CONFORMALLY FLAT METRICS, CONSTANT MEAN CURVATURE SURFACES IN PRODUCT SPACES AND R-STABILITY OF HYPERSURFACES

Tese Apresentada à Banca Examinadora
do Instituto Nacional de Matemática Pura e Aplicada
como Requerimento Parcial para Obtenção do Grau de
Doutor em Ciências

por

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O conhecimento incha; o Amor é que constrói.

(I Cor 8, 1b)

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Conteúdo

Prefácio

Nesta tese apresentamos resultados em três linhas distintas da área de Geometria Diferencial.

No primeiro capítulo provamos que qualquer bola contida na bola fechada menos um conjunto finito de pontos é estritamente convexa em qualquer métrica Riemanniana, completa conforme à métrica Euclidiana, que tem curvatura escalar constante positiva e curvatura média do bordo não negativa com respeito à normal que aponta para o interior. A demonstração deste teorema é baseada no estudo de uma equação a derivadas parciais não linear onde, após o uso de uma transformação conveniente, aplicamos o Método dos Planos Móveis.

No segundo capítulo, que faz parte de um trabalho feito em colaboração com J. H. de Lira, estudamos superfícies de curvatura média constante em variedades tri-dimensionais que são produtos de esferas ou planos hiperbólicos pela reta ou pelo espaço Lorentziano unidimensional. Inicialmente fazemos um estudo qualitativo das superfícies de curvatura média constante do tipo espaço que são invariantes por rotações. A seguir mostramos que a diferencial de Abresch-Rosenberg para superfícies imersas nesses espaços pode ser obtida como uma combinação linear de duas diferenciais de Hopf; como subproduto deste fato damos uma nova prova de que essa diferencial é holomorfa quando a superfície tem curvatura média constante. Também apresentamos uma nova prova de que esferas de curvatura média constante nestes espaços são esferas de rotação, bem como a versão Lorentziana deste resultado. Finalizamos apresentando dois teoremas de classificação de discos de curvatura média constante nestes espaços como aplicações do fato que a diferencial de Abresch-Rosenberg é holomorfa.

No terceiro e último capítulo melhoramos um teorema de Alencar, do Carmo e Elbert que dá condições suficientes para que um domínio de uma hipersuperfície r-mínima e r-especial do espaço Euclidiano com curvatura de Gauss-Kronecker diferente de zero seja r-estável. Mostramos que as mesmas condições ainda são suficientes no caso em que a curvatura de Gauss-Kronecker se anula num conjunto de capacidade zero. Para mostrar esse teorema provamos antes uma relação entre auto-valores de certos operadores elípticos de domínios onde retiramos um subconjunto pequeno e a capacidade do conjunto removido.

M.P.A.C.

Chapter 1

Convex Balls in Locally Conformally Flat Metrics

1.1 Introduction

Let B_1 denote the open unit ball of $\mathbb{R}^n$, $n \geq 3$. Given a finite set of points $\Lambda = \{p_1, \ldots, p_k\} \subset B_1$, $k \geq 1$, we will consider a complete Riemannian metric g on $\overline{B_1} \setminus \Lambda$ of constant positive scalar curvature $R(g) = n(n-1)$ and conformally related to the Euclidean metric δ. We will also assume that g has nonnegative boundary mean curvature. Here, and throughout this chapter, second fundamental forms will be computed with respect to the inward unit normal vector.

In this chapter we prove

Theorem 1.1 *If $B \subset B_1 \setminus \Lambda$ is a standard Euclidean ball, then ∂B is convex with respect to the metric g.*

Here, we say that ∂B is *convex* if its second fundamental form is positive definite. Since ∂B is umbilical in the Euclidean metric and the notion of an umbilical point is conformally invariant, we know that ∂B is also umbilic in the metric g. In that case ∂B is convex if its mean curvature h is positive everywhere.

This theorem is motivated by an analogous one on the sphere due to R. Schoen [35]. He shows that if $\Lambda \subset S^n$ $n \geq 3$, is nonempty and g is a complete Riemannian metric on $S^n \setminus \Lambda$, conformal to the standard round metric g_0 and with constant positive scalar curvature $n(n-1)$, then every standard ball $B \subset S^n \setminus \Lambda$ is convex with respect to the metric g. Schoen used this geometrical result to prove the compactness of the solutions to the Yamabe problem in the locally conformally flat case. Later, D. Pollack also used Schoen's theorem to prove a compactness result for the singular

Yamabe problem on the sphere where the singular set is a finite colection of points $\Lambda = \{p_1, \ldots, p_k\} \subset S^n$, $n \geq 3$ (see [31]). In this sense our results can be viewed as the first step to prove compactness for the singular Yamabe problem with boundary conditions.

We shall point out that to find a metric satisfying the hypotheses of Theorem 1.1 is equivalent to finding a positive solution to an elliptic equation with critical Sobolev exponent. The idea of the proof is to get geometrical information from that equation by applying the Moving Planes Method as in [21].

1.2 Preliminaries and Examples

In this section we will introduce some notations and we shall recall some classical results that will be used in the proof of Theorem 1.1. We will also describe an useful example.

Let (M^n, g_0) be a smooth compact orientable Riemannian manifold possibly with boundary, $n \geq 3$. Let us denote by $R(g_0)$ its scalar curvature and by $h(g_0)$ its boundary mean curvature. Let $g = u^{\frac{4}{n-2}} g_0$ be a metric conformal to g_0. Then the positive function u satisfies the following nonlinear elliptic partial differential equation of critical Sobolev exponent

$$\begin{cases} \Delta_{g_0} u - \frac{n-2}{4(n-1)} R(g_0) u + \frac{n-2}{4(n-1)} R(g) u^{\frac{n+2}{n-2}} = 0 & \text{in } M, \\ \frac{\partial u}{\partial \nu} - \frac{n-2}{2} h(g_0) u + \frac{n-2}{2} h(g) u^{\frac{n}{n-2}} = 0 & \text{on } \partial M, \end{cases} \tag{1.1}$$

where ν is the inward unit normal vector field to ∂M.

The problem of existence of solutions to (1.1), when $R(g)$ and $h(g)$ are constants, is referred to as the *Yamabe problem*. It was completely solved when $\partial M = \emptyset$ in a sequence of works, beginning with H. Yamabe himself [38], passing by N. Trudinger [37] and T. Aubin [4], and finally by R. Schoen [36]. In the case of nonempty boundary, J. Escobar solved almost all the cases (see [15], [16]) followed by Z. Han and Y. Li [22], F. Marques [26] and others. In this article, however, we wish to study solutions of (1.1), with $R(g)$ constant, which become singular on finite set of points $\Lambda = \{p_1, \ldots, p_k\} \subset M$. In the case that $\Lambda \subset M$ is a general closed set this is the so called *singular Yamabe problem*. This singular behavior is equivalent, at least in the case that g_0 is conformally flat, to requiring g to be complete on $M \backslash \Lambda$. The existence problem (with $\partial M = \emptyset$) displays a relationship between the size of Λ and the sign of $R(g)$. It is known that for a solution with $R(g) < 0$ to exist, it is necessary and sufficient that $\dim(\Lambda) > \frac{n-2}{2}$ (see [5], [27] and [19]), while if a solution exists with

$R(g) \geq 0$, then $\dim(\Lambda) \leq \frac{n-2}{2}$. Here $\dim(\Lambda)$ stands for the Hausdorff dimension of Λ. Those existence problems are more difficult when $R(g) > 0$ and the most well-known examples are given by the Fowler solutions which we will now discuss briefly.

Example. Let $u > 0$ be such that

$$\begin{cases} \Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}^n \setminus \{0\}, n \geq 3, \\ 0 \text{ is an isolated singularity.} \end{cases} \tag{1.2}$$

In this case $g = u^{\frac{4}{n-2}} \delta$ is a metric of constant scalar curvature $n(n-1)$. Using the invariance under conformal transformations we may work in different background metrics. The most convenient one here is the cylindrical metric $g_{cyl} = d\theta^2 + dt^2$ on $S^{n-1} \times \mathbb{R}$. In this case $g = v^{\frac{4}{n-2}} g_{cyl}$, where v is defined in the whole cylinder and satisfies

$$\frac{d^2 v}{dt^2} + \Delta_\theta v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0. \tag{1.3}$$

One easily verifies that the solutions to equation (1.2) and (1.3) are related by

$$u(x) = |x|^{\frac{2-n}{2}} v(x/|x|, -\log|x|). \tag{1.4}$$

By a deep theorem of Caffarelli, Gidas and Spruck (see [9], Theorem 8.1) we know that v is rotationally symmetric, that is $v(\theta, t) = v(t)$, and therefore the PDE (1.3) reduces to the following ODE:

$$\frac{d^2 v}{dt^2} - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0.$$

Setting $w = v'$ this equation is transformed into a first order Hamiltonian system

$$\begin{cases} \frac{dv}{dt} = w, \\ \frac{dw}{dt} = \frac{(n-2)^2}{4} v - \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}}, \end{cases}$$

whose Hamiltonian energy is given by

$$H(v, w) = w^2 - \frac{(n-2)^2}{4} v^2 + \frac{(n-2)^2}{4} v^{\frac{2n}{n-2}}.$$

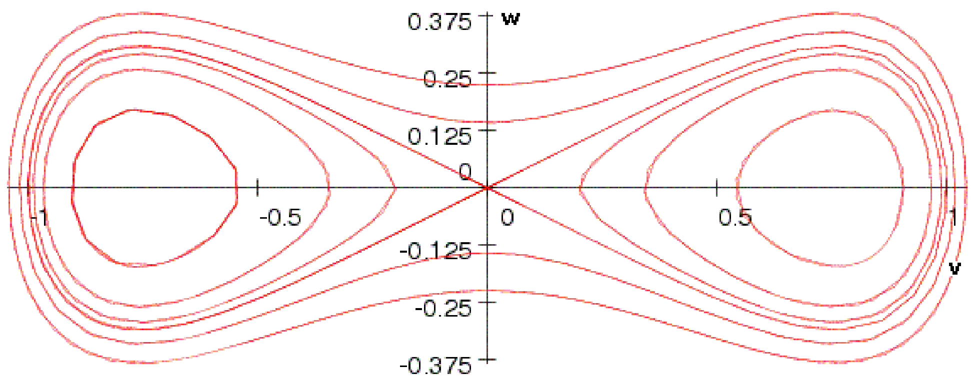

Figure 1.1: Typical phase plane of H

The solutions $(v(t), v'(t))$ describe the level sets of H and we note that $(0,0)$ and $(\pm v_0, 0)$, where $v_0 = (\frac{n-2}{n})^{\frac{n-2}{4}}$, are the equilibrium points. We restrict ourselves to the half-plane $\{v > 0\}$ where $g = v^{\frac{4}{n-2}} g_{cyl}$ has geometrical meaning. On the other hand we are looking for complete metric. Those will be generated by the *Fowler solutions*: the periodic solutions around the equilibrium point $(v_0, 0)$. They are symmetric with respect to v-axis (see figure 1.1) and can be parametrized by the minimum value ε attained by v, $\varepsilon \in (0, v_0]$, (and a translation parameter T). We will denote them by v_ε. We point out that v_0 corresponds to the scaling of g_{cyl} which makes the cylinder have scalar curvature $n(n-1)$. We observe that one obtains the Fowler solutions u_ε in $\mathbb{R}^n \setminus \{0\}$ by using the relation (1.4).

We can now construct metrics satisfying the hypotheses of Theorem 1.1 (with $\Lambda = \{0\}$ an $R(g) = n(n-1)$) from the Fowler solutions. To do this, we just take a Fowler solution v defined for $t \geq t_0$, where t_0 is such that we have $w = \frac{dv}{dt} \leq 0$, or equivalently,

$$h(g) = -\frac{2}{n-2} v^{-\frac{n}{n-2}} \frac{dv}{dt} \geq 0.$$

This concludes our Example.

By another result of Caffarelli, Gidas and Spruck (see Theorem 1.2 in [9]) it is known that, given a positive solution u to

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \tag{1.5}$$

which is defined in the punctured ball $B_1 \setminus \{0\}$ and which is singular at the origin, there exists a unique Fowler solution u_ε such that

$$u(x) = (1 + o(1)) u_\varepsilon(|x|) \text{ as } |x| \to 0.$$

Also, from Theorem 2 in [25], either u extends as a smooth solution to the ball, or there exist positive constants C_1, C_2 such that

$$C_1 |x|^{(2-n)/2} \le u(x) \le C_2 |x|^{(2-n)/2}. \tag{1.6}$$

From now on we will work in the Euclidean space with the metric $g = u^{\frac{4}{n-2}} \delta$. In that context, u is a positive function on $\mathbb{R}^n \setminus \{0\}$ which satisfies (1.5). In the study of the equation (1.5) we will make use of the inversion map in $\mathbb{R}^n$ and some related properties. We begin with the definition:

The map $I : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ defined by $I(x) = \frac{x}{|x|^2}$ is called *inversion with respect to* $S^{n-1}(1)$. The inversion is a *conformal map* that takes a neighborhood of infinity onto a neighborhood of the origin. It follows immediately that I is its own inverse and is the identity on $S^n(1)$.

We will now describe the *Kelvin Transform*, which is closely related to the inversion map. We define the Kelvin transform of a function u as the function $\tilde{u} = \mathcal{K}[u]$ given by

$$\tilde{u}(x) = |x|^{2-n} u(|x|^{-2} x).$$

The Kelvin transform appears naturally when we consider the pull back of $g = u^{\frac{4}{n-2}} \delta$ by I. In fact, since I is conformal we get $I^* g = v^{\frac{4}{n-2}} \delta$ and therefore

$$\begin{aligned} v^{\frac{4}{n-2}}(x) &= I^* g(x)(\partial_{x^i}, \partial_{x^i}) \\ &= u^{\frac{4}{n-2}}(I(x)) \langle I_* \partial_{x^i}, I_* \partial_{x^i} \rangle \\ &= u^{\frac{4}{n-2}}(I(x)) \frac{1}{|x|^4}. \end{aligned}$$

That is $v = \mathcal{K}[u]$.

The main property of the Kelvin transform is that it preserves harmonic functions. Actually, a computation gives that

$$\Delta\tilde{v}(x) = \mathcal{K}[|x|^4 \Delta v].$$

As a consequence we get

Proposition 1.2 *The Kelvin transform preserves the equation (1.5).*

Proof:

Suppose that $\Delta v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0$ on $\Omega \subset \mathbb{R}^n \setminus \{0\}$. Then we get

$$\begin{aligned}
\Delta\tilde{v} &= \mathcal{K}[|x|^4 \Delta v] = \mathcal{K}[-\frac{n(n-2)}{4}|x|^4 v^{\frac{n+2}{n-2}}] \\
&= |x|^{2-n}\Big(-\frac{n(n-2)}{4}|x|x|^{-2}|^4 v^{\frac{n+2}{n-2}}(x|x|^{-2})\Big) \\
&= -\frac{n(n-2)}{4}|x|^{-(n+2)} v^{\frac{n+2}{n-2}}(x|x|^{-2}) \\
&= -\frac{n(n-2)}{4}\Big(|x|^{-(n-2)} v(x|x|^{-2})\Big)^{\frac{n+2}{n-2}} \\
&= -\frac{n(n-2)}{4}\tilde{v}^{\frac{n+2}{n-2}},
\end{aligned}$$

where $\tilde{v} = \mathcal{K}[v]$. □

Now assume that $g = u^{\frac{4}{n-2}}\delta$ is defined on a neighborhood of origin and consider the Taylor expansion of u

$$u(x) = a + \sum b_i x_i + O(|x|^2), \tag{1.7}$$

where $a = u(0) > 0$.

Using the expansion (1.7) for $v = \mathcal{K}[u]$ we obtain the asymptotic expansion:

$$v(x) = |x|^{2-n}\Big(a + \sum b_i x^i |x|^{-2}\Big) + O(|x|^{-n}), \tag{1.8}$$

in a neighborhood of infinity.

And consequently we also find

$$v_{x^k}(x) = (2-n)a\frac{x^k}{|x|^n} + O(|x|^{-n}), \tag{1.9}$$

$$v_{x^k x^l}(x) = O(|x|^{-n}). \tag{1.10}$$

These asymptotic expansions are necessary to start the process of moving planes. We use the notation $x_\lambda = (x', 2\lambda - x^n)$ to denote the reflection of the point $x = (x', x^n)$ with respect to the plane $x^n = \lambda$.

