HIGH DIMENSION DIFFEOMORPHISMS EXHIBITING INFINITELY MANY STRANGE ATTRACTORS

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ABSTRACT. In this work we show, on a manifold of any dimension, that arbitrarily near any smooth diffeomorphism with a homoclinic tangency associated to a sectionally dissipative fixed (or periodic) point (i.e. the product of any pair of eigenvalues has norm less than 1), there exists a diffeomorphism exhibiting infinitely many Hénon-like strange attractors. In the two-dimensional case this has been proved in [2]. We also show a parameteric version of this result is true.

1. INTRODUCTION

The two-parameter Hénon family of transformations of the plane

$$h_{a,b}(x, y) = (1 - ax^2 + y, bx)$$

was studied by Hénon [3] to show, via a numerical approach, how a simple model of an invertible dynamical system suggests the presence of a non-hyperbolic strange attractor. However, the possibility that the attractor observed by Hénon was just a periodic orbit with very high period could not be excluded. In a remarkable work Benedicks and Carleson ([1]) showed that this is not the case and they exhibited a positive Lebesgue measure subset of parameters (a, b) for which the map $h_{a,b}$ has a nonhyperbolic strange attractor.

An important application of Benedicks-Carleson's methods ([1]) was done by Mora and Viana in [5] in the setting of homoclinic bifurcation on surfaces. More precisely, they showed that generic one-parameter families of surfaces diffeomorphisms unfolding a homoclinic tangency always include the presence, for a Lebesgue positive measure set of parameter values, of Hénon-like strange attractors or repellers.

The result in [5] was extended by Viana ([15]) to homoclinic bifurcations on manifolds of any dimension. Later on, Colli [2] showed a diffeomorphism of surfaces having a homoclinic tangency can be approximated by diffeomorphisms exhibiting not only a strange attractor, but also by diffeomorphisms displaying infinitely many of such strange attractors.

Our purpose in the present work is to extend the existence of infinitely many strange attractors in [2] to higher dimensions in its full generality of dissipative homoclinic bifurcations. Our main result is as follows

Theorem A. *Let $\varphi : M \mapsto M$ be a smooth diffeomorphism on any manifold with a homoclinic tangency associated to a sectionally dissipative point.*

Then, there exists an open set $\mathcal{U}$ of $\mathrm{Diff}^\infty(M)$ *containing φ in its closure, such that every $\psi \in \mathcal{U}$ can be approximated by a diffeomorphism exhibiting infinitely many nonhyperbolic strange attractors.*

In the statement above, *smooth* means that $\varphi : M \mapsto M$ is C^∞, M being a n-dimensional manifold. We also recall that a homoclinic tangency is just a tangency between the stable and unstable manifolds of a saddle periodic point. The saddle is called (*codimension-one*) *sectionally or strongly dissipative* if it has just one expanding eigenvalue and the product of any two eigenvalues has norm less than one. As in [15], we define *attractor* of a transformation φ to be a compact, φ-invariant and transitive set Λ whose basin $W^s(\Lambda) = \{z \in M : \mathrm{dist}(\varphi^n(z), \Lambda) \to 0 \text{ as } n \to \infty\}$ has nonempty interior. We call the attractor *strange* if it contains a dense orbit $\{\varphi^n(z_1) : n \geq 0\}$ displaying exponential growth of the derivative, that is,

$$\|D\varphi^n(z_1)\| \geq e^{cn} \text{ for all } n \geq 0 \text{ and some } c > 0$$

We also obtain a one-parameter version of Theorem A. More precisely,

Theorem B. *For a generic subset of smooth one-parameter families $\{\varphi_\mu\}$ of diffeomorphisms, on any manifold, that unfold a homoclinic tangency at parameter value $\mu = 0$ associated to a sectionally dissipative fixed (or periodic) point there exist sequences $I_n \to 0$ of intervals and dense subsets $E_n \subset I_n$ such that for all $\mu \in E_n$, the corresponding map φ_μ displays infinitely many nonhyperbolic strange attractors.*

By *smooth one-parameter family of diffeomorphism* we mean that $\Phi : \mathbb{R} \times M \mapsto M$, $\Phi(\mu, x) = (\mu, \varphi_\mu(x))$ is a C^∞ map and φ_μ is a diffeomorphism for all μ.

It is worth to point out that diffeomorphisms with the homoclinic tangencies are not only approximated by ones displaying the phenomenon described before but also for different ones. For instance, it has been shown that homoclinic tangencies are approximated by Newhouse's infinitely many sinks (attracting periodic orbits) ([6],[7]) and cascades of period doubling bifurcation ([16]). Still, it is conjectured that such an important phenomenon concerning infinitely many attractors might be rare for parameterized families of diffeomorphisms going through bifurcation of homoclinic tangencies: a conjecture in [8] and [9] states that for most parameter values, the corresponding diffeomorphisms display only finitely many attractors.

It is worth to point out also, that in the direction of existence of infinitely many strange attractors, some particular results have been found. In 1990 [4], Gambaudo-Tresser constructed an example of C^2 diffeomorphism in the two-dimensional disk exhibiting infinitely many hyperbolic strange attractors. In 2000 [12], Pumariño-Rodriguez exhibited a C^∞ family of vector fields in $\mathbb{R}^3$, related to a saddle-focus connection, which, with a positive Lebesgue measure set in the parameter values, displays infinitely many Henón-like strange attractors.

Among the difficulties to extend the result in [2] from two to higher dimensions we have that projections along the invariant foliations (in our case unstable foliations) of a basic set may not have a much regular metric behavior: in general, these projections are not Lipschitz but just Hölder continuous. We follow some ideas presented in [11] to bypass these difficulties and also to obtain further estimates necessary to prove Theorems A and B. On the other hand, to construct strange attractors we need to display a high dimensional renormalization scheme for heteroclinic tangencies in 2-cycles and then apply results in [15].

This work is organized as follows. In Section 2, we review the construction used to prove that infinitely many coexisting attracting periodic orbits for diffeomorphisms in high dimensions as presented in [11]. We take special care with the expansion and contraction rates of the basic sets involved. This chapter finishes with the Theorem 1.1 which summarizes the facts established in the previous sections. In Section 3, we proved some preliminary machinery to show the main theorems. In Section 3.1, we describe a higher dimension version of the renormalization scheme in 2-cycles of periodic points with a heteroclinic tangency in [2], following ideas in [10] and [15]. This renormalization scheme depends on a delicate relation between the contracting and expanding eigenvalue of periodic points involved. In section 3.2, we give a brief summary of the main result in [15] and derive several consequences of its proof. In section 3.3, we make a special perturbation for one-parameter families of diffeomorphisms to obtain new families which have linearizing coordinates in a neighborhood of the periodic points, as in section 3.1. Such perturbation is necessary since in the renormalization scheme of section 3.1 we assume that there exist linearizing coordinates in a neighborhood of the periodic points. In Section 4, we prove Theorems A and B. The proofs are consequence of the a main lemma showed in section 4.4. The proof of Theorem B is more delicate and we have to be more careful in applying the main lemma.

2. Preliminaries

In this section, we follow ideas and rewrite some results in [11] to create a language which we shall use in the proof of the main theorems. We start by giving a formal definition of stable thickness for a hyperbolic basic set whose stable foliation have codimension one. We show a condition given in [11] to obtain a basic hyperbolic set with "intrinsically" C^1 unstable foliations. Moreover, the projection along leaves of $W^u(\Lambda_1)$ is intrinsically C^1. In the next section we give a formal definition of unstable thickness for a hyperbolic basic set Λ_1 whose unstable foliation has codimension bigger than one. In this case we assume that Λ_1 has a periodic point displaying a unique weakest contracting eigenvalue. Later on, we show that we can obtain such a condition.

2.1. ***Cantor sets and thickness.*** A *Cantor set* in $\mathbb{R}$, is a compact, perfect and totally disconnected set. Let K be a Cantor set and I its *convex hull*, *i.e.* the minimal (closed) interval of $\mathbb{R}$ containing K. A *gap* of K is a connected component of $\mathbb{R} \setminus K$. A *presentation* of K is an ordering $\mathcal{U} = \{U_n\}_{n\geq 1}$ of the bounded gaps. An *ordered presentation* of K is a presentation $\mathcal{U}$ such that $\ell(U_n) \leq \ell(U_m)$ for all $n > m$, where $\ell(U_n)$ denoted of length of U_n. The *bridge* at $u \in \partial U_n$, $U_n \in \mathcal{U}$, is the component of $I \setminus (U_1 \cup \cdots \cup U_n)$ that contains u. The *thickness* of K is the number

$$\tau(K) = \inf\{\tau(K, \mathcal{U}, u) \ : \ u \in K\},$$

where $\mathcal{U}$ is any ordered presentation of K,

$$\tau(K, \mathcal{U}, u) = \frac{\ell(C)}{\ell(U_n)},$$

and where C is the bridge at $u \in \partial U_n$. this definition of thickness not depend on the ordered presentation $\mathcal{U}$ (see [10]).Let $k \in K$. The *local thickness* of K at k is the number

$$\tau(K, k) = \lim_{\varepsilon \to 0}\big(\sup\{\tau(\tilde{K}) \ : \ \tilde{K} \subset K \cap B_\varepsilon(k) \text{ a Cantor set }\}\big)$$

Let K_1, K_2 be Cantor sets and I_1, I_2 their convex hulls. We say that the pair $\langle K_1, K_2 \rangle$ is *linked* if $I_1 \cap I_2 \neq \emptyset$. I_1 is not inside a gap of K_2 and I_2 is not inside a gap of K_1.The link is called *stable* if the same condition are verified by the interiors $\mathrm{int}(I_1)$, $\mathrm{int}(I_2)$ of I_1, I_2.

Let Λ be a nontrivial basic set of a C^2 diffeomorphism $\varphi : M \mapsto M$, whose stable foliation is of codimension one, i.e., such that $\dim W^s(x) = n - 1$, $n = \dim M$, for all $x \in \Lambda$. Let $z \in W^s(\Lambda)$ and $\phi : [-a, a] \mapsto M$ be a C^1 embedding transverse to $W^s(\Lambda)$ at $z = \phi(0)$. The *local stable thickness of* Λ *at* z is $\tau^s(\Lambda, z) = \tau(\phi^{-1}(W^s(\Lambda)), 0)$. This is independent of the choice of ϕ, as a consequence of the fact that (under codimension-one assumption) the holonomy maps (i.e., the projections along the leaves) of the stable foliation of Λ can be extended to C^1 maps. Actually, this smoothness of the holonomy of $W^s(\Lambda)$, together with the transitivity of $\varphi|_\Lambda$, also implies that $\tau^s(\Lambda, z)$ has the same value for every $z \in W^s(\Lambda)$. We denote by $\tau^s(\Lambda)$ this constant value and call it the *local stable thickness* of Λ. This is a strictly positive (finite) number and depends continuously on the diffeomorphism, in the sense that if Λ_ψ denotes the smooth continuation of Λ for a diffeomorphism ψ which is C^2-close to φ, then $\tau^s(\Lambda_\psi)$ is close to $\tau^s(\Lambda)$. *Local unstable thickness* $\tau^u(\Lambda, z)$ and $\tau^u(\Lambda)$ are defined in a similar way, when $W^u(\Lambda)$ has codimension one. In particular, both the stable thickness and unstable thickness are well-defined if M is a surface.

In the proof of the main theorems we will use the following two important results involving thick Cantor sets,

Proposition 2.1. (Newhouse's Gap Lemma) *Let K_1, K_2 be Cantor sets in $\mathbb{R}$ such that $\tau(K_1) \cdot \tau(K_2) > 1$ and $\langle K_1, K_2 \rangle$ is linked. Then, $K_1 \cap K_2 \neq \emptyset$.*

The next result is used by Colli [2] to show the existence of infinitely many strange attractors for diffeomorphisms on a manifold of dimension two.

Proposition 2.2. (Linking Lemma) *Let K_1, K_2 be Cantor sets in $\mathbb{R}$, with $\tau(K_1) \cdot \tau(K_2) > 1$, and I_1, I_2 the convex hull of K_1, K_2, respectively. Let $\vartheta_\beta : I_1 \mapsto \mathbb{R}$ and $\widetilde{\vartheta}_\beta : I_2 \mapsto \mathbb{R}$ be such that*

a.- *ϑ_β and $\widetilde{\vartheta}_\beta$ are topological embedding, for all $\beta \in \mathbb{R}$;*

b.- *$\vartheta_\beta(x)$ and $\widetilde{\vartheta}_\beta(y)$ are differentiable with respect to β, for all $x \in K_1$ and $y \in K_2$;*

c.- $\partial_\beta\big[v_\beta(x) - \widetilde{v}_\beta(y)\big] \geq c > 0$, *for all* $x \in K_1$ *and* $y \in K_2$;

d.- *if* $\widetilde{K}_1 \subset K_1$ *and* $\widetilde{K}_2 \subset K_2$ *are Cantor subsets and let* $\beta_0 \in \mathbb{R}$ *be such that the pair* $\langle \vartheta_{\beta_0}(\tilde{K}_1)\,,\, \widetilde{\vartheta}_{\beta_0}(\tilde{K}_2)\rangle$ *is linked. Then, for any* $\varepsilon > 0$, *there is* β *such that*

i) $|\beta - \beta_0| < \varepsilon$;

ii) the pair $\langle \vartheta_\beta(\widetilde{K}_1)\,,\, \widetilde{\vartheta}_\beta(\widetilde{K}_2)\rangle$ *has two (stable) sublinks.*

2.2. ***Intrinsically smooth foliations of hyperbolic sets.*** Let $X \subset \mathbb{R}^m$ be a compact set and $\varphi : X \mapsto \mathbb{R}^n$ be continuous. We say that φ is *intrinsically* C^1 on X if there exists a continuous map $\Delta\varphi : X \times X \mapsto \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that

$$\varphi(x) - \varphi(z) = \Delta\varphi(x, z) \cdot (x - z) \quad \text{for all} \quad x, z \in X.$$

Such a $\Delta\varphi$ (which is, in general, far from unique) is called an *intrinsic derivative* of φ. We say that φ is *intrinsically* $C^{1+\gamma}$ on X if it admits some γ-Hölder continuous intrinsic derivative.

Remark 1: Let $\varphi : X \mapsto \mathbb{R}^n$ be Lipschitz continuous and $U \subset X \times X$ be such that $\{\|x - z\| : (x, z) \in U\}$ is bounded away from zero. Then, there is a Lipschitz continuous map $\Delta : U \mapsto \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that $\varphi(x) - \varphi(z) = \Delta(x, z) \cdot (x - z)$ for every $(x, z) \in U$.

Let q_0 be a transverse homoclinic point associated to some hyperbolic fixed (or periodic) saddle point p of a C^2 diffeomorphism $\varphi : M \mapsto M$. We assume $q_0 \notin W^{ss}(p)$ and another mild (open and dense) transversally condition to be stated in (1) below. Then, our goal, in this section, is to prove that there exists a hyperbolic basic set Λ_1 containing p and q_0 and whose unstable foliation is intrinsically C^1. We assume that φ is C^2 linearizable on a neighborhood U of p.

