. If we suppose that $g^{-m}(M^{l_k+m}\upsilon_k) \geq (1+\delta)^m$, then we have that for every $0 < s < m$

$$\begin{aligned}(1+\delta)^m \leq g^{-m}(M^{l_k+m}\upsilon_k) &= g^{-s}(M^{l_k+m}\upsilon_k) \cdot g^{-(m-s)}(M^{l_k+m-s}\upsilon_k) \\ &< (1+\delta)^{m-s} \cdot g^{-s}(M^{l_k+m}\upsilon_k).\end{aligned}$$

This implies that $g^{-s}(M^{l_k+m}\upsilon_k) > (1+\delta)^s$, and in particular implies that for $0 < n \leq m$, $g^{-n}(M^{l_k+m}\upsilon_k) \geq (1+\delta)^n > \alpha^{-1}(1+\delta)^n$. To end, for $m < n < m + k$ we have that

$$g^{-n}(M^{l_k+m}\upsilon_k) = g^{-m}(M^{l_k+m}\upsilon_k) \cdot g^{-(n-m)}(M^{l_k}\upsilon_k) \geq \alpha^{-1}(1+\delta)^n,$$

and this contradicts the maximality of l_k. This end the proof of the Claim.

Continuing with the proof of the proposition, as $g^{-n}(M^{n_k+n}(\upsilon_k)) < \alpha(1+\delta)^n$ for $0 < n \leq k$, it follows from the previous claim that $g^{-n}(M^{l_k+n}\upsilon_k) \leq \alpha(1+\delta)^n$ for any $0 < n \leq k + l_k$. Then, if we take $z_k = f^{l_k}(y_k)$ and $\omega_k = M^{l_k}(\upsilon_k)$, we have that for each $0 < n \leq k$

$$g^{-n}(\omega_k) \geq \alpha^{-1}(1+\delta)^n \quad \text{and} \quad g^{-n}(M^n(\omega_k)) \leq \alpha(1+\delta)^n.$$

To end, take (z, ω) an adherence point of (z_k, ω_k), and we have that for $n \geq 0$

$$g^{-n}(\omega) \geq \alpha^{-1}(1+\delta)^n \quad \text{and} \quad g^{-n}(M^n(\omega)) \leq \alpha(1+\delta)^n.$$

It follows form the Corollary 3.3.2 that $z \in \alpha^{-1}H^-(\delta)$ with critical direction ω and that $(z, \omega) \in C(\alpha, \delta)$. ■

We denote by $\operatorname{supp}(X)$ the closed subset of X that support all measure f-invariant, i.e.,

$$\operatorname{supp}(X) = \overline{\cup\{\operatorname{supp}(\nu) \,:\, \nu \text{ is } f\text{-invariant }\}}.$$

Lemma 3.9. *Let* $0 < \delta < b^{-1} - 1$ *and* $1 \leq \alpha < \alpha_0(\delta) = b^{-1}/(1+\delta)$. *Then* $\operatorname{supp}(X) \subset \omega(\alpha^{-1}H^-(\delta))$.

Proof. Any point in the support of an invariant measure ν is approximated by regular points. By the proof of Theorem C, any $x \in \mathcal{R}(A, \nu)$ has infinitely many iterates in $\alpha^{-1}H^{-}(\delta)$. The previous remark and the Poincaré recurrence theorem implies that

$$\operatorname{supp}(\nu) \subset \omega(\alpha^{-1}H^{-}(\delta)) = \overline{\bigcup_{z\in\alpha^{-1}H^{-}(\delta)} \omega(z)},$$

and this implies that $\operatorname{supp}(X) \subset \omega(\alpha^{-1}H^{-}(\delta))$. ■

Lemma 3.10. *If there exist $\delta \in (0, b^{-1} - 1)$ and $\alpha_0(\delta) > \alpha \geq 1$ such that the property $P(\alpha, \delta)$ is not satisfied, then the set* $\operatorname{supp}(X)$ *has dominated splitting.*

Proof. As the property $P(\alpha, \delta)$ is not satisfied, then there exists $k > 0$ such that for every $x \in X$ and $\upsilon \in \overline{\mathbb{C}}_x$ both $g^{-n}(\upsilon) < \alpha^{-1}(1 + \delta)^n$ for $1 \leq n \leq k$, or $g^k(M^m\upsilon) < 1$ for every $m \geq 1$. In particular, for points $x \in \alpha^{-1}H^{-}(\delta)$ with critical direction ξ, can not pass the first condition, then

$$g^k(M^m\xi) < 1. \tag{3.15}$$

We assert that $\omega(\alpha^{-1}H^{-}(\delta))$ has dominated splitting. By the equation (3.15), and taking $0 < \delta' < 1 - b$, for every $x \in \alpha^{-1}H^{-}(\delta)$ with critical direction ξ we have that $g^k(M^m(\xi)) < (1 + \delta')^k$. Let $z \in \omega(x)$ and $(m_l)_l$ a sequence of natural numbers goes to infinity, such that $f^{m_l}(x) \to z$. Taking a subsequence if necessary, there exist a direction $\xi_z \in \overline{\mathbb{C}}_z$ such that $M^{m_l}(\xi) \to \xi_z$. It follows by continuity of g, that

$$(1 + \delta')^k > g^k(M^{m+m_l}(\xi)) = g^k(M^m(M^{m_l}(\xi))) \to g^k(M^m(\xi_z)),$$

and this property is true for every point $z \in \omega(x)$ with $x \in \alpha^{-1}H^{-}(\delta)$, so this implies that $\omega(\alpha^{-1}H^{-}(\delta))$ satisfies the hypothesis of the Criteria of Domination I. In particular $\operatorname{supp}(X)$ has dominated splitting. ■

We can rewrite the Lemma 3.7 to obtain the following Lemma.

Lemma 3.11. *Let β, $\delta > 0$ and $\Lambda \subset X$ compact f-invariant set such that $\Lambda \subset \beta H^{\pm}(\delta)$. Then linear cocycle A has dominated splitting in Λ.*

Corollary 3.3.3. *Let $\Lambda \subseteq X$ compact f-invariant. The linear cocycle A has dominated splitting in Λ if and only if there exist β, $\delta_0 > 0$ such that $\Lambda \subset \beta^{-1}\mathring{H}^{-}(\delta) \cap \beta^{-1}\mathring{H}^{+}(\delta)$, for every $\delta \in (0, \delta_0)$.*

Proof. Let $T\Lambda = E \oplus F$ be a dominated splitting. By Proposition 3.1.6, we have that there exist constants $C > 0$ and $0 < \lambda < 1$ such that $g^{-n}(F_z) \geq C^{-1}\lambda^{-n}$ and

$g^n(E_z) \geq C^{-1}\lambda^{-n}$. It is only necessary to take $C = \beta$ and δ_0 small enough such that $\lambda^{-1} = 1 + \delta_0$, to obtain the first direction.

The reciprocal is immediate by the previous Lemma. ■

Proposition 3.3.6. *If* supp (X) *has dominated splitting but X does not have dominated splitting, then there exists* δ_0 *such that the property* $P(\alpha, \delta)$ *is satisfied for every* $\alpha \geq 1$ *and* $\delta \in (0, \delta_0)$.

Proof. Since that X does not have a dominated splitting and contradicting the Criteria of Domination I, follows that for every positive integer k, there exists a point $x_k \in X$, and a integer $m > 0$ such that for every direction $\omega \in \overline{\mathbb{C}}_{x_k}$ we have that

$$g^k(M^m(\omega)) \geq 1. \tag{3.16}$$

On the other hand, the set α-limit of x_k, $\alpha(x_k)$ support an invariant measure for f. Take $z_0 \in \alpha(x_k) \cap \operatorname{supp}(X)$, then by the previous Corollary there exists one direction ξ_0 such that

$$g^{-n}(\xi_0) > \beta^{-1}(1 + \delta')^n, \quad \text{for every} \quad n \geq 1,$$

and all $\delta' \in (0, \delta_0)$. Let us take $\delta < \delta'' < \delta' < \delta_0$ and k fixed.

Let $(n_t)_t \nearrow \infty$ such that $f^{-n_t}(x_k) \to z_0$. For every positive integer s, we can find some neighborhood $U_s \subset \mathbb{P}(X)$ of ξ_0 such that for every $\xi \in U_s$, holds that $g^{-n}(\xi) \geq \beta^{-1}(1+\delta')^n$ for all $1 \leq n \leq s$. If we take t great enough, $f^{-n_t}(x_k)$ is inside of the projection in X of neighborhood U_s. So, there exists $\xi_s \in \overline{\mathbb{C}}_{f^{-n_t}(x_k)}$ such that $g^{-n}(\xi_s) \geq \beta^{-1}(1 + \delta')^n$ for all $1 \leq n \leq s$. Note that for s great enough we have that $g^{-s}(\xi_s) \geq (1 + \delta'')^s$, hence we have in the hypothesis of the Corollary 3.2.2.

