On the topological invariance of the algebraic multiplicity of a holomorphic vector field.

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Resumo

Nesta tese estudamos a equivalência topológica entre germes de campos de vetores holomorfos com uma singularidade isolada na origem de $\mathbb{C}^n$ $(n \geq 2)$. Assim, se a equivalência topológica é de classe C^1, provamos que a multiplicidade algébrica do campo (na origem) é invariante. Alem disso, se considerarmos somente campos de vetores em $\mathbb{C}^2$, é suficiente supor que a equivalência topológica é diferenciável na origem de $\mathbb{C}^2$.

Palavras chaves: vector field, holomorphic, multiplicity.

Preface

Given a curve $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, singular at $0 \in \mathbb{C}^2$, we define its *algebraic multiplicity* as the degree of the first nonzero jet of f, that is, $\nu(f) = \nu$ where

$$f = f_\nu + f_{\nu+1} + \cdots$$

is the Taylor development of f and $f_\nu \neq 0$. A well known result by Burau [5] and Zariski [6] states that ν is a *topological invariant*, that is, given $\widetilde{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 2)$ and a homeomorphism $\mathrm{h} : U \to \widetilde{U}$ between neighborhoods of $0 \in \mathbb{C}^2$ such that $\mathrm{h}(f^{-1}(0) \cap U) = \widetilde{f}^{-1}(0) \cap V$ then $\nu(f) = \nu(\widetilde{f})$. Consider now a holomorphic vector field Z in $\mathbb{C}^2$ with a singularity at $0 \in \mathbb{C}^2$. If

$$Z = Z_\nu + Z_{\nu+1} + \cdots$$

we define $\nu = \nu(Z)$ as the *algebraic multiplicity* of Z. A natural question, posed by J.F.Mattei is: is $\nu(Z)$ a topological invariant of $\mathcal{F}_Z$?. In [2], the authors give a positive answer if $\mathcal{F}_Z$ is a *generalized curve*, that is, if the desingularization of Z does not contain complex saddle-nodes. In this work, we impose conditions on the topological equivalence $\mathrm{h} : U \to \widetilde{U}$. Thus, in Part II we prove the following:
Theorem. *Let* $\mathrm{h} : U \to \widetilde{U}$ *be a topological equivalence between* $\mathcal{F}_Z$ *and* $\mathcal{F}_{\widetilde{Z}}$ *and assume that* h *preserves the orientation of* $\mathbb{C}^2$. *Suppose that* h *is differentiable at* $0 \in \mathbb{C}^2$ *and such that* $d\,\mathrm{h}(0) : \mathbb{R}^4 \to \mathbb{R}^4$ *is a real isomorphism. Then the algebraic multiplicities of* Z *and* $\widetilde{Z}$ *are the same.*

In Part I, we consider holomorphic vector fields Z and $\widetilde{Z}$ in $\mathbb{C}^n\,(n \geq 2)$ with isolated singularity at $0 \in \mathbb{C}^n$ and prove the following:
Theorem. *Let* U *and* $\widetilde{U}$ *be neighborhoods of* $0 \in \mathbb{C}^n$ *and let* $\mathrm{h} : U \to \widetilde{U}$ *be a* C^1 *equivalence between* $\mathcal{F}_Z$ *and* $\mathcal{F}_{\widetilde{Z}}$, *that is, a* C^1 *diffeomorphism taking leaves of* $\mathcal{F}_Z$ *to leaves of* $\mathcal{F}_{\widetilde{Z}}$. *Then the algebraic multiplicities of* Z *and* $\widetilde{Z}$ *are equal.*
The two parts of this thesis are independent and can be read in any order.

Contents

Part I

The C^1 case

1 Introduction

Let $\mathcal{F}$ be a holomorphic foliation by curves of a neighborhood U of $0 \in \mathbb{C}^n$ with a unique singularity at $0 \in \mathbb{C}^n \, (n \geq 2)$. We assume that $\mathcal{F}$ is generated by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}, \; a_i \in \mathcal{O}_U, \; g.c.d.(a_1, a_2, \ldots, a_n) = 1.$$

The algebraic multiplicity of $\mathcal{F}$ (at $0 \in \mathbb{C}^n$) is the minimum vanishing order at $0 \in \mathbb{C}^n$ of the functions a_i. Let $\widetilde{\mathcal{F}}$ be another holomorphic foliation by curves of a neighborhood $\widetilde{U}$ of $0 \in \mathbb{C}^n$ and let $\mathrm{h} : U \to \widetilde{U}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, that is, a homeomorphism taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. A natural question, posed by J.F.Mattei is: are the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ the same?. In [2], the authors give a positive answer if $n = 2$ and $\mathcal{F}$ is a *generalized curve*, that is, if its desingularization does not contain complex saddle-nodes. In this work we give a sufficient condition on the topological equivalence $\mathrm{h} : U \to \widetilde{U}$ for the algebraic multiplicity to be invariant. Let $\pi : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ be the quadratic blow up with center at $0 \in \mathbb{C}^n$. Clearly the map $h := \pi^{-1} \, \mathrm{h} \, \pi$ is a homeomorphism between $\pi^{-1}(U \backslash \{0\})$ and $\pi^{-1}(\widetilde{U} \backslash \{0\})$. Then we prove the following:

Theorem 1.1. *Suppose that h extends to the divisor $\pi^{-1}(0)$ as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$. Then the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are the same.*

If h is a C^1 diffeomorphism, we prove that h extends to the divisor. Thus, we obtain that the algebraic multiplicity is invariant by C^1 equivalences:

Theorem 1.2. *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be two foliations by curves of neighborhoods U and $\widetilde{U}$ of $0 \in \mathbb{C}^n$, $n \geq 2$. Let $\mathrm{h} : U \to \widetilde{U}$ be a C^1 equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, that is, a C^1 diffeomorphism taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. Then the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are equal.*

It is known that there exists a unique way of extending the pull back foliation $\pi^*(\mathcal{F}|_{U \backslash \{0\}})$ to a singular analytic foliation $\mathcal{F}_0$ on $\pi^{-1}(U)$ with singular set of codimension ≥ 2. We say that $\mathcal{F}_0$ is the strict transform of $\mathcal{F}$ by π. Let $\widetilde{\mathcal{F}}_0$ be the strict transform of $\widetilde{\mathcal{F}}$ by π. In order to prove Theorem 1.1 we show that the algebraic multiplicity of $\mathcal{F}$ depends on the Chern class of the tangent bundle of $\mathcal{F}_0$. To relate the Chern classes of the tangent bundles of $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ we use the following theorem (see [1]).

Theorem 1.3. *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be foliations by curves on the complex manifolds M and $\widetilde{M}$ respectively. Let $c(T\mathcal{F})$ denote the Chern class of the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$. Let $h : M \to \widetilde{M}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ and consider the map $h^* : H^2(M, \mathbb{Z}) \to H^2(\widetilde{M}, \mathbb{Z})$ induced in the cohomology. Then $h^*(c(T\mathcal{F})) = c(T\widetilde{\mathcal{F}})$.*

Clearly the homeomorphism $h : \pi^{-1}(U \backslash \{0\}) \to \pi^{-1}(\widetilde{U} \backslash \{0\})$ is a topological equivalence between $\mathcal{F}_0|_{\pi^{-1}(U \backslash \{0\})}$ and $\widetilde{\mathcal{F}}_0|_{\pi^{-1}(\widetilde{U} \backslash \{0\})}$. To be able to apply Theorem 1.3 we show that h extends as a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. This is the non trivial part of the proof. Thus, we prove the following.

Theorem 1.4. *Let* V *and* $\widetilde{V}$ *be complex manifolds, let* $Y \subset V$ *and* $\widetilde{Y} \subset \widetilde{V}$ *be analytic subvarieties of codimension* ≥ 1 *and, let* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *be holomorphic foliations by curves on* V *and* $\widetilde{V}$ *respectively. Suppose there is a homeomorphism* h *between* V *and* $\widetilde{V}$ *with* $h(Y) = \widetilde{Y}$ *and such that* $h|_{V\setminus Y}$ *is a topological equivalence between* $\mathcal{F}|_{V\setminus Y}$ *and* $\widetilde{\mathcal{F}}|_{\widetilde{V}\setminus\widetilde{Y}}$*. Then* h *is a topological equivalence between* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$*.*

This part is organized as follows. In section 2, we state and prove a lemma used in section 3 to prove Theorem 1.4. In section 4 we prove Theorem 1.1. Finally, section 5 discusses the case C^1.

2 A fundamental lemma

Let $\mathbb{D} = \{z \in \mathbb{C} : ||z|| < 1\}$ and $\mathbb{B} = \{z \in \mathbb{C}^{n-1} : ||z|| < 1\}$ where $n \geq 2$. Let M be a complex manifold of complex dimension n and let D be a subset of M homeomorphic to a disc. We say that D is a *singular disc* if for all $x \in D$ there exist a neighborhood $\mathcal{D}$ of x in D, and an injective holomorphic function $f : \mathbb{D} \to M$ such that $f(\mathbb{D}) = \mathcal{D}$. If $f'(0) = 0$ we say that x is a *singularity* of D, otherwise x is a *regular point* of D (this does not depend on f). The set S of singularities of D is discrete and closed in D and we have that $D\setminus S$ is a complex submanifold of M. Thus, if x is a regular point of D, there is a neighborhood U of x in M and holomorphic coordinates (w, z), $w \in \mathbb{B}$, $z \in \mathbb{D}$ on U such that $D \cap U$ is represented by $(w = 0)$. If D does not have singularities we say that it is a *regular disc*. In this case, by uniformization, there is a holomorphic map $f : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that f is a biholomorphism between E and D. *Example.* Let $\mathcal{F}$ be a holomorphic foliation by curves on the complex manifold M and let $D \subset M$ be homeomorphic to a disc. If D is contained in a leaf of $\mathcal{F}$ then it is a regular disc.

Lemma 2.1. *Let* $F : \mathbb{D}\times[0, 1] \to \mathbb{C}^n$ *be a continuous map such that for all* $t \in [0, 1]$, *the map* $F(*, t) : \mathbb{D} \to \mathbb{C}^n$ *is a homeomorphism onto its image. Thus, we have a continuous family of discs* $D_t := F(\mathbb{D} \times \{t\})$*. Suppose* D_t *is a regular disc for each* $t > 0$*. Then* D_0 *is a singular disc.*

Remark. Actually, we may only assume that $\mathcal{D}_t$ is a singular disc for all $t > 0$.
We now state the lemmas used in the proof of Lemma 2.1.

Lemma 2.2. *Let* U *be a simply connected domain in the complex plane such that* $\partial\,\mathrm{U}$ *is a Jordan curve. Then any uniformization* $f : \mathbb{D} \to \mathrm{U}$ *extends as a homeomorphism between* $\overline{\mathbb{D}}$ *and* $\overline{\mathrm{U}}$*.*

Lemma 2.3. *For each* $k \in \mathbb{N}$, *let* $\phi_k : \mathbb{S}^1 \to \mathbb{S}^1$ *be a homeomorphism. Then* $\{\phi_k\}$ *has a subsequence which converges almost everywhere with respect to the Lebesgue measure in the circle.*

Lemma 2.4. *Let* $\phi : \mathbb{S}^1 \to \mathbb{C}$ *be a bounded measurable function. Suppose that* $\int_{\mathbb{S}^1} z^n\phi(z)dz = 0$ *for all* $n \in \mathbb{Z}$*. Then, with respect to the Lebesgue measure in the circle,* ϕ *vanishes almost everywhere.*

Lemma 2.5. *Let $\mathcal{D} \subset \mathbb{C}^n$ be a bounded set homeomorphic to a disc. Let $p \in \mathcal{D}$ and suppose that $\mathcal{D}\backslash\{p\}$ has a complex structure such that the inclusion $\mathcal{D}\backslash\{p\} \to \mathbb{C}^n$ is holomorphic. Then there is a holomorphic injective map $g : \mathbb{D} \to \mathbb{C}^n$ with $g(\mathbb{D}) = \mathcal{D}$, $g(0) = p$.*

Proof of Lemma 2.1. Let $p = F(x_0, 0)$ be any point in D_0. Let $\mathrm{U} \subset \mathbb{D}$ be a disc centered at x_0 and such that $\overline{\mathrm{U}} \subset \mathbb{D}$. Let $t_k > 0$ be such that $t_k \to 0$ as $k \to \infty$ an define $\mathcal{D}_k = F(\mathrm{U} \times \{t_k\})$. Clearly $\overline{\mathcal{D}}_k \subset \overline{D}_{t_k}$. By uniformization, D_{t_k} is equivalent to a subset of $\mathbb{C}$ and, since $\mathcal{D}_k$ is a proper subset of D_{t_k}, we have (again by uniformization) that $\mathcal{D}_k$ is holomorphically equivalent to the unitary disc $\mathbb{D}$. Then there is a holomorphic map $f_k : \mathbb{D} \to \mathbb{C}^n$ which is a biholomorphism between $\mathbb{D}$ and $\mathcal{D}_k$. If we think that D_{t_k} is a subset of $\mathbb{C}$, by applying Lemma 2.2, we conclude that f_k extends as a homeomorphism $f_k : \overline{\mathbb{D}} \to \overline{\mathcal{D}}_k$. We may assume that $f_k(0) = F(x_0, t_k)$ for all k; otherwise we compose f_k with a suitable Moebius transformation. Observe that $f_k(\mathbb{D})$ is contained in the compact set $F(\overline{\mathrm{U}} \times [0, 1])$, hence $\{f_k\}$ is uniformly bounded and, by Montel's theorem, we can assume that f_k converges uniformly on compact sets to a holomorphic function $f : \mathbb{D} \to \mathbb{C}^n$. Note that $f(0) = p$, since

$$f(0) = \lim_{k\to\infty} f_k(0) = \lim_{k\to\infty} F(x_0, t_k) = F(x_0, 0) = p.$$

Let $\mathbb{S}^1 = \partial\mathbb{D}$ and consider for each k the homeomorphism

$$\varphi_k := f_k|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial\mathcal{D}_k.$$

Define

$$\pi : F(\partial\,\mathrm{U} \times [0, 1]) \to \mathbb{S}^1$$

$$\pi(F(\zeta, t)) = \zeta.$$

Clearly, π maps $\partial\mathcal{D}_k = F(\partial\,\mathrm{U} \times \{t_k\})$ homeomorphically onto $\mathbb{S}^1$. Therefore for each k, the map

$$\phi_k := \pi \circ \varphi_k : \mathbb{S}^1 \to \mathbb{S}^1$$

is a homeomorphism. By taking a subsequence, we may assume that ϕ_k converges a.e. to a function $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ (Lemma 2.3). Therefore φ_k converges a.e. to

$$\varphi := \pi^{-1} \circ \phi : \mathbb{S}^1 \to \partial\mathcal{D}_0.$$

Fix $x \in \mathbb{D}$. Since $\{\varphi_k\}$ is uniformly bounded, by the dominated convergence theorem we have that

$$\frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi_k(w)}{w - x}dw \to \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi(w)}{w - x}dw \tag{1}$$

as $k \to \infty$. By the Cauchy's integral formula, the left part of (1) is equal to $f_k(x)$ and, since $f_k(x) \to f(x)$, we conclude that

$$f(x) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\varphi(w)}{w - x}dw. \tag{2}$$

Assertion 1. *$f : \mathbb{D} \to \mathbb{C}^n$ is not constant.*

Proof. Assume by contradiction that f is a constant function. Then $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, where $f^{(n)}$ is the nth derivative of f. From (2), by induction on n, it is not difficult to prove that

$$f^{(n)}(0) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi(w)}{w^{n+1}} dw = 0$$

for all $n \geq 1$. Hence

$$\int_{\mathbb{S}^1} w^n \varphi(w) dw = 0 \quad \text{for all} \quad n \leq -2. \tag{3}$$

On the other hand, for each k and any $n \geq 0$ we have that $\int_{\mathbb{S}^1} w^n \varphi_k(w) dw = 0$ because $w^n \varphi_k(w)$ extends holomorphically to $\mathbb{D}$ as $w^n f_k(w)$. Then, by the dominated convergence theorem we have that

$$\int_{\mathbb{S}^1} w^n \varphi(w) dw = \lim_{k \to \infty} \int_{\mathbb{S}^1} w^n \varphi_k(w) dw = 0 \quad \text{for all} \quad n \geq 0. \tag{4}$$

Thus, from (3) and (4):

$$\int_{\mathbb{S}^1} w^n \varphi(w) dw = 0 \quad \text{for all} \quad n \in \mathbb{Z} \backslash \{-1\}.$$

And easy computation shows that each coordinate of the function $\phi := \varphi - p$ satisfies the hypothesis of Lemma 2.4. Then $\phi = 0$ a.e. and therefore $\varphi = p$ a.e., which is a contradiction because $\varphi(\mathbb{S}^1) \subset \partial \mathcal{D}_0$ and $p \notin \partial \mathcal{D}_0$.

Assertion 2. $f(\mathbb{D}) \subset D_0$.