Lemma 1.3 *(see [9], page 227.) Let v be a function in a neighborhood of infinity satisfying the asymptotic expansions (1.8) to (1.10). Then there exist large positive constants $\bar\lambda$, R such that, if $\lambda \geq \bar\lambda$,*

$$v(x) > v(x_\lambda) \;\; \textit{for } x^n < \lambda, \;\; |x| > R.$$

To end this section we present another geometrical result concerning conformally flat metrics.

Lemma 1.4 *Small Euclidean balls B_r in $\mathbb{R}^n$ away from the singular set are convex with respect to $g = u^{\frac{4}{n-2}}\delta$.*

Proof:

We know that ∂B_r is umbilical with respect to g and so, we just need to show that the mean curvature of ∂B_r is positive with respect to the inward unit normal vector ν. From the boundary condition in problem (1.1) we have that the mean curvature can be computed as

$$h(g) = \frac{2}{n-2}u^{-\frac{n}{n-2}}\Big(-\frac{\partial u}{\partial \nu} + \frac{n-2}{2r}u\Big). \tag{1.11}$$

On the other hand, there exist $c_1 > 0$ and $c_2 > 0$ such that $|\frac{\partial u}{\partial \nu}| < c_1$ and $u > c_2$ in $\overline{B_r}$. Therefore,

$$-\frac{\partial u}{\partial \nu} + \frac{n-2}{2r}u > -c_1 + \frac{n-2}{2r}c_2.$$

Thus, if r is sufficiently small we have that $h(g) > 0$ on ∂B_r.

□

1.3 Proof of Theorem 1.1

The proof will be by contradiction. Suppose ∂B is not convex. Then there exists a point $q \in \partial B$ such that the mean curvature of ∂B in q with respect to g is $h(q) \leq 0$. If we write $g = u^{\frac{4}{n-2}}\delta$ we have that u is a positive smooth function on $\overline{B}_1 \setminus \Lambda$ satisfying

$$\begin{cases} \Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 & \text{in } B_1 \setminus \Lambda, \\ \frac{\partial u}{\partial \nu} - \frac{n-2}{2} u + \frac{n-2}{2} h u^{\frac{n}{n-2}} = 0 & \text{on } \partial B_1. \end{cases} \tag{1.12}$$

We point out that the condition for the metric g to be complete should be equivalent to requiring that the set Λ consist of non removable singularities of u.

Now, we will choose a point $p \in \partial B$, $p \neq q$ and let us consider the inversion

$$I : \mathbb{R}^n \setminus \{p\} \to \mathbb{R}^n \setminus \{p\}.$$

This map takes $\overline{B_1} \setminus (\{p\} \cup \Lambda)$ on $\mathbb{R}^n \setminus (B(\bar{a}, r) \cup \Lambda)$, where $B(\bar{a}, r)$ is an open ball of center $\bar{a} \in \mathbb{R}^n$ and radius $r > 0$ and Λ still denotes the singular set. Let us denote by Σ the boundary of $B(\bar{a}, r)$, that is, $\Sigma = I(\partial B_1)$.

The image of $\partial B \setminus \{p\}$ is a hyperplane Π and by a coordinate choice we may assume $\Pi = \Pi_0 := \{x \in \mathbb{R}^n : x^n = 0\}$. We may suppose that the center of the ball $B(\bar{a}, r)$ lies on the x^n-axis, below Π_0. Notice also that Λ lies below Π_0.

Since I is a conformal map we have $I^*g = v^{\frac{4}{n-2}}\delta$, where v is the Kelvin transform of u on $\mathbb{R}^n \setminus (B(\bar{a}, r) \cup \Lambda)$.

This metric has constant positive scalar curvature $n(n-1)$ in $\mathbb{R}^n \setminus (B(\bar{a}, r) \cup \Lambda)$ and nonnegative mean curvature h on Σ.

As before v is a solution of the following problem

$$\begin{cases} \Delta v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}^n \setminus (B(\bar{a}, r) \cup \Lambda), \\ \frac{\partial v}{\partial \nu} + \frac{n-2}{2r} v + \frac{n-2}{2} h v^{\frac{n+2}{n-2}} = 0 & \text{on } \Sigma. \end{cases}$$

Furthermore, we know that Π_0 is umbilical and we are also assuming $h_{I^*g}(I(q)) \leq 0$, where the mean curvature is computed with respect to $\frac{\partial}{\partial x^n}$. On the other hand we have $\frac{\partial v}{\partial x^n}(I(q)) + \frac{n-2}{2} h_{I^*g}(I(q)) v^{\frac{n}{n-2}} = 0$. Hence, $\frac{\partial v}{\partial x^n}(I(q)) \geq 0$.

Now we start with the Moving Planes Method. We will denote by x_λ the reflection of x with respect to the hyperplane $\Pi_\lambda := \{x \in \mathbb{R}^n : x^n = \lambda\}$ and set $\Omega_\lambda = \{x \in \mathbb{R}^n \setminus (B(\bar{a}, r) \cup \Lambda) : x^n \leq \lambda\}$. We define

$$w_\lambda(x) = v(x) - v_\lambda(x) \text{ for } x \in \Omega_\lambda,$$

where $v_\lambda(x) := v(x_\lambda)$.

Since the infinity is a regular point to I^*g, the expansions (1.8) to (1.10) hold to v. It follows from Lemma 1.3 that there exist $R > 0$ and $\bar{\lambda} > 0$ such that $w_\lambda > 0$ in $\Omega_\lambda \setminus B(0,R)$, if $\lambda \geq \bar{\lambda}$. Without loss of generality we can choose $R > 0$ such that $B(\overline{a}, r) \cup \Lambda \subset B(0,R)$.

Now we remark that v has a positive infimum $v > v_0 > 0$ in $B(0,R)\setminus(B(\bar{a},r)\cup\Lambda)$. It follows from the fact that v is a classical solution to (1.5) in $B(0,R)\setminus(B(\bar{a},r)\cup\Lambda)$ and $v(x)$ blows up as x approaches to some $p_j \in \Lambda$ as we seen in (1.6). So, by using (1.8), we may choose $\overline{\lambda} > 0$ large enough such that $v_\lambda(x) < v_0/2$, for $x \in B(0,R)$ and for $\lambda \geq \overline{\lambda}$.

Thus, for sufficiently large λ we get $w_\lambda > 0$ in $\mathrm{int}(\Omega_\lambda)$, and we may write

$$\Delta w_\lambda + c_\lambda(x) w_\lambda = 0 \text{ in } \mathrm{int}(\Omega_\lambda), \tag{1.13}$$

where

$$c_\lambda(x) = \frac{n(n-2)}{4} \frac{v(x)^{\frac{n+2}{n-2}} - v_\lambda(x)^{\frac{n+2}{n-2}}}{v(x) - v_\lambda(x)}.$$

Notice that, by definition, w_λ always vanishes on Π_λ. In particular, setting $\lambda_0 = \inf\{\overline{\lambda} > 0 : w_\lambda > 0 \text{ on } \mathrm{int}(\Omega_\lambda), \forall \lambda \geq \overline{\lambda}\}$ we obtain by continuity that w_{λ_0} satisfies (1.13), $w_{\lambda_0} \geq 0$ in Ω_{λ_0} and $w_{\lambda_0} = 0$ on Π_{λ_0}. Hence, by applying the strong maximum principle, we conclude that either $w_{\lambda_0} > 0$ in $\mathrm{int}(\Omega_{\lambda_0})$ or $w_{\lambda_0} = v - v_{\lambda_0}$ vanishes identically. But in the second case Π_{λ_0} is a symmetry hyperplane to v, which is a contradiction, since there are no singularities above Π_{λ_0}. Thus $w_{\lambda_0} > 0$ in $\mathrm{int}(\Omega_{\lambda_0})$.

Now, by the E. Hopf maximum principle,

$$\frac{\partial w_{\lambda_0}}{\partial x^n} = 2\frac{\partial v}{\partial x^n} < 0 \text{ in } \Pi_{\lambda_0}, \tag{1.14}$$

and since $\frac{\partial v}{\partial x^n}(I(q)) \geq 0$, we have $\lambda_0 > 0$. In this case, by definition of λ_0, we can choose sequences $\lambda_k \uparrow \lambda_0$ and $x_k \in \Omega_{\lambda_k}$ such that $w_{\lambda_k}(x_k) < 0$.

We recall that v_λ satisfies (1.5) on $\overline{\Omega}_\lambda$ in the classical sense, while v satisfies (1.5) on $\overline{\Omega}_\lambda$ in the distributional sense. In particular we conclude that w_λ is a weak supersolution on $\overline{\Omega}_\lambda$, and thus can be redefined on a set of measure zero so as to be upper semicontinuos. Hence w_λ achieves its infimum.

Then we may assume, without loss of generality, that x_k is a minimum of w_{λ_k} in Ω_{λ_k}. We have that $x_k \notin \Pi_k$ because w_k always vanishes on Π_k. So, either $x_k \in \Sigma$ or is an interior point. Even when x_k is an interior point we claim that $(x_k)_k$ is a bounded sequence. More precisely,

Claim 1.5 *[see §2 in [12]] There exists $R_0 > 0$, independent of λ, such that if w_λ is negative somewhere in $int(\Omega)$, and $x_0 \in int(\Omega)$ is a minimum point of w_λ, then $|x_0| < R_0$.*

For completeness we present a proof in the Appendix A.

So, we can take a convergent subsequence $x_k \to \bar{x} \in \Omega_{\lambda_0}$. Since $w_{\lambda_k}(x_k) < 0$ and $w_{\lambda_0} \geq 0$ in Ω_{λ_0} we necessarily have $w_{\lambda_0}(\bar{x}) = 0$ and therefore $\bar{x} \in \partial\Omega_{\lambda_0} = \Pi_{\lambda_0} \cup \Sigma$. We point out that $\nabla w_{\lambda_k}(x_k) = 0$ because x_k is a interior minimum point to w_{λ_k} and hence $\nabla w_{\lambda_0}(\bar{x}) = 0$. In particular, by inequality (1.14), $x_k \notin \Pi_{\lambda_0}$. Therefore we have $\bar{x} \in \Sigma$ and by E. Hopf maximum principle again,

$$\frac{\partial w_{\lambda_0}}{\partial \eta}(\bar{p}) = \frac{\partial v}{\partial \eta}(\bar{p}) - \frac{\partial v}{\partial \eta}(\bar{p}_{\lambda_0}) < 0, \tag{1.15}$$

where $\eta := -\nu$ is the inward unit normal vector to Σ. In the following we will denote by Σ_λ the reflection of Σ with respect to Π_λ.

Since Σ_λ is an umbilical sphere in $(\mathbb{R}^n, \delta)$ we have Σ_λ is umbilical in $(\mathbb{R}^n, g)$, consequently the second fundamental form of Σ_{λ_0} with respect to η is $II^\eta = h^\eta Id$.

Now, we recall that

$$\frac{\partial v}{\partial \nu} + \frac{n-2}{2r} v + \frac{n-2}{2} h v^{\frac{n+2}{n-2}} = 0 \text{ on } \Sigma \tag{1.16}$$

So, since $v(\bar{x}) = v(\bar{x}_{\lambda_0})$ we have from (1.15) and (1.16) that

$$h^\eta(\bar{x}_{\lambda_0}) < h^\eta(\bar{x}) = -h,$$

Since $h \geq 0$, we have that $\bar{x}_{\lambda_0}$ is a non convex point in Σ_{λ_0}. Considering the problem back in B_1, we denote by P_1 the ball corresponding to $\Pi^+_{\lambda_0}$ and by K_1 the ball corresponding to $B(\bar{a}, r)_{\lambda_0}$. Then we get $K_1 \subset P_1 \subset B$. Furthermore, ∂K_1 is the reflection of ∂B_1 with respect to ∂P_1.

We have shown that if ∂B has a non convex point, the smaller ball K_1 has a non convex point on ∂K_1. We repeat this argument to obtain a sequence of balls with non convex points on the boundaries, $B \supset K_1 \supset \cdots \supset K_j \supset \cdots$.

This sequence cannot converge to a point, since, from Lemma 1.4 small balls are always convex. On the other hand, if $K_j \to K_\infty$ where K_∞ is not a point, then $K_\infty \subset B$ is a ball in $B_1 \setminus \Lambda$ such that its boundary is the reflection of ∂B_1 with respect to to itself, that is a contradiction.

Chapter 2

CMC Surfaces of Genus Zero in $\mathbb{M}^2(\kappa) \times \mathbb{R}$

This chapter is part of a joint work with J. H. de Lira [11].

2.1 Introduction

U. Abresch and H. Rosenberg have recently proved that there exists a quadratic differential for an immersed surface in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ which is holomorphic when the surface has constant mean curvature. Here, $\mathbb{M}^2(\kappa)$ denotes the two-dimensional simply connected space form with constant curvature κ. This differential Q plays the role of the usual Hopf differential in the theory of constant mean curvature surfaces immersed in space forms. Thus, they were able to prove the following theorem:

Theorem. (Theorem 2, p. 143, [1]) *Any immersed cmc sphere $S^2 \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ in a product space is actually one of the embedded rotationally invariant cmc spheres $S^2_H \subset \mathbb{M}^2(\kappa) \times \mathbb{R}$.*

The rotationally invariant spheres referred to above were constructed by W.-Y. Hsiang and W.-T. Hsiang in [24] for $\kappa < 0$ and by R. Pedrosa and M. Ritore in [30] for any value of κ. The theorem quoted above proves affirmatively a conjecture stated by Hsiang and Hsiang in their paper [24]. More importantly, it indicates that some tools often used for surface theory in space forms could be redesigned to more general three dimensional homogeneous spaces, the more natural ones after space forms being $\mathbb{M}^2(\kappa) \times \mathbb{R}$. The price to be paid in abandoning space forms is that the technical difficulties are quite involved. The method in [1] is to study very closely the revolution surfaces in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ in order to find out the suitable differential.

Our idea here is to relate the Q differential with the usual Hopf differential after embedding $\mathbb{M}^2(\kappa) \times \mathbb{R}$ in some Euclidean space $\mathbb{E}^4$. We prove that Q is written as a linear combination of the two Hopf differentials Ψ^1 and Ψ^2 associated to the two normal directions to the surface in $\mathbb{E}^4$. More precisely

Theorem 2.2 *Let $x : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be an isometric immersion with mean curvature H. If H is constant, the quadratic differential $Q = 2H\Psi^1 - \frac{\epsilon}{r}\Psi^2$ is holomorphic on Σ.*

Here, ϵ and r are constants such that $\kappa = \epsilon/r^2$.

After that, we present another proof of the result of Abresch-Rosenberg (see Theorem 2.7). We also prove an extension of the well-known Nitsche's theorem about free boundary surfaces.

Theorem 2.10 *Let $X : \mathbb{D} \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be an immersion with constant mean curvature such that $X(\partial\mathbb{D})$ lies in some slice $\mathbb{M}^2(\kappa) \times \{t_0\}$ and makes constant angle along its boundary. Then $X(\Sigma)$ is part of a rotationally invariant surface.*

Finally, we obtain a characterization of stable CMC discs with circular boundary on $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

Theorem 2.11 *Let Σ be a disc type surface immersed with constant mean curvature H in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Suppose that $\partial\Sigma$ is contained in a geodesic circle in some slice $\mathbb{M}^2(\kappa) \times \{t_0\}$ and that the immersion is stable. Then Σ is part of a rotationally invariant surface. Moreover, if $H = 0$ then Σ is a totally geodesic disc.*

These results have counterparts in Lorentzian product spaces. In fact, the rotationally invariant surfaces are the predominant examples in our cases and although these surfaces are well known in the Riemannian product spaces some new properties appear when we consider the Lorentzian product. We begin with a qualitative study of such surfaces.

2.2 Rotationally Invariant CMC Discs

Let $(\mathbb{M}^2(\kappa), \mathrm{d}\sigma^2)$ be a two dimensional surface endowed with a complete metric with constant sectional curvature κ. We fix the product metric $\mathrm{d}\sigma^2 + \varepsilon \mathrm{d}t^2$, $\varepsilon \in \{-1, 1\}$, on the product $\mathbb{M}^2(\kappa) \times \mathbb{R}$. This metric is Lorentzian if $\varepsilon = -1$ and Riemannian if $\varepsilon = 1$. In the Lorentzian products we will consider only *space-like* surfaces, i.e., surfaces for which the metric induced on them is a Riemannian metric.

