Instituto Nacional de Matemática Pura e Aplicada

SINGULAR LEVI-FLAT HYPERSURFACES AN APPROACH THROUGH HOLOMORPHIC FOLIATIONS

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Con cariño, a mi hijo.......

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ABSTRACT

The aim of this Thesis is to investigate germs at $0 \in \mathbb{C}^n$, $n \geq 2$ of real analytic Levi-flat hypersurfaces with singularities. Inspired by a recent work of Cerveau-Lins Neto [12], we generalize a result of Burns-Gong [7] on Levi-flat hypersurface with Morse type singularity. We also obtain in certain cases normal forms of Levi-flat hypersurface defined by the vanishing of the real part of complex quasihomogeneous polynomials. Finally we study germs at $0 \in \mathbb{C}^n$ of singular k-webs tangent to Levi-flat hypersurfaces, generalizing a result of [12] for codimension one holomorphic foliations tangent to Levi-flat hypersurfaces.

Keywords: Levi-flat Hypersurfaces, Holomorphic Foliations, Singular Webs.

RESUMO

O objetivo desta tese é investigar germes em $0 \in \mathbb{C}^n$, $n \geq 2$ de hipersuperfícies Levi-flat reais analíticas com singularidades. Inspirado pelo recente trabalho de Cerveau-Lins Neto [12], generalizamos um resultado de Burns-Gong [7], sobre hipersuperfícies Levi-flat com singularidade do tipo Morse. Encontramos também em certos casos formas normais de hipersuperfícies Levi-flat definidas pela anulação da parte real de polinômios complexos quase-homogêneos. Finalmente estudamos germes em $0 \in \mathbb{C}^n$ de k-webs singulares tangente a hipersuperfícies Levi-flat, generalizando um resultado de [12] para folheações holomorfas de codimensão um tangentes a hipersuperfícies Levi-flat.

Palvras-chave: Hipersuperfícies Levi-flat, Folheações Holomorfas, Webs singulares.

Contents

1 Notations and Results **5**

1.1 Complex variables background 5
1.2 Levi-flat hypersurfaces 6
1.3 Singular holomorphic foliations 7
1.4 Levi-flat hypersurfaces and foliations 8
1.5 The reduced singularities in dimension two 11
1.6 Examples 14

2 Normal forms of Levi-Flat hypersurfaces **15**

2.1 Tougeron's lemma on finite determinacy 16
2.2 Proof of Theorem 1 17
2.3 Quasihomogeneous polynomials 21
2.4 Proof of corollary 1 22
2.5 Applications 22

3 Levi-flat hypersurfaces with A_k, D_k, E_k singularities **24**

3.1 Normal forms of Levi-flat in $\mathbb{C}^n$, $n \geq 3$ 26
3.2 Proof of Theorem 2 27

4 Levi-flat hypersurfaces and webs **45**

4.1 Local webs 45
4.2 Webs as closures of meromorphic multi-sections 46
4.3 First integrals for webs 48
4.4 Levi-flat hypersurfaces and webs 49
4.5 Proof of Theorem 3 53

INTRODUCTION

In this work we consider germs at $0 \in \mathbb{C}^n$, $n \geq 2$ of real analytic Levi-flat hypersurfaces with singularities. A well-known theorem of E.Cartan says that a real analytic smooth hypersurface M in $\mathbb{C}^n$ has no local holomorphic invariants, if M is Levi-flat, i.e, it is foliated by smooth holomorphic hypersurfaces of $\mathbb{C}^n$. In suitable local coordinates such a hypersurface is given by $\mathcal{R}e(z_n) = 0$. On the other hand, if M is not Levi-flat, the invariants of M are given by the theory of Cartan [9], Chern-Moser [13].

A real analytic hypersurface M in $\mathbb{C}^n$ can be decomposed into M^* and $sing(M)$, where M^* is a smooth real analytic hypersurface and $sing(M)$, the singular locus, is contained in a proper analytic subvariety of lower dimension. A real analytic hypersurface M with singularities is said to be *Levi-flat* if its smooth part M^* is Levi-flat.

Singular Levi-flat hypersurfaces have been previously studied by E.Bedford [6], X.Gong [15], M.Brunella [8]. Local questions about Levi-flat hypersurfaces with quadratic singularities have been studied by Burns-Gong [7] and most recently Cerveau-Lins Neto [12] have studied Local Levi-flat hypersurfaces invariants by codimension one holomorphic foliations. This new approach using methods from the theory of holomorphic foliations, inspired this work.

This work has three purposes. First, we will prove a generalization of a result due to Burns-Gong [7].

Theorem 1. *Let $M = F^{-1}(0)$, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$, $n \geq 2$, be a germ of irreducible real analytic function such that*

(a). $F(z_1, \ldots, z_n) = \mathcal{R}e(P(z_1, \ldots, z_n)) + h.o.t$, where P is a homogeneous polynomial of degree k with an isolated singularity at $0 \in \mathbb{C}^n$.

(b). The Milnor number of P at $0 \in \mathbb{C}^n$ is μ.

(c). M is Levi-flat.

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\phi(M) = (\mathcal{R}e(h) = 0)$, where $h(z)$ is a polynomial of degree $\mu + 1$ and $j_0^k(h) = P$.

In the second contribution of this work, we obtain in certain cases, normal forms for real analytic Levi-flat hypersurfaces which are defined by the vanishing of the real part of a quasihomogeneous polynomial. The quasihomogeneous polynomials that we will consider is a special class of germs, the famous A_k, D_k, E_k singularities or simple singularities (cf. [1]). More precisely, our result is the following :

Theorem 2. *Let $M = F^{-1}(0)$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of irreducible real analytic Levi-flat hypersurface. Suppose that F is of one of following types:*

(a). $F(z) = \mathcal{R}e(z_1^2 + z_2^{k+1} + \ldots + z_n^2) + H(z, \bar{z})$, *where* $k \geq 3$ *and*

$$H(z, \bar{z}) = 0(|z|^{k+2}), \ H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

(b). $F(z) = \mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, *where* $k \geq 6$ *and*

$$H(z, \bar{z}) = 0(|z|^{k}), \ H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

(c). $F(z) = \mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, *where*

$$H(z, \bar{z}) = 0(|z|^{5}), \ H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

Then there exists a germ of biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *such that*

$$\varphi(M) = (\mathcal{R}e(z_1^2 + z_2^{k+1} + \ldots + z_n^2) = 0),$$

$$\varphi(M) = (\mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) = 0),$$

$$\varphi(M) = (\mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0),$$

respectively.

We find the following list:

Name	Normal form	Conditions
A_k	$\mathcal{R}e(z_1^2 + z_2^{k+1} + \ldots + z_n^2) = 0$	$k = 1$ or $k \geq 3$
D_k	$\mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) = 0$	$k = 4$ or $k \geq 6$
E_6	$\mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	

The third contribution of this work is a generalization of a result due to Cerveau-Lins Neto [12]. More precisely, we have the following :

Theorem 3. *Let $\mathcal{W}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$ of k-web tangent to a germ at $0 \in \mathbb{C}^n$ of an irreducible real-analytic Levi-flat hypersurface M. Assume that $\mathcal{W}$ is irreducible and has a finite number of invariant analytic leaves through the origin. Denote by X the variety associated to $\mathcal{W}$.*

(a). If $n = 2$. Then $\mathcal{W}$ has a non-constant holomorphic first integral.

(b). If $n \geq 3$, and $cod_{X_{reg}}(sing(X)) \geq 2$. Then $\mathcal{W}$ has a non-constant holomorphic first integral.

In both cases the web $\mathcal{W}$ has a non-constant holomorphic first integral of the form

$$f_0(x) + z.f_1(x) + \ldots + z^{k-1}.f_{k-1}(x) + z^k,$$

where $f_0, f_1, \ldots, f_{k-1} \in \mathcal{O}_n$.

We would like to observe that if $n = 2$ and $k = 1$, $\mathcal{W}$ is a non-dicritical holomorphic foliation at $(\mathbb{C}^2, 0)$ tangent to a germ at $0 \in \mathbb{C}^2$ of an irreducible real analytic Levi-flat hypersurface M, then a theorem due to Cerveau-Lins Neto says that $\mathcal{W}$ has a non-constant holomorphic first integral. In this sense, Theorem 3 is a generalization of Cerveau-Lins Neto's theorem.
This work is organized as follows:

1. Notations and Results. We begin with the basic definitions and results concerning Levi-flat hypersurfaces and holomorphic foliations. Those result will be used later.

2. Normal forms of Levi-flat hypersurfaces. In this chapter we obtain normal forms for Levi-flat hypersurfaces which are defined by the vanishing of the real part of a homogeneous polynomial. We will also give applications and some examples of our main theorem.

3. Levi-flat hypersurfaces with A_k, D_k, E_k singularities. We will give a list due to V.I.Arnold of A_k, D_k, E_k singularities and we recall some properties. We obtain in certain cases normal forms for Levi-flat hypersurfaces defined by the vanishing of the real part of A_k, D_k, E_k types.

4. Levi-flat hypersurfaces and webs. We investigate germs at $0 \in \mathbb{C}^n$ of codimension one k-webs tangent to germs at $0 \in \mathbb{C}^n$ of real analytic Levi-flat hypersurfaces. In particular, our main theorem generalizes a result of Cerveau-Lins Neto for holomorphic foliation in the non-dicritical case.

Chapter 1

Notations and Results

1.1 Complex variables background

First, we fix some terminology. We will be working in $\mathbb{C}^n$, and we will frequently write the coordinates as $z = (z_1, \ldots, z_n)$. Note that, if $z \in \mathbb{C}$ then we can write $z = x + iy$, where $x, y \in \mathbb{R}$ are the real and imaginary parts of z. Therefore, we can think of $\mathbb{C}^n$ as $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ by writing $z_k = x_k + iy_k$. The complex conjugation is defined by $\bar{z}_k = x_k - iy_k$, and

$$dz_k = dx_k + idy_k \quad \text{and} \quad d\bar{z}_k = dx_k - idy_k. \tag{1.1}$$

A (smooth) *real hypersurface* in $\mathbb{C}^n$ is a subset M of $\mathbb{C}^n$ such that for every point $p_0 \in M$ there is a neigborhood U of p_0 in $\mathbb{C}^n$ and a smooth real-valued function ρ defined in U such that

$$M \cap U = \{Z \in U : \rho(Z) = 0\}, \tag{1.2}$$

with differential $d\rho$ nonvanishing in U. Such a function ρ is called a *local defining function* for M near p_0. The hypersurface M is *real-analytic* if the defining function ρ in (1.2) can be chosen to be real-analytic.

Example 1.1. The hypersurface in $\mathbb{C}^n$ given by the equation $Im(z_n) = 0$ is a "flat" real hyperplane in $\mathbb{C}^n$.

Example 1.2. The hypersurface in $\mathbb{C}^n$ given by the equation

$$Im(z_n) - \sum_{j=1}^{n-1} |z_j|^2 = 0 \tag{1.3}$$

is called the Lewy hypersurface.

Example 1.3. The unit sphere in $\mathbb{C}^n$ given by $\sum_{j=1}^{n} |z_j|^2 = 1$ is a compact hypersurface. The reader can check that the holomorphic rational mapping $H(z) = (H_1(z), \ldots, H_n(z))$ given by

$$H_j(z) := \frac{iz_j}{1 - z_n}, \quad j = 1, \ldots, n-1, \quad H_n(z) := \frac{i(z_n + 1)}{1 - z_n},$$

takes the unit sphere minus the point $(0, 0, \ldots, 1)$ bijectively to the Lewy hypersurface given in example 1.2.

Remark 1.4. Given a smooth real analytic hypersurface M and $p \in M$, there exists a local real analytic coordinates $(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ such that $M = (x_1 = 0)$ in a neigborhood of p. However, in the general there is no holomorphic change of coordinates which performs this equivalence. For instance, as in example 1.2.

1.2 Levi-flat hypersurfaces

A smooth real hypersuperface $M \subset \mathbb{C}^n$ is said to be **Levi-flat** if the codimension one distribution

$$T^{\mathbb{C}}M = TM \cap i(TM) \subset TM$$

is integrable, in Frobenius' sense. It follows that M is smoothly foliated by immersed complex manifolds of complex dimension $n - 1$. The foliation defined by this distribution is called the Levi foliation and will be denoted by $\mathcal{L}_M$.

If M is real analytic, then according to E. Cartan, around each $p \in M$ we can find local holomorphic coordinates $z_1, \ldots, z_n$ such that $M = \{\mathcal{R}e(z_1) = 0\}$, and consequently the leaves of $\mathcal{L}_M$ are locally $\{z_1 = ic\}$, $c \in \mathbb{R}$. In particular, the Levi foliation $\mathcal{L}_M$ extends to a codimension one holomorphic foliation defined in a neighborhood of M, with leaves $\{z_1 = c\}$, $c \in \mathbb{C}$.

A real analytic subset M is irreducible if it cannot be expressed as $M = M_1 \cup M_2$, with both M_1 and M_2 real analytic and different from M. Any real analytic subset can be decomposed (on relatively compact open subsets) into a finite collection of irreducible components.

An irreducible real analytic subset M has a well defined dimension $dim_{\mathbb{R}} M$, and it can be decomposed as a disjoint union $M = M^* \cup sing(M)$, where:

(i). M^* is nonempty and open in M, and it is formed by those points of M around which M is a smooth real analytic submanifold of $\mathbb{C}^n$ of dimension $dim_{\mathbb{R}} M$.

(ii). $sing(M)$ is a real analytic subset, all of whose irreducible components have dimension strictly smaller that $dim_{\mathbb{R}} M$.

When $dim_{\mathbb{R}} M = 2n-1$, or more generally each irreducible component of M has dimension $2n-1$, we call M a real analytic hypersurface. In this case, we say that M is Levi-flat if M^* is Levi-flat.

1.3 Singular holomorphic foliations

In this section we define codimension one singular holomorphic foliations.

Definition 1.5. Let X be a complex manifold of dimension $n \geq 2$. A codimension one singular holomorphic foliation on X is an object $\mathcal{F}$ given by collections $\{\omega_\alpha\}_{\alpha \in A}$, $\{U_\alpha\}_{\alpha \in A}$ and $\{g_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$, such that:

(i). $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X.

(ii). ω_α is a holomorphic integrable 1-form not identically zero in $\{U_\alpha\}$. (That is $\omega_\alpha \wedge d\omega_\alpha = 0$).

(iii). If $U_\alpha \cap U_\beta \neq \emptyset$ then $\{g_{\alpha\beta}\} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $\omega_\alpha = g_{\alpha\beta}\omega_\beta$ in $U_\alpha \cap U_\beta$.

For each form ω_α, we define the singular set as

$$sing(\omega_\alpha) = \{p \in U_\alpha : \omega_\alpha(p) = 0\} := S_\alpha. \tag{1.4}$$

Note that S_α is an analytic sub-variety of U_α. It follows from (iii) that $S_\alpha \cap U_\alpha \cap U_\beta = S_\beta \cap U_\alpha \cap U_\beta$. Therefore, the union of the sets S_α, defines an analytic sub-variety

S on X. This set, that we will denote by $sing(\mathcal{F})$, is called the singular set of $\mathcal{F}$. In particular, $\mathcal{F}$ defines a codimension one foliation (non-singular) in the open set $U = X \setminus sing(\mathcal{F})$, a leaf of $\mathcal{F}$ is by definition, a leaf of the restriction of $\mathcal{F}|_U$. See [17] for the complete bibliography.

1.4 Levi-flat hypersurfaces and foliations

In this section we give some basic definitions and state the results of [12], we also give some examples. Let us fix some notations that will be used from now on.

1. $\mathcal{O}_n$: The ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. $\mathcal{O}(U)$ = set of holomorphic functions in the open set $U \subset \mathbb{C}^n$.
2. $\mathcal{O}_n^* = \{f \in \mathcal{O}_n / f(0) \neq 0\}$. $\mathcal{O}^*(U) = \{f \in \mathcal{O}(U)/f(z) \neq 0, \forall z \in U\}$.
3. $\mathcal{M}_n = \{f \in \mathcal{O}_n / f(0) = 0\}$ maximal ideal of $\mathcal{O}_n$.
4. $\mathcal{A}_n$: The ring of germs at $0 \in \mathbb{C}^n$ of complex valued real analytic functions.
5. $\mathcal{A}_{n\mathbb{R}}$: The ring of germs at $0 \in \mathbb{C}^n$ of real valued real analytic functions. Note that $F \in \mathcal{A}_n$ is in $\mathcal{A}_{n\mathbb{R}}$ if and only if $F = \bar{F}$.
6. $Diff(\mathbb{C}^n, 0)$: The group of germs at $0 \in \mathbb{C}^n$ of holomorphic diffeomorphisms $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with the operation of composition.
7. $j_0^k(f)$: The k-jet at $0 \in \mathbb{C}^n$ of $f \in \mathcal{O}_n$.

Let M be a germ at $(\mathbb{C}^n, 0)$ of a real codimension one irreducible analytic set. For the sake of simplicity we will denote germs and representative of germs by the same letter. Since M is real analytic of codimension one and irreducible, it can be defined by $(F = 0)$, where F is an irreducible germ of real analytic function. The singular set of M is defined by $sing(M) = (F = 0) \cap (dF = 0)$ and its smooth part $(F = 0) \setminus (dF = 0)$ will be denoted by M^*. In this case, the Levi distribution L on M^* is defined by

$$L_p := ker(\partial F(p)) \subset T_pM^* = ker(dF(p)), \text{ for any } p \in M^*.$$

In this situation, M is Levi-flat if the Levi distribution L on M^* is integrable.

Remark 1.6. If the hypersurface M is defined by $(F = 0)$ then the Levi distribution L on M can be defined by the real analytic 1-form $\eta = i(\partial F - \bar{\partial} F)$, which will be called the Levi 1-form of F. The integrability condition is equivalent to $(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F|_{M^*} = 0$

Example 1.7. If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is holomorphic and non constant then the analytic set defined by $M = (Im(f) = 0)$ is Levi-flat. The leaves of the Levi foliation on M are the real levels of f.