Proof. Let $z \in \mathbb{C}^n$ be such that $z = f(x)$. Then $z = \lim f_k(x)$. Since $f_k(x)$ is contained in $\mathcal{D}_k = F(\mathrm{U} \times \{t_k\})$, we have that $f_k(x) = F(x_k, t_k)$ with $x_k \in \mathrm{U}$. By taking a subsequence, we may assume that $x_k \to \bar{x} \in \overline{\mathrm{U}}$. Then

$$z = \lim_{k \to \infty} f_k(x) = \lim_{k \to \infty} F(x_k, t_k) = F(\lim_{k \to \infty} x_k, 0) = F(\bar{x}, 0) \in D_0.$$

Therefore $f(\mathbb{D}) \subset D_0$.

It follows from Assertion 1 that f' does not vanish. Then, we know that the zero set of f' is discrete and closed in $\mathbb{D}$. Hence, there exists a disc $\Omega \subset \mathbb{D}$ centered at 0 such that $f' \neq 0$ on $\Omega \backslash \{0\}$. Since f is not constant, 0 is an isolated point in $f^{-1}(p)$. Thus, we assume Ω to be small enough such that $\overline{\Omega} \cap f^{-1}(p) = \{0\}$. In particular, $f(\partial \Omega)$ does not pass through p.

Assertion 3. There is a disc $\mathcal{D} \subset D_0$ with $p \in \mathcal{D}$ and such that $\mathcal{D} \subset f(\Omega)$.

Proof. Let $x \in \Omega \backslash \{0\}$. Then $f'(x) \neq 0$ and there exists a disc $\Delta \subset \Omega$ with $x \in \Delta$ and such that $f|_{\overline{\Delta}} : \overline{\Delta} \to \mathbb{C}^n$ is injective, hence a homeomorphism onto its image, since $\overline{\Delta}$ is compact. Then $f(\Delta)$ is homeomorphic to a disc and, since $f(\Delta) \subset D_0$, we have that $f(\Delta)$ is open in D_0. Then $f(x)$ is an interior point of $f(\Omega)$ as a subset of D_0. It follows that every point $z \in f(\Omega \backslash \{0\})$ is an interior point of $f(\Omega) \subset D_0$. Thus if

z is a point in the boundary of $f(\Omega)$, since z is not an interior point, we have that $z \notin f(\partial\Omega \cup \{0\}) = f(\partial\Omega) \cup \{p\}$. Therefore:

$$\partial f(\Omega) \subset f(\partial\Omega) \cup \{p\}.$$

Since $f(\partial\Omega)$ does not pass through p, we may take a disc $\mathcal{D} \subset D_0$ containing p and such that $\mathcal{D}$ is disjoint of $f(\partial\Omega)$. Finally, we claim that $\mathcal{D} \subset f(\Omega)$. Let $z \in \mathcal{D}$ and suppose that $z \notin f(\Omega)$. Since $\mathcal{D}$ contains p, we may take $x \neq 0$, close enough to 0, such that $z' := f(x) \in \mathcal{D}$. We have $z' \neq p$ because $x \neq 0$, hence we may take a path γ in $\mathcal{D}\backslash\{p\}$ connecting z and z'. Since $z \notin f(\Omega)$ and $z' \in f(\Omega)$, there exists $z'' \in \gamma$ such that $z'' \in \partial f(\Omega)$. Then, since $\partial f(\Omega) \subset f(\partial\Omega) \cup \{p\}$, we have $z'' \in f(\partial\Omega) \cup \{p\}$. But this is a contradiction because $z'' \in \gamma$ is contained in $\mathcal{D}\backslash\{p\}$, which is disjoint of $f(\partial\Omega) \cup \{p\}$.

Let $z \in \mathcal{D}\backslash\{p\}$. By Assertion 3, $z = f(x)$ with $x \in \Omega$. Since $z \neq p$ we have $x \neq 0$, hence $f'(x) \neq 0$. Then, from the proof of Assertion 3, there exists a disc $\Delta \subset \Omega$, $x \in \Delta$, such that $f|_\Delta : \Delta \to \mathbb{C}^n$ is injective and $f(\Delta)$ is a neighborhood of z in $\mathcal{D}\backslash\{p\}$. Since f is holomorphic it follows that $\mathcal{D}\backslash\{p\}$ is a Riemann surface and the inclusion $\mathcal{D}\backslash\{p\} \to \mathbb{C}^n$ is a holomorphic map. Then, by Lemma 2.5, there is a holomorphic injective map $g : \mathbb{D} \to \mathbb{C}^n$ with $g(\mathbb{D}) = \mathcal{D} \subset D_0$. Since p was arbitrary, it follows that D_0 is a singular disc, which finishes the proof of Lemma 2.1. □

Proof of Lemma 2.2. See [7] p.310. □

Proof of Lemma 2.3. We give a sketch of the proof. By taking a subsequence we may assume that ϕ_k converges on a dense subset of $\mathbb{S}^1$. Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be a covering. For each k, we may choose a lifting $f_k : \mathbb{R} \to \mathbb{R}$ of ϕ_k by π. Since ϕ_k is a homeomorphism, f_k is monotone and we may assume that f_k is increasing for all k. We may also assume that f_k converges on a dense subset R of $\mathbb{R}$. For all $y \in R$, we define $f(y) = \lim f_k(y)$. Observe that f is increasing, since so is f_k for all k. We extend f to $\mathbb{R}$ as

$$f(x) = \limsup_{y \in R, y < x} f(y).$$

It is not difficult to see that $f : \mathbb{R} \to \mathbb{R}$ is increasing. Then f is continuous on a set A of total measure. Now, it is not difficult to prove that for all $x \in A$, the sequence $f_k(x)$ converges to $f(x)$ and the lemma follows. □

Proof of Lemma 2.4. We claim that $\int_{\mathbb{S}^1} f(z)\phi(z)dz = 0$ for all continuous function $f : \mathbb{S}^1 \to \mathbb{C}$. By the Stone -Weierstrass approximation theorem (see [9]), f can be uniformly approximated by a sequence of functions $P_k = A_k + iB_k$, $k \in \mathbb{N}$, where $A_k, B_k : \mathbb{R}^2 \to \mathbb{R}$ are real polynomials. Let $z = x + iy$ and observe that $\bar{z} = 1/z$ if $z \in \mathbb{S}^1$. Then

$$\begin{aligned} P_k(z) &= A_k(x,y) + iB_k(x,y) = A_k(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}) + iB_k(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}) \\ &= A_k(\frac{z+1/z}{2}, \frac{z-1/z}{2}) + iB_k(\frac{z+1/z}{2}, \frac{z-1/z}{2}). \end{aligned}$$

Hence $P_k(z) = \sum a_j z^j$ where j runs on the integers, the sum is finite and the coefficients a_j depend only on the coefficients of A_k and B_k. Then

$$\int_{\mathbb{S}^1} P_k(z)\phi(z)dz = \int_{\mathbb{S}^1} (\sum_j a_j z^j)\phi(z)dz = \sum_j a_j \int_{\mathbb{S}^1} z^j \phi(z)dz = 0,$$

by hypothesis. Since P_k converges uniformly to f and ϕ is bounded we have that

$$\int_{\mathbb{S}^1} P_k(z)\phi(z)dz \to \int_{\mathbb{S}^1} f(z)\phi(z)dz$$

as $k \to \infty$ and therefore $\int_{\mathbb{S}^1} f(z)\phi(z)dz = 0$.

Now, we take an uniformly bounded sequence of continuous functions $f_k : \mathbb{S}^1 \to \mathbb{C}$ which converges a.e. to $\bar{\phi}$. Then $\{f_k\phi\}$ is uniformly bounded and converges a.e. to $\bar{\phi}\phi = |\phi|^2$. Thus by the dominated convergence theorem we have that

$$\int_{\mathbb{S}^1} f_k(z)\phi(z)dz \to \int_{\mathbb{S}^1} |\phi(z)|^2 dz$$

as $k \to \infty$. Therefore $\int_{\mathbb{S}^1} |\phi(z)|^2 dz = 0$ and it follows that $\phi = 0$ almost everywhere. □

Proof of Lemma 2.5. Let A_r denote the annulus $\{z \in \mathbb{C},\ r < |z| < 1\}$ where $r \geq 0$. Since $\mathcal{D} \subset \mathbb{C}^n$ is bounded and $\mathcal{D}\backslash\{p\}$ is homeomorphic to an annulus we have (see [10]) that there exist a biholomorphism

$$g : A_r \to \mathcal{D}\backslash\{p\},$$

such that $g(z) \to p$ as $|z| \to r$. Take R with $r < R < 1$ and let Γ_r and Γ_R be denote the circles $|z| = r$ and $|z| = R$ respectively. For $r < |z| < R$ we have the formula

$$g(z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\Gamma_r} \frac{p}{w-z}dw,$$

since g extends continuously to Γ_r as $g|_{\Gamma_r} = p$. But the second integral is equal to zero because $p/(w-z)$ is holomorphic on the disc $|w| < r$, then

$$g(z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(w)}{w-z}dw.$$

Therefore g extends to the disc $|z| < R$. Since g is not constant and $g|_{\Gamma_r} = p$ we necessarily have $r = 0$ and the lemma follows. □

3 An extension theorem

This section is devoted to prove Theorem 1.4. We show first that Theorem 1.4 is a consequence of the following theorem.

Theorem 3.1. *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M. Let $X \subset M$ be an analytic subvariety of codimension ≥ 1. Suppose that:*

- *(i) $\mathcal{F}$ is generated by a holomorphic vector field.*
- *(ii) There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.*
- *(iii) If $D_z := h(\{z\} \times D)$ then for all z: either D_z is contained in X, or $D_z \cap X$ is discrete and $D_z\backslash X$ is contained in a leaf of $\mathcal{F}$.*

Then $\mathcal{F}$ is regular and the sets D_z are the leaves of $\mathcal{F}$.

Proof of Theorem 1.4. Let p be a point in Y which is regular for $\mathcal{F}$. Let Σ denote a ball in $\mathbb{C}^{n-1}$ and D a disc in $\mathbb{C}$. Consider a neighborhood W of p on which $\mathcal{F}$ is a product foliation, that is, $W \simeq \Sigma \times D$ and the sets $\{z\} \times D$ are the leaves of $\mathcal{F}|_W$. We take W small enough such that $\widetilde{\mathcal{F}}$ restricted to $M := h(W)$ is generated by a holomorphic vector field. Let X be the intersection between M and $\widetilde{Y}$. We will show that the hypothesis of Theorem 3.1 hold for $\widetilde{\mathcal{F}}$ restricted to M. Hypothesis (i) and (ii) of 3.1 evidently hold. Let $D_z = h(\{z\} \times D)$.

Assertion 1. For all $z \in \Sigma$, either $\{z\} \times D$ is contained in Y, or $S'_z := (\{z\} \times D) \cap Y$ is discrete and closed in $\{z\} \times D$.

Proof. Since Y is closed, we have that $A = \{z\} \times D \cap Y$ is closed in $\{z\} \times D$. Suppose that A is not discrete. Let $x \in A$ be an accumulation point of A and let f be a holomorphic function which defines Y on a neighborhood of x. Let $\mathcal{D}$ be a disc in $\{z\} \times D$, with $x \in \mathcal{D}$, and such that f is defined on $\mathcal{D}$. Since $A \subset Y$ we have that f vanishes on $A \cap \mathcal{D}$. Then f vanishes on $\mathcal{D}$ because $A \cap \mathcal{D}$ has $x \in \mathcal{D}$ as an acummulation point. Thus, $\mathcal{D} \subset Y$ and we have therefore $\mathcal{D} \subset A$. It follows that A is open in $\{z\} \times D$ and, by connectedness, $A = \{z\} \times D \cap Y$. Thus $\{z\} \times D$ is contained in Y.

Suppose that D_z is not contained in X. Let $S_z = h(S'_z)$, where S'_z is given by Assertion 1. Then S_z is discrete in D_z. Observe that $(\{z\} \times D)\backslash S'_z$ is contained in a leaf of $\mathcal{F}|_{M\backslash Y}$. Then, since $h|_{M\backslash Y}$ is a topological equivalence between $\mathcal{F}|_{V\backslash Y}$ and $\widetilde{\mathcal{F}}|_{\widetilde{V}\backslash\widetilde{Y}}$, it follows that

$$D_z\backslash S_z = h((\{z\} \times D)\backslash S'_z)$$

is contained in a leaf of $\widetilde{\mathcal{F}}$. Thus, hypothesis (iii) of 3.1 holds. Then $\widetilde{\mathcal{F}}$ is regular on $M = h(W)$ and every D_z is contained in a leaf of $\widetilde{\mathcal{F}}$. Therefore we conclude:

Assertion 2. If p is a point in Y which is regular for $\mathcal{F}$, then p is mapped by h to a regular point of $\widetilde{\mathcal{F}}$. Moreover, there exists a neighborhood Ω of p in its leaf which is mapped by h onto a neighborhood of $h(p)$ in its leaf.

Now, by using Assertion 2 for h and h^{-1}, we deduce that p is regular for $\mathcal{F}$ if and only if $h(p)$ is regular for $\widetilde{\mathcal{F}}$. Hence

$$h(\mathrm{Sing}(\mathcal{F})) = \mathrm{Sing}(\widetilde{\mathcal{F}}).$$

It remains to prove that h maps any leaf of $\mathcal{F}$ onto a leaf of $\widetilde{\mathcal{F}}$. Let p be a regular point of $\mathcal{F}$. Let L be the leaf of $\mathcal{F}$ passing through p and let $\widetilde{L}$ be the leaf of $\widetilde{\mathcal{F}}$ passing through $h(p)$. Let A be the set of points in L which are mapped by h into $\widetilde{L}$. By Assertion 2, if $x \in A$ there exists a neighborhood of x in L_p contained in A. Therefore A is open. Now, let $x \notin A$. Then $h(x) \notin \widetilde{L}$. Thus, if $L' \neq L$ is the leaf of $\widetilde{\mathcal{F}}$ passing through $h(x)$ it follows by Assertion 2 that there exists a neighborhood Ω of x in L which is mapped by h into $L' \neq \widetilde{L}$, hence Ω is contained in $L\backslash A$. Then A is also closed and it follows by conectedness that $A = L$, that is, $h(L) \subset \widetilde{L}$. Analogously, we prove that $h^{-1}(\widetilde{L}) \subset L$. Therefore $h(L) = \widetilde{L}$. □

We proceed now to prove Theorem 3.1.

Proposition 3.2. *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M. Let $X \subset M$ be an analytic subvariety of codimension ≥ 1. Suppose that:*

- *(i) There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.*
- *(ii) If $D_z := h(\{z\} \times D)$ then for all z: either D_z is contained in X, or D_z is contained in a leaf of $\mathcal{F}$.*

Consider $z' \in \Sigma$ and suppose that $D_{z'}$ is a singular disc. Let $S_{z'}$ the set of singularities of $D_{z'}$. Then $D_{z'} \backslash S_{z'}$ is contained in a leaf of $\mathcal{F}$.

Proof. It is sufficient to prove the following.

Assertion. If $p \in D_{z'} \backslash S_{z'}$ then p has a neighborhood in $D_{z'} \backslash S_{z'}$ contained in a leaf of $\mathcal{F}$.

Suppose Assertion holds. let L be a leaf of $\mathcal{F}$ and let $x \in (D_{z'} \backslash S_{z'}) \cap L$. By Assertion, there is a neighborhood Δ of x in $D_{z'} \backslash S_{z'}$ such that $\Delta \subset L$. Then $\Delta \subset (D_{z'} \backslash S_{z'}) \cap L$ and it follows that the intersection of $D_{z'} \backslash S_{z'}$ with any leaf is open in $D_{z'} \backslash S_{z'}$. Then, since $D_{z'} \backslash S_{z'}$ is connected, we have that it is contained in a unique leaf.