Let us denote by $\sigma_1, \dots, \sigma_u, \lambda_1, \dots, \lambda_s,\ u + s = m$, the eigenvalues of $D\varphi(p)$, with $|\sigma_u| \geq \cdots \geq |\sigma_1| > 1 > |\lambda_1| \geq \cdots \geq |\lambda_s|$. We define $1 \leq w \leq s$ by $|\lambda_1| = \cdots = |\lambda_w|$ and let $E^s = E^w \oplus E^{ss}$ be the invariant splitting such that $D\varphi(p)|_{E^w}$ has eigenvalues $\lambda_1, \dots, \lambda_w$ and $D\varphi(p)|_{E^{ss}}$ has eigenvalues $\lambda_{w+1}, \dots, \lambda_s$. We choose C^2 linearizing coordinates $(\xi_1, \dots, \xi_u, \zeta_1, \dots, \zeta_s)$ on a neighborhood U of p and, furthermore, we may assume that

C1.- $W^u(p) \subset \{\zeta_1 = \cdots = \zeta_s = 0\}$ and $W^s(p) \subset \{\xi_1 = \cdots = \xi_u = 0\}$;

C2.- $E^w = \{0^u\} \times \mathbb{R}^w \times \{0^{s-w}\}$ and the strong manifold (tangent to E^{ss} at p) satisfies $W^{ss}_{loc}(p) \subset \{\xi_1 = \cdots = \xi_u = \zeta_1 = \cdots = \zeta_w = 0\}$.

Up to a convenient choice of riemannian metric we have, for $\sigma = |\sigma_1|$, $\lambda = \lambda_1| = |\lambda_w|$ and $\theta = |\lambda_{w+1}|$,

C3.- $(\sigma - \varepsilon)\|v\| \leq \|D\varphi(p)v\|$, for all $v \in E^u$

C4.- $(\lambda - \varepsilon)\|v\| \leq \|D\varphi(p)v\| \leq (\lambda + \varepsilon)\|v\|$, for all $v \in E^w$

C5.- $\|D\varphi(p)v\| \leq (\theta + \varepsilon)\|v\|$, for all $v \in E^{ss}$

where $\varepsilon > 0$ is fixed small enough so that $\theta + 2\varepsilon < \lambda - 2\varepsilon < \lambda + 2\varepsilon < \sigma - 2\varepsilon$. (In the case $w = s$, i.e., if all contracting eigenvalues have the same norm, $E^{ss} = \{0\}$, $W^{ss}(p) = \{p\}$ and we leave θ undefined).

Now we will construct a hyperbolic set whose unstable foliation is intrinsically C^1 using the tranversality between $W^s(p)$ and $W^u(p)$ at q_0.

Fix $q, r \in U$ in the orbit of q_0 in such a way that $q \in W^s(p)_{loc}$ and $r = \varphi^{-N}(q) \in W^u(p)_{loc}$. Take

$$V = V_\delta = \left\{\|(\xi_1, \dots, \xi_u)\| \leq \delta\right\} \times \left\{\|(\zeta_1, \dots, \zeta_s)\| \leq \rho\right\}$$

where $\delta > 0$ is small and $\rho > 0$ is fixed in such a way that $\{q, r\} \subset \mathrm{int}(V) \subset V \subset U$. Let $n = n(\delta)$ be minimum such that $r \in \mathrm{int}(\varphi^n(V))$. (We suppose that δ is conveniently adjusted so that $\varphi^{N+n}(V)$ cuts V in two cylinders.

We define

$$\Lambda = \bigcap_{k \in \mathbb{Z}} \varphi^{(N+n)k}(V) \quad \text{and} \quad \Lambda_1 = \bigcup_{i=1}^{N+n} \varphi^i(V).$$

It is well know that Λ_1 is a nontrivial hyperbolic basic set, see [13]. We assume the following generic (open and dense) condition

$$D_{uw}\phi_{uw}(r) \quad \textit{is an isomorphism.} \tag{1}$$

Here D denotes the usual derivative and (1) means that *unstable/weak-stable* directions are not sent to *strong-stable* directions by $\phi = \varphi^N$. With this condition it is shown in [11] that for every point $x \in \Lambda_1$, the *intrinsic tangent space* to Λ_1

$$\mathrm{IT}_x\Lambda_1 = \mathrm{span}\left\{v : \text{ there is } (x_n)_n \in \Lambda_1^{\mathbb{N}} \text{ so that } x_n \to x \text{ and } \frac{x_n - x}{\|x_n - x\|} \to v\right\}$$

(for simplicity we consider here $M = \mathbb{R}^n$) is contained in $(u+w)$-dimensional space. Moreover the intrinsic tangent space to $W^u_{loc}(\Lambda_1)$ at every point $x \in W^u_{loc}(\Lambda_1)$ is contained $(u+w)$-dimensional space. In particular, $\mathrm{IT}_pW^u(\Lambda_1) \subset E^u \oplus E^w$. The fact that we have a good property for the unstable foliation is showed in the following result.

Proposition 2.3. (1) *Suppose that φ^N satisfies condition (1) above and consider Λ_1 also as above. Then, the map $\mathcal{F} : W^u(\Lambda_1) \ni\mapsto T_xW^u(x)$ is intrinsically C^1 on compact parts of $W^u(\Lambda_1)$.*

(2) *Let Σ_0, Σ_1 be (small) C^1 sections transverse to $W^u(x)$ for some $x \in W^u(\Lambda_1)$ and let $\pi : \Sigma_0 \cap W^u(\Lambda_1) \to \Sigma \cap W^u(\Lambda_1)$ be to denote the projection along the leaves of $W^u(\Lambda_1)$. Then, π is intrinsically C^1.*

2.3. ***Thickness in higher dimension.*** In this subsection we want to define the *local unstable thickness* of a basic set with unstable foliation of codimension greater than one.

Consider Λ_1 as it was constructed in the previous section. And we suppose that for a periodic point p, $D\varphi(p)$ has a unique (necessarily real) weakest contracting eigenvalue $\lambda = \lambda_1$, and φ is C^2 linearizable near p. Then, we consider $\pi : \Lambda_1 \cap W^s_{loc}(p) \mapsto \mathbb{R}$ to be an arbitrary intrinsically C^1 map such that $\ker\Delta(\pi(p,p))$ does not contain $\mathrm{IT}(\Lambda_1 \cap W^s_{loc}(p)) = E^w$ (i.e., $\Delta\pi(p,p)|E^w$ is bijective) and we define

$$\tau^u(\Lambda_1, p) = \tau(\pi(\Lambda_1 \cap W^s_{loc}(p)), \pi(p))$$

the *local unstable thickness* of Λ_1 at p. It is shown in [11] that the definition above does not depends on π as taken above, also it is strictly positive and varies continuously with the diffeomorphism: if ψ is a C^2-small perturbation of φ, $\tau^u(\Lambda_1(\psi), p)$ is a small variation of $\tau^u(\Lambda_1, p)$.

Let $\pi_w : \Lambda_1 \cap W^s_{\text{loc}}(p) \to \mathbb{R}$ be the restriction to $\Lambda_1 \cap W^s_{\text{loc}}(p) \subset \{0^u\} \times \mathbb{R}^s$ of the projection $(\xi_1, \dots, \xi_u, \zeta_1, \dots, \zeta_s) \longmapsto \zeta_1$. π_w is a homeomorphism onto its image K^w and moreover π_w^{-1} is intrinsically $C^{1+\gamma}$ on K^w, see [11]. The fact that $\pi \circ \pi_w^{-1}$ is an intrinsically C^1 map with

$$\Delta(\pi \circ \pi_w^{-1})(0,0) = \Delta\pi(p,p) \cdot \Delta\pi_w^{-1}(0,0) \neq 0.$$

Then, $\tau(\pi(\Lambda_1 \cap W^s_{\text{loc}}(p)), \pi(p)) = \tau(K^w, 0)$ as a consequence of $\pi(\Lambda_1 \cap W^s_{loc}(p)) = (\pi \circ \pi_w^{-1})(K^w)$ and the following result. See [11],

Lemma 2.1. *Let $K \subset \mathbb{R}$ be a Cantor set, $y \in K$ and $g : K \to \mathbb{R}$ be an intrinsically C^1 map with $\Delta g(y,y) \neq 0$. Then, $\tau(g(K), g(y)) = \tau(K, y)$.*

It is also shown that K^w is *dynamically defined Cantor set*, in the same sense as in ([10], Ch.IV), i.e., $\tau(K^w) > 0$. Moreover, if ψ is a diffeomorphism C^2-close to φ, $\tau(K^w(\psi), 0)$ is close to $\tau(K^w, 0)$.

The following result shows that the definition of unstable thickness does not depend on transverse section to $W^u(\Lambda_1)$. We will use such fact in the section 4.

Proposition 2.4. (a) *Let $q \in W^u(p)$, Σ be a C^1 section transverse to $W^u(p)$ at the point q and $\pi : W^u(\Lambda_1) \cap \Sigma \to \mathbb{R}$ be an intrinsically C^1 map such that $\mathrm{IT}_q(W^u(\Lambda_1) \cap \Sigma)$ is not contained in $\ker(\Delta\pi(q,q))$. Then, $\tau(\pi(W^u(\Lambda_1) \cap \Sigma), \pi(q)) = \tau^u(\Lambda_1, p)$.*

(b) *More generality, given $z \in W^u(\Lambda_1)$, Σ a transverse section to $W^u(\Lambda_1)$ at z and $\pi : W^u(\Lambda_1)\Sigma \to \mathbb{R}$ a submersion with $\mathrm{IT}_z(W^u(\Lambda_1) \cap \Sigma) \nsubseteq \ker(\Delta\pi(z,z))$. Then, $\tau(\pi(W^u(\Lambda_1)\cap), \pi(z)) = \tau^u(\Lambda_1, p)$.*

2.4. ***Unique least contracting eigenvalue.*** Let $\{\varphi_\mu\}$ be a C^∞ one-parameter family of diffeomorphisms generically unfolding at $\mu = 0$ a quadratic homoclinic tangency associated to saddle fixed (or periodic) point p of φ_0. We also assume once more that there are C^2 μ-dependent coordinates $(\xi_1, \dots, \xi_u, \zeta_1, \dots, \zeta_s)$ linearizing the φ_μ, for μ near zero, on a neighborhood U of the analytic continuation p_μ of p. Moreover, these coordinates can be taken to satisfy conditions (C1)-(C5) of Section 2.2.

We assume in this section that $D\varphi_0(p)$ has exactly two weakest contracting eigenvalues and these are complex conjugate numbers, this means that, $w = 2$, $\lambda_1 = \lambda e^{-i\gamma}$, $\lambda_2 = \lambda e^{i\gamma}$ with $\lambda > |\lambda_3|$ and $\gamma \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$. Here we may even assume that $D\varphi_\mu(p_\mu)|_{E^w}$ is conformal with respect to Euclidean metric introduce by coordinates ζ_1, ζ_2. On the other hand, we may take, say for $\mu \geq 0$, points $q_\mu W^s_{loc}(p_\mu)$, $r_\mu \in W^u_{loc}(p_\mu)$ depending continuously on μ, such that $\varphi_\mu^N(r_\mu) = q_\mu$ for some fixed $N \geq 1$, r_0, q_0 belong to the orbit of the tangency and r_μ, q_μ are points of the transverse intersection of $W^u(p_\mu)$ and $W^s(p_\mu)$ for every $\mu > 0$. Recall that, moreover, there exists a sequence of parameter value $\mu_j \to 0$ such that $W^u(p_\mu)$ and $W^s(p_\mu)$ also have point of tangential intersection.

For each fixed $\mu = \mu_j$ and every sufficiently large $n \geq 1$, there is a neighborhood of $V = V(j,n)$ of $\{p_\mu, q_\mu\}$, as in section 2.2, such that

$$\Lambda(j,n) = \bigcap_{k\in\mathbb{Z}} \varphi_\mu^{(n+N)k}(V)$$

is a φ_μ^{N+n}-invariant hyperbolic set and $\varphi_\mu^{N+n}|\Lambda(j,n)$ is conjugate to the 2-shift. Moreover, given any periodic point $\widetilde{p} \in \Lambda(j,n)$, there are parameter values $\widetilde{\mu}$ arbitrarily close to μ_j for which $\varphi_{\widetilde{\mu}}$ has homoclinic tangencies associated to (the analytic continuation of) $\widetilde{p}$. We consider $\widetilde{p} = \widetilde{p}(j,n)$ to be the unique φ_μ^{n+N}-fixed point in $\Lambda(j,n) \setminus \{p_\mu\}$. Clearly, the orbit of $\widetilde{p}$ passes arbitrarily close to p_μ if j and n are sufficiently large. The following result show our goal in this subsection,

Proposition 2.5. *Suppose that φ_0 satisfy the condition (2) below. Given j sufficiently large. Then, there exist values of $n = n(j)$ arbitrarily large such that $D\varphi_\mu^{N+n}(\widetilde{p})$ has a unique weakest contracting eigenvalue.*

Consider

$$D\varphi_\mu^N = \begin{pmatrix} A_{uu} & A_{uw} & A_{us} \\ A_{wu} & A_{ww} & A_{ws} \\ A_{su} & A_{sw} & A_{ss} \end{pmatrix}, \quad \Delta_\mu = \begin{pmatrix} A_{uu} & A_{uw} \\ A_{wu} & A_{ww} \end{pmatrix}$$

where the expression of $D\varphi_\mu^N$ with respect to the splitting $E^u \times E^w \times E^s = \mathbb{R}^u \times \mathbb{R}^2 \times \mathbb{R}^{s-2}$. We also denote

$$D\varphi_\mu^{-N} = \begin{pmatrix} A^-_{uu} & A^-_{uw} & A^-_{us} \\ A^-_{wu} & A^-_{ww} & A^-_{ws} \\ A^-_{su} & A^-_{sw} & A^-_{ss} \end{pmatrix}$$

The generic assumption in the proposition above is (cf. (1))

$$\Delta_{\mu=0}(r_0), \quad \text{and so also } A^-_{ss}(\mu = 0, q_0) \text{ is an isomorphism.} \tag{2}$$

Of the proof of the Proposition 2.5 above we can obtain that $\dim W^u(\widetilde{p}) = \dim W^u(p)$, $\dim W^s(\widetilde{p}) = \dim W^s(p)$ and $D\varphi_\mu^{n+N}(\widetilde{p})$ is sectionally dissipative if $D\varphi_0(p)$ is.

We conclude that there exist a sequence of parameter values $\widetilde{\mu}_j \to 0$ such that $\varphi_{\widetilde{\mu}_j}$ exhibit homoclinic tangencies associated to $\widetilde{p}_j \to p$ and $D\varphi_{\widetilde{\mu}_j}^{k_j}(\widetilde{p}_j)$, k_j is the period of $\widetilde{p}_j$,has a unique weakest contracting eigenvalue.

2.5. ***Thick invariant Cantor sets.*** Let φ be a C^∞ diffeomorphism with a quadratic homoclinic tangency at q_0 associated to a fixed (or periodic) point p. We suppose that $\dim W^u(p) = 1$ and $D\varphi(p)$ is sectionally dissipative, i.e., the product of any two of its eigenvalues has norm less than one.