We conclude that, we can find s and l_s such that $s - l_s > k$ and

$$g^{-n}(M^{-l_s}(\xi_s)) \geq (1 + \delta)^n, \quad \text{for every} \quad 0 < n \leq s - l_s,$$

in particular, and calling $\upsilon_k = M^{-l_s}(\xi_s)$, we have that

$$g^{-n}(\upsilon_k) \geq \alpha^{-1}(1 + \delta)^n, \quad \text{for every} \quad 0 < n \leq k.$$

To end, by equation (3.16) we have that there exist m_k such that $g^k(m^{m_k}(\upsilon_k)) \geq 1$, so the property $P(\alpha, \delta)$ is satisfied. ■

With this in mind, we can prove one direction of the Theorem D. Later, we present their proof of the other direction of this Theorem.

Proof of Theorem D:. *If X does not have dominated splitting, then $C(\alpha,\delta) \neq \emptyset$ for some $0 < \delta < 1 - b$ and $1 \leq \alpha < \alpha_0(\delta)$:*
It is sufficient to prove that X satisfies the property $P(\alpha,\delta)$ for some $0 < \delta < 1 - b$ and $1 \leq \alpha < \alpha_0(\delta)$ (Proposition 3.3.5). But if for some pair (α,δ) as before, the property $P(\alpha,\delta)$ is not satisfied, we have that $\operatorname{supp}(X)$ has dominated splitting (Lemma 3.10). Since that X does not have dominated splitting, then by Proposition 3.3.6, for every $0 < \delta' < \min(\delta_0, 1 - b)$ and $\alpha_0(\delta') > \alpha' \geq 1$, $P(\alpha',\delta')$ is satisfied. So we have a contradiction. ■

Now we work to proof the opposite direction of the Theorem D. For this, we use the fact that for every critical point, there exists a critical value intrinsically linked with him. This is the notion of critical pair that we introduce in the following paragraph.

Definition 3.3.5. *We say that a pair $(x,y) \in X \times X$ is a (α,δ)-critical pair if there exists $\alpha \geq 1$ and $\delta > 0$ such that:*

1. *$x \in C(\alpha,\delta)$ with critical direction ξ,*
2. *$y \in V(\alpha,\delta)$ with critical direction ϖ,*
3. *there exist a sequence of positive integer l_k such that*

$$f^{l_k}(x) \to y \quad \text{and} \quad M^{l_k}(\xi) \to \varpi.$$

It follows directly of the previous definition the following Proposition.

Proposition 3.3.7. *If X has dominated splitting, then X does not have a (α,δ)-critical pair.*

Proof. If A have dominated splitting $TX = E \oplus F$, then the angle of the invariant splitting is great of some $\alpha > 0$. If (x,y) is critical pair, the direction F_x is defined by ξ, and E_y is defined by ϖ, but by the third condition on the previous definition we have that $M^{l_k}(F_x) \to E_y$, and this say that $F_y = E_y$; a contradiction. ■

The following Proposition, related each (α,δ)-critical point, with a (α,δ)-critical value.

Proposition 3.3.8. *Let $0 < \delta < 1 - b$ and $1 \leq \alpha < \alpha_1(\delta)$. For every (α,δ)-critical point x, there exists a (α,δ)-critical value y such that the pair (x,y) is a (α,δ)-critical pair.*

The proof is given after.

Remark 18. *Given a critical point x, the critical value y is not, a priori, uniquely defined, can be occurs that for different critical values y and y', makes (x, y) and (x, y') critical pairs.*

Now we will conclude the proof of Theorem D.

Proof of Theorem D:. *If $C(\alpha, \delta) \neq \emptyset$ with $0 < \delta < 1 - b$ and $1 \leq \alpha < \alpha_1(\delta)$, then X does not have Dominated Splitting:*
If there exist critical point, then there exist a critical pair, so by Proposition 3.3.7, X does not have dominated splitting. ∎

Only remains to proof the Proposition 3.3.8. For this, we need the following lemma.

Lemma 3.12. *Let $0 < \delta < 1 - b$, $1 \leq \alpha < \alpha_1(\delta)$ and $x \in C(\alpha, \delta)$ with critical direction ξ. Then there exist $\delta < \delta' < 1 - b$ and $k_0 \geq 1$ satisfying the following property: for every $k \geq k_0$ there exists $m_k \geq 1$ such that $g^k(M^{m_k}(\xi)) \geq (1 + \delta')^k$.*

Proof. First one, we take δ' as in the Remark 17, and we recall that this constant satisfies the inequalities $\delta < \delta' < 1 - b$ and $\alpha(1 + \delta) < 1 + \delta$.

Now by contradiction, we suppose that for every k, there exist $n_k \geq k$ such that for every $m \geq 1$

$$g^{n_k}(M^m \xi) < (1 + \delta')^{n_k}.$$

On the other hand, since x is a (α, δ)-critical point, then $g^{-n}(M^n(\xi)) \leq \alpha^{n+1}(1 + \delta)^n$ for every $n \geq 0$. It follows that for n great enough $g^n(\xi) \geq \alpha^{-1}(1 + \delta')^{-n} > (1 - \delta')^n$.

Form the previous inequalities, we conclude that there exist a sequence $(n_k)_k$ satisfying

$$\log(1 - \delta') \leq \lim_{k \to \infty} \frac{1}{n_k} \log(g^{n_k}(\xi)) \leq \log(1 + \delta'), \tag{3.17}$$

that is the key in the proof of the Criteria of Negative Exponent (see Lemma 3.8). Now the proof follows by arguing as in the proof of Lemma 3.8:

1. Let us take a subsequence if necessary, and we can assume that the sequence of measure
$$\mu_k = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \delta_{M^i(\xi_x)},$$
converge to a measure μ.

2. The projection in X of the measure μ, is a f-invariant measure μ', and denote by $\mathcal{R}(A, \mu')$ the set of regular points in the support of μ'.

3. We have $\mu(\mathrm{supp}\,(\mu) \cap pr^{-1}(\mathcal{R}(A, \mu'))) = 1$.

4. For every $(z, w) \in \mathrm{supp}\,(\mu) \cap pr^{-1}(\mathcal{R}(A, \mu'))$ the limit

$$I(z, w) = \lim_{k\to\infty} \frac{1}{n_k} \log(g^{n_k}(w)) = \pm(\lambda^-(z) - \lambda^+(z)) < \log(1 + \delta').$$

5. Since that μ' is partially hyperbolic, then $\lambda^-(z) - \lambda^+(z) \leq \log(b)$, so we have two possibilities, either $I(z, w) \leq \log(b)$ or $I(z, w) > \log(b^{-1})$.

6. If $I(z, w) = -(\lambda^-(z) - \lambda^+(z)) \geq \log(b^{-1}) > \log(1 + \delta')$ that is a contradiction.

7. Then it follows that

$$\log(b) \geq \int I(z, w) d\mu(z, w) = \lim_{k\to\infty} \frac{1}{n_k} \log(g^{n_k}(M(\xi)) \geq \log(1 - \delta)$$

   that is a contradiction.

This implies the assertion of the Lemma. ■

Remark 19. *Now we justify the item (4) in the definition of critical point (Definition 3.3.3). Let* (x, ξ) *a pair with* $x \in \alpha^{-1}H^-(\delta)$*,* $f^n(x) \notin \alpha\mathring{H}^-(\delta)$ *for every* $n \geq 1$ *and* ξ *the critical direction. Then arguing as in the previous proof, we have that for n great enough* $g^n(\xi) > (1 - \delta')^n$ *where* $\delta' > 0$ *is as in the Remark 17.*

On the other hand if we assume that there are not a maximal element in the orbit of x, then there exist $n_k \nearrow \infty$ *such that* $f^{n_k}(x) \in \alpha^{-1}H^-(\delta)$*. This implies that* $g^{-n_k}(M^{n_k}(\xi)) \geq \alpha^{-1}(1 + \delta)^{n_k}$*, so it follows that*

$$g^{n_k}(\xi) < \frac{\alpha}{(1 + \delta)^{n_k}} < \alpha(1 + \delta)^{n_k} < (1 + \delta')^{n_k}.$$

Hence from the previous inequalities we conclude that there exist a sequence $(n_k)_k$ *satisfying the equation (3.17),and arguing as in the previous proof, this is a contradiction.*

An immediate Corollary from the previous Lemma is the equivalence between critical points and the property $P(\alpha, \delta)$.