Proof of Assertion. Let p in $D_{z'} \backslash S_{z'}$. Since p is a regular point of the singular disc $D_{z'}$, on a neighborhood $U \subset M$ of p we may consider coordinates (w, y), $w \in \mathbb{B}$, $y \in \mathbb{D}$ with $p = (0, 0)$ and such that $D_{z'} \cap U$ is represented by $(w = 0)$. Suppose that $p = h(z', t')$. Let Σ' be a ball in Σ containing z' and let D' be a disc in D containing t'. Then $W = \Sigma' \times D'$ is a neighborhood of (z', t') and, by taking W small enough, we assume $h(\overline{W}) \subset U$. Let $D'_z = h(\{z\} \times D')$. Note that $D'_{z'} \subset D_{z'} \cap U$, hence $D'_{z'}$ is contained in $(w = 0)$. Let $g : U \to \mathbb{D}$ be the projection $g(w, y) = y$. Consider $z \in \Sigma'$ and suppose $D_z \backslash X \neq \emptyset$. By hypothesis (ii), D_z is contained in a leaf of $\mathcal{F}$. Therefore D'_z is contained in leaf of $\mathcal{F}$ and we have that $g|_{D'_z} : D'_z \to \mathbb{D}$ is a holomorphic map. Remember that $D'_{z'} \subset (w = 0)$. Then $g|_{D'_{z'}} : D'_{z'} \to \mathbb{D}$ is given by $(0, y) \to y$ and is therefore a one to one map. Then $g(D'_{z'})$ is a disc in $\mathbb{D}$ with $g(\partial D'_{z'})$ as boundary. Note that $p = (0, 0) \in D'_{z'}$, hence 0 is contained in the disc $g(D'_{z'})$. Therefore the curve $g(\partial D'_{z'})$ winds once around 0. By the continuity of h we assume Σ' small enough such that $g(\partial D'_z)$ is homotopic to $g(\partial D'_{z'})$ in $\mathbb{D} \backslash \{0\}$ for all $z \in \Sigma'$. Then $g(\partial D'_z)$ winds once around 0 and $g|_{D'_z}$ has therefore a unique zero. In other words, the plaque D'_z intersects $Y = \mathbb{B} \times \{0\} \subset U$ at a unique point. Thus, we can define the map $f : h(W) \backslash X \to Y$ by $f(D'_z \backslash X) = D'_z \cap Y$ whenever $D'_z \backslash X \neq \emptyset$. We have that f is holomorphic because it is constant along the leaves and, restricted to any transversal, is a holonomy map. Since f is bounded and X has codimension ≥ 1, by the generalized Riemann's extension theorem, f extends to a holomorphic function on $h(W)$. Observe that f restricted to Y is the identity map, then f is a submersion in a neighborhood V of Y. Hence f defines a regular foliation $\mathcal{N}$ on V. It is easy to see that $\mathcal{N}$ coincides with $\mathcal{F}$ on $V \backslash X$, thus $\mathcal{N} = \mathcal{F}$. Therefore $p \in Y$ is a regular point of $\mathcal{F}$.

Now, by reducing the neighborhood $W = \Sigma' \times D'$ of (z', t'), we may assume that $h(W)$ is contained in a neighborhood of p where $\mathcal{F}$ is given by a submersion f. Obviously

$D'_{z'}$ is a neighborhood of p in D_z. We shall prove that $D'_{z'}$ is contained in a leaf of $\mathcal{F}$ (the leaf passing through p). If $D'_{z'}$ is not contained in X, so is $D_{z'}$ and, by hypothesis (ii), we have that $D'_{z'}$ is contained in a leaf of $\mathcal{F}$. On the other hand, suppose that $D'_{z'}$ is contained in X. Then there exists a sequence of points $z_k \to z'$ such that $h(\{z_k\} \times D)$ is not contained in X, otherwise $h(\Sigma'' \times D) \subset X$ for some neighborhood $\Sigma'' \subset \Sigma$ of z', which is a contradiction because X has codimension ≥ 1. Thus, by (ii), we have that D'_{z_k} is contained in a leaf of $\mathcal{F}$ for all k. Recall $D'_{z_k} \subset h(W)$ is contained in a domain where $\mathcal{F}$ is given by the submersion f. Then f is constant over $D'_{z_k} = h(\{z_k\} \times D')$ and in particular, for all $t \in D'$ we have $f(h(z_k, t)) = f(h(z_k, t'))$. Then:

$$\begin{aligned} f(h(z', t)) &= f(h(\lim_{k\to\infty} z_k, t)) = \lim_{k\to\infty} f(h(z_k, t)) \\ &= \lim_{k\to\infty} f(h(z_k, t')) = f(h(\lim_{k\to\infty} z_k, t')) \\ &= f(h(z', t')). \end{aligned}$$

Therefore, for all $t \in D'$ we have that $h(z', t)$ and $h(z', t')$ are contained in the same Leaf. It follows that $D'_{z'}$ is contained in the leaf passing trough $h(z', t')$. Thus, Assertion is proved. ☐

Proposition 3.3. *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M such that:*

(i) $\mathcal{F}$ is generated by a holomorphic vector field.

(ii) There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.

(iii) For all z, there is a discrete closed set $S_z \subset D_z := h(\{z\} \times D)$ such that $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$.

Then $\mathcal{F}$ is regular and the sets D_z are the leaves of $\mathcal{F}$.

We need the following Lemmas.

Lemma 3.4. *Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be smooth, and holomorphic on $\mathbb{D}$. Suppose that f is regular on $\mathbb{S}^1$. Then f is a regular map if and only if the curve $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{C}$ has degree 1*[1].

Proof. If ζ is a point in $\mathbb{S}^1$, then $i\zeta$ is the unitary vector tangent to $\mathbb{S}^1$ at ζ. Thus the velocity vector of the curve $f|_{\mathbb{S}^1}$ in $f(\zeta)$ is the image by $f'(\zeta)$ of $i\zeta$, that is, $f'(\zeta)i\zeta$. The winding number of the curve $f'(\zeta)i\zeta$ is the number of zeros on $\mathbb{D}$ of the function $izf'(z)$, and this number is equal to 1 if and only if $f' \neq 0$ on $\mathbb{D}$. ☐

Lemma 3.5. *Let M be a complex manifold and $D \subset M$ a singular disc. Then there exists a holomorphic injective map $g : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that $g(E) = D$.*

---

[1] The degree of a parameterized regular curve in the plane is defined as the winding number around 0 of its velocity vector.

Proof. Let S be the set of singularities of D. Observe that $D\backslash S$ is homeomorphic to a subset of the plane. Then, since $D\backslash S$ is a Riemann surface, we have that $D\backslash S$ is actually holomorphically equivalent to a proper subset of the Riemann sphere (see [10]). Thus, there is a biholomorphism

$$g : U \to D\backslash S$$

where U is a domain in $\mathbb{C}P^1$. Consider $p \in S$. Since D is a singular disc, there is a holomorphic injective map $h : \mathbb{D} \to \mathcal{D}$, $h(0) = p$, where $\mathcal{D}$ is disc in D. We may assume that h is regular on $\overline{\mathbb{D}}\backslash\{0\}$. Then p is the unique singularity in $\overline{\mathcal{D}} \subset D$ and therefore the holomorphic function

$$g^{-1} \circ h : \overline{\mathbb{D}}\backslash\{0\} \to U$$

is well defined. Since $g^{-1} \circ h$ is injective, we have that $g^{-1} \circ h(\mathbb{S}^1)$ is a Jordan curve, hence it divides the sphere $\mathbb{C}P^1$ in two domains both homeomorphic to a disc. Then the image of $g^{-1} \circ h$ is contained in one of these domains and it follows by Riemann's extension theorem that $g^{-1} \circ h$ extends to $\mathbb{D}$ as a one to one map. Then, since $h : \mathbb{D} \to \mathcal{D}$ is a homeomorphism, we have that g^{-1} extends to $\mathcal{D}$ as a homeomorphism. Therefore, g^{-1} extends to D as a homeomorphism onto a simply connected domain $\Omega \subset \mathbb{C}$. Then $U = \Omega\backslash S'$, where $S' = g^{-1}(S)$ is a discrete closed set and we have therefore that $g : \Omega\backslash S' \to M$ extends to Ω as a holomorphic injective function $g : \Omega \to M$. Since $\Omega \subset \mathbb{C}P^1$ is simply connected and clearly $\Omega \neq \mathbb{C}P^1$, the lemma follows from Riemann's Uniformization Theorem. ☐

Proof of Proposition 3.3.

Assertion 1. For all z, we have that D_z is a singular disc and the sets $D_z\backslash \operatorname{Sing}(\mathcal{F})$ are the nonsingular leaves of $\mathcal{F}$.

Proof. Let $x \in D_z$. Since S_z is a discrete closed subset of D_z, there is a disc $\mathcal{D} \subset D_z$ with $x \in \mathcal{D}$ such that $\mathcal{D}\backslash\{x\} \subset D_z\backslash S_z$. Then, from hypothesis (*iii*), $\mathcal{D}\backslash\{x\}$ is contained in a leaf of $\mathcal{F}$. If $\mathcal{D}$ is small enough, we may think that $\mathcal{D}$ is contained in $\mathbb{C}^n$. Hence, by applying Lemma 3.5, there exists a holomorphic injective map $g : \mathbb{D} \to M$ with $g(\mathbb{D}) = \mathcal{D}$. Since that $x \in D_z$ was arbitrary, it follows that D_z is a singular disc.

Let L be a leaf of $\mathcal{F}$ and suppose that $x \in L \cap (D_z\backslash \operatorname{Sing}(\mathcal{F}))$ for some z. Take $\mathcal{D} \subset D_z$ as above. We assume $\mathcal{D}$ small enough such that it is contained in a neighborhood U of x where $\mathcal{F}$ is trivial and given by the submersion f. Then $\mathcal{D}\backslash\{x\}$ is contained in a leaf of $\mathcal{F}|_U$ and f is therefore constant over $\mathcal{D}\backslash\{x\}$. Hence, by continuity, f is constant over $\mathcal{D}$. Then $\mathcal{D}$ is contained in a leaf of $\mathcal{F}|_U$ and we have therefore $\mathcal{D} \subset L$. Thus we have $\mathcal{D} \subset L \cap (D_z\backslash \operatorname{Sing}(\mathcal{F}))$. It follows that $L \cap (D_z\backslash \operatorname{Sing}(\mathcal{F}))$ is an open subset of both L and $D_z\backslash \operatorname{Sing}(\mathcal{F})$ for all L and z. Now, fix a leaf L. Since the intersection of L with any $D_z\backslash \operatorname{Sing}(\mathcal{F})$ is open in L, it follows by connectedness that L is contained in a unique $D_z\backslash \operatorname{Sing}(\mathcal{F})$. For this $D_z\backslash \operatorname{Sing}(\mathcal{F})$, we also have that its intersection with any leaf is open in $D_z\backslash \operatorname{Sing}(\mathcal{F})$. Again by connectedness $D_z\backslash \operatorname{Sing}(\mathcal{F})$ is contained in a unique leaf, thus we necessarily have $D_z\backslash \operatorname{Sing}(\mathcal{F}) \subset L$ and it follows that $D_z\backslash \operatorname{Sing}(\mathcal{F}) = L$. Therefore Assertion 1 is proved.

Fix $p \in M$. We have $p \in D_{z'}$ for some $z' \in \Sigma$. Take p' in $D_{z'} \backslash S_{z'}$. From hypothesis (iii), p' is a regular point of $\mathcal{F}$. We have $p' = h(z', t')$ with $t' \in D$. If $B \subset \Sigma$ is a ball containing z', then $\Sigma_0 := B \times \{t'\}$ is a $(n-1)$ ball passing through (z', t'). We assume B small enough such that $\overline{\Sigma}_0$ is mapped by h into a neighborhood W of p' where $\mathcal{F}$ is equivalent to a product foliation. Let $\widetilde{\Sigma}$ (submanifold of W) be a global transversal to $\mathcal{F}|_W$. If w is a point contained in $\overline{h(\Sigma_0)}$, the leaf of $\mathcal{F}|_W$ passing through it intersects $\widetilde{\Sigma}$ in a unique point $\psi(w)$. We claim that ψ is a homeomorphism of $\overline{h(\Sigma_0)}$ onto its image. Since $\overline{h(\Sigma_0)}$ is compact, it suffices to prove that ψ is injective on $\overline{h(\Sigma_0)}$. Suppose that w_1 and w_2 are two points in $\overline{h(\Sigma_0)}$ contained in the same leaf L of $\mathcal{F}|_W$. From Assertion 1, we have that $L \subset D_z$ for some z. Then $h^{-1}(L) \subset \{z\} \times D$, hence $h^{-1}(w_1)$ and $h^{-1}(w_2)$ are two different points in the intersection of $(z \times D)$ with $\overline{\Sigma}_0$, which is a contradiction because $\overline{\Sigma}_0 \subset \Sigma \times \{t'\}$ intersects $(z \times D)$ only at (z, t').

We redefine $\widetilde{\Sigma}$ as $\widetilde{\Sigma} = \psi(h(\Sigma_0))$. Then for all $z \in B$ we have that D_z intersects $\widetilde{\Sigma}$ at the unique point $\psi(h(z, t_0))$; otherwise, if D_z intersects $\widetilde{\Sigma}$ in two different points x_1 and x_2, then $h^{-1}\psi^{-1}(x_1)$ and $h^{-1}\psi^{-1}(x_2)$ would be two different points in the intersection of $\{z\} \times \mathbb{D}$ with Σ_0.

We may define the map

$$g : V = h(B \times \mathbb{D}) \to \widetilde{\Sigma},$$

$$g(D_z) = D_z \cap \widetilde{\Sigma}.$$

By Assertion 1, each leaf of $\mathcal{F}$ is contained in some D_z. Then g is constant along the leaves. Therefore, since the restriction of g to any transversal is a holonomy map, we have that g is holomorphic on $V \backslash \operatorname{Sing}(\mathcal{F})$. Actually, since $\operatorname{Sing}(\mathcal{F})$ has codimension ≥ 2, g is holomorphic on V.

Consider $x \in \widetilde{\Sigma} \backslash g(\operatorname{Sing}(\mathcal{F}))$. Then $D = g^{-1}(x)$ does not intersect $\operatorname{Sing}(\mathcal{F})$. Clearly D is equal to some D_z. Then, by Assertion 1, $D \backslash \operatorname{Sing}(\mathcal{F}) = D$ is a leaf of $\mathcal{F}$. Thus, we conclude that for all $x \in \widetilde{\Sigma} \backslash g(\operatorname{Sing}(\mathcal{F}))$, the leaf passing through x is simply connected. Moreover, since $\operatorname{Sing}(\mathcal{F})$ has codimension ≥ 2, we have that $g(\operatorname{Sing}(\mathcal{F}))$ has codimension ≥ 1 in $\widetilde{\Sigma}$ and we have therefore that:

Assertion 2. For all x in a dense subset of $\widetilde{\Sigma}$, the leaf passing through x is simply connected.

Let Z be a holomorphic vector field which generates $\mathcal{F}$ on V and φ the local complex flow of Z. Let L be a leaf of $\mathcal{F}|_V$ and let x_L be its intersection with $\widetilde{\Sigma}$ ($g(L) = \{x_L\}$). There exists $\varepsilon_L > 0$ such that $\varphi(x_L, *)$ maps the disc $|t| < \varepsilon_L$ biholomorphically onto a neighborhood D_L of x_L in L. Thus, given any x in D_L there exists a unique $\tau_L(x)$ with $|\tau_L(x)| < \varepsilon_L$ such that $\varphi(x_L, \tau_L(x)) = x$. The function $\tau_L : D_L \to \mathbb{C}$ is the complex time between x_L and x. Clearly τ_L is holomorphic on D_L.

Assertion 3. The function τ_L can be analytically continued on L along any path $\gamma : [0,1] \to L$ with $\gamma(0) = x_L$.

Proof. Since γ does not intersect $\operatorname{Sing}(\mathcal{F})$ there exists $\delta > 0$ such that for all x in $\gamma([0,1])$, the map $\varphi(x, *)$ is a biholomorphism between $\mathbb{D}_{2\delta}$ and its image. Denote x_L

by x_0 and let $0 = s_0 < s_1 < \cdots < s_r = 1$ and $x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r)$ be such that:

(i) The open sets $\varphi(x_i, \mathbb{D}_\delta)$ for $i = 0, \ldots, r$ cover $\gamma([0,1])$.

(ii) x_i is contained in $\varphi(x_{i-1}, \mathbb{D}_\delta)$ for $i = 1, \ldots, r$.

For each $i = 0, \ldots, r$ let $\tau_i' : \varphi(x_i, \mathbb{D}_{2\delta}) \to \mathbb{D}_{2\delta}$ be defined by $\varphi(x_i, \tau_i'(x)) = x$. Let $x \in \varphi(x_{i-1}, \mathbb{D}_\delta) \cap \varphi(x_i, \mathbb{D}_\delta)$. Let $t_i = \tau_{i-1}'(x_i)$ for $i = 1, \ldots, r$ and define $t_0 = 0$. Clearly, $|t_i|$ and $|\tau_i'(x)|$ are less than δ, hence $|t_i + \tau_i'(x)| < 2\delta$ and we have that

$$\begin{aligned} \varphi(x_{i-1}, t_i + \tau_i'(x)) &= \varphi(\varphi(x_{i-1}, t_i), \tau_i'(x)) \\ &= \varphi(\varphi(x_{i-1}, \tau_{i-1}'(x_i)), \tau_i'(x)) \\ &= \varphi(x_i, \tau_i'(x)) \\ &= x. \end{aligned}$$

Then, by definition of τ_{i-1}' we obtain:

$$t_i + \tau_i'(x) = \tau_{i-1}'(x). \tag{5}$$

For each $i = 1, \ldots, r$ let τ_i be the holomorphic function on $\varphi(x_i, \mathbb{D}_\delta)$ defined by

$$\tau_i = \tau_i' + t_0 + \cdots + t_i.$$

By using (5) we deduce that $\tau_{i-1} = \tau_i$ on $\varphi(x_{i-1}, \mathbb{D}_\delta) \cap \varphi(x_i, \mathbb{D}_\delta)$. Moreover, it follows from the definition that τ_0 is equal to τ_L in a neighborhood of $x_0 = x_L$. Therefore, τ_0,...,τ_r give an analytic continuation of τ_L along γ.