Let $\{\varphi_\mu\}$ be a C^∞ one-parameter family of diffeomorphisms with $\varphi_0 = \varphi$, such that generically unfold the homoclinic tangency. We suppose once more that the φ_μ, μ near zero, admit C^2 μ-dependent linearizing coordinates $(\xi, Z) \in \mathbb{R}\times\mathbb{R}^{n-1}$ on a neighborhood U of p. We fix these coordinates in such a way that $W^u_{loc}(p_\mu) \subset \{Z = 0\}$ and $W^s_{loc}(p_\mu) \subset \{\xi = 0\}$. The assumption on the eigenvalues of $D\varphi_0(p)$ means that we may choose a norm in $\mathbb{R}^n$ to be such that

$$|\sigma_\mu| \cdot \|S_\mu\| < 1 \quad \text{for every } \mu \text{ near zero}$$

where σ_μ is the expanding eigenvalue of $D\varphi_\mu(p_\mu)$ and $S_\mu = D\varphi_\mu|E^s(p_\mu)$.

It is shown in [11] that, there are a constant N (positive integer) and for each positive integer n, reparametrization $\mu = M_n(\nu)$ of the variable μ and (μ, n)-dependent coordinates transformation

$$(\nu, x, Y) \mapsto (M_n(\nu), \Theta_{n,\nu)}(x, Y))$$

such that the map

$$(\nu, x, Y) \mapsto (\nu, \Theta_{n,\nu}^{-1} \circ \varphi_{M_n(\nu)}^{n+N} \circ \Theta_{n,\nu}(x, Y)),$$

converge, in C^2-topology, to the map $(\nu, x, Y) \mapsto (\nu, x^2 + \nu, Ax)$, where $A \in \mathbb{R}^{n-1}$.

The existence of a hyperbolic basic set Λ_2 with arbitrarily large thickness follows of the fact that for the map $x \mapsto x^2 + \nu$, and also for $\psi_{-2} : (x, Y) \mapsto (x^2 - 2, Ax)$, there exist invariant expanding Cantor sets K_j with thickness $\tau(K_j) \to +\infty$ as $j \to +\infty$. Moreover, these K_j are transitive and have a dense subset of periodic orbits. It is follows that each K_j has , for n large, $\mu = M_n(\nu)$ and ν close to -2, an analytic continuation as a hyperbolic basic set $K_j(n, \mu)$ of

$$(\Theta_{n,\nu}^{-1} \circ \varphi_{M_n(\nu)}^{n+N} \circ \Theta_{n,\nu}(x, Y)).$$

In particular, the set $K_j(n, \mu)$ has codimension-1 stable foliation and stable thickness $\tau(K_j(n, \mu))$ close to $\tau(K_j) \gg 1$. Then, we just take $\Lambda_2 = \Lambda_2(\mu) = \Theta_{n,\nu}(K_j(n, \mu))$ with j and n large and $\mu = M_n(\mu)$, ν close to -2. It is also shown that parameter values $\nu_n \to -2$ can be taken in such a way that

$$f_{\nu_n}^{(n)} = \Theta_{n,\nu_n}^{-1} \circ \varphi_{M_n(\nu_n)}^{n+N} \circ \Theta_{n,\nu_n}$$

have periodic points $P(n, \nu_n)$ and $Q(n, \nu_n) \in K_j(n, \mu)$, $\mu = M_n(\nu_n)$, which are heteroclinic related and $W^u(Q(n, \nu_n))$ also has nontransverse intersections with $W^s(P(n, \nu_n))$.

Now for $f = f_{\nu_n}^{(n)}$, there are $0 < \underline{\lambda} = \underline{\lambda}(n) < \overline{\lambda} = \overline{\lambda}(n) < 1$, $1 < \underline{\sigma} < \overline{\sigma}$ and $c = c(n) > 0$ such that

(1) $c^{-1}\underline{\sigma}^i\|u\| \le \|Df^i(x) \cdot u\| \le c\overline{\sigma}^i\|u\|$;

(2) $c^{-1}\underline{\lambda}^i\|v\| \le \|Df^i(x) \cdot v\| \le c\overline{\lambda}^i\|v\|$,

for all $x \in K_j(n, \mu)$, $u \in E_x^u$, $v \in E_x^s$ and $i \ge 0$. If $\Lambda_2 = \Lambda(n, \mu) = \Theta_{n,\nu_n}(K_j(n, \mu))$ is a hyperbolic basic set for φ_μ^{n+N}, where $\mu = M_n(\nu_n)$ and $z \in \Lambda_2$ is a periodic point of φ_μ^{n+N} of period $k = (n + N)j$. Then, $z = \Theta_{n,\nu_n}(x)$, where x is a periodic point of f of period j. We conclude that if σ_2 is the expanding eigenvalue of $D\varphi_\mu^k(z)$. Then,

$$\sigma_2^k = \sigma_2^{(n+N)j} \le \overline{\sigma}^j \le (\sqrt[n+N]{\overline{\sigma}})^j,$$

therefore, $\sigma_2 \le \sqrt[n+N]{\overline{\sigma}} \to 1$, as $n \to +\infty$. For μ near zero and y near $\varphi_\mu^{-N}(q_0)$ in U, we have $\|D\varphi_\mu^N(z)\| \le \overline{k}$, for a large constant $\overline{k}$, and if $S_{2\mu}^k = D\varphi_\mu^k(z)|E_z^s$, we have

$$\begin{aligned} \|S_{2\mu}^k\| = \|D\varphi_\mu^k|E^s(z)\| &\le \|D\varphi_\mu^{Nj}|E^s(\varphi_\mu^{nj}(z)) \circ D\varphi_\mu^{nj}|E^s(z)\| \\ &\le (\overline{k})^j\|S_\mu\|^{nj} < 1, \end{aligned}$$

for n sufficiently large, that mens, $\|S_{2\mu}\| < \lambda_0 < 1$ for n large, where λ_0 does not depend on n.

For the discussion above together with section 2-4 we concluded this section with the following result which is a summarized of this chapter.

Theorem 2.1. *Let φ_0 be a smooth diffeomorphism having a homoclinic tangency associated to a sectionally dissipative saddle fixed (or periodic) point. Then, there exists a smooth diffeomorphism φ arbitrarily near φ_0 such that*

a.- *φ has hyperbolic basic set Λ_1 and Λ_2 with $\tau^s_{\text{loc}}(\Lambda_2) \cdot \tau^u_{\text{loc}}(\Lambda_1) > 1$;*
b.- *there are periodic points $p_1 \in \Lambda_1$ and $p_2 \in \Lambda_2$ such that $W^u(p_2)$ has a transversal intersection with $W^s(p_1)$ and $W^u(p_1)$ meet quadratically in a point q with $W^s(p_2)$;*
c.- *the hyperbolic basic set Λ_1 has intrinsically C^1 unstable foliation and $p_1 \in \Lambda_1$ has a unique least contracting eigenvalue;*
d.- *there exists $c > 0$ such that if $Q_1 \in \Lambda_1$ and $Q_2 \in \Lambda_2$ are periodic points of period k_1 and k_2 respectively, denote $\lambda_i = \|S_i = D\varphi|_{E^s_{Q_i}}\|$ and $\sigma_i^{k_i}$ the unstable eigenvalue of $D\varphi^{k_i}$, $i = 1, 2$. Then,*
   d1) $|\lambda_2 \cdot \sigma_2| < 1$;
   d2) $|\sigma_1^{2c} \cdot \lambda_2| < 1$;
   d3) *σ_2 is so small that* $|\sigma_2 \cdot (\lambda_1\sigma_1)^{c/2}| < 1$.

3. Renormalization scheme and quadratic-like families

In this section we describe a higher-dimensional version of the renormalization scheme in 2-cycles of periodic points with a heteroclinic tangency following ideas from [15] and [10]. We Also state and comment about *quadratic-like* families as considered in [15]. Finally we make a delicate discussion on how to perturb a one-parameter families of diffeomorphisms to obtain linearizability.

3.1. ***Renormalization scheme in 2-cycles.*** Let φ be a C^∞ diffeomorphism having basic sets Λ_1, Λ_2 and fixed (or periodic) points $p_1 \in \Lambda_1$ and $p_2 \in \Lambda_2$, such that $\dim W^u(p_1) = W^s(p_2) = 1$; $W^s(p_1)$ and $W^u(p_2)$ have a transverse intersection in a point r_0 and $W^u(p_1)$ have a nontransverse contact (i.e. tangency) with $W^s(p_2)$ in a point q, see figure 1. We suppose that $D\varphi(p_1)$ is sectionally dissipative, (*i.e.* the product of any two of its eigenvalues has norm less than one). We also suppose that the tangency is quadratic.

Let $\{\varphi_\mu\}$ be a C^∞ one-parameter family of diffeomorphisms with $\varphi_0 = \varphi$ and generically unfolding the tangency. We assume that φ_0 is C^4 linearizable near p_1 and p_2. As C^k-linearizable is an open condition (see [14]). We assume that the φ_μ , μ close to zero, admit C^4 μ-dependent linearizing coordinates in a neighborhood of p_1 and p_2, that means, there are neighborhoods U_1 of p_1 and U_2 of p_2 such that the expression of φ_μ, μ small, in U_1 is $(\xi, H) \mapsto (\sigma_{1\mu}\xi, S_{1\mu}H)$, in U_2 is $(\eta, J) \mapsto (\sigma_{1\mu}2\eta, S_{2\mu}J)$ where $\sigma_{1\mu}$ and $\sigma_{2\mu}$ are the expanding eigenvalue of $D\varphi_\mu(p_1)$ and $D\varphi_\mu(p_2)$ respectively and $S_{i\mu} = D\varphi_\mu|_{E^s(p_i)}$, $i = 1, 2$. We may suppose that $q = (1, 0^{n-1}) \in U_1$, therefore, there exists $N > 1$ such that $\varphi_\mu^N(q) = (0, J_0) \in U_2$, see figure 1. We assume that for (μ, ξ, H) close to $(0, 1, 0^{n-1})$ we may write $\varphi_\mu^N(\xi, H)$ as

$$\Big(\alpha(\xi-1)^2 + \beta\cdot H + a\mu + r(\mu, \xi-1, H)\,,\; J_0 + \gamma(\xi-1) + R(\mu, \xi-1, H)\Big)$$

where we have α,$a \in \mathbb{R}$, $\beta \in \mathcal{L}(\mathbb{R}^{n-1}, \mathbb{R})$, $\gamma \in \mathcal{L}(\mathbb{R}, \mathbb{R}^{n-1})$ and

$$r, R, Dr, DR, \partial_{\xi\xi}r, \partial_{\mu\xi}r \quad \text{and} \quad \partial\mu\mu r \quad \text{vanish} \quad \text{at} \quad (0, 1, 0^{n-1}) \tag{3}$$

The hypothesis of nondegeneracy of the tangency amount to having $\alpha \neq 0$ and $a \neq 0$. Moreover, using a μ-reparametrization and μ-dependent linear changes of the space of coordinates, we may even assume $a = 1$, $r(\mu, 0, 0) = 0$, $R(\mu, 0, 0) = 0^{n-1}$ and $\partial_\xi(\mu, 0, 0) = 0$.

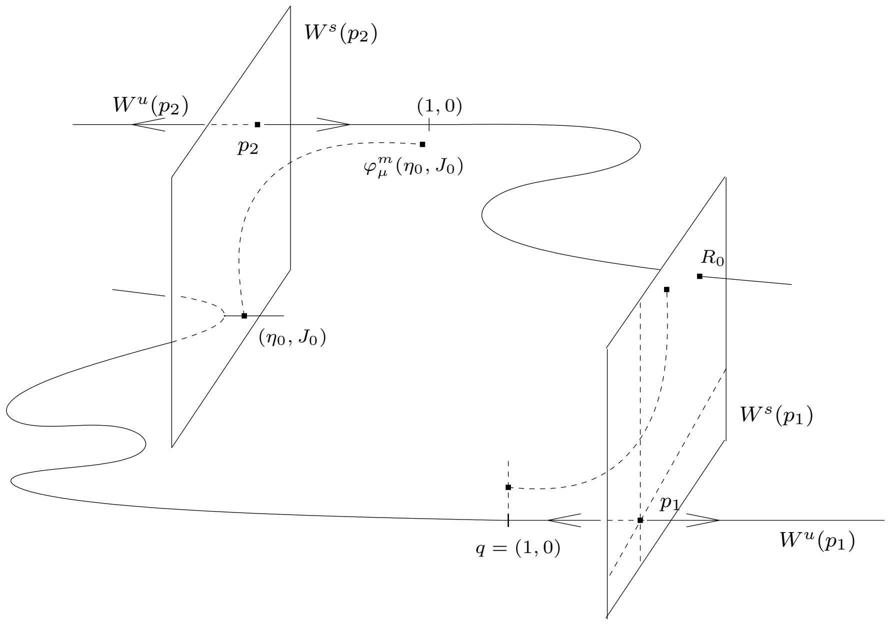

FIGURE 1. *Renormalization scheme*

We still have to consider the transition map among the neighborhoods U_2 of p_2 and U_1 of p_1 and their "transverse" intersection. We may suppose that $r_0 = (1, 0) \in U_2$, then there is $N_1 > 0$ such that $\varphi_\mu^{N_1}(r_0) = (0, R_0) \in U_1$, for μ small. Suppose that $\varphi_\mu^{N_1}$, for (η, J) near $(1, 0)$, has the form

$$\varphi_\mu^{N_1}(\eta, J) = (0, R_0) + \begin{pmatrix} a_\mu & B_\mu \\ c_\mu & D_\mu \end{pmatrix} \begin{pmatrix} \eta - 1 \\ J \end{pmatrix} + \Big(\theta(\mu, \eta-1, J), \Theta(\mu, \eta-1, J)\Big)$$

where $a_\mu \in \mathbb{R}$, $B_\mu \in \mathcal{L}(\mathbb{R}^{n-1}, \mathbb{R})$, $c_\mu \in \mathcal{L}(\mathbb{R}, \mathbb{R}^{n-1})$, $D_\mu \in \mathcal{L}(\mathbb{R}^{n-1})$, $\theta(\mu, 0, 0) = 0$, $\Theta(\mu, 0, 0) = 0$ and

$$D\theta,\ D\Theta \quad \text{vanish} \quad \text{at} \quad (\mu, \eta-1, J) = (\mu, 0, 0). \tag{4}$$

By transversality between $W^u(p_2)$ and $W^s(p_1)$ we have $a_\mu \neq 0$, for μ small.

Now we fix $A_0 \leq 3$ a real constant. Fix N and N_1 as above. We denote $\Phi : \mathbb{R} \times M \longrightarrow \mathbb{R} \times M$, C^∞ map, $\Phi(\mu, x) = (\mu, \varphi_\mu)$.

Theorem 3.1. *Let N, N_1 positive integer as above and let $0 < c < 1$ be a small constant such that the following hold $|\sigma_1^{2c} \cdot \lambda_2| < 1$ and $|\sigma_2 \cdot (\lambda_1 \cdot \sigma_1)^{c/2}| < 1$. Choose $n = n(m)$ such that $(c/2) \cdot m \leq n(m) \leq c \cdot m$. Then, there exists a sequence $\Theta_{n,m} : [1/A_0, A_0] \times [-A_0, A_0] \longrightarrow \mathbb{R} \times M$ of C^k diffeomorphisms such that the sequence $f_{n,m} = \Theta_{n,m}^{-1} \circ \Phi^{N+n+N_1+m} \circ \Theta_{n,m}$ converge to the map*

$$\phi(a, x, y_1, \cdots, y_{n-1}) = (a, 1 - ax^2, 0^{n-1})$$

in C^k topology, as $n, m \to \infty$.