Definition 1.8. Let $\mathcal{F}$ and $M = F^{-1}(0)$ be germs at $(\mathbb{C}^n, 0)$ of a codimension one singular holomorphic foliation and of a real Levi-flat hypersurface, respectively. We say that $\mathcal{F}$ and M are tangent, if the leaves of the Levi foliation $\mathcal{L}$ on M are also leaves of $\mathcal{F}$.

D. Cerveau and Lins Neto [12], proved the following result, concerning the situation of definition 1.8.

Theorem 1.9. *Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of holomorphic codimension one foliation tangent to a germ at $0 \in \mathbb{C}^n$ of real codimension one and irreducible analytic variety M. Then $\mathcal{F}$ has a non-constant meromorphic first integral. In the case $n = 2$ we have:*

(a). *If $\mathcal{F}$ is dicritical then it has a non-constant meromorphic first integral f/g, where $f, g \in \mathcal{O}_2$ and $f(0) = g(0) = 0$.*

(b). *If $\mathcal{F}$ is non-dicritical then it has a non-constant holomorphic first integral.*

Recall that a germ of foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is dicritical if it has infinitely many analytic separatrices through the origin. Otherwise, it is called non-dicritical.

1.4.1 The complexification

Given $H \in \mathcal{A}_n$ we can write its Taylor series at $0 \in \mathbb{C}^n$ as

$$H(z) = \sum_{\mu,\nu} H_{\mu\nu} z^{\mu} \bar{z}^{\nu}, \tag{1.5}$$

where $H_{\mu\nu} \in \mathbb{C}$, $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, $z^\mu = z_1^{\mu_1} \ldots z_n^{\mu_n}$, $\bar{z}^\nu = \bar{z}_1^{\nu_1} \ldots \bar{z}_n^{\nu_n}$. When $H \in \mathcal{A}_{n\mathbb{R}}$ then the coefficients $H_{\mu\nu}$ satisfy

$$\bar{H}_{\mu\nu} = H_{\nu\mu}.$$

The complexification $H_{\mathbb{C}} \in \mathcal{O}_{2n}$ of H is defined by the series

$$H_{\mathbb{C}}(z, w) = \sum_{\mu,\nu} H_{\mu\nu} z^\mu w^\nu. \tag{1.6}$$

If the series in (1.5) converges in polydisk $D_r = \{z \in \mathbb{C}^n / |z_j| \leq r\}$ then the series in (1.6) converges in the polydisk $D_r \times D_r$. Moreover, $H(z) = H_{\mathbb{C}}(z, \bar{z})$.

Let $F \in \mathcal{A}_{n\mathbb{R}}$, $F(0) = 0$, be irreducible and such that $M = F^{-1}(0)$ is Levi-flat. If the Taylor series of F is

$$F(z) = \sum_{\mu,\nu} F_{\mu\nu} z^\mu \bar{z}^\nu,$$

the complexification $F_{\mathbb{C}} \in \mathcal{O}_{2n}$ of F is defined by the series

$$F_{\mathbb{C}}(z, w) = \sum_{\mu,\nu} F_{\mu\nu} z^\mu w^\nu. \tag{1.7}$$

In particular $F_{\mathbb{C}}(z, \bar{z}) = F(z)$. The complexification $\eta_{\mathbb{C}}$ of its Levi 1-form $\eta = i(\partial F - \bar{\partial} F)$ can be written as

$$\eta_{\mathbb{C}} = i(\partial_z F_{\mathbb{C}} - \partial_w F_{\mathbb{C}}) = i \sum_{\mu,\nu} (F_{\mu\nu} w^\nu d(z^\mu) - F_{\mu\nu} z^\mu d(w^\nu)).$$

The complexification $M_{\mathbb{C}}$ of M is defined as $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$ and its smooth part is $M_{\mathbb{C}}^* = M_{\mathbb{C}} \backslash (dF_{\mathbb{C}} = 0)$. The integrability condition of $\eta = i(\partial F - \bar{\partial} F)|_{M^*}$ implies that $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ is integrable. Therefore $\eta_{\mathbb{C}} = 0$ defines a foliation $\mathcal{L}_{\mathbb{C}}$ on $M_{\mathbb{C}}^*$ that will be called the complexification of $\mathcal{L}$.

We will assume that the Taylor series of F converges in the polydisk D_r^n. The following result was proved in [12].

Lemma 1.10. *Let F, M, M^* and $F_{\mathbb{C}}$ be as above. Then for any $z_0 \in M^*$ the leaf L_{z_0} of $\mathcal{L}$ through z_0 is contained in the hypersurface $\{z \in D_r^n | F_{\mathbb{C}}(z, \bar{z}_0) = 0\}$. In particular, L_{z_0} is closed in M^*.*

Now we consider a germ at $0 \in \mathbb{C}^2$ of real analytic Levi-flat $M = (F = 0)$, where F is irreducible in $\mathcal{A}_{2\mathbb{R}}$. Let $F_{\mathbb{C}}$, $M_{\mathbb{C}} = (F_{\mathbb{C}} = 0) \subset (\mathbb{C}^4, 0)$ and $M^*_{\mathbb{C}}$ be as before. We will assume that the power series that defines $F_{\mathbb{C}}$ converges in a neighborhood of $\bar{\triangle} = \{(z, w) \in \mathbb{C}^4/|z|, |w| \leq 1\}$, so that $F_{\mathbb{C}}(z, \bar{z}) = F(z)$ for all $|z| \leq 1$.

Let $V := M^*_{\mathbb{C}} \backslash sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$ and denote L_p the leaf of $\mathcal{L}_{\mathbb{C}}$ through p, where $p \in V$. In this situation the following lemma is proved in [12].

Lemma 1.11. *For any $p = (z_0, w_0) \in V$ the leaf L_p is closed in $M^*_{\mathbb{C}}$.*

Definition 1.12. The algebraic dimension of $sing(M)$ is the complex dimension of the singular set of $M_{\mathbb{C}}$.

The second result of [12] concerns the existence of a foliation tangent to the singular Levi-flat hypersurface.

In a certain sense, the next result asserts that if the singularities of M are sufficiently small (in the algebraic sense) then M is given by the zeroes of the real part of a holomorphic function.

Theorem 1.13. *Let $M = F^{-1}(0)$ be a germ of an irreducible analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, with Levi 1-form $\eta = i(\partial F - \bar{\partial} F)$. Assume that the algebraic dimension of $sing(M) \leq 2n - 4$. Then there exists an unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, if one of the following conditions is fulfilled:*

(a). *$n \geq 3$ and $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 3$.*

(b). *$n \geq 2$, $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 2$ and $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.*

Moreover, in both cases the foliation $\mathcal{F}_M$ has a non-constant holomorphic first integral f such that $M = (Re(f) = 0)$.

1.5 The reduced singularities in dimension two

Let M and $\mathcal{F}$ be germs at $(\mathbb{C}^2, 0)$ of a real analytic Levi-flat hypersurface and of a holomorphic foliation, respectively, where $\mathcal{F}$ is tangent to M. We will assume that:

(i). $\mathcal{F}$ is defined by a germ at $0 \in \mathbb{C}^2$ of holomorphic vector field X with an isolated singularity at 0.

(ii). M is irreducible and defined by $(F = 0)$, where $F \in \mathcal{A}_{2\mathbb{R}}$ is irreducible.

Let us assume that 0 is a reduced singularity of X, in the sense of Seidenberg. Denote the eigenvalues of $DX(0)$ by λ_1, λ_2. We have two possibilities:

(a). $\lambda_1, \lambda_2 \neq 0$ and $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$. In this case, X has exactly two analytic separatrices through 0, both smooth. It can be written in a suitable coordinate system (u, v), as

$$X = \lambda_1.u(1 + R_1(u, v))\partial_u + \lambda_2.v(1 + R_2(u, v))\partial_v. \tag{1.8}$$

where $R_1(0,0) = R_2(0,0) = 0$. The separatrices are $S_1 := \{v = 0\}$ and $S_2 := \{u = 0\}$.

(b). $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case, X has a saddle-node at 0. We will suppose without lost of generality that $\lambda_1 = 1$. It can be written in a suitable coordinate system (u, v), as

$$X = u^{m+1}\partial_u + [v(1 + \lambda.u^m) + h.o.t]\partial_v, \tag{1.9}$$

where $\lambda \in \mathbb{C}$, $m \geq 1$ (cf. [20]). In this case, X has one or two analytic separatrices through the origin.

The following lemma is proved in [12].

Lemma 1.14. *Suppose that X has a reduced singularity at $0 \in \mathbb{C}^2$ and is tangent to $M = F^{-1}(0)$ Levi-flat hypersurface. Then $\lambda_1, \lambda_2 \neq 0$, $\lambda_2/\lambda_1 \in \mathbb{Q}_-$ and X has a holomorphic first integral.*
In particular, in a suitable coordinates system (x, y) around $0 \in \mathbb{C}^2$, $X = \phi.Y$, where $\phi(0) \neq 0$ and

$$Y = q.x\partial_x - p.y\partial_y \text{ , } gcd(p, q) = 1. \tag{1.10}$$

In this coordinate system, $f(x, y) := x^p.y^q$ is a first integral of X.

1.5.1 Saddle singularities with first integral

We consider the following situation: Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^2$ of a non-dicritical foliation and consider a resolution $\pi : (\widetilde{\mathbb{C}}^2, D) \to (\mathbb{C}^2, 0)$ of the foliation $\mathcal{F}$. Let $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$ and D be the exceptional divisor. Since $\mathcal{F}$ is non-dicritical, all irreducible components of D are $\mathcal{F}$-invariants. Assume that for any $p \in sing(\widetilde{\mathcal{F}}) \subset D$ there exists a local coordinate system $(W, (u, v))$ such that $\widetilde{\mathcal{F}}|_W$ has a first integral of the form $u^m v^n$, where $m, n \in \mathbb{N}$ and $gcd(m, n) = 1$. We will call this type of singularity a saddle with first integral.

Another result that we will use is the following, (cf. [18] pg. 162):

Theorem 1.15. *Let $\mathcal{F}$ be a non-dicritical foliation, $\pi : (\tilde{\mathbb{C}}^2, D) \to (\mathbb{C}^2, 0)$ be a minimal resolution and $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$. Assume that all singularities of $\tilde{\mathcal{F}}$ in D are saddles with first integral. Fix a transversal $\sum$ through a point $p \in \sum \cap D$, which is not a singularity of $\tilde{\mathcal{F}}$. Then:*

(a). The transversal is complete, in the sense that there is a neighborhood U_0 of p in $\sum$ such that for any smaller neigborhood $p \in U \subset U_0$ then $V_U := int(\overline{sat_{\tilde{\mathcal{F}}}(U)})$ is a neighborhood of D, where int denotes the interior and

$$sat_{\tilde{\mathcal{F}}}(U) := \cup_{q \in U} L_q,$$

L_q=leaf of $\tilde{\mathcal{F}}$ through q.

(b). There exist a finite ramified covering $\Pi : (\mathbb{D}, 0) \to (\sum, 0)$ and a subgroup $G \subset Diff(\mathbb{C}, 0)$ which covers the pseudo-group of holonomy of the germ $\tilde{\mathcal{F}}_D$ of $\tilde{\mathcal{F}}$ at D.

For a precise definition of the pseudo-group of holonomy of the germ $\tilde{\mathcal{F}}_D$, we refer to [18]. The group G is usually called the global holonomy group of $\tilde{\mathcal{F}}$. In particular in [18] the following result is proved:

Corollary 1.16. *In the situation of theorem 1.15 the foliation $\mathcal{F}$ has a first integral if, and only if, the group G is finite.*

1.6 Examples

D.Burns and X.Gong [7] have classified all singular quadratic Levi-flat hypersurfaces (hypersurfaces defined by the vanishing of a real analytic quadratic polynomial) in $\mathbb{C}^n$. They have proved the following result.

Theorem 1.17. *If $M \subset \mathbb{C}^n$ is a quadratic Levi-flat hypersurface, then it is biholomorphically equivalent to a hypersurface with one of the following five defining functions.*

(i). $\mathcal{R}e(z_1^2 + \ldots + z_k^2) = 0,\ k = 1, \ldots, n.$

(ii). $z_1^2 + 2z_1\bar{z}_1 + \bar{z}_1^2 = 0$

(iii). $z_1^2 + 2\lambda z_1\bar{z}_1 + \bar{z}_1^2 = 0,$ *where* $0 < \lambda < 1.$

(iv). $(z_1 + \bar{z}_1)(z_2 + \bar{z}_2) = 0$

(v). $z_1\bar{z}_2 - z_2\bar{z}_1 = 0$

The hypersurface (i) is defined by the vanishing of the real part of a holomorphic function, the hypersurface (v) is defined by the vanishing of the imaginary part of a meromorphic function.

On the other hand, the hypersurface of $\mathbb{C}^n$ defined by

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n / \mathcal{R}e(z_1)^2 - Im(z_1)^3 = 0\}$$

is irreducible Levi-flat and not defined by the vanishing of the imaginary part of a meromorphic function. For instance see [7], proposition [5.4].

More complicated examples can be derived by pull-back of a Levi-flat hypersurface by a holomorphic mapping. That is, if $M \subset \mathbb{C}^n$ is a hypersurface defined by $g = 0$, and $f : \mathbb{C}^N \to \mathbb{C}^n$ is a nontrivial holomorphic mapping, then the set $\tilde{M} \subset \mathbb{C}^N$, defined by $g \circ f = 0$ is a Levi-flat hypersurface. This can be seen by pulling back the Levi foliation of M which becomes the Levi foliation of $\tilde{M}$.

Chapter 2

Normal forms of Levi-Flat hypersurfaces

In this chapter we study normal forms of real analytic Levi-flat hypersurfaces. An interesting class of Levi-flat hypersurfaces are those real analytic varieties defined by the vanishing of the real part of a holomorphic function. As, we have remarked before, Levi-flat hypersurfaces are not always of this type.

In the case of a real analytic smooth Levi-flat hypersurface M of $\mathbb{C}^n$, its local structure is very well understood, according to E. Cartan (see for instance [5] §1.7), around each $p \in M$ we can find local holomorphic coordinates $z_1, \ldots, z_n$ such that $M = \{\mathcal{R}e(z_1) = 0\}$.

More recently D. Burns and X. Gong [7] have proved an analogous result in the following case:

Let $M = F^{-1}(0)$ be a Levi-flat, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$, $n \geq 2$, is a germ of real analytic function such that

$$F(z_1, \ldots, z_n) = \mathcal{R}e(z_1^2 + \ldots + z_n^2) + H(z, \bar{z}) \tag{2.1}$$

with

$$H(z, \bar{z}) = 0(|z|^3)\,,\ H(z, \bar{z}) = \overline{H}(\bar{z}, z). \tag{2.2}$$

They show that there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\phi(M) = (\mathcal{R}e(z_1^2 + \ldots + z_n^2) = 0).$$

In [12], the authors prove the above result by using the theory of holomorphic foliations.

We are interested in finding similar normal forms in a situation more general. Our main result is the following :

Theorem 1. *Let* $M = F^{-1}(0)$*, where* $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$*,* $n \geq 2$*, be a germ of irreducible real analytic function such that*

(a). $F(z_1, \ldots, z_n) = \mathcal{R}e(P(z_1, \ldots, z_n)) + h.o.t$*, where* P *is a homogeneous polynomial of degree* k *with an isolated singularity at* $0 \in \mathbb{C}^n$*.*

(b). The Milnor number of P *at* $0 \in \mathbb{C}^n$ *is* μ*.*

(c). M *is Levi-flat.*

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *such that* $\phi(M) = (\mathcal{R}e(h) = 0)$*, where* $h(z)$ *is a polynomial of degree* $\mu + 1$ *and* $j_0^k(h) = P$*.*

Remark 2.1. Any homogeneous polynomial of degree 2 in $\mathbb{C}[z_1, \ldots, z_n]$ with isolated singularity at $0 \in \mathbb{C}^n$ is equivalent to $z_1^2 + \ldots + z_n^2$. In particular, we obtain the result of [7].

The following result is a consequence of the proof of theorem 1.

Corollary 1. *Let* Q *be a quasihomogeneous polynomial of degree* d *with an isolated singularity at* $0 \in \mathbb{C}^n$*,* $n \geq 3$ *and* $F(z) = \mathcal{R}e(Q(z)) + h.o.t.$ *Assume that* $M = F^{-1}(0)$ *is Levi-flat. Then there exists a germ of biholomorphism* $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *such that*

$$\phi(M) = (\mathcal{R}e(Q(z) + \sum_j c_j e_j(z)) = 0),$$

where $e_1, \ldots, e_s$ *are the elements of the monomial basis of the local algebra* A_Q *such that* $deg(e_j) > d$ *and* $c_j \in \mathbb{C}$*.*

2.1 Tougeron's lemma on finite determinacy

Definition 2.2. Two germs $f, g \in \mathcal{O}_n$ are said to be right equivalent, if there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $f \circ \phi^{-1} = g$. In other words, this means that g can be obtained from f by a local change of coordinates.

Morse Lemma can now be rephrased by saying that if $0 \in \mathbb{C}^n$ is an isolated singularity of f with Milnor number $\mu(f,0) = 1$ then f is right equivalent to its second jet. The next lemma is a generalization of Morse's Lemma. We refer to [4], pg.121.

Lemma 2.3 (Tougeron's lemma). *Suppose $0 \in \mathbb{C}^n$ is an isolated singularity of $f \in \mathcal{M}_n$ with Milnor number μ. Then f is right equivalent to $j_0^{\mu+1}(f)$.*

2.2 Proof of Theorem 1

Let $M = F^{-1}(0) \subset (\mathbb{C}^n, 0)$ be a Levi-flat, where $F(z) = \mathcal{R}e(P(z)) + h.o.t$ with P a homogeneous polynomial of degree $k \geq 2$ with an isolated singularity at $0 \in \mathbb{C}^n$ and Milnor number μ. We want to prove that there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $\phi(M) = (\mathcal{R}e(h) = 0)$, where h is a polynomial of degree $\mu + 1$.