Assertion 4. Let L be any leaf of $\mathcal{F}|_V$ and let $\gamma', \gamma'' : [0,1] \to L$ be paths such that $\gamma'(0) = \gamma''(0) = x_L$ and $\gamma'(1) = \gamma''(1) = x \in L$. Let τ_L' be the analytic continuation of τ_L along γ' and let τ_L'' be the analytic continuation of τ_L along γ''. Then $\tau_L'(x) = \tau_L''(x)$. Thus, τ_L extends as a holomorphic function on L. Therefore we may define $\tau : V \setminus \mathrm{Sing}(\mathcal{F}) \to \mathbb{C}$ by $\tau = \tau_L$ on L. Then τ is holomorphic on $U \setminus \mathrm{Sing}(\mathcal{F})$ and extends to U because Sing($\mathcal{F}$) *has codimension ≥ 2. Moreover, if restricted to a leaf, τ is a regular map. In particular, τ is a submersion on $U \setminus \mathrm{Sing}(\mathcal{F})$.*

Proof. Fix L and denote x_L by x_0. Let $0 = s_0 < \cdots < s_r = 1$, let $\Sigma_0, \ldots, \Sigma_r$ be transversals to the foliation at the points $x_0, x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r)$ respectively, and let $\delta > 0$ with the following properties:

(i) $\Sigma_0 \subset \widetilde{\Sigma}$.

(ii) The flow φ maps $\Sigma_i \times \mathbb{D}_{2\delta}$ biholomorphically onto its image, for all $i = 0, \ldots, r$.

(iii) The transversal Σ_i is contained in $\varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$, for all $i = 1, \ldots, r$.

(iv) For all $i = 1, \ldots, r$ we have that $\Sigma_i = h_i(\Sigma_0)$, where h_i is the holonomy map along γ.

Denote by V' the union of the sets $\varphi(\Sigma_i \times \mathbb{D}_\delta)$ for $i = 0, \ldots, r$. Consider $x \in V'$ and let L_x be the leaf passing through x. Let $k \in \{0, \ldots, r\}$ be such that $x \in \varphi(\Sigma_k \times \mathbb{D}_\delta)$. Then L_x intersects Σ_k and it follows from hypothesis (iv) that L_x intersects each Σ_i. Since $\Sigma_0 \subset \widetilde{\Sigma}$ we have that L_x intersects Σ_0 in a unique point and, by (iv), the same holds for each Σ_i. Then we may define $\rho_i : V' \to \Sigma_i$ such that $\rho_i(x)$ is the point of intersection between L_x and Σ_i. Let $\tau_i'(x) \in \mathbb{D}_\delta$ be defined by $\varphi(\rho_i(x), \tau_i'(x)) = x$. Since $\rho_i(x) \in \Sigma_i$, by hypothesis (iii) we have that $\rho_i(x) \in \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$ for $i = 1, \ldots, r$. Then for $i = 1, \ldots, r$ we may define $t_i : V' \to \mathbb{D}_\delta$ as $t_i = \tau_{i-1}' \circ \rho_i$. Define $t_0 : V' \to \mathbb{D}_\delta$ as the zero function. Clearly, ρ_i, τ_i and t_i are holomorphic functions. We proceed as in the proof of Assertion 3. Let $x \in \varphi(\Sigma_i \times \mathbb{D}_\delta) \cap \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$. Since $|t_i(x)|$ and $|\tau_i'(x)|$ are less than δ, then $|t_i(x) + \tau_i'(x)| < 2\delta$. Thus, by hypothesis (ii), $\varphi(\rho_{i-1}(x), t_i(x) + \tau_i'(x))$ is well defined and:

$$\begin{aligned}
\varphi(\rho_{i-1}(x), t_i(x) + \tau_i'(x)) &= \varphi(\varphi(\rho_{i-1}(x), t_i(x)), \tau_i'(x)) \\
&= \varphi(\varphi(\rho_{i-1}(x), \tau_{i-1}' \circ \rho_i(x)), \tau_i'(x)) \\
&= \varphi(\rho_i(x), \tau_i'(x)) \\
&= x.
\end{aligned}$$

Then by definition of τ_{i-1}' we deduce that

$$t_i(x) + \tau_i'(x) = \tau_{i-1}'(x).$$

Thus, the holomorphic functions on $\varphi(\Sigma_i \times \mathbb{D}_\delta)$ defined as

$$\tau_i(x) = \tau_i'(x) + t_0(x) + \cdots + t_i(x) \tag{6}$$

for each $i = 0, \ldots, r$ are such that

$$\tau_i = \tau_{i-1}$$

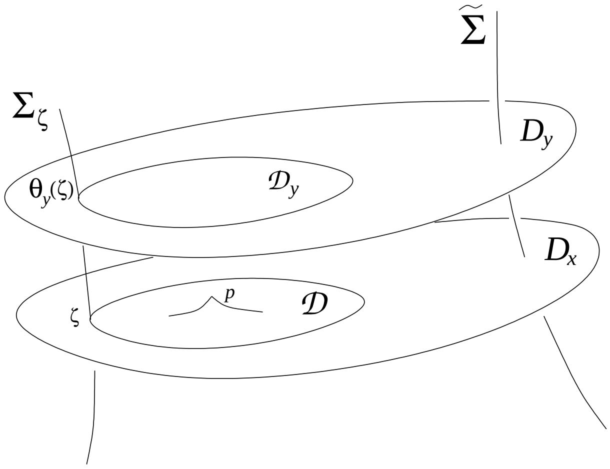
on $\varphi(\Sigma_i \times \mathbb{D}_\delta) \cap \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$. Observe that for any leaf L', the restriction $\tau_0|_{L'}$ coincides with $\tau_{L'}$ on a neighborhood of $x_{L'}$. Then $\tau_0|_{L'}, \cdots, \tau_r|_{L'}$ give an analytic continuation of $\tau_{L'}$. Thus, $\tau_r|_L$ is the analytic continuation of τ_L along γ', hence $\tau_r(x) = \tau_L'(x)$. We denote τ_r by τ'. Analogously we construct τ'' for γ''. Then we have that $\tau''|_{L'}$ is an analytic continuation of $\tau_{L'}$ and, $\tau''|_L$ is the analytic continuation of τ_L along γ'', hence $\tau''(x) = \tau_L(x)$. By Assertion 2, we may take a sequence $\{x_k\}$ of points in Σ_0 with $x_k \to x$ as $k \to \infty$ and such that the leaf L_k passing through x_k is simply connected for all k. From above $\tau'|_{L_k}$ and $\tau''|_{L_k}$ are analytic continuations of τ_{L_k}. Since L_k is simply connected and, by Assertion 2, τ_{L_k} has an analytic continuation along any path, then $\tau'|_{L_k}$ and $\tau''|_{L_k}$ coincide on a neighborhood of x_k. In particular, $\tau'(x_k) = \tau''(x_k)$. Making $k \to \infty$ it follows by continuity that $\tau'(x) = \tau''(x)$, that is, $\tau_L'(x) = \tau_L''(x)$. Therefore, τ_L extends to L.

We define $\tau : V \setminus \operatorname{Sing}(\mathcal{F}) \to \mathbb{C}$ by $\tau|_L = \tau_L$. From above, τ coincides with the holomorphic function τ' on a neighborhood of the point x (arbitrary point). Then τ is holomorphic. Finally, remember (equation 6) that on a neighborhood of any non singular point, τ is expressed as

$$\tau_r(x) = \tau_r'(x) + t_0(x) + \cdots + t_r(x).$$

If we restrict x to a leaf, the first term of the sum above is a regular map and the other terms are constants. Hence τ is a regular map of any leaf. This finishes the proof of Assertion 4.

Given $x \in \widetilde{\Sigma}$, we know that $g^{-1}(x)$ is equal to D_z for some z. We denote $g^{-1}(x)$ by D_x. Thus, we have $p \in D_x$ for $x = g(p)$. It follows from hypothesis (iii) that there is a disc $\mathcal{D}' \subset D_x$ containing p such that $\mathcal{D}' \backslash \{p\}$ is contained in a leaf. Lemma 2.5 implies that there is a holomorphic bijective map $f : \Omega \to \mathcal{D}'$, $f(0) = p$, where $\Omega \subset \mathbb{C}$ is a disc containing $\mathbb{D}$. Thus if $\mathcal{D} = f(\mathbb{D})$, we have that $f : \overline{\mathbb{D}} \to \overline{\mathcal{D}}$ is holomorphic and regular on $\overline{\mathbb{D}} \backslash \{0\}$. Since $\overline{\mathcal{D}} \backslash \{p\}$ is contained in a leaf and by Assertion 3 we have that τ is a submersion on $U \backslash \operatorname{Sing}(\mathcal{F})$, then there exists a neighborhood V of $\partial\Delta$ on which τ defines a foliation by transversal balls along $\partial\Delta$. If we denote by Σ_ζ the transversal passing trough $\zeta \in \partial\Delta$ we have that τ is constant along Σ_ζ. Recall that $y \in \widetilde{\Sigma}$ is the unique point in the intersection of D_y and $\widetilde{\Sigma}$. It follows from the transversal uniformity of the foliation that if $y \in \widetilde{\Sigma}$ is close to x then D_y intersects only one time each transversal Σ_ζ. Let $\theta_y(\zeta)$ be the intersection of D_y with Σ_ζ. Since $\theta_y(\zeta)$ and ζ are both contained in Σ_ζ, we have that $\tau(\theta_y(\zeta)) = \tau(\zeta)$ for all $\zeta \in \partial\Delta$. Note that $\theta_y := \theta_y(\partial\Delta)$ is a smooth Jordan curve in D_y. By Assertion 2, we may choose y such that D_y is a leaf. We consider $\mathcal{D}_y \subset D_y$, the regular disc bounded by θ_y.



Let $f_y : \mathbb{D} \to \mathcal{D}_y$ be a uniformization map. Since θ_y is a smooth Jordan curve, f_y extends as a diffeomorphism $f_y : \overline{\mathbb{D}} \to \overline{\mathcal{D}}_y$ (see [8], p.323). By Assertion 3, we have that τ is regular on $\overline{\mathcal{D}}_y$. It follows that $\tau \circ f_y : \overline{\mathbb{D}} \to \mathbb{C}$ is a regular map. Therefore, by Lemma 3.4, the curve $\tau \circ f_y : \mathbb{S}^1 \to \mathbb{C}$ has degree 1. Remember that $\tau(\theta_y(\zeta)) = \tau(\zeta)$ for all $\zeta \in \partial\Delta$, thus $\tau(\partial\mathcal{D}_y) = \tau(\partial\mathcal{D})$. Then

$$\tau \circ f_y(\mathbb{S}^1) = \tau(\partial\mathcal{D}_y) = \tau(\partial\mathcal{D}) = \tau \circ f(\mathbb{S}^1).$$

Therefore $\tau \circ f : \mathbb{S}^1 \to \mathbb{C}$ is only a reparametrization of $\tau \circ f_y : \mathbb{S}^1 \to \mathbb{C}$, hence $\tau \circ f : \mathbb{S}^1 \to \mathbb{C}$ is regular and has degree 1. Again by Lemma 3.4, $\tau \circ f : \mathbb{D} \to \mathbb{C}$ is also a regular map and in particular, $\tau \circ f$ is locally injective. Therefore there exists a disc $\mathrm{U} \subset \mathbb{D}$, centered at 0, such that $\tau \circ f$ is injective on $\overline{\mathrm{U}}$. Then

$$\tau \circ f(\partial\,\mathrm{U})$$

is a Jordan curve in $\mathbb{C}$. We also denote $f(\mathrm{U})$ by $\mathcal{D}$. Again, let Σ_ζ be the transversal ball through $\zeta \in \partial\mathcal{D}$. Proceeding as above, if Σ' is a small enough ball in $\widetilde{\Sigma}$ containing $x = g(p)$, we obtain that for all $y \in \Sigma'$ the set D_y intersects each Σ_ζ at the unique point $\theta_y(\zeta)$. Thus we have the Jordan curve θ_y in D_y such that $\tau(\theta_y) = \tau(\partial D)$. Remember that $\tau(\partial D) = \tau \circ f(\partial\,\mathrm{U})$ is a Jordan curve in $\mathbb{C}$. It follows that $\tau(\theta_y)$ is Jordan curve in $\mathbb{C}$ for all y. Let $\mathcal{D}_y \subset D_y$ be the disc bounded by θ_y. Since D_y is a singular disc, by Lemma 3.5, there is an injective holomorphic map $f_y : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that $f_y(E) = D_y$. Let $\Omega_y \subset E$ be such that $f_y(\Omega_y) = \mathcal{D}_y$. Clearly Ω_y is a disc and $f_y(\partial\Omega_y) = \partial\mathcal{D}_y$. Then

$$\tau \circ f_y(\partial\Omega_y) = \tau(\partial\mathcal{D}_y)$$

is, from above, a Jordan curve in $\mathbb{C}$. Hence we deduce that the holomorphic function $\tau \circ f_y : \Omega_y \to \mathbb{C}$ is injective on $\overline{\Omega}_y$. Thus, since f_y is injective, we conclude that

$$\tau : \overline{\mathcal{D}}_y \to \mathbb{C}$$

is injective for all $y \in \Sigma'$.

Denote by W the union of the discs $\mathcal{D}_y$ for all $y \in \Sigma'$. It is easy to see that W is a neighborhood of p. Define

$$F : \overline{W} \to \widetilde{\Sigma} \times \mathbb{C}$$

$$F(w) = (g(w), \tau(w))$$

Assertion 5. F is a biholomorphism between W and its image.

Proof. Clearly F is holomorphic on W. We shall prove that F injective on $\overline{W}$. Suppose $F(w) = F(w')$. Then $g(w) = g(w') = y$, hence $w, w' \in D_y$ and, since $\overline{W} \cap D_y = \overline{\mathcal{D}}_y$, we have $w, w' \in \overline{\mathcal{D}}_y$. On the other hand $\tau(w) = \tau(w')$ and since τ is injective on $\overline{\mathcal{D}}_y$ we conclude that $w = w'$. Now, since $\overline{W}$ is compact, F is a homeomorphism onto its image and it follows that F is a biholomorphism.

Now, we will prove that $p \in W$ is regular for $\mathcal{F}$. Let $\mathcal{N}$ be the regular foliation on $\widetilde{\Sigma} \times \mathbb{C}$ whose leaves are the sets $\{*\} \times \mathbb{C}$. Let $\mathcal{F}'$ be the pull-back foliation of $\mathcal{N}$ by the biholomorphism F. Then $\mathcal{F}'$ is regular and it is easy to see that $\mathcal{F}'$ coincides with $\mathcal{F}$ out on a open set of W (out of $\mathrm{Sing}(\mathcal{F})$). Then $\mathcal{F}' = \mathcal{F}$ on W and $\mathcal{F}$ is therefore regular at p. Since $p \in U$ was arbitrary, we have proved that $\mathrm{Sing}(\mathcal{F})$ is empty. Then, from Assertion 1, the sets D_z are the leaves of $\mathcal{F}$. The proof of Proposition 3.3 is complete. ☐

Lemma 3.6. *Let M be a complex manifold and let $X \subset M$ be a subvariety of codimension $k \geq 1$. Let p be a point in X. Then there exist a path x_t, $t \in [0, 1)$, with $x_0 = p$ and such that x_t is out of X for all $t > 0$.*

Proof. Let U be a neighborhood of p in M and $f : U \to \mathbb{C}^r$ be such that $f^{-1}(0) = X \cap U$. There is a complex disc D passing through p such that $f|_D$ does not vanish, otherwise

$f^{-1}(0) \subset X$ contains a neighborhood of p in M. By reducing D, we may assume that $p \in D$ is the unique zero of $f|_D$. Then $D\backslash\{p\}$ is disjoint of X and it is sufficient to take any path x_t in D with $x_0 = p$ and $x_t \neq p$ for all $t > 0$. $\square$

Proof of Theorem 3.1.

Assertion 1. Let $z \in \Sigma$ such that D_z is not contained in X. Then D_z is contained in a leaf of $\mathcal{F}$.

Proof. Take $t_0 \in D_z$ such that $h(z, t_0) \notin X$. Since X is closed in M, if Σ' is a small enough neighborhood (ball) of z in Σ, we have that $h(z', t_0) \notin X$ for all $z' \in \Sigma'$. Hence, for all $z' \in \Sigma'$ we have that $D_{z'}$ is not contained in X. Then, by hypothesis (ii), $S_{z'} := D_{z'} \cap X$ is discrete and $D_{z'}\backslash S_{z'}$ is contained in a nonsingular leaf of $\mathcal{F}$. Therefore, $\mathcal{F}$ restricted to $M' := h(\Sigma' \times D)$ satisfies the hypothesis of Proposition 3.3 and we have therefore that D_z is contained in a leaf of $\mathcal{F}$.