Proof. We first describe a construction of $\Theta_{n,m}$. We start observing that if one look at $\varphi_\mu^{-N_1}(W^s(p_1))$ in U_2 coordinates near $(1, 0)$. Then, it is the graph of a function $x \mapsto \Gamma_\mu(x)$. Analogously, $W^u(p_2)$ near $(0, R_0)$ is the graph of a function $x \mapsto \Delta_\mu(x)$ in U_1 coordinates. For n and m sufficiently large, we also define the functions $x \mapsto \Gamma_\mu^{(m)}(x)$ and $y \mapsto \Delta_\mu^{(n)}(y)$ whose graphs correspond to $\varphi_\mu^{-N_1}\left(\{\xi = \sigma_{1\mu}^{-n}\}\right)$ and $\varphi_\mu^{N_1}\left(\{J = S_{2\mu}^m \cdot J_0\}\right)$ respectively.

Using the notation above, we take $\eta_0 = \eta_0^{(n,m)}(\mu) = \sigma_{2\mu}^{-m} \Gamma_\mu^{(m)}(S_{2\mu}^m \cdot J_0)$, such that, $(\sigma_{1\mu}^{-n}, \Delta_\mu^{(n)}(\sigma_{1\mu}^{-n})) = \varphi_\mu^{N_1} \circ \varphi_\mu^m(\eta_0, J_0)$, *i.e.*

$$\sigma_{1\mu}^{-n} = a_\mu(\sigma_{2\mu}^m \eta_0 - 1) + B_\mu S_{2\mu}^m J_0 + \theta(\mu, \sigma_{2\mu}^m \eta_0 - 1, S_{2\mu}^m J_0) \quad \text{and} \tag{5}$$

$$\Delta_\mu^{(n)}(\sigma_{1\mu}^{-n}) = R_0 + c_\mu(\sigma_{2\mu}^m \eta_0 - 1) + D_\mu S_{2\mu}^m J_0 + \Theta(\mu, \sigma_{2\mu}^m \eta_0 - 1, S_{2\mu}^m J_0). \tag{6}$$

Consider the (n, m)-dependent reparametrization

$$\mu = \mu_{n,m}(a) = -\frac{a}{\alpha} \sigma_{1\mu}^{-2n} \sigma_{2\mu}^{-2m} + \eta_0 - \sigma_{1\mu}^{-2n} \sigma_{2\mu}^{-2m} \beta \cdot S_{1\mu}^n \Delta_\mu^{(n)}(\sigma_{1\mu}^{-n}) \tag{7}$$

Recall that $\beta \in \mathcal{L}(\mathbb{R}^{n-1}, \mathbb{R})$. From (5) we have

$$a = a_{n,m}(\mu) = -\alpha \sigma_{1\mu}^{2n} \sigma_{2\mu}^{2m} \mu - \sigma_{1\mu}^{2n} \sigma_{2\mu}^{2m} \eta_0 + \beta \cdot S_{1\mu}^n \Delta_\mu^{(n)}(\sigma_{1\mu}^{-n}). \tag{8}$$

It is easy to check that any constant given $A_0 > 0$ for (n, m) sufficiently large $a_{n,m}(\mu)$ maps a small interval I_n, in μ-space, close $\mu = 0$ diffeomorphically onto $[-A_0, A_0]$. Then, we introduce (μ, n, m) dependent coordinates (x, Y) given by

$$\widehat{\Theta}_{n,m}\big(a, x, Y\big) = \Big(\mu_{n,m}(a) = \mu\,, -\sigma_{1\mu}^{-2n} \sigma_{2\mu}^{-2m} \frac{a}{\alpha} x + \eta_0, \sigma_{2\mu}^{-m} \alpha_n Y + J_0\Big)$$

with $\alpha_n = \sigma_{1\mu}^{-n} \cdot \omega^n$, where $1 < \omega < \min\{\sigma_{1\mu}, (\sqrt{\lambda_{1\mu} \cdot \sigma_{1\mu}})^{-1}\}$. Denote $(\mu\,, \eta\,, J) = \Theta_{n,m}(a, x, Y)$. Then, the return map Φ^{N+n+N_1+m} in the $(\mu\,, \eta\,, J)$-coordinates is given by

$$(\mu\,, \eta\,, J) \longrightarrow \Big(\mu\,, \quad \alpha(\xi - 1)^2 + \beta \cdot H + \mu + r(\mu\,, \xi - 1, H)\,, \\ J_0 + \gamma(\xi - 1) + R(\mu\,, \xi - 1, H)\Big),$$

where $\xi(\eta\,,J)=\sigma^n_{1\mu}\big[a_\mu(\sigma^m_{2\mu}\eta-1)+B_\mu S^m_{2\mu}J+\theta(\mu\,,\sigma^m_{2\mu}\eta-1\,,S^m_{2\mu}J)\big]$ and $H(\eta\,,J)=S^n_{1\mu}\cdot\big[R_0+c_\mu(\sigma^m_{2\mu}\eta-1)+D_\mu S^m_{2\mu}J+\Theta(\mu\,,\sigma^m_{2\mu}\eta-1\,,S^m_{2\mu}J)\big]$.
Then, the return map in (a,x,Y)-coordinates is given by

$$(a\,,x,Y)\to\Big(a\,,(-\alpha/a)\sigma^{2n}_{1\mu}\sigma^{2m}_{2\mu}\big[\alpha(\xi-1)^2+\beta\cdot H+\mu+r(\mu\,,\xi-1,H)-\eta_0\big]\,,$$
$$\sigma^m_{2\mu}(\alpha_n)^{-1}\big[\gamma(\xi-1)+R(\mu\,,\xi-1,H)\big]\Big)$$

where

$$\begin{aligned}\xi(x,Y)\ &=\sigma^n_{1\mu}\big\{a_\mu\big[\sigma^m_{2\mu}(-\sigma^{-2n}_{1\mu}\sigma^{-2m}_{2\mu}\tfrac{a}{\alpha}x+\eta_0)-1\big]\\ &+B_\mu S^m_{2\mu}(\sigma^{-m}_{2\mu}\alpha_nY+J_0)+\theta(\mu\,,\sigma^m_{2\mu}\eta-1\,,S^m_{2\mu}J)\big\}\end{aligned}$$

and

$$\begin{aligned}H(x,Y)\ &=S^n_{1\mu}\big\{R_0+c_\mu\big[\sigma^m_{2\mu}(-\sigma^{-2n}_{1\mu}\sigma^{-2m}_{2\mu}\tfrac{a}{\alpha}x+\eta_0)-1\big]\\ &+D_\mu S^m_{2\mu}(\sigma^{-m}_{2\mu}\alpha_nY+J_0)+\Theta(\mu\,,\sigma^m_{2\mu}\eta-1\,,S^m_{2\mu}J)\big\}.\end{aligned}$$

Using the definition of $\eta_0=\eta_0^{n,m}(\mu)$ and $\mu_{n,m}(a)$, *i.e.* using (5),(6) and (7) we have

$$\begin{aligned}f_{n,m}(a,x,Y\,)\ &=\widehat{\Theta}^{-1}_{n,m}\circ\Phi^{N+n+N_1+m}\circ\widehat{\Theta}_{n,m}(a,x,Y\,)\\ &=\big(a\,,H_1(a,x,Y\,),H_2(a,x,Y\,)\big)\end{aligned}$$

where

$$\begin{aligned}H_1(a,x,Y\,)\ &=(-\alpha^2/a)\sigma^{2n}_{1\mu}\sigma^{2m}_{2\mu}\big[-a_\mu\sigma^{-n}_{1\mu}\sigma^{-m}_{2\mu}\tfrac{a}{\alpha}x+\sigma^n_{1\mu}B_\mu S^m_{2\mu}\sigma^{-m}_{2\mu}\alpha_nY\\ &+\sigma^n_{1\mu}\overline{\theta}_{n,m}(a,x,Y\,)\big]^2+\ \sigma^{2n}_{1\mu}\sigma^{2m}_{2\mu}(-\alpha/a)\big[\beta S^n_{1\mu}D_\mu S^m_{2\mu}\sigma^{-m}_{2\mu}\alpha_nY\\ &-\beta S^n_{1\mu}c_\mu\sigma^{-2n}_{1\mu}\sigma^{-m}_{2\mu}\tfrac{a}{\alpha}x-\tfrac{a}{\alpha}\sigma^{-2n}_{1\mu}\sigma^{-2m}_{2\mu}+\beta S^n_{1\mu}\overline{\Theta}_{n,m}(a,x,Y\,)\\ &+r(\mu\,,\xi-1,H)\big]\end{aligned}$$

and

$$\begin{aligned}H_2(a,x,Y)=\ &\ \sigma^m_{1\mu}(\alpha_n)^{-1}\big[\gamma\big(-a_\mu\sigma^{-n}_{1\mu}\sigma^{-m}_{2\mu}\tfrac{a}{\alpha}x+\sigma^n_{1\mu}B_\mu S^m_{2\mu}\sigma^{-m}_{2\mu}\alpha_nY\\ &+\sigma^n_{1\mu}\overline{\theta}(a,x,Y\,)\big)+R(\mu,\xi-1,H)\big]\end{aligned}$$

where

$$\begin{aligned}\overline{\theta}_{n,m}(a,x,Y)\ &=\theta\big(\mu\,,\sigma^m_{2\mu}(-\sigma^{-2n}_{1\mu}\sigma^{-2m}_{2\mu}\tfrac{a}{\alpha}x+\eta_0)-1,\ S^m_{2\mu}(\sigma^{-m}_{2\mu}\alpha_nY+J_0)\big)\\ &-\theta\big(\mu\,,\sigma^m_{1\mu}\eta_0-1,\,S^m_{2\mu}J_0)\big)\end{aligned}$$

and

$$\begin{aligned}\overline{\Theta}_{n,m}(a,x,Y)\ &=\Theta\big(\mu\,,\sigma^m_{2\mu}(-\sigma^{-2n}_{1\mu}\sigma^{-2m}_{2\mu}\tfrac{a}{\alpha}x+\eta_0)-1,\ S^m_{2\mu}(\sigma^{-m}_{2\mu}\alpha_nY+J_0)\big)\\ &-\Theta\big(\mu\,,\sigma^m_{1\mu}\eta_0-1,\,S^m_{2\mu}J_0)\big)\end{aligned}$$

We have to show the following convergence:

(1) $\sigma^n_{1\mu}\sigma^m_{2\mu}\big[-a_\mu\sigma^{-n}_{1\mu}\sigma^{-m}_{2\mu}\frac{a}{\alpha}x+\sigma^n_{1\mu}B_\mu S^m_{2\mu}\sigma^{-m}_{2\mu}\alpha_nY$
$$+\sigma^n_{1\mu}\overline{\theta}_{n,m}(a,x,Y\,)\big]\longrightarrow-\,a_0\frac{a}{\alpha}x;$$

(2) $(-\alpha/a)\sigma^{2n}_{1\mu}\sigma^{2m}_{2\mu}\big[\beta S^n_{1\mu}D_\mu S^m_{2\mu}\sigma^{-m}_{2\mu}\alpha_nY-\beta S^n_{1\mu}c_\mu\sigma^{-2n}_{1\mu}\sigma^{-m}_{2\mu}\frac{a}{\alpha}x$
$$-\frac{a}{\alpha}\sigma^{-2n}_{1\mu}\sigma^{-2m}_{2\mu}+\beta S^n_{1\mu}\overline{\Theta}_{n,m}(a,x,Y\,)+r(\mu\,,\xi-1,H)\big]\longrightarrow1;$$

(3) $\sigma^m_{2\mu}(\alpha_n)^{-1}\big[-a_\mu\sigma^{-n}_{1\mu}\sigma^{-m}_{2\mu}\frac{a}{\alpha}x+\sigma^n_{1\mu}B_\mu S^m_{2\mu}\sigma^{-m}_{2\mu}\alpha_nY\big]\longrightarrow0;$
(4) $\sigma^{2n}_{1\mu}\sigma^{2m}_{1\mu}r(\mu\,,\xi(x,Y)-1,H(x,Y))\longrightarrow0;$
(5) $\sigma^m_{2\mu}(\alpha_n)^{-1}R(\mu\,,\xi(x,Y)-1,H(x,Y))\longrightarrow0$

To obtain the convergence, we choose a compact part of $\mathbb{R}^{n+1}$, so that $\|(a,x,Y)\| \leq$ const., where the convergence will take place and let K be a sufficiently large constant(there will be some slight abuse of notation when dealing with K).

Observe that the hypothesis imply that, for μ small,

$$\sigma_2^m(\lambda_1 \cdot \sigma_1)^{n(m)} \longrightarrow 0 \quad \text{as} \quad m \to +\infty \tag{9}$$

$$\sigma_1^{2n(m)} \cdot \lambda_2^m \longrightarrow 0 \quad \text{as} \quad m \to +\infty \tag{10}$$

In the proof of the convergence of the items (1) to (5), we will make use of (9) and (10) or their weaker versions. Recall that $|\sigma_{2\mu} \cdot \lambda_{2\mu}| < 1$.

We start estimating part 1,2 and 3. Observe first that $\sigma_{1\mu}^{-n}(\alpha_n)^{-1} \to 0$ and $\|\sigma_{1\mu}^{2n} S_{1\mu}^n \alpha_n Y\| \leq K|\sigma_{1\mu}^{2n}\lambda_{1\mu}^n\alpha_n| \leq K|(\sqrt{\lambda_{1\mu}\sigma_{1\mu}})^n| \to 0$ as $n \to +\infty$. It is clear that

$$\begin{array}{l}
\|\sigma_{1\mu}^n\sigma_{2\mu}^m\sigma_{1\mu}^n B_\mu S_{2\mu}^m \sigma_{2\mu}^{-m}\alpha_n Y\| \leq K|\sigma_{1\mu}^{2n}\lambda_{2\mu}^m\alpha_n|,\\
\|\sigma_{1\mu}^{2n}\sigma_{2\mu}^{2m}\beta \cdot S_{1\mu}^n c_\mu \sigma_{1\mu}^{-2n}\sigma_{2\mu}^{-m}x\| \leq K|\sigma_{2\mu}^m\lambda_{1\mu}^n|,\\
\|\sigma_{1\mu}^{2n}\sigma_{2\mu}^{2m}\beta \cdot S_{1\mu}^n D_\mu S_{2\mu}^m\sigma_{2\mu}^{-m}\alpha_n Y\| \leq K|\sigma_{1\mu}^{2n}\sigma_{2\mu}^m\lambda_{1\mu}^n\lambda_{2\mu}^m\alpha_n|,\\
\|\sigma_{2\mu}^m(\alpha_n)^{-1}\sigma_{1\mu}^{-n}\sigma_{2\mu}^{-m}x\| \leq K|(\alpha_n)^{-1}\sigma_{1\mu}^{-n}| \quad \text{and}\\
\|\sigma_{1\mu}^m(\alpha_n)^{-1}\sigma_{1\mu}^n B_\mu S_{2\mu}^m\sigma_{2\mu}^{-m}\alpha_n Y\| \leq K|\sigma_{1\mu}^n\lambda_{2\mu}^m|
\end{array}$$

converges to zero as $n, m \to +\infty$.