The idea is to use theorem 1.13 to prove that there exists a germ $f \in \mathcal{O}_n$ such that the foliation $\mathcal{F}$ defined by $df = 0$ is tangent to M and $M = (\mathcal{R}e(f) = 0)$. The foliation $\mathcal{F}$ can viewed as an extension to a neighborhood of $0 \in \mathbb{C}^n$ of the Levi foliation $\mathcal{L}$ on M^*.

Suppose for a moment that $M = (\mathcal{R}e(f) = 0)$ and let us conclude the proof. Without lost of generality, we can suppose that f is not a power in $\mathcal{O}_n$. In this case $\mathcal{R}e(f)$ is irreducible (cf. [12]). This implies that $\mathcal{R}e(f) = U.F$, where $U \in \mathcal{A}_{n\mathbb{R}}$ and $U(0) \neq 0$. Let $\sum_{j \geq k} f_j$ be the taylor series of f, where f_j is a homogeneous polynomial of degree j, $j \geq k$. Then

$$\mathcal{R}e(f_k) = j_0^k(\mathcal{R}e(f)) = j_0^k(U.F) = U(0).\mathcal{R}e(P(z_1, \ldots, z_n)).$$

Hence $f_k(z_1, \ldots, z_n) = U(0).P(z_1, \ldots, z_n)$. We can suppose that $U(0) = 1$, so that

$$f(z) = P(z) + h.o.t \tag{2.3}$$

In particular, $\mu = \mu(f,0) = \mu(P,0)$, $f \in \mathcal{M}_n$, because P has isolated singularity at $0 \in \mathbb{C}^n$. Hence by lemma 2.3, f is right equivalent to $j_0^{\mu+1}(f)$, i.e. there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $h := f \circ \phi^{-1} = j_0^{\mu+1}(f)$. Therefore, $\phi(M) = (\mathcal{R}e(h) = 0)$ and this will conclude the proof of theorem 1.

Let us prove that we can apply theorem 1.13. We can write

$$F(z) = \mathcal{R}e(P(z_1, \dots, z_n)) + H(z_1, \dots, z_n),$$

where $H : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$ is a germ of real-analytic function and $j_0^k(H) = 0$. For simplicity, we assume that P has real coefficients. Then we get the complexification

$$F_{\mathbb{C}}(z, w) = \frac{1}{2}(P(z) + P(w)) + H_{\mathbb{C}}(z, w)$$

and $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0) \subset (\mathbb{C}^{2n}, 0)$. In the general case, replacing $P(w) = \sum a_j w^j$ by $\tilde{P}(w) = \sum \bar{a}_j w^j$, we will recover each step of proof.

Since $P(z)$ has an isolated singularity at $0 \in \mathbb{C}^n$, we get $sing(M_{\mathbb{C}}) = \{0\}$, and so the algebraic dimension of $sing(M)$ is 0. On other hand, the complexification of $\eta = i(\partial F - \bar{\partial} F)$ is

$$\eta_{\mathbb{C}} = i(\partial_z F_{\mathbb{C}} - \partial_w F_{\mathbb{C}}).$$

Recall that $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$. Now we compute $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$. We can write $dF_{\mathbb{C}} = \alpha + \beta$, with

$$\alpha = \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j := \frac{1}{2} \sum_{j=1}^{n} (\frac{\partial P}{\partial z_j}(z) + A_j) dz_j$$

and

$$\beta = \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j := \frac{1}{2} \sum_{j=1}^{n} (\frac{\partial P}{\partial w_j}(w) + B_j) dw_j,$$

where $\frac{1}{2}\sum_{j=1}^{n} A_j dz_j = \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial z_j} dz_j$ and $\frac{1}{2}\sum_{j=1}^{n} B_j dw_j = \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial w_j} dw_j$.

Then $\eta_{\mathbb{C}} = i(\alpha - \beta)$, and so

$$\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*} = (\eta_{\mathbb{C}} + i dF_{\mathbb{C}})|_{M_{\mathbb{C}}^*} = 2i\alpha|_{M_{\mathbb{C}}^*} = -2i\beta|_{M_{\mathbb{C}}^*}. \tag{2.4}$$

In particular, $\alpha|_{M_{\mathbb{C}}^*}$ and $\beta|_{M_{\mathbb{C}}^*}$ define $\mathcal{L}_{\mathbb{C}}$. Therefore $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$ can be splited in two parts. Let $M_1 = \{(z, w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial w_j} \neq 0 \text{ for some } j = 1, \dots, n\}$ and $M_2 = \{(z, w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial z_j} \neq 0 \text{ for some } j = 1, \dots, n\}$, note that $M_{\mathbb{C}} = M_1 \cup M_2$. Set

$$X_1 := M_1 \cap \{\frac{\partial P}{\partial z_1}(z) + A_1 = \dots = \frac{\partial P}{\partial z_n}(z) + A_n = 0\}$$

and

$$X_2 := M_2 \cap \{\frac{\partial P}{\partial w_1}(w) + B_1 = \ldots = \frac{\partial P}{\partial w_n}(w) + B_n = 0\}.$$

Then $sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) = X_1 \cup X_2$. Since $P \in \mathbb{C}[z_1, \ldots, z_n]$ has an isolated singularity at $0 \in \mathbb{C}^n$, we conclude that $cod_{M_{\mathbb{C}}^*} sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}) = n$.

If $n \geq 3$, we can directly apply Theorem 1.13 and the proof ends. In the case $n = 2$, we are going to prove that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

We begin by a blow-up at $0 \in \mathbb{C}^4$. Let $F(x,y) = \mathcal{R}e(P(x,y)) + h.o.t$ and $M = F^{-1}(0)$ Levi-flat. Its complexification can be written as

$$F_{\mathbb{C}}(x,y,z,w) = \frac{1}{2}P(x,y) + \frac{1}{2}P(z,w) + H_{\mathbb{C}}(x,y,z,w).$$

We take the exceptional divisor $D = \mathbb{P}^3$ of the blow-up $\pi : (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$ with homogeneous coordinates $[a : b : c : d]$, $(a,b,c,d) \in \mathbb{C}^4 \backslash \{0\}$. The intersection of the strict transform $\tilde{M}_{\mathbb{C}}$ of $M_{\mathbb{C}}$ by π with the divisor $D = \mathbb{P}^3$ is the surface

$$Q = \{[a : b : c : d] \in \mathbb{P}^3 / P(a,b) + P(c,d) = 0\},$$

which is an irreducible smooth surface.

Consider for instance the chart $(W, (t,u,z,v))$ of $\tilde{\mathbb{C}}^4$ where

$$\pi(t,u,z,v) = (t.z, u.z, z, v.z) = (x,y,z,w).$$

We have

$$F_{\mathbb{C}} \circ \pi(t,u,z,v) = z^k(\frac{1}{2}P(t,u) + \frac{1}{2}P(1,v) + zH_1(t,u,z,v)),$$

where $H_1(t,u,z,v) = H(tz, uz, z, vz)/z^{k+1}$, which implies that

$$\tilde{M}_{\mathbb{C}} \cap W = (\frac{1}{2}P(t,u) + \frac{1}{2}P(1,v) + zH_1(t,u,z,v) = 0)$$

and so $Q \cap W = (z = P(t,u) + P(1,v) = 0)$.

On the other hand, as we have seen in (3.2), the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M_{\mathbb{C}}^*} = 0$, where

$$\alpha = \frac{1}{2}\frac{\partial P}{\partial x}dx + \frac{1}{2}\frac{\partial P}{\partial y}dy + \frac{\partial H_{\mathbb{C}}}{\partial x}dx + \frac{\partial H_{\mathbb{C}}}{\partial y}dy.$$

In particular, we get

$$\pi^*(\alpha) = z^{k-1}(\frac{1}{2}\frac{\partial P}{\partial x}(t,u)zdt + \frac{1}{2}\frac{\partial P}{\partial y}(t,u)zdu + \frac{1}{2}kP(t,u)dz + z\theta),$$

where $\theta = \pi^*(\frac{\partial H_{\mathbb{C}}}{\partial x}dx + \frac{\partial H_{\mathbb{C}}}{\partial y}dy)/z^k$.

Hence, $\tilde{\mathcal{L}}_{\mathbb{C}}$ is defined by

$$\alpha_1 = \frac{1}{2}\frac{\partial P}{\partial x}(t,u)zdt + \frac{1}{2}\frac{\partial P}{\partial y}(t,u)zdu + \frac{1}{2}kP(t,u)dz + z\theta. \tag{2.5}$$

Since $Q \cap W = (z = P(t,u) + P(1,v) = 0)$, we see that Q is $\tilde{\mathcal{L}}_{\mathbb{C}}$-invariant. In particular, $S := Q \backslash sing(\tilde{\mathcal{L}}_{\mathbb{C}})$ is a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$. Fix $p_0 \in S$ and a transverse section $\sum$ through p_0. Let $G \subset Diff(\sum, p_0)$ be the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$. Since $dim(\sum) = 1$, we can think that $G \subset Diff(\mathbb{C}, 0)$. Let us prove that G is finite and linearizable.

At this part we use that the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ are closed (see lemma 1.11). Let $G' = \{f'(0)/f \in G\}$ and consider the homomorphism $\phi : G \to G'$ defined by $\phi(f) = f'(0)$. We assert that ϕ is injective. In fact, assume that $\phi(f) = 1$ and by contradiction that $f \neq id$. In this case $f(z) = z + a.z^{r+1} + \ldots$, where $a \neq 0$. According to [18], the pseudo-orbits of this transformation accumulate at $0 \in (\sum, 0)$, contradicting that the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ are closed. Now, it suffices to prove that any element $g \in G$ has finite order (cf. [19]). In fact, if $\phi(g) = g'(0)$ is a root of unity then g has finite order because ϕ is injective. On the other hand, if $g'(0)$ was not a root of unity then g would have pseudo-orbits accumulating at $0 \in (\sum, 0)$ (cf. [18]). Hence, all transformations of G have finite order and G is linearizable.

This implies that there is a coordinate system w on $(\sum, 0)$ such that $G = \langle w \to \lambda w \rangle$, where λ is a d^{th}-primitive root of unity (cf. [19]). In particular, $\psi(w) = w^d$ is a first integral of G, that is $\psi \circ g = \psi$ for any $g \in G$.

Let Z be the union of the separatrices of $\mathcal{L}_{\mathbb{C}}$ through $0 \in \mathbb{C}^4$ and $\tilde{Z}$ be its strict transform under π. The first integral ψ can be extended to a first integral $\varphi : \tilde{M}_{\mathbb{C}} \backslash \tilde{Z} \to \mathbb{C}$ be setting

$$\varphi(p) = \psi(\tilde{L}_p \cap \sum),$$

where $\tilde{L}_p$ denotes the leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$ through p. Since ψ is bounded (in a compact neighborhood of $0 \in \sum$), so is φ. It follows from Riemann extension theorem that

φ can be extended holomorphically to $\tilde{Z}$ with $\varphi(\tilde{Z}) = 0$. This provides the first integral and finishes the proof of theorem 1.

2.3 Quasihomogeneous polynomials

In this section, we state some general facts about normal forms of quasihomogeneous polynomials.

The local algebra of $f \in \mathcal{O}_n$ is by definition

$$A_f = \mathcal{O}_n/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

Recall that $\mu(f, 0) = dim A_f$.

Definition 2.4. The Newton support of germ $f = \sum a_{ij} x^i y^j$ is defined as $supp(f) = \{(i, j) : a_{ij} \neq 0\}$.

Definition 2.5. A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be quasihomogeneous of degree d with indices $\alpha_1, \ldots, \alpha_n$, if for any $\lambda \in \mathbb{C}$ and $(z_1, \ldots, z_n) \in \mathbb{C}^n$, we have

$$f(\lambda^{\alpha_1} z_1, \ldots, \lambda^{\alpha_n} z_n) = \lambda^d f(z_1, \ldots, z_n).$$

The index α_s is also called the weight of the variable z_s.

In the above situation, if $f = \sum a_k x^k$, $k = (k_1, \ldots, k_n)$, $x^k = x_1^{k_1} \ldots x^{k_n}$, then $supp(f) \subset \Gamma = \{k : a_1 k_1 + \ldots + a_n k_n = d\}$. The set Γ is called the diagonal. Usually one takes $\alpha_i \in \mathbb{Q}$ and $d = 1$.

One can define the quasihomogeneous filtration of the ring $\mathcal{O}_n$. It consists of the decreasing family of ideals $\mathcal{A}_d \subset \mathcal{O}_n$, $\mathcal{A}_{d'} \subset \mathcal{A}_d$ for $d < d'$. Here $\mathcal{A}_d = \{Q :$ degrees of monomials from $supp(Q)$ are $deg(Q) \geq d\}$; (the degree is quasihomogeneous).

When $\alpha_1 = \ldots = \alpha_n = 1$, this filtration coincides with the usual filtration by the usual degree.

Definition 2.6. A function f is called semiquasihomogeneous if $f = Q + F'$, where Q is quasihomogeneous of degree d of finite multiplicity and $F' \in \mathcal{A}_{d'}$, $d' > d$.

We will use the following result (cf. [1]).

Theorem 2.7. *Let f be a semiquasihomogeneous function, $f = Q + F'$ with quasihomogeneous Q of finite multiplicity. Then f is right equivalent to the function $Q + \sum_j c_j e_j(z)$, where $e_1, \ldots, e_s$ are the elements of the monomial basis of the local algebra A_Q such that $deg(e_j) > d$ and $c_j \in \mathbb{C}$.*

Example 2.8. If $f = Q + F'$ is semiquasihomogeneous and $Q(x, y) = x^2 y + y^k$, then f is right equivalent to Q. Indeed, the base of the local algebra $\mathcal{O}_2/(xy, x^2 + ky^{k-1})$ is $1, x, y, y^2, \ldots, y^{k-1}$ and lies below the diagonal Γ. Here $\mu(Q, 0) = k + 1$.

2.4 Proof of corollary 1

Let $M = F^{-1}(0)$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$ of real analytic Levi-flat hypersurface, where $F(z) = \mathcal{R}e(Q(z)) + h.o.t$ and Q is a quasihomogeneous polynomial with an isolated singularity at $0 \in \mathbb{C}^n$. It is easily seen that $sing(M_{\mathbb{C}}) = \{0\}$ and $cod_{M^*_{\mathbb{C}}} sing(\mathcal{L}_{\mathbb{C}}) \geq 3$. The argument is essentially the same of the proof of theorem 1. In this way, there exists an unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, moreover $\mathcal{F}_M$: $dh = 0$, $h(z) = Q(z) + h.o.t$ and $M = (\mathcal{R}e(h) = 0)$. Acoording to theorem 2.7, there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $h \circ \phi^{-1}(w) = Q(w) + \sum_k c_k e_k(w)$, where c_k and e_k as above. Hence

$$\phi(M) = (\mathcal{R}e(Q(w) + \sum_k c_k e_k(w)) = 0).$$

2.5 Applications

Here we give some applications of theorem 1.

Example 2.9. $Q(x, y) = x^2 y + y^3$ is a homogeneous polynomial of degree 3 with an isolated singularity at $0 \in \mathbb{C}^2$ and Milnor number $\mu(Q, 0) = 4$. According to [4] pg. 184, any germ $f(x, y) = x^2 y + y^3 + h.o.t$ is right equivalent to $x^2 y + y^3$.

In particular, if $F(z) = \mathcal{R}e(x^2 y + y^3) + h.o.t$ and $M = (F = 0)$ is a germ of real analytic Levi-flat at $0 \in \mathbb{C}^2$, Theorem 1 implies that there exists a holomorphic change of coordinate such that

$$M = (\mathcal{R}e(x^2 y + y^3) = 0).$$

Example 2.10. If $Q(x,y) = x^5 + y^5$ then $f(x,y) = Q(x,y) + h.o.t$ is right equivalent to $x^5 + y^5 + c.x^3y^3$, where $c \neq 0$ is a constant (see [4] pg. 194). Let $F(z) = \mathcal{R}e(x^5 + y^5) + h.o.t$ be such that $M = (F = 0)$ is Levi-flat, Theorem 1 implies that there exists a holomorphic change of coordinate such that

$$M = (\mathcal{R}e(x^5 + y^5 + c.x^3y^3) = 0).$$

Example 2.11. About normal forms of Parabolic singularities [4] pg. 246, we have two interesting families $P_8 : x^3 + y^3 + z^3 + a.xzy$, where $a^3 + 27 \neq 0$. and $X_9 : x^4 + y^4 + a.x^2y^2$, where $a^2 \neq 4$. In this case, we get the following normal forms of Levi-flat hypersurfaces.

$$M = (\mathcal{R}e(x^3 + y^3 + z^3 + a.xzy) = 0).$$

$$M = (\mathcal{R}e(x^4 + y^4 + a.x^2y^2) = 0).$$

Chapter 3

Levi-flat hypersurfaces with A_k, D_k, E_k singularities

An important problem in Singularity theory is the classification of holomorphic germs $f \in \mathcal{O}_n$ with respect to holomorphic change of coordinates in $\mathbb{C}^n$. When we consider only germs f with an isolated singularity at $0 \in \mathbb{C}^n$, the list starts with the famous A_k, D_k, E_k singularities, see for instance Arnold's papers [1], [2]:

Name	Normal form	Conditions
A_k	$z_1^2 + z_2^{k+1} + \ldots + z_n^2$	$k \geq 1$
D_k	$z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2$	$k \geq 4$
E_6	$z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2$	
E_7	$z_1^3 z_2 + z_2^3 + z_3^2 + \ldots + z_n^2$	
E_8	$z_1^5 + z_2^3 + z_3^2 + \ldots + z_n^2$	

Table 1

Several characterizations of the A_k, D_k, E_k singularities are well known, see for instance Durfee [14].

