Robust transitivity and ergodicity of transformations on the real line and the real plane.

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Introduction

In this work we describe the dynamics of a large class of transitive transformations on the real line and the real plane. We introduce the notion of Alternating Systems to understand problems like robust transitivity and ergodicity with respect to the Lebesgue measure. Then, we look for sufficient conditions to obtain robust expansivity, existence of ergodic absolutely continuous probabilities in the unidimensional case and physical measure in the bidimensional case (In both cases, with respect to induced transformations on bounded parts of the space). Important examples like the Boole transformation ([1],[2],[4],[5],[9]) and the Hénon-Devaney map ([12],[13],[14]) can be treated using this approach.

We begin by discussing the unidimensional theory.

Unidimensional Theory.

1. Unidimensional Alternating Systems.

Let f :Dom$(f) \subset \mathbb{R} \to \mathbb{R}$ be a continuous transformation defined in an open and dense subset of $\mathbb{R}$. We say that f is transitive if there exists $x \in$Dom(f) such that $o^+(x) = \{f^n(x)/n \in \mathbb{N}\}$ (the positive orbit of x) is well defined and it is dense in $\mathbb{R}$. We say that f is C^1 robustly transitive if f is C^1 and there exists $\epsilon > 0$ such that for every g :Dom$(f) \to \mathbb{R}$ ϵ-C^1 close to f (i.e. $|f(x) - g(x)| < \epsilon$ and $|f'(x) - g'(x)| < \epsilon$ for all $x \in$Dom(f)) we have that g is transitive.

To study the problem of robust transitivity for differentiable transformations on the real line, we observe that the natural context is the set of transformations without critical points (a Critical point is a point where the derivative is zero). This is due to the well known fact that a function with a critical point can be C^1 perturbed to create an attracting periodic point ([19]) and this situation destroys the transitivity on the whole line. Therefore we set the following question:

What is the dynamics of transitive differentiable transformations without critical points ?

Observe that if f :Dom$(f)\subset \mathbb{R} \longrightarrow \mathbb{R}$ is a transitive transformation such that $f'(x) \neq 0$ for all $x \in$ Dom(f), then f cannot be defined in the whole line, that is, there exists at least a point $p_0 \in \mathbb{R}$ such that $p_0 \notin$ Dom(f). Indeed, if f is defined in the whole line then f is increasing or decreasing, and so it cannot be transitive. In fact f cannot be extended continuously to $\mathbb{R}$. We will study transformations with only a discontinuity, since we believe that this is the central part to understand all transitive transformations in the whole line. The techniques developed in this work can be used to understand transitive differentiable functions with many discontinuities (in any case this has to be proved). Models of transitive transformations with many discontinuities have been considered in ([4],[6]).

We want to formulate a definition such that: (1) It contains a large class of transitive transformations with a discontinuity (say in $p_0 = 0$). (2) It can be extended to higher dimensions. (3) It does not depend on differentibility. In view of this let us consider an example that contains this information. The Boole transformation

$B : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ defined by the formula $B(x) = x - \frac{1}{x}$

is transitive. In fact, the Lebesgue measure is an ergodic σ-finite B-invariant measure ([2],[9]).

Let $A = \{a_1, \cdots, a_n\}$ be a colection of n different numbers and denote by $\mathbb{P}_A$ the partition induced by A on $\mathbb{R}$. We say that the colection $\{b_1, \cdots, b_{n+1}\}$ of $n+1$ diferent numbers is *between* A if for all $E \in \mathbb{P}_A$ there exists a unique $i \in \{1, \cdots, n, n+1\}$ such that $b_i \in E$. Also, for every $z \in \mathbb{R} - \{0\}$ define sign$(z) = 1$ if $z > 0$ and sign$(z) = -1$ if $z < 0$.

B has essentially two features:

(AS1) For each $n \geq 1$, $f^{-n}(0)$ consists of 2^n numbers between the $2^n - 1$ previous $f^{-n+1}(0) \cup \cdots \cup f^{-1}(0) \cup \{0\}$.

(AS2) Every $x \in \mathbb{R}$ such that $f^n(x) \neq 0$ for all $n \geq 1$ has the following property: its orbit $o^+(x)$ changes sign infinitely many times, that is, for all $N \geq 0$ there exists $M > N$ such that $\text{sign}(f^M x) = -\text{sign}(f^N x)$ (see Figure 1).

We will call any continuous function $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ with properties (AS1) and (AS2) an *alternating system.*

Theorem A: Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a differentiable function such that:
(1) f is transitive.
(2) there exists $a > 0$ such that $f'(x) \geq a$ for all $x \neq 0$ (i.e. f is increasing in each connected component of its domain)
Then f is an alternating system.

2. Transitive Alternating Systems.

We will construct alternating systems such that they are not transitive, so the next step is to identify transitivity. Any increasing alternating system f induces a structure, it will be called the Markov structure induced by f, say

$$(f, \mathbb{P}_f, I, f_I : I \to I)$$

where:

(1) $\mathbb{P}_f$ is a partition on $\mathbb{R}$, it will be called the partition induced by f on $\mathbb{R}$.