To by remining to estimate convergence of $\overline{\theta}_{n,m}$ and $\overline{\Theta}_{n,m}$ to complete (1) and (2). We have

$$|\overline{\theta}_{n,m}(a,x,Y)| \leq K|\partial_x\theta(a,\tilde{x},\tilde{Y})|\cdot|\sigma_{1\mu}^{-2n}\sigma_{2\mu}^{-m}| + K\|\partial_Y\theta(a,\tilde{x},\tilde{Y})\|\cdot|\sigma_{2\mu}^{-m}\lambda_{2\mu}^m\alpha_n|$$

$$\begin{aligned}
\|\overline{\Theta}_{n,m}(a,x,Y)\| \;&\leq K|\partial_x\Theta(a,\tilde{x},\tilde{Y})|\cdot|\sigma_{1\mu}^{-2n}\sigma_{2\mu}^{-m}|\\
&+K\|\partial_Y\Theta(a,\tilde{x},\tilde{Y})\|\cdot|\sigma_{2\mu}^{-m}\lambda_{2\mu}^m\alpha_n|
\end{aligned}$$

for some $(a,\tilde{x},\tilde{Y})$ between the points $(a, -\sigma_{1\mu}^{-2n}\sigma_{2\mu}^{-m}\frac{a}{\alpha}x + \sigma_{2\mu}^m\eta_0 - 1, S_{2\mu}^m(\sigma_{2\mu}^{-m}\alpha + J_0))$ and $(\sigma_{2\mu}^m\eta_0 - 1, S_{2\mu}^m J_0)$. From inequalities above and using (4) we have that $|\sigma_{1\mu}^{2n}\sigma_{2\mu}^m\overline{\theta}_{n,m}(a,x,y)|$, $\|\sigma_{1\mu}^{2n}\sigma_{2\mu}^{2m}S_{1\mu}^n\overline{\Theta}_{n,m}(a,x,y)\| \leq \|\sigma_{1\mu}^{2n}\sigma_{2\mu}^{2m}\lambda_{1\mu}^n\overline{\Theta}_{n,m}(a,x,y)\|$ and $|\sigma_{1\mu}^n\sigma_{2\mu}^m(\alpha_n)^{-1}\overline{\theta}(a,x,Y)|$ converges to zero as $n, m \to +\infty$.

On the other hand, it is not difficult to see that

$$|\xi(a,x,Y)| \leq K|\sigma_{1\mu}^{-n}\sigma_{2\mu}^{-m}|, \;\; |H(a,x,Y)| \leq K|\lambda_{1\mu}^n| \text{ and } |\mu| \leq K|\sigma_{2\mu}^{-m}|.$$

Finally, we want to see that

$$|\sigma_{1\mu}^{2n}\sigma_{2\mu}^{2m}r(\mu,\xi(a,x,Y)-1,H(a,x,Y))| \quad \text{and}$$

$$\|\sigma_{2\mu}^m(\alpha_n)^{-1}R(\mu,\xi(a,x,Y),H(a,x,Y))\|$$

converges to zero as $n, m \to +\infty$, for that, we write Taylor expansion of r, up to order 4 near $(\mu, 0.0)$. We recall that, $\partial_\xi r$ and $\partial_H r$ are null at $(\mu,0,0)$,

$$r(\mu,\xi-1,H) = \sum_{j=1}^{4}\sum_{\beta_1+\beta_2=j}\frac{\partial^j}{\partial_\xi^{\beta_1}\partial_H^{\beta_2}}r(\mu,\xi-1,H)(\xi-1)^{\beta_1}H^{\beta_2} + R_4(\mu,\widehat{\xi},\widehat{H})$$

where

$$\frac{R_4(\mu,\widehat{\xi},\widehat{H})}{\|(\mu,\widehat{\xi},\widehat{H})\|}\to 0 \quad \text{and} \quad \|(\mu,\widehat{\xi},\widehat{H})\|\to 0,$$

H^{β_2} is a homogeneous polinomy of degree β_2 in the coordinates of $H=(h_1,\dots,h_{n-1})$. Then, $\left|\sigma_{1\mu}^{2n}\sigma_{2\mu}^{2m}r\big(\mu,\xi(a,x,Y)-1,H(a,x,Y)\big)\right| \to 0$ as $n,m\to\infty$ as a consequence of the estimative of $\xi(a,x,Y)$, $H(a,x,Y)$, $|\mu|$ and (3), (9) and (10).

We also write, Taylor expansion of R near $(\mu,0,0)$ up to order 2 and we use essentially the same argument as above applies to R, we have that

$$\left\|\sigma_{2\mu}^{m}(\alpha_n)^{-1}R\big(\mu,\xi(a,x,Y),H(a,x,Y)\big)\right\| \longrightarrow 0 \quad \text{as} \quad n,m\to\infty.$$

Then, this proves that

$$f_{n,m}(a,x,Y)\longrightarrow \widetilde{\phi}(a,x,Y)=\big(a,1-aa_0^2x^2\,,0^{n-1}\big)$$

as $n,m\to+\infty$ (uniformly on $[-A_0,A_0]\times[-A_0,A_0]^n$). Moreover, the same kind of estimates apply to all derivatives up to order k, $k\geq 3$, it proves that this convergence (items (1) to (5)) holds in the C^k topology.

Since, $\widetilde{\phi}$ as above is conjugated to $\phi(a,x,Y)=(a,1-ax^2,0^{n-1})$ by $h(a,x,Y)=(a,\frac{1}{a_0}x,Y)$. Then, taking $\Theta_{n,m}=\widehat{\Theta}_{n,m}\circ h$ we have $\Theta_{n,m}^{-1}\circ\Phi^{N+n+N_1+m}\circ\Theta_{n,m}$ converge to the map

$$\phi(a,x,y_1,\cdots,y_{n-1})=(a,1-ax^2,0^{n-1})$$

in C^k topology, as $n,m\to\infty$.

$\square$

3.2. ***Quadratic-like families.*** Being motived by the Theorem 3.1 above we will consider quadratic (or Hénon)-like families as considered in [15].

We say that $\Psi=\{\psi_a\}$ is a *quadratic (or Hénon)-like family* if $\{\psi_a\}$ is a C^r one-parameter family of diffeomorphisms, $r\geq 3$, and $\{\psi_a\}$ is sufficiently close to $\{\phi_a\}=\Phi$, where $\phi(a,x,Y)=(a,\phi_a(x,Y))$ and $\phi_a(x,Y)=(1-ax^2,0^{n-1})$, for all a.

Theorem 3.2. (Viana [15]) *Let $0<c<\log(2)$ and $\Psi=\{\psi_a\}$ be a quadratic (or Hénon)-like family. Then, there exists a set $E=E(c,\Psi)\subset(1,2)$, with $m(E)>0$ such that for every $a\in E$, there is a compact, ψ_a-invariant set $\Lambda=\Lambda_a$ satisfying that $W^s(\Lambda)$ has nonempty interior and there is $Z_1\in\Lambda$ such that $\{\psi^n(Z_1):n\geq 0\}$ is dense in Λ and $\|D\psi_a^n(Z_1)\|\geq e^{cn}$, for all $n\geq 0$ and some $c>0$.*

In the theorem m denote the Lebesgue measure and the set Λ above is called a *strange attractor*. In the proof of the Theorem 3.2 the point Z_1 is taken to be *critical* in the sense that there exists a direction in the tangent space to M at Z_1 which is exponentially contracted by both positive and negative iterates of $D\psi_a$. Clearly, the presence of such point is an obstruction to (uniform) hyperbolicity of the attractor.

From the proof of the Theorem 3.2 above can be derived another properties of the set $E=E(c,\Psi)$: E is constructed from exclusions of parameters of a *host interval* (compact interval) $\Omega_0\subset(1,2)$ which depends only of the quadratic family $\{\phi_a\}$. In fact the interval Ω_0 can be chosen near 2, such that if $\{\psi_a\}$ is sufficiently close to $\{\phi_a\}$, $m(E) \geq (1-\delta)|\Omega_0|$ for chosen $\delta > 0$. However, if we consider only a finite number of exclusions of parameters of Ω_0, we can see from the proof that they varies continuously with $\Phi = \{\phi_a\}$. Considering this comment about properties of the set E we conclude the following

Lemma 3.1. *Let $E(\Psi) \subset \Omega_0$ be the set obtained in the Theorem 2.2. Let $I \subset \Omega_0$ be an interval such that $m(E \cap I) \geq c|I|$, for $c > 0$. Then, given $\varepsilon > 0$, for all $\widetilde{\Psi} = \{\widetilde{\psi}_a\}$ sufficiently close to $\{\psi_a\}$, there exists a set $\widetilde{E} = \widetilde{E}(\widetilde{\Psi})$ such that $m(\widetilde{E} \cap I) \geq (c-\varepsilon)|I|$ and for $a \in \widetilde{E}$, $\widetilde{\psi}_a$ has a nonhyperbolic strange attractor.*

Let $\{\varphi_\mu\}$ be a C^∞ one-parameter family of diffeomorphisms unfolding a heteroclinic tangency at $\mu = 0$ in 2-cycles involving periodic points p_1 and p_2 as considered in the first part of this section. Then, by Theorem 3.1 there exists a sequence of host intervals $\Omega_{n,m}$ in the μ-space, going to zero as n, m go to infinity, each one corresponding to Ω_0 by (μ, n, m)-reparametrization. Moreover, if we embed the family $\{\varphi_\mu\}$ in a C^∞ two-parameter family $\varphi_{\mu,\alpha}\}$ we have that, for each α sufficiently small, there is a sequence $\Omega_{n,m}(\alpha)$ of host intervals going to $\mu_T(\alpha)$, where $\mu_T(\alpha)$ is the value of the tangency between $W^s(p_2(\alpha))$ and $W^u(p_1(\alpha))$. In addition, by the form of the (μ, n, m)-reparametrization, given in the Theorem 3.1, it is easy to see that $\Omega_{n,m}(\alpha)$ depends continuously on α. And also, the convergence of the families in the Theorem 3.1 is uniform in α. So for each α small there is a set $E_{n,m}(\alpha) \subset \Omega_{n,m}(\alpha)$ with $m(E_{n,m}(\alpha) > 0$ and for all $\mu \in E_{n,m}(\alpha)$, $\varphi_{\mu,\alpha}$ has a strange attractor, by application of the Theorem 3.2. These assumptions imply the following

Remark 2: Fix $\alpha_0 > 0$ small. Then, given $\varepsilon > 0$, there are $n_0 = n_0(\alpha_0)$, $m_0 = m_0(\alpha_0)$ such that for all $\Omega_{n,m}(\alpha)$ with $0 < \alpha < \alpha_0$, $n > n_0$ and $m > m_0$ we have

2.1 $\sup\{|\mu - \mu_T(\alpha)| \,:\, \mu \in \Omega_{n,m}(\alpha)\} < \varepsilon$;
2.2 $m(E_{n,m}(\alpha) \cap \Omega_{n,m}(\alpha)) \geq \frac{3}{4}|\Omega_{n,m}(\alpha)|$;
2.3 $\Omega_{n,m}(\alpha)$ varies continuously with respect to α.

3.3. ***Special perturbation.*** Let $\{\varphi_\mu\}$ be C^∞ one-parameter family of diffeomorphisms, we want to show that if a saddle fixed (or periodic) point p_0 of φ_0, which is sectionally dissipative (i.e. the product of any pair of eigenvalues has norm less than 1), is not C^4-linearizable, that means, the eigenvalues of $D\varphi_0(p_0)$ are resonant, see [14], there exists an appropriate arbitrarily small perturbation of the family $\{\varphi_\mu\}$ such that it is possible to destroy the resonance and turn p_μ the continuation of the point p_0 in C^4-linearizable one for almost every μ near zero. To be more specific,

Lemma 3.2. *Let $\{\varphi_\mu\}_{\mu\in I}$ be a one-parameter family of diffeomorphisms having a saddle periodic point p_0 of φ_0 which is not C^k-linearizable, where I is an small interval around zero. Then, There exist a one-parameter family of diffeomorphisms $\{\psi_\mu\}_{\mu\in I}$ arbitrarily close to $\{\varphi_\mu\}$ and a subinterval $I' \subset$*

I around zero such that for almost every value $\mu \in I'$, ψ_μ is C^k-linearizable near $p(\psi_\mu)$, $k \geq 2$, where $p(\psi_\mu)$ is the continuation of p_0.

Remark 3: The family $\{\psi_\mu\}$ in the theorem which is arbitrarily near to $\{\varphi_\mu\}_{\mu \in I}$ does not depend on the interval I.

Proof. Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $D\varphi_0(p_0)$. Suppose that p_0 is not C^k linearizable, $k \geq 2$. Then the eigenvalues satisfy the resonant conditions of Sternberg, see [14], *i.e.* there exists j with $1 \leq j \leq n$ such that

$$\lambda_j = \lambda_1^{k_1} \cdot \lambda_2^{k_2} \cdots \lambda_n^{k_n} \quad \text{for} \quad 2 \leq \sum_{i=1}^{k} k_i \leq k\,, \ \text{ with } \ k_i \geq 0,\ 1 \leq i \leq n.$$

Consider the following holomorphic functions

$$h_j(Z) = (1+Z)\lambda_j \quad \text{and} \quad H(Z) = \prod_{i=1}^{n} \big[(1+Z)\lambda_i\big]^{k_i}, \quad Z \in \mathbb{C}.$$

Note that $h_j(Z) = H(Z)$ at $Z = 0$. In addition, $h_j'(Z) = \lambda_j = h_j(0)$, for all $Z \in \mathbb{C}$ and $H'(0) \neq h_j'(0)$ since $H'(0) = \sum_{i=1}^{n} k_1 \lambda_i \prod_{j=1}^{n} \lambda_j^{k_j - \delta_{ij}} = \sum_{i=1}^{n} k_i H(0) = h_j'(0) \sum_{i=1}^{n} k_i$, where δ_{ij} is 1 if $i = j$ and 0 if $i \neq j$. Then, $h_j(Z) \neq H(Z)$, for all $Z \in B_\varepsilon(0)$ and $Z \neq 0$, for some $\varepsilon > 0$, where $B_\varepsilon(0)$ is the ball in $\mathbb{C}$ the radius ε and center 0. ε depends on j and k_i, $i = 1, \cdots, n$ with $2 \leq \sum_{i=1}^{n} k_i \leq k$ and $1 \leq j \leq n$. Then, ε depends on a finite number of conditions. Therefore, we can take ε sufficiently small such that $h_j(Z) \neq H(Z)$ for all $Z \in B_\varepsilon(0)$ and $Z \neq 0$. In fact, for ε small enough $h_j'(Z) \neq H'(Z)$ for all $Z \in B_\varepsilon(0)$.