In this chapter, we are interested in obtaining normal forms of Levi-flat hypersurfaces which are defined by the vanishing of the real part of quasihomogeneous polynomials. The polynomials that we will be consider are the A_k, D_k, E_k singularities. In this sense, we remark the following: let $f \in \mathcal{O}_n$ be of A_1 type and $F = \mathcal{R}e(f) + h.o.t$ be such that $M = F^{-1}(0)$ is Levi-flat. Then there exists a germ of biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(M) = (\mathcal{R}e(f) = 0)$ (see [7]). When f is of D_4 type, we have an analogous result (see Chapter 1, example 2.9). We will prove the following:

Theorem 2. *Let $M = F^{-1}(0)$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of irreducible real analytic Levi-flat hypersurface. Suppose that F is of one of the following types:*

(a). $F(z) = \mathcal{R}e(z_1^2 + z_2^{k+1} + \ldots + z_n^2) + H(z, \bar{z})$, where $k \geq 3$ and

$$H(z, \bar{z}) = 0(|z|^{k+2}), \ H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

(b). $F(z) = \mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, where $k \geq 6$ and

$$H(z, \bar{z}) = 0(|z|^{k}), \ H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

(c). $F(z) = \mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, where

$$H(z, \bar{z}) = 0(|z|^{5}), \ H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

Then there exists a germ of biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\varphi(M) = (\mathcal{R}e(z_1^2 + z_2^{k+1} + \ldots + z_n^2) = 0),$$

$$\varphi(M) = (\mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) = 0),$$

$$\varphi(M) = (\mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0),$$

respectively.

We find the following list:

Name	Normal form	Conditions
A_k	$\mathcal{R}e(z_1^2 + z_2^{k+1} + \ldots + z_n^2) = 0$	$k = 1$ or $k \geq 3$
D_k	$\mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) = 0$	$k = 4$ or $k \geq 6$
E_6	$\mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	

Table 2

For A_2, D_5, E_7, E_8 the problem remains open.

3.1 Normal forms of Levi-flat in $\mathbb{C}^n$, $n \geq 3$

We would like to observe that the normal forms of A_k, D_k, E_k singulatities due to V.I.Arnold are polynomials with an isolated singularity at $0 \in \mathbb{C}^n$, and are stable under deformations. For instance, given a germ $f \in \mathcal{O}_n$ of A_k type and if we set $g = f + h.o.t$, then g is right equivalent to f, i.e. there exists $\varphi \in Diff(\mathbb{C}^n, 0)$ such that $g \circ \varphi^{-1} = f$. We send the reader to the reference [26], pg. 32 for the complete bibliography.

The following proposition is a consequence of the proof of corollary 1 (Chapter 2).

Proposition 3.1. *Let Q be a quasihomogeneous polynomial with an isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 3$ such that*

(a). $F(z_1, \ldots, z_n) = \mathcal{R}e(Q(z_1, \ldots, z_n)) + H(z, \bar{z})$, *with*

$$H(z, \bar{z}) = 0(|z|^{deg(Q)+1}), \; H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

where $deg(Q)$ is the degree (as polynomial) of Q

(b). $M = F^{-1}(0)$ *is Levi-flat.*

Then there exists an unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M. Moreover, the foliation $\mathcal{F}_M$ has a non-constant holomorphic first integral $f(z) = Q(z) + h.o.t$ and $M = (\mathcal{R}e(f) = 0)$.

The above proposition implies theorem 2 for $n \geq 3$.

Corollary 3.2. *Let g be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of A_k, D_k or E_k type and $F(z) = \mathcal{R}e(g(z)) + H(z, \bar{z})$, where*

$$H(z, \bar{z}) = 0(|z|^{deg(g)+1}), \; H(z, \bar{z}) = \overline{H}(\bar{z}, z).$$

Assume that $M = F^{-1}(0)$ is Levi-flat. Then there exists a germ of biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\varphi(M) = (\mathcal{R}e(g(z)) = 0).$$

Proof. Let g be as in table 1. By proposition 3.1 there exists $f \in \mathcal{O}_n$ such that $f(z) = g(z) + h.o.t$ and $M = (\mathcal{R}e(f) = 0)$. Since g is stable by deformations, f is right equivalent to g, i.e. there exists $\varphi \in Diff(\mathbb{C}^n, 0)$ such that $f \circ \varphi^{-1} = g$. Therefore, $\varphi(M) = (\mathcal{R}e(g) = 0)$. □

Finally, observe that for $n \geq 3$, the table 2 is complete.

Name	Normal form	Conditions
A_k	$\mathcal{R}e(z_1^2 + z_2^{k+1} + \ldots + z_n^2) = 0$	$k \geq 1$
D_k	$\mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) = 0$	$k \geq 4$
E_6	$\mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	
E_7	$\mathcal{R}e(z_1^3 z_2 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	
E_8	$\mathcal{R}e(z_1^5 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	

Table 3

3.2 Proof of Theorem 2

If $n \geq 3$, by corollary 3.2, Theorem 2 is proved. We give the proof for $n = 2$. The idea is to use Theorem 1.13. Let us assume for a moment that there exists a foliation $\mathcal{F}_M$ with a non-constant holomorphic first integral f and $M = (\mathcal{R}e(f) = 0)$. Since $F(z) = \mathcal{R}e(h(z)) + H(z, \bar{z})$, where h is a germ at $0 \in \mathbb{C}^2$ of A_k, D_k or E_6 types, and $M = F^{-1}(0)$, we must have $f(z) = h(z) + h.o.t$. Then there exists $\varphi \in Diff(\mathbb{C}^n, 0)$ such that $f \circ \varphi^{-1} = h$, finally $\varphi(M) = (\mathcal{R}e(h) = 0)$.

Let us mention two remarks:

Remark 3.3. Let $\eta = i(\partial F - \bar{\partial} F)$ and $\eta_{\mathbb{C}}$ be as before. Recall that $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$, respectively. Set $\alpha = \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j$ and $\beta = \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j$. Hence $dF_{\mathbb{C}} = \alpha + \beta$ and $\eta_{\mathbb{C}} = i(\alpha - \beta)$, so that

$$\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = 2i\alpha|_{M^*_{\mathbb{C}}} = -2i\beta|_{M^*_{\mathbb{C}}} \tag{3.1}$$

In particular, $\alpha|_{M^*_{\mathbb{C}}}$ and $\beta|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}_{\mathbb{C}}$.

Remark 3.4. Let $F(z) = \mathcal{R}e(h(z)) + h.o.t$ be such that $M = F^{-1}(0)$ is Levi-flat and $h(z)$ is a germ at $0 \in \mathbb{C}^2$ of A_k, D_k or E_k types. It is easy to check that $M^*_{\mathbb{C}} = M_{\mathbb{C}} \setminus \{0\}$ and $cod_{M^*_{\mathbb{C}}} sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = 2$.

Let us prove that we can apply theorem 1.13. We are going to prove directly that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral. For convenience, the proof will be divided into the following cases:

1. Case A_k, $k \geq 3$.

2. Case D_k, $k \geq 6$.

3. Case E_6.

3.2.1 Case A_k, $k \geq 3$

Let $(x, y) \in \mathbb{C}^2$. Write

$$F(x, y) = \mathcal{R}e(x^2 + y^{k+1}) + H(x, y, \bar{x}, \bar{y}).$$

Therefore, the complexification $F_{\mathbb{C}}$ of F can be written as

$$F_{\mathbb{C}}(x, y, z, w) = \frac{1}{2}(x^2 + y^{k+1}) + \frac{1}{2}(z^2 + w^{k+1}) + H_{\mathbb{C}}(x, y, z, w) \tag{3.2}$$

and $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0) \subset (\mathbb{C}^4, 0)$. Note that $sing(M_{\mathbb{C}}) = \{0\}$.

The resolution of singularities of $M_{\mathbb{C}}$ will be a detailed analysis. First of all, we begin by a blow-up at $0 \in \mathbb{C}^4$, $\pi : (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$. Let $\tilde{M}_{\mathbb{C}}$ denote the strict transform of $M_{\mathbb{C}}$ by π. We take the divisor $\mathbb{P}^3$ of the blow-up π with coordinates $[x : y : z : w]$, $(x, y, z, w) \in \mathbb{C}^4 \backslash \{0\}$. The intersection of $\tilde{M}_{\mathbb{C}}$ with the divisor $\mathbb{P}^3$ is the singular algebraic surface

$$Q := \{[x : y : z : w] \in \mathbb{P}^3 | x^2 + z^2 = 0\}.$$

1. Consider for instance the chart $(U, (t, u, z, v))$ of $\tilde{\mathbb{C}}^4$, where

$$\pi(t, u, z, v) = (z.t, z.u, z, z.v) = (x, y, z, w).$$

From (3.2) we have

$$F_{\mathbb{C}} \circ \pi(t, u, z, v) = z^2.(\frac{1}{2} + \frac{1}{2}t^2 + \frac{1}{2}z^{k-1}v^{k+1} + \frac{1}{2}z^{k-1}u^{k+1} + zH_1),$$

where $H_1 = H_{\mathbb{C}}(zt, zu, z, zv)/z^3$, which implies that $\tilde{M}_{\mathbb{C}} \cap U = \tilde{F}_{\mathbb{C}}^{-1}(0)$, where

$$\tilde{F}_{\mathbb{C}}(t,u,v,w) = 1 + t^2 + z^{k-1}v^{k+1} + z^{k-1}u^{k+1} + 2zH_1,$$
$$\Longrightarrow Q_1 := Q \cap U = (z = t^2 + 1 = 0).$$

On the other hand, as we have seen in the remark 3.3, the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\alpha = xdx + \frac{(k+1)}{2}y^k dy + \theta, \tag{3.3}$$

and θ is a 1-form with $j_0^k(\theta) = 0$. Therefore, the foliation $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$ is defined by $\alpha_1|_{\tilde{M}_{\mathbb{C}}} = 0$, where

$$\alpha_1 = (t^2 + \frac{(k+1)}{2}u^{k+1}z^{k-1})dz + \frac{(k+1)}{2}u^k z^k du + ztdt + z\theta_1, \tag{3.4}$$

and $\theta_1 = \pi^*(\theta)/z^2$, which implies that Q_1 is $\tilde{\mathcal{L}}_{\mathbb{C}}$-invariant. We would like to remark that

$$sing(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap U = (\alpha_1 \wedge d\tilde{F}_{\mathbb{C}} = 0, \tilde{F}_{\mathbb{C}} = 0).$$

As the reader can check, (3.4) implies that $sing(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap Q_1 = \emptyset$. In particular, Q_1 is the union of two leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ isomorphic to $\mathbb{C}^2$, say L_1 and L_2.

2. Consider now the chart $(V, (t,u,v,w))$ of $\tilde{\mathbb{C}}^4$, where

$$\pi(t,u,v,w) = (t.w, u.w, v.w, w) = (x,y,z,w).$$

From (3.2) we have

$$F_{\mathbb{C}} \circ \pi(t,u,z,v) = w^2.(\frac{1}{2}t^2 + \frac{1}{2}v^2 + \frac{1}{2}w^{k-1} + \frac{1}{2}w^{k-1}u^{k+1} + wG_1),$$

where $G_1 = H_{\mathbb{C}}(wt, wu, wv, w)/w^3$, which implies that $\tilde{M}_{\mathbb{C}} \cap V = \tilde{F}_{\mathbb{C}}^{-1}(0)$, where

$$\tilde{F}_{\mathbb{C}}(t,u,v,w) = \frac{1}{2}t^2 + \frac{1}{2}v^2 + \frac{1}{2}w^{k-1} + \frac{1}{2}w^{k-1}u^{k+1} + wG_1, \tag{3.5}$$
$$\Longrightarrow Q \cap V = (w = t^2 + v^2 = 0).$$

This implies that $sing(\tilde{M}_{\mathbb{C}}) \cap \mathbb{P}^3$ is a line L which in this coordinate system is $L \cap V = (w = t = v = 0)$. Notice that $\bar{L}_1 \cap \bar{L}_2 = L$.

We need more blow-ups along L to resolve this hypersurface. The process involves $\frac{(k-1)}{2}$ more explosions if k is odd and $\frac{k}{2}$ if k is even.

(a) If k is odd. We do $(k-1)/2$ explosions in the u-axis, obtaining divisors $D_1, \ldots, D_{(k-1)/2}$. In the appropiate chart, we have the equations

$$\begin{cases} w = w \\ u = u \\ t_{i-1} = w.t_i \\ v_{i-1} = w.v_i, \end{cases}$$

where $t_0 = t$, $v_0 = v$ and $1 \leq i \leq (k-1)/2$. Let $(V_{(k-1)/2}, (t, u, v, w))$ be the chart of the last explosion, we obtain

$$\pi_{(k-1)/2}(t, u, v, w) = (w^{(k-1)/2}.t, u, w^{(k-1)/2}.v, w),$$

where $t = t_{(k-1)/2}$, $v = v_{(k-1)/2}$. Denote by $\hat{M}_{\mathbb{C}}$ the strict transform of $\tilde{M}_{\mathbb{C}}$ under $\pi_{(k-1)/2}$. From (3.5), we get

$$\tilde{F}_{\mathbb{C}} \circ \pi_{(k-1)/2}(t, u, v, w) = w^{k-1}.(\frac{1}{2} + \frac{1}{2}t^2 + \frac{1}{2}v^2 + \frac{1}{2}u^{k+1} + wG_2),$$

where $G_2 = \pi^*_{(k-1)/2}(wG_1)/w^k$, so that $\hat{M}_{\mathbb{C}} \cap V_{(k-1)/2} = \hat{F}_{\mathbb{C}}^{-1}(0)$, where

$$\hat{F}_{\mathbb{C}}(t, u, v, w) = 1 + t^2 + v^2 + u^{k+1} + 2wG_2,$$

$$\Longrightarrow \hat{Q} := \hat{M}_{\mathbb{C}} \cap V_{(k-1)/2} \cap D_{(k-1)/2} = (w = 1 + t^2 + v^2 + u^{k+1} = 0). \quad (3.6)$$

Notice that $\hat{M}_{\mathbb{C}} \cap V_{(k-1)/2}$ is a smooth hypersurface.

At this part, we will see that $\hat{Q}$ is invariant by the strict transform of $\tilde{\mathcal{L}}_{\mathbb{C}}$ under $\pi_{(k-1)/2}$. In fact, the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by (3.3), so that the foliation $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$ in the chart V is defined by $\alpha_1|_{\tilde{M}_{\mathbb{C}}} = 0$, where

$$\alpha_1 = (t^2 + \frac{(k+1)}{2}u^{k+1}w^{k-1})dw + \frac{(k+1)}{2}u^k w^k du + twdt + w\eta_1,$$

and $\eta_1 = \pi^*(\theta)/w^2$. Therefore, the foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi^*_{(k-1)/2}(\tilde{\mathcal{L}}_{\mathbb{C}})$ in the chart $V_{(k-1)/2}$ is defined by $\alpha_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\alpha_2 = \frac{(k+1)}{2}(t^2 + u^{k+1})dw + \frac{(k+1)}{2}wu^k du + twdt + w\eta_2, \qquad (3.7)$$

and $\eta_2 = \pi^*_{(k-1)/2}(w\eta_1)/w^k$, (here $t = t_{(k-1)/2}$, $v = v_{(k-1)/2}$).

Hence, $\hat{Q}$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant. As we have already remarked

$$sing(\hat{\mathcal{L}}_{\mathbb{C}}) \cap V_{(k-1)/2} = (\alpha_2 \wedge d\hat{F}_{\mathbb{C}} = 0, \hat{F}_{\mathbb{C}} = 0).$$

It follows from (3.6) and (3.7) that

$$C := \hat{Q} \cap sing(\hat{\mathcal{L}}_{\mathbb{C}}) = (w = t^2 + u^{k+1} = v^2 + 1 = 0). \tag{3.8}$$

Therefore, C has the following irreducible components

$$(w = t + iu^{(k+1)/2} = v + i = 0),\ (w = t - iu^{(k+1)/2} = v + i = 0),$$

$$(w = t + iu^{(k+1)/2} = v - i = 0),\ (w = t - iu^{(k+1)/2} = v - i = 0).$$

In order to study the singular set of $\hat{\mathcal{L}}_{\mathbb{C}}$, we will work in the first explosion, for instance in the chart $(V_1, (t, u, s, p))$, where

$$\begin{cases} t = t \\ u = u \\ v = s.t \\ w = p.t, \end{cases}$$

we obtain $\pi_1(t, u, s, p) = (t, u, s.t, p.t) = (t, u, v, w)$. In this chart, we can see other two rules of $sing(\hat{\mathcal{L}}_{\mathbb{C}})$. In fact, it follows from (3.5) that the strict transform of $\tilde{M}_{\mathbb{C}}$ under π_1 is given by

$$\hat{M}_{\mathbb{C}} \cap V_1 = (1 + s^2 + p^{k-1}t^{k-3} + u^{k+1}p^{k-1}t^{k-3} + 2tG_3 = 0),$$

where $G_3 = \pi_1^*(wG_1)/t^3$, which implies that

$$\hat{Q}_1 := \hat{M}_{\mathbb{C}} \cap V_1 \cap D_1 = (t = s^2 + 1 = 0). \tag{3.9}$$

The foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi_1^*(\tilde{\mathcal{L}}_{\mathbb{C}})$ in this chart is defined by $\omega_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned} \omega_2 = &(2p + \frac{(k+1)}{2}u^{k+1}p^k t^{k-3})dt + \\ &+ \frac{(k+1)}{2}u^k p^k t^{k-2}du + (1 + \frac{(k+1)}{2}u^{k+1}p^{k-1}t^{k-3})tdp + t\eta_3, \end{aligned} \tag{3.10}$$

and $\eta_3 = \pi^*_{(k-1)/2}(w\eta_1)/t^3$. Observe that if $k > 3$, (3.9) and (3.10) implies that $\hat{Q}_1$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and

$$B := \hat{Q}_1 \cap sing(\hat{\mathcal{L}}_{\mathbb{C}}) = (t = p = s^2 + 1 = 0).$$