A tangent vector v to $\mathbb{M}^2(\kappa) \times \mathbb{R}$ is projected in a horizontal component v^h and a vertical component v^t that belong, respectively, to the $T\mathbb{M}^2(\kappa)$ and $T\mathbb{R}$ factors. We denote by $\langle , \rangle$ and D respectively the metric and covariant derivative in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. The curvature tensor in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ is denoted by $\bar{R}$.

Let (ρ, θ) be polar coordinates centered at some point p_0 of $\mathbb{M}^2(\kappa)$. So, we obtain cylindrical coordinates (ρ, θ, t) in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. We point out that the metric of $\mathbb{M}^2(\kappa)$ is given in these polar coordinates by $\mathrm{d}\sigma^2 = \mathrm{d}\rho^2 + \mathrm{sn}_\kappa^2(\rho)\mathrm{d}\theta^2$ and therefore the metric on $\mathbb{M}^2(\kappa) \times \mathbb{R}$ can be written as $\mathrm{d}\rho^2 + \mathrm{sn}_\kappa^2(\rho)\mathrm{d}\theta^2 + \varepsilon \mathrm{d}t^2$. Here

$$\mathrm{sn}_\kappa(\rho) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\rho), & \text{if } \kappa > 0, \\ \rho, & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\rho), & \text{if } \kappa < 0. \end{cases}$$

Let us fix a curve $s \mapsto (\rho(s), 0, t(s))$ in the plane $\theta = 0$. If we rotate this curve along the t-axis we obtain a *rotationally invariant surface* Σ in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ whose axis is $\{p_0\} \times \mathbb{R}$. This means that this surface has a parametrization X, in terms of the cylindrical coordinates defined above, of the following form

$$(s, \theta) \mapsto (\rho(s), \theta, t(s)).$$

The tangent plane to Σ at a point (s, θ) is spanned by the coordinate vector fields

$$X_s = \dot{\rho}\, \partial_\rho + \dot{t}\, \partial_t \quad \text{and} \quad X_\theta = \partial_\theta,$$

and an unit normal vector field to X is given by

$$n = \frac{1}{W} (\dot{t}\, \partial_\rho - \varepsilon \dot{\rho}\, \partial_t),$$

where $W^2 := (\dot{\rho}^2 + \varepsilon \dot{t}^2)$ and so $\langle n, n \rangle = \varepsilon$. Notice that when $\varepsilon = -1$ Σ is space-like and $W^2 > 0$.

The induced metric on Σ is given by

$$\begin{aligned} \langle \mathrm{d}X, \mathrm{d}X \rangle &:= E\mathrm{d}s^2 + 2F\mathrm{d}s\mathrm{d}\theta + G\mathrm{d}\theta^2 \\ &= (\dot{\rho}^2 + \varepsilon \dot{t}^2)\, \mathrm{d}s^2 + \mathrm{sn}_\kappa^2(\rho)\, \mathrm{d}\theta^2. \end{aligned}$$

Using the fact that $s \mapsto \rho(s)$ parametrizes a geodesic on $\mathbb{M}^2(\kappa)$ and the vector field ∂_t is parallel we can compute the derivatives

$$\begin{aligned} X_{ss} &:= D_{X_s} X_s \\ &= \ddot{\rho}\, \partial_\rho + \dot{\rho} D_{\partial_\rho} \partial_\rho + \ddot{t}\, \partial_t = \ddot{\rho}\, \partial_\rho + \ddot{t}\, \partial_t, \\ X_{s\theta} &:= D_{X_s} X_\theta = 0, \\ X_{\theta\theta} &:= D_{X_\theta} X_\theta = D_{\partial_\theta} \partial_\theta. \end{aligned}$$

Now we can compute the coefficients of the second fundamental form in these cylindrical coordinates. First, we obtain,

$$e := \langle X_{ss}, n\rangle = \frac{\ddot{\rho}\dot{t} - \ddot{t}\dot{\rho}}{W}.$$

And since $\langle D_{\partial_\theta}\partial_\theta, \partial t\rangle = 0$, we have,

$$\begin{aligned} g &:= \langle X_{\theta\theta}, n\rangle = \frac{\dot{t}}{W}\langle D_{\partial_\theta}\partial_\theta, \partial_\rho\rangle \\ &= -\frac{\dot{t}}{W}\langle \partial_\theta, D_{\partial_\theta}\partial_\rho\rangle = -\frac{\dot{t}}{W}\langle \partial_\theta, D_{\partial_\rho}\partial_\theta\rangle \\ &= -\frac{\dot{t}}{2W}\partial_\rho\langle \partial_\theta, \partial_\theta\rangle = -\frac{\dot{t}}{2W}\partial_\rho(\mathrm{sn}_\kappa^2(\rho)) \\ &= -\frac{\dot{t}}{W}\mathrm{sn}_\kappa(\rho)\mathrm{cs}_\kappa(\rho), \end{aligned}$$

where $\mathrm{cs}_\kappa(\rho)$ stands for the derivative of function $\mathrm{sn}_\kappa(\rho)$. Of course, the mixed term $f := \langle X_{s\theta}, n\rangle$ is null.

Using the formula $H = \frac{eG-2fF+gE}{2(EG-F^2)}$ for the mean curvature of X, we get

$$2W^3\mathrm{sn}_\kappa(\rho)H = (\ddot{\rho}\dot{t} - \ddot{t}\dot{\rho})\mathrm{sn}_\kappa(\rho) - W^2\dot{t}\mathrm{cs}_\kappa(\rho). \tag{2.1}$$

We now assume that s is arc length of the profile curve. Thus, we have $W^2 = \dot{\rho}^2 + \varepsilon\dot{t}^2 = 1$. We denote by φ the angle with the horizontal axis ∂_ρ. When $\varepsilon = 1$ we have $\dot{\rho} = \cos\,\varphi$ and $\dot{t} = \sin\,\varphi$, while for $\varepsilon = -1$ we have $\dot{\rho} = \cosh\,\varphi$ and $\dot{t} = \sinh\,\varphi$. So, we get $\ddot{\rho}\dot{t} - \ddot{t}\dot{\rho} = -\dot{\varphi}\,\sin^2(\varphi) - \dot{\varphi}\,\cos^2(\varphi)$, for $\varepsilon = 1$ and $\ddot{\rho}\dot{t} - \ddot{t}\dot{\rho} = \dot{\varphi}\,\sinh^2(\varphi) - \dot{\varphi}\,\cosh^2(\varphi)$, for $\varepsilon = -1$. In both cases, $\ddot{\rho}\dot{t} - \ddot{t}\dot{\rho} = -\dot{\varphi}$. Therefore, the equation (2.1) takes the following form

$$2\mathrm{sn}_\kappa(\rho)H = -\dot{\varphi}\mathrm{sn}_\kappa(\rho) - \dot{t}\mathrm{cs}_\kappa(\rho).$$

We restrict ourselves to the analysis of the Lorentzian case, since the Riemannian case was extensively studied in [30], [24] and [1]. Taking the value of $\dot{\varphi}$ in the last equation we conclude that Σ has constant mean curvature H if and only if the profile curve $(\rho(s), t(s), \varphi(s))$ is solution to the following ODE system

$$\begin{cases} \dot{\rho} = \cosh\varphi, \\ \dot{t} = \sinh\varphi, \\ \dot{\varphi} = -2H - \sinh\varphi\,\mathrm{ct}_\kappa(\rho), \end{cases} \tag{2.2}$$

where $\mathrm{ct}_\kappa(\rho) = \mathrm{cs}_\kappa(\rho)/\mathrm{sn}_\kappa(\rho)$ is the geodesic curvature of the geodesic circle centered at p_0 with radius ρ in $\mathbb{M}^2(\kappa)$. The first integral for this system (given by the flux through an horizontal plane $\mathbb{M}^2(\kappa) \times \{t\}$) is

$$I = \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + 2H \int_0^\rho \mathrm{sn}_\kappa(\tau)\,\mathrm{d}\tau. \tag{2.3}$$

By definition, $I = I(s)$ is constant along any solution of (2.2). We can integrate the last term in (2.3). For simplicity, we fix $\kappa > 0$. We obtain

$$\begin{aligned} I &= \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + \frac{2H}{\sqrt{\kappa}} \int_0^\rho \sin(\sqrt{\kappa}\tau)\,\mathrm{d}\tau \\ &= \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + 2H\Big(- \frac{1}{\kappa}\cos\sqrt{\kappa}\rho + \frac{1}{\kappa}\Big) \\ &= \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + 2H\Big(- \frac{1}{\kappa}(\cos^2\frac{\sqrt{\kappa}\rho}{2} - \sin^2\frac{\sqrt{\kappa}\rho}{2}) + \frac{1}{\kappa}\Big) \\ &= \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + \frac{4H}{\kappa}\sin^2\sqrt{\kappa}\,\frac{\rho}{2}. \end{aligned}$$

In the general case the equation (2.3) can be written as

$$I = \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + 4H\mathrm{sn}^2_\kappa\frac{\rho}{2}. \tag{2.4}$$

It is clear that solutions to the system (2.2) are defined on the whole real line and, since $\cosh\varphi$ never vanishes, *the profile curve may be written as a graph over the ρ-axis*.

Now, we will describe the *maximal surfaces*, i.e., solutions for $H = 0$. Given a fixed value for I we obtain for $H = 0$ that

$$I = \sinh\varphi\,\mathrm{sn}_\kappa(\rho). \tag{2.5}$$

We point out that the horizontal planes are the unique maximal revolution surfaces with $I = 0$. In fact if we put $I = 0$ in (2.5) we necessarily have $\sinh\varphi = 0$ for $\rho > 0$. Thus, $\dot{t} = 0$ and we conclude that the solution is an horizontal plane. Hence we may assume $I \neq 0$. In this case, since $\mathrm{sn}_\kappa(\rho) \to 0$ when $\rho \to 0$ it follows that $\sinh\varphi \to \infty$ when $\rho \to 0$. So, Σ has a singularity and is asymptotic to the light cone at p_0 (the light cone corresponds to $\varphi = \infty$). Moreover $\sinh\varphi \to 0$ if $\rho \to \infty$ in the case $\kappa \leq 0$. This means that these maximal surfaces are asymptotic to an horizontal plane for $\rho \to \infty$, i.e., these surfaces have planar ends. We refer to these singular surfaces as *Lorentzian catenoids*. These examples are not complete in the spherical case $\kappa > 0$, since we have $\sinh\varphi \to \infty$ if $\rho \to \frac{\pi}{\sqrt{\kappa}}$.

Consider now the case $H \neq 0$. We observe that the solutions for (2.2) have no positive minimum $\rho_m > 0$ for ρ. Otherwise, either the solutions must have vertical tangent plane at the minimum points (this is impossible since the solutions are space-like and, in fact, are graphs over the horizontal axis), or they are not complete (they have a singularity at the line $\rho = \rho_m$). The rotation of the profile curve in this last case generates a line $t = t_0$ of singularities for the solution. So, the unique possibility for the existence of an isolated singularity instead of a line of singularities is that $\rho \to 0$. In this case the solutions are regular if and only if $\varphi \to 0$ as $\rho \to 0$ what implies that $\sinh\varphi \to 0$ as $\rho \to 0$. So, necessarily $I = 0$ as we can see taking the limit $\rho \to 0$ in (2.4). So, examples of solutions for the system above, which meet orthogonally the revolution axis, have $I = 0$. Conversely, if we put $I = 0$ in (2.4) we get

$$0 = \sinh\varphi\, \mathrm{sn}_\kappa(\rho) + 4H\mathrm{sn}_\kappa^2(\frac{\rho}{2}).$$

Dividing the above expression by $\mathrm{sn}_\kappa^2(\frac{\rho}{2})$ (recall that $\rho > 0$ by definition) we have

$$-4H = \sinh\varphi \frac{2\mathrm{sn}_\kappa(\rho/2)\mathrm{cs}_\kappa(\rho/2)}{\mathrm{sn}_\kappa^2(\rho/2)},$$

that is,

$$-2H = \sinh\varphi\, \mathrm{ct}_\kappa(\frac{\rho}{2}). \tag{2.6}$$

Since $\mathrm{ct}_\kappa(\frac{\rho}{2}) > 0$, we conclude that $-H$ and $\sinh\varphi$ have the same sign. Moreover, one sees that $\sinh\varphi \to 0$ if $\rho \to 0$. So, *all solutions for (2.2) with $I = 0$ meet the revolution axis orthogonally*, as we claimed. Thus, these solutions correspond to the initial conditions $t(0) = t_0$ (we may assume $t_0 = 0$ after translating the solution along the revolution axis), $\rho(0) = 0$ and $\varphi(0) = 0$ for the system (2.2). Now, using (2.6) we obtain

$$\begin{aligned}
\mathrm{ct}_\kappa(\rho) \quad &:= \quad \frac{\mathrm{cs}_\kappa \rho}{\mathrm{sn}_\kappa \rho} = \frac{\mathrm{cs}_\kappa^2 \frac{\rho}{2}}{2\mathrm{sn}_\kappa \frac{\rho}{2}\, \mathrm{cs}_\kappa \frac{\rho}{2}} - \kappa \frac{\mathrm{sn}_\kappa^2 \frac{\rho}{2}}{2\mathrm{sn}_\kappa \frac{\rho}{2}\, \mathrm{cs}_\kappa \frac{\rho}{2}} \\
&= \quad \frac{1}{2}\left(\frac{\mathrm{cs}_\kappa \frac{\rho}{2}}{\mathrm{sn}_\kappa \frac{\rho}{2}} - \kappa \frac{\mathrm{sn}_\kappa \frac{\rho}{2}}{\mathrm{cs}_\kappa \frac{\rho}{2}}\right) = \frac{1}{2}\left(\mathrm{ct}_\kappa \frac{\rho}{2} - \kappa \frac{1}{\mathrm{ct}_\kappa \frac{\rho}{2}}\right) \\
&= \quad \frac{1}{2}\left(\frac{-2H}{\sinh\varphi} + \kappa \frac{\sinh\varphi}{2H}\right) \\
&= \quad \frac{-4H^2 + \kappa \sinh^2\varphi}{4H\sinh\varphi}.
\end{aligned}$$

Replacing this on the third equation in (2.2) we see that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{1}{4H}\left(-4H^2 - \kappa \sinh^2 \varphi\right) \tag{2.7}$$

or equivalently that $\dot{\varphi} = -H - \kappa \frac{\sinh^2 \varphi}{4H}$. (We observe that $\dot{\varphi} = -H$ is the corresponding equation for the case $\kappa = 0$, i.e., for $\mathbb{L}^3$. This can be obtained as a limiting case if we take $\kappa \to 0$.) So, in the case $\kappa < 0$, the range for the angle φ is $\varphi < \varphi_0 = \operatorname{arcsinh}(\frac{-2H}{\sqrt{-\kappa}})$. This can be seen taking $\frac{\mathrm{d}}{\mathrm{d}s}\varphi_0 = 0$ in (2.7). The surface necessarily is asymptotic to a space like cone with angle φ_0. There are no complete solutions for $\kappa > 0$ and $H \neq 0$, since the angle at $\rho = 0$ and at $\rho = \frac{\pi}{\sqrt{\kappa}}$ are not the same unless we have $H = 0$.

Next, we study the case when $\varphi \to \varphi_0$ as $\rho \to 0$ for some positive value of φ_0. This means that the solution is asymptotic to a space-like cone at p_0. In this case $\sinh \varphi \to \sinh \varphi_0 < \infty$ as $\rho \to 0$. Thus taking the limit $\rho \to 0$ in (2.4) we obtain $I = 0$. So, as we seen above, necessarily $\varphi_0 = 0$. This contradiction implies that there are no examples with $\varphi_0 > 0$.