(2) I is a bounded subset of $\mathbb{R}$ such that the transformation $f_I : I \to I$ defined by $f_I(x) = f^{n(x)}x$, where $n(x) = \min\{n \geq 1 : f^n x \in I\}$, is well defined, for every x such that $f^n(x) \neq 0$ for all $n \geq 0$. The subset I will be called the distributor interval of f and f_I will be called the transformation induced by f on I.

We will see that the following definition characterizes the transitivity in the set of increasing alternating systems (increasing means that f is increasing in each connected component of its domain).

Definition: Let f be an alternating system. We say that f is expansive if for all $x \neq y$ (and such that $f^n(x), f^n(y)$ are defined for all $n \geq 1$) there exist $N \geq 1$ and $A \neq B \in \mathbb{P}_f$ such that $f^N x \in A$ and $f^N y \in B$.

Theorem B: Let f be an increasing alternating system. Then,

f is expansive if and only if f is transitive.

Examples.

Definition: Let f be an alternating system. We say that f is affine if f is a segment of straight line in each atom of the partition $\mathbb{P}_f$ induced by f on $\mathbb{R}$.

Corollary A: If f is an affine symmetric (i.e. $f(x) = -f(-x)$ for all $x \neq 0$) alternating system then f is transitive.

Let us consider the following examples:

Define $f_a : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ as follows: take $x_0, a \in \mathbb{R}$ such that x_0 is greater than 1 and $1 - \epsilon < a < 1$ (Think ϵ as a very small positive number). Set $u_1^- = -\frac{1}{x_0}$ and $x_1^+ = \frac{x_0}{a}$.

$$f_a(x) = \begin{cases} \text{segment joining } (-x_0, 0) \text{ with } (u_1^-, x_0) \;, & if \;\; u_1^- \geq x \geq -x_0 \\ -\frac{1}{x} & if \;\; 0 > x \geq u_1^- \\ \text{segment joining } (x_0, 0) \text{ with } (x_1^+, x_0) \;, & if \;\; x_1^+ \geq x \geq x_0 \\ ax & if \;\; x \geq x_1^+ \end{cases}$$

Define f_a in $(-\infty, u_0] \cup (0, x_0]$ so that f_a is symmetric (i.e. $f_a(x) = -f_a(-x)$).

Corollary B: f_a is a transitive alternating systems.

3. Robustly Transitive Alternating Systems.

Remark: It is easy to see that the Boole transformation is an expansive alternating system such that it is not C^1 robustly transitive (with respect to the topology defined before).

In view of this, we will define hyperbolicity of an alternating system using its induced transformation, that is: an alternating system f is expanding (with respect to the transformation f_I induced by f on the distributor interval I) if f_I is expanding. This is placed as follows:

Definition: Let f be a C^1 alternating system. We say that f is expanding (with respect to f_I) if f is expansive and there exist constants $\sigma > 1$ and $C > 0$ such that:

$$|f_I'(x)| \geq C\sigma^{n(x)}, \text{ for every } x \in I \text{ such that } f^n(x) \neq 0 \text{ for all } n \geq 0.$$

where $n(x) = \min\{n \geq 1 : f^n x \in I\}$ (i.e. $n(x)$ is the return time of x to the distributor interval I).

Remark: In the context of alternating sytems, the definition of hyperbolicity (with respect to the induced transformation) is not enough to guarantee robust transitivity. This fact places an important difference between dynamics on an interval and dynamics on the real line. Essentially, if f is an alternating system without oblique asymptots, the distortion, due to the behaviour of f at ∞, obstructs the possibility of a differentiable compactification of f to a bounded interval, therefore the compactification techniques can not be used in these cases.

We will look for sufficient conditions to obtain C^1 robust transitivity for alternating systems. We will specify this conditions in Section 3 of Chapter 1, in this introduction we just call them distortion conditions.

Theorem C: Let f be an increasing alternating system. Suppose that:
(1) f is expanding.
(2) Distortion conditions (see Theorem 1.3.1)
Then f is C^1 robustly transitive.

Example:
Corollary C: Define $f_a : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ by $f_a(x) = ax - \frac{1}{x}$ $(a < 1)$. If a is sufficiently close to 1 then f_a is robustly transitive.

4. Ergodic Alternating systems.

Let $(\mathbb{R}, \mathbb{B}, \lambda)$ be the set of real numbers equipped with the Borel σ-algebra and the Lebesgue measure. Denote by $f : \mathbb{R}-\{0\} \longrightarrow \mathbb{R}$ a measurable transformation. Some basic definitions are the following:

(a) f is a non-singular transformation if it preserves subsets of Lebesgue measure zero, that is, $\lambda(E) = 0 \Longleftrightarrow \lambda(f^{-1}E) = 0,\ E \in \mathbb{B}$.

(b) f is ergodic (with respect to the Lebesgue measure) if f is a non-singular transformation such that $f^{-1}E = E \Longrightarrow \lambda(E) = 0$ or $\lambda(E^c) = 0,\ E \in \mathbb{B}$.

Usual objetives in infinite ergodic theory are:

(1) To look for σ-finite invariant measures with respect to non-singular transformations, if they exist ([7],[9],[16],[17]).

(2) To prove ergodicity with respect to non-singular transformations ([2],[4],[6],[9]).