Assertion 2. Let $z \in \Sigma$ such that D_z is contained in X. Then D_z is a singular disc.

Proof. Let $x \in D_z$, $x = h(z, t)$. Let $\Sigma' \subset \Sigma$ be a neighborhood (a ball) of z and $D' \subset D$ be a neighborhood (a disc) of t. If Σ' and D' are small enough, we may assume that $M' := h(\Sigma' \times D')$ is a domain in $\mathbb{C}^n$. Since X has codimension ≥ 1, by Lemma 3.6, there is a path $x_s = h(z_t, t_s)$ in M' such that $x_0 = x$ and $x_s \notin X$ for all $s > 0$. Then $D_s := D_{z_s}$ is not contained in X for all $s > 0$ and it follows by Assertion 1 that D_s is contained in a leaf. Hence D_s is a regular disc for all $s > 0$. Then, we may apply Lemma 2.1 to the family of discs D_s and conclude that $D_z = D_0$ is a singular disc.

Assertion 3. Let z be such that $D_z \subset X$. Let S_z be the set of singularities of the singular disc D_z. Then $D_z\backslash S_z$ is contained in a leaf of $\mathcal{F}$.

Proof. By Assertion 2, if D_z is not contained in X we have that D_z is contained in a leaf of $\mathcal{F}$. Therefore, the hypothesis of Proposition 3.2 holds for $\mathcal{F}$ and Assertion 3 follows.

Let z be such that D_z is not contained in X. By hypothesis (iii) of 3.1, we have that $S_z := D_z \cap X$ is discrete and $D_z\backslash S_z$ is contained in a leaf of $\mathcal{F}$. From this and Assertion 3 we conclude: for all z there is a discrete set S_z such that $D_z\backslash S_z$ is contained in a leaf of $\mathcal{F}$. Therefore the hypothesis of Proposition 3.3 holds and Theorem 3.1 follows. $\square$

4 The algebraic multiplicity and the Chern class of the tangent bundle of the strict transform

Let $\mathcal{F}_0$, $\widetilde{\mathcal{F}}_0$ and h as in §1.

Proposition 4.1. *If h extends to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$, then the extension also denoted by h is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$.*

Proof. Is a direct application of Theorem 1.4. $\square$

Proof of Theorem 1.1. Suppose that $\mathcal{F}$ is generated on U by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}, \ a_i \in \mathcal{O}_U, \ g.c.d.(a_1, a_2, \ldots, a_n) = 1.$$

For each $j = 1, 2, \ldots, n$, let $U_j = (x_j \neq 0)$ and $U_j' = \pi^{-1}(U_j)$. Let $V_j = \pi^*(V|_{U_j})$. If $(x_1^j, \ldots, x_n^j)$ are coordinates on U_j' such that

$$\pi(x_1^j, \ldots, x_n^j) = (x_j^j x_1^j, \ldots, x_j^j, \ldots, x_j^j x_n^j),$$

then

$$V_j = a_j \frac{\partial}{\partial x_j^j} + \sum_{i=1, i \neq j}^{n} \frac{a_i - x_i^j a_j}{x_j^j} \frac{\partial}{\partial x_i^j},$$

where $a_i = a_i \circ \pi$ for $i = 1, \ldots, n$. On U_j', $\mathcal{F}_0$ is defined by the vector field

$$W_j = \frac{1}{(x_j^j)^{r-\xi}} V_j,$$

where r is the algebraic multiplicity of V at $0 \in \mathbb{C}^n$ and $\xi = 1$ or 0 depending on the divisor being invariant or not by $\mathcal{F}_0$. Evidently $V_i = V_j$ on $U_i' \cap U_j'$. Then

$$W_i = \left(x_j^j / x_i^i\right)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j'.$$

It follows from this equation that the tangent bundle $T\mathcal{F}_0$ of $\mathcal{F}_0$ is isomorphic to $L^{\xi - r}$, where L is the line bundle associated to the divisor $E = \pi^{-1}(0)$. Then the Chern class $c(T\mathcal{F}_0)$ of $T\mathcal{F}_0$ is equal to $(\xi - r)c(L)$. It is natural consider E as an element in $H_{n-2}(U', \mathbb{Z})$, where $U' = \pi^{-1}(U)$. We know that $c(L)$ is equal to $d(E) \in H^2(U', \mathbb{Z})$, the dual of E. Therefore

$$c(T\mathcal{F}_0) = (\xi - r)d(E).$$

On the other hand, make $\widetilde{U}' = \pi^{-1}(\widetilde{U})$ and observe that the divisor E is invariant by $\mathcal{F}_0$ if and only if it is by $\widetilde{\mathcal{F}}_0$. Then analogously we have

$$c(T\widetilde{\mathcal{F}}_0) = (\xi - \widetilde{r})\widetilde{d}(E),$$

where $\widetilde{r}$ is the algebraic multiplicity of $\widetilde{\mathcal{F}}$ and $\widetilde{d}(E) \in H^2(\widetilde{U}', \mathbb{Z})$ is the dual of E. By Proposition 4.1 we have that $h : U' \to \widetilde{U}'$ is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. Then Theorem 1.3 implies that

$$(\xi - r)h^*(d(E)) = (\xi - \widetilde{r})\widetilde{d}(E). \tag{7}$$

We may assume that U is a ball in $\mathbb{C}^n$. Thus, we have that U' is a tubular neighborhood of E and therefore $H^2(U', \mathbb{Z}) \simeq \mathbb{Z}$. Since the cohomology is invariant by homeomorphisms, we also have $H^2(\widetilde{U}', \mathbb{Z}) \simeq \mathbb{Z}$. Can be proved that $d(E)$ and $\widetilde{d}(E)$ are generators of $H^2(U', \mathbb{Z})$ and $H^2(\widetilde{U}', \mathbb{Z})$ respectively. Then we have that $h^*(d(E)) = \widetilde{d}(E)$ or $h^*(d(E)) = -\widetilde{d}(E)$. By using this in (7) we obtain $r = \widetilde{r}$ or $r + \widetilde{r} = 2\xi$. The second possibility implies $r = \widetilde{r} = \xi$, since $r \geq 1$, $\widetilde{r} \geq 1$ and $\xi = 1$ or 0. Therefore we conclude that $r = \widetilde{r}$. $\square$

Remark. Under the hypothesis of Theorem 1.1, we have another invariants. The restriction of $\mathcal{F}_0$ to the divisor is a foliation with $\mathrm{Sing}(\mathcal{F}_0)$ as singular set. It is well known that this foliation coincides out of the singular set with a unique foliation $\mathcal{N}$ of codimension ≥ 2 in the divisor (the saturated foliation). We will say that $\mathcal{N}$ is the foliation induced by $\mathcal{F}_0$ in the divisor. Let $\widetilde{\mathcal{N}}$ be the foliation induced by $\widetilde{\mathcal{F}}_0$ in the divisor. It follows from Theorem 1.4 that $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are topologically equivalent. Thus, since the divisor is isomorphic to $\mathbb{P}^{n-1}$, Theorem 1.3 implies that $d(\mathcal{N}) = d(\widetilde{\mathcal{N}})$. In other words, the degree of the foliation induced in the divisor is invariant.

From above, $\mathcal{F}_0$ is generated by the holomorphic vector fields W_i and

$$W_i = \left(x_j^j/x_i^i\right)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j',$$

where $\xi = 1$ or 0. Let $x \in U_i' \cap U_j'$. Let $x^i = (x_1^i, \ldots, x_n^i)$ be the coordinates of x in U_i' and let $x^j = (x_1^j, \ldots, x_n^j)$ be the coordinates of x in U_j'. Since $\pi(x^i) = \pi(x^j)$, we have that

$$(x_i^i x_1^i, \ldots, x_i^i, \ldots, x_i^i x_n^i) = (x_j^j x_1^j, \ldots, x_j^j, \ldots, x_j^j x_n^j),$$

hence $x_j^j/x_i^i = x_j^i$. Replacing in last equation we obtain:

$$W_i = \left(x_j^i\right)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j'. \tag{8}$$

Observe that $\pi^{-1}(0) \cap U_i'$ is represented by $(x_i^i = 0)$. Recall that $\pi^{-1}(0)$ is canonically isomorphic to $\mathbb{P}^{n-1}$. A point p in $\pi^{-1}(0) \cap U_i'$ given by

$$(x_1^i(p), \ldots, 0_i, \ldots, x_n^i(p))$$

is represented in homogeneous coordinates by

$$[z_1 : \cdots : z_n](p) = [x_1^i(p) : \cdots : 1_i : \cdots : x_n^i(p)],$$

hence $x_j^i(p) = (z_j/z_i)(p)$. Thus, if $\mathcal{U}_i = U_i' \cap \pi^{-1}(0)$ and $J_i = W_i|_{\mathcal{U}_i}$, it follows from (8) that

$$J_i = (z_j/z_i)^{r-\xi} J_j \quad \text{on} \quad \mathcal{U}_i \cap \mathcal{U}_j. \tag{9}$$

Let S be the union of the components of codimension 2 of $\mathrm{Sing}\,(\mathcal{F}_0)$. Then S is the codimension 1 part (respect to the divisor) of the zero set of $\{J_i\}$. Each J_i may be expressed as $J_i = f_i Z_i$, where f_i is a holomorphic function on $\mathcal{U}_i$ and the vector field Z_i has singular set of codimension ≥ 2. It follows from (9) that

$$Z_i = (f_j/f_i)\,(z_j/z_i)^{r-\xi} Z_j \ \text{ on } \ \mathcal{U}_i \cap \mathcal{U}_j.$$

From this equation, it is not difficult to conclude that

$$r = d(\mathcal{N}) - \deg(S) - 1 + \xi,$$

where $\deg(S)$ is the degree of S as a divisor of $\pi^{-1}(0)$. Then, since the algebraic multiplicity y and the degree of the foliation induced in the divisor are invariants, we deduce that the degree of the codimension 1 part of the singular set of the strict transform is also an invariant. Moreover it is not difficult to see that $h(S) = \widetilde{S}$, where $\widetilde{S}$ is the union of the components of codimension 2 of $\mathrm{Sing}(\widetilde{\mathcal{F}}_0)$.

5 The case C^1

In this section we prove Theorem 1.2. In view of Theorem 1.1, it is sufficient to show the following.

Proposition 5.1. *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be two foliations by curves of neighborhoods U and $\widetilde{U}$ of $0 \in \mathbb{C}^n$. Let $\mathrm{h} : U \to \widetilde{U}$ be a C^1 equivalence. Let $h : \pi^{-1}(U\backslash\{0\}) \to \pi^{-1}(\widetilde{U}\backslash\{0\})$ be as before. Then h can be extended to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$.*

We start the proof.

Proposition 5.2. *Under the conditions of Proposition 5.1, we have that $d\,\mathrm{h}(0) : \mathbb{C}^n \to \mathbb{C}^n$ maps complex lines onto complex lines. Furthermore, if $J : \mathbb{C}^n \to \mathbb{C}^n$ is the conjugation $J(z) = \bar{z}$, then either $d\,\mathrm{h}(0) : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear isomorphism, or $d\,\mathrm{h}(0) = Q \circ J$, where $Q : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear isomorphism. Thus, $d\,\mathrm{h}(0)$ induces a diffeomorphism of $\mathbb{P}^{n-1}$ onto itself.*

Proof. let L be a complex line, $0 \in L \subset \mathbb{C}^n$. There exists $\mathbb{C}$-linear functions $A_i : \mathbb{C}^n \to \mathbb{C}$ for $i = 1, \ldots, (n-1)$, such that

$$L = \{z \in \mathbb{C}^n \ : \ A_i(z) = 0, \ \text{ for all } \ i = 1, 2, \ldots, (n-1)\}.$$

Let $V : U \to \mathbb{C}^n$ be a holomorphic vector field which generates $\mathcal{F}$. The set:

$$B = \{z \in \mathbb{C}^n \ : \ A_i \circ V(z) = 0, \ \text{for all} \quad i = 1, 2, \ldots, (n-1)\}$$

is an analytic variety and it is easy to see that $0 \in B$. Then, there exists a complex curve contained in B and passing through 0. In particular there exists a sequence of points $z_k \in \mathbb{C}^n\backslash\{0\}$, $z_k \to 0$, such that $A_i \circ V(z_k) = 0$ for all $k \in \mathbb{N}$ and all $i = 1, 2, \ldots, (n-1)$. In other words, $T_{z_k}\mathcal{F} = L$ for all $k \in \mathbb{N}$. Now, since h is a C^1 equivalence, $d\,\mathrm{h}_{z_k}(T_{z_k}\mathcal{F}) = T_{\mathrm{h}(z_k)}\widetilde{\mathcal{F}}$, that is, $d\,\mathrm{h}_{z_k}(L) = T_{\mathrm{h}(z_k)}\widetilde{\mathcal{F}}$ is a complex line for all $k \in \mathbb{N}$. Making $k \to \infty$, since $\mathrm{h} \in C^1$ and the space of complex lines of $\mathbb{C}^n$ is compact, we obtain that $d\ h_0(L)$ is also a complex line. The second part of the proposition is an immediate consequence of the following lemma. ☐

Lemma 5.3. *Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a $\mathbb{R}$-linear isomorphism. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and assume that A maps complex lines onto complex lines. Then, either A is a $\mathbb{C}$-linear isomorphism, or $A = Q \circ J$ with $Q : \mathbb{C}^n \to \mathbb{C}^n$ a $\mathbb{C}$-linear isomorphism.*

Proof. Since A maps any complex line onto a complex line, for all $v \in \mathbb{C}^n\backslash\{0\}$ there exists $\theta(v) \in \mathbb{C}\backslash\{0\}$ such that $A(iv) = \theta(v)A(v)$. Let v_1 and v_2 be two $\mathbb{C}$-linearly independent vectors. Then

$$A(iv_1 + iv_2) = A(iv_1) + A(iv_2) = \theta(v_1)A(v_1) + \theta(v_2)A(v_2).$$

Moreover:

$$\begin{aligned} A(iv_1 + iv_2) &= A(i(v_1 + v_2)) = \theta(v_1 + v_2)A(v_1 + v_2) \\ &= \theta(v_1 + v_2)A(v_1) + \theta(v_1 + v_2)A(v_2). \end{aligned}$$

From the equations above, we obtain:

$$(\theta(v_1) - \theta(v_1 + v_2))A(v_1) + ((\theta(v_2) - \theta(V_1 + v_2))A(v_2) = 0. \tag{10}$$

Let L_1 and L_2 be the complex lines generated by v_1 and v_2 respectively. Since v_1 and v_2 are $\mathbb{C}$-linearly independent, we have that L_1 and L_2 are different. This implies, since A is an isomorphism, that $A(L_1)$ and $A(L_2)$ are different complex lines. Then, since $A(L_1)$ and $A(L_2)$ are generated by $A(v_1)$ and $A(v_2)$ respectively, we have that $A(v_1)$ and $A(v_2)$ are $\mathbb{C}$-linearly independent. Thus, it follows from equation (10) that

$$\theta(v_1) = \theta(v_1 + v_2) = \theta(v_2).$$

It is now easy to see that $\theta(v) = \theta_0, \ \forall v \in \mathbb{C}^n \backslash \{0\}$. We know that there exists two $\mathbb{C}$-linear transformations $P : \mathbb{C}^n \to \mathbb{C}^n$ and $Q : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$A(z) = P(z) + Q(\bar{z}), \ \text{ for all } \ z \in \mathbb{C}^n.$$

Then

$$A(iz) = iP(z) - iQ(\bar{z}).$$

On the other hand

$$A(iz) = \theta_0 A(z) = \theta_0 P(z) + \theta_0 Q(\bar{z}), \ \text{ for all } \ z \in \mathbb{C}^n.$$

consequently

$$(\theta_0 - i)P(z) + (\theta_0 + i)Q(\bar{z}) = 0.$$

Since, as functions of z, $(\theta_0 - i)P$ and $(\theta_0 + i)Q \circ J$ are holomorphic and anti-holomorphic respectively, we have that

$$(\theta_0 - i)P \equiv 0, \ \ (\theta_0 + i)Q \circ J \equiv 0.$$

From this it is easy to see that either $P = 0$, or $Q = 0$. This proves the lemma. □

Definition 5.4. *Let $\{z_k\}$ be a sequence of points in $\mathbb{C}^n \backslash \{0\}$. Let L be a complex line in $\mathbb{C}^n$. We say that $\{z_k\}$ is tangent to L at 0 if $z_k \to 0$ and every accumulation point of $\{z_k/||z_k||\}$ is contained in L.*

Let $\pi : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ be the blow up at $0 \in \mathbb{C}^n$. We know that $\pi^{-1}(0)$ is naturally isomorphic to $\mathbb{P}^{n-1}$. Thus, for each $p \in \pi^{-1}(0)$ we denote by L_p the respective complex line in $\mathbb{C}^n$.