Corollary 3.3.4. *Let* $0 < \delta < 1 - b$ *and* $1 \leq \alpha < \alpha_1(\delta)$*. Then the set of critical points* $C(\alpha, \delta)$ *is not empty if and only if the cocycle M has the property* $P(\alpha, \delta)$*.*

Proof. The property $P(\alpha, \delta)$ implies the existence of critical point from the Proposition 3.3.5, and the other direction it follows from the previous Lemma. ■

Now we proceed to give the proof of the Proposition 3.3.8.

Proof of Proposition 3.3.8: Let x be a (α, δ)-critical point, with critical direction ξ.

We will apply the Corollary 3.2.2. Let $k > 0$, $\gamma_0 = (1 + \delta')$ and $\gamma_1 = (1 + \delta)$. Also take n_0 and $\delta_0 > 0$ the number given by this Corollary. Let us take $s > \max(n_0, k_0)$ such that $s\delta_0 - 1 > k$ where k_0 is given in the Lemma 3.12. Since $g^s(M^{m_s}(\xi)) \geq (1+\delta')^s$, then there exists j with $s - j > s\delta_0 - 1 > k$ such that $g^n(M^{m_s+j}(\xi)) \geq (1 + \delta)^n > \alpha^{-1}(1 + \delta)^n$ for every $0 \leq n \leq s - j$. We call $n_k = m_s + j$, $x_k = f^{n_k}(x)$ and $\upsilon_k = M^{n_k}(\xi)$, then

$$g^n(\upsilon_k) \geq \alpha^{-1}(1 + \delta)^n$$

for every $0 < n \leq k$.

Since that x a critical point $g^{-k}(\xi) \geq \alpha^{-1}(1 + \delta) \geq 1$ for k great enough. We conclude that there exist k' such that for every $k > k'$ holds:

1. $g^n(\upsilon_k) \geq \alpha^{-1}(1 + \delta)^n$, for every $1 \leq n \leq k$,
2. $g^{-k}(M^{-n_k}(\upsilon_k)) \geq 1$.

this is, the property $P(\alpha, \delta)$ is satisfied for the inverse cocycle M^{-1}. It follows for the Proposition 3.3.5 that there exist a (α, δ)-critical value y with critical direction ϖ; and a sequence $(l_k)_k$ such that $f^{l_k}(x)$ converge to y and $M^{l_k}(\xi)$ converge to ϖ, then the pair (x, y) is a (α, δ)-critical pair. ■

3.4 Properties of the Critical Point

In this section we discuss the main properties of the critical point. In fact, these properties justify the notion of critical point and highlight its meaning. Moreover, the properties show how the notion of critical point is an intrinsic notion of the dynamics.

First we give a series of property that do not depend on working with holomorphic dynamics.

Later, in subsection 3.4.2, we give a series of properties strongly hinge on the fort that we are dealing with polynomial automorphisms.

3.4.1 General Context

• **Compactness:** Compactness of the set of critical point, it follows from the following elemental observation:

If (x, ξ) is a $(\alpha, \delta$-critical point, then for every $n \geq 1$, $g^{-n}(M^n(\xi)) < (1+\delta)^n$.

The proof of this fact is by induction, and the idea is that assuming that the inequality not hold, we have a contradiction with the fact that $f^n(x) \notin \alpha^{-1}H(\delta)$.

Now if $(x_m, \xi_m) \to (x, \xi)$ where (x_m, ξ_m) are (α, δ)-critical points, it is follows that

$$g^{-n}(\xi) \geq \alpha^{-1}(1+\delta)^n, \quad \text{and} \quad g^{-n}(M^n(\xi)) \leq (1+\delta)^n < \alpha(1+\delta)^n,$$

for every $n \geq 1$, so (x, ξ) is a (α, δ)-critical point. This proof that the set of critical point is closed in the compact bundle $\mathbb{P}(X)$, hence is compact.

• **Invariance by the change metric:** The invariance by the change of metric it is follows from the following Lemma.

Lemma 3.13. *Let $\|\cdot\|$ and $\|\cdot\|'$ two metrics in TX. Let g and h, the norm of the multiplier defined as in the equation (3.9) with the metric $\|\cdot\|$ and $\|\cdot\|'$ respectively. If x is a (α_1, δ)-critical point in the metric $\|\cdot\|$, then there exist α_2 such that is (α_2, δ)-critical point in the metric $\|\cdot\|'$.*

Proof. From compactness, there exits a constant $\beta > 0$ such that for every $z \in X$ and $v \in T_z$ hold that $\beta^{-1}\|v\| \leq \|v\|' \leq \beta\|v\|$. If we define $t : \mathbb{P}(X) \to \mathbb{R}^+$ by $t(z, [v]) = \|v\|'_z/\|v\|_z$, then for every $z \in X$ and $v \in T^*_z$ we have that

$$\beta^{-1} \leq t(z, [v]) \leq \beta, \text{ that is equivalent with } \beta^{-1} \leq t(z, [v])^{-1} \leq \beta.$$

We remark that

$$g^n(\xi) = \frac{|\det(A^n_z)|}{\|A^n_z v_\xi\|^2_{f^n(z)}}$$

where $\|v_\xi\|_z = 1$ and $[v_\xi] = \xi$, and that

$$h^n(\xi) = \frac{|\det(A^n_z)|}{\|A^n_z u_\xi\|'^2_{f^n(z)}}$$

where $\|u_\xi\|'_z = 1$ and $[u_\xi] = \xi$. Since that $u_\xi = \lambda v_\xi$ with $\lambda \neq 0$, then

$$1 = \|u_\xi\|'_z = t(z, \xi)|\lambda| \cdot \|v_\xi\|_z = t(z, \xi)|\lambda|,$$

and this implies that

$$|\lambda| = t(z, \xi)^{-1}.$$

From this equation, and by the definition of g and h, is not difficult to see that for every $n \in \mathbb{Z}$ we have the equality

$$h^n(\xi) = T(z, \xi, n)^2 g^n(\xi),$$

where

$$T(z,\xi,n) = \frac{t(z,\xi)}{t(f^n(z), M^n(\xi))}.$$

Note also that T have the property of cocycle as g, that is

$$T(z,\xi,n+m) = T(f^m(z), M^m(\xi), n)\cdot T(z,\xi,m);$$

and that $\beta^{-2} \leq T \leq \beta^2$. We conclude that

$$\beta^{-4} g^n(\xi) \leq h^n(\xi) \leq \beta^4 g^n(\xi).$$

To end, is only necessary to take $\alpha_2 = \alpha_1\beta^4$ to conclude the Lemma. ■

• **Invariance by Conjugation:** Also we have that the following proposition.

Lemma 3.14. *Critical points are invariant by conjugation, whenever the conjugation is close to the identity.*

Proof. Let $A = (f, A_*)$, $\widehat{A} = (\widehat{f}, \widehat{A_*})$ and $H = (h, H_*)$ linear cocycles and M, $\widehat{M}$ and N, the respective projective cocycles related with them, such that $H \circ A = \widehat{A} \circ H$. The is clear that if we denote $\widehat{z} = h(z)$

$$\widehat{A}^n_{\widehat{z}} = H_{f^n(z)} \circ A^n_z \circ H^{-1}_{\widehat{z}}.$$

Now consider $\widehat{\xi} \in \overline{\mathbb{C}}_{\widehat{z}}$ and $v_{\widehat{\xi}}$ some unitary vector that define the direction $\widehat{\xi}$. Let us take $\xi \in \overline{\mathbb{C}}_z$ such that $N(\xi) = \widehat{\xi}$ and denote by

$$v_\xi = \frac{H^{-1}_{\widehat{x}} v_{\widehat{x}}}{\|H^{-1}_{\widehat{x}} v_{\widehat{x}}\|}$$

and

$$w_{M^n(\xi)} = \frac{A^n_z v_\xi}{\|A^n_z v_\xi\|}$$

we conclude that

$$\widehat{A}^n_{\widehat{z}} v_{\widehat{x}} = \|H^{-1}_{\widehat{x}} v_{\widehat{x}}\| \cdot \|A^n_z v_\xi\| \cdot H_{f^n(z)} w_{M^n(\xi)}.$$

Hence we have that

$$\det(\widehat{A}^n_{\widehat{z}}) = \det(H_{f^n(z)}) \cdot \det(A^n_z) \cdot \det(H^{-1}_{\widehat{z}})$$

and

$$\|\widehat{A}^n_{\widehat{z}} v_{\widehat{x}}\| = \|H_{f^n(z)} w_{M^n(\xi)}\| \cdot \|H^{-1}_{\widehat{x}} v_{\widehat{x}}\| \cdot \|A^n_z v_\xi\|,$$

and we conclude that

$$g^n(\widehat{\xi}, \widehat{M}) = g(M^n(\xi), N) \cdot g^n(\xi, M) \cdot g^{-1}(\widehat{\xi}, N),$$

or equivalently

$$g(M^n(\xi), N) \cdot g^n(\xi, M) = g^n(N(\xi), \widehat{M}) \cdot g(\xi, N),$$

since that when $N \approx id$, $g(\cdot, N) \approx 1$, it follows the Lemma. ∎

• **Far from set with Domination:** As a Corollary from Theorem D, we can give some information in the case for a cocycle A, does not have dominated. We remark that in particular, the set $C(\alpha, \delta)$ is not empty, for every $0 < \delta < 1 - b$ and $1 \leq \alpha < \alpha_1(\delta)$, and this is compact.