We are interested in the problem (2) for non-singular alternating systems. Essentially, there are two ways to resolve this problem: the first one is to consider a differentiable compactification, when it is possible. For example in [8] it is shown that the Boole transformation $f(x) = x - \frac{1}{x}$ is ergodic, for that it can be used the variable $u = \arctan x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and then study $\tilde{f}(u) = \arctan\big(\tan u - \frac{1}{\tan u}\big)$. It can be shown that $\tilde{f}$ (and so f) is ergodic with respect to the Lebesgue measure. The second method is to consider the transformation induced by the respective map on the distributor interval. For example in [2] it is shown that the Boole transformation is ergodic and this is done showing that the transformation induced by f on $[-1, 1]$ (and then f) is ergodic.

In the case of alternating systems the problem of ergodicity can be treated using clasical argumets like bounded distortion and the Folklore theorem for Markov expanding maps ([3],[8],[11]). We will look for sufficient conditions to guarantee the ergodicity of hyperbolic alternating systems. These conditions are placed in Section 4 of Chapter 1, in this introduction we just call them distortion conditions.

<u>*Theorem D*</u>: Let f be an increasing alternating system. Suppose that:
(1) f is expanding (with respect to the transformation f_I induced by f on the distributor interval I).
(2) Distortion conditions (see Theorem 1.4.1).

Then f is ergodic with respect to the Lebesgue measure.

Examples.
Corollary D: If f is an affine symmetric alternating system then f is ergodic with respect to the Lebesgue measure.

Remark: (About the generality of alternating system)
Let $f : \mathbb{R} - \{p_1, \cdots, p_N\} \longrightarrow \mathbb{R}$ be a differentiable transitive transformation such that $f'(x) > 0$ for all x (N can be ∞ and points $p_1, \cdots, p_N$ are called discontinuities). It can be proved that if p is a fixed point of f then f has a discontinuity $> p$. So if f has a single discontinuity then f does not have fixed points. In general, if f has n discontinuities then f has $n-1$ fixed points. All of them, with the adequate definition, are alternating systems. For example, let us suppose that:

(1) $f : \mathbb{R} - \{p_1, p_2\} \longrightarrow \mathbb{R}$ is a differentiable transitive function without critical points.
(2) $\{x = p_1\}, \{x = p_2\}$ are vertical asymptots.

Then f has two basic properties:

(AS1) The inverse images of $\{p_1, p_2\}$ are, successively, between the previous ones.

(AS2) Lebesgue a.e $x \in \mathbb{R}$ changes sign infinitely many times.

An example of a transitive ergodic transformation with n discontinuities is the generalized Boole's transformation $x \longrightarrow x - (\frac{a_1}{x-b_1} + ... + \frac{a_n}{x-b_n})$, where $a_i > 0$ for all $i \in \{1, 2, ..., n\}$ and $b_1 < b_2 < .... < b_n$ ([4]). An example of a transitive ergodic transformation with infinite discontinuities is $x \longrightarrow \tan x$ ([6]).

Now we will describe our results in the bidimensional case.

Bidimensional Theory.

1. Bidimensional Alternating Systems.

In Chapter 2, we generalize the ideas described in the unidimensional context to study transitive diffeomorphisms on the plane with a curve of discontinuities. Why a curve of discontinuities ? To answer this question we proceed as follows: Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ $[t \to \alpha(t) = (\alpha_1(t), \alpha_2(t))]$ be a smooth curve. We will say that α is horizontal if $\frac{d\alpha_1}{dt}(t_0) \neq 0$ for all $t_0 \in \mathbb{R}$ and α is vertical if $\frac{d\alpha_2}{dt}(t_0) \neq 0$ for all $t_0 \in \mathbb{R}$. If $\alpha, \beta : \mathbb{R} \to \mathbb{R}^2$ are horizontal smooth curves, we say that α *is over* β ($\alpha > \beta$) if for any vertical straight line l we have that any intersection of l with $\alpha(\mathbb{R})$ is greater than any intersection of l with $\beta(\mathbb{R})$ (Make a similar definition to order vertical smooth curves) Suppose that $F : \overline{Dom(F)} = \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a diffeomorphism with $\mathcal{F}, \mathcal{G}$ transversal F-invariant foliations such that leaves of $\mathcal{F}$ are horizontal smooth curves and leaves of $\mathcal{G}$ are vertical smooth curves . Suppose that F is increasing, that is, $\alpha > \beta \Rightarrow F(\alpha) > F(\beta)$. Fix a leaf $g \in \mathcal{G}$ and suppose that each leaf of $\mathcal{F}$ meets g in a single point. If Dom(F)=$\mathbb{R}^2$ then the projection on g along $\mathcal{F}$, say $P_g : g \hookleftarrow$, is a monotone unidimensional transformation. Therefore, using the unidimensional theory (see Lemma 1.1.1 and Theorem 1.1.1), if F is transitive then P_g has, at least, an explosive discontinuity p_0 (explosive means that $\{x = p_0\}$ is a vertical asymptote for P_g). Hence the leaf of $\mathcal{F}$ containing p_0 is an explosive discontinuity curve of F. Consequently, a large class of bidimensional transitive diffeomorphisms have an explosive discontinuity curve (think of diffeomorphisms such that a full Lebesgue measure subset of the plane is hyperbolic, partially hyperbolic or it has a dominated splitting)