It remains to look at the case $\varphi \to \infty$ as $\rho \to 0$ (since the case $\varphi \to -\infty$ as $\rho \to 0$ is similarly). In this case, the solution is asymptotic to the light cone at p_0. For any non zero value of I and for $\kappa < 0$, we obtain after dividing (2.4) by $\mathrm{sn}^2_\kappa(\rho/2)$ that,

$$\begin{aligned} \frac{I}{\mathrm{sn}_\kappa(\rho/2)} &= \frac{2\sinh\varphi \mathrm{sn}_\kappa(\rho/2)\mathrm{cs}_\kappa(\rho/2)}{\mathrm{sn}^2_\kappa(\rho/2)} + 4H \\ &= 2\sinh\varphi\, \mathrm{ct}_\kappa(\frac{\rho}{2}) + 4H. \end{aligned}$$

and so, taking limit for $\rho \to \infty$ we have that $\sinh\varphi \to -2H/\sqrt{-k}$.

We summarize the facts above in the following theorem.

Theorem 2.1 *Let Σ be a rotationally invariant space like surface with constant mean curvature H in the Lorentzian product $\mathbb{M}^2(\kappa) \times \mathbb{R}$ with $\kappa \leq 0$. Then Σ is a vertical graph. Moreover,*

1. *If $H = 0$, either Σ is a horizontal plane $\mathbb{M}^2(\kappa) \times \{t\}$ or Σ is asymptotic to a light cone with vertex at some point p_0 of the rotation axis. In this case, Σ has a singularity at p_0 and has horizontal planar ends.*

2. *If $H \neq 0$ either Σ is a complete disc-type surface meeting orthogonally the rotation axis or Σ is asymptotic to a light cone with vertex p_0 at the rotation axis. In the last case, the surface is singular at p_0 and is asymptotic to a space-like cone with vertex at p_0 and slope φ_0, where $\sinh\varphi_0 = \frac{-2H}{\sqrt{-\kappa}}$.*

2.3 Hopf Differentials in Some Product Spaces

Let Σ be a Riemannian surface and $X\colon \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ an isometric immersion. If $\kappa \geq 0$, we may consider Σ as immersed in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. If $\kappa < 0$, we immerse Σ in $\mathbb{L}^3 \times \mathbb{R}$. In fact, we may write $X = (p, t)$, with $t \in \mathbb{R}$ and $p \in \mathbb{M}^2(\kappa) \subset \mathbb{R}^3$ for $\kappa \geq 0$ and $p \in \mathbb{M}^2(\kappa) \subset \mathbb{L}^3$ for $\kappa < 0$. By writing $\mathbb{M}^2(\kappa) \times \mathbb{R} \subset \mathbb{E}^4$ we mean all these possibilities. The metric and covariant derivative in $\mathbb{E}^4$ and are also denoted by $\langle , \rangle$ and D respectively.

Let (u, v) be local coordinates in Σ for which $X(u, v)$ is a conformal immersion inducing the metric $E\,(\mathrm{d}u^2 + \mathrm{d}v^2)$ in Σ. Let ∂_u, ∂_v be the coordinate vectors and $e_1 = E^{-\frac{1}{2}}\partial_u$, $e_2 = E^{-\frac{1}{2}}\partial_v$ the associated local orthonormal frame tangent to Σ. The unit normal directions to Σ in $\mathbb{E}^4$ are denoted by $n_1, n_2 = p/r$, where $r^2 = \operatorname{sgn}\kappa\langle p, p\rangle$. We denote by h^k_{ij} the components of h^k, the second fundamental form of Σ with respect to n_k, $k = 1, 2$. Then

$$h^k_{ij} = \langle D_{e_i} e_j, n_k\rangle.$$

It is clear that the h^1_{ij} are the components of the second fundamental form of the immersion $\Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$. We have for the components of h^2 that

$$\begin{aligned} h^2_{ij} &= \langle D_{e_i} e_j, n_2\rangle = \langle D_{e_i} e_j, p/r\rangle = -\frac{1}{r}\langle e^h_i, e^h_j\rangle = \frac{1}{r}(\varepsilon\langle e^t_i, e^t_j\rangle - \delta_{ij}) \\ &= \frac{1}{r}(\varepsilon\langle e_i, \partial_t\rangle\langle e_j, \partial_t\rangle - \delta_{ij}). \end{aligned}$$

In what following we will consider $\kappa \neq 0$. Denoting $\epsilon = \operatorname{sgn}\kappa$, we remark that $\kappa = \epsilon/r^2$. The components of h^1 and h^2 in the frame ∂_u, ∂_v are given by

$$\begin{aligned} &e = h^1(\partial_u, \partial_u) = Eh^1_{11},\ f = h^1(\partial_u, \partial_v) = Eh^1_{12},\ g = h^1(\partial_v, \partial_v) = Eh^1_{22}, \\ &\tilde{e} = h^2(\partial_u, \partial_u) = Eh^2_{11},\ \tilde{f} = h^2(\partial_u, \partial_v) = Eh^2_{12},\ \tilde{g} = h^2(\partial_v, \partial_v) = Eh^2_{22}. \end{aligned}$$

The *Hopf differential* associated to h^k is defined by $\Psi^k = \psi^k \mathrm{d}z^2$, where $z = u + iv$ and the coefficients ψ^1, ψ^2 are given by

$$\psi^1 = \frac{1}{2}(e - g) - i\,f, \quad \psi^2 = \frac{1}{2}(\tilde{e} - \tilde{g}) - i\,\tilde{f}.$$

Now we compute the derivatives,

$$E_u = 2\langle D_u\partial_u, \partial_u\rangle = -2\langle D_v\partial_v, \partial_u\rangle = 2\langle D_u\partial_v, \partial_v\rangle, \tag{2.8}$$

$$E_v = 2\langle D_v\partial_v, \partial_v\rangle = -2\langle D_u\partial_u, \partial_v\rangle = 2\langle D_u\partial_v, \partial_u\rangle. \tag{2.9}$$

We point out that $D_u n_1 = -\frac{e}{E}\partial_u - \frac{f}{E}\partial_v$ and $D_v n_1 = -\frac{f}{E}\partial_u - \frac{g}{E}\partial_v$. So, we have,

$$\begin{aligned}
e_v &= -\partial_v\langle D_u n_1, \partial_u\rangle = -\langle D_v D_u n_1, \partial_u\rangle - \langle D_u n_1, D_v\partial_u\rangle \\
&= -\langle D_v D_u n_1, \partial_u\rangle + \frac{e}{E}\langle \partial_u, D_v\partial_u\rangle + \frac{f}{E}\langle \partial_v, D_v\partial_u\rangle \\
&= -\langle D_v D_u n_1, \partial_u\rangle + \frac{eE_v}{2E} + \frac{fE_u}{2E}, \\
f_v &= -\partial_v\langle D_u n_1, \partial_v\rangle = -\langle D_v D_u n_1, \partial_v\rangle - \langle D_u n_1, D_v\partial_v\rangle \\
&= -\langle D_v D_u n_1, \partial_v\rangle + \frac{e}{E}\langle \partial_u, D_v\partial_v\rangle + \frac{f}{E}\langle \partial_v, D_v\partial_v\rangle \\
&= -\langle D_v D_u n_1, \partial_v\rangle - \frac{eE_u}{2E} + \frac{fE_v}{2E}, \\
f_u &= -\partial_u\langle D_v n_1, \partial_u\rangle = -\langle D_u D_v n_1, \partial_u\rangle - \langle D_v n_1, D_u\partial_u\rangle \\
&= -\langle D_u D_v n_1, \partial_u\rangle + \frac{f}{E}\langle \partial_u, D_u\partial_u\rangle + \frac{g}{E}\langle \partial_v, D_u\partial_u\rangle \\
&= -\langle D_u D_v n_1, \partial_u\rangle + \frac{fE_u}{2E} - \frac{gE_v}{2E}, \\
g_u &= -\partial_u\langle D_v n_1, \partial_v\rangle = -\langle D_u D_v n_1, \partial_v\rangle - \langle D_v n_1, D_u\partial_v\rangle \\
&= -\langle D_u D_v n_1, \partial_v\rangle + \frac{f}{E}\langle \partial_u, D_u\partial_v\rangle + \frac{g}{E}\langle \partial_v, D_u\partial_v\rangle \\
&= -\langle D_u D_v n_1, \partial_v\rangle + \frac{fE_v}{2E} + \frac{gE_u}{2E}.
\end{aligned}$$

Thus,

$$\begin{aligned}
e_v - f_u &= -\langle D_v D_u n_1, \partial_u\rangle + \frac{eE_v}{2E} + \langle D_u D_v n_1, \partial_u\rangle + \frac{gE_v}{2E} \\
&= \langle D_u D_v n_1, \partial_u\rangle - \langle D_v D_u n_1, \partial_u\rangle + \frac{E_v}{2E}(e+g) \\
&= -\langle \bar{R}(\partial_u, \partial_v)n_1, \partial_u\rangle + HE_v, \qquad (2.10) \\
f_v - g_u &= -\langle D_v D_u n_1, \partial_v\rangle - \frac{eE_u}{2E} + \langle D_u D_v n_1, \partial_v\rangle - \frac{gE_u}{2E} \\
&= \langle D_u D_v n_1, \partial_v\rangle - \langle D_v D_u n_1, \partial_v\rangle - \frac{E_u}{2E}(e+g) \\
&= -\langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle - HE_u, \qquad (2.11)
\end{aligned}$$

where in the last equalities of (2.10) and (2.11) we used the Codazzi Equation (see (B.5) and (B.6) in Appendix B).

Now since $e+g=2HE$ we get,

$$\begin{aligned} e_u+g_u &= 2HE_u+2EH_u \\ &= -2f_v+2g_u-2\langle\bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle+2EH_u, \\ e_v+g_v &= 2HE_v+2EH_v \\ &= 2e_v-2f_u+2\langle\bar{R}(\partial_u,\partial_v)n_1,\partial_u\rangle+2EH_v. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{e_u-g_u}{2} &= -f_v-\langle\bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle+EH_u, \\ \frac{e_v-g_v}{2} &= f_u-\langle\bar{R}(\partial_u,\partial_v)n_1,\partial_u\rangle-EH_v. \end{aligned}$$

On the other hand, it is easy to see that (see (B.3) and (B.4) in Appendix B):

$$\begin{aligned} \langle\bar{R}(\partial_u,\partial_v)n_1,\partial_u\rangle &= -\kappa\, E\langle\partial_v^h,n_1^h\rangle, \\ \langle\bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle &= \kappa\, E\langle\partial_u^h,n_1^h\rangle, \end{aligned}$$

that is, we obtain that

$$\partial_u\Re\psi^1=\partial_v\Im\psi^1-\kappa\, E\langle\partial_u^h,n_1^h\rangle+EH_u, \tag{2.12}$$

$$\partial_v\Re\psi^1=-\partial_u\Im\psi^1+\kappa\, E\langle\partial_v^h,n_1^h\rangle-EH_v. \tag{2.13}$$

We also calculate

$$\begin{aligned} &\partial_u\Re\psi^2=\frac{\varepsilon}{2r}\,\partial_u\Big(\langle\partial_u,\partial_t\rangle^2-\langle\partial_v,\partial_t\rangle^2\Big)=\frac{\varepsilon}{r}\Big(\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u,\partial_t\rangle \\ &-\langle\partial_v,\partial_t\rangle\,\langle D_{\partial_u}\partial_v,\partial_t\rangle\Big)=\frac{\varepsilon}{r}\Big(\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u,\partial_t\rangle-\langle\partial_v,\partial_t\rangle\,\langle D_{\partial_v}\partial_u,\partial_t\rangle\Big) \\ &=\frac{\varepsilon}{r}\Big(\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u,\partial_t\rangle-\langle\partial_v,\partial_t\rangle\,\partial_v\langle\partial_u,\partial_t\rangle\Big) \\ &=\frac{\varepsilon}{r}\Big(\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u,\partial_t\rangle-\partial_v(\langle\partial_v,\partial_t\rangle\,\langle\partial_u,\partial_t\rangle)+\langle D_{\partial_v}\partial_v,\partial_t\rangle\,\langle\partial_u,\partial_t\rangle\Big) \\ &=\frac{\varepsilon}{r}\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u+D_{\partial_v}\partial_v,\partial_t\rangle-\frac{\varepsilon}{r}\,\partial_v(\langle\partial_u,\partial_t\rangle\,\langle\partial_v,\partial_t\rangle). \end{aligned}$$

On the other hand, by (2.8) and (2.9) we get

$$\begin{aligned} &D_{\partial_u}\partial_u+D_{\partial_v}\partial_v=\langle D_{\partial_u}\partial_u+D_{\partial_v}\partial_v,\partial_u\rangle\partial_u+ \\ &\langle D_{\partial_u}\partial_u+D_{\partial_v}\partial_v,\partial_v\rangle\partial_v+\varepsilon\langle D_{\partial_u}\partial_u+D_{\partial_v}\partial_v,n_1\rangle n_1 \\ &=\Big(\frac{1}{2}E_u-\langle D_u\partial_v,\partial_v\rangle\Big)\partial_u+\Big(-\langle D_u\partial_v,\partial_u\rangle+\frac{1}{2}E_v\Big)\partial_v+\varepsilon(e+g)n_1 \\ &=\varepsilon(e+g)n_1=2\varepsilon HEn_1. \end{aligned}$$

Therefore, we have

$$\begin{aligned}\partial_u \Re\psi^2 &= \frac{\varepsilon}{r}\langle\partial_u,\partial_t\rangle\,\langle 2\varepsilon HEn_1,\partial_t\rangle - \frac{\varepsilon}{r}\,\partial_v(\langle\partial_u,\partial_t\rangle\,\langle\partial_v,\partial_t\rangle)\\ &= -\frac{2\varepsilon HE}{r}\langle\partial_u^h, n_1^h\rangle + \,\partial_v\Im\psi^2.\end{aligned}$$

Similarly, we prove that

$$\partial_v\Re\psi^2 = -\partial_u\Im\psi^2 + \frac{2\varepsilon HE}{r}\langle\partial_v^h, n_1^h\rangle.$$

Then using the above mentioned fact that $\kappa = \epsilon/r^2$, we conclude that the function $\psi := 2H\psi^1 - \varepsilon\frac{\epsilon}{r}\psi^2$ satisfies

$$\begin{aligned}\partial_u\Re\psi &= \partial_v\Im\psi + 2\Re\psi^1 H_u - 2\Im\psi^1 H_v + 2EHH_u\\ &= \partial_v\Im\psi + 2eH_u + 2fH_v\\ \partial_v\Re\psi &= -\partial_u\Im\psi + 2\Re\psi^1 H_v + 2\Im\psi^1 H_u - 2EHH_v\\ &= -\partial_u\Im\psi - 2gH_v - 2fH_u.\end{aligned}$$

Now, using the complex differentiation $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$ we get

$$\begin{aligned}\partial_{\bar z}\psi &= \frac{1}{2}(\partial_u\Re\psi - \partial_v\Im\psi) + \frac{i}{2}(\partial_v\Re\psi + \partial_u\Im\psi)\\ &= eH_u + fH_v - ifH_u - igH_v\\ &= 2\psi^1 H_{\bar z} + 2EH_z.\end{aligned}$$

That is, defining the quadratic differential $Q := 2H\,\Psi^1 - \varepsilon\frac{\epsilon}{r}\,\Psi^2$ we have that Q is holomorphic on Σ if H is constant. Namely,

Theorem 2.2 *Let $x : \Sigma \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ be an isometric immersion with mean curvature H. If H is constant the quadratic differential $Q = 2H\Psi^1 - \varepsilon\frac{\epsilon}{r}\Psi^2$ is holomorphic on Σ.*

If Q is holomorphic then

$$eH_u + fH_v = 0, \quad fH_u + gH_v = 0.$$

This implies that $A\nabla H = 0$, where, A is the shape operator for X and ∇H is the gradient of H on Σ. If $\nabla H = 0$, i.e., $H_u = H_v = 0$ on Σ, then H is constant. On the other hand, if $\nabla H \neq 0$ on an open set $\Sigma' \subset \Sigma$ we have $K_{\text{ext}} = \det A = 0$ on Σ'. We have that $e_1 := \nabla H/|\nabla H|$ is a principal direction with principal curvature $\kappa_1 = 0$. Moreover, $H = \kappa_2$, where, κ_2 is the principal curvature of Σ calculated on a direction e_2 perpendicular to e_1. So, the only planar (umbilical) points on Σ are the points where H vanishes. Moreover, the integral curves of e_2 are the level curves for $H = \kappa_2$ since they are orthogonal to ∇H. Thus H is constant along such each line.