Proposition 5.5. *Let $\{p_k\}$ be a sequence of points in $\widehat{\mathbb{C}^n} \backslash \pi^{-1}(0)$. Then $p_k \to p \in \pi^{-1}(0)$ if and only if $\{\pi(p_k)\}$ is tangent to L_p at 0.*

Proof. Let

$$U_1 = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \, : \, z_1 \neq 0\}.$$

There exist coordinates $(x_1, x_2, \ldots, x_n)$ on $\pi^{-1}(U_1)$ such that

$$\pi(x_1, x_2, \ldots, x_n) = (x_1, x_1 x_2, \ldots, x_1 x_n)$$

and $\pi^{-1}(0)$ is represented by $x_1 = 0$. Without lost of generality we may assume that $p = (0, 0, \ldots, 0)$, $L_p = \{z_2 = 0, ..., z_n = 0\}$, and $p_k = ({x_1}^k, {x_2}^k, \ldots, {x_n}^k) \in \pi^{-1}(U_1)$ for all $k \in \mathbb{N}$. Assume first that $p_k \to p$. Evidently, $\pi(p_k) \to 0$. On the other hand

$$\frac{\pi(p_k)}{||\pi(z_k)||} = \frac{{x_1}^k}{|{x_1}^k|} \frac{(1, {x_2}^k, \ldots, {x_n}^k)}{||(1, {x_2}^k, \ldots, {x_n}^k)||}$$

and since

$$\frac{(1, {x_2}^k, \ldots, {x_n}^k)}{||(1, {x_2}^k, \ldots, {x_n}^k)||} \to (1, 0, 0, \ldots, 0),$$

any accumulation point of $\{\pi(p_k)/||\pi(p_k)||\}$ have the form $\lambda(1, 0, \ldots, 0)$ with $\lambda \in \mathbb{C}\backslash\{0\}$. Therefore $\{\pi(p_k)\}$ is tangent to L_p in 0. Conversely, suppose that $\{\pi(p_k)\}$ is tangent to L_p in 0. Let $\pi(p_k) = z^k = (z_1^k, ..., z_n^k)$. Since any accumulation point of

$$\frac{z^k}{||z^k||} = (\frac{z_1^k}{||z^k||}, ..., \frac{z_n^k}{||z^k||})$$

is contained in $L_p = \{z_2 = 0, ..., z_n = 0\}$, Then for k big enough and all $j = 2, ..., n$ we have that $\frac{z_j^k}{||z^k||} < \epsilon$, with $\epsilon > 0$ arbitrary. Then

$$||z^k|| \leq |z_1^k| + \ldots + |z_n^k| \leq |z_1^k| + (n-1)\epsilon||z^k||,$$

hence $|z_1^k| \geq (1 - (n-1)\epsilon)||z^k||$ and therefore $z_1^k \neq 0$ if ϵ is small enough. Thus, we may assume that $p_k = (x_1^k, ..., x_n^k) \in \pi^{-1}(U_1)$. Then $(x_1^k, x_1^k x_2^k, \ldots, x_1^k x_n^k) \to 0$ and therefore ${x_1}^k \to 0$. On the other hand, since any accumulation point of

$$\frac{\pi(p_k)}{||\pi(p_k)||} = \frac{{x_1}^k}{|{x_1}^k|} \frac{(1, {x_2}^k, \ldots, {x_n}^k)}{||(1, {x_2}^k, \ldots, {x_n}^k)||}$$

is contained in $L_p \subset \mathbb{C}^n$, then the same property holds for the sequence

$$\frac{(1, {x_2}^k, \ldots, {x_n}^k)}{||(1, {x_2}^k, \ldots, {x_n}^k)||},$$

hence ${x_i}^k \to 0$ for all $i = 2, 3, \ldots, n$ and therefore $p_k \to p$.

proof of Proposition 5.1. Let $p \in \pi^{-1}(0)$ and $\{p_k\}$ any sequence of points in $\pi^{-1}(U)\backslash\pi^{-1}(0)$ such that $p_k \to p$.

Since $h \in C^1$, we have

$$\mathrm{h}(\pi(p_k)) = d\,\mathrm{h}_0(\pi(p_k)) + r(\pi(p_k)), \text{ where } \frac{r(\pi(p_k))}{||\pi(p_k)||} \to 0 \text{ as } k \to 0.$$

Then

$$\frac{\mathrm{h}(\pi(p_k))}{||\pi(p_k)||} = d\,\mathrm{h}_0\left(\frac{(\pi(p_k))}{||\pi(p_k)||}\right) + \frac{r(\pi(p_k))}{||\pi(p_k)||}. \tag{11}$$

By proposition 5.5, $\pi(p_k)$ is tangent to L_p at 0, hence any point of accumulation of the sequence $\{(\pi(p_k))/\|\pi(p_k)||\}$ is contained in L_p. Thus, it is easy to see from equation

(11) that any point of accumulation of the sequence $\{\mathrm{h}(\pi(p_k))/||\pi(p_k)||\}$ is contained in $d\,\mathrm{h}_0(L_p)$ and the same holds for the sequence

$$\frac{\mathrm{h}(\pi(p_k))}{||\,\mathrm{h}(\pi(p_k))||} = \frac{\mathrm{h}(\pi(p_k))}{||\pi(p_k)||}\frac{||\pi(p_k)||}{||\,\mathrm{h}(\pi(p_k))||}.$$

From proposition 5.2, we have that $d\,\mathrm{h}_0(L_p)$ is a complex line. Then $\{\mathrm{h}(\pi(p_k)\})$ is tangent to $d\,\mathrm{h}_0(L_p)$ at 0. It follows by proposition 5.5 that $\pi^{-1}\circ\mathrm{h}\circ\pi(p_k) = h(p_k) \to q$, where $q \in \pi^{-1}(0)$ is such that $L_q = d\,\mathrm{h}_0(L_p)$. We remark that $q = H(p)$, where H is the diffeomorphism of $\pi^{-1}(0) \simeq \mathbb{P}^{n-1}$ onto itself induced by $d\,\mathrm{h}_0$ (5.2). We define $h(p) = H(p)$ for all p in $\pi^{-1}(0)$. Finally, we prove that $h : \pi^{-1}(0) \to \pi^{-1}(0)$ is a homeomorphism. Let $p \in \pi^{-1}(0)$ and let p_k in $\widehat{\mathbb{C}^n}$ such that $p_k \to p$ as $k \to \infty$. Let x be a limit point of the sequence $\{h(p_k)\}$, that is, $x = \lim h(q_k)$, where $\{q_k\}$ is a subsequence of $\{p_k\}$. Clearly, there exists a subsequence $\{r_k\}$ of $\{q_k\}$ such that: either $\{r_k\}$ is contained in $\pi^{-1}(0)$, or $\{r_k\}$ is contained in $\pi^{-1}(U)\backslash\pi^{-1}(0)$. If $\{r_k\}$ is contained in $\pi^{-1}(0)$ we have that $h(r_k) = H(r_k)$ for all k and

$$x = \lim_{k\to\infty} h(q_k) = \lim_{k\to\infty} h(r_k) = \lim_{k\to\infty} H(r_k) = H(p) = h(p),$$

since H is continuous. On the other hand, if $\{r_k\}$ is contained in $\pi^{-1}(U)\backslash\pi^{-1}(0)$, we have from above that

$$x = \lim_{k\to\infty} h(r_k) = H(p) = h(p).$$

Therefore h is continuous. Analogously we prove that h^{-1} is also continuous.

Part II

The differentiable case

6 Introduction

Let $\mathcal{F}$ be a holomorphic foliation by curves of a neighborhood U of $0 \in \mathbb{C}^2$ with a unique singularity at $0 \in \mathbb{C}^2$. We assume that $\mathcal{F}$ is generated by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i},\ a_i \in \mathcal{O}_U,\ g.c.d.(a_1, a_2, \ldots, a_n) = 1.$$

The algebraic multiplicity of $\mathcal{F}$ (at $0 \in \mathbb{C}^2$) is the minimum vanishing order at $0 \in \mathbb{C}^2$ of the functions a_i. Let $\widetilde{\mathcal{F}}$ be a holomorphic foliation by curves of a neighborhood $\widetilde{U}$ of $0 \in \mathbb{C}^2$ and Let $\mathrm{h} : U \to \widetilde{U}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, that is, a homeomorphism taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. A natural question, posed by J.F.Mattei is: are the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ the same?. In [2], the authors give a positive answer if $\mathcal{F}$ is a *generalized curve*, that is, if the desingularization of $\mathcal{F}$ does not contain complex saddle-nodes. If $\mathcal{F}$ is *dicritical*, that is, after a blow up the exceptional divisor is not invariant by the strict transform of $\mathcal{F}$, the conjecture is also true: in this case, it is not difficult to show that the algebraic multiplicity of $\mathcal{F}$ is equal to the index of $\mathcal{F}$ (as defined in [2]) along a generic separatrix. Then the topological invariance of the algebraic multiplicity of a dicritical singularity is a consequence of the topological invariance of the index along a curve, which is proved in [2]. Thus, from now on we assume that $\mathcal{F}$ is nondicritical. In this work we impose conditions on the topological equivalence $\mathrm{h} : U \to \widetilde{U}$ and prove the following.

Theorem 6.1. *Let* $\mathrm{h} : U \to \widetilde{U}$ *be a topological equivalence between* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *and assume that* h *preserves the orientation of* $\mathbb{C}^2$. *Suppose that* h *is differentiable at* $0 \in \mathbb{C}^2$ *and such that* $d\,\mathrm{h}(0) : \mathbb{R}^4 \to \mathbb{R}^4$ *is a real isomorphism. Then the algebraic multiplicities of* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *are the same.*

Let $\pi : \widehat{\mathbb{C}^2} \to \mathbb{C}^2$ be the blow up at $0 \in \mathbb{C}^2$. Given a complex line P passing through $0 \in \mathbb{C}^2$, we say that P is *regular for* $\mathcal{F}$, if the strict transform of P by π intersects the divisor E at a regular point of the strict transform of $\mathcal{F}$. The following theorem is a key step in the proof of Theorem 6.1.

Theorem 6.2. *Let* $\mathrm{h} : U \to \widetilde{U}$ *be a topological equivalence between* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *and assume that* h *preserves the orientation of* $\mathbb{C}^2$. *Let* P *and* $\widetilde{P}$ *be two complex lines passing through* $0 \in \mathbb{C}^2$ *which are regular for* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *respectively. Suppose that* $P \cap U$ *is homeomorphic to a disc and* $\mathrm{h}(P \cap U) = \mathrm{h}(\widetilde{P} \cap \widetilde{U})$. *Then the algebraic multiplicities of* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *are equal.*

The paper is organized as follows. In section 7 we prove a weaker version of Theorem 6.2. In section 8 we stay and prove a topological lemma, fundamental for the following sections. We prove Theorem 6.2 in section 9. Finally, in section 10 we prove Theorem 6.1.

7 A first theorem.

Let $\mathrm{h} : U \to \widetilde{U}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Let $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ be the strict transforms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively. Let W and $\widetilde{W}$ be denote the sets $\pi^{-1}(U)$ and

$\pi^{-1}(\widetilde{U})$ respectively. Let

$$h : W\backslash E \to \widetilde{W}\backslash E$$

be the homeomorphism defined by $h = \pi^{-1}\,\mathrm{h}\,\pi$. We have a natural fibration ρ on $\widehat{\mathbb{C}^2}$ which fibers are the strict transforms of the complex lines passing through $0 \in \mathbb{C}^2$. Consider $p, \widetilde{p} \in E$ and let L_p and $L_{\widetilde{p}}$ be the fibers of ρ passing through p and $\widetilde{p}$ respectively. This section is devoted to prove the following.

Theorem 7.1. *Suppose that* p *and* $\widetilde{p}$ *are regular points of* $\mathcal{F}_0$ *and* $\widetilde{\mathcal{F}}_0$ *respectively. Let* Ω *be a neighborhood of* p *in* $\widehat{\mathbb{C}^2}$. *Suppose that* h *extends to* $(W\backslash E) \cup \Omega$ *as a homeomorphism onto its image, such that* $h(L_p \cap W) = L_{\widetilde{p}} \cap \widetilde{W}$. *Then the algebraic multiplicities of* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *are the same.*

Let ν be the algebraic multiplicity of $\mathcal{F}$ at 0 and let $p_1, ..., p_k$ be the singularities of $\mathcal{F}_0$ on E. We have the following relation due to Van Den Essen:

$$\sum_{i=1}^{k} \mu(\mathcal{F}_0, p_i) = \mu(\mathcal{F}, 0) - \nu^2 + \nu + 1,$$

where $\mu(\mathcal{F}, p)$ is the Milnor number of $\mathcal{F}$ at p. Let $s = \sum_{i=1}^{k} \mu(\mathcal{F}_0, p_i)$. In the same way, let $\widetilde{s}$ be the sum of the Milnor numbers of the singularities on E of $\widetilde{\mathcal{F}}_0$. Then, since the Milnor number is a topological invariant, it is sufficient to prove that $s = \widetilde{s}$.

Let $\mathcal{D} \subset E \cap \Omega$ be a closed disc containing p, which does not contain singularities of $\mathcal{F}_0$ and such that $h(\mathcal{D})$ does not contains singularities of $\widetilde{\mathcal{F}}_0$. Let D and $\widetilde{D}$ be the closed discs in E equal to the closure of $E\backslash\mathcal{D}$ and $E\backslash h(\mathcal{D})$ respectively. Then h maps $W\backslash D$ homeomorphically onto $\widetilde{W}\backslash\widetilde{D}$, and the interiors of D and $\widetilde{D}$ contain all the singularities of $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ respectively. Observe that h is a topological equivalence between $\mathcal{F}_0|_{W\backslash D}$ and $\widetilde{\mathcal{F}}_0|_{\widetilde{W}\backslash\widetilde{D}}$. Since $h(L_p \cap W) = L_{\widetilde{p}} \cap \widetilde{W}$, we have the homeomorphism

$$h : (W\backslash D)\backslash L_p \to (\widetilde{W}\backslash\widetilde{D})\backslash L_{\widetilde{p}}.$$

We know that $W\backslash L_p$ and $\widetilde{W}\backslash L_{\widetilde{p}}$ are isomorphic to $\mathbb{C}^2$, where the divisor can be represented by the vertical line $\{z_1 = 0\}$ and the sets $W\backslash L_p$ and $\widetilde{W}\backslash L_{\widetilde{p}}$ give neighborhoods V and $\widetilde{V}$ of $\{z_1 = 0\}$. Thus, we may think that the foliations $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ are defined on the sets V and $\widetilde{V}$ in $\mathbb{C}^2$, and that

$$h : V\backslash D \subset \mathbb{C}^2 \to \widetilde{V}\backslash\widetilde{D} \subset \mathbb{C}^2$$

is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. Observe that $\mathcal{F}_0$ is globally defined by a holomorphic vector field on V and the same holds for $\widetilde{\mathcal{F}}_0$ on $\widetilde{V}$. The disc D is contained in $\{z_1 = 0\}$ and we may assume that $D = \{(0, z_2) : |z_2| \leq r\}$, where $r > 0$.

We proceed now to compute s. Let Z be a holomorphic vector field, which generates the foliation $\mathcal{F}_0$ on V. Let B be a neighborhood of D homeomorphic to a ball, such that ∂B is homeomorphic to S^3 and $\overline{B} \subset V$. Since all the singularities of $\mathcal{F}_0$ are contained in $D \subset B$, it can be proved that the sum of the Milnor numbers of the singularities of $\mathcal{F}_0$ is equal to the degree of the map

$$\frac{Z}{||Z||} : \partial B \to \mathbb{S}^3,$$

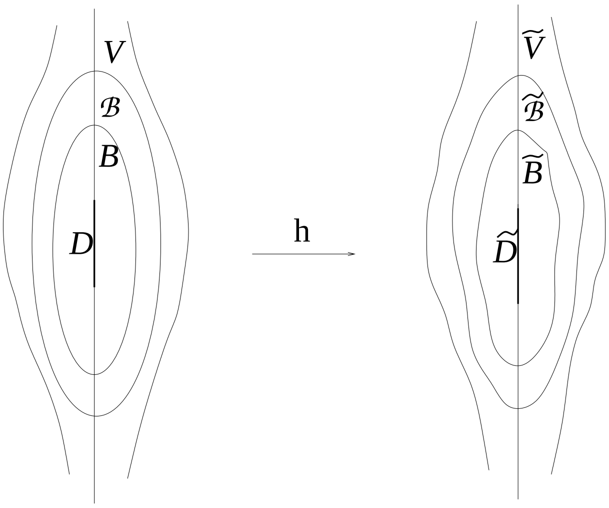
$$\frac{Z}{||Z||}(z) = \frac{Z(z)}{||Z(z)||}.$$

Let $\mathcal{B}$ be a neighborhood of $\overline{B}$ homeomorphic to a ball and such that $\overline{\mathcal{B}} \subset V$. Since V is a neighborhood of $\{z_1 = 0\}$, for $\varepsilon > 0$ small enough, the set $\{|z_1| < 2\varepsilon, |z_2| < 4r\}$, which contains D, is contained in V. Then, we may chose B and $\mathcal{B}$ such that

$$\mathcal{B} \subset \{|z_1| < \varepsilon, |z_2| < 2r\}.$$

The last hypothesis will be used only in the proof of Lemma 7.5.