This implies that B has the following irreducible components

$$(t = p = s + i = 0), \quad (t = p = s - i = 0).$$

In the particular case $k = 3$, we need 1 blow-up along L to resolve $\tilde{M}_{\mathbb{C}}$. For instance in the coordinate system V_1, we have

$$\hat{M}_{\mathbb{C}} \cap V_1 = (1 + s^2 + p^2 + u^4p^2 + 2tG_2 = 0),$$

where $G_3 = \pi_1^*(wG_1)/t^3$, which implies that

$$\hat{Q}_1 = \hat{M}_{\mathbb{C}} \cap V_1 \cap D_1 = (t = p^2u^4 + p^2 + s^2 + 1 = 0).$$

Notice that $\hat{M}_{\mathbb{C}} \cap V_1$ is a smooth hypersurface.
On the other hand, (3.10) implies that the foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi_1^*(\tilde{\mathcal{L}}_{\mathbb{C}})$ in V_1 is defined by $\omega_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\omega_2 = 2p(1 + p^2u^4)dt + (1 + 2u^4p^2)tdp + 2u^3p^3tdu + t\eta_3, \tag{3.11}$$

and $\eta_3 = \pi_1^*(w\eta_1)/t^3$. Note that (3.11) implies that $\hat{Q}_1$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and $\hat{Q}_1 \cap sing(\hat{\mathcal{L}}_{\mathbb{C}})$ has the following irreducible components

$$(t = p = s + i = 0),\ (t = s + ip = pu^2 - i = 0),\ (t = s + ip = pu^2 + i = 0),$$

$$(t = p = s - i = 0),\ (t = s - ip = pu^2 + i = 0),\ (t = s - ip = pu^2 - i = 0).$$

(b) If k is even. We do $k/2$ explosions in the u-axis, obtaining divisors $D_1, \ldots, D_{k/2}$. As we have seen in (a), in an appropiate chart, we have the equations

$$\begin{cases} w = w \\ u = u \\ t_{i-1} = w.t_i \\ v_{i-1} = w.v_i, \end{cases}$$

where $t_0 = t$, $v_0 = v$ and $1 \leq i \leq k/2$. Let $(V_{k/2}, (t, u, v, w))$ be the chart of the last explosion, we obtain

$$\pi_{k/2}(t, u, v, w) = (w^{k/2}.t, u, w^{k/2}.v, w), \tag{3.12}$$

where $t_{k/2} = t$ and $v_{k/2} = v$. Denote $\hat{M}_{\mathbb{C}}$ be as before. It follows from (3.5) and (3.12) that

$$\hat{M}_{\mathbb{C}} \cap V_{k/2} = (1 + t^2 w + v^2 w + u^{k+1} + 2wG_4 = 0),$$

where $G_4 = \pi^*_{k/2}(wG_1)/w^k$, which implies that

$$\tilde{Q} := \hat{M}_{\mathbb{C}} \cap V_{k/2} \cap D_{k/2} = (w = 1 + u^{k+1} = 0). \tag{3.13}$$

From (3.7), the foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi^*_{k/2}(\tilde{\mathcal{L}}_{\mathbb{C}})$ in $V_{k/2}$ is defined by $\beta_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned} \beta_2 =& \frac{(k+1)}{2} w u^k du + \\ &+ (\frac{(k+2)}{2} t^2 w + \frac{(k+1)}{2} u^{k+1}) dw + t w^2 da + w\eta_4, \end{aligned} \tag{3.14}$$

and $\eta_4 = \pi^*_{k/2}(w\eta_1)/w^k$. Note that (3.13) and (3.14) implies that $\tilde{Q}$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and

$$\tilde{Q} \cap sing(\hat{\mathcal{L}}_{\mathbb{C}}) = \emptyset.$$

In particular, $\tilde{Q}$ is the union of $k+1$ leaves of $\hat{\mathcal{L}}_{\mathbb{C}}$ isomorphic to $\mathbb{C}^2$.

In the others charts, the study is similar to case k odd.

3. The study in the other charts is analogous, because there is a symmetry of the variables in the definition of the hypersurface $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$.

Let us prove that $\hat{\mathcal{L}}_{\mathbb{C}}$ has a non-constant holomorphic first integral. Let D be the global exceptional divisor of the resolution of singularities of $M_{\mathbb{C}}$, as we have seen before, all irreducible components of D are $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant. Set $Z := D \backslash sing(\hat{\mathcal{L}}_{\mathbb{C}})$. Fix $p_0 \in Z$ and a transversal $\sum$ to Z. For instance in the case k odd, we work in the chart $(V_{(k-1)/2}, (t, u, v, w))$, take $p_0 = (0, 0, 0, 0)$ and the section $\sum = \{(0, 0, 0, w) | w \in \mathbb{C}\}$, parametrized by w. Call G the holonomy group of the leaf Z of $\hat{\mathcal{L}}_{\mathbb{C}}$ in the section $\sum$. As we have seen in (3.6) and (3.8), we have

$$Z \cap V_{(k-1)/2} = \hat{Q} \backslash (w = t^2 + u^{k+1} = v^2 + 1 = 0).$$

Note that if we set $Z_1 = \hat{Q} \backslash (w = t^2 + u^{k+1} = v + i = 0)$ and $Z_2 = \hat{Q} \backslash (w = t^2 + u^{k+1} = v - i = 0)$ then $Z \cap V_{(k-1)/2} = Z_1 \cap Z_2$. The fundamental group $\Pi_1(Z_1, p_0)$ is generated by two loops δ_1, δ_2. These loops as follows: δ_1, δ_2 are loops that turns around $(w = t^2 + u^{k+1} = v + i = 0)$. Analogously, $\Pi_1(Z_2, p_0)$ is generated by two loops γ_1, γ_2. Acoording to Zariski [25], we get

$$\Pi_1(Z_1, p_0) = \langle [\delta_1], [\delta_2] : \delta_1.\delta_2^{(k+1)/2} = \delta_2^{(k+1)/2}.\delta_1 \rangle,$$

$$\Pi_1(Z_2, p_0) = \langle [\gamma_1], [\gamma_2] : \gamma_1.\gamma_2^{(k+1)/2} = \gamma_2^{(k+1)/2}.\gamma_1 \rangle.$$

Then $\Pi_1(Z \cap V_{(k-1)/2}, p_0)$ is generated by δ_1, δ_2, γ_1, γ_2. Therefore $G = \langle f_1, f_2, g_1, g_2 \rangle$, where f_i corresponding to $[\delta_i]$, and g_i to $[\gamma_i]$, for $i = 1, 2$. We get from (3.7) that $f_1'(0) = 1$, $f_2'(0) = e^{-2\pi i/k+1}$, $g_1'(0) = 1$, $g_2'(0) = e^{-2\pi i/k+1}$, so that $f_1(w) = w + w^2 r_1$, $f_2(w) = e^{-2\pi i/k+1}.w + w^2 r_2$, and $g_1(w) = w + w^2 s_1$, $g_2(w) = e^{-2\pi i/k+1}.w + w^2 s_2$. Since all leaves of $\mathcal{L}_{\mathbb{C}}$ are closed, we get $f_1(w) = w$ and $g_1(w) = w$, therefore $G = \langle f_2, g_2 \rangle$. Observe that $G' := \{g'(0) | g \in G\}$ is a finite group, and by similar arguments of the proof of theorem 1, the homomorphism $\phi : G \to G'$ defined by $\phi(g) = g'(0)$ is an isomorphism. This implies that G is finite and linearizable: in a some holomorphic coordinate system z of $(\sum, 0)$ we have $f_2(z) = g_2(z) = e^{-2\pi i/k+1}.z$, so that $G = \langle f_2 \rangle$. The function $H(z) = z^{k+1} \in \mathcal{O}_1$ satisfies $H \circ f_2 = H$. By [19] it can be extended to a non-constant holomorphic first integral, say $\hat{h}$, of $\hat{\mathcal{L}}_{\mathbb{C}}$, defined in some neighborhood of $\hat{Q}$ in $\hat{M}_{\mathbb{C}}$, which provides a first integral for $\mathcal{L}_{\mathbb{C}}$.

In the case k even, the proof is similar.

3.2.2 Case D_k, $k \geq 6$

Write

$$F(x, y) = \mathcal{R}e(x^2 y + y^{k-1}) + H(x, y, \bar{x}, \bar{y}).$$

Therefore, the complexification $F_{\mathbb{C}}$ of F can be written as

$$F_{\mathbb{C}}(x, y, z, w) = \frac{1}{2}(x^2 y + y^{k-1}) + \frac{1}{2}(z^2 w + w^{k-1}) + H_{\mathbb{C}}(x, y, z, w), \tag{3.15}$$

and $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0) \subset (\mathbb{C}^4, 0)$. Note that $sing(M_{\mathbb{C}}) = \{0\}$.

First of all, we begin by a blow-up at $0 \in \mathbb{C}^4$, $\pi : (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$. We take the divisor $\mathbb{P}^3$ of the blow-up π with coordinates $[x : y : z : w]$, $(x, y, z, w) \in \mathbb{C}^4 \backslash \{0\}$. Denote $\tilde{M}_{\mathbb{C}}$ be as before. The intersection of $\tilde{M}_{\mathbb{C}}$ with the divisor $\mathbb{P}^3$ is the singular algebraic surface

$$R := \{[x : y : z : w] | x^2 y + z^2 w = 0\}.$$

1. Consider for instance the chart $(W_1, (t, u, z, v))$ of $\tilde{\mathbb{C}}^4$, where

$$\pi(t, u, z, v) = (zt, zu, z, zv) = (x, y, z, w)$$

From (3.15) we have

$$F_{\mathbb{C}} \circ \pi(t, u, z, v) = z^3 (\frac{1}{2} v + \frac{1}{2} u t^2 + \frac{1}{2} z^{k-4} u^{k-1} + \frac{1}{2} z^{k-4} v^{k-1} + z H_1),$$

where $H_1 = H_{\mathbb{C}}(zt, zu, z, zv)/z^4$, which implies that $\tilde{M}_{\mathbb{C}} \cap W_1 = \tilde{F}_{\mathbb{C}}^{-1}(0)$, where

$$\tilde{F}_{\mathbb{C}}(t, u, z, v) = \frac{1}{2} v + \frac{1}{2} u t^2 + \frac{1}{2} z^{k-4} u^{k-1} + \frac{1}{2} z^{k-4} v^{k-1} + z H_1,$$

$$\Longrightarrow R_1 := R \cap W_1 = (z = u.t^2 + v = 0).$$

On the other hand, the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M_{\mathbb{C}}^*} = 0$, where

$$\alpha = xy dx + \frac{1}{2}(x^2 + (k-1) y^{k-2}) dy + \theta, \tag{3.16}$$

and θ is a 1-form with $j_0^{k-2}(\theta) = 0$. Therefore, the foliation $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$ in this chart is defined by

$$\begin{aligned} \alpha_1 = &\frac{1}{2} z^2 (t^2 + (k-1) u^2) du + \\ &+ (\frac{3}{2} u t^2 + \frac{(k-1)}{2} z^{k-4} u^{k-1}) dz + z t u dt + z \theta_1, \end{aligned} \tag{3.17}$$

where $\theta_1 = \frac{\pi^*(\theta)}{z^3}$. Note that R_1 is $\tilde{\mathcal{L}}_{\mathbb{C}}$-invariant. As we have already remarked in the case A_k, we have

$$sing(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap W_1 = (\alpha_1 \wedge d\tilde{F}_{\mathbb{C}} = 0, \tilde{F}_{\mathbb{C}} = 0).$$

Now, as the reader can check, (3.17) implies that

$$sing(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap R_1 = (z = t = v = 0) \cup (z = u = v = 0).$$

2. Consider now the chart $(W_2, (t, u, v, w))$ of $\tilde{\mathbb{C}}^4$, where

$$\pi(t, u, v, w) = (t.w, u.w, v.w, w) = (x, y, z, w),$$

we have

$$F_{\mathbb{C}} \circ \pi(t, u, v, w) = w^3.(\frac{1}{2}ut^2 + \frac{1}{2}v^2 + \frac{1}{2}u^{k-1}w^{k-4} + \frac{1}{2}w^{k-4} + wG_1),$$

where $G_1 = H_{\mathbb{C}}(wt, wu, wv, w)/w^4$, which implies that $\tilde{M}_{\mathbb{C}} \cap W_2 = \tilde{F}_{\mathbb{C}}^{-1}(0)$, where

$$\tilde{F}_{\mathbb{C}}(t, u, v, w) = \frac{1}{2}ut^2 + \frac{1}{2}v^2 + \frac{1}{2}u^{k-1}w^{k-4} + \frac{1}{2}w^{k-4} + wG_1, \tag{3.18}$$

$$\Longrightarrow R_2 := R \cap W_2 = (w = ut^2 + v^2 = 0).$$

This implies that $sing(\tilde{M}_{\mathbb{C}}) \cap \mathbb{P}^3$ is a line L which in this coordinate system is $(w = t = v = 0)$.

We need more blow-ups along L to resolve $\tilde{M}_{\mathbb{C}}$. The process involves $(k-4)/2$ more explosions if k is even and $(k-3)/2$ if k is odd.

(a) We do $(k-4)/2$ explosions the u-axis, obtaining divisors $D_1, \ldots, D_{(k-4)/2}$. In the appropiate chart, we have the equations

$$\begin{cases} w = w \\ u = u \\ t_{i-1} = w.t_i \\ v_{i-1} = w.v_i, \end{cases}$$

where $t_0 = t$, $v_0 = v$ and $1 \leq i \leq (k-4)/2$. Let $(U_{(k-4)/2}, (t, u, v, w))$ be the chart of the last explosion, we obtain

$$\pi_{(k-4)/2}(t, u, v, w) = (w^{(k-4)/2}.t, u, w^{(k-4)/2}.v, w),$$

where $t_{(k-4)/2} = t$ and $v_{(k-4)/2} = v$. Denote by $\hat{M}_{\mathbb{C}}$ the strict transform of $\tilde{M}_{\mathbb{C}}$ under $\pi_{(k-4)/2}$. From (3.18), we get

$$\tilde{F}_{\mathbb{C}} \circ \pi_{(k-4)/2}(t, u, v, w) = w^{k-4}.(\frac{1}{2} + \frac{1}{2}a^2u + \frac{1}{2}b^2 + \frac{1}{2}u^{k-1} + wG_2),$$

where $G_2 = \pi^*_{(k-4)/2}(wG_1)/w^{k-3}$, which implies that $\hat{M}_{\mathbb{C}} \cap U_{(k-4)/2} = \hat{F}_{\mathbb{C}}^{-1}(0)$, where

$$\hat{F}_{\mathbb{C}}(t,u,v,w) = \frac{1}{2} + \frac{1}{2}a^2u + \frac{1}{2}b^2 + \frac{1}{2}u^{k-1} + wG_2,$$

$$\Longrightarrow \hat{R} := \hat{M}_{\mathbb{C}} \cap U_{(k-4)/2} \cap D_{(k-4)/2} = (w = 1 + a^2u + b^2 + u^{k-1} = 0). \quad (3.19)$$

Notice that $\hat{M}_{\mathbb{C}} \cap U_{(k-4)/2}$ is a smooth hypersurface.

At this part, we will see that $\hat{R}$ is invariant by the strict transform of $\tilde{\mathcal{L}}_{\mathbb{C}}$ under $\pi_{(k-4)/2}$. In fact, the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by (3.16), so that the foliation $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$ in the chart W_2 is defined by $\alpha_1|_{\tilde{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned}\alpha_1 =& (\frac{3}{2}ut^2 + \frac{(k-1)}{2}u^{k-1}w^{k-4})dw + \\ &+ twudt + \frac{1}{2}(t^2 + (k-1)u^{k-2}w^{k-4})wdu + w\eta_1,\end{aligned}$$

and $\eta_1 = \pi^*(\theta)/w^3$. Therefore, the foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi^*_{(k-4)/2}(\tilde{\mathcal{L}}_{\mathbb{C}})$ is defined by $\alpha_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned}\alpha_2 =& \frac{(k-1)}{2}u(t^2 + u^{k-2})dw + \\ &+ \frac{1}{2}(t^2 + (k-1)u^{k-2})wdu + tuwdt + w\eta_2,\end{aligned} \quad (3.20)$$

and $\eta_2 = \pi^*_{(k-4)/2}(w\eta_1)/w^{k-3}$, (here $t_{(k-4)/2} = t$ and $v_{(k-4)/2} = v$). From (3.20), we have that $\hat{R}$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and

$$K = \hat{R} \cap sing(\hat{\mathcal{L}}_{\mathbb{C}}) = (w = u(t^2 + u^{k-2}) = v^2 + 1 = 0). \quad (3.21)$$

Note that K is composed of six components

$$\begin{aligned}&(w = v + i = u = 0),\ (w = v + i = t + iu^{(k-2)/2} = 0),\\ &(w = v - i = u = 0),\ (w = v + i = t - iu^{(k-2)/2} = 0),\\ &(w = v - i = t + iu^{(k-2)/2} = 0),\ (w = v - i = t - iu^{(k-2)/2} = 0).\end{aligned}$$

In order to study of singular set of $\hat{\mathcal{L}}_{\mathbb{C}}$, we will work in the first explosion, for instance in the chart $(U_1, (t, u, s, p))$ where

$$\begin{cases} t = t \\ u = u \\ v = s.t \\ w = p.t. \end{cases}$$

We obtain $\pi_1(t, u, s, p) = (t, u, s.t, p.t) = (t, u, v, w)$. In this chart, we can see other irreducible components of $sing\hat{\mathcal{L}}_{\mathbb{C}}$. In fact, it is easy to check that

$$\hat{M}_{\mathbb{C}} \cap U_1 = (u + s^2 + u^{k-1}p^{k-4}t^{k-6} + p^{k-4}t^{k-6} + 2tG_4 = 0),$$

where $G_4 = \pi_1^*(wG_1)/t^3$, which implies that

$$\hat{R}_1 := \hat{M}_{\mathbb{C}} \cap U_1 \cap D_1 = (t = u + s^2 = 0).$$

Note that $\hat{M}_{\mathbb{C}} \cap U_1$ is a smooth hypersurface.
On the other hand, the foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi_1^*(\tilde{\mathcal{L}}_{\mathbb{C}})$ in this chart is defined by $\omega_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned} \omega_2 = &(\frac{5}{2}up + \frac{(k-1)}{2}u^{k-1}p^{k-3}t^{k-6})dt + \\ &+ (\frac{3}{2}u + \frac{(k-1)}{2}u^{k-1}p^{k-4}t^{k-6})tdp + \\ &+ \frac{1}{2}(1 + (k-1)u^{k-2}p^{k-4}t^{k-6})tpdu + t\eta_4, \end{aligned} \tag{3.22}$$

and $\eta_4 = \pi_1^*(w\eta_1)/t^3$. Observe that if $k > 6$, (3.22) implies that $\hat{R}_1$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and

$$K_1 := \hat{R}_1 \cap sing(\hat{\mathcal{L}}_{\mathbb{C}}) = (t = pu = u + s^2 = 0).$$