Corollary 3.4.1. *Suppose that there exist a compact subset $\Lambda \subset X$, f-invariant that has dominated splitting. Then $d(\Lambda, C(\alpha, \delta)) > 0$.*

Proof. It is immediate from the compactness of $C(\alpha, \delta)$, and that the set of point (α, δ)-critical point in Λ is empty. ∎

In the context of cocycles dynamically defined, we also can give another similarities consequences. We consider f be a biholomorphisms in some complex two dimensional manifold, that have a compact invariant set X. We consider he natural linear cocycle $Df_{\#} = (f, Df)$.

Remark 20. *In the one dimensional context (real or complex), critical point are far from hyperbolic set, however they can be accumulated by hyperbolic sets.*

Corollary 3.4.2. *Suppose that there exist a compact subset $\Lambda \subset X$, f-invariant such that is hyperbolic. Then $d(\Lambda, C(\alpha, \delta)) > 0$.*

We also have that critical points are not regular points.

Lemma 3.15. *A critical point is not a regular point.*

Proof. If we assume that a critical point x (with critical direction ξ) is regular, then the fiber $T_x = E^+ \oplus E^-$ and the exponents related with this splitting satisfies the inequalities

$$\lambda^- \leq \log(b) < 0 \leq \lambda^+,$$

and

$$\lambda^- - \lambda^+ \leq \log(b).$$

We assert that the related direction of ξ, is the subspace E^+. If not, we have

$$\lim_{n\to-\infty}\frac{1}{n}\log\|A_z^n v_\xi\| = -\lambda^-,$$

where v_ξ is unitary and define the direction ξ. From the previous equation and since that $g^n(\xi) \geq \alpha^{-1}(1+\delta)^n$ for $n \leq 0$, we have that

$$\lim_{n\to-\infty}\frac{1}{n}\log(g^n(\xi)) = \lambda^- - \lambda^+ \geq \log(1+\delta)$$

that is a contradiction.

Now as E^+ define the direction ξ, we have that

$$\lim_{n\to\infty}\frac{1}{n}\log(g^n(\xi)) = \lambda^- - \lambda^+.$$

On the other hand, since that $g^n(\xi) \geq \alpha^{-(n+1)}(1+\delta)^{-n}$ we conclude hat

$$\log(b^{-1}) \leq \log(\alpha(1+\delta)) < \log(1+\delta'),$$

where δ' is as in the Remark 17, but this is a contradiction. ■

Moreover, when the cocycle is dynamically defined, critical point are disjoint to *"hyperbolic blocks"*.

More precisely, let f be a Hénon map. We denote by $\mathcal{R} \subset J$ the set of all regular points. Given $C > 0$ fixed, consider the set

$$B(0,C) = \{z \in \mathcal{R} \,:\, |Df^n|E^-(z)| \leq C\exp(n\lambda^-(z)) \ \text{ and } |Df^{-n}|E^+(z)| \leq C\exp(-n\lambda^+(z))\}.$$

This set is a closed set, and given $l \in \mathbb{N}$ we define the *hyperbolic block* of large l

$$B(l,C) = \cup_{k=-l}^{l} f^k(B(0,C)).$$

Since that all point in the hyperbolic point is regular, it is follows from the previous lemma, that critical points are disjoint to hyperbolic blocks.

• **Tangencies of Periodic Contain Critical Points:** Let f be Hénon map with $b = |\det(Df)| < 1$. Without loss of generality, we can assume that p is a fixed point. Let λ^s and λ^u the eigenvalues of Df in p, then $b = |\lambda^s| \cdot |\lambda^u|$. Note that

$$g^{-n}(E^u_p) = \frac{b^{-n}}{|Df^{-n}|E^u_p|^2} = \left(\frac{(\lambda^u)^2}{b}\right)^n > b^{-n}.$$

On the other hand,

$$g^n(E^s_p) = \frac{b^n}{|Df^n|E^s_p|^2} = \frac{|\lambda^u|^n}{|\lambda^s|^n} = \left(\frac{(\lambda^u)^2}{b}\right)^n > b^{-n}.$$

Then there exist $\lambda > b^{-1}$ such that $g^{-n}(E^u_p) > \lambda^n$ and $g^n(E^s_p) > \lambda^n$. We conclude that for every $0 < \delta < 1 - b$, $p \in \mathring{H}^-(\delta)$ and $p \in \mathring{H}^+(\delta)$. Now we fix δ. Since the previous inequalitys open properties, we can find $\varepsilon > 0$ small such that if $z \in W^u_p(\varepsilon)$, then $z \in \mathring{H}^-(\delta)$.

On the other hand, provided that ε small enough, it is follows that for any tangency point $z \in W^u_p(\varepsilon)$, we have that $z \notin \mathring{H}^+(\delta)$. Otherwise, we denote by F_z the tangent direction to $W^u_p(\varepsilon)$ in z that in fact is the critical direction. Since that if $z \in \mathring{H}^+(\delta)$, we conclude that there exist another direction E_z transversal to F_z such that $Df^n_\#(E_z) \to E^u_p$ has $n \to \infty$. Here $Df_\#$ denote the projective cocycle induced by Df.

Now we have that

$$g^n(E^u_p) < \left(\frac{1}{(1+\delta)}\right)^n$$

for n great enough, it is follows that $g^n(E_z)$ contracts for the future, that is a contradiction.

• **Stable and center unstable sets of Critical value and points:** We recall some basic definitions. The *unstable set* of a point x for f, is the set

$$W^u(x) = \{y \in \mathbb{C}^n \ : \ d(f^{-n}(x), f^{-n}(y)) \to 0, \ \text{when } n \to \infty\},$$

where d is the euclidean distance. Similarly, the *local unstable set* of size ε is the set

$$W^u_\varepsilon(x) = \{y \in W^u(x) \ : \ d(f^{-n}(x), f^{-n}(y)) \le \varepsilon, \ \text{for every } n \ge 0\}.$$

Equivalently we can define *stable set* and *local stable set* in the same fashion above, but taking forward iterate.

In the [P-RH], the authors proof the following two result. It is not difficult to see, that it is possible to adapt the proof for the holomorphic context. See [P-RH] for details.

We take α and δ as in the Theorem C.

Lemma 3.16. *If $y \in \alpha H^+(\delta)$ and $W^s_\varepsilon(y)$ is a C^1-injective immersed submanifolds, then $E_y = T_y W^s_\varepsilon(y)$ where E_y is the critical direction related with y. Similarly, $y \in \alpha^{-1}H^-(\delta)$ and $W^s_\varepsilon(u)$ is a C^1-injective immersed submanifolds, then $F_y = T_y W^s_\varepsilon(y)$, where F_y is the critical direction related with y.*

Also we have

Lemma 3.17. *Let Λ be a compact invariant dissipative set. There exist a continuous functions $\varphi : \alpha H^+(\delta) \to Emb^1(\mathbb{D}, \mathbb{C}^2)$, such that if $W^s_\varepsilon(x) = \varphi^s(x)(\varepsilon\mathbb{D})$, the following properties hold:*

1. *$T_y W^s_\varepsilon(y) = E_y$, where x is the critical direction related with y,*

2. *$W^s_\varepsilon(x) = \{y \in \mathbb{C}^2 \,:\, \mathrm{dist}\,(f^n(x), f^n(y)) < \varepsilon\}$,*

3. *there exists $\lambda < 1$ and $C > 0$ such that for any $x \in \alpha H^+(\delta)$*

   (a) *$|Df^n(E_x)| < C\lambda^n$.*

   (b) *$\mathrm{dist}\,(f^n(x), f^n(y)) < C\lambda^n$, for every $y \in W^s_\varepsilon(x)$.*

Now appear a question.