On the other hand, let $\psi_\mu : W \to \mathbb{R}^n$ be a C^∞ family of local charts defined in a neighborhood W of p_0 with $\psi_\mu(p_0) = 0$, for all $\mu \in I$. We take W sufficiently small so that $\varphi_0^j(W) \cap W = \emptyset$ for all $0 < j < n_0$, where n_0 is the period of p_0. Let ξ be a C^∞ bump function on $\mathbb{R}$ satisfying

$$\begin{cases} \xi(s) = 0, & \text{if} \quad s \geq 2 \\ \xi(s) = 1, & \text{if} \quad s \leq 1 \\ 0 \leq \xi(s) \leq 1, & \quad \forall s \in \mathbb{R}. \end{cases}$$

Let $\gamma > 0$ be a small constant such that $B_\gamma(0) \subset \psi_\mu(W)$. We define the perturbed families $\{\varphi_{\mu,t}\}$ by $\varphi_{\mu,t} = f_{\mu,t} \circ \varphi_\mu$, where

$$\begin{cases} f_{\mu,t}(x) = x, \ \text{if} & x \in M \setminus W \\ f_{\mu,t}(x) = \psi_\mu^{-1}\big(\big[1 + t \cdot \widetilde{\xi}(\|\psi_\mu(x)\|)\big] \cdot \psi_\mu(x)\big), \ \text{if} & x \in W. \end{cases}$$

and $\widetilde{\xi}(y) = \xi\left(\frac{4\|y\|}{\gamma}\right)$, for all $y \in W$. First observe that the eigenvalues of $D\varphi_{\mu,t}^{n_0}(p_\mu)$ are $(1+t)\lambda_{1\mu}$, $(1+t)\lambda_{2\mu}, \ldots, (1+t)\lambda_n$,where $\lambda_{i\mu}$ and p_μ are the continuation of λ_i and p_0 respectively, $i = 1, \ldots, n$. We also have that $\varphi_{0,t}$ is C^k linearizable near p_0 for all $0 < |t| < \varepsilon$, where ε is as above. We define

$$\Gamma_{j,\overline{k}}(\mu, Z) = \prod_{i=1}^{n} \big[(1+Z)\lambda_{i\mu}\big]^{k_i} - (1+Z)\lambda_{j\mu}$$

where $\overline{k} = (k_1, \ldots, k_n)$.

Claim: *There exist intervals $I' \subset I$ around $\mu = 0$ and $J \subset [-\varepsilon_0, \varepsilon_0]$ around $t = 0$, $\varepsilon > \varepsilon_0 > 0$ such that if for each $t \in J$ define $Z_t = \{\mu \in I' : \Gamma_{j,\overline{k}}(\mu, t) = 0\}$. Then, the set $L = \{t \in J : m(Z_t) > 0\}$ is countable.*

By the claim, we conclude that for all $t \in J \setminus \{\text{countable set}\}$, $\varphi^{n_0}_{\mu,t}$ is C^k linearizable for almost every $\mu \in I'$.

□

Proof. (*of the claim*) Recall that $\partial_Z \Gamma_{j,\overline{k}}(0, Z) \neq 0$, for all $Z \in B_\varepsilon(0)$ and $\Gamma_{j,\overline{k}}(0, Z) \neq 0$, for all $Z \in B_\varepsilon(0)\backslash\{0\}$. Then, by Implicit Function, there exist $0 < \varepsilon_0 \leq \varepsilon$ and $I' \subset I$ a subinterval with $0 \in I'$ such that if $\Gamma_{j,\overline{k}}(\mu', t') = 0$. Then, $\Gamma_{j,\overline{k}}(\mu', t) \neq 0$, for all $t \in [-\varepsilon_0, \varepsilon_0] \setminus \{t'\}$. Define

$$Z_t = \{\, \mu \in I' \,:\, \Gamma_{j,\overline{k}}(\mu, t) = 0 \,\} \quad \text{and} \quad L_n = \{t \in [-\varepsilon_0, \varepsilon_0] \,:\, m(Z_t) > \frac{1}{n}\,\}$$

Observe that $Z_t \cap Z_{t'} = \emptyset$ if $t \neq t'$ for all $t, t' \in [-\varepsilon_0, \varepsilon_0]$. So, L_n is a finite set, *i.e.* $L = \{t \in J \,:\, m(Z_t) > 0\}$ is countable set. □

4. Proof of the main result

4.1. ***Fixing some notation.*** Let $\widetilde{\varphi}$ be a C^∞ diffeomorphism that has a homoclinic tangency associated to sectionally dissipative periodic point p_0. Then, by the Theorem 2.1 there exists φ a C^∞ diffeomorphism, arbitrarily close to $\widetilde{\varphi}$, exhibiting hyperbolic basic sets Λ_1, Λ_2 and periodic points $p_1 \in \Lambda_1$ and $p_2 \in \Lambda_2$ satisfying items (a) to (d) of the Theorem 1.1. Let $\mathcal{U}$ be a neighborhood of φ of hyperbolic continuations of Λ_1 and Λ_2, it means that, there exists a C^∞ function

$$\begin{array}{rcl} \Phi_i : \mathcal{U} & \longrightarrow & C^0(\Lambda_i, M) \\ \psi & \longrightarrow & \Phi_i(\psi) \end{array}$$

such that $\Lambda_i(\psi) = \Phi_i(\psi)(\Lambda_i)$ is a basic set for $\psi \in \mathcal{U}$, where $C^0(\Lambda_i, M)$ is the space of injective and continuous functions $h : \Lambda_i \to M$. In fact, $\Phi_i(\psi)$ conjugates $\varphi|_{\Lambda_i}$ to $\psi|_{\Lambda_i(\psi)}$, $i = 1, 2$.

We denote by r_0 the point of transversal intersection between $W^u(p_2)$ and $W^s(p_1)$. Let $\delta > 0$ be a small constant such that for all $\psi \in \mathcal{U}$, $x \in B_\delta(p_1) \cap \Lambda_1$ and $y \in B_\delta(p_2) \cap \Lambda_2$, $W^u(y, \psi)$ meet transversally in a neighborhood of r_0, where $B_\delta(p_i)$ is the ball of radius δ centered at p_i, $i = 1, 2$.

Let U be a sufficiently small neighborhood of q which is the quadratic tangent point between $W^u(p_1, \varphi)$ and $W^s(p_2, \varphi)$. We take C^∞ coordinates $(V, u) \in [-1, 1]^{n-1} \times [-1, 1]$ in U in such a way that

1. q has coordinates $(0^{n-1}, 0)$;
2. The connected component of $W^s(p_2) \cap U$ containing q is given by $\{u = 0\}$;
3. for $\psi \in \mathcal{U}$ and $y \in B_\delta(p_2) \cap \Lambda_2$ the connected component of $W^s(y, \psi) \cap U$ is given by $\{u = A_2(y)(V < \psi) : v \in [-1, 1]^{n-1}\}$;
4. for $\psi \in \mathcal{U}$ and $x \in B_\delta(p_1) \cap \Lambda_1$ the connected component of $W^u(x, \psi) \cap U$ corresponding to the obvious way to the connected component of

$W^u(p_1,\varphi)\cap U$ containing q is given by
$\{(V(x),u(x))(t,\psi) : t\in[-1,1]\}$;

(5) $\big(V(p_1),u(p_1)\big)(0,\varphi) = (0^{n-1},0)$ and $\partial_t u(p_1)(0,\varphi)=0$.

Furthermore, for each $\psi\in\mathcal{U}$ the maps

$$y \longrightarrow A_2(y)\big([-1,1]^{n-1},\psi\big) \quad \text{and} \quad x \longrightarrow \big(V(x),u(x)\big)\big([-1,1],\psi\big)$$

are continuous in the C^∞ topology and the maps

$$A_2(y) : [-1,1]^{n-1}\times\mathcal{U} \longrightarrow [-1,1] \text{ and } \big(V(x),u(x)\big) : [-1,1]\times\mathcal{U} \longrightarrow [-1,1]^n$$

are C^∞, for all $y\in B_\delta(p_2)\cap\Lambda_2$ and for all $x\in B_\delta(p_1)\cap\Lambda_1$.

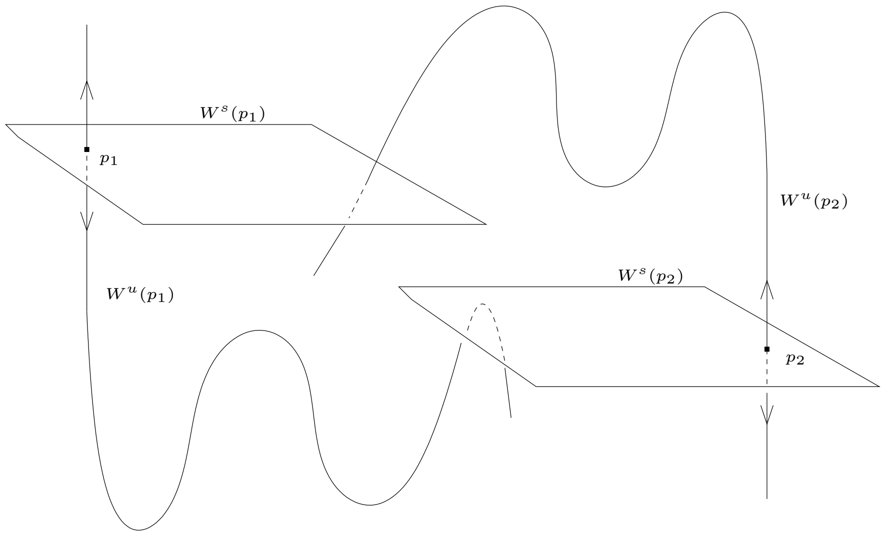

FIGURE 2. *Heteroclinic tangency*

4.2. ***Control of the orbits.*** As in the case of dimension two, we need some control of the orbits of strange attractors.

Suppose that there is $\widehat{\varphi}\in\mathcal{U}$ with periodic points $Q_1\in B_\delta(p_1)\cap\Lambda_1$ and $Q_2\in B_\delta\cap\Lambda_2$ of periods k_1 and k_2 respectively such that $W^u(Q_1)$ and $W^s(Q_2)$ are tangent (quadratically) inside U. Assume that $\widehat{\varphi}^{k_1}$ linearizable near Q_1 and $\widehat{\varphi}^{k_2}$ linearizable near Q_2. Take a one-parameter family $\{\varphi_\mu\}\subset\mathcal{U}$ with $\varphi_0=\widehat{\varphi}$ generically unfolding the tangency. By the Theorem 3.1 and the Theorem 3.2 there are a sequence of host intervals $\Omega_n\to 0$, as $n\to+\infty$, subsets $E_n\subset\Omega_n$ with $m(E_n)>0$ and integers $k_n\to+\infty$ as $n\to+\infty$, such that for $\mu\in E_n$, $\widehat{\varphi}_\mu^{k_n}$ has a nonhyperbolic strange attractor $A_n=A_n(\mu)$ inside U. Then, as in two-dimensional case, we can take U, $\mathcal{U}$ and δ sufficiently small such that for some $n_0>0$ sufficiently large we have that for all $n\geq n_0$,

$$\varphi_\mu^j(A_n)\cap U=\emptyset,\;\; 0<j<k_n,$$

and $\varphi_\mu(U)\cap U=\emptyset$ and $\varphi_\mu^{-1}(U)\cap U=\emptyset$. This implies that any perturbation done inside U but outside a neighborhood of the strange attractor A_n does not affect the remaining of the orbit.

4.3. ***Persistence of the tangency.*** In this subsection we take $\mathcal{U}$, U, and δ as in the previous section. Recall that φ have hyperbolic basic sets Λ_1 and Λ_2 with periodic points $p_1\in\Lambda_1$ and $p_2\in\Lambda_2$ satisfying items (a) to (d) of the Theorem 2.1. Let U_2 be a neighborhood of Λ_2 such that $W^s(\Lambda_2)$ admits an extension to a C^1 foliation $\mathcal{F}_2^s=\mathcal{F}_2^s(\psi)$ defined in U_2. By C^1 we mean here that the tangent spaces to the leaves $T_z\mathcal{F}_2^s(z)$ vary in C^1 fashion with the point z. $\mathcal{F}_2^s$ depends continuously on $\psi\in\mathcal{U}$. Clearly we can take q, the point of tangency between $W^s(p_2)$ and $W^u(p_1)$, to belong to U_2 and $U\subset U_2$.

Now we need the following implicit function,

Lemma 4.1. (Implicit Function) *Let $X\subset\mathbb{R}^n$ be a compact set and $I\subset\mathbb{R}$ be a compact interval. Let $F:X\times I\to\mathbb{R}$ be a intrinsically C^1 map and $(x_0,t_0)\in X\times\mathrm{int}(I)$ be such that*

$$F(x_0,t_0)=0 \text{ and } \Delta F_x(t_0,t_0)\neq 0. \tag{11}$$

Then, there exist $V\subset X$ a compact neighborhood of x_0 and unique intrinsically C^1 map $f:V\to I$ such that $f(x_0)=t_0$ and $F(x,f(x))=0$, for all $x\in V$.

We apply this lemma of the following way, we define $\xi_s=\xi_s(\psi)$ a C^1 vector field on U orthogonal to the leaves of $\mathcal{F}_2^s(\psi)$. By the Proposition 2.3, $W^u(\Lambda_1)\cap U$ contains an intrinsically C^1 diffeomorphic image Y of $X\times I$, where X is a small neighborhood of p_1 in $W^s_{loc}(p_1)\cap\Lambda_1$ and I is a compact interval. Let $\xi_u=\xi_u(\psi)$ be some intrinsically C^1 vector field on Y tangent to the leaves of $W^u(\Lambda_1(\psi))\cap U$ and finally we define $F(y,\psi)=\xi_u(y,\psi)\cdot\xi_s(y,\psi)$, which is a intrinsically C^1 map and the hypotheses (11) in the Lemma 4.1 correspond to have a quadratic tangency at q between $W^u(p_1,\varphi)$ and $W^s(p_2,\varphi)$. Observe that $F(q,\varphi)=0$. Then, by the lemma we get that there exist V_1 a compact neighborhood of p_1 in $W^s_{loc}(p_1)\cap\Lambda_1(\psi)$ and $\pi_{1\varphi}:V_1\to W^u(\Lambda_1(\psi))\cap U$ an intrinsically C^1 map such that each $\pi_{1\varphi}(x)$, $x\in V_1$, is a point of tangency between $W^u(x)$ and some leaf of $\mathcal{F}_2^s(\varphi)$.

On the other hand, we also introduce $\pi^s_\psi:U\to W^u_{loc}(p_2)$ the projection along the leaves of $\mathcal{F}_2^s(\psi)$ onto $W^u_{loc}(p_2)$, for all $\psi\in\mathcal{U}$. We identify $W^u_{loc}(p_2)$ with an interval in $\mathbb{R}$ by the following C^1 diffeomorphism $\mathcal{X}_\psi:W^u_{loc}(p_2)\to\mathbb{R}$ with $\mathcal{X}_\psi(p_2)=0$, for all $\psi\in\mathcal{U}$. If it is necessary, we perturbed φ, so that $\Delta\pi_{1\varphi}(p_1,p_1)\cdot IT_{p_1}(\Lambda_1\cap W^s_{loc}(p_1))$ is not tangent to the stable leaf $\mathcal{F}_2^s(q)$, see section 7 in [11]. Then, $\mathcal{X}_\varphi\circ\pi^s_\varphi\circ\pi_{1\varphi}$ is a intrinsically C^1 map and $\Delta(\mathcal{X}_\varphi\circ\pi^s_\varphi\circ\pi_{1\varphi})(p_1,p_1)|E^w$ is bijective, that means, by Proposition 2.4, that $\tau^u(\Lambda_1,p_1)=\tau(\mathcal{X}_\varphi\circ\pi^s_\varphi\circ\pi_{1\varphi}(V_1),0)$. We put $K^u_\varphi=\mathcal{X}_\varphi\circ\pi^s_\varphi\circ\pi_{1\varphi}(V_1)$, *i.e.* $\tau(K^u_\varphi,0)=\tau^u(\Lambda_1,p_1)$.