Remark: Following [18], an Anosov splitting for a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ consists of a complete Riemannian metric μ on $T\mathbb{R}^2$, a continuous invariant splitting $T\mathbb{R}^2 = E^s \oplus E^u$, and a constant $\lambda \in (0, 1)$ such that:

$\|Df^n(v)\| < \lambda^n \|v\|$, for $v \in E^s$

$\|Df^n(v)\| < \lambda^{-n} \|v\|$, for $v \in E^u$

where $\|.\|$ is the norm induced by μ. An Anosov diffeomorphism is a diffeomorphism admitting an Anosov splitting such that E^s and E^u are integrable line fields on the plane. In [18] it is proved the following: If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is Anosov then the nonwandering set of f is empty or it is just a point. Therefore, we can not hope transitivity (i.e, a dense full orbit in $\mathbb{R}^2$) for diffeomorphism defined on the whole plane (At least in the hyperbolic context) but we will see that the presence of an explosive discontinuity curve will be a source of complexity for a large class of (hyperbolic) diffeomorphisms on the plane.

To illustrate the definition of bidimensional alternating systems let us consider a clasical example. The Hénon map is defined as follows:

$$F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{y = -x\},\ F(x, y) = (x + \tfrac{1}{y}, y - \tfrac{1}{y} - x).$$

F is a transitive area preserving diffeomorphism ([13]). It was introduced by Hénon as an asymptotic form of the equations of motion of the restricted three body problem ([12]). As in the unidimensional case, the fundamental features of F are:

Let $A = \{\alpha_1, \cdots, \alpha_n\}$ be a collection of n smooth horizontal (or vertical) curves ($\alpha_i : \mathbb{R} \to \mathbb{R}^2$) without intersections, if $B = \{\beta_1, \cdots, \beta_{n+1}\}$ is a collection of $n+1$ smooth curves, we say that B is *between* A if for every element E in the partition induced by A on $\mathbb{R}^2$ there exists a unique $i \in \{1, \cdots, n+1\}$ such that $\beta_i \in E$. Also, we can define the sign of a point with respect to a curve as follows: Let γ be a horizontal curve. For every $p \in \mathbb{R}^2 - \{\gamma\}$ define $\mathrm{sign}_\gamma(p) = 1$ if $p \in R^+$ and $\mathrm{sign}_\gamma(p) = -1$ if $p \in R^-$, where $R^\pm$ is the superior (respectively, inferior) connected component of $\mathbb{R}^2 - \{\gamma\}$. Similarly, we define the sign of a point with respect to a vertical curve δ. In the case of F (the Hénon map) $\gamma = \{y = 0\}$ is a horizontal curve of discontinuities and $\delta = \{y = -x\}$ is a vertical curve of discontinuities for F^{-1}.

(AS1) For each $n \geq 1$, $F^{-n}(\{y = 0\})$ consists of 2^n horizontal curves between the $2^n - 1$ previous $F^{-n+1}(\{y = 0\}) \cup \cdots \cup F^{-1}(\{y = 0\}) \cup \{y = 0\}$. Similarly, $F^n(\{y = -x\})$ consists of 2^n vertical curves between the $2^n - 1$ previous $F^{n-1}(\{y = -x\}) \cup \cdots \cup F(\{y = -x\}) \cup \{y = -x\}$.

(AS2) For every $p \in \mathbb{R}^2$ such that $F^n p$ exists for all $n \in \mathbb{Z}$ we have: for every $M \geq 1$ there exists $N \geq 1$ such that $\mathrm{sign}_\gamma(F^M p) = -\mathrm{sign}_\gamma(F^N p)$ (i.e $F^n p$ is over and below the curve γ indefinitely) and for every $M \geq 1$ there exists $N \geq 1$ such that $\mathrm{sign}_\delta(F^{-M} p) = -\mathrm{sign}_\delta(F^{-N} p)$ (i.e $F^{-n} p$ is on the right and on the left of the the curve δ indefinitely).

If $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a horizontal curve and $\delta : \mathbb{R} \to \mathbb{R}^2$ is a vertical curve, we will call any homeomorphism $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ with properties (AS1) and (AS2) a bidimensional alternating system.

Theorem E: Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be a diffeomorphism. Suppose that:

(1) F is transitive.
(2) $\mathbb{R}^2 - \{\gamma\}$ admits transversal F-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that:
(i) Every leaf of $\mathcal{F}^s$ intersects every leaf of $\mathcal{F}^u$ in a single point.
(ii) $\mathcal{F}^s$ is horizontal and $\mathcal{F}^u$ is vertical.
(iii) $\mathcal{F}^s$ and $\mathcal{F}^u$ are continuous (i.e. the holonomy maps are continuous).
(3) F and F^{-1} are increasing in each connected component of their domains.
Then F is an alternating system.

2. Transitive Alternating Systems.

There are alternating systems which are not transitive, so the next step is to identify transitivity. Like unidimensional alternating systems, any increasing (bidimensional) alternating system F induces a structure, it will be called the Markov structure induced by F, say

$$(F, \mathbb{P}_F, R, F_R : R \to R)$$

where

(1) $\mathbb{P}_F$ is a partition on $\mathbb{R}^2$, it will be called the partition induced by F on $\mathbb{R}^2$.