Question: *Points in $\alpha^{-1}H^-(\delta)$, has tangent manifolds?*

It is natural to ask of critical points are related to unstable direction. Moreover, assuming an affirmative answer for the previous question, in the context of Hénon maps appear:

Question: *Let x be a critical point and let $W^u_{loc}(x)$ be a local submanifold tangent to the critical direction F_x, does it hold that $W^u_{loc}(x) \cap U^+ \neq \emptyset$?.*

3.4.2 Critical Point in the Holomorphic Context

Recall that for polynomial (or rational maps) in $\mathbb{C}$, always exists critical point. Moreover they determine the global dynamics. In this direction recall the following statement: *The Julia set $J_p \subset \mathbb{C}$ is hyperbolic, if and only if $PC(p) \cap J_p = \emptyset$.* Here $PC(p)$ denote the postcritical set defined by

$$PC(p) = \overline{\cup_{n\geq 1} p^n\left(\{z \,:\, p'(z) = 0\}\right)}.$$

Following these result, we wonder (in two dimensional dynamics) if always exist critical point (even outside of J) and if they determine the global dynamics. In fact, recall that we have proved: *If $CP \cap J = \emptyset$,if and only if J has dominated splitting*, where CP is any $C(\alpha, \delta)$ as in the Theorem D.

But, do they always exist outside of J, when J has dominated splitting?. In fact, we can formulate the following questions.

Question A: *Do always exists critical point in $\mathbb{C}^2$?*

Question B: *If K^+ has interior, always exists critical point in K^+ ?*

We can answer positively the Question B, for a polynomial automorphisms close to one dimensional polynomial p. Let

$$f_\delta(x, y) = (y, p(y) - \delta x),$$

with $|\delta|$ small. When we refer to critical point of p, i.e., $p'(x) = 0$ we denote them as one dimensional critical point. With the words critical point, we are referring to the critical point we have introduced.

Observe that to introduce of critical point, we only need to deal with compact invariant set, so the notion can be extended to K^+.

Let us assume that the polynomial p satisfies:

1. there are not one dimensional critical point in J_p,
2. J_p is connected,
3. the filled Julia set K_p has interior.

The item 3, implies that the set K_p^+ associated with the two dimensional map $f_0 : (x, y) \mapsto (y, p(y))$, has non empty interior. In fact, is easy to see that $K_p^+ = \mathbb{C} \times K_p$. We recall that since $|\delta|$ small, $f = f_\delta$ is close to f_0, hence J_f is close to the set $J_0 = \{(y, p(y)) \ : , y \in J_p\}$.

Proposition 3.4.1. *Under the previous hypothesis, there are a critical point in the interior of K^+.*

Proof. If there are not critical points in K^+, then this has dominated splitting, so K^+ is foliated by holomorphic stable leaves

$$K^+ = \sup_{x \in K^+} W^s(x).$$

On the other hand, the map

$$z = (x, y) \hookrightarrow (y, p(y)) \hookrightarrow p(y),$$

is holomorphic and the image of K_p^+ is K_p, that is contained in the y-axis. So, for $|\delta|$ small enough, there exists a holomorphic disc D, close to the y-axis and transversal to the stable foliation of f in K^+.

We define π^s the projection to D, by the stable foliation. Now we define

$$z \in D \cap K^+ \hookrightarrow f(z) \hookrightarrow \pi^s(f(z)) \in D.$$

Then the map $(\pi^s \circ f) : D \to D$ is a holomorphic one dimensional map. We denote by pr_2 the projection in the second variable. Since π^s close to pr_2 nearby the Julia set, and f close to f_0, then $\pi^s \circ f$ is close to p nearby the Julia set, thus it is follows that $\pi^s \circ f$ has degree equal to degree of polynomial p.

From the previous observation, there exist $c \in D$ such that $(\pi^s \circ f)'(c) = 0$. Now, Df does not have kernel, it is follows that

$$(Df)(T_c D) \subset \text{ Kernel}(D\pi^s).$$

Since that $\text{Kernel}(D\pi^s) = E^s$, we conclude that $(Df)(T_c D) \subset E^s$ but this is a contradiction, because $T_c D \pitchfork E^s$. ∎

3.5 Another Proof of Theorem 2.4.2

In this subsection we present an independent proof of the Theorem 2.4.2, that not use the Mañé Theorem. A first important remark is the following.

Remark 21. *Is useful remark, since the $F(x) = E^+_x$ in every regular point, is easy to see that for $x \in \mathcal{R}$*

$$\lim_{n\to\infty} \frac{1}{n} \log(g^n(F(x)) = \pm(\lambda^-(x) - \lambda^+(x)).$$

Since $g^n(F(x)) \leq \lambda^n$, the previous limit is negative, thus is equal to $\lambda^-(x) - \lambda^+(x)$.

For the proof we recall two classical results. The first of them, is a characterization for hyperbolicity, and the second is a corollary of the Pliss Lemma.

Lemma 3.18. *If $\|Df^{-n}|_{F(x)}\| \to 0$ for every $x \in J^*$, then J^* is hyperbolic.*

Lemma 3.19. *Given $0 < \gamma_1 < \gamma_0$, there exist N_0 and δ_0 satisfying the following property: If there exist $x \in J^*$ such that $\|Df^{-n}|_{F(x)}\| > \gamma_0^n$, with $n \geq N_0$, then there exists $0 \leq j < n$ such that $n - j > n\delta_0 - 1$, and*

$$\|Df^{-i}|_{F(f^{-j}(x))}\| > \gamma_1^i,$$

for every $0 < i \leq n - j$.

Proof of Theorem 2.4.2. We suppose that $J^* \cap J_0 = \emptyset$. In particular this implies that any invariant measure supported in J^* is hyperbolic.

On he other hand, if J^* is not hyperbolic, then there exists $x_0 \in J^*$ such that

$$\|Df^{-n}|_{F(x_0)}\| \nrightarrow 0.$$

It follows that there exists a constant $c > 0$ such that for every n, there exist $N(n) \geq n$ such that

$$\|Df^{-N(n)}|_{F(x_0)}\| = \prod_{j=1}^{N(n)} \|Df^{-j}|_{F(f^{1-j}(x))}\| > c.$$

We define $\alpha_n = \sqrt[n]{c_0}$ and $\beta_n = \alpha_n - 1/n$. It is clear that $\alpha_n \to 1$ and $\beta_n \to 1$.

We fix $k > 0$. Let us take $\gamma_0 = \alpha_k$ and $\gamma_1 = \beta_k$. Let N_0 and δ_0 the constants from the Pliss Lemma. Also we take n great enough such that:

- $N(n) > \max(N_0, k)$,
- $N(n)\delta_0 - 1 > k$.

Since $(\alpha_{N(n)})^{N(n)} > (\alpha_k)^k$, we conclude that: *there exists z_k in the past orbit of x_0, such that for all $0 < n \leq k$ we have that*

$$\|Df^{-n}|_{F(z_k)}\| > \beta_k^n.$$

We take $x_k = f^{-k}(z_k)$ then we have that for every $0 < n \leq k$

$$\|Df^n|_{F(x_k)}\| < \beta_k^{-n}.$$

Take $x \in J^*$ such that $x_n \to x$, then we have that

$$\|Df^n|_{F(x)}\| \leq 1$$

for every n.

From the previous inequality, and the property of ρ-hyperbolicity with constant μ_0, we conclude that

a. $b^n \leq g^n(F(x))$, for every $n \geq 1$, and

b. $g^n(F(f^m(x))) < \lambda^n$, for every m, $n \geq 1$.

Form the previous inequalities, we conclude that

$$\log(b) \leq \lim_{n\to\infty} \frac{1}{n} \log(g^n(F(x))) \leq \log(\lambda) < 0.$$

Now the proof follows by arguing as in the proof of Criteria of Negative Exponent (Lemma 3.8):

1. Let us take a subsequence $(n_k)_k$, such that the sequence of measure

$$\nu'_k = \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{Df^i_\#(F(x))},$$

   converge to a measure ν', where $Df_\#$ is the projective cocycle in the trivial bundle $J \times \overline{\mathbb{C}}$, of the linear cocycle Df. Note that the support of ν' is contained in the omega limits set of $(x, F(x))$, by the cocycle $Df_\#$.

2. The projection in J^* of the measure ν' is a f-invariant measure ν. Note that the support of ν is contained in the omega limits set of x.