Then K_1 is composed of one line and one curve:

$$(t = u = s = 0) \text{ and } (t = p = u + s^2 = 0).$$

In the particular case $k = 6$, we need 1 blow-up along L to resolve $\tilde{M}_{\mathbb{C}}$. For instance in the coordinate system U_1, we have

$$\hat{M}_{\mathbb{C}} \cap U_1 = (u + s^2 + u^5p^2 + p^2 + 2tG_4 = 0),$$

where $G_4 = \pi_1^*(wG_1)/t^3$, which implies that

$$\hat{R}_1 = \hat{M}_{\mathbb{C}} \cap U_1 \cap D_1 = (t = u + s^2 + u^5p^2 + p^2 = 0).$$

The foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi_1^*(\tilde{\mathcal{L}}_{\mathbb{C}})$ in U_1 is defined by $\alpha_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned} \alpha_2 =& \frac{5}{2}(up + u^5p^3)dt + \\ &+ (\frac{3}{2}u + \frac{5}{2}u^5p^2)tdp + \frac{1}{2}(1 + 5u^2p^2)tpdu + t\eta_4, \end{aligned} \tag{3.23}$$

and $\eta_4 = \pi_1^*(w\eta_1)/t^3$. Note that (3.23) implies that $\hat{R}_1$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and $\hat{R}_1 \cap sing(\hat{\mathcal{L}}_{\mathbb{C}})$ has the following components

$$(t = p = u + s^2 = 0),\ (t = u = s - ip = 0),\ (t = s - ip = pu^2 - i = 0),$$

$$(t = u = s + ip = 0),\ (t = s - ip = pu^2 + i = 0),$$

$$(t = s + ip = pu^2 - i = 0), (t = s + ip = pu^2 - i = 0).$$

(b) If k is odd. We do $(k-3)/2$ explosions in the u-axis, obtaining divisors $D_1, \ldots, D_{(k-3)/2}$. Let $(U_{(k-3)/2}, (t, u, v, w))$ be the chart of the last explosion, we obtain

$$\pi_{(k-3)/2}(t, u, v, w) = (w^{(k-3)/2}.t, u, w^{(k-3)/2}.v, w), \tag{3.24}$$

where $t = t_{(k-3)/2}$ and $v = v_{(k-3)/2}$. Denote $\hat{M}_{\mathbb{C}}$ be as before. It follows from (3.18) and (3.24) that

$$\hat{M}_{\mathbb{C}} \cap U_{(k-3)/2} = (1 + a^2uw + b^2w + u^{k-1} + 2wG_3 = 0),$$

where $G_3 = \pi_1^*(wG_1)/w^{k-4}$, which implies that

$$\tilde{R} = \hat{M}_{\mathbb{C}} \cap U_{(k-3)/2} \cap D_{(k-3)/2} = (w = 1 + u^{k-1} = 0).$$

The foliation $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$ is defined by (3.20). Therefore, $\hat{\mathcal{L}}_{\mathbb{C}} = \pi^*_{(k-3)/2}(\tilde{\mathcal{L}}_{\mathbb{C}})$ in this chart is defined by $\alpha_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned}\alpha_2 =&(\frac{k}{2}uwa^2 + \frac{(k-1)}{2}u^{k-1})dw+ \\ &+\frac{1}{2}(wa^2 + (k-1)u^{k-2})wdu + auw^2da + w\eta_3,\end{aligned} \tag{3.25}$$

and $\eta_3 = \pi^*_{(k-3)/2}(w\eta_1)/w^{k-5}$. From (3.25), $\tilde{R}$ is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and

$$\tilde{R} \cap sing(\hat{\mathcal{L}}_{\mathbb{C}}) = \emptyset.$$

In particular, $\tilde{R}$ is a union of $k-1$ leaves of $\hat{\mathcal{L}}_{\mathbb{C}}$ isomorphic to $\mathbb{C}^2$.

In the others charts, the study is similar to case k even.

3. The study in the other charts is analogous.

Let us prove that $\hat{\mathcal{L}}_{\mathbb{C}}$ has a non-constant holomorphic first integral. Let D be the global exceptional divisor of the resolution of singularities of $M_{\mathbb{C}}$, as we have seen before, all irreducible components of D are $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant. Set $Z := D \backslash sing(\hat{\mathcal{L}}_{\mathbb{C}})$. Fix $p_0 \in Z$ and a transversal $\sum$ to Z. For instance for case k even, we work in the chart $(U_{(k-4)/2}, (t, u, v, w))$, take $p_0 = (0, 0, 0, 0)$ and the section $\sum = \{(0, 0, 0, w) | w \in \mathbb{C}\}$, parametrized by w. Call G the holonomy group of the leaf Z of $\hat{\mathcal{L}}_{\mathbb{C}}$ in the section $\sum$. As we have seen in (3.19) and (3.21), we have

$$Z \cap U_{(k-4)/2} = \hat{R} \backslash (w = u(t^2 + u^{k-2}) = v^2 + 1 = 0).$$

Note that if we set $Z_1 = \hat{R} \backslash (w = u(t^2 + u^{k-2}) = v + i = 0)$, $Z_2 = \hat{R} \backslash (w = u(t^2 + u^{k-2}) = v - i = 0)$ then $Z \cap U_{(k-4)/2} = Z_1 \cap Z_2$. The fundamental group $\Pi_1(Z_1, p_0)$ is generated by three loops δ_1, δ_2 and δ_3. These loops as follows: δ_1, δ_2 are loops that turns around $(w = t^2 + u^{k-2} = v + i = 0)$, and δ_3 is a loop that turns around $(w = u = v + i = 0)$. Analogously, $\Pi_1(Z_2, p_0)$ is generated by two loops γ_1, γ_2 and γ_3. According to Zariski [25], we get

$$\Pi_1(Z_1, p_0) = \langle [\delta_1], [\delta_2], [\delta_3] : \delta_1^{(k-2)/2}.\delta_2 = \delta_2.\delta_1^{(k-2)/2} \rangle,$$

$$\Pi_1(Z_2, p_0) = \langle [\gamma_1], [\gamma_2], [\gamma_3] : \gamma_1^{(k-2)/2}.\gamma_2 = \gamma_2.\gamma_1^{(k-2)/2} \rangle.$$

Then $\Pi_1(Z \cap U_{(k-4)/2}, p_0)$ is generated by δ_1, δ_2, δ_3, γ_1, γ_2, γ_3. Therefore

$$G = \langle f_1, f_2, f_3, g_1, g_2, g_3 \rangle,$$

where f_i corresponding to $[\delta_i]$, and g_i to $[\gamma_i]$, for $i = 1, 2, 3$. We get from (3.20) that $f_1'(0) = e^{-2\pi i/k-1}$, $f_2'(0) = e^{-4\pi i/k-1}$, $f_3'(0) = 1$, $g_1'(0) = e^{-2\pi i/k-1}$, $g_2'(0) = e^{-4\pi i/k-1}$, $g_3'(0) = 1$ so that $f_1(w) = e^{-2\pi i/k-1}w + w^2 r_1$, $f_2(w) = e^{-4\pi i/k-1}.w + w^2 r_2$, $f_3(w) = w + w^2 r_3$, and $g_1(w) = e^{-2\pi i/k-1}w + w^2 s_1$, $g_2(w) = e^{-4\pi i/k-1}.w + w^2 s_2$, $f_3(w) = w + w^2 s_3$. Since all leaves of $\mathcal{L}_{\mathbb{C}}$ are closed, we get $f_3(w) = w$ and $g_3(w) = w$, therefore $G = \langle f_1, f_2, g_1, g_2 \rangle$. Observe that $G' := \{g'(0) | g \in G\}$ is a finite group, and by similar arguments of the proof of theorem 1, the homomorphism $\phi : G \to G'$ defined by $\phi(g) = g'(0)$ is an isomorphism. This implies that G is finite, it follows that G is linearizable: in a some holomorphic coordinate system z of $(\sum, 0)$ we have $f_1(z) = g_1(z) = e^{-2\pi i/k-1}.z$ and $f_2(z) = g_2(z) = e^{-4\pi i/k-1}.z$, so that $G = \langle f_1 \rangle$, because $f_1 \circ f_1 = f_2 = g_2$. The function $H(z) = z^{k-1} \in \mathcal{O}_1$ satisfies $H \circ f_1 = H$. By [19] it can be extended to a non-constant holomorphic first integral, say $\hat{h}$, of $\hat{\mathcal{L}}_{\mathbb{C}}$, defined in some neighborhood of $\hat{R}$ in $\hat{M}_{\mathbb{C}}$, which provides a first integral for $\mathcal{L}_{\mathbb{C}}$.

In the case k odd, the proof is similar.

3.2.3 Case E_6

Write

$$F(x, y) = \mathcal{R}e(x^4 + y^3) + H(x, y, \bar{x}, \bar{y}).$$

Therefore, the complexification $F_{\mathbb{C}}$ of F can be written as

$$F_{\mathbb{C}}(x, y, z, w) = \frac{1}{2}(x^4 + y^3) + \frac{1}{2}(z^4 + w^3) + H_{\mathbb{C}}(x, y, z, w), \tag{3.26}$$

and $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0) \subset (\mathbb{C}^4, 0)$. Note that $sing(M_{\mathbb{C}}) = \{0\}$.

We begin by a blow-up at $0 \in \mathbb{C}^4$, $\pi : (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$. Let $\tilde{M}_{\mathbb{C}}$ be as before. We take the divisor $\mathbb{P}^3$ of the blow-up π with coordinates $[x : y : z : w]$, $(x, y, z, w) \in \mathbb{C}^4 \backslash \{0\}$. The intersection of $\tilde{M}_{\mathbb{C}}$ with the divisor $\mathbb{P}^3$ is the singular algebraic surface

$$N := \{[x : y : z : w] | y^3 + w^3 = 0\}.$$

1. Consider for instance the chart $(W_1, (t, u, v, w))$ of $\tilde{\mathbb{C}}^4$, where

$$\pi(t, u, z, v) = (wt, wu, wv, w) = (x, y, z, w).$$

We have

$$F_{\mathbb{C}} \circ \pi(t, u, v, w) = w^3(\frac{1}{2} + \frac{1}{2}wt^4 + \frac{1}{2}u^3 + \frac{1}{2}v^4w + z.H_1),$$

where $H_1 = H_{\mathbb{C}}(wt, wu, wv, w)/w^4$, which implies that

$$\tilde{M}_{\mathbb{C}} \cap W_1 = (1 + wt^4 + u^3 + v^4w + 2.zH_1 = 0)$$

$$\Longrightarrow N_1 = N \cap W_1 = (w = u^3 + 1 = 0).$$

Note that $\tilde{M}_{\mathbb{C}} \cap W_1$ is a smooth hypersurface on $\tilde{\mathbb{C}}^4 \cap W_1$. The foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\alpha = 2x^3dx + \frac{3}{2}y^2dy + \theta, \tag{3.27}$$

where θ is a 1-form with $j^3_0(\theta) = 0$. The foliation $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$ in this chart is defined by $\alpha_1|_{\tilde{M}_{\mathbb{C}}} = 0$, where

$$\alpha_1 = 2w^2t^3dt + (2wt^4 + \frac{3}{2}u^3)dw + \frac{3}{2}wu^2du + w\eta_1, \tag{3.28}$$

and $\eta_1 = \pi^*(\theta)/w^3$. Note that N_1 is $\tilde{\mathcal{L}}_{\mathbb{C}}$-invariant and from (3.28), we get $N_1 \cap sing(\tilde{\mathcal{L}}_{\mathbb{C}}) = \emptyset$. In particular, N_1 is a union of three leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ isomorphic to $\mathbb{C}^2$, say L_1, L_2, L_3.

2. In the chart $(W_2, (t, u, z, v))$ of $\tilde{\mathbb{C}}^4$, where

$$\pi(t, u, z, v) = (z.t, z.u, z, z.v) = (x, y, z, w).$$

From (3.26) we have

$$F_{\mathbb{C}} \circ \pi(t, u, z, v) = z^3(\frac{1}{2}z + \frac{1}{2}z.t^4 + \frac{1}{2}u^3 + \frac{1}{2}v^3 + z.H_1),$$

where $H_1 = H_{\mathbb{C}}(zt, zu, z, zv)/z^4$, which implies that

$$\tilde{M}_{\mathbb{C}} \cap W_2 = (z + zt^4 + u^3 + v^3 + 2.zH_1 = 0) \tag{3.29}$$

$$\Longrightarrow N_2 = N \cap W_2 = (z = u^3 + v^3 = 0).$$

This implies that $sing(\tilde{M}_{\mathbb{C}}) \cap \mathbb{P}^3$ is a line L which in this coordinate system is $\{z = v = u = 0\}$. Notice that $\bar{L}_1 \cap \bar{L}_2 \cap \bar{L}_3 = L$. We need more blow-ups along L to resolve $\tilde{M}_{\mathbb{C}}$. The process involves 3 explosions.

We do 3 explosions in the t-axis, obtaining a sequence of divisors D_1, D_2, D_3. In the appropiate chart, we have the equations

$$\begin{cases} v = v \\ t = t \\ z_{i-1} = v.z_i \\ u_{i-1} = v.u_i, \end{cases}$$

where $z_0 = z$, $u_0 = u$ and $1 \leq i \leq 3$. Let $(U_3, (t, u_3, z_3, v))$ be the chart in the last explosion, we obtain

$$\pi_3(t, u_3, z_3, v) = (t, v^3.u_3, v^3.z_3, v) = (t, u, z, v).$$

Denote by $\hat{M}_{\mathbb{C}}$ the strict transform of $\tilde{M}_{\mathbb{C}}$ under π_3. From (3.29), we get

$$\hat{M}_{\mathbb{C}} \cap U_3 = (1 + z + zt^4 + v^6u^3 + 2.vG_3 = 0),$$

where $G_3 = \pi_3^*(zH_1)/v^4$, (here $z_3 = z$ and $u_3 = u$) which implies that

$$B =: \hat{M}_{\mathbb{C}} \cap U_3 \cap D_3 = (v = 1 + z + zt^4 = 0). \tag{3.30}$$

We will see that B is invariant by the strict transform of $\tilde{\mathcal{L}}_{\mathbb{C}}$ under π_3, where $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$.

In fact, the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by (3.27). Therefore, the foliation $\tilde{\mathcal{L}}_{\mathbb{C}} = \pi^*(\mathcal{L}_{\mathbb{C}})$ in the chart W_2 is defined by $\alpha_1|_{\tilde{M}_{\mathbb{C}}} = 0$, where

$$\alpha_1 = 2z^2t^3dt + (2zt^4 + \frac{3}{2}u^3)dz + \frac{3}{2}zu^2du + z\eta_1,$$

and $\eta_1 = \pi^*(\theta)/z^3$. Therefore, the foliation $\hat{\mathcal{L}}_{\mathbb{C}} = \pi_3^*(\tilde{\mathcal{L}}_{\mathbb{C}})$ in the chart U_3 is defined by $\alpha_2|_{\hat{M}_{\mathbb{C}}} = 0$, where

$$\begin{aligned}\alpha_2 = & 2vz^2t^3dt + 6(z^2t^4 + zu^3v^6)dv + \\ & + (2zt^4 + \frac{3}{2}u^3v^6)vda + \frac{3}{2}zu^2v^7du + v\eta_4,\end{aligned} \tag{3.31}$$

and $\eta_4 = \pi_1^*(z\eta_1)/v^6$, (here $z_3 = z$ and $u_3 = u$).

From (3.31), B is $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant and

$$B \cap sing(\hat{\mathcal{L}}_{\mathbb{C}}) = (v = t = z + 1 = 0). \tag{3.32}$$

3. Finally, the study in the other charts is analogous.

Let us prove that $\hat{\mathcal{L}}_{\mathbb{C}}$ has a non-constant holomorphic first integral. Let D be the global exceptional divisor of the resolution of singularities of $M_{\mathbb{C}}$, as we have seen before, all irreducible components of D are $\hat{\mathcal{L}}_{\mathbb{C}}$-invariant. Set $Z := D \backslash sing(\hat{\mathcal{L}}_{\mathbb{C}})$. Fix $p_0 \in Z$ and a transversal $\sum$ to Z. For instance, we work in the chart $(\tilde{V}, (t, u, z, v))$, take $p_0 = (0, 0, 0, 0)$ and the section $\sum = \{(0, 0, 0, v) | v \in \mathbb{C}\}$, parametrized by w. Call G the holonomy group of the leaf Z of $\hat{\mathcal{L}}_{\mathbb{C}}$ in the section $\sum$. As we have seen in (3.30) and (3.32), we have

$$Z \cap U_3 = B \backslash (v = t = z + 1 = 0).$$

The fundamental group $\Pi_1(Z \cap U_3, p_0)$ is generated by a loop δ that turns around of $(v = t = z + 1 = 0)$. Therefore $G = \langle f \rangle$, where f corresponding to $[\delta]$, from (3.31), we have $f'(0) = e^{-2\pi i/3}$, so that $f(v) = e^{-2\pi i/3}.v + v^2 r$. Since all leaves of $\mathcal{L}_{\mathbb{C}}$ are closed, the group G is finite, it follows that G is linearizable: in a some holomorphic coordinate system z of $(\sum, 0)$ we have $f(z) = e^{-2\pi i/3}.z$. The function $H(z) = z^3 \in \mathcal{O}_1$ satisfies $H \circ f = H$. By [19] it can be extended to a non-constant holomorphic first integral, say $\hat{h}$, of $\hat{\mathcal{L}}_{\mathbb{C}}$, defined in some neighborhood of B in $\hat{M}_{\mathbb{C}}$.