These considerations imply that if there exist examples of surfaces with holomorphic Q and non constant mean curvature, these examples cannot be compact, have zero extrinsic Gaussian curvature and are foliated by curvature lines along which H is constant. Recently, I. Fernández and P. Mira announced that they had constructed such examples (see [18]).

For the case $\varepsilon = 1$ this quadratic form coincides with that one obtained by U. Abresch and H. Rosenberg in [1]. If we denote by q the quadratic form $q = 2H\,h^1 - \varepsilon\frac{\epsilon}{r}h^2$ then it is clear that Q is the complexification of the traceless part of q.

A simple computation shows that $Q = 0$ if only if $q = \frac{1}{2}\text{trace}(q)I$. As a consequence we have the following rigidity result for minimal surfaces.

Lemma 2.3 *If $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ is a minimal immersion with $Q = 0$, then $X(\Sigma)$ is part of a slice $\mathbb{P}_t := \mathbb{M}^2(\kappa) \times \{t\}$.*

Proof:

Given an arbitrary local orthonormal frame field $\{e_1, e_2\}$ we get,

$$\begin{aligned} 2Hh^1_{12} - \varepsilon\frac{\epsilon}{r}h^2_{12} &= 0 \\ 2Hh^1_{11} - \varepsilon\frac{\epsilon}{r}h^2_{11} &= \frac{1}{2}(4H^2 - \varepsilon\frac{\epsilon}{r}(h^2_{11} + h^2_{22})) \\ 2Hh^1_{22} - \varepsilon\frac{\epsilon}{r}h^2_{22} &= \frac{1}{2}(4H^2 - \varepsilon\frac{\epsilon}{r}(h^2_{11} + h^2_{22})) \end{aligned}$$

Thus,

$$\begin{aligned} 2Hh^1_{12} &= \varepsilon\frac{\epsilon}{r}h^2_{12} \\ &= \varepsilon\frac{\epsilon}{r}\frac{1}{r}(\varepsilon\langle e_1, \partial_t\rangle\langle e_2, \partial_t\rangle) \\ &= \kappa\langle e_1, \partial_t\rangle\langle e_2, \partial_t\rangle \qquad (2.14) \\ 2Hh^1_{11} &= 2H^2 + \varepsilon\frac{\epsilon}{2r}h^2_{11} - \varepsilon\frac{\epsilon}{2r}h^2_{22} \end{aligned}$$

$$
\begin{aligned}
&= 2H^2 + \varepsilon\frac{\epsilon}{2r}\frac{1}{r}(\varepsilon\langle e_1,\partial_t\rangle^2 - 1) - \varepsilon\frac{\epsilon}{2r}\frac{1}{r}(\varepsilon\langle e_2,\partial_t\rangle^2 - 1) \\
&= 2H^2 + \frac{\kappa}{2}\langle e_1,\partial_t\rangle^2 - \frac{\kappa}{2}\langle e_2,\partial_t\rangle^2 \qquad (2.15) \\
2Hh^1_{22} &= 2H^2 - \varepsilon\frac{\epsilon}{2r}h^2_{11} + \varepsilon\frac{\epsilon}{2r}h^2_{22} \\
&= 2H^2 - \varepsilon\frac{\epsilon}{2r}\frac{1}{r}(\varepsilon\langle e_1,\partial_t\rangle^2 - 1) + \varepsilon\frac{\epsilon}{2r}\frac{1}{r}(\varepsilon\langle e_2,\partial_t\rangle^2 - 1) \\
&= 2H^2 - \frac{\kappa}{2}\langle e_1,\partial_t\rangle^2 + \frac{\kappa}{2}\langle e_2,\partial_t\rangle^2 \qquad (2.16)
\end{aligned}
$$

Now, if $H = 0$ it follows from these equations that the vector field ∂_t is always normal to Σ. So, the surface is part of a slice $\mathbb{P}_t := \mathbb{M}^2(\kappa)\times\{t\}$, for some $t \in \mathbb{R}$. □

We also say something about umbilical points of such immersions. We recall that here $\kappa \neq 0$.

Lemma 2.4 *Let $X : \Sigma \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ be a CMC immersion with $Q = 0$. Then $p_0 \in \Sigma$ is an umbilical point of X if and only if Σ has a horizontal plane at p_0.*

Proof:

Without loss of generality we may suppose that $H \neq 0$. At p_0 we have that for an arbitrary frame, $h^1_{12} = 0$. So, either $\langle e_1,\partial_t\rangle = 0$ or $\langle e_2,\partial_t\rangle = 0$ at p_0. Since $h^1_{11} = h^1_{22} = H$ at that point the equations (2.15) and (2.16) imply that both angles $\langle e_i,\partial_t\rangle$ are null, since $\kappa \neq 0$. So, Σ has horizontal tangent plane, and conversely. □

The next lemma is a fundamental tool in the proof of Abresch-Rosenberg's Theorem.

Lemma 2.5 *If $X : \Sigma \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ is an immersion with $Q = 0$ and constant nonzero mean curvature, then each intersection $\Sigma\cap\mathbb{P}_t$ occurs with constant angle. Moreover, this angle is zero if and only if all points of this intersection are umbilical. In particular, the umbilical points are isolated.*

Proof:

If we are at a non-umbilical point, we may choose the frame $\{e_1, e_2\}$ as a locally defined principal frame field (on a neighborhood Σ' of that point). Thus, $h^1_{12} = 0$ and therefore $\langle e_1,\partial_t\rangle = 0$ or $\langle e_2,\partial_t\rangle = 0$ on Σ'. We fix $\langle e_1,\partial_t\rangle = 0$.

We conclude that on Σ' the lines of curvature of Σ with direction e_1 are locally contained in the slices $\mathbb{P}_t$. Conversely, the connected components σ of $\Sigma'\cap\mathbb{P}_t$ are lines of curvature of $\mathbb{P}_t$ with tangent direction given by e_1. Thus, if we parametrize such a line by its arc length, we have

$$
\frac{\mathrm{d}}{\mathrm{d}s}\langle n_1,\partial_t\rangle = \langle D_{\sigma'}n_1,\partial_t\rangle = \langle D_{e_1}n_1,\partial_t\rangle = h^1_{11}\langle e_1,\partial_t\rangle = 0. \qquad (2.17)
$$

We conclude that, for a fixed t, Σ' and $\mathbb{P}_t$ make a constant angle $\theta(t)$ along each connected component of their intersection. We recall that these conclusions hold whenever we are away from the umbilical points of $X(\Sigma)$. So, if a connected component of the intersection between $\mathbb{P}_t$ and Σ has a non umbilical point, then the angle is constant, non zero, along this component, unless that there exists also an umbilical point on this same component. However, at this point the angle is necessarily zero. So, by continuity of the angle function, either all points on a connected component $\Sigma \cap \mathbb{P}_t$ are umbilical and the angle is zero, or all points are non umbilical and the angle is non zero. However, if all points on a connected component σ are umbilical points for h^1, then, as we noticed above, Σ is tangent to $\mathbb{P}_t$ along σ. So, along σ we have $n_1 = \partial_t$ and consequentely,

$$h^1_{11} = \langle D_{e_1} e_1, n_1\rangle = -\langle e_1, D_{e_1} n_1\rangle = 0.$$

Thus, $h^1_{22} = h^1_{11} = 0$ and $H = 0$. From this contradiction, we conclude that there are no umbilical points on *curves* $\Sigma \cap t^{-1}(a, b)$, where $[a, b]$ is the maximal interval for which Σ intersects some slice $\mathbb{P}_t$. The only possibility is that there exists isolated umbilical points (the umbilical points may not be on *any* curve on $\Sigma \cap \mathbb{P}_t$) as may occur on the top and bottom levels. $\square$

Before we present another consequence we will fix some notation. First, as we indicated in the previous Lemma, θ is the positive angle between n_1 and ∂_t, so that $\cos\theta(t) = \langle n_1, \partial_t\rangle$ for $\varepsilon = 1$ and $\cosh\theta(t) = -\langle n_1, \partial_t\rangle$ for $\varepsilon = -1$. We denote

$$\mathrm{sn}(t) = \begin{cases} \sin\theta(t), & \varepsilon = 1 \\ \sinh\theta(t), & \varepsilon = -1 \end{cases}, \quad \mathrm{cs}(t) = \begin{cases} \cos\theta(t), & \varepsilon = 1 \\ \cosh\theta(t), & \varepsilon = -1. \end{cases}$$

Thus, if we denote by τ the tangential part $\partial_t - \varepsilon\langle \partial_t, n_1\rangle n_1$ of the field ∂_t, then $\tau = \langle e_2, \partial_t\rangle\, e_2$ and we may write $\partial_t = \tau + \varepsilon\langle\partial_t, n_1\rangle n_1 = \mathrm{sn}(t)e_2 + \mathrm{cs}(t)n_1$.

Lemma 2.6 *If $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ is an immersion with $Q = 0$ and nonzero mean curvature, then each curve of $\Sigma \cap \mathbb{P}_t$ has constant geodesic curvature on $\mathbb{P}_t$.*

Proof:

By Lemma 2.5 we have that there exists an orthonormal principal frame field $\{e_1, e_2\}$ on a dense subset of Σ (the subset obtained by taking away the isolated umbilical points of Σ) and on this dense subset we have $\tau \neq 0$. Then we may choose a positive sign for $\mathrm{sn}(t)$. Now, we calculate the geodesic curvature $\langle D_{e_1} e_1, \nu\rangle$ of the curvature lines on $\mathbb{P}_t$. Here $\nu := Je_1$ is the positively oriented (with respect to ∂_t) unit vector field on $\mathbb{P}_t$ normal to such a curve. We have

$$e_2 = \frac{\tau}{|\tau|} = \frac{1}{\mathrm{sn}(t)}\,\tau = \frac{1}{\mathrm{sn}(t)}\,(\partial_t - \mathrm{cs}(t)n_1).$$

Since $\langle n_1, \partial_t \rangle$ is constant along this curve, and therefore $\mathrm{sn}(t)$ and $\mathrm{cs}(t)$ are constants, we conclude that

$$D_{e_1} e_2 = \frac{1}{\mathrm{sn}(t)} (D_{e_1} \partial_t - \mathrm{cs}(t)\, D_{e_1} n_1) = \frac{\mathrm{cs}(t)}{\mathrm{sn}(t)}\, h^1_{11} e_1.$$

So, the geodesic curvature $\langle D_{e_1} e_1, e_2 \rangle$ of the lines of curvature in the direction e_1, is given by $-(\mathrm{cs}(t)/\mathrm{sn}(t))\, h^1_{11}$. Now we write

$$\nu = Je_1 = \langle \nu, n_1 \rangle\, n_1 + \langle \nu, e_2 \rangle\, e_2 = \sin\theta(t)\, n_1 - \cos\theta(t)\, e_2,$$

for $\varepsilon = 1$ and

$$\nu = Je_1 = -\langle \nu, n_1 \rangle\, n_1 + \langle \nu, e_2 \rangle\, e_2 = -\sinh\theta(t)\, n_1 - \cosh\theta(t)\, e_2,$$

for $\varepsilon = -1$. In general

$$\nu = Je_1 = \varepsilon \mathrm{sn}(t)\, n_1 - \mathrm{cs}(t)\, e_2.$$

We calculate

$$\langle D_{e_1} \nu, e_1 \rangle = -\varepsilon \mathrm{sn}(t) h^1_{11} - \mathrm{cs}(t) \frac{\mathrm{cs}(t)}{\mathrm{sn}(t)}\, h^1_{11} = -\frac{1}{\mathrm{sn}(t)}\, h^1_{11}.$$

Thus, since h^1_{11} is constant along the e_1-curve, we have that the geodesic curvature of this line relatively to the slice $\mathbb{P}_t$ is also constant and equal to $h^1_{11}/\mathrm{sn}(t)$. We conclude that for each t, $\Sigma \cap \mathbb{P}_t$ consists of constant geodesic curvature lines of $\mathbb{P}_t$. □

For topological spheres $\Sigma \simeq S^2$ it is well known that holomorphic differentials vanish everywhere. Using this fact, we can present another proof of the Theorem of Abresch and Rosenberg quoted in the introduction of this chapter.

Theorem 2.7 *(Abresch-Rosenberg): Any immersed cmc sphere $S^2 \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ in a Riemannian product space is an embedded rotationally invariant cmc sphere.*

Proof:

Let $[a, b]$ be the maximal interval of points t such that $\Sigma \cap \mathbb{P}_t$ is nonempty. Then Σ is tangent to $\mathbb{P}_b$. Suppose that $\Sigma \cap \mathbb{P}_b$ contains two distinct points. For small $\delta > 0$ we have by Lemma 2.6 that, for $t \in (b - \delta, b)$, $\Sigma \cap \mathbb{P}_t$ is the union of two disconnected geodesic circles of $\mathbb{P}_t$. Let t_1 be the smaller $t \in [a, b]$ with this property. At t_1 only two possibilities may occur. First, one of this geodesic circles disappear at a point and in this case we conclude that Σ is a disconnected surface. Second, the two geodesic circles meet at a point and in this case we do not have a curve with constant geodesic curvature.

A similar argument shows that each nonempty intersection $\Sigma \cap \mathbb{P}_t$ is single closed curve with constant geodesic curvature on $\mathbb{P}_t$; This implies that Σ is foliated by circles.

To conclude the proof we fix a geodesic circle σ of $\mathbb{P}_{t_0}$ contained on Σ. Let p_0 be the center of this circle in $\mathbb{P}_{t_0}$ and denote by s the length in $\mathbb{P}_{t_0}$ measured starting from some meridian passing through p_0. So, each value for s in $[0, 2\pi)$ determines a plane Π_s containing some meridian of $\mathbb{P}_{t_0}$ and the t-axis. Let us denote by γ_s the curve of intersection between the plane Π_s and Σ. Then $\gamma_s(t)$ is determined by its initial data $\gamma_s'(t_0)$ and by its curvature $\theta'(t)$. Changing the point on σ, the initial data differ by a rigid motion (an isometry on $\mathbb{P}_{t_0}$) and the angle function $\theta(t)$ remains the same at points of equal height t. Then, by the uniqueness of the fundamental theorem on planar curves, the two curves differ by the same rigid motion that fix σ. Therefore, all the curves γ_s differ by a rotation about the t-axis (because all these curves pass through a point of the same circle σ with congruent initial tangent vector). Thus, Σ is contained on a surface of revolution in the sense we defined in §2.2. Since Σ is complete and connected, it follows that Σ is rotationally invariant. $\square$

In the case that the product is Lorentzian, we can say a little more about immersed discs.

Theorem 2.8 *Let $X : \mathbb{D} \to \mathbb{M}^2(\kappa) \times \mathbb{L}^1$ be a complete space like cmc immersed disc in the Lorentzian product with $\kappa < 0$. If $Q = 0$, then $X(\mathbb{D})$ is rotationally invariant graphic over some $\mathbb{P}_t$.*

Proof:

Since the immersion is space like, the projection $\pi : (p, t) \mapsto p$ of $X(\Sigma)$ over $\mathbb{M}^2(\kappa)$ is a local diffeomorphism. Furthermore, it increases Riemannian distances

$$\begin{aligned} \langle \mathrm{d}\pi V, \mathrm{d}\pi V \rangle &= \langle V^h, V^h \rangle \\ &= \langle V, V \rangle + \langle V^t, V^t \rangle \\ &\geq \langle V, V \rangle. \end{aligned}$$

Thus by an standard reasoning (see e.g. [10]), this projection is a covering map. Since $X(\Sigma)$ is a disc, then in fact we have a global diffeomorphism between $X(\Sigma)$ and $\mathbb{M}^2(\kappa)$. So, we conclude that $X(\Sigma)$ is a graphic over $\mathbb{M}^2(\kappa)$ and, in particular, over each $\mathbb{P}_t$. Now, since $X(\Sigma)$ is space-like, it is not possible that $t \to \pm\infty$ when we approaches $X(\partial\Sigma)$. Thus $X(\Sigma)$ lies in a slab $\mathbb{M}^2(\kappa) \times [a, b]$ with finite $[a, b]$. So $X(\partial\Sigma)$ is a closed curve on $\partial_\infty \mathbb{M}^2(\kappa) \times \mathbb{R}$ and, consequently, the interior intersections

$X(\Sigma) \cap \mathbb{P}_t$ do not meet the asymptotic boundary (since $X(\Sigma)$ is a graph). Again by Lemma 2.6, is a round circle in $\mathbb{P}_t$ and by Lemma 2.5 has constant angle function $\theta(t)$ and the proof follows as we before. □

To conclude this section we state another uniqueness result. This is a direct consequence of the proof of Theorem 2.7 and will be used in the proof of Theorem 2.10 below.