3. We have $\nu'(\text{supp}\,(\nu) \cap pr^{-1}(\mathcal{R}(\nu))) = 1$.

4. For every $(z, w) \in \text{supp}\,(\mu') \cap pr^{-1}(\mathcal{R}(\nu))$, from the Remark 21 we have that

$$I(z, w) = \lim_{k\to\infty} \frac{1}{n_k} \log(g^{n_k}(w)) = \lambda^-(z) - \lambda^+(z).$$

5. Since that ν is hyperbolic, then $\lambda^-(z) - \lambda^+(z) < \log(b)$, it follows from (a), that

$$\log(b) > \int I(z, w) d\nu'(z, w) = \lim_{k\to\infty} \frac{1}{n_k} \log(g^{n_k}(Df_\#(F(x))) \geq \log(b)$$

that is a contradiction.

This conclude the proof of the Theorem. ■

Chapter 4

Some Open Questions

In this Chapter we summarize, the list of question that appear a long of the work, for contextualize them, we refer the page that they appear.

1. Can we find non-planar dominated splitting and non-hyperbolic Hénon maps? (pag. 11).
2. Is the dynamics conjugated to a one dimensional Siegel disk or Herman ring multiplied by a uniform contraction? (pag. 14).
3. Under which conditions the hypothesis of dominated splitting implies hyperbolicity on J? (pag. 17).
4. If f is a dissipative Hénon map with dominated splitting in J^*, and all periodic point in J^* are hyperbolic, Is f hyperbolic in J? (pag. 17).
5. Let f be a dissipative Hénon map with dominated splitting in J. If f is also expansive in J, is the set J_0 empty? (pag. 41).
6. Let f be a dissipative Hénon map with dominated splitting in J. If the set J_0 is not empty, there exist some condition under which $J_0 \setminus J^* \neq \emptyset$? (pag. 41).
7. Points in $\alpha^{-1}H^-(\delta)$, has tangent manifolds? (pag. 89).
8. Let x be a critical point and let $W^u_{loc}(x)$ be a local submanifold tangent to the critical direction F_x, does it hold that $W^u_{loc}(x) \cap U^+ \neq \emptyset$? (pag. 89).
9. Always exists critical point in $\mathbb{C}^2$? (pag. 89).
10. If K^+ has interior, always exists critical point in K^+ ? (pag. 89).

Another questions: ??

Appendix A

Introduction

The principal motivation for this Appendix, is explain how is defined the measure of maximal entropy μ, for Hénon maps. However, this Appendix is not fundamental to read, for the comprehension of the principal topics presented in this works. The reader can skip this Chapter.

In this Appendix we present in the first section, how Potential Theory is an important tool, in the study of dynamics of polynomial in one complex variable. We use this how a motivation to introduce the notions of Pluripotential Theory, that is the natural extension on higher dimension.

A.1 Potential Theory and Polynomial

It know of the study of dynamics in one complex variable for polynomials, that the Potential Theory plays a fundamental role to describe the metrics properties related with the Julia set. We briefly describe these results.

For a finite Borel measure ν, we define its *(logarithmic) potential* as the function

$$p_\nu(z) = \int \log|z-w| d\nu(w).$$

We also consider the energy $I(\nu)$, of ν as the integral

$$I(\nu) = \int\int \log|z-w| d\nu(z)d\nu(w) = \int p_\nu(z)d\nu(z).$$

For a compact set $K \subset \mathbb{C}$, denote by $\mathcal{P}(K)$ the set of all finite Borel measures with support within K. If there exist $\mu \in \mathcal{P}(K)$ such that

$$I(\mu) = V = \sup_{\mathcal{P}(K)} I(\nu),$$

then μ is called an *equilibrium measure* for K. To end, we define the *(logarithmic) capacity* for K by

$$c(K) = e^V,$$

and we say that a set K is polar if $c(K) = 0$.

We list a series of result of potential theory, relating with non-polar set. One reference for this topics is the book of Thomas Ransford [R] (see also [M]).

1. The potential p_ν is subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus \operatorname{supp}(\nu)$.

2. The Laplacian (in the sense of distribution) of p_ν is equal to $\Delta p_\nu = 2\pi\nu$.

3. Every compact set K in $\mathbb{C}$ has an equilibrium measure, that we denote by μ.

4. If K is non-polar, μ is unique and $\operatorname{supp}(\mu) \subset \partial_e K$ where ∂_e denote the exterior boundary of K (Also in the case non-polar, μ can be obtained as an *harmonic measure*, see Definition 4.3.1 and Theorem 4.3.14 in [R] for details).

5. Let D_∞ the connected component of $\overline{\mathbb{C}} \setminus K$ which contains ∞. The Green function G_K on D_∞ with a logarithmic pole at ∞ is (by definition) the infimum of the Perron family $\mathcal{G}$ consisting of all non-negative superharmonic functions $s(z)$ such that $s(z) - \log|z|$ is bounded near ∞. When K is non-polar set, the Green function always exist and satisfy the equalities

$$G_K(z) = p_\mu(z) - I(\mu) = p_\mu(z) - \log c(K) = \log|z| - \log c(K) + o(1), \;\; as\ z \to \infty. \tag{A.1}$$

Denote K_f be the filled Julia set for a polynomial f of degree not less than 2, and $J_f = \partial K$ be the Julia set related with f. If we write $f(z) = a_d z^d + O(z^{d-1})$, is possible to show that $c(K_f) = 1/|a_d|^{1/(d-1)} > 0$. Its follows that J_f support an unique equilibrium (harmonic) measure μ_f such that

$$\mu_f = \frac{1}{2\pi}\Delta G_f,$$

where G_f is the Green function on $\mathbb{C} \setminus K_f$, and the Laplacian is in the sense of distributions.

All the previous results for are uniquely based in the potential theory.

A first dynamical result that give a new focus of attention, is that the Green function has a explicitly expression in terms of the dynamics of the polynomial f. First one, we can prove that $G(f(z)) = dG(z)$, and this combining with the equation (A.1), we obtain that the Green function is dynamically defined by the equation

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ |z| = \max\{0, \log |z|\}$. The second important fact, is proper to Brolin (see [Br]), that establish that the equilibrium measure is the weak limit, of asymptotic mass distribution related with the dynamics of f. Fixed a point $z_0 \in \mathbb{C}$ be a not exceptional point, we define:

$$\mu_n = \frac{1}{d^n} \sum_{f^n(z)=z_0} \delta_z.$$

Theorem A.1.1 (Brolin). *The sequence $(\mu_n)_n$ converge to μ_f in the weak topology, hence μ_f is a f invariant measure. Moreover, the equilibrium measure is strongly mixing (so ergodic).*

Also we have the following ergodic result (see [G] and [L]).

Theorem A.1.2. *The measure μ_f is the unique measure of maximal entropy.*

A.2 Pluripotential Theory

For a simple exposition of the topic of current and Pluripotential Theory, oriented for the study of Hénon maps we suggest the book *Holomorphic dynamics* of Morasawa *Et.al.*, in [MNTU]. Other recommendations for Pluripotential Theory are, [K] and the Appendix (A.1–A.7) on *Dynamic of Rational Map on* $\mathbb{P}^k$, of Nessim Sibony in [C]. The following subsection, are based in the Appendix of Nessin Sibony, refer above.

A.2.1 Plurisubharmonic Function and Smooth Approximation

Definition A.2.1. *Let U be a open set of $\mathbb{R}^n$. A function $u : \mathbb{R}^n \to [-\infty, \infty)$ is subharmonic if:*

1. *u is not identically $-\infty$ in any component of U,*

2. *u is uppersemicontinuous (u.s.c.),*

3. *u satisfy the sub-mean value property: for any* $x_0 \in U$ *and* $r > 0$ *such that* $B_r(x_0) \Subset U$*,*

$$u(x_0) \leq M(x_0, r) = \int_{|y|=1} u(x_0 + ry)\frac{d\sigma(y)}{c_n},$$

where $c_n = \int_{|y|=1} d\sigma(y)$ *and* σ *is Lebesgue measure in the sphere.*

It can be shown that is u is subharmonic, then $u \in L^1_{loc}(U)$ and $\Delta u \geq 0$ in the sense of distributions. If $v \in L^1_{loc}(U)$ and $\Delta v \geq 0$, then v is equal to a subharmonic function (a.e.).