In all cases, we have seen that the foliation $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral. This finishes the proof of theorem 2.

Chapter 4

Levi-flat hypersurfaces and webs

In this chapter, we investigate germs at $0 \in \mathbb{C}^n$ of codimension one k-webs tangent to germs at $0 \in \mathbb{C}^n$ of real analytic Levi-flat hypersurfaces.

4.1 Local webs

We refer the terminology used in [22]. A germ of singular codimension one k-web on $(\mathbb{C}^n, 0)$, $n \geq 2$, is an equivalence class $[\omega]$ of germs of k-symmetric 1-forms, that is sections of $Sym^k\Omega^1(\mathbb{C}^n, 0)$, modulo multipilication by $\mathcal{O}^*(\mathbb{C}^n, 0)$ such that a suitable representative ω defined in a connected neighborhood U of the origin satisfies the following conditions:

1. The zero set of ω has codimension at least two.
2. The 1-form ω, seen as a homogeneous polynomial of degree k in the ring $\mathcal{O}(\mathbb{C}^n, 0)[dx_1, \ldots, dx_n]$, is square-free.
3. (Brill's condition) For a generic $p \in U$, $\omega(p)$ is a product of k linear forms.
4. (Frobenius's condition) For a generic $p \in U$, the germ of ω at p is the product of k germs of integrable 1-forms.

Both conditions (3) and (4) are automatic for germs of webs on $(\mathbb{C}^2, 0)$ and non-trivial for germs on $(\mathbb{C}^n, 0)$ when $n \geq 3$.

We can think k-webs as first order differential equations of degree k. There exists an alternative definition for germs of singular webs that is in a certain sense more geometric. The idea is to consider the germ of web as a meromorphic section of the projectivization of the cotangent bundle. This is a classical point view in the theory of differential equations, which has been recently explored in Web-geometry. For instance see Cavalier-Lehmann [10], [11], J. Yartey [23]. In this section, we will use both definitions.

4.1.1 The contact distribution

Let us denote $\mathbb{P} := \mathbb{P}T^*(\mathbb{C}^n, 0)$ the projectivization of the cotangent bundle of $(\mathbb{C}^n, 0)$ and $\pi : \mathbb{P}T^*(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ the natural projection. Over a point p the fiber $\pi^{-1}(p)$ parametrizes the one-dimensional subspaces of $T^*_p(\mathbb{C}^n, 0)$. On $\mathbb{P}$ there is a canonical codimension one distribution, the so called contact distribution $\mathcal{D}$. Its description in terms of a system of coordinates $x = (x_1, \ldots, x_n)$ of $(\mathbb{C}^n, 0)$ goes as follows: let $dx_1, \ldots, dx_n$ be the basis of $T^*(\mathbb{C}^n, 0)$ associated to the coordinate system $(x_1, \ldots, x_n)$. Given a point $(x, y) \in T^*(\mathbb{C}^n, 0)$, we can write $y = \sum_{j=1}^n y_j dx_j$, $(y_1, \ldots, y_n) \in \mathbb{C}^n$. In this way, if $(y_1, \ldots, y_n) \neq 0$ then we set $[y] = [y_1, \ldots . y_n] \in \mathbb{P}^{n-1}$ and $(x, [y]) \in (\mathbb{C}^n, 0) \times \mathbb{P}^{n-1} \cong \mathbb{P}$. In the affine coordinate system $y_n \neq 0$ of $\mathbb{P}$, the distribiution $\mathcal{D}$ is defined by $\alpha = 0$, where

$$\alpha = dx_n - \sum_{j=1}^{n-1} p_j dx_j, \qquad p_j = -\frac{y_i}{y_n} \qquad (1 \leq j \leq n-1). \tag{4.1}$$

The 1-form α is called the contact form.

4.2 Webs as closures of meromorphic multi-sections

Now consider $X \subset \mathbb{P}$ a sub-variety, not necessarily irreducible, but of pure dimension n. Let $\pi_X : X \to (\mathbb{C}^n, 0)$ be the restriction to X of the projection $\pi : \mathbb{P} \to (\mathbb{C}^n, 0)$. Suppose also that X satisfies the following conditions:

1. The image under π of every irreducible component of X has dimension n.
2. The generic fiber of π intersects X in k distints smooth points and at these the differential $d\pi_X : T_pX \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is surjective. Note that $k = deg(\pi_X)$.

3. The restriccion of the contact form α to the smooth part of every irreducible component of X is integrable. We denote $\mathcal{F}_X$ the foliation defined by $\alpha|_X = 0$.

We can then define $\mathcal{W}$ a germ at $0 \in \mathbb{C}^n$ of k-web as a triple $(X, \pi_X, \mathcal{F}_X)$. This definition is equivalent to the one laid down in Section 4.1. Denote by X the variety associated to $\mathcal{W}$. The singular set of X will be denoted by $sing(X)$ and its the smooth part by X_{reg}.

Here and subsequently, $\mathcal{W}$ denotes a germ at $0 \in \mathbb{C}^n$ of codimension one k-web, X the contact variety associated to $\mathcal{W}$, π_X the restriction to X of the projection $\pi : \mathbb{P} \to (\mathbb{C}^n, 0)$ and $\mathcal{F}_X$ the foliation defined by $\alpha|_X = 0$.

Definition 4.1. Let R be the set of points $p \in X$ where

- either X is singular,
- or the differential $d\pi_X : T_pX_{reg} \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is not an isomorphism.

The analytic set R is called the criminant set of $\mathcal{W}$ and $\triangle_{\mathcal{W}} = \pi(R)$ the discriminant of $\mathcal{W}$. Note that $dim(R) \leq n-1$.

4.2.1 The foliation $\mathcal{F}_X$

Since the restriction of $\mathcal{D}$ to X_{reg} is integrable, it defines a foliation $\mathcal{F}_X$, which in general is a singular foliation. Given $p \in (\mathbb{C}^n, 0) \backslash \triangle_{\mathcal{W}}$, $\pi_X^{-1}(p) = \{q_1, \ldots, q_k\}$, where $q_i \neq q_j$, if $i \neq j$, $(deg(\pi_X) = k)$, denote by $\mathcal{F}_X^i$ the germ of $\mathcal{F}_X$ at q_i, $i = 1, \ldots, k$.

The projections $\pi_*(\mathcal{F}_X^i) := \mathcal{F}_p^i$ defines k germs of codimension one foliations at p.

Definition 4.2. A leaf of the web $\mathcal{W}$ is, by definition, the projection on $(\mathbb{C}^n, 0)$ of a leaf of $\mathcal{F}_X$.

Remark 4.3. Given $p \in (\mathbb{C}^n, 0) \backslash \triangle_{\mathcal{W}}$, and $q_i \in \pi_X^{-1}(p)$, the projection $\pi_X(L_i)$ of the leaf L_i of $\mathcal{F}_X$ through q_i, gives origen to a leaf of $\mathcal{W}$ through p. In particular, $\mathcal{W}$ has at most k leaves through p.

Remark 4.4. Let $\omega \in Sym^k\Omega_1(\mathbb{C}^n, 0)$ and assume that it defines a k-web $\mathcal{W}$ with contact variety X. Then X is irreducible if, and only if, ω is irreducible in the ring $\mathcal{O}_n[dx_1, \ldots, dx_n]$. In this case we say the web is irreducible.

4.3 First integrals for webs

Let $\mathcal{O}(X)$ denote the ring of holomorphic functions on X.

Definition 4.5. We say that $\mathcal{W}$ a k-web has a meromorphic first integral if, and only if, there exists

$$P(z) = f_0 + z.f_1 + \ldots + z^k.f_k \in \mathcal{O}_n[z],$$

where $f_0, \ldots, f_k \in \mathcal{O}_n$, such that every irreducible component of the hypersurface $(P(z_0) = 0)$ is a leaf of $\mathcal{W}$, for all $z_0 \in (\mathbb{C}, 0)$.

Definition 4.6. We say that $\mathcal{W}$ a k-web has a holomorphic first integral if, and only if, there exists

$$P(z) = f_0 + z.f_1 + \ldots + z^{k-1}.f_{k-1} + z^k \in \mathcal{O}_n[z],$$

where $f_0, \ldots, f_{k-1} \in \mathcal{O}_n$ and such that every irreducible component of the hypersurface $(P(z_0) = 0)$ is a leaf of $\mathcal{W}$, for all $z_0 \in (\mathbb{C}, 0)$.

We will use the following proposition (cf. [16] Th. 5, pg. 32).

Proposition 4.7. *Let V be an analytic variety. If $\pi : V \to W$ is a finite branched holomorphic covering of pure order k over an open subset $W \subseteq \mathbb{C}^n$, then to each holomorphic function $f \in \mathcal{O}(V)$ there is canonically associated a monic polynomial $P_f(z) \in \mathcal{O}_n[z] \subseteq \mathcal{O}(V)[z]$ of degree k such that $P_f(f) = 0$ in $\mathcal{O}(V)$.*

We have now the following lemma. The proof is an easy adaption of an argument of I.Pan (cf. [21]).

Lemma 4.8. *Suppose that $(X, \pi_X, \mathcal{F}_X)$ defines a k-web $\mathcal{W}$ on $(\mathbb{C}^n, 0)$, $n \geq 2$, where X is an irreducible sub-variety of $\mathbb{P}$. If $\mathcal{F}_X$ has a non-constant holomorphic first integral then $\mathcal{W}$ also has a holomorphic first integral.*

Proof. Let $g \in \mathcal{O}(X)$ be the first integral for $\mathcal{F}_X$. By proposition 4.7, there exists a monic polynomial $P_g(z) \in \mathcal{O}_n[z]$ of degree k such that $P_g(g) = 0$ in $\mathcal{O}(X)$. Write

$$P_g(z) = g_0 + z.g_1 + \ldots + z^{k-1}.g_{k-1} + z^k,$$

where $g_0, \ldots, g_{k-1} \in \mathcal{O}_n$.

Assertion.– P_g defines a holomorphic first integral for $\mathcal{W}$.

Let $U \subseteq (\mathbb{C}^n, 0) \backslash \triangle_{\mathcal{W}}$ be an open subset and set $\varphi : X \to (\mathbb{C}^n, 0) \times \mathbb{C}$ be defined by $\varphi = (\pi_X, g)$. Take a leaf L of $\mathcal{W}|_U$. Then there is $z \in \mathbb{C}$ such that the following diagram

$$\begin{tikzcd} \pi_X^{-1}(U) \cap \varphi^{-1}(L \times \{z\}) \arrow[rr, "\varphi"] \arrow[dr, "\pi_X"'] & & L \times \{z\} \arrow[dl, "pr_1"] \\ & L & \end{tikzcd}$$

is commutative, where pr_1 is the projection to first coordinate. One deduce that L is a leaf of $\mathcal{W}$ if and only if g is constant along of each connected component of $\pi_X^{-1}(L)$ contained in $\varphi^{-1}(L \times \{z\})$.

Consider now the hypersurface $G = \varphi(X) \subset (\mathbb{C}^n, 0) \times \mathbb{C}$ which is the closure of set

$$\{(x, s) \in U \times \mathbb{C} : g_0(x) + s.g_1(x) + \ldots + s^{k-1}.g_{k-1}(x) + s^k = 0\}.$$

Let $\psi : (\mathbb{C}^n, 0) \times \mathbb{C} \to (\mathbb{C}^n, 0)$ be the usual projection and denote by $Z \subset (\mathbb{C}^n, 0)$ the analytic subset such that the restriction to G of ψ not is a finite branched covering. Notice that for all $x_0 \in (\mathbb{C}^n, 0) \backslash Z$, the equation

$$g_0(x) + s.g_1(x) + \ldots + s^{k-1}.g_{k-1}(x) + s^k = 0$$

defines k analytic hypersurfaces pairwise transverse in x_0 and therefore correspond to leaves of $\mathcal{W}$.

$\square$

4.4 Levi-flat hypersurfaces and webs

Let M be a germ at $0 \in \mathbb{C}^n$ of real analytic Levi-flat hypersurface. Denote by M_{reg}, the smooth part of M.

Definition 4.9. We say that M is tangent to $\mathcal{W}$ if any leaf of the Levi foliation $\mathcal{L}$ on M_{reg} is also a leaf of $\mathcal{W}$.

We will see that there exists germs of real analytic Levi-flat hypersurfaces which are not tangent to foliations, even in the case $n = 2$. For instance, the following example is tangent to a web.

Example 4.10. ([12]) Let $f_0, f_1, \ldots, f_k \in \mathcal{O}_n$, $n \geq 2$, be irreducible germs of holomorphic functions, where $k \geq 2$. Consider the family of hypersurfaces

$$G := \{G_s := f_0 + s.f_1 + \ldots + s^k f_k / s \in \mathbb{R}\}.$$

By eliminating the real variable s in the system $G_s = \bar{G}_s = 0$, we obtain a real analytic germ $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$ such that any complex hypersurface $(G_s = 0)$ is contained in the real hypersurface $(F = 0)$. For instance, in the case $k = 2$, we obtain

$$F = det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ \bar{f}_0 & \bar{f}_1 & \bar{f}_2 & 0 \\ 0 & \bar{f}_0 & \bar{f}_1 & \bar{f}_2 \end{pmatrix} =$$

$$= f_0^2.\bar{f}_2^2 + \bar{f}_0^2.f_2^2 + f_0.f_2.\bar{f}_1^2 + \bar{f}_0.\bar{f}_2.f_1^2 + |f_1|^2(f_0.\bar{f}_2 + \bar{f}_0.f_2) - 2|f_0|^2.|f_2|^2. \quad (4.2)$$

which comes from the elimination of s in the system

$$f_0 + s.f_1 + s^2.f_2 = \bar{f}_0 + s.\bar{f}_1 + s^2.\bar{f}_2 = 0.$$

We would like to observe that the examples of this type are tangent to singular webs. The web is obtained by the elimination of s in the system given by

$$\begin{cases} f_0 + s.f_1 + s^2.f_2 + \ldots + s^k.f_k = 0 \\ df_0 + s.df_1 + s^2.df_2 \ldots + s^k.df_k = 0 \end{cases}$$

In the case we get a 2-web given by the implicit differential equation $\Omega = 0$, where

$$\Omega = det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ df_0 & df_1 & df_2 & 0 \\ 0 & df_0 & df_1 & df_2 \end{pmatrix}$$

This example shows that, although $\mathcal{L}$ is a foliation on $M_{reg} \subset M = (F = 0)$, in general it is not tangent to a germ of holomorphic foliation at $(\mathbb{C}^n, 0)$. In fact, M. Brunella [8] in has proved that in the general situation a germ of real analytic Levi-flat hypersurface is "almost" like that. He proves that there exist a complex manifold Y together with a codimension one divisor D, a real analytic Levi-flat hypersurface $N \subset Y$, an open subset $N_0 \subset N$, a codimension one singular foliation $\mathcal{F}$ in Y tangent to N and a holomorphic map $\pi : (Y, D) \to (\mathbb{C}^n, 0)$ such that

(a). $\pi|_{N_0} : N_0 \to M_{reg}$ is an isomorphism.

(b). $\pi|_{\overline{N_0}} : \overline{N_0} \to \overline{M_{reg}}$ is a proper map.

In particular, the Levi foliation $\mathcal{L}$ on M_{reg} satisfies $\pi^*(\mathcal{L}) = \mathcal{F}|_{N_0}$, but in general there is no germ of foliation $\mathcal{G}$ at $0 \in \mathbb{C}^n$ such that $\pi^*(\mathcal{G}) = \mathcal{F}$, whereas sometimes there are webs as the example above.

Example 4.11. [Clairaut's equations] The Clairaut's equations are tangent to Levi-flat hypersurfaces. Consider the first-order implicit differential equation:

$$y = xp + f(p), \tag{4.3}$$

where $(x, y) \in \mathbb{C}^2$, $p = \frac{dy}{dx}$ and $f \in \mathbb{C}[p]$ is a polynomial of degree k. The equation (4.3) defines a k-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$. Let $S = F^{-1}(0)$, where $F(x, y, p) = y - xp - f(p)$.

Let $\alpha = dy - pdx$ be the contact 1-form and $\mathcal{F}_S$ the foliation defined by $\alpha|_S = 0$, in the chart (x, p) of S, we get $\alpha|_S = (x + f'(p))dp$.

The criminant set of $\mathcal{W}$ is given by

$$R = (y - xp - f(p) = x + f'(p) = 0),$$

note that $\mathcal{F}_S$ is tangent to S along R.

On the other hand, $\mathcal{F}_S$ has a non-constant first integral $g(x, p) = p$. Let $\pi_S : S \to (\mathbb{C}^2, 0)$ be the restriction to S of the usual projection $\pi : \mathbb{P} \to (\mathbb{C}^2, 0)$. The leaves of $\mathcal{F}_S$ project by π_S in leaves of $\mathcal{W}$. Those leaves are as follows:

$$-y + s.x + f(s) = 0, \tag{4.4}$$

where $s \in \mathbb{C}$. By the elimation of the variable s in the system:

$$\begin{cases} -y + s.x + f(s) = 0 \\ -\bar{y} + s.\bar{x} + \overline{f(s)} = 0, \end{cases}$$

we obtain a Levi-flat hypersurface tangent to $\mathcal{W}$. In particular, the Clairaut's equation has a holomorphic first integral.

The following **Problem** was proposed by Cerveau-Lins Neto in [12].

"Let M be a real analytic germ of a Levi-flat hypersurface at $0 \in \mathbb{C}^n$. Assume that there exists a singular codimension one k-web $\mathcal{W}$, such that $\mathcal{W}$ is tangent to M. Does the web has a non-constant meromorphic first integral as in example 4.10.?"

We are unable to prove the above problem in full generality. More precisely, we will prove the following.