Consider the sets $\widetilde{B} = h(B\backslash D) \cup \widetilde{D}$, $\widetilde{\mathcal{B}} = h(\mathcal{B}\backslash D) \cup \widetilde{D}$ and $\widetilde{V} = h(V\backslash D) \cup \widetilde{D}$. It is easy to see that $\widetilde{B}$, $\widetilde{\mathcal{B}}$ and $\widetilde{V}$ are neighborhoods of $\widetilde{D}$ in $\mathbb{C}^2$.



Let

$$\varphi : \overline{\mathbb{D}}_\varepsilon \times \overline{\mathcal{B}} \to V \subset \mathbb{C}^2$$

and

$$\widetilde{\varphi} : \overline{\mathbb{D}}_\varepsilon \times \widetilde{\mathcal{B}} \to \widetilde{V} \subset \mathbb{C}^2$$

be the local complex flows of Z and $\widetilde{Z}$ respectively, where $\mathbb{D}_\varepsilon = \{T \in \mathbb{C} : ||T|| < \varepsilon\}$ with ε small enough. Now, we follow the ideas used in [2] to prove the topological invariance of the Milnor number.

Lemma 7.2. *There exists continuous functions $\tau : \mathcal{B}\backslash D \to (0, \varepsilon)$ and $\widetilde{\tau} : h(\mathcal{B}\backslash D) \to \mathbb{D}_\varepsilon\backslash\{0\}$ such that for all $z \in \mathcal{B}\backslash D$ we have:*

(i) $\varphi(\tau(z), z) \in \mathcal{B}\backslash D$.

(ii) $\varphi(t\tau(z), z) \neq z$, *for any* $t \in (0, 1]$.

(iii) $h(\varphi(\tau(z), z)) = \widetilde{\varphi}(\widetilde{\tau}(h(z)), h(z))$.

We say that a function $f : U \to \mathbb{R}$ is *lower(upper) semi-continuous* if given $\epsilon > 0$ and $x_0 \in U$, there is a neighborhood Ω of x_0 in U such that $f(x) \geq f(x_0) - \epsilon$ ($f(x) \leq f(x_0) + \epsilon$) for all $x \in \Omega$. We need the following lemma.

Lemma 7.3. *Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ be an upper and a lower semicontinuous function respectively. Suppose that $f < g$. Then there exists a continuous function $h : U \to \mathbb{R}$ such that $f < h < g$. In particular, if g is a strictly positive lower semicontinuous function, then there exists a continuous function h such that $0 < h < g$.*

Proof of Lemma 7.2. Clearly, given $z \in \mathcal{B}\backslash D$ there exists $\delta > 0$ such that $\varphi(*, z)$ is injective on $\mathbb{D}_\delta$. Then define $\delta(z) > 0$ as the supreme of $\delta' \leq \varepsilon$ such that $\varphi(*, z)$ is injective on $\mathbb{D}_{\delta'}$.

Assertion 1. *The function $\delta : \mathcal{B}\backslash D \to (0, \varepsilon]$ is lower semicontinuous.*

Proof. Fix $z_0 \in \mathcal{B}\backslash D$ and let $\epsilon > 0$. We will prove that for z close enough to z_0 we have $\delta(z) \geq \delta(z_0) - \epsilon$. Suppose by contradiction that for $z_k \to z_0$ we have that $\varphi(*, z_k)$ is not injective on $\mathbb{D}_{\delta(z_0)-\epsilon}$. Then there are points t_k, t'_k in $\mathbb{D}_{\delta(z_0)-\epsilon}$, with $t_k \neq t'_k$ and such that $\varphi(t_k, z_k) = \varphi(t'_k, z_k)$ for all k. By taking a subsequence we may assume that $t_k \to a$ and $t'_k \to a'$ with $a, a' \in \overline{\mathbb{D}}_{\delta(z_0)-\epsilon} \subset \mathbb{D}_{\delta(z_0)}$. By continuity we have

$$\varphi(a, z_0) = \lim_{k\to\infty} \varphi(t_k, z_k) = \lim_{k\to\infty} \varphi(t'_k, z_k) = \varphi(a', z_0)$$

and, since $\varphi(*, z_0)$ is injective on $\mathbb{D}_{\delta(z_0)}$, we deduce that $a = a'$. Let $z' = \varphi(a, z_0)$ and take a neighborhood Ω of z' and $\delta_0 > 0$ such that $\varphi(*, z)$ is injective on $\mathbb{D}_{\delta_0}$ for all $z \in \Omega$. For k big enough we have that $\varphi(a, z_k) \in \Omega$ and $(t_k - a), (t'_k - a') \in \mathbb{D}_{\delta_0}$. Then, since

$$\varphi(t_k - a, \varphi(a, z_k)) = \varphi(t_k, z_k) = \varphi(t'_k, z_k) = \varphi(t'_k - a', \varphi(a', z_k)),$$

we have that $t_k - a = t'_k - a'$, hence $t_k = t'_k$, which is a contradiction.

Assertion 2. *Consider $\bar{\delta} : \mathcal{B}\backslash D \to (0, \varepsilon]$, where $\bar{\delta}(z)$ is the supreme of $\delta' < \varepsilon$ such that $\varphi(\mathbb{D}_{\delta'}, z) \subset \mathcal{B}\backslash D$. Then $\bar{\delta}$ is a lower semicontinuous function.*

Proof. Fix z_0 and let $\epsilon > 0$. The set $\varphi(\overline{\mathbb{D}}_{\bar{\delta}(z_0)-\epsilon}, z_0)$ is compact and is contained in $\mathcal{B}\backslash D$. If z is close enough to z_0 we have that $\varphi(\overline{\mathbb{D}}_{\bar{\delta}(z_0)-\epsilon}, z)$ is also contained in $\mathcal{B}\backslash D$. Then $\bar{\delta}(z) \geq \bar{\delta}(z_0) - \epsilon$ and it follows that $\bar{\delta}$ is lower semicontinuous.

Consider $\widetilde{\delta} : h(\mathcal{B}\backslash D) \to (0, \varepsilon]$, where $\widetilde{\delta}(w)$ is the supreme of $\delta' < \varepsilon$ such that $\widetilde{\varphi}(*, w)$ is injective on $\mathbb{D}_{\delta'}$. As in Assertion 1, we can prove that $\widetilde{\delta}$ is a lower semicontinuous function.

Assertion 3. *Define $\hat{\delta} : \mathcal{B}\backslash D \to (0, \varepsilon]$, where $\hat{\delta}(z)$ is the supreme of $\delta' < \varepsilon$ such that $h(\varphi(\mathbb{D}_{\delta'}, z))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z))}, h(z))$. Then $\hat{\delta}$ is a lower semicontinuous function.*

Proof. Fix z_0 and let $\epsilon > 0$. Since $h(\varphi(\mathbb{D}_{\hat{\delta}(z_0)}, z_0))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))}, h(z_0))$, there is $\epsilon' > 0$ such that $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z_0))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'}, h(z_0))$. Let Σ be a disc passing through $h(z_0)$ and transverse to the foliation. Since $\widetilde{\delta}$ is lower semicontinuous, we may take Σ small enough such that $\widetilde{\varphi}(*, z)$ is injective on $\overline{\mathbb{D}}_{\widetilde{\delta}(h(z_0))-\epsilon'}$ for all $z \in \Sigma$. Moreover, we may take Σ small enough such that $\widetilde{\varphi}$ is injective on

$\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'} \times \Sigma$. Let M denote the open set $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'} \times \Sigma)$ and let $M' = \widetilde{\varphi}(\mathbb{D}_{\epsilon'/2} \times \Sigma)$. We may take a neighborhood Ω of z_0 such that $h(\Omega) \subset M'$ and $\widetilde{\delta}(h(z)) \geq \widetilde{\delta}(h(z_0)) - \epsilon'/2$ for all $z \in \Omega$, because $\widetilde{\delta}$ is lower semicontinuous. Since $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z_0))$ is compact and is contained in M, we may assume Ω small enough such that $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$ is contained in M for all $z \in \Omega$. Fix $z \in \Omega$. Since $h(z) \in M'$, there is $w' \in \Sigma$ and t', with $|t'| < \epsilon'/2$, such that $h(z) = \widetilde{\varphi}(t', w')$. Since $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$ is contained in M, we deduce that it is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z_0))-\epsilon'}, w')$. Then, given w in $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$, we have that $w = \widetilde{\varphi}(t'', w')$ with $|t''| < \widetilde{\delta}(h(z_0)) - \epsilon'$. Thus

$$w = \widetilde{\varphi}(t'', w') = \widetilde{\varphi}(t'' - t', \widetilde{\varphi}(t', w')) = \widetilde{\varphi}(t'' - t', h(z)),$$

where $|t'' - t'| \leq |t''| + |t'| < \widetilde{\delta}(h(z_0)) - \epsilon' + \epsilon'/2 = \widetilde{\delta}(h(z_0)) - \epsilon'/2 \leq \widetilde{\delta}(h(z))$. Then $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z))}, h(z))$ and it follows that $\hat{\delta}$ is lower semicontinuous.

It is easy to see that the function $g = \min\{\delta, \bar{\delta}, \hat{\delta}\}$ is also lower semicontinuous. Then, by Lemma 7.3, there exists a positive continuous function τ on $\mathcal{B}\backslash D$ such that $\tau < \delta, \bar{\delta}, \hat{\delta}$. By the definition of $\bar{\delta}$, (i) is satisfied. Since $\varphi(*, z)$ is injective on $\mathbb{D}_{\bar{\delta}}$ and $\tau(z) \in \mathbb{D}_{\bar{\delta}}$, we have that (ii) holds. Now, we shall define $\widetilde{\tau}$. Let $w = h(z) \in h(\mathcal{B}\backslash D)$. Since $\tau < \hat{\delta}$, we have that $h(\varphi(\tau(z), z))$ is contained in $\widetilde{\varphi}(\mathbb{D}_{\widetilde{\delta}(h(z))}, h(z))$ and by injectivity there exists a unique $\widetilde{\tau}(h(z))$ in $\mathbb{D}_{\widetilde{\delta}(h(z))}$ such that $h(\varphi(\tau(z), z)) = \widetilde{\varphi}(\widetilde{\tau}(h(z)), h(z))$. Now, it is easy to see that $\widetilde{\tau}$ is continuous and therefore (iii) holds. $\square$

Proof of Lemma 7.3. Consider $x \in U$ and $a_x \in \mathbb{R}$, such that $f(x) < a_x < g(x)$. It follows from the definition of lower and upper semicontinuous function that there exists a neighborhood V_x of x in U such that $f(y) < a_x < g(y)$ for all $y \in V_x$. We may take a subset $I \subset U$, such that $U \subset \bigcup_{i \in I} V_i$ and $\{V_i\}_{i \in I}$ is locally finite. Thus, we have $f(x) < a_i < g(x)$ for all $x \in V_i$. Let $\{\psi_i\}_{i \in I}$ be a partition of the unity subordinate to $\{V_i\}_{i \in I}$. Then, we define $h : U \to \mathbb{R}$ by

$$h(x) = \sum_{i \in I} \psi_i(x) a_i.$$

Clearly, h is continuous. If $x \in V_i$, then $f(x) < a_i < g(x)$, hence $\psi_i(x) f(x) < \psi_i(x) a_i < \psi_i(x) g(x)$ and it follows that $f < h < g$. $\square$

From Lemma 7.2, we have the maps

$$f : \mathcal{B}\backslash D \to \mathcal{B}\backslash D,$$

$$f(z) = \varphi(\tau(z), z)$$

and

$$\widetilde{f} : \widetilde{\mathcal{B}}\backslash \widetilde{D} \to \widetilde{\mathcal{B}}\backslash \widetilde{D},$$

$$\widetilde{f}(w) = \widetilde{\varphi}(\widetilde{\tau}(w), w)$$

with

$$h \circ f = \widetilde{f} \circ h$$

and such that f and $\widetilde{f}$ are without fixed points.

There exists $\psi, \widetilde{\psi} : \mathbb{C}^2 \to \mathbb{C}^2$ with the following properties:

(*i*) $\psi(D) = 0$ and $\widetilde{\psi}(\widetilde{D}) = 0$.

(*ii*) $\psi : \mathbb{C}^2 \backslash D \to \mathbb{C}^2 \backslash \{0\}$ and $\widetilde{\psi} : \mathbb{C}^2 \backslash \widetilde{D} \to \mathbb{C}^2 \backslash \{0\}$ are homeomorphisms.

(*iii*) ψ and $\widetilde{\psi}$ are equal to the identity out of B and $\widetilde{B}$ respectively.

We define

$$\begin{array}{rcl} f' & = & \psi f \psi^{-1} : \mathcal{B} \backslash \{0\} \to \mathcal{B} \backslash \{0\} \subset \mathbb{C}^2, \\ \widetilde{f}' & = & \widetilde{\psi} \widetilde{f} \widetilde{\psi}^{-1} : \widetilde{\mathcal{B}} \backslash \{0\} \to \widetilde{\mathcal{B}} \backslash \{0\} \subset \mathbb{C}^2, \\ h' & = & \widetilde{\psi} h \psi^{-1} : V \to \widetilde{V}. \end{array}$$

Then we have the following:

(*i*) f' and $\widetilde{f}'$ do not have fixed points.

(*ii*) On ∂B, we have $f' = f$ and $\widetilde{f}' = \widetilde{f}$.

(*iii*) h' is a homeomorphism with $h'(0) = 0$ and such that $h' \circ f' = \widetilde{f}' \circ h'$.

Thus, there are well defined maps:

$$\begin{array}{rcl} (f' - \mathrm{id}) & : & \mathcal{B} \backslash \{0\} \to \mathbb{C}^2 \backslash \{0\}, \\ (\widetilde{f}' - \mathrm{id}) & : & \widetilde{\mathcal{B}} \backslash \{0\} \to \mathbb{C}^2 \backslash \{0\}. \end{array}$$

Observe that $H_3(\mathcal{B} \backslash \{0\}) \subset H_3(\mathbb{C}^2 \backslash \{0\})$ and this inclusion is an isomorphism between the groups. Then $(f' - \mathrm{id})$ induces a map

$$(f' - \mathrm{id})_* : H_3(\mathbb{C}^2 \backslash \{0\}) \to H_3(\mathbb{C}^2 \backslash \{0\})$$

at the homology level.

Lemma 7.4. *$(f' - \mathrm{id})_*$ is the multiplication by s.*

Proof. We have that $\partial B \subset \mathcal{B}$ is a generator of $H_3(\mathbb{C}^2 \backslash \{0\})$. It is known that, homologically:

$$(f' - \mathrm{id})(\mathbb{S}^3) = (f' - \mathrm{id})(\partial B) = n \mathbb{S}^3,$$

where n is the degree of the map:

$$g : \partial B \to \mathbb{S}^3,$$

$$g(z) = \frac{(f' - \mathrm{id})}{||(f' - \mathrm{id})||}(z).$$

Thus, it is sufficient to prove that $\deg(g) = s$. Observe that $g = \frac{(f-\mathrm{id})}{||(f-\mathrm{id})||}$, since $f' = f$ on ∂B. By (*ii*) of Lemma 7.2 the map

$$G : [0,1] \times \partial B \to \mathbb{S}^3,$$

$$G(t,z) = \frac{\varphi(t\tau(z),z) - z}{||\varphi(t\tau(z),z) - z||}, \quad t \neq 0,$$

$$G(0,z) = \frac{\tau(z)}{||\tau(z)||} \cdot \frac{Z(z)}{||Z(z)||}$$

is well defined. Evidently, $G(1,z) = g(z)$. On the other hand:

$$\begin{aligned} \lim_{t\to 0} G(t,z) &= \frac{\tau(z)}{||\tau(z)||} \lim_{t\to 0} \left|\left| \frac{\varphi(t\tau(z),z) - z}{t\tau(z)} \right|\right|^{-1} \cdot \lim_{t\to 0} \frac{\varphi(t\tau(z),z) - z}{t\tau(z)} \\ &= \frac{\tau(z)}{||\tau(z)||} \lim_{s\to 0} \left|\left| \frac{\varphi(s,z) - z}{s} \right|\right|^{-1} \cdot \lim_{s\to 0} \frac{\varphi(s,z) - z}{s} \\ &= \frac{\tau(z)}{||\tau(z)||} \cdot \frac{Z(z)}{||Z(z)||}. \end{aligned}$$

It follows that G is continuous and therefore is a homotopy between $g(z)$ and $G(0,z) = \frac{\tau(z)}{||\tau(z)||} \cdot \frac{Z(z)}{||Z(z)||}$. Now, since $\pi_3(\mathbb{S}^1) = \{0\}$, the map $\tau/|\tau| : \partial B \to \mathbb{S}^1$ is homotopic to the constant $1 \in \mathbb{S}^1$ and g is homotopic to $Z/||Z||$. Therefore $\deg(g) = \deg(Z/||Z||) = s$.