Now, we define $K^s_\psi=\mathcal{X}_\psi(W^u_{loc}(p_2)\cap\Lambda_2)$ and $K^u_\psi=\mathcal{X}_\psi\circ\pi_{2\psi}\circ\pi_{1\psi}(V_1)$, for $\psi\in\mathcal{U}$, which are near K^s_φ and K^u_φ, respectively, if ψ is near to φ. By the Section 2.3 we have that $\tau^u(\Lambda_1(\psi),p_1)=\tau(K^w_\psi,0)$, for all $\psi\in\mathcal{U}$, where $K^w_\psi=\pi^w_\psi(V_1)$, taking π^w_ψ as we defined it in the section 2.3 and V_1 sufficiently small compact neighborhood of $W^s_{loc}(p_1, \psi) \cap \Lambda_1$. The value $\tau(K^w_\psi, 0)$ varies continuously with the diffeomorphism $\psi \in \mathcal{U}$ in C^2 topology and the sets K^s_ψ and K^w_ψ are dynamically defined Cantor sets, see [11].

The applications $h^u_\psi : K^w_\varphi \to K^w_\psi$ defined by $h^u_\psi(x) = \pi^w_\psi \circ \Phi_1(\psi) \circ (\pi^w_\varphi)^{-1}(x)$ and $h^s_\psi : K^s_\varphi \to K^s_\psi$ defined by $h^s_\psi(x) = \mathcal{X}_\psi \circ \Phi_2(\psi) \circ (\mathcal{X}_\varphi)^{-1}(x)$ are the natural equivalence between K^w_φ and K^w_ψ, and, K^s_φ and K^s_ψ, respectively. By the Theorem 1.1 we have

$$\tau(K^w_\varphi, 0) \cdot \tau(K^s_\varphi, 0) \geq 1 + t_0, \text{ for some } t_0 > 0$$

By continuity of thickness, the definition of local thickness and considering $\mathcal{U}$ small enough, there is $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$ we can find Cantor sets $\widetilde{K}^w_\varphi \subset K^w_\varphi \cap B_\delta(0)$ and $\widetilde{K}^s_\varphi \subset K^s_\varphi \cap B_\delta(0)$ whose continuations $\widetilde{K}^w_\psi$ of $\widetilde{K}^w_\varphi$ and $\widetilde{K}^s_\psi$ of $\widetilde{K}^s_\varphi$ satisfy

$$\tau(\widetilde{K}^w_\psi) \cdot \tau(\widetilde{K}^s_\psi) \geq 1 + t_0/2 \ \forall \psi \in \mathcal{U}$$

Now define the following functions $\vartheta^u_\psi : K^w_\varphi \to \mathbb{R}$ by

$$\vartheta^u_\psi(x) = \mathcal{X}_\psi \circ \pi^s_\psi \circ \pi_{1\psi} \circ \Phi_1(\psi) \circ (\pi^w_\varphi)^{-1}(x)$$

and $\vartheta^s_\psi : K^s_\varphi \to \mathbb{R}$ by $\vartheta^s_\psi(x) = \mathcal{X}_\psi \circ \Phi_2(\psi) \circ \mathcal{X}^{-1}_\varphi(x)$. Then,

$$\tau((\vartheta^u_\psi(\widetilde{K}^w_\varphi)) \cdot \tau(\vartheta^s_\psi(\widetilde{K}^s_\varphi)) \geq 1 + \frac{t_0}{2}, \ \ \forall \psi \in \mathcal{U}.$$

Let $\{\widehat{\varphi}_\mu\}_{\mu \in [-1,1]} \subset \mathcal{U}$ be a one-parameter family of diffeomorphisms, with $\widehat{\varphi}_0 = \varphi$, generically unfolding the tangency between $W^u(p_1, \varphi)$ and $W^s(p_2, \varphi)$. Then, for $\delta_0 > 0$ sufficiently small and considering that Cantor sets $\widetilde{K}^w_\varphi$ and $\widetilde{K}^s_\varphi$ as we defined above, there exists a parameter value μ_0 close to $\mu = 0$ such that the pair $\langle \vartheta^u_{\widehat{\varphi}_{\mu_0}}(\widetilde{K}^w_\varphi)\, , \, \vartheta^s_{\widehat{\varphi}_{\mu_0}}(\widetilde{K}^s_\varphi) \rangle$ is a stable linked.

Let $\mathcal{Z}$ be a small neighborhood of $\{\widehat{\varphi}_\mu\}$ in the space of one-parameter families of diffeomorphisms and I an interval such that for each family $\{\varphi_\mu\} \in \mathcal{Z}$ we have that $\langle \vartheta^u_{\varphi_\mu}(\widetilde{K}^w_\varphi)\, , \, \vartheta^s_{\varphi_\mu}(\widetilde{K}^s_\varphi) \rangle$ is a linked pair, for all $\mu \in I$. We define

$$\mathcal{W} = \left\{ \varphi_\mu \in \mathcal{U} \, : \, \{\varphi_\mu\} \in \mathcal{Z} \ \text{ and } \ \mu \in I \right\}$$

which is an open set by the openess of linking property. Observe that $\mathcal{Z}$ is arbitrarily close to φ. $\mathcal{W}$ is called *open set of persistence of tangencies.*

Lemma 4.2. (Main Lemma) *Let $I' \subset I$ be any subinterval. Then, there exists a residual subset $\mathcal{R}$ of $\mathcal{Z}$ such that for each family $\Psi = \{\psi_\mu\} \in \mathcal{R}$, there is a parameter value $\overline{\mu} \in I'$ such that the corresponding map $\psi_{\overline{\mu}}$ exhibit infinitely many nonhyperbolic strange attractors.*

Proof of Theorem A: Let $\widetilde{\varphi}$ be a C^∞ diffeomorphism with a homoclinic tangency associated to a sectionally dissipative saddle point. Then, by the Theorem 2.1 there exists φ arbitrarily near to $\widetilde{\varphi}$ and as we see above there exists $\mathcal{W}$ an open set arbitrarily near to φ, which, by the Main Lemma, satisfies that every diffeomorphism $\psi \in \mathcal{W}$ can be approximated by a diffeomorphism displaying infinitely many nonhyperbolic strange attractors. Taking $\mathcal{U}_{\varphi_0}$ the union of this open sets we obtain Theorem A.

Corollary 4.1. *There exists a residual subset $\mathcal{R}$ of $\mathcal{Z}$ such that for each family $\Psi = \{\psi_\mu\} \in \mathcal{R}$ the set of parameter values $\mu \in I$, for which ψ_μ has infinitely many nonhyperbolic strange attractors, is dense in I.*

Proof of Theorem B: First, we see the following remark of the Theorem 2.1

Remark 4: Let $\Phi = \{\varphi_\mu\}$ be a C^∞ one-parameter family of diffeomorphisms such that φ_0 has a homoclinic tangency associated to sectionally dissipative saddle point. Among the families with this property, there exist a residual subset which satisfies the following conditions: C^2 linearizability of saddle point, quadratic tangency at φ_0, generic unfolding as μ varies through 0, and conditions (1),(2) of the Chapter 2. Furthermore, we can see that, in the considerations done above, the Theorem 2.1 holds for a generic subset of C^∞ families of diffeomorphisms (see [11], Sect. 7), that means, if $\{\varphi_\mu\}$ belongs in this generic subset, there exists a sequence of parameter values $\mu_n \to 0$ such that $\varphi = \varphi_\mu$ satisfies items (a) to (d) of the Theorem 3.1 and the subfamilies $\{\psi_\nu\}$ with $\psi_\nu = \varphi_{\mu+\nu}$, ν near zero, generically unfold the heteroclinic tangency of item (b) of the Theorem 2.1.

Then, proof of the Theorem B is followed by of the Corollary 4.1 above and, keep in mind, that countable intersection of residual subsets is a residual subset.

4.4. ***Proof of the Main Lemma.*** The proof of the Main Lemma will be done by induction. In this subsection, B_r denotes the ball of radius r, $B_r(x)$ denotes the ball of radius r and center $x \in M$ and m denote the Lebesgue measure in $\mathbb{R}$. We also denote $\pi_{2\psi_\mu}$ the restriction of π^s_ψ to $\pi_{1\psi}(V_1)$. Let $U, \mathcal{U}$ as in the previous section and $\mathcal{Z} \supset \mathcal{R}_1 \supset \mathcal{R}_2 \supset \cdots \supset \mathcal{R}_N \supset \cdots$ be a sequence of sets satisfy that

a.- for $N \geq 1$ and each family $\Psi = \{\psi_\mu\} \in \mathcal{R}_N$, there exists a compact set $E_N = E_N(\Psi) \subset I'$, $m(E_N) > 0$, such that for $\mu \in E_N$, ψ_μ has N distinct strange attractors $S_1 = S_1(\Psi), \ldots, S_N = S_N(\Psi)$; furthermore,
  - a.1) each attractor S_i, $i = 1, \cdots, N$, is generated as in Chapter 2, (Theorems 2.1 and 2.2) together with the section 3.2 and the orbit of S_i intersects U only once, inside $B_{r_i} \subset U$, where $B_{r_i} \cap B_{r_j} = \emptyset$, $i \neq j$;
  - a.2) $E_{N+1}(\Psi) \subset E_N(\Psi)$;

b.- for each $\Psi = \{\psi_\mu\} \in \mathcal{R}_N$ and μ in a neighborhood of the convex hull of $E_N(\Psi)$, there are bridges C^s_i, D^s_N of $\widetilde{K}^s_\varphi$ and C^u_i, D^u_N of $\widetilde{K}^w_\varphi$, $i = 1, \cdots, N$, such that
  - b.1) their images $\vartheta^s_{\psi_\mu}(C^s_i) = C^s_i(\Psi;\mu)$ and $\vartheta^u_{\psi_\mu}(C^u_i \cap \widetilde{K}^w_\varphi) = C^u_i(\Psi;\mu)$ form a stable linked pair, see figure 3;
  - b.2) images of their intersections in U satisfy
$$\widetilde{C}_N(\Psi;\mu) = (\pi_{2\psi_\mu})^{-1} \circ \mathcal{X}^{-1}_{\psi_\mu}(C^s_i(\Psi;\mu) \cap C^u_i(\Psi;\mu)) \subset B_{r_i};$$
  - b.3) images of D^s_N, $\vartheta^s_{\psi_\mu}(C^s_N) = C^s_N(\Psi;\mu)$ and D^u_N, $\vartheta^s_{\psi_\mu}(D^u_N \cap \widetilde{K}^w_\varphi) = D^u_N(\Psi;\mu)$ form a stable linked pair;

b.4) images of their intersections in U, satisfy

$$\widetilde{D}_N(\Psi;\mu) = (\pi_{2\psi_\mu})^{-1} \circ \mathcal{X}_{\psi_\mu}^{-1}(D_N^s(\Psi;\mu) \cap D_N^u(\Psi;\mu)) \subset B_{\varepsilon_N},$$

where $B_{\varepsilon_N} \subset U$ and $B_{r_i} \cap B_{\varepsilon_N} = \emptyset$.

We will show that $\mathcal{R}_1$ is open and dense in $\mathcal{Z}$ and $\mathcal{R}_{N+1}$ is open and dense in $\mathcal{R}_N$, for all $N \geq 1$. Then, the proof of the Main Lemma follows taking $\mathcal{R} = \bigcap_{N\geq 1} \mathcal{R}_N$ which is a residual subset of $\mathcal{Z}$ and for each $\Psi = \{\psi_\mu\} \in \mathcal{R}$, there exists a sequence of the following way, $I' \supset E_1 \supset E_2 \supset \cdots \supset E_N \supset \cdots$ of compact sets as item (a) above. Therefore, for each $\overline{\mu} \in \bigcap_{N\geq 1}$, ψ_μ exhibits infinitely many strange attractors.

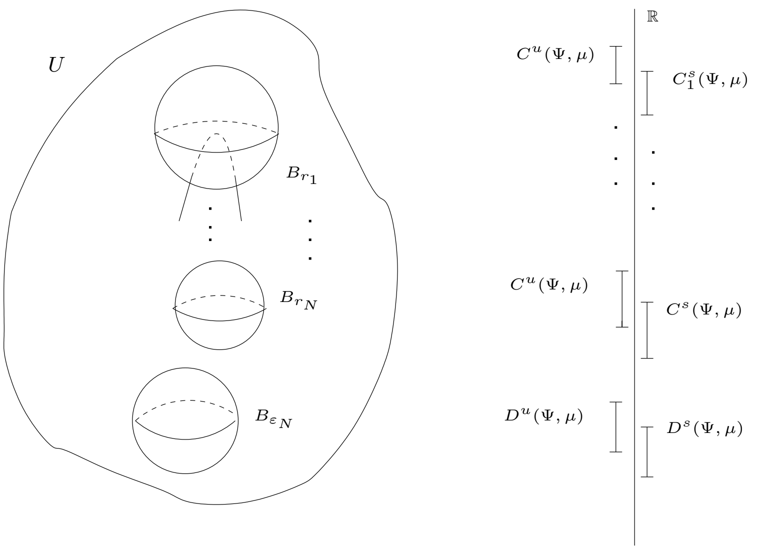

FIGURE 3. *induction*

The openness of $\mathcal{R}_N$, is a consequence of the following fact, linking property is an open condition (i.e. item (b) is open) and applying Lemma 3.1 to item (a)(i.e. it is open). Now we will prove that $\mathcal{R}_{N+1}$ is dense in $\mathcal{R}_N$, $N \geq 1$ (the proof also shows that $\mathcal{R}_1$ is dense in $\mathcal{Z}$; for that, for $\Psi = \{\psi_\mu\} \in \mathcal{Z}$ we take $E_0(\Psi) = I'$, D_0^s the convex hull of K_φ^s, $D_0^u = K_\varphi^u$ and proceed as below with $N = 0$).

Let $\Psi = \{\psi_\mu\} \in \mathcal{R}_N$. We show that after four perturbation of the family $\{\psi_\mu\}$, to be described below, we get a family $\{\varphi_\mu\} \in \mathcal{R}_{N+1}$ C^∞ arbitrarily near to Ψ.