Corollary 2.9 *Let $x : \mathbb{D} \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be a cmc immersed disc with $x(\partial\mathbb{D}) \subset \mathbb{M}^2(\kappa) \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. Then, if $Q \equiv 0$, Σ is part of a rotationally invariant surface.*

2.4 Free Boundary Surfaces in Product Spaces

A classical result of J. Nitsche (see, e.g., [28] and [34]) characterized discs and spherical caps as equilibria solutions for the free boundary problem in space forms. We will be concerned now how to reformulate this problem in the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

Let Σ be an orientable compact surface with non empty boundary and $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be an isometric immersion. By a *volume-preserving variation of* X we mean a family $X_s : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$, with small s, of isometric immersions such that $X_0 = X$ and $\int \langle \partial_s X_s, n_s \rangle \mathrm{d}A_s = 0$, where $\mathrm{d}A_s$ and n_s represent, respectively, the element of area and an unit normal vector field to X_s. In the sequel we write $\xi_s = \partial_s X_s$ for the variational field and write $f_s = \langle \partial_s X_s, n_s \rangle$. For simplicity we set $\xi = \xi_0$ and $f = f_0$. We say that X_s is an *admissible variation* if it is volume-preserving and at each time s the boundary $X_s(\partial\Sigma)$ of $X_s(\Sigma)$ lies on a slice $\mathbb{M}^2(\kappa) \times \{t_0\}$. This mean that the variation X_s has free boundary on $\mathbb{M}^2(\kappa) \times \mathbb{R}$. We denote by Ω_s the compact domain in $\mathbb{M}^2(\kappa) \times \{t_0\}$ whose boundary is $\partial\Sigma$ (in the spherical case $\kappa > 0$, we choose one of the two domains bounded by $X(\partial\Sigma)$). A *stationary surface* is by definition a critical point for the following functional

$$E(s) = \int_\Sigma \mathrm{d}A_s + \alpha \int_{\Omega_s} \mathrm{d}\Omega,$$

for some constant α, where $\mathrm{d}\Omega$ is the volume element for Ω_s induced from $\mathbb{M}^2(\kappa)$. The first variation formula for this functional is (see ([33] for this formula in general Riemannian manifolds)

$$E'(0) = -2 \int_\Sigma H f dA + \int_{\partial\Sigma} \langle \xi, \nu + \alpha\bar{\nu} \rangle \, \mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ is the line element for $\partial\Sigma$ and ν, $\bar{\nu}$ are the unit co-normal vector fields to $\partial\Sigma$ relatively to Σ and to $\mathbb{M}^2(\kappa)$, respectively. If we prescribe $\alpha = -\cos\theta$ in the Riemannian case and $\alpha = -\cosh\theta$ in the Lorentzian case, we conclude that *a stationary surface Σ has constant mean curvature and makes constant angle θ along $\partial\Sigma$ with the horizontal plane and conversely.*

As before, we suppose that the surface is space like when we consider immersion in Lorentzian products. Now we have,

Theorem 2.10 *Let $X : \overline{\mathbb{D}} \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be an immersion with constant mean curvature such that $X(\partial\mathbb{D})$ lies in some slice $\mathbb{M}^2(\kappa) \times \{t_0\}$ and makes constant angle along its boundary. Then $X(\mathbb{D})$ is part of a rotationally invariant surface.*

Proof:

The proof follows the idea of the proof of Nitsche's Theorem in $\mathbb{R}^3$ (see [28]). We write $z = u+iv \in \overline{\mathbb{D}}$. If we put $\partial_z = \frac{1}{2}(\partial_u - i\partial_v)$, then the $\mathbb{C}$-bilinear complexification of q satisfies

$$q_{\mathbb{C}}(\partial_z, \partial_z) := q(\partial_u, \partial_u) - q(\partial_v, \partial_v) - 2iq(\partial_u, \partial_v) = 2Q(\partial_z, \partial_z).$$

Now if $X(\partial\mathbb{D})$ is contained in a horizontal slice $t = t_0$ then $q(\tau, \nu) = 0$ on $X(\partial\mathbb{D})$. Here $\tau = E^{-1}(-v\partial_u + u\partial_v)$ is the unit tangent vector to $\partial\mathbb{D}$ and $\nu = E^{-1}(u\partial_u + v\partial_v)$ is the unit outward co-normal to $\partial\Sigma$. In fact $h^2(\tau, \nu) = 0$ since τ is a horizontal vector and $h^1(\tau, \nu) = 0$ since $X(\partial\Sigma)$ is a line of curvature for $X(\mathbb{D})$ by Joachimstahl's Theorem.

On the other hand, we have on $\partial\Sigma$ that

$$0 = q(\tau, \nu) = (u^2 - v^2)q(\partial_u, \partial_v) - uvq(\partial_u, \partial_u) + uvq(\partial_v, \partial_v) = \Im(z^2 Q(\partial_z, \partial_z))$$

From this we conclude that $\Im(z^2\Psi) \equiv 0$ on $\partial\Sigma$. Since $z^2\Psi$ is holomorphic on $x(\mathbb{D})$, then $\Im(z^2\Psi)$ is harmonic. So, $z^2\Psi \equiv 0$ on Σ and hence $Q \equiv 0$ on $X(\mathbb{D})$. Thus, $X(\mathbb{D})$ is part of a CMC rotationally invariant surface. □

We also obtain a result about stable CMC discs in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Here stability for a CMC surface Σ means that the quadratic form

$$J[f] := \int_\Sigma \Big(\Delta f + \varepsilon(|h^1|^2 + \mathrm{Ric}(n_1, n_1))f \Big) f \,\mathrm{d}A,$$

is non-negative with respect to the variational fields f generating volume - preserving variations. In the formula above, Ric denotes the Ricci curvature of $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

In other words, the first eigenvalue of the stability operator $L := \Delta + \varepsilon(|h^1|^2 + \mathrm{Ric}(n_1, n_1))$ is greater than or equal to 0.

The result we will state below was proved for surfaces in space forms by Alias, Lopez and Palmer in [3]. They had used a classical result due to R. Courant about nodal domains of eigenfunctions. We recall that a *nodal set* of a C^0 function $f : \Sigma \to \mathbb{R}$ is a connected component of the complementar of $f^{-1}(0)$. Now, given an operator $L = \Delta + q$, where q is a bounded function, let $\lambda_1 < \lambda_2 \leq \ldots$ denote the eigenvalues of the Dirichlet problem of L, repeated according to its multiplicity, and $\{f_1, f_2, \ldots\}$ a complete orthonormal basis of eigenfunctions associated to $\{\lambda_1, \lambda_2, \ldots\}$. The Courant's Nodal Domain Theorem says that *the number of nodal domains of f_k is less than equal to k*, $k = 1, 2, \ldots$. We will apply this Theorem to the stability operator.

Theorem 2.11 *Let Σ be a disc type surface immersed with constant mean curvature H in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Suppose that $\partial\Sigma$ is contained in a geodesic circle in some slice $\mathbb{M}^2(\kappa) \times t_0$ and that the immersion is stable. Then Σ is part of a rotationally invariant surface. Moreover, if $H = 0$ then Σ is a totally geodesic disc.*

Proof:

We consider the vector field $V(t, p) = a \wedge \partial_t \wedge p$, where a is the vector in $\mathbb{R}^3$ ($\mathbb{L}^3$, if $\varepsilon = -1$) perpendicular to the plane where $\partial\Sigma$ lies and $\wedge$ stands for the vector cross product in $\mathbb{E}^4$, which is determined by

$$\langle v_1 \wedge v_2 \wedge v_3, v\rangle = \det(v_1, v_2, v_3, v),$$

where det is calculated in the canonical basis.

It is easy to see that V is is a Killing field in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Then $f := \langle V, n_1\rangle$ satisfies trivially $J[f] = 0$. On the other hand, since $\mathrm{div}V = 0$ in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ and $\partial\Sigma \subset \mathbb{M}^2(\kappa) \times \{t_0\}$ we have,

$$\int_\Sigma \langle V, n_1\rangle dA = \int_\Sigma f dA = 0.$$

Moreover, since $X(\partial D)$ is a geodesic circle on a slice $\mathbb{P}_{t_0}$ we have $f = 0$ on $X(\partial D)$.

Now let ν be the exterior unit conormal direction to Σ along the boundary $\partial\Sigma$. The normal derivative of f is calculated as

$$\begin{aligned} \nu(f) &= \nu\langle V, n_1\rangle = \langle a \wedge \partial_t \wedge D_\nu p, n_1\rangle + \langle a \wedge \partial_t \wedge p, D_\nu n_1\rangle \\ &= \langle a \wedge \partial_t \wedge \nu, n_1\rangle + \langle a \wedge \partial_t \wedge p, D_\nu n_1\rangle = \langle \tau, D_\nu n_1\rangle = -h^1(\tau, \nu), \end{aligned}$$

where, $\tau = p \wedge a \wedge \partial_t$ is the tangent positively oriented unit vector to $\partial\Sigma$. Since $\langle \tau, \partial_t \rangle = 0$ and $\langle \tau, \nu \rangle = 0$ it follows that

$$h^2(\tau, \nu) = -\langle \tau^h, \nu^h \rangle = 0.$$

Thus,

$$2H\, \nu(f) = -2H\, h^1(\tau, \nu) = -q(\tau, \nu).$$

However, if u, v denote the usual cartesian coordinates on Σ then

$$q(\tau, \nu) = E^{-1}\, q(u\partial_u + v\partial_v, -v\partial_u + u\partial_v) = -\Im(z^2 Q)$$

on $\partial\Sigma$. We conclude that $2H\, \nu(f) = \Im(z^2 Q)$. Proceeding as in ([3]) we verify that $\nu(f)$ vanishes at least three times, since Σ is disc type. We point out that, since $J[f] = 0$, f is an eigenfunction to the stability operator L whose eigenvalue is 0. To conclude the proof we need to show that f vanishes identically.

Let us suppose that $f \neq 0$ on Σ and let $\{\lambda_1, \lambda_2, \ldots\}$ the eigenvalues of the Dirichlet problem of the stability operator L as above. Then f does not correspond to λ_1, since f changes of sign in Σ. We claim that f does not correspond to λ_2 either. It follows by applying Courant's theorem on nodal domains since f can partition Σ into at most two nodal domains whose common boundary can intersect $\partial\Sigma$ in either two or zero points. By E. Hopf maximum principle, f has non-vanishing normal derivative at all other points on $\partial\Sigma$. Thus $\lambda_1 < \lambda_2 < 0$ and this implies that the immersion is unstable. So this allows us to conclude that f vanishes on the whole disc. So, $X(\Sigma)$ is foliated by the flux lines of V, i.e. by horizontal geodesic circles. Moreover, $Q = 0$ on $X(\Sigma)$. Thus, $X(\Sigma)$ is a rotationally invariant surface $\square$

Chapter 3

The r-stability of Hypersurfaces with Zero Gauss-Kronecker Curvature

3.1 Introduction

Let M be an oriented hypersurface of the $(n+1)$-dimensional Euclidean space and let $g : M \to S^n$ denote its Gauss map. The shape operator of M is the self-adjoint map given by $B := -dg$, that is, for each $p \in M$,

$$B_p : T_pM \to T_pM, \ B_p(X) = -dg_p(X).$$

The eigenvalues of B_p are called the *principal curvatures of* M *at* p. We denote them by $k_1(p), \ldots, k_n(p)$ and we define the *r-mean curvature of* M as the normalized r-elementary symmetric function of the principal curvatures of M, namely,

$$H_0 = 1, \ H_r = \binom{n}{r}^{-1} S_r, \ r = 1, \ldots, n,$$

where

$$S_r = \sum_{i_1<\ldots<i_r} k_{i_1} \ldots k_{i_r}.$$

Notice that H_1, H_2 and H_n are the mean, scalar and Gauss-Kronecker curvatures of M, respectively.

We say that M is *r-minimal* when $H_{r+1} = 0$. It is well known that the r-minimal hypersurfaces of the Euclidean space are critical points of the r-area functional $A_r = \int_M S_r dM$ for compactly supported variations of M.

In order to state our results we need introduce more notations. Let P_r be the *Newton transformations of* B, which can be defined inductively by

$$P_0 = I,\ P_r = S_r I - BP_{r-1}, \quad r = 1, 2, \dots, n.$$

Let $C_0^\infty(M)$ denote the set of smooth functions with compact support on M. Using the Newton transformations we define *the linearized operator*

$$L_r(f) = \mathrm{div}(P_r \nabla f), \quad \text{ for } f \in C_0^\infty(M). \tag{3.1}$$

We denote by T_r *the Jacobi operator*,

$$T_r f = L_r f - (r+2) S_{r+2} f,$$

and by $I_r(f_1, f_2) = -\int_M f_1 T_r(f_2) dM$ the associated bilinear symmetric form.

Let D be a regular domain on M, that is, D is bounded and has piecewise smooth boundary. We say that D is *r-stable* if $I_r(f,f) \geq 0$ for all $f \in C_0^\infty(D)$ or if $I_r(f,f) \leq 0$ for all $f \in C_0^\infty(D)$. Otherwise we say that D is *r-unstable*.

In the study of r-stability we need to suppose L_r is elliptic. This is equivalent to P_r being positive definite or negative definite everywhere. On the other hand, by a Theorem of Hounie-Leite in [23], it is known that, when $H_{r+1} = 0$, then L_r is elliptic if and only if $\mathrm{rank}(B) > r$. In the following, without loss of generality, we will fix $P_r > 0$. Also, the eigenvalues of the operator $\sqrt{P_r} B$ appear naturally and we will denote them by $\theta_1(r), \theta_2(r), ..., \theta_n(r)$.

In [2] Alencar, do Carmo and Elbert gave sufficient conditions for a regular domain on a r-minimal hypersurface of the Euclidean space to be r-stable. Their general result assumes that the quotient $\frac{|H_n|}{\|\sqrt{P_r} B\|^2}$ is constant. In this case the hypersurface is said to be *r-special*.

Theorem A (Theorem 1.3 of [2]) *Let $x : M^n \to \mathbb{R}^{n+1}$ be an oriented r-special hypersurface with $H_{r+1} = 0$ and $H_n \neq 0$ everywhere. Let $D \subset M$ be a regular domain such that the area of $g(D) \subset S^n$ is smaller than the area of a spherical cap whose first eigenvalue for the spherical Laplacian is*

$$\tau = \max_{i,D} \Big(\frac{\sum_j \theta_j^2(r)}{\theta_i^2(r)} \Big).$$

Then D is r-stable.

This Theorem is a generalization of a classical result to minimal surfaces of $\mathbb{R}^3$ due to Barbosa and do Carmo in [8]. We point out that the hypothesis of M to be r-special occurs naturally when $r = n-2$. In fact, a computation shows that $\frac{|S_n|}{\|\sqrt{P_{n-2}}B\|^2} = \frac{1}{n}$.

In this chapter we are interested to improve the condition on the Gauss-Kronecker curvature considering points on M for which $H_n = 0$. A simple example, like a cylinder over a plane curve, shows that H_n cannot be identically zero. In [2], remark 4.2, it was conjectured that Theorem A holds if the set of zeros of H_n is contained in a submanifold of codimension ≥ 2. Here we answer this conjecture affirmatively as a consequence of a more general result. We will consider hypersurfaces with $H_n = 0$ on a subset of *capacity* zero (see below the definition of capacity). It is known ([32] §2, p. 35) that submanifolds of codimension $d \geq 2$ have capacity zero. Later, in Proposition 3.5 we will present another proof of this fact for completeness. In section 3.2 we will give definitions and develop some facts about capacity. To state our result we will denote by A the set $A = \{p \in M : H_n(p) = 0\}$ and by $\lambda_1(D)$ the *first eigenvalue* for the Laplacian on D. We also point out that, since symmetrization of domains in the sphere does not increase eigenvalues, the hypothesis on the first eigenvalue of the spherical image of D in Theorem A implies that $\lambda_1(g(D)) \geq \tau$. For simplicity we will use this condition on our result below.