In the same way, we have that

$$(\widetilde{f'} - \mathrm{id})_* : H_3(\mathbb{C}^2\backslash\{0\}) \to H_3(\mathbb{C}^2\backslash\{0\})$$

is the multiplication by $\widetilde{s}$.

Let

$$h'_* : H_3(\mathbb{C}^2\backslash\{0\}) \to H_3(\mathbb{C}^2\backslash\{0\})$$

be the isomorphism induced by h'. Clearly, the following lemma implies Proposition 6.2.

Lemma 7.5. *The following diagram commutes:*

$$\begin{tikzcd} H_3(\mathbb{C}^2\backslash\{0\}) \arrow[r, "(f'-\mathrm{id})_*"] \arrow[d, "h'_*"] & H_3(\mathbb{C}^2\backslash\{0\}) \arrow[d, "h'_*"] \\ H_3(\mathbb{C}^2\backslash\{0\}) \arrow[r, "(\widetilde{f'}-\mathrm{id})_*"] & H_3(\mathbb{C}^2\backslash\{0\}) \end{tikzcd}$$

Proof. Recall that $\mathcal{B}$ was chosen such that

$$\mathcal{B} \subset \{|z_1| < \varepsilon, |z_2| < 2r\} \subset \{|z_1| < 2\varepsilon, |z_2| < 4r\} \subset V.$$

Since $h' \circ f' = \widetilde{f'} \circ h'$ we have $(\widetilde{f'} - \mathrm{id}) \circ h' = \widetilde{f'} \circ h' - h' = h' \circ f' - h'$. It is sufficient to prove that $h' \circ f' - h'$ and $h' \circ (f' - \mathrm{id}) : \mathcal{B}\backslash\{0\} \to \mathbb{C}^2\backslash\{0\}$ are homotopic. For any

$z \in \mathcal{B}\backslash\{0\}$ and $t \in [0,1]$ we have that $f'(z), (1-t)z \in \mathbb{D}_\epsilon \times \mathbb{D}_{2r}$. Then $(f'(z)+(1-t)z)$ is contained in $\mathbb{D}_{2\epsilon} \times \mathbb{D}_{4r} \subset V$. Therefore, the map:

$$F : [0,1] \times (\mathcal{B}\backslash\{0\}) \to \mathbb{C}^2\backslash\{0\},$$

$$F(t,z) = h'(f'(z) - (1-t)z) - h'(tz)$$

is well defined. F is continuous and $F(t,z) \neq 0$ for all $(t,z) \in [0,1] \times (\mathcal{B}\backslash\{0\})$ because $F(t,z) = 0$ implies $h'(f'(z) - (1-t)z) = h'(tz)$ and since h' is a homeomorphism $f'(z) - (1-t)z = tz$, hence $f'(z) = z$, which contradicts $f'(z) \neq z$. Thus F is a homotopy between $h' \circ f' - h'$ and $h' \circ (f' - \mathrm{id})$. ☐

8 A topological fact.

Let $\pi : \widehat{\mathbb{C}^2} \to \mathbb{C}^2$ be the blow up at $0 \in \mathbb{C}^2$ and let $E = \pi^{-1}(0)$. Let $\rho : \widehat{\mathbb{C}^2} \to E$ be the natural projection. If $x \in E$, the set $\rho^{-1}(x)$ is the strict transform of a complex line passing through $0 \in \mathbb{C}^2$ and $x \in \rho^{-1}(x)$. Let M be a complex manifold. We say that $\mathcal{D}$ is a complex disc in M, if $\mathcal{D} \subset M$ and there is a map $f : \overline{\mathbb{D}} \to M$, which is a homeomorphism onto $\mathcal{D}$ and is holomorphic on $\mathbb{D}$. Let V be any subset of M containing $\partial\mathcal{D}$. The map $f|_{S^1} : S^1 \to \partial\mathcal{D} \subset M$ defines a 1-cycle in V and represents an element in $H_1(V)$ which does not depend on f. We denote this 1-cycle by $\partial\mathcal{D}$ independly of the set V. For simplicity, we write $\gamma = \gamma'$ in $H_1(M)$ for means that the 1-cycles γ and γ' represents the same element in the group $H_1(M)$. The following Lemma is a reason for assuming that the topological equivalence h preserves the orientation of $\mathbb{C}^2$.

Lemma 8.1. *Let* $\mathrm{h} : U \to U'$ *be a homeomorphism, where* U *and* U' *are neighborhoods of* $0 \in \mathbb{C}^2$ *homeomorphic to balls. Let* P *and* P' *be two complex lines passing through* $0 \in \mathbb{C}^2$. *Suppose that* $P \cap U$ *is homeomorphic to a disc and* $\mathrm{h}(P \cap U) = P' \cap U'$. *Let* L *and* L' *be the strict transforms of* P *and* P' *respectively. Let* p *and* p' *be the points of intersection of* L *and* L' *with* E *respectively. Denote by* W *and* W' *the sets* $\pi^{-1}(U)$ *and* $\pi^{-1}(U')$ *in* $\widehat{\mathbb{C}^2}$ *and let* $h : W\backslash E \to W'\backslash E$ *be the homeomorphism defined by* $h = \pi^{-1}\,\mathrm{h}\,\pi$. *Let* $V \subset W$ *be a neighborhood of* p *and let*

$$\varphi : \mathbb{D} \times \mathbb{D} \to V$$

be a biholomorphism such that $\varphi(\{0\} \times \mathbb{D}) = L \cap V$ *and* $\varphi(\mathbb{D} \times \{0\}) = E \cap V$. *Let* r *with* $0 < r < 1$ *and consider the disc* $\mathcal{B}_w = \varphi(w, |z| \leq r)$, *where* $w \in \mathbb{D}$. *Let* Ω *be a neighborhood of* p' *in* E, *homeomorphic to a disc. Let* $V' \subset \widehat{\mathbb{C}^2}$ *be the set* $\rho^{-1}(\Omega)$. *Let* $\mathcal{A}' \subset V'\backslash E$ *and* $\mathcal{B}' \subset V'\backslash L'$ *be complex discs transverse to* L' *and* E *respectively. Then, for* $|w|$ *small enough we have the following:*

(i) If h *preserves the orientation of* $\mathbb{C}^2$, *then*

$$h(\partial\mathcal{B}_w) = \xi\partial\mathcal{B}' \quad \textit{in} \quad H_1(V'\backslash(L' \cup E)),$$

where $\xi = +1$ *or* -1.

(ii) If h *inverts the orientation of* $\mathbb{C}^2$*, then*

$$h(\partial\mathcal{B}_w) = -2\xi\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \textit{in} \quad H_1(V'\backslash(L'\cup E)),$$

where $\xi = +1$ *or* -1.

Remark. With some hypothesis on the foliation $\mathcal{F}$, we have in fact that the topological equivalence h necessarily preserves the orientation of $\mathbb{C}^2$. Precisely, we have the following.

Proposition 8.2. *Let* $\mathcal{F}$ *be a holomorphic foliation by curves on* U *which has* $0 \in \mathbb{C}^2$ *as its unique singularity. Suppose that* $\mathcal{F}$ *has three smooth and transverse separatrices. Suppose that* $\widetilde{\mathcal{F}}$ *is another holomorphic foliation of a neighborhood* $\widetilde{U}$ *of* $0 \in \mathbb{C}^2$ *and let*

$$\mathrm{h} : U \to \widetilde{U}$$

be a topological equivalence between $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$*. Then* h *preserves the orientation of* $\mathbb{C}^2$.

Let $U \subset \mathbb{C}^2$ be an open set homeomorphic to a ball. Let P be a complex line in $\mathbb{C}^2$ and suppose that $U \cap P$ is homeomorphic to a disc. It follows by Alexander's duality theorem that $H_1(U\backslash P) \simeq \mathbb{Z}$. Let $\mathcal{D} \subset \mathbb{C}^2$ be a complex disc transverse to P. The 1-cycle $\partial\mathcal{D}$ represents an element in $H_1(U\backslash P) \simeq \mathbb{Z}$, which does not depends on the disc $\mathcal{D}$. We know that $\partial\mathcal{D}$ is a generator of the group and we say that it is the *positive generator of* $H_1(U\backslash P)$. Given a homeomorphism $f : M \to M'$, where M and M' are oriented manifolds, we define $\deg(f)$ to be 1 or -1 depending on whether f preserves or reverses orientation.

Lemma 8.3. *Let* $\mathrm{h} : U \to U'$ *be a homeomorphism, where* U *and* U' *are neighborhoods of* $0 \in \mathbb{C}^2$ *homeomorphic to balls. Let* P *and* P' *be two complex lines passing through* $0 \in \mathbb{C}^2$*. Suppose that* $P \cap U$ *is homeomorphic to a disc and* $\mathrm{h}(P \cap U) = P' \cap U'$*. Let* a *and* a' *be 1-cycles in* $U\backslash P$ *and* $U'\backslash P'$ *representing the positive generators of* $H_1(U\backslash P)$ *and* $H_1(U'\backslash P')$ *respectively. Then*

$$\mathrm{h}(a) = \deg(\mathrm{h})\deg(\mathrm{h}|_P)a' \quad \textit{in} \quad H_1(U'\backslash P').$$

Proof of Lemma 8.1. If $\mathcal{B}'' \subset V'\backslash L'$ is any complex disc transverse to E, we have that $\partial\mathcal{B}''$ is homologous $\partial\mathcal{B}'$ in $H_1(V'\backslash(L'\cup E))$. Thus, we may change the disc $\mathcal{B}'$ if necessary and assume that it is contained in W'. Let b' be the 1-cycle defined by $b' = \pi(\partial\mathcal{B}')$. Then, since $\pi(\mathcal{B}') \subset U'$ is a complex disc transverse to P' and $\pi(\partial\mathcal{B}') = \partial\pi(\mathcal{B}')$, we have that b' is a positive generator of $H_1(U'\backslash P')$. Analogously, if $b = \pi(\partial\mathcal{B}_w)$, we deduce that b is a positive generator of $H_1(U\backslash P)$. It follows from Lemma 8.3 that:

$$\mathrm{h}(b) = \psi\xi b' \quad \text{in} \quad H_1(U'\backslash P'),$$

where $\psi = \deg(\mathrm{h})$ and $\xi = \deg(\mathrm{h}|_P)$. Then, since $\pi^{-1} : U'\backslash P' \to W'\backslash(L'\cup E)$ is well defined, we have that

$$\pi^{-1}(\mathrm{h}(b)) = \psi\xi\pi^{-1}(b') \quad \text{in} \quad H_1(W'\backslash(L'\cup E))$$

and thus

$$h(\partial\mathcal{B}_w) = \psi\xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L'\cup E)). \tag{12}$$

Observe that $\pi(\mathcal{A}')$ is a complex disc transverse to P'. Then the cycle $\partial\pi(\mathcal{A}') = \pi(\partial\mathcal{A}')$ represents the positive generator of $H_1(U'\backslash P')$. Thus, we deduce that $\pi(\partial\mathcal{A}') = \pi(\partial\mathcal{B}')$ in $H_1(U'\backslash P')$ and therefore

$$\partial\mathcal{A}' = \partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L' \cup E)). \tag{13}$$

Let $\mathcal{C}$ be the disc $\varphi(0, |z| \le r)$ in L. Let $\mathcal{C}'$ be a disc in L' containing p'. Since h maps $\mathcal{C}$ homeomorphically into L' with $h(p) = p'$, the cycle $h(\partial\mathcal{C})$ is a generator of the group $H_1(L'\backslash\{p'\})$ and we have $h(\partial\mathcal{C}) = \deg(h|_L)\partial\mathcal{C}'$. Thus, since $h|_L$ preserves orientation if an only if h$|_P$ does, we have that $h(\partial\mathcal{C}) = \xi\partial\mathcal{C}'$ in $H_1(L'\backslash\{p'\})$. Since $L'\backslash\{p'\}$ is contained in $V'\backslash E$, we conclude that

$$h(\partial\mathcal{C}) = \xi\partial\mathcal{C}' \quad \text{in} \quad H_1(V'\backslash E). \tag{14}$$

Observe that $\partial\mathcal{C}' = \partial\mathcal{B}'$ in $H_1(V'\backslash E)$. Moreover, $\partial\mathcal{C} = \varphi(0, |z| = r)$ is homologous to $\partial\mathcal{B}_w = \varphi(w, |z| = r)$ in the set $T = \varphi(|z| \le |w|, |z| = r)$. It is easy to see that for $|w|$ small enough, the set $h(T)$ is contained in $V'\backslash E$. Then $h(\partial\mathcal{C})$ and $h(\partial\mathcal{B}_w)$ are homologous in $V'\backslash E$. It follows from (14) and the observations above that for $|w|$ small enough:

$$h(\partial\mathcal{B}_w) = \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash E). \tag{15}$$

We know that there exists integers n and m such that

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + m\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash(L' \cup E)).$$

Then, since $V'\backslash(L' \cup E) \subset V'\backslash E$:

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + m\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash E),$$

hence

$$h(\partial\mathcal{B}_w) = m\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash E),$$

because $\partial\mathcal{A}' = 0$ in $H_1(V'\backslash E)$. From this and (15) we conclude that $m = \xi$. Then

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(V'\backslash(L' \cup E))$$

and, since $V'\backslash(E \cup L')$ is contained in $W'\backslash(E \cup L')$, we have that

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L' \cup E)). \tag{16}$$

From (13) we have $\partial\mathcal{A}' = \partial\mathcal{B}'$ in $H_1(W'\backslash(L' \cup E))$. Replacing in (16) we obtain:

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{B}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L' \cup E)).$$

Thus, from (12) we have:

$$\psi\xi\partial\mathcal{B}' = n\partial\mathcal{B}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W'\backslash(L' \cup E))$$

and therefore $n = (\psi - 1)\xi$. This proves the Lemma. $\square$

Proof of Proposition 8.2. It is known that the germ of three smooth and transverse curves is equivalent to the germ given by its tangents lines. Therefore we may assume that $\mathcal{F}$ has three transverse complex lines P_1, P_2 and P_3 as separatrices. Then $\mathrm{h}(P_1)$, $\mathrm{h}(P_2)$ and $\mathrm{h}(P_3)$ are smooth and transverse separatrices of $\widetilde{\mathcal{F}}$ and we can also assume that they are contained in complex lines $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$. By reducing U we may assume that $U \cap P_1$, $U \cap P_2$ and $U \cap P_3$ are homeomorphic to discs. We may take a neighborhood $\widetilde{U}' \subset \mathrm{h}(U)$ of $0 \in \mathbb{C}^2$ such that $\widetilde{U}' \cap \widetilde{P}_1$, $\widetilde{U}' \cap \widetilde{P}_2$ and $\widetilde{U}' \cap \widetilde{P}_3$ are homeomorphic to discs and are contained in $\mathrm{h}(U \cap P_1)$, $\mathrm{h}(U \cap P_2)$ and $\mathrm{h}(U \cap P_3)$ respectively. Then if we make $U' = \mathrm{h}^{-1}(\widetilde{U}')$, it is easy to see that $U' \cap P_1$, $U' \cap P_2$ and $U' \cap P_3$ are homeomorphic to discs and $\mathrm{h}(U' \cap P_1) = \widetilde{U}' \cap \widetilde{P}_1$, $\mathrm{h}(U' \cap P_2) = \widetilde{U}' \cap \widetilde{P}_2$, $\mathrm{h}(U' \cap P_3) = \widetilde{U}' \cap \widetilde{P}_3$. We may choose two of the complex lines P_1, P_2 and P_3, say us P_1 and P_2, such that $\deg(\mathrm{h}\,|_{P_1}) = \deg(\mathrm{h}\,|_{P_2})$. Let $\mathcal{D} \subset P_1$ be a disc containing $0 \in \mathbb{C}^2$. Then $\mathrm{h}(\partial \mathcal{D}) = \deg(\mathrm{h}\,|_{P_1}) \partial\, \mathrm{h}(\mathcal{D})$ in $H_1(\widetilde{P}_1 \cap \widetilde{U}' \backslash \{0\})$ and, since $\widetilde{P}_1 \cap \widetilde{U}' \backslash \{0\} \subset \widetilde{U}' \backslash \widetilde{P}_2$, we have that

$$\mathrm{h}(\partial \mathcal{D}) = \deg(\mathrm{h}\,|_{P_1}) \partial\, \mathrm{h}(\mathcal{D}) \quad \text{in} \quad H_1(\widetilde{U}' \backslash \widetilde{P}_2).$$

On the other hand, since $\partial \mathcal{D}$ and $\partial\, \mathrm{h}(\mathcal{D})$ are positive generators of $H_1(U' \backslash P_2)$