(2) R is a bounded subset of $\mathbb{R}^2$ such that the transformation $F_R : R \to R$ defined by $F_R(p) = F^{n(p)}p$, where $n(p) =\min\{n \geq 1 : F^n p \in R\}$, is well defined, for every p such that $F^n(p) \notin \{\gamma\} \cup \{\delta\}$ for all $n \in \mathbb{Z}$. The subset R will be called the distributor subset of F and F_R will be called the transformation induced by F on R.

We will see that the following definition characterizes transitivity in the set of increasing bidimensional alternating systems.
<u>Definition</u>: Let F be an alternating system. We say that F is expansive if for all $p \neq q$ (and such that $F^n p, F^n q$ are defined for all $n \in \mathbb{Z}$) there exist $N \geq 1$ and $A \neq B \in \mathbb{P}_F$ such that $F^N p \in A$ and $F^N q \in B$.

<u>*Theorem F*</u>: Let F be an increasing alternating system. Then,

F is expansive if and only if F is transitive.

<u>Examples.</u>

Definition: Let F be an alternating system. We say that F is affine if F is affine in every atom of the partition $\mathbb{P}_F$ induced by F on $\mathbb{R}^2$.

Corollary E: If F is an affine symmetric (i.e, $F(p) = -F(-p)\ \forall p$) alternating system then F is transitive.

Define $F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{y = -ax\}$ $(1 - \epsilon < a < 1)$ by

$F_a(x, y) = (x + \frac{1}{y}, ay - \frac{a}{y} - ax)$.

Theorem G: F_a is a transitive alternating system.

3. Robustly Transitive Alternating Systems.

Definition: Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be a C^1 alternating system. We say that F is C^1 robustly transitive if there exists $\epsilon > 0$ such that $\forall$ $G : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ ϵ-C^1 close to F, i.e

$\|G(p) - F(p)\| < \epsilon$ and $\|D_pG - D_pF\| < \epsilon\ \forall p \in \mathbb{R}^2 - \{\gamma\}$

we have that G is transitive.

Remark: The Henon map is an expansive bidimensional alternating system such that it is not C^1 robustly transitive (with respect to the topology defined before).

Definition: Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be a C^1 alternating system. We say that F is Anosov if F is an expansive increasing alternating system such that the transformation $F_R : R \hookleftarrow$, induced by F on the distributor subset R, admits a continuous DF_R-splitting $E^s \oplus E^u$ such that for some constants $C > 0$ and $\sigma > 1$ we have:

$\|D_pF_R(v)\| \geq C\sigma^{n(p)}\|v\|$, for all $v \in E^u$ and
$\|D_p(F_R)^{-1}(v)\| \geq C\sigma^{n(p)}\|v\|$, for all $v \in E^s$

for all $p \in R$ such that $F^n(p) \neq 0$ for all $n \in \mathbb{Z}$, where

$n(p) = \min\{n \geq 1 : F^n p \in R\}$ (i.e. $n(p)$ is the return time of p to R).

We think that this is the right definition of Anosov diffeomorphisms for diffeomorphisms on the plane with an explosive discontinuity curve. At this point, it is clear that to obtain robust transitivity for an Anosov alternating system F: (1) we have to preserve the structure of alternating system (i.e any C^1 perturbation sufficiently close to F is again an alternating system). (2) we have to preserve the expansivity due to the hyperbolicity of the transformation induced by F on the distributor subset R. In this section we just will set some questions (see Section 3 of Chapter 2).

4. Ergodic Alternating systems.

Let $F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{y = -ax\}$ $(1 \geq a \geq 0)$ be an increasing alternating system. Suppose that $\mathbb{R}^2 - \{y = 0\}$ admits a F-invariant stable folation, say $\mathcal{F}^s$, such that every leaf of $\mathcal{F}^s$ intersect $\{y = -ax\}$ in a single point. Let $f = P_\delta : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be the projection on the discontinuity line $\delta = \{y = -ax\}$ along $\mathcal{F}^s$ (with respect to F).

Remark: f is an unidimensional increasing alternating system and its differentiability depends on the differentiability of the foliation $\mathcal{F}^s$. Denote by f_I the transformation induced by f on the distributor interval I.

Remark: In section 4 of Chapter 2 we will see that if F is an alternating system in the previous conditions such that $\mathcal{F}^s$ is a C^1 foliation (i.e. the holonomy maps are C^1) and f_I admits an ergodic f_I-invariant probability μ_I absolutely continuous (with respect to Lebesgue measure) then the transformation $F_R : R \hookleftarrow$, induced by F on the distributor subset R, admits a F_R-invariant probability μ on the Borel σ-algebra of R such that the basin of μ has full Lebesgue measure .

Example.

Corollary F: If F is an affine symmetric alternating system then the transformation $F_R : R \hookleftarrow$, induced by F on the distributor subset R, admits a physical measure μ such that the basin of μ has full Lebesgue measure.

Finally, in Chapter 3 we will present some questions and further works.

Chapter 1

Unidimensional Alternating Systems.