Theorem 3.1 *Let M be an oriented r-minimal hypersurface of $\mathbb{R}^{n+1}$, which is r-special on $M \backslash A$. Let $D \subset M$ be a regular domain such that $A \subset D$ and $\lambda_1(g(D)) \geq \tau$. Then, if* $\mathrm{Cap}(A) = 0$, *D is r-stable.*

The idea of the proof is to use a relation between the eigenvalues of domains from which we remove a subset and the capacity of the removed subset. Actually, we need just a comparison between the first eigenvalues of P_r on D and on $D \setminus A$. This relation is well known (for the Laplacian) and we can find in [32], for domains of the Euclidean space, and in [13], for domains of a closed Riemannian manifold. Here we obtain such results for an elliptic operator L in divergence form on a bounded domain D of a n-dimensional Riemannian manifold M. Given $A \subset D$, let $\lambda_k(D)$ and $\lambda_k(D \setminus A)$ denote the k-th eigenvalue of the Dirichlet problem of L on D and on $D \setminus A$, respectively. We have the following result that will be proved in the next section

Theorem 3.2 *In the above conditions there exist positive constants ε_k and C_k, such that if* $\mathrm{Cap}\, A \leq \varepsilon_k$, *then*

$$\lambda_k(D) \leq \lambda_k(D \setminus A) \leq \lambda_k(D) + C_k \,\mathrm{Cap}\, A^{\frac{1}{2}}.$$

In particular, $\lambda_k(D) = \lambda_k(D \setminus A)$ *if* $\operatorname{Cap} A = 0$.

3.2 The Spectrum of Domains and the Capacity

In this section we introduce the notion of capacity and prove the Theorem 3.2.

Let $(M^n, \langle\, ,\, \rangle)$ be a smooth Riemannian manifold and $D \subset M$ a bounded domain. As usual, we define $H_0^1(D)$ as the closure of $C_0^\infty(D)$ with respect to the norm H^1:

$$|u|_{H^1}^2 := \int_D u^2 dM + \int_D |\nabla u|^2 dM, \text{ for } u \in C_0^\infty(D)$$

where ∇ is the gradient, $|.|$ is the norm of vector, and dM is the volume element with respect to the metric $\langle\, ,\, \rangle$. Now, given $A \subset D$, we set $\mathfrak{H}(D; A) = \{u \in H_0^1(D) : \exists\, U \subset D$ open, $A \subset U$ and $u = 1$ in $U\}$ and $H(D; A)$ as the *closure* of $\mathfrak{H}(D; A)$ with respect to H^1. With this notation we define the *Capacity of* A as

$$\operatorname{Cap} A = \inf \Big\{ \int_D |\nabla u|^2 dM : u \in H(D; A) \Big\}.$$

Below we have some known consequences of this definition (see, e.g. [17], §4.7).

Proposition 3.3 *For any* $A \subset D$ *we have,*
i) $\operatorname{Cap} A = \inf\{\operatorname{Cap} U : U$ *open,* $A \subset U\}$*;*
ii) If $A_1 \subset \ldots \subset A_k \subset \ldots$ *are compact subsets of* D*, then*

$$\lim_{k\to\infty} \operatorname{Cap} A_k = \operatorname{Cap} (\cap_k A_k).$$

We will use the notation $V^\varepsilon(A)$ to denote *the tubular neighborhood of* A of radius ε, that is, $V^\varepsilon(A) = \{x \in M : \operatorname{dist}(x, A) < \varepsilon\}$, where $\operatorname{dist}(\, ,\,)$ stands for the distance function on M. Using this notation we have the following consequence of the above proposition:

Corollary 3.4 *If* A *is compact, then*

$$\lim_{\varepsilon\to 0} \operatorname{Cap} V^\varepsilon(A) = \operatorname{Cap} A.$$

In the case that A is a submanifold of codimension $d \geq 2$ we can say more:

Proposition 3.5 *If* A *is an embedded submanifold of codimension* $d \geq 2$ *then*

$$\operatorname{Cap} V^\varepsilon(A) \to 0 \textit{ as } \varepsilon \to 0.$$

In particular, $\operatorname{Cap} A = 0$.

Proof:

We will show that there exist sequences $\varepsilon_k \to 0$ and $\zeta_k \in C_0^\infty(D)$ with $\operatorname{supp}\zeta_k \subset V^{\varepsilon_k}$ and $\zeta = 1$ in some neighborhood of A, whose energy $\int_D |\nabla\zeta_k|^2 dM \to 0$, as $k \to \infty$.

Fix $\alpha > 0$ and consider a smooth function $\psi : D \to \mathbb{R}$ such that

$$\psi(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, A) < \alpha/2, \\ 0, & \text{if } \operatorname{dist}(x, A) > \alpha, \end{cases}$$

and $\psi(x) = \psi(y)$, if $\operatorname{dist}(x, A) = \operatorname{dist}(y, A)$. Naturally, $\psi \in C_0^\infty(D)$ if α is small enough and $|\nabla\psi|$ is uniformly bounded on D. Now, as $\psi(x)$ depends only on distance of x to A, we can define a sequence $\psi_k : D \to \mathbb{R}$, given by $\psi_k(x) = \psi(kx)$, where kx denotes a point on D whose distance to A is $\operatorname{dist}(kx, A) = k\operatorname{dist}(x, A)$.

Then ψ_k is an uniformly bounded sequence whose support is contained in the neighborhood of radius α/k of A, $k = 1, 2, \ldots$. Namely, $\operatorname{supp}\psi_k \subset V^{\frac{\alpha}{k}}(A)$.

In particular,

$$|\psi_k|^2_{L^2(D)} := \int_D \psi_k^2 dM \to 0 \text{ as } k \to \infty. \tag{3.2}$$

Since A is a submanifold of codimension d, the Jacobian of the change of variables given by $h(y) := \frac{1}{k}y = x$ has range d. Thus,

$$\begin{aligned} \int_D |\nabla\psi_k(x)|^2 dM &= k^2 \int_D |\nabla\psi(kx)|^2 dM = k^2 \int_D |\nabla\psi(y)|^2 k^{-d} dM \\ &= k^{2-d} \int_D |\nabla\psi(y)|^2 dM. \end{aligned}$$

We conclude that when $d \geq 2$, the sequence ψ_k is bounded in H^1. In particular there exists a constant $C > 0$, independent of k, such that

$$\int_D |\nabla\psi_k|^2 dM \leq C.$$

We point out that $\operatorname{supp}|\nabla\psi_k| \subset V^{\frac{\alpha}{k}}(A) \setminus V^{\frac{\alpha}{2k}}(A)$. Hence, passing to a subsequence, we may suppose that $\cup_j \operatorname{supp}\nabla\psi_j \subset D$ and $\operatorname{supp}\nabla\psi_i \cap \operatorname{supp}\nabla\psi_j = \emptyset$, if $i \neq j$.

Now set $S_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$ and define $\zeta_k = \frac{1}{S_k}\sum_{j=1}^k \frac{\psi_j}{j}$. It is clear that ζ_k is a smooth function whose support is contained in $V^{\frac{\alpha}{k}}(A)$ and $\zeta_k = 1$ in $V^{\frac{\alpha}{2k}}(A)$.

Therefore,

$$\begin{aligned}\int_D |\nabla\zeta_k|^2 dM &\leq \frac{1}{S_k^2}\int_D \Big(\sum_{j=1}^{k}\frac{|\nabla\psi_j|}{j}\Big)^2 dM\\ &\leq \frac{1}{S_k^2}\sum_{j=1}^{k}\frac{1}{j^2}\int_D |\nabla\psi_j|^2 dM\\ &\leq \frac{C}{S_k^2}\sum_{j=1}^{k}\frac{1}{j^2}\to 0,\ \text{as } k\to\infty.\end{aligned}$$

$\square$

Remark: *Sets of zero capacity in a Riemannian manifold could not be smooth submanifolds.*

Now recall that, $H_0^1(D)$ is a Hilbert space with respect to

$$\langle u_1, u_2\rangle_* = \int_D \langle\nabla u_1, \nabla u_2\rangle dM.$$

In fact $\langle\, ,\,\rangle_*$ and $\langle\, ,\,\rangle_{H^1}$ are equivalents, as we can see by using the *Poincaré inequality*

$$|u|_{L^2(D)} \leq C|\nabla u|_{L^2(D)}.$$

Hence $H(D;A)$ is a closed (affine) subspace of $H_0^1(D)$ with respect to $\langle\, ,\,\rangle_*$. Let u_A be the orthogonal projection of 0 on $H(D;A)$. By definition we get $|\nabla u_A|^2_{L^2(D)} = \mathrm{Cap}\, A$.

Now we start the proof of Theorem 3.2 following some ideas contained in [13]. Here we will consider nonempty boundary domains and general elliptic operators in the divergence form.

For each $x\in D$, let $P_x : T_xM \to T_xM$ be a symmetric, positive (or negative) defined operator. We define, for each$f\in C_0^\infty(D)$,

$$Lf = \mathrm{div}(P\nabla f) + qf,$$

where $q : D\to\mathbb{R}$ is a bounded function.

We consider the unique extension of L to $H_0^1(D)$. Then, L is an elliptic operator and let us denote by $\{\lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$ the spectrum of the Dirichlet problem of L on D, repeated according to its multiplicity.

We recall the Courant's minmax principle for the eigenvalues of L on domains of M:

$$\lambda_k(D \setminus A) = \min_{E \in \mathcal{E}_k} \max_{f \in E \setminus \{0\}} \frac{\int_{D\setminus A}(\langle P\nabla f, \nabla f\rangle + qf^2)dM}{\int_{D\setminus A} f^2 dM},$$

where $\mathcal{E}_k$ is the set of k-dimensional subspaces of $H^1_0(D \setminus A)$. The quotient above is called *Rayleigh quotient* for f.

By using the analogous characterization for $\lambda_k(D)$ we easily obtain the first inequality of Theorem 3.2. In order to obtain the second inequality we choose $f_1, \dots, f_k$, an orthonormal basis of eigenfunctions associated to $\lambda_1(D), \dots, \lambda_k(D)$, and set F_k the space generated by $f_1, \dots, f_k$. Then $F_k \subset H^1_0(D)$ and

$$\lambda_k(D) = \max_{f \in F_k \setminus \{0\}} \frac{\int_D(\langle P\nabla f, \nabla f\rangle + qf^2)dM}{\int_D f^2 dM}.$$

We now define $E_k = \{g = f(1-u_A) : f \in F_k\}$. It is clear that $E_k \subset H^1_0(D \setminus A)$ is a finite dimensional subspace. We will see that, when A has small capacity, then E_k has dimension equal to k. In fact, the functions $g_j = f_j(1-u_A), j = 1, \dots, k$ form a basis for E_k. We have that

$$\langle g_i, g_j\rangle_{L^2(D)} = \delta_{ij} - 2\int_D f_i f_j u_A dM + \int_D f_i f_j u_A^2 dM.$$

Thus, using the Cauchy-Schwarz and Poincaré inequalities we have

$$\begin{aligned}
&|\langle g_i, g_j\rangle_{L^2(D)} - \delta_{ij}| \le 2\Big|\int_D f_i f_j u_A dM\Big| + \Big|\int_D f_i f_j u_A^2 dM\Big| \\
&\le 2\Big(\int_D (f_i f_j)^2 dM\Big)^{\frac{1}{2}}\Big(\int_D u_A^2 dM\Big)^{\frac{1}{2}} + \max_D (f_i f_j)\int_D u_A^2 dM \\
&\le C_k\Big(\int_D u_A^2 dM\Big)^{\frac{1}{2}} + C'_k\int_D u_A^2 dM \\
&\le B_k(Cap\, A^{\frac{1}{2}} + Cap\, A),
\end{aligned}$$

where B_k is a positive constant depending only on $\operatorname{vol} D$ and $\max_{i=1,\dots,k}|f_i|_{L^\infty(D)}$. On the other hand, we may choose $\varepsilon_k > 0$ sufficiently small such that, if $\operatorname{Cap} A < \varepsilon_k$ then

$$B_k(Cap\, A + Cap\, A^{\frac{1}{2}}) < \min\{\langle g_i, g_j\rangle_{L^2(D)} - \delta_{ij} : i, j = 1, \dots, k\}.$$

For such ε_k we have that $g_1, \ldots, g_k$ form an orthonormal basis and consequently E_k has dimension equal to k as we claimed. Now we look for estimates of the numerator of the Rayleigh quotient for $g = f(1-u_A) \in E_k$ such that $|f|_{L^2(D)} = 1$. We first observe that for any $A \subset D$ with $Cap\, A < \varepsilon_k$,

$$|g|^2_{L^2(D)} = 1 - 2\int_D f^2 u_A dM + \int_D f^2 u_A^2 dM \geq 1 - B'_k Cap\, A^{\frac{1}{2}}. \tag{3.3}$$

Now we have,

$$\begin{aligned}
&\int_D \langle P\nabla g, \nabla g\rangle + qg^2 dM = \\
&\int_D \langle P(\nabla f - u_A\nabla f - f\nabla u_A), \nabla f - u_A\nabla f - f\nabla u_A\rangle dM + \\
&\int_D q(f^2 - 2f^2u_A + f^2u_A)dM \\
&= \int_D \langle P\nabla f, \nabla f\rangle - \langle P\nabla f, u_A\nabla f\rangle - \langle P\nabla f, f\nabla u_A\rangle - u_A\langle P\nabla f, \nabla f\rangle dM + \\
&\int_D u_A^2\langle P\nabla f, \nabla f\rangle + u_A f\langle P\nabla f, \nabla u_A\rangle - f\langle P\nabla f, \nabla u_A\rangle dM + \\
&\int_D (u_A f\langle P\nabla f, \nabla u_A\rangle + f^2\langle P\nabla u_A, \nabla u_A\rangle + qf^2 - 2qf^2u_A + qf^2u_A^2)dM \\
&= \int_D ([\langle P\nabla f, \nabla f\rangle + qf^2] - 2\langle P\nabla f, \nabla f\rangle u_A + \langle P\nabla f, \nabla f\rangle u_A^2)dM + \\
&+ \int_D (\langle P\nabla u_A, \nabla u_A\rangle f^2 - 2\langle P\nabla f, \nabla u_A\rangle f(1-u_A) - 2f^2u_A + f^2u_A^2)dM \\
&\leq \lambda_k + D_k^1 \operatorname{Cap} A^{1/2} + D_k^2 \operatorname{Cap} A + D_k^3 \operatorname{Cap} A + D_k^4 \operatorname{Cap} A + \\
&D_k^5 \operatorname{Cap} A^{1/2} + D_k^6 \operatorname{Cap} A \leq \lambda_k(D) + B''_k(\operatorname{Cap} A + \operatorname{Cap} A^{1/2}),
\end{aligned}$$

where B''_k depends only on vol$D$, $\max_{i=1,\ldots,k} |f_i|_{L^\infty(D)}$, $\max_{i=1,\ldots,k} |\nabla f_i|_{L^\infty(D)}$ and $\max_D ||P||$.