Theorem A.2.1. *Let* $(v_j)_j$ *be a sequence of subharmonic functions on a domain* $U \subset \mathbb{R}^n$*. Suppose that the sequence* $(v_j)_j$ *is bounded above on every compact subset of* U*.*

1. *If* $(v_j)_j$ *does not converge to* $-\infty$ *on compact subsets of* U*, then there is a sequence* $(v_{j_k})_k$ *that converge in* $L^1_{loc}(U)$ *to subharmonic function.*

2. *If* v *is subharmonic and* $v_j \to v$ *in* $L^1_{loc}(U)$*, then*

$$\limsup_{j\to\infty} \sup_K (v_j - f) \leq \sup_K (v - f)$$

for every compact $K \subset U$ *and every function* f *that is continuous on* K*.*

Definition A.2.2. *Let* U *be a open set of* $\mathbb{C}^n$*. Let* $u : \mathbb{C}^n \to [-\infty, \infty)$ *be a u.s.c. function that is not identically* $-\infty$ *in any component of* U*. We say that* u *is plurisubharmonic (psh) if*

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + we^{i\theta})d\theta,$$

Remark 22. *Any psh function is subharmonic has function of* $\mathbb{R}^{2n}$*.*

For any differentiable function $v : U \subset \mathbb{C}^n \to \mathbb{C}$, we can express his differential dv has a a sum of a $\mathbb{C}$-linear part and an anti $\mathbb{C}$-linear part $dv = \partial v + \overline{\partial} v$. Using the standard notation

$$z_j = x_j + iy_j, \quad dz_j = dx_j + idy_j, \quad d\overline{z}_j = dx_j - idy_j$$

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

one has the following formula:

$$dv = \sum_{j=1}^{2} \left(\frac{\partial v}{\partial x_j} dx_j + \frac{\partial v}{\partial y_j} dy_j \right);$$

$$\partial v = \sum_{j=1}^{2} \frac{\partial v}{\partial z_j} dz_j;$$

$$\overline{\partial} v = \sum_{j=1}^{2} \frac{\partial v}{\partial \overline{z}_j} d\overline{z}_j.$$

We have that $d = \partial + \overline{\partial}$, and define $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$ so that $dd^c = \frac{1}{\pi}\partial\overline{\partial}$.

Plurisubharmonic function have the following properties.

Properties A.2.1. *1. If f is holomorphic in U, then $\log \|f\|$ is psh in U.*

2. *A function u is* **pluriharmonic** *if $dd^c u = 0$. In this case, it is the real part of a holomorphic function. One can also write $u = \log \|h\|$, where h is a nonvanishing holomorphic function.*

3. *If $g : U \to U'$ is a holomorphic map between two open set in $\mathbb{C}^n$ and u is psh in U', then $u \circ g$ is either psh in U or $-\infty$.*

4. *A function $v \in L^1_{loc}$ is equal almost everywhere to a psh function if and only if*

$$\sum_{j,k=1} \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0,$$

for every $w \in \mathbb{C}^n$. This means that the left-hand side define a positive measure. Since that

$$dd^c v = \frac{1}{\pi} \sum_{j,k=1} \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k} dz_j d\overline{z}_k,$$

we say that v is psh if and only if $dd^c v \geq 0$.

For any two real functions $u, v : \mathbb{C}^n \to \mathbb{R}$ we define their *convolution* has the function $u * v : \mathbb{C}^n \to \mathbb{R}$ given by

$$u * v(z) = \int_{\mathbb{C}^n} u(z - w) v(w) dV(w).$$

Proposition A.2.1. *Let u be a psh function on $U \subset \mathbb{C}^n$, with $u \not\equiv -\infty$. Let $\chi : \mathbb{C}^n \to \mathbb{R}$ be a function satisfying:*

$$\chi \in C^\infty, \quad \chi \geq 0, \quad \chi(z) = \chi(\|z\|), \quad \operatorname{supp}(\chi) \subset B(0,1), \quad \int_{\mathbb{C}^n} \chi dV = 1.$$

For every $\varepsilon > 0$ define

$$\chi_\varepsilon(z) = \frac{1}{\varepsilon^{2n}}\chi\left(\frac{z}{\varepsilon}\right).$$

Then $u_\varepsilon = u * \chi_\varepsilon$ is a C^∞ psh function in $\{z \in U : d(z, \partial U) > \varepsilon\}$, for each $\varepsilon > 0$. Moreover, $u_\varepsilon \downarrow u$, and $u_\varepsilon \to u$ in $L^1_{loc}(U)$, as $\varepsilon \downarrow 0$.

A.2.2 Current

We denote by $\mathcal{D}^p$ the space of compactly supported smooth form of degree p on $\mathbb{R}^n$. A sequence $(\varphi_j)_j \subset \mathcal{D}^p$ converges to 0 if:

1. write $\varphi_j = \sum_{|I|=p} \varphi_I^j dx_I$, where I is a ordered multi-index, with $I = \{i_1 < \cdots < i_p\}$,
2. there exists a compact $K \subset \mathbb{R}^n$ such that $\operatorname{supp}(\varphi_I^j) \subset K$ for all I and j,
3. $D^\alpha(\varphi_I^j) \to 0$ uniformly as $j \to \infty$, for all I and $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}^+)^n$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \text{with } |\alpha| = \sum_{j=1}^n \alpha_j.$$

We denote by $\mathcal{D}_p$ the dual space $(\mathcal{D}^p)'$. An element $T \in \mathcal{D}_p$ is said a *current of dimension p* (or current of degree $n-p$). We say that a sequence of current of dimension p, $(T_j)_j$ converge to a current T, and denote by $T_j \to T$; if for every $\varphi \in \mathcal{D}^p$ we have

$$\langle T_j, \varphi \rangle = \langle T, \varphi \rangle.$$

Also, we define the set $\operatorname{supp}(T)$ as the set $\operatorname{supp}(T) := (\mathcal{N}(T))^c$ where

$$\mathcal{N}(T) = \operatorname{Int}\{z \in \mathbb{C}^n : \exists U \text{ neighborhood of } z, \text{ such that } T(\varphi) = 0, \forall \varphi, \text{ with } \operatorname{supp}(\varphi) \Subset U\}.$$

We can think a current of dimension p as a $(n-p)$-form with distribution as coefficients. For this, first note that for each ordered multi-index J with $|J| = p$ there exist a unique ordered multi-index J^* with $|J^*| = n - p$ such that $J \cup J^* = \{1, \ldots, n\}$. Let I, J are ordered multi-indexes with $|I| = n - p$, $|J| = p$, and let S be a distribution and ϕ a compact supported smooth function. We define

$$\langle S\, dx_I, \phi dx_J \rangle = \begin{cases} (-1)^{\sigma(I,J)} \langle S, \phi \rangle & , \text{ if } I = J^* \\ 0 & , \text{ if } I \neq J^* \end{cases},$$

where $\sigma(I, J)$ is the signature of the permutation

$$(i_1, \ldots, i_{n-p}, j_1, \ldots, j_p) \mapsto (1, \ldots, n).$$

To end, if T is a current of dimension p and we define the distribution T_I by the equation

$$\langle T_I, \phi dx_J\rangle = (-1)^{\sigma(I,J)}\langle T, \phi dx_J\rangle,$$

is not difficult to see that

$$\left\langle \sum_{|I|=n-p} T_I dx_I, \sum_{|J|=p} \varphi_J dx_J \right\rangle = \langle T, \varphi\rangle$$

when $\varphi = \sum_{|J|=p} \varphi_J dx_J$. With this in mind, we say that

$$T = \sum_{|I|=n-p} T_I dx_I,$$

and we denote

$$\langle T, \varphi\rangle = \int T \wedge \varphi.$$

Example 8. The space $\mathcal{D}^0 = C_0^\infty$ is the space of all smooth compactly supported function in $\mathbb{R}^n$, then $\mathcal{D}_0 = \mathcal{D}'$ the space of distributions.

Example 9. Let M be a smooth manifold in $\mathbb{R}^n$ of dimension p, for $\varphi \in \mathcal{D}^p(M)$ define

$$\langle [M], \varphi\rangle = \int_M \varphi,$$

define a current of dimension p.

Example 10. Let $g : N \to M$ be a proper smooth map between the manifolds N and M, then

$$\langle g_*[N], \varphi\rangle = \int_N g^*\varphi,$$

is a current of dimension p in M.

Example 11. Let $T \in \mathcal{D}_p$ and $\alpha \in \mathcal{D}^k$, then we define $T \wedge \alpha$ be the relation

$$\langle T \wedge \alpha, \varphi\rangle = \langle T, \alpha \wedge \varphi\rangle,$$

this define a current of dimension $p - k$.

Example 12. Given a current T of dimension q, we define the current dT of degree $q + 1$ be the relation

$$\langle dT, \varphi\rangle = (-1)^{p+1}\langle T, d\varphi\rangle,$$

for $\varphi \in \mathcal{D}^{n-q-1}$. We say that the current T is closed, if $dT = 0$.