Part 1. Let μ_N be a total density point of E_N, *i.e.*

$$m\big(E_N \cap [\mu_N - \delta, \mu_N + \delta]\big)/(2\delta) \longrightarrow 1, \quad \text{as} \quad \delta \to 0.$$

Let d_0 be the distance from

$$\bigcap_{\mu \in E_N} \widetilde{D}_N(\Psi,\mu) \quad \textit{to} \quad \mathbb{R}^n \setminus B_{\varepsilon_N}.$$

Take $0 < \gamma_1 < d_0/2$ and q_N be the center of B_{ε_N}. Define the following function

$$\xi_N(V,u) = \xi(\frac{3}{\gamma_1}[\|(V,u) - q_N)\| - (\varepsilon_N - \gamma_1)])$$

where ξ is a C^∞ bump function satisfying

$$\begin{cases} \xi(s) = 0, & \text{if} \quad s \geq 2 \\ \xi(s) = 1, & \text{if} \quad s \leq 1 \\ 0 \leq \xi(s) \leq 1, & \quad \forall s \in \mathbb{R}. \end{cases}$$

for α small, we define the C^∞ diffeomorphism

$$\begin{array}{rcll} G_\alpha : M & \longrightarrow & M & \\ x & \longrightarrow & x, \text{ if} & x \in M \setminus U \\ (V,u) & \longrightarrow & (V, u + \alpha \cdot \xi_N(V,u)), & \text{if } x \in U \end{array}$$

First, note that $G_\alpha \circ \psi_\mu = \psi_\mu$. Then, for $\alpha = 0$ and for all μ. For each α small, denote $G_\alpha \circ \Psi$ the family $\{G_\alpha \circ \psi_\mu\}$. The Cantor sets $D^s_N(G_\alpha \circ \Psi; \mu_N)$, $D^u_N(G_\alpha \circ \Psi; \mu_N)$ have $0 \neq \alpha$-velocity with respect each other. By item (b) of induction hypothesis $D^s_N(\Psi; \mu_N)$, $D^u_N(\Psi; \mu_N)$ form a linked pair, we get that all the hypotheses of the Proposition 2.2 (Linking Lemma) are satisfied. Then, there is α_0 arbitrarily small such that the linked pair above has two stable sublinked. We also have,

$$\|G_{\alpha_0} \circ \Psi - \Psi\|_{C^r} \leq \text{Const.}|\alpha_0| \cdot \|\xi_N\|_{C^r} \leq \text{Const.}|\alpha_0| \cdot \left(\frac{3}{\gamma_1}\right)^r.$$

Observe that the perturbation above does not affect $U \setminus B_{\varepsilon_N}$, *i.e.* does not affect items (b.1) and (b.2) of induction hypothesis. Take $\Psi^1 = \{G_{\alpha_0} \circ \psi_\mu\} = \{\psi^1_\mu\}$ and let

$$\langle C^s_{N+1}(\Psi^1; \mu_N), C^u_{N+1}(\Psi^1; \mu_N)\rangle \text{ and } \langle D^s_{N+1}(\Psi^1; \mu_N), D^u_{N+1}(\Psi^1; \mu_N)\rangle$$

be the sublinks pairs of $\langle D^s_N(\Psi^1; \mu_N), D^u_N(\Psi^1; \mu_N)\rangle$. where (for $\beta = s, u$)

$$C^\beta_{N+1}(\Psi^1; \mu_N) = \vartheta^\beta_{\psi^{(1)}_{\mu_N}}(C^\beta_{N+1} \cap \widetilde{K}^s_\varphi) \text{ and } D^\beta_{N+1}(\Psi^1; \mu_N) = \vartheta^\beta_{\psi^{(1)}_{\mu_N}}(D^\beta_{N+1} \cap \widetilde{K}^w_\varphi),$$

for some bridges C^s_{N+1}, C^u_{N+1} of $\widetilde{K}^s_\varphi$ and D^s_{N+1}, D^u_{N+1} of $\widetilde{K}^w_\varphi$. Since the sublinks are distinct, there exist $r_{N+1} > 0$ and $\varepsilon_{N+1} > 0$ such that

$$\widetilde{C}_{N+1}(\Psi^1; \mu) = (\mathcal{X}_{\psi_\mu} \circ \pi_{2\psi_\mu})^{-1}\big(\widehat{C}^s_{N+1}(\Psi^1, \mu) \cap C^u_{N+1}(\Psi^1; \mu)\big) \subset B_{r_{N+1}} \subset B_{\varepsilon_N}$$

and

$$\widetilde{D}_{N+1}(\Psi^1; \mu) = (\mathcal{X}_{\psi_\mu} \circ \pi_{2\psi_\mu})^{-1}\big(\widehat{D}^s_{N+1}(\Psi^1, \mu) \cap D^u_{N+1}(\Psi^1; \mu)\big) \subset B_{\varepsilon_{N+1}} \subset B_{\varepsilon_N}$$

and $B_{r_{N+1}} \cap B_{\varepsilon_{N+1}} = \emptyset$.

Part 2. Take $\gamma_2 > 0$ small and $B_{r_{N+1} - 2\gamma_2} \subset B_{r_{N+1}}$ concentric to the ball $B_{r_{N+1}}$, *i.e.* they have the same center. On the other hand, by gap lemma, $C^s_{N+1}(\Psi^1; \mu_N) \cap C^u_{N+1}(\Psi^1; \mu_N) \neq \emptyset$. For $\gamma_2 > 0$ sufficiently small we obtain that the tangency between $W^u(x)$ and $W^s(y)$, for some $x \in \Lambda_1(\psi^{(1)}_{\mu_N}) \cap B_\delta(p_1)$ and $y \in \Lambda_2(\psi^{(1)}_{\mu_N}) \cap B_\delta(p_2)$, is inside $B_{r_{N+1} - 2\gamma_2}$. Then, there are periodic points $Q_1 \in \Lambda_1(\psi^{(1)}_{\mu_N})$ near x and $Q_2 \in \Lambda_2(\psi^{(1)}_{\mu_N}) \cap B_\delta(p_2)$ near y such that $W^u(Q_1, \psi^{(1)}_{\mu_N})$ and $W^s(Q_2, \psi^{(1)}_{\mu_N})$ cross $B_{r_{N+1}-2\gamma_2} \subset B_{r_{N+1}}$ and

$$|A(Q_2)(V, \psi^{(1)}_{\mu_N}) - A(x)(V, \psi^{(1)}_{\mu_N})| < \frac{1}{2}\delta_1;$$

$$\|(V(Q_1), u(Q_1))(t, \psi^{(1)}_{\mu_N}) - (V(y), u(y))(t, \psi^{(1)}_{\mu_N})\| < \frac{1}{2}\delta_1$$

for every $V \in [-1,1]^{n-1}$, $t \in [-1,1]$ and $0 < 2\delta_1 < \frac{1}{2}\gamma_2$. Let n_1 and n_2 be the periods of Q_1 and Q_2, respectively, and fix $\beta > 0$ small. Then, by the Lemma 2.2 we obtain a one-parameter family of diffeomorphisms $\Psi^2 = \{\psi^{(2)}_\mu\}$ arbitrarily near Ψ^1, independently β, such that $(\psi^{(2)}_\mu)^{n_1}$ is C^k linearizable near Q_1 and $(\psi^{(2)}_\mu)^{n_2}$ is C^k linearizable near Q_2, $k \geq 4$, for almost every point $\mu \in [\mu_N - \beta, \mu_N + \beta]$. Since Ψ^2 is arbitrarily near to Ψ^1, and by the Lemma 2.1 there exists a compact set $E_N(\Psi^2)$ with $m(E_N(\Psi^2) > 0$ and $E_N(\Psi^2) \subset [\mu_N - \beta, \mu_N + \beta]$ such that $E_N(\Psi^2)$ satisfies item (a) of induction hypothesis. Then, we consider $\mu'_N \in E_N(\Psi^2)$ a total density point such that $(\psi^{(2)}_{\mu'_N})^{n_1}$ is C^k linearizable near Q_1 and $(\psi^{(2)}_{\mu'_N})^{n_2}$ is C^k linearizable near Q_2.

The family Ψ^2 can be chosen arbitrarily close to Ψ^1 and μ'_N sufficiently near to μ_N such that $D^u_{N+1}(\Psi^2; \mu'_N)$ and $D^s_{N+1}(\Psi^2; \mu'_N)$ form still a linked pair,

$$\begin{aligned}\widetilde{D}_{N+1}(\Psi^2; \mu'_N) &= (\mathcal{X}_{\psi^2_{\mu'_N}} \circ \pi_{2\psi^2_{\mu'_N}})^{-1}\big(\widehat{D}^s_{N+1}(\Psi^2, \mu'_N) \cap D^u_{N+1}(\Psi^2; \mu'_N)\big)\\ &\subset B_{\varepsilon_{N+1}}\end{aligned}$$

and $W^u(Q_1, \psi^{(2)}_{\mu'_N})$ and $W^s(Q_2, \psi^{(2)}_{\mu'_N})$ cross $B_{r_{N+1}-2\gamma_2}$. Moreover,

$$|A(Q_2)(V, \psi^{(2)}_{\mu'_N}) - A(Q_2)(V, \psi^{(1)}_{\mu_N})| < \frac{1}{2}\delta_1;$$

$$\|(V(Q_1), u(Q_1))(t, \psi^{(2)}_{\mu'_N}) - (V(Q_1), u(Q_1))(t, \psi^{(1)}_{\mu_N})\| < \frac{1}{2}\delta_1$$

where $\delta_1 + \beta < \frac{1}{2}\gamma_2$.

Part 3. Let $\widetilde{q}_N$ be the center of the ball $B_{r_{N+1}}$, and define the following map

$$\widetilde{\xi}_N(V, u) = \xi(\frac{3}{\gamma_2}[\|(V, u) - \widetilde{q}_N)\| - (r_{N+1} - \gamma_2)])$$

Equal to the first perturbation, we define the diffeomorphism $\widetilde{G}_\alpha$, for α small, by

$$\begin{aligned}\widetilde{G}_\alpha : M &\longrightarrow M\\ x &\longrightarrow x, \text{ if } x \in M \setminus U\\ (V, u) &\longrightarrow (V, u + \alpha \cdot \widetilde{\xi}_N(V, u)), \text{ if } x \in U.\end{aligned}$$

Then, there is α_1, with $|\alpha_1| \leq \text{const.}(2\delta_1 + \beta) < \frac{1}{2}\gamma_2$, such that $W^u(Q_1, \widetilde{G}_{\alpha_1} \circ \Psi^2)$ and $W^s(Q_2, \widetilde{G}_{\alpha_1} \circ \Psi^2)$ have a tangency inside $B_{r_{N+1}-\gamma_2}$. Take $\Psi^3 = \widetilde{G}_{\alpha_1} \circ \Psi^2$ observe that $E_N(\Psi^2) = E_N(\Psi^3)$ and $(\psi^{(3)}_{\mu'_N})^{n_1}$ is C^k linearizable near Q_1 and $(\psi^{(3)}_{\mu'_N})^{n_2}$ is C^k linearizable near Q_2. Also,

$$\|\Psi^2 - \Psi^3\|_{C^r} \leq \text{Const.}|\alpha_1| \left(\frac{3}{\gamma_2}\right)^r.$$

Part 4. Define $\widetilde{G}_\alpha \circ \Psi^3 = \widetilde{G}_{\alpha+\alpha_1} \circ \Psi^2$. As the family $\{\widetilde{G}_{\alpha_1} \circ \psi^{(2)}_\mu\}$ generically unfolds the tangency for the parameter value $\mu = \mu'_N$, for each α small, there exists $\mu_T(\alpha)$ such that $W^u(Q_1, \widetilde{G}_\alpha \circ \psi^{(3)}_{\mu_T(\alpha)})$ and $W^s(Q_2, \widetilde{G}_\alpha \circ \psi^{(3)}_{\mu_T(\alpha)})$ are tangent. Although, the family $\{\widetilde{G}_\alpha \circ \psi^{(3)}_\mu\}$ generically unfolds this tangency. Observe that $\mu_T(0) = \mu'_N$ and if α is sufficiently small, $(\widetilde{G}_\alpha \circ \psi^{(3)}_\mu)^{n_1}$ is C^k linearizable near Q_1 and $(\widetilde{G}_\alpha \circ \psi^{(3)}_\mu)^{n_2}$ is C^k linearizable near Q_2, for α near to $\alpha = 0$ and μ near to $\mu = \mu'_N$. As $\mu'_N \in E_N(\Psi^3)$ is a total density point, there is $t_0 > 0$ such that

$$m(E_N(\Psi^3) \cap [\mu'_N - t, \mu'_N + t]) \geq t, \ \ \forall 0 < t \leq t_0 \tag{12}$$

Let Ω be a host interval of strange attractors in the μ-space for the family Ψ^3 such that $|\Omega| < t_0$ and $m(E(\Psi^3)) > \frac{3}{4}|\Omega|$. Take Ω satisfying control of the orbits as in Section 4.2. Then, by the discussion in section 3.2 (summarized in Remark 2) consider $\Omega(\alpha)$ be the natural continuation of $\Omega = \Omega(0)$ arbitrarily near $\mu_T(\alpha)$ (i.e. $|\Omega(\alpha)| \leq t_0$) corresponding to the family $\{\widetilde{G}\alpha \circ \psi^{(3)}_\mu\}$ such that the relative measure of $E(\alpha) \subset \Omega(\alpha)$ of strange attractors satisfies $m(E(\alpha)) \geq \frac{3}{4}|\Omega(\alpha)|$. We may suppose, without loss of generality, that $\Omega(\alpha)$ is on the right of $\mu_T(\alpha)$, for α small, and $\mu_T(\alpha)$ decreases as α increases. So that, we can choose $\alpha_2 > 0$ close to $\alpha = 0$ and $\Omega = \Omega(0)$ near $\mu_T(0) = \mu'_N$ such that

$$\mu_T(\alpha_2) < \mu < \mu_T(0), \ \ \forall \mu \in \Omega(\alpha_2)$$

If we denote by $\mu_c(\alpha)$ the center of the host interval $\Omega(\alpha)$. Then, there exists α_3 with $0 < \alpha_3 < \alpha_2$ such that $\mu_c(\alpha_3) = \mu_T(0) = \mu'_N$. From this and (12) follow (even using that μ'_N is a total density point of $E_N(\Psi^3) = E_N(\widetilde{G}_\alpha \circ \Psi^3)$, for all α small) that

$$m(E_N(\widetilde{G}_{\alpha_3} \circ \Psi^3) \cap E(\alpha_3)) \geq \left(\frac{3}{4} - \frac{1}{2}\right)|\Omega(\alpha_3)| > 0.$$

Finally, we take $\Phi = \{\varphi_\mu\} = \widetilde{G}_{\alpha_3} \circ \Psi^3$ and $E_{N+1} = E_N(\Phi) \cap E(\alpha_3)$. Also,

$$|\Phi - \Psi^3\|_{C^r} \leq \text{Const.} \left(|\alpha_3| \left(\frac{3}{\gamma_2}\right)^r\right)$$

We conclude that

$$\begin{aligned} \|\Phi - \Psi\|_{C^r} &\leq \text{Const.}\left(|\alpha_0|\left(\tfrac{3}{\gamma_1}\right)^r + |\alpha_1|\left(\tfrac{3}{\gamma_2}\right)^r + |\alpha_3|\left(\tfrac{3}{\gamma_2}\right)^r\right) \\ &+ \|\Psi^1 - \Psi^2\|_{C^r} \end{aligned}$$

α_0 can be taken arbitrarily small with respect to γ_1, α_1 and α_3 can be taken also arbitrarily small with respect to γ_2 and by the Lemma 3.1, $\|\Psi^1 - \Psi^2\|_{C^r}$ is arbitrarily small for any r. Then, $\|\Phi - \Psi\|_{C^r}$ is arbitrarily small for any r. This concludes the proof of the Main Lemma.

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