In this Chapter we begin the study of transitive transformations in the real line. We introduce the definition of alternating systems to understand problems like robust transitivity and ergodicity with respect to the Lebesgue measure. Two good reasons to study unidimensional alternating systems are: the first one is that a large class of transitive differentiable functions are alternating systems, the second and not less important is that projections along invariant foliations relative to an important class of bidimensional dynamics (see Chapter 2) are alternating systems.

1.1 Alternating Systems.

We begin with some basic definitions.

If $f : \mathbb{R} - \{p_0\} \longrightarrow \mathbb{R}$ is a continuous function then we say that f is transitive if there exists $x \in \mathbb{R}$ such that $o^+(x)$, the positive orbit of x, is dense in $\mathbb{R}$ (Remember that $o^+(x) = \{f^n(x) : n \in \mathbb{N}\}$).

If $B = \{b_1, b_2, \cdots, b_n\}$ is a collection of n different numbers then we say that B is *between* the collection of $n-1$ different numbers $A = \{a_1, a_2, \cdots, a_{n-1}\}$ if for all E in the partition induced by A on $\mathbb{R}$ there exists a unique $i \in \{1, \cdots, n\}$ such that $b_i \in E$. Also, for every $z \in \mathbb{R} - \{0\}$ define $\text{sign}(z) = 1$

if $z > 0$ and $\text{sign}(z) = -1$ if $z < 0$.

We begin the study of monotone endomorphisms with a discontinuity, let us say in $p_0 = 0$.

Definition 1.1.1 *Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a continuous transformation, we say that f is an alternating system if:*

(AS1) For every $n \geq 1$, $f^{-n}(0)$ consists of 2^n different numbers between the previous $2^n - 1$ different numbers $f^{-n+1}(0) \cup \cdots f^{-1}(0) \cup \{0\}$.

(AS2) Every $x \in \mathbb{R}$ such that $f^n(x) \neq 0$ $\forall n \geq 1$ has the following property: $o^+(x_0)$ changes sign infinitely many times, that is, $\forall N \geq 0$ there exists $M > N$ such that $sign(f^M x) = -sign(f^N x)$.

Lemma 1.1.1 *Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a transformation such that:*

(1) f is continuous.
(2) f is increasing in each connected component of $\mathbb{R} - \{0\}$, i.e, if $x < y$ then $f(x) < f(y)$
(3) $f(0, +\infty) = f(-\infty, 0) = \mathbb{R}$.
(4) The point $\{p_0 = 0\}$ is an explosive discontinuity of f, i.e, the line $\{x = 0\}$ is a vertical asymptote of f such that $f(x) \longrightarrow +\infty$, when $x \longrightarrow 0^-$ and $f(x) \longrightarrow -\infty$, when $x \longrightarrow 0^+$.
(5) $f(x) \longrightarrow -\infty$, when $x \longrightarrow -\infty$ and $f(x) \longrightarrow +\infty$, when $x \longrightarrow +\infty$.
(6) f is over the diagonal in $(-\infty, 0)$ and below the diagonal in $(0, +\infty)$, i.e, $f(x) < x$ $\forall x > 0$ and $f(x) > x$ $\forall x < 0$.

If f satisfies (1),...,(6) then f is an alternating system.

Proof: Consider $f_- : (-\infty, 0) \longrightarrow \mathbb{R}$ and $f_+ : (0, +\infty) \longrightarrow \mathbb{R}$ the left branch (respectively the right branch) of f. Due to our hypothesis f_- and f_+ are continuous increasing bijective functions without fixed points. Take two numbers x_0^-, x_0^+ such that $x_0^- < 0 < x_0^+$ and $f(x_0^-) = f(x_0^+) = 0$ (observe that f has only two roots). Observe that from the monotony of f_+ and f_- property (AS1) of Definition 1.1.1 follows. Define the following sequences:

$\{x_1^+ = f_+^{-1}(x_0^+),\ x_n^+ = f_+^{-1}(x_{n-1}^+),\ n \geq 2\}$
$\{x_1^- = f_-^{-1}(x_0^-),\ x_n^- = f_-^{-1}(x_{n-1}^-),\ n \geq 2\}$
$\{u_1^+ = f_+^{-1}(x_0^-),\ u_n^+ = f_+^{-1}(x_{n-1}^-),\ n \geq 2\}$
$\{u_1^- = f_-^{-1}(x_0^+),\ u_n^- = f_-^{-1}(x_{n-1}^+),\ n \geq 2\}$

Observe that (AS2) of Definition 1.1.1 is a consequence of the following claim:

Claim. $x_n^+ \longrightarrow +\infty,\ \ x_n^- \longrightarrow -\infty,\ \ u_n^\pm \longrightarrow 0,\ $ when $n \longrightarrow +\infty$.

Proof of Claim: From the properties of f_+ and f_-, $\{x_n^+\}$, $\{u_n^-\}$ are increasing sequences and $\{x_n^-\}$, $\{u_n^+\}$ are decreasing sequences. If there exist a such that $0 < a < \infty$ and $x_n^+ \longrightarrow a$ then $f_+(x_n^+) = x_{n-1}^+ \longrightarrow f_+(a)$ (because f_+ is continuous) hence $f_+(a) = a$. But f_+ does not have fixed points, this is a contradition. So $x_n^+ \longrightarrow +\infty$. Similarly we prove $x_n^- \longrightarrow -\infty$. On the other hand, observe that $f_-(u_n^-) = x_{n-1}^+ \longrightarrow +\infty$. If $u_n^- \longrightarrow a < 0$ then $f_-(u_n^-) = x_{n-1}^+ \longrightarrow f_-(a) \neq +\infty$ and from this a contradiction. Similarly we prove that $u_n^+ \searrow 0$. Therefore, property (AS2) of Definition 1.1.1 follows. Hence f is an alternating system.