Theorem 3. *Let $\mathcal{W}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$ of k-web tangent to a germ at $0 \in \mathbb{C}^n$ of an irreducible real-analytic Levi-flat hypersurface M. Assume that $\mathcal{W}$ is irreducible and has a finite number of invariant analytic leaves through the origin. Denote by X the variety associated to $\mathcal{W}$.*

- *(a). If $n = 2$. Then $\mathcal{W}$ has a non-constant holomorphic first integral*
- *(b). If $n \geq 3$, and $cod_{X_{reg}}(sing(X)) \geq 2$. Then $\mathcal{W}$ has a non-constant holomorphic first integral*

Remark 4.12. The condition of finiteness of the number of analytic leaves through $0 \in \mathbb{C}^n$ will be used only on M. Since the leaves of $\mathcal{L}$ are analytic (see Lemma 1.10), this condition is equivalent to say that $\mathcal{W}|_M$ is non-dicritical, (in the sense of foliations).

Observe that for $n = 2$ and $k = 1$, we obtain Theorem 1 of [12] in the non-dicritical case.

Remark 4.13. When we consider $\mathcal{W}$ a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of a smooth k-web tangent to a germ at $0 \in \mathbb{C}^n$ of irreducible real codimension one submanifold M; i.e, $\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$ is a generic superposition of k germs at $0 \in \mathbb{C}^n$ of regular foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$. The irreducibility and tangency conditions to M implies that there exists an unique $i \in \{1, \ldots, k\}$ such that $\mathcal{F}_i$ is tangent to M. Therefore we can find a coordinates system $z_1, \ldots, z_n$ of $\mathbb{C}^n$ such that $\mathcal{F}_i$ is defined by $dz_n = 0$ and $M = (\mathcal{R}e(z_n) = 0)$.

4.4.1 Lifting of Levi-flat hypersurfaces to the cotangent bundle

In this section we give some remarks about the lifting of a Levi-flat hypersurface to the cotangent bundle of $\mathbb{C}^n$.

Let $\mathbb{P}$ be as before, the projectivised cotangent bundle of $(\mathbb{C}^n, 0)$ and M an irreducible real analytic Levi-flat at $(\mathbb{C}^n, 0)$, $n \geq 2$. Note that $\mathbb{P}$ is a $\mathbb{P}^{n-1}$-bundle over $(\mathbb{C}^n, 0)$, whose fibre $\mathbb{P}T^*_z\mathbb{C}^n$ over $z \in \mathbb{C}^n$ will be thought as the set of complex hyperplanes in $T^*_z\mathbb{C}^n$. Let $\pi : \mathbb{P} \to (\mathbb{C}^n, 0)$ be the usual projection.

The regular part M_{reg} of M can be lifted to $\mathbb{P}$: just take, for every $z \in M_{reg}$, the complex hyperplane

$$T^{\mathbb{C}}_z M_{reg} = T_z M_{reg} \cap i(T_z M_{reg}) \subset T_z\mathbb{C}^n. \tag{4.5}$$

We call

$$M'_{reg} \subset \mathbb{P} \tag{4.6}$$

this lifting of M_{reg}. We remark that it is no more a hypersurface: its (real) dimension $2n-1$ is half of the real dimension of $\mathbb{P}T^*\mathbb{C}^n$. However, it is still "Levi-flat", in a sense which will be precised below.

Take now a point y in the closure $\overline{M'_{reg}}$ projecting on $\mathbb{C}^n$ to a point $x \in \overline{M}$. The following lemma was proved by Brunella [8].

Lemma 4.14. *There exist, in a germ of neighbourhood $U_y \subset \mathbb{P}T^*\mathbb{C}^n$ of y, a germ of real analytic subset N_y of dimension $2n-1$ containing $M'_{reg} \cap U_y$.*

We will use the result of [8].

Proposition 4.15. *In the above situation, there exists, in a germ of neighbourhood $V_y \subset U_y$ of y, a germ of complex analytic subset Y_y of (complex) dimension n containing $N_y \cap V_y$.*

4.5 Proof of Theorem 3

The proof will be divided into two parts. First, we give the proof for $n = 2$. The proof in dimension $n \geq 3$ will be done by reduction to the case of dimension two.

4.5.1 Planar webs

Consider $n = 2$. A k-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$ can be given in coordinates $(x, y) \in \mathbb{C}^2$ by

$$\omega = a_0(x,y)(dy)^k + a_1(x,y)(dy)^{k-1}.(dx) + \ldots + a_k(x,y)(dx)^k = 0,$$

where the coefficients $a_j \in \mathcal{M}_2$, $j = 1, \ldots, k$.

We set

$$U = \{(x, y, [adx + bdy]) \in \mathbb{P}T^*(\mathbb{C}^2, 0) : a \neq 0\}$$

and

$$V = \{(x, y, [adx + bdy]) \in \mathbb{P}T^*(\mathbb{C}^2, 0) : b \neq 0\}.$$

Observe that $\mathbb{P}T^*(\mathbb{C}^2, 0) = U \cup V$.

- Let S be the surface associated to $\mathcal{W}$. In the coordinates $(x, y, p) \in U$, where $p = \frac{dy}{dx}$, we get

  $$S = \{(x, y, p) \in \mathbb{P}T^*(\mathbb{C}^2, 0) : F(x, y, p) = 0\},$$

  where $F(x, y, p) = a_0(x, y)p^k + a_1(x, y)p^{k-1} + \ldots + a_k(x, y)$. Note that S is possibly singular at 0.

- Let $\mathcal{F}_S$ be the foliation associated to $\mathcal{W}$. In the coordinates $(x, y, p) \in U$, $\mathcal{F}_S$ is defined by $\alpha|_S = 0$, where $\alpha = dy - pdx$.

- In the coordinates $(x, y, p) \in U$, the criminant set R is defined by the equations $F(x, y, p) = F_p(x, y, p) = 0$.

In V the coordinate system is $(x, y, q) \in \mathbb{C}^3$, where $q = \frac{1}{p}$, the equations are similar.

4.5.2 Proof in dimension two

Let $\mathcal{W}$ be a k-web tangent to M Levi-flat. Assume that $\mathcal{W}$ satisfies the hypothesis of theorem 3 (see pg. 52) . Let S be as before, and $\pi : \mathbb{P}T^*\mathbb{C}^2 \to \mathbb{C}^2$ the usual projection. The idea is to use lemma 4.8. We will be assume that $\mathcal{W}$ is defined by

$$\omega = a_0(x,y)(dy)^k + a_1(x,y)(dy)^{k-1}.dx + \ldots + a_k(x,y)(dx)^k = 0, \tag{4.7}$$

where the coefficients $a_j \in \mathcal{M}_2$, $j = 1, \ldots, k$.

Lemma 4.16. *In the hypothesis of theorem 3, the surface S is irreducible and $S \cap \pi^{-1}(0)$ contains just a number finite of points.*

Proof. Since $\mathcal{W}$ is irreducible so is S. On the other hand, $S \cap \pi^{-1}(0)$ is finite because $\mathcal{W}$ is non-dicritical. $\square$

We can assume without lost of generality that $S \cap \pi^{-1}(0)$ contains just one point, in case general, the idea of the proof is the same. In this situation, we can suppose that $a_0(0,0) = 1$ in (4.7). Then in the coordinate system $(x, y, p) \in \mathbb{C}^3$, where $p = \frac{dy}{dx}$, we have $\pi^{-1}(0) \cap S = \{p_0 = (0,0,0)\}$, which implies that S is singular at $p_0 \in \mathbb{P}T^*(\mathbb{C}^2, 0)$. In particular, S is defined by $F^{-1}(0)$, where

$$F(x, y, p) = p^k + a_1(x, y)p^{k-1} + \ldots + a_k(x, y),$$

and $a_1, \ldots, a_k \in \mathcal{M}_2$. Let $\mathcal{F}_S$ be the foliation defined by $\alpha|_S = 0$. The hypothesis implies that $\mathcal{F}_S$ is a non-dicritical foliation with an isolated singularity at p_0.

Let M'_{reg} be the lifting of M_{reg} by π_S, and denote by $\sigma : (\tilde{S}, D) \to (S, p_0)$ the resolution of singularities of S at p_0. Let $\tilde{\mathcal{F}} = \sigma^*(\mathcal{F}_S)$ be the pull-back of $\mathcal{F}_S$ under σ.

Lemma 4.17. *In the above situation. The foliation $\tilde{\mathcal{F}}$ has only singularities of saddle with first integral type in D.*

Proof. Let $y \in \overline{M'_{reg}}$, it follows from lemma 4.14 there exist, in a neighbourhood $U_y \subset \mathbb{P}T^*\mathbb{C}^2$ of y, a real analytic subset N_y of dimension 3 containing $M'_{reg} \cap U_y$. Then by proposition 4.15, there exists, in a neighbourhood $V_y \subset U_y$ of y, a complex analytic subset Y_y of (complex) dimension 2 containing $N_y \cap V_y$. As germs at y, we get $Y_y = S_y$ then $N_y \cap V_y \subset S_y$, we have that $N_y \cap V_y$ is a real analytic hypersurface in S_y, and it is Levi-flat because each irreducible component contains a Levi-flat piece (cf. [7], Lemma 2.2).

Let us denote $M'_y = N_y \cap V_y$. The hypothesis implies that $\mathcal{F}_S$ is tangent to M'_y. These local constructions are sufficiently canonical to be patched together, when y varies on $\overline{M'_{reg}}$: if $S_{y_1} \subset V_{y_1}$ and $S_{y_2} \subset V_{y_2}$ are as above, with $M'_{reg} \cap V_{y_1} \cap V_{y_2} \neq \emptyset$, then $S_{y_2} \cap (V_{y_1} \cap V_{y_2})$ and $S_{y_1} \cap (V_{y_1} \cap V_{y_2})$ have some common irreducible components containing $M'_{reg} \cap V_{y_1} \cap V_{y_2}$, so that M'_{y_1}, M'_{y_2} can be glued by identifying those components. In this way, we obtain a Levi-flat hypersurface N on S tangent to $\mathcal{F}_S$.

Since $\mathcal{F}_S$ is non-dicritical, all irreducible components of D are $\tilde{\mathcal{F}}$-invariants. Let $\tilde{N}$ be the strict transform of N under σ, then $\tilde{N} \supset D$. In particular, $\tilde{N}$ contains all singularities of $\tilde{\tilde{\mathcal{F}}}$ in D. It follows from lemma 1.14 (see Chapter 1) that all singularities of $\tilde{\mathcal{F}}$ are saddle with first integral.

$\square$

End of the proof of theorem in dimension two. The idea is to prove that $\mathcal{F}_S$ has a holomorphic first integral. Since D is $\tilde{\mathcal{F}}$-invariant, we have $S := D \backslash sing(\tilde{\mathcal{F}})$ is a leaf of $\tilde{\mathcal{F}}$. Now, fix $p \in S$ and a transverse section $\sum$ through p. By lemma 4.17, the singularities of $\tilde{\mathcal{F}}$ in D are saddle with first integral types. Therefore the transverse section $\sum$ is complete. (See theorem 1.15). Let $G \subset Diff(\sum, p)$ be the holonomy group of the leaf S of $\tilde{\mathcal{F}}$. It follows from lemma 1.10 that all leaves of $\mathcal{F}_S$ through points of N_{reg} are closed in N_{reg}. This implies that G is a finite group by the same arguments of the proof of theorem 1. By corollary 1.16, $\mathcal{F}_S$ has a non-constant holomorphic first integral. Finally from Lemma 4.8, $\mathcal{W}$ has a first integral as follows:

$$f_0(x,y) + z.f_1(x,y) + \ldots + z^{k-1}.f_{k-1}(x,y) + z^k,$$

where $f_0, f_1, \ldots, f_{k-1} \in \mathcal{O}_2$.

4.5.3 Proof in the dimension $n \geq 3$

Let us give an idea of the proof. First of all, we will prove that there is a holomorphic embedding $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ with the following properties:

(i). $i^{-1}(M)$ has real codimension one on $(\mathbb{C}^2, 0)$.

(ii). $i^*(\mathcal{W})$ is a k-web on $(\mathbb{C}^2, 0)$ and $i^*(\mathcal{W})$ is tangent to $i^{-1}(M)$.

Set $E := i(\mathbb{C}^2, 0)$. The above conditions and theorem 3 in dimension two imply that $\mathcal{W}|_E$ has a non-constant holomorphic first integral, say $g = f_0 + z.f_1 + \ldots + z^{k-1}.f_{k-1} + z^k$, where $f_0, \ldots, f_{k-1} \in \mathcal{M}_2$. After that we will use a lemma to prove that g can be extended to a holomorphic germ g_1, which is a first integral of $\mathcal{W}$.

On the other hand, let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of a holomorphic codimension one foliation, tangent to a real analytic hypersurface M. Let us suppose that $\mathcal{F}$ is defined by $\omega = 0$, where ω is a germ at $0 \in \mathbb{C}^n$ of an integrable holomorphic 1-form with $cod_{\mathbb{C}^n}(sing(\omega)) \geq 2$. We say that a holomorphic embedding $i : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ is transverse to ω if $cod_{\mathbb{C}^n}(sing(\omega)) = 2$, which means in fact that, as a germ of set, we have $sing(i^*(\omega)) = \{0\}$. Note that the concept is independent of the particular germ of holomorphic 1-form which represents $\mathcal{F}$. Therefore, we will say that the embedding i is transverse to $\mathcal{F}$ if it is transverse to some holomorphic 1-form ω representing $\mathcal{F}$.

The following lemma is proved in [12].

Lemma 4.18. *In the above situation, there exists a 2-plane $E \subset \mathbb{C}^n$, transverse to $\mathcal{F}$, such that the germ at $0 \in E$ of $M \cap E$ has real codimension one.*

We say that a embedding i is transverse to $\mathcal{W}$ if it is transverse to all k-foliations which defines $\mathcal{W}$. Now, one deduces the following:

Lemma 4.19. *There exists a 2-plane $E \subset \mathbb{C}^n$, transverse to $\mathcal{W}$, such that the germ at $0 \in E$ of $M \cap E$ has real codimension one.*

Proof. First of all, note that outside of the singular part of $\mathcal{W}$, we can suppose that $\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$, where $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are germs of codimension one regular foliations. Since $\mathcal{W}$ is tangent to M, there is a foliation $\mathcal{F}_j$ such that is tangent to a Levi foliation $\mathcal{L}$ on M_{reg}. Lemma 4.18 implies that we can find a 2-plane E_0 tranverse to M and to $\mathcal{F}_j$. Clearly the set of linear mappings tranverse to $\mathcal{F}_1, \ldots, \mathcal{F}_k$ simultaneously is open and dense in the set of linear mappings from $\mathbb{C}^2$ to $\mathbb{C}^n$, by transversality theory, there exists a linear embedding i such that $E = i(\mathbb{C}^2, 0)$ is transverse to M_{reg} and to $\mathcal{W}$ simultaneously. $\square$

Let E be a 2-plane as in lemma 4.19. It easy to check that $\mathcal{W}|_E$ satisfies the hypothesis of theorem 3. By the two dimensional case $\mathcal{W}|_E$ has a non-constant first integral:

$$g_0 + z.g_1 + \ldots + z^{k-1}.g_{k-1} + z^k, \tag{4.8}$$

where $g_0, \ldots, g_{k-1} \in \mathcal{O}_2$.

Let X be the contact variety associated to $\mathcal{W}$ and set S be the contact surface associated to $\mathcal{W}|_E$. Observe that $\mathcal{F}_S$ has a non-constant holomorphic first integral g defined on S.

Lemma 4.20. *In the above situation, we get $\mathcal{F}_X|_S = \mathcal{F}_S$ and $\mathcal{F}_X$ has a non-constant holomorphic first integral g_1 on X, such that $g_1|_S = g$.*

Proof. It is easily seen that $S \subset X$ which implies that $\mathcal{F}_X|_S = \mathcal{F}_S$. Let us extend g to X. Fix $p \in X_{reg} \backslash sing(\mathcal{F}_X)$. It is possible to find a small neighborhood $W_p \subset X$ of p and a holomorphic coordinate chart $\varphi : W_p \to \triangle$, where $\triangle \subset \mathbb{C}^n$ is a polydisc, such that:

(i). $\varphi(S \cap W_p) = \{z_3 = \ldots = z_n = 0\} \cap \triangle$.

(ii). $\varphi_*(\mathcal{F}_X)$ is given by $dz_n|_\triangle = 0$.

Let $\pi_n : \mathbb{C}^n \to \mathbb{C}^2$ be the projection defined by $\pi_n(z_1, \ldots, z_n) = (z_1, z_2)$ and set $\tilde{g}_p := g \circ \varphi^{-1} \circ \pi_n|_\triangle$. We obtain that $\tilde{g}$ is a holomorphic function defined in $\triangle$ and is a first integral of $\varphi_*(\mathcal{F}_X)$. Let $g_p = \tilde{g}_p \circ \varphi$. Notice that, if $W_p \cap W_q \neq \emptyset$, p and q being regular points for $\mathcal{F}_X$, then we have $g_p|_{W_p \cap W_q} = g_q|_{W_p \cap W_q}$. This follows easily form the identity principle for holomorphic functions. In particular, g can be extended to

$$W = \bigcup_{p \in X_{reg} \backslash sing(\mathcal{F}_X)} W_p,$$

which is a neighborhood of $X_{reg} \backslash sing(\mathcal{F}_X)$. Call g_W this extension.

Since $cod_{X_{reg}} sing(\mathcal{F}_X) \geq 2$, by a theorem of Levi (cf. [24]), g_W can be extended to X_{reg}, as $cod_{X_{reg}}(sing(X)) \geq 2$ this allows us to extend g_W to g_1 as holomorphic first integral for $\mathcal{F}_X$, in whole X.

$\square$

End of the proof of theorem in dimension $n \geq 3$. Since $\mathcal{F}_X$ has a non-constant holomorphic first integral on X, lemma 4.8 implies that $\mathcal{W}$ has a non-constant holomorphic first integral. This finishes the proof of theorem 3.

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