Therefore, choosing $\varepsilon_k > 0$ such that $|g|^2_{L^2(D)} > 0$ in (3.3), we have

$$\begin{aligned}
&\frac{\int_D \langle P\nabla g, \nabla g\rangle + qg^2 dM}{\int_D g^2 dM} \leq \frac{\lambda_k(D)}{1 - B'_k \operatorname{Cap} A^{1/2}} + \frac{B''_k(\operatorname{Cap} A + \operatorname{Cap} A^{1/2})}{1 - B'_k \operatorname{Cap} A^{1/2}} \\
&= \lambda_k(D) + \frac{\lambda_k(D)B'_k \operatorname{Cap} A^{1/2} + B''_k(\operatorname{Cap} A + \operatorname{Cap} A^{1/2})}{1 - B'_k \operatorname{Cap} A^{1/2}}.
\end{aligned}$$

We observe now that, if $\operatorname{Cap} A < \varepsilon_k$, then $1 - B'_k \operatorname{Cap} A^{1/2} > 1 - B'_k \varepsilon_k^{1/2}$ and therefore we can estimate the second term above

$$\frac{\lambda_k(D) B'_k \operatorname{Cap} A^{1/2} + B''_k(\operatorname{Cap} A + \operatorname{Cap} A^{1/2})}{1 - B'_k \operatorname{Cap} A^{1/2}} \leq \frac{\lambda_k(D) B'_k \operatorname{Cap} A^{1/2}}{1 - B'_k \varepsilon_k^{1/2}} +$$
$$\frac{(B''_k \operatorname{Cap} A^{1/2} + B''_k) \operatorname{Cap} A^{1/2}}{1 - B'_k \varepsilon_k^{1/2}} \leq \frac{(\lambda_k(D) B'_k + B''_k \varepsilon_k^{1/2} + B''_k) \operatorname{Cap} A^{1/2}}{1 - B'_k \varepsilon_k^{1/2}}$$
$$= C_k \operatorname{Cap} A^{1/2}.$$

So,

$$\frac{\int_D \langle P\nabla g, \nabla g\rangle + qg^2 dM}{\int_D g^2 dM} \leq \lambda_k(D) + C_k \operatorname{Cap} A^{1/2}.$$

This implies that

$$\lambda_k(D \setminus A) = \max_{g \in E_k \setminus \{0\}} \frac{\int_D (\langle P\nabla g, \nabla g\rangle + qg^2) dM}{\int_D g^2 dM} \leq \lambda_k(D) + C_k \operatorname{Cap} A^{1/2},$$

and we conclude the proof of Theorem 3.2.

Corollary 3.6 *Given a closed subset $A \subset D$ with $\operatorname{Cap} A = 0$ let λ_k^ε be the k-th eigenvalue of the Dirichlet problem of the operator L in $D \setminus V^\varepsilon(A)$. Then*

$$\lambda_k^\varepsilon \to \lambda_k(D).$$

In particular, if $\lambda_k^\varepsilon \geq 0$ for all $\varepsilon > 0$, then $\lambda_k(D) \geq 0$.

3.3 Proof of Theorem 3.1

We start presenting an equivalent condition for r-stability. A simple computation shows that

$$\operatorname{trace}(B^2 P_r) = ||\sqrt{P_r} B||^2 := \sum_{i=1}^n \theta_i^2(r).$$

On the other hand, Lemma 2.1 of [7] says that $\operatorname{trace}(B^2 P_r) = S_1 S_{r+1} - (r+2) S_{r+2}$. Thus, for r-minimal immersions, the Jacobi operator can be written as

$$T_r = L_r + \operatorname{trace}(B^2 P_r) = L_r + ||\sqrt{P_r} B||^2.$$

Using integration by parts, we have that

$$\begin{aligned} I_r(f,f) &= \int_D \langle P_r \nabla f, \nabla f\rangle - ||\sqrt{P_r}B||^2 f^2 dM \\ &= \int_D |\sqrt{P_r}\nabla f|^2 - ||\sqrt{P_r}B||^2 f^2 dM. \end{aligned}$$

So, in the case $P_r > 0$, to check that a regular domain $D \subset M$ is r-stable we just need to show that the last term above is always nonpositive or nonnegative for all $f \in C_0^\infty(D)$. Similarly in the case $P_r < 0$.

Now we fix $\varepsilon > 0$ and denote $D^\varepsilon = D \setminus V^\varepsilon(A)$. Then, the Gauss map g is a local diffeomorphism on D^ε. Let $\varphi : g(D^\varepsilon) \to \mathbb{R}$ be the positive first eigenfunction of the spherical Laplacian $\widetilde{\Delta}$ on $g(D^\varepsilon)$, that is,

$$\begin{cases} \widetilde{\Delta}\varphi + \lambda_1^\varepsilon \varphi = 0 & \text{in } g(D^\varepsilon), \\ \varphi > 0 & \text{in } g(D^\varepsilon), \\ \varphi = 0 & \text{on } \partial g(D^\varepsilon), \end{cases}$$

where $\lambda_1^\varepsilon = \lambda_1(D^\varepsilon)$ is the first eigenvalue of $\widetilde{\Delta}$ on $g(D^\varepsilon)$. Recall that, since $D^\varepsilon \subset D$, we have $\lambda_1^\varepsilon \geq \lambda_1$.

In the following we will consider the pull back metric $\widetilde{s}$ by g on D^ε and denote by $\widetilde{\nabla}$ the gradient, by $[\,]$ the norm of a vector, and by dS the volume element in this metric. By Lemma 2.9 in [2] one have $\widetilde{\nabla} f = B^{-2}\nabla f$, for smooth functions f on M and a simple computation gives $[X] = |BX|$, for any tangent vector X. Also we point out that, in the metric $\widetilde{s}$, the Gauss map $g : D^\varepsilon \to S^n$ is a local isometry. Now let $\psi = \varphi \circ g$ defined in D^ε. Then ψ is positive and satisfies $\widetilde{\Delta}\psi + \lambda_1^\varepsilon\psi = 0$ in D^ε. Thus, by Corollary 1 in [20], we have that the first eigenfunction of the operator $\widetilde{\Delta} + \lambda_1^\varepsilon$ is nonnegative:

$$0 \leq \inf\Big\{ \int_{D^\varepsilon} ([\widetilde{\nabla} f]^2 - \lambda_1^\varepsilon f^2) dS : f \in C_0^\infty(D^\varepsilon), \int_{D^\varepsilon} f^2 dS = 1\Big\}. \tag{3.4}$$

Now, since $\det g = S_n$ and the immersion is r-special on D^ε, we have $dS = |S_n| dM = c||\sqrt{P_r}B||^2 dM$, where c is positive constant. Also, by hypothesis, $\lambda_1^\varepsilon \geq \lambda_1 \geq \tau$. Thus

$$\int_{D^\varepsilon} ([\widetilde{\nabla} f]^2 - \lambda_1^\varepsilon f^2) dS \leq c \int_{D^\varepsilon} ([\widetilde{\nabla} f]^2 - \tau f^2) ||\sqrt{P_r}B||^2 dM. \tag{3.5}$$

Observe now that $\rho_i = \frac{\theta_i^2(r)}{\sum_j \theta_j^2(r)}$, $i = 1, \ldots, n$, are the eigenvalues of the operator $\frac{\sqrt{P_r}B}{||\sqrt{P_r}B||}$. Therefore, by definition

$$\tau := \max_{i,\,D} \frac{1}{\rho_i^2} = \max_{D} \left| \left(\frac{\sqrt{P_r}B}{||\sqrt{P_r}B||} \right)^{-1} \right|^2 .$$

So,

$$[\widetilde{\nabla} f]^2 \leq \left[\left(\frac{\sqrt{P_r}B}{||\sqrt{P_r}B||} \right)^{-1} \right]^2 \left[\left(\frac{\sqrt{P_r}B\widetilde{\nabla} f}{||\sqrt{P_r}B||} \right) \right]^2 \leq \tau \left[\left(\frac{\sqrt{P_r}B\widetilde{\nabla} f}{||\sqrt{P_r}B||} \right) \right]^2 .$$

Using this on (3.5), we have

$$\begin{aligned} c\int_{D^\varepsilon} ([\widetilde{\nabla} f]^2 - \tau f^2) ||\sqrt{P_r}B||^2 dM &\leq c\tau \int_{D^\varepsilon} \Big(\frac{[\sqrt{P_r}B\widetilde{\nabla} f]^2}{||\sqrt{P_r}B||^2} - f^2 \Big) ||\sqrt{P_r}B||^2 dM \\ &= c\tau \int_{D^\varepsilon} \Big([\sqrt{P_r}B\widetilde{\nabla} f]^2 - ||\sqrt{P_r}B||^2 f^2 \Big) dM. \end{aligned}$$

Now, we recall that $\widetilde{\nabla} f = B^{-2}\nabla f$ and $[X] = |BX|$ to find

$$[\sqrt{P_r}B\widetilde{\nabla} f]^2 = |\sqrt{P_r}\nabla f|^2.$$

We conclude that, for any $f \in C_0^\infty(D^\varepsilon)$ with $\int_{D^\varepsilon} f^2 dS = 1$,

$$0 \leq c\tau \int_{D^\varepsilon} \Big(|\sqrt{P_r}\nabla f|^2 - ||\sqrt{P_r}B||^2 f^2 \Big) dM.$$

This implies that the first eigenvalue of the Jacobi operator T_r is nonnegative on D^ε and thus, by Corollary 3.6, we have that first eigenvalue of the Jacobi operator T_r is nonnegative in all D. This shows that D is r-stable.

Remark: *In the case $P_r < 0$ we may set $Q_r = -P_r$ in the above computations to conclude that*

$$0 \leq c\tau \int_{D^\varepsilon} \Big(|\sqrt{Q_r}\nabla f|^2 - ||\sqrt{Q_r}B||^2 f^2 \Big) dM.$$

On the other hand the Jacobi operator is given by

$$\begin{aligned} T_r f &= \operatorname{div}(P_r \nabla f) + \mathit{trace}(B^2 P_r) \\ &= -\operatorname{div}(Q_r \nabla f) + \mathit{trace}(B^2 Q_r). \end{aligned}$$

We conclude that the first eigenvalue of T_r is nonpositive on D and therefore we obtain r-stability according to our definition.

Appendix A

Proof of Claim 1.5

First write (1.13) setting $c_\lambda(x) = 0$ when $w_\lambda(x) = 0$. Fix $0 < \mu < n-2$ and define $g(x) = |x|^{-\mu}$ and $\phi(x) = \frac{w_\lambda(x)}{g(x)}$. Then, using the equation (1.13),

$$\Delta\phi + \frac{2}{g}\langle\nabla g, \nabla\phi\rangle + \left(c_\lambda(x) + \frac{\Delta g}{g}\right)\phi = 0.$$

By a computation we get $\Delta g = -\mu(n-2-\mu)|x|^{-\mu-2}$, that is,

$$\frac{\Delta g}{g} = -\mu(n-2-\mu)|x|^{-2}.$$

On the other hand, the expansion (1.8) implies that $w_\lambda(x) = O(|x|^{2-n})$ and consequently $c_\lambda(x) = O(|x|^{-n-2-2+n}) = O(|x|^{-4})$. Hence we obtain

$$c_\lambda(x) + \frac{\Delta g}{g} \leq C(|x|^{-4} - \mu(n-2-\mu))|x|^{-2}).$$

In particular $c(x) + \frac{\Delta g}{g} < 0$ for large $|x|$. Choose R_0 with $B(\bar{a}, r) \cup \Lambda \subset B(0, R_0)$ such that

$$C(|x|^{-4} - \mu(n-2-\mu))|x|^{-2}) < 0, \text{ for } |x| \geq R_0. \tag{A.1}$$

Now let $x_0 \in \text{int}(\Omega_\lambda)$ so that $w_\lambda(x_0) = \inf_{\text{int}(\Omega_\lambda)} w_\lambda < 0$.

Since $\lim_{|x|\to+\infty} \phi(x) = 0$ and $\phi(x) \geq 0$ on $\partial\Omega_\lambda$, there exists $\bar{x}_0$ such that ϕ has its minimum at $\bar{x}_0$. By applying the maximum principle for ϕ at $\bar{x}_0$ we get $c_\lambda(\bar{x}_0) + \frac{\Delta g(\bar{x}_0)}{g} \geq 0$ and by (A.1), $|\bar{x}_0| < R_0$. Now we have

$$\begin{aligned} \frac{w_\lambda(x_0)}{g(\bar{x}_0)} &\leq \frac{w_\lambda(\bar{x}_0)}{g(\bar{x}_0)} = \phi(\bar{x}_0) \\ &\leq \phi(x_0) = \frac{w_\lambda(x_0)}{g(x_0)}. \end{aligned}$$

This implies $|x_0| \leq |\bar{x}_0| \leq R_0$ and proves the claim.

Appendix B

Some Basic Formulas for Immersed Surfaces in $\mathbb{M}^2(\kappa) \times \mathbb{R}$

In this appendix we deduce formulas (B3)-(B6) used in § 2.3.

For a Riemannian or Lorentzian product $M^n \times \mathbb{R}$ we have that

$$\langle \overline{R}(X,Y)Z,W\rangle = \langle \overline{R}(X^h,Y^h)Z^h,W^h\rangle,$$

where $\overline{R}$ stands for the Riemannian tensor of $M \times \mathbb{R}$ and we are using the notation $X = X^h + X^t \in TM \times T\mathbb{R}$. Thus, if M has constant sectional curvature κ,

$$\langle \overline{R}(X,Y)Z,W\rangle = \kappa(\langle X^h,Z^h\rangle\langle Y^h,W^h\rangle - \langle X^h,W^h\rangle\langle Y^h,Z^h\rangle) \tag{B.1}$$

Now, given an isometric immersion $\Sigma^n \to M \times \mathbb{R}$ of an oriented Riemannian manifold, let $n_1 \in \Sigma^\perp$ be an unit normal vector field. Then $D_X n_1$ is a tangent vector and

$$(\overline{R}(X,Y)n_1)^\top = -D_X D_Y n_1 + D_Y D_X n_1 + D_{[X,Y]} n_1, \quad \text{for } X,Y \in TM, \tag{B.2}$$

where D is Levi-Civita connection of $M \times \mathbb{R}$.

In the following we will fix $n = 2$ for simplicity. Let $\partial_u, \partial_v \in T\Sigma$ tangent vectors such that $\langle \partial_u, \partial_u\rangle = \langle \partial_v, \partial_v\rangle =: E$ and $\langle \partial_u, \partial_v\rangle = 0$. If we denote by ∂_t the paralel unit vector field tangent to the factor $\mathbb{R}$ we can write

$$\begin{aligned} E &= \langle \partial_u^h, \partial_u^h\rangle + \varepsilon\langle \partial_u, \partial_t\rangle^2 \\ &= \langle \partial_v^h, \partial_v^h\rangle + \varepsilon\langle \partial_v, \partial_t\rangle^2 \end{aligned}$$

and

$$0 = \langle \partial_u, \partial_v\rangle = \langle \partial_u^h, \partial_v^h\rangle + \varepsilon\langle \partial_u, \partial_t\rangle\langle \partial_v, \partial_t\rangle$$

Also,

$$\langle \partial_v^h, n_1 \rangle = -\varepsilon \langle \partial_v, \partial_t \rangle \langle n_1, \partial_t \rangle,$$
$$\langle \partial_v^h, n_1 \rangle = -\varepsilon \langle \partial_v, \partial_t \rangle \langle n_1, \partial_t \rangle.$$

Then, using the above identities in (B.1) we obtain

$$\begin{aligned}
\langle \overline{R}(\partial_u, \partial_v) n_1, \partial_u \rangle &= \kappa(\langle \partial_u^h, n_1^h \rangle \langle \partial_v^h, \partial_u^h \rangle - \langle \partial_u^h, \partial_u^h \rangle \langle \partial_v^h, n_1^h \rangle) \\
&= \kappa(\langle \partial_u^h, n_1^h \rangle (-\varepsilon \langle \partial_u, \partial_t \rangle \langle \partial_v, \partial_t \rangle) - (E - \varepsilon \langle \partial_u^h, \partial_t \rangle^2) \langle \partial_v^h, n_1^h \rangle) \\
&= \kappa(-\varepsilon \langle \partial_u^h, n_1^h \rangle \langle \partial_u, \partial_t \rangle \langle \partial_v, \partial_t \rangle - E \langle \partial_v^h, n_1^h \rangle + \varepsilon \langle \partial_u^h, \partial_t \rangle^2 \langle \partial_v^h, n_1^h \rangle) \\
&= -\kappa E \langle \partial_v^h, n_1^h \rangle \qquad \text{(B.3)}
\end{aligned}$$

and similarly

$$\langle \overline{R}(\partial_u, \partial_v) n_1, \partial_v \rangle = \kappa E \langle \partial_u^h, n_1^h \rangle. \qquad \text{(B.4)}$$

Finally, from (B.2), we obtain the Codazzi Equations

$$\langle \bar{R}(\partial_u, \partial_v) n_1, \partial_u \rangle = -\langle D_u D_v n_1, \partial_u \rangle + \langle D_v D_u n_1, \partial_u \rangle \qquad \text{(B.5)}$$

and

$$\langle \bar{R}(\partial_u, \partial_v) n_1, \partial_v \rangle = -\langle D_u D_v n_1, \partial_v \rangle + \langle D_v D_u n_1, \partial_v \rangle. \qquad \text{(B.6)}$$

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