Lemma 1.1.1 says that any function with a graph satisfying the hypothesis of Lemma 1.1.1 is an alternating system. An important example is the well known Boole's transformation, $x \longrightarrow B(x) = x - \frac{1}{x}$. It is known B is an alternating system preserving Lebesgue measure and it is a transitive ergodic transformation ([2],[5],[8],[9]).

Theorem 1.1.1 *Let $f : \mathbb{R} - \{0\}$ be a differentiable function such that:*

(1) f is transitive.
(2) There exists $a > 0$ such that $f'(x) \geq a\ \forall x \neq 0$.

Then f is an alternating system.

Proof: f satisfies the hypothesis of Lemma 1.1.1. Indeed, hypothesis (1) and (2) of Lemma 1.1.1 are clear. Hypothesis (5) of Lemma 1.1.1 follows from (2). To prove hypothesis (6) of Lemma 1.1.1 observe that f can not have a fixed point. Indeed , if there exists $p \in \mathbb{R}$ such that $f(p) = p$ then, due to (2) we have that $f[p, +\infty) = [p, +\infty)$ (if $p > 0$) or $f(-\infty, p] = (-\infty, p]$ (if

$p < 0$). Therefore, f is not transitive and this is a contradiction. So, keeping in mind that f does not have fixed points, if f does not satisfy hypothesis (6) of Lemma 1.1.1 we have two possibilities:
(i) $f(x) > x \; \forall x > 0$, or
(ii) $f(x) < x \; \forall x < 0$.
In any case, due to the fact f is increasing (in each connected component of its domain), we will have $f(0, +\infty) \subset (0, +\infty)$ or $f(-\infty, 0) \subset (-\infty, 0)$ and so f is not transitive. This is a contradiction.
We will prove hypothesis (4) of Lemma 1.1.1. Indeed, if the line $\{x = 0\}$ is not a vertical asymptot of f then the possibilities are:
(i) There exists $c_1 \in \mathbb{R}$ such that $f(x) \to c$, when $x \to 0^-$, or
(ii) There exists $c_2 \in \mathbb{R}$ such that $f(x) \to c$, when $x \to 0^+$.
In any case, the graph of f does not correspond to a transitive one. To finish observe that hypothesis (3) of Lemma 1.1.1 is clear from (2). This completes the proof.

1.2 Transitive Alternating Systems.

There are alternating systems which are not transitive. Indeed, just consider a differentiable function $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ satisfying the hypotesis of Lemma 1.1.1 such that x_0 is a periodic point of period 2 and $f'(x_0) < 1$, $f'(f(x_0)) < 1$. In this case x_0 is an attracting periodic point with respect to f (There exists a subset A such that $\mathrm{int}(A) \neq \emptyset$ and $f(A) \subset A$) It is clear that this kind of functions are not transitive. In view of this we do the following definitions.

Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be an alternating system satisfying the hypothesis of Lemma 1.1.1, we will always say that f is an *increasing alternating system*. Any increasing alternating system f induces a structure, it will be called the Markov structure induced by f, say

$$(f, \{x_n^{\pm}\}_{n\geq 0}, \{u_n^{\pm}\}_{n\geq 0}, \mathbb{P}_f, I, f_I : I \to I)$$

where:

(1) $\{x_n^-\}_{n\geq 0}$, $\{x_n^+\}_{n\geq 0}$, $\{u_n^-\}_{n\geq 0}$, $\{u_n^+\}_{n\geq 0}$ are the sequences constructed in Lemma 1.1.1 (they are, for each $n \geq 1$, the exterior and interior numbers of $f^{-n}(0)$). This sequences will be called the sequences induced by f. Observe

that they have the following dynamics:

$$
\begin{array}{l}
u_{n+1}^{-} \longrightarrow x_n^{+} \longrightarrow x_{n-1}^{+} \longrightarrow \cdots \longrightarrow x_1^{+} \longrightarrow x_0^{+} \longrightarrow 0 \\
u_{n+1}^{+} \longrightarrow x_n^{-} \longrightarrow x_{n-1}^{-} \longrightarrow \cdots \longrightarrow x_1^{-} \longrightarrow x_0^{-} \longrightarrow 0.
\end{array}
$$