Example 13. Let $f : M \to N$ be a smooth proper submersion. Given $\psi \in \mathcal{D}^{n-p}$ we denote $f_*\psi$ the pushforward of ψ. This is given by the relation

$$\langle f_*\psi, \varphi\rangle = \langle \psi, f^*\varphi\rangle = \int \psi \wedge f^*\varphi.$$

When T is a current, we define f^*T be the dual formula, this is,

$$\langle f^*T, \varphi\rangle = \langle T, f_*\varphi\rangle.$$

Note that when T is smooth, f^*T is the usual pullback of the form T. The pullback operation has the following properties.

(a) $\deg(f^*T) = \deg(T)$.

(b) If ψ is a smooth form, then $f^*(T \wedge \psi) = f^*T \wedge f^*\psi$.

(c) $d(f^*T) = f^*(dT)$.

(d) $\operatorname{supp}(f^*T) \subset f^{-1}(\operatorname{supp} T)$.

(e) If $T_j \to T$, the $f^*T_j \to f^*T$.

A.2.3 Positive Currents

Now we consider current acting in the space $\mathcal{D}^{p,q}(\mathbb{C}^n)$ of the smooth forms with compact support of bidegree (p, q). We denote this space by $\mathcal{D}_{p,q}$ and an element of $\mathcal{D}_{p,q}$ is a current of bidimension (p, q) or of bidegree $(n - p, n - q)$. As before, we can think a current of dimension (p, q) as a differential form of bidegree $(n - p, n - q)$, whit distribution as coefficients.

Also we consider the Poincaré operator $d = \partial + \overline{\partial}$ and the operator $d^c = i/2\pi(\overline{\partial} - \partial)$. If T is a current of bidimension (p, p), then we have the following relations:

$$\langle dT, \varphi\rangle = -\langle T, d\varphi\rangle$$
$$\langle d^cT, \varphi\rangle = -\langle T, d^c\varphi\rangle$$
$$\langle dd^cT, \varphi\rangle = \langle T, dd^c\varphi\rangle.$$

Definition A.2.3. *Let T be a current of bidimension (p, p). We say that T is a positive current, and denote by $T \geq 0$, if the following properties holds: for every collection $\{\alpha_1, \ldots, \alpha_p\} \subset \mathcal{D}^{1,0}$ define*

$$\varphi = i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_1 p \wedge \overline{\alpha}_p,$$

then $\langle T, \varphi\rangle \geq 0$.

Definition A.2.4. *We say that a current $T = \sum_I T_i dx_I$ is representable by integration, if each distribution T_I is a regular measure.*

Proposition A.2.2. *Any positive current of bidimension (p, p) on an open set of $\mathbb{C}^n$, is representable by integration.*

Remark 23. *The previous Proposition generalizes a result of distribution:*

"If T is a positive distribution in an open set U
(i.e. $T(\varphi) \geq 0$ for any non-negative test function),
then T is a measure."

Note also that if T is representable by integration, then the current can be extended to the space of all continuous forms.

Example 14. Let u be a psh function. Since that

$$\langle dd^c u w, w\rangle = \sum_{j,k=1} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) w_j \overline{w}_k \geq 0,$$

for every $w \in \mathbb{C}^n$, it follows that $dd^c u$ is a closed positive current of bidegree $(1, 1)$.

Example 15. Let Z be an analytic subset of an open set U of $\mathbb{C}^n$, of pure dimension p. Let Reg Z the set of regular point of Z. Lelong has shown that the current $[Z]$, of bidimension (p, p) defined by

$$\langle [Z], \varphi\rangle = \int_{\text{Reg } Z} \varphi,$$

is a closed positive current of bidimension (p, p).

Proposition A.2.3. *Let T be a closed positive current of bidegree $(1, 1)$, on a open set $U \subset \mathbb{C}^n$. Then for every $z_0 \in U$ there exist a neighborhood $U_0 \subset U$ and a psh function on U_0, such that $T = dd^c u$.*

Remark 24. *If u is a psh function as in the previous Proposition, we say that u is a potential of T. In the case that U is equal to $\mathbb{C}^n$, the potential function is globally defined. If u_1 and u_2 are two potential of T, in the same open set, then $u_1 - u_2$ is a pluriharmonic function.*

A.2.4 Exterior Product of Currents

Is easy to see, that for two smooth functions u and v, is valid the following formula:

$$(dd^c u) \wedge (dd^c v) = dd^c(u \cdot dd^c v).$$

Moreover, the previous formula is true inductively for more functions, that is,

$$dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_p = dd^c(u_1 \cdot dd^c u_2 \wedge \cdots \wedge dd^c u_p).$$

This fact make sense in the context of positive current.

Let T be a closed positive current of bidimension (p, p) in an open set $U \subset \mathbb{C}^n$. Set $\beta = dd^c\|z\|^2$. We denote by $|T|$ the positive measure defined by

$$|T| = S \wedge \beta^p.$$

If $u \in \mathrm{Psh}(U) \cap L^1_{loc}(|T|)$, then uT is a current in U. We define the current $dd^c u \wedge T$ as before, that is, $dd^c u \wedge T = dd^c(uT)$. We have the following results.

Theorem A.2.2. *Let T be a closed positive current of bidimension (p, p) in an open set $U \subset \mathbb{C}^n$.*

(1) If $u \in \mathrm{Psh}(U) \cap L^1_{loc}(|T|)$, then the current $dd^c u \wedge T$ is a closed positive current in U. Moreover, if $u_j \to u$ in $L^1_{loc}(U)$, then $dd^c u_j \wedge T \to dd^c u \wedge T$ in the sense of current.

(2) Let $u_k \in \mathrm{Psh}(U) \cap L^\infty_{loc}(|T|)$, for $k = 1, \ldots, q$. For every k, let u_k^j be a decreasing sequence of psh functions that converge pointwise to u_k. Then

(a) $u_1^j dd^c u_2^j \wedge \cdots \wedge dd^c u_q^j \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_q$,

(b) $dd^c u_1^j \wedge dd^c u_2^j \wedge \cdots \wedge dd^c u_q^j \to dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_q$,

in the sense of currents.

Remark 25. *In particular, when u and u_k are continuous and bounded in the neighborhood U, then the previous results of convergence, are valid.*

A.2.5 The Invariant measure for Hénon maps

Now we back with the description of the invariant measure for Hénon map.

Definition A.2.5. *The Green function's related with a Hénon map f of degree d, are defined be the limit*

$$G^{\pm}(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(z)\|,$$

that always exist (see [BS1]).

Remark 26. *The Green function is a measure of the rate of logarithmic scape points.*

We recommend review the definitions of filtration, exposed in the Chapter [**?**]. The Green functions are the following properties:

Properties A.2.2.

1. *$G^{\pm}$ are continuous plurisubharmonic function in $\mathbb{C}^2$, and pluriharmonic function in $U^{\pm} \cup \operatorname{int}(K^{\pm})$;*
2. *$K^{\pm} = \{z \in \mathbb{C}^2 \,:\, G^{\pm}(z) = 0\}$ and $G^{\pm} \geq 0$ in $\mathbb{C}^2$.*
3. *$G^{\pm}(f(z)) = d^{\pm 1} G^{\pm}$.*
4. *Let $M_x = \{x\} \times \mathbb{C}$ and $Y_x = K^+ \cap M_x$. The function $y \mapsto G^+(x,y)$ is the Green function G_{Y_x}. A symmetrical result holds for the function G^-.*
5. *$G^+(x,y) - \log|y|$ is pluriharmonic and bounded in V^-.*
6. *$G^-(x,y) - \log|x|$ is pluriharmonic and bounded in V^+.*

So we can consider the closed positive currents, called the *stable/unstable currents* defined by

$$\mu^{\pm} = dd^c G^{\pm}. \tag{A.2}$$

Since that $G^{\pm}$ are pluriharmonic in the complement of $J^{\pm}$, it follows that for every point z of $U^+ \cup \operatorname{int}(K^+)$, we have that $dd^c G^+(z) = 0$; this imply that

$$\operatorname{supp}(dd^c G^{\pm}) = J^{\pm}. \tag{A.3}$$

Another property of the currents $\mu^{\pm}$ is the following: in case that a current is defined by the equation $T = dd^c u$ where u is a locally integrable function in $\mathbb{C}^2$, then we can define the pullback f^*T, by $dd^c(u \circ f)$. From the previous observation it follows that

$$f^*\mu^{\pm} = dd^c(G^{\pm} \circ f) = d^{\pm}\mu^{\pm}. \tag{A.4}$$

With all this, the measure

$$\mu = \mu^+ \wedge \mu^- \tag{A.5}$$

is f-invariant, since

$$f^*\mu = f^*(\mu^+ \wedge \mu^-) = f^*\mu^+ \wedge f^*\mu^- = (d\mu^+) \wedge (d^{-1}\mu^-) = \mu.$$

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