(2) $\mathbb{P}_f$ is the partition induced by sequences $\{x_n^{\pm}\}_{n\geq 0}$, $\{u_n^{\pm}\}_{n\geq 0}$ and it will be called the partition induced by f on $\mathbb{R}$. Consider the atoms of the partition $\mathbb{P}_f$, i.e

$$
\begin{array}{l}
\{B_{n+1}^{-} = (x_{n+1}^{-}, x_n^{-})\}_{n\geq 0}, \ \ \{B_{n+1}^{+} = (x_n^{+}, x_{n+1}^{+})\}_{n\geq 0} \\
\{I_{n+1}^{-} = (u_n^{-}, u_{n+1}^{-})\}_{n\geq 0}, \ \ \{I_{n+1}^{+} = (u_{n+1}^{+}, u_n^{+})\}_{n\geq 0}
\end{array}
$$

observe that they have the following dynamics:

$$
\begin{array}{ccccccccccc}
I_{n+1}^{-} & \longrightarrow & B_n^{+} & \longrightarrow & B_{n-1}^{+} & \longrightarrow & \cdots\cdots\cdots\cdots\longrightarrow & B_1^{+} & \longrightarrow & (0, x_0^{+}) \\
I_{n+1}^{+} & \longrightarrow & B_n^{-} & \longrightarrow & B_{n-1}^{-} & \longrightarrow & \cdots\cdots\cdots\cdots\longrightarrow & B_1^{-} & \longrightarrow & (u_0^{-}, 0).
\end{array}
$$

(3) $I = [x_0^{-}, x_0^{+}]$. This subset will be call the distributor interval of f. Observe that each real number not equal to 0 visits I after some positive iterate, that is

$$
\mathbb{R} - \{0\} = \bigcup_{n\geq 1} f^{-n}(I).
$$

(4) $f_I : I \to I$ is the transformation defined by $f_I(x) = f^{n(x)}(x)$ for every $x \in I - \{f^{-n}(0) : n \geq 0\}$, where $n(x) =\min\{n \geq 1 : f^n x \in I\}$. Observe that f_I is well defined (except in a subset of zero Lebesgue measure), it will be called the transformation induced by f on the distributor interval I.

Also observe that if $x \in I_n^{-} \cup I_n^{+}$, $n \geq 1$, then $f_I(x) = f^n(x)$. This fact will be used several times in this work.

Definition 1.2.1 *Let f be an alternating system. We say that f is expansive if given $x, y \in \mathbb{R} - \{0\}$ such that $f^n(x), f^n(y)$ are defined $\forall n \geq 0$ there exist $A \neq B$ atoms of the partition $\mathbb{P}_f$ induced by f on $\mathbb{R}$ and $N \geq 1$ such that $f^N(x) \in A$ and $f^N(y) \in B$.*

Theorem 1.2.1 *Let f be an increasing alternating system. Then,*

f is expansive $\Longleftrightarrow$ f is transitive.

Proof: ($\Rightarrow$) We will construct a conjugacy between f and the unilateral shifts of two symbols. Let Σ be the space of unilateral sequences of two symbols $\{1, -1\}$ with its usual topology, that is

$$d(a,b) = \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^n}, \text{ where } a = \{a_n\}_{n\geq 0},\ b = \{b_n\}_{n\geq 0} \in \Sigma.$$

Σ is compact and the shift $\sigma : \Sigma \to \Sigma$, $\sigma(a_0 a_1 \cdots) = (a_1 a_2 \cdots)$ is continuous and transitive, therefore if a conjugacy between f and σ is posible then f is transitive. We will construct a homeomorphism $h : A \longrightarrow \Sigma'$ such that $h \circ F = \sigma \circ h$, where $A = \mathbb{R}$ (Mod 0) and $\Sigma' = \Sigma$ (Mod 0). Take $A = \mathbb{R} - \{x/f^n(x) = 0 \text{ for some } n \geq 1\}$ and Σ' the subset of sequences in Σ except those which terminate with an infinite string of $1'$s or $-1'$s. Define $h : A \longrightarrow \Sigma'$ by $h(x) = (a_0 a_1 \cdots)$, where

$$a_n = \begin{cases} 1 & if\ f^n(x) > 0 \\ -1 & if\ f^n(x) < 0 \end{cases}$$

From (AS2) of Definition 1.1.1 if $x \in A$ there exists $a \in \Sigma'$ such that $h(p_0) = a$. The hypotheses imply that if $x \neq y$ then there exists a positive iterate n such that $f^n(x) < 0 < f^n(y)$, in other words $x \neq y \Longrightarrow h(x) \neq h(y)$. Therefore h is well defined and it is one to one.
h is onto Σ'. For that take $a = (a_0 a_1 \cdots) \in \Sigma'$ and denote $Q_+ = \{x \in \mathbb{R}/x > 0\}$, $Q_- = \{x \in \mathbb{R}/x < 0\}$. Define:

$$\begin{aligned} A_1 &= \overline{F^{-1}(Q_{a_1}) \cap Q_{a_0}} \\ A_n &= \overline{F^{-n}(Q_{a_n}) \cap F^{-n+1}(Q_{a_{n-1}}) \cap \cdots \cap Q_{a_0}}\ ,\ \ n \geq 1 \end{aligned}$$

If $a \in \Sigma'$ then there exists $N \geq 1$ such that $a_1, a_2, \cdots, a_{N-1} = a_0$ but $a_N = -a_0$. Therefore, from (AS1) of Definition 1.1.1, $\{A_n\}_{n\geq 1}$ is a nested sequence of compact intervals. Take $x \in \bigcap_{n\geq 1}(A_n)$, we have $h(x) = a$.

The continuity of h is a consequence of the continuity of f.

($\Leftarrow$) Suppose that f is not expansive. In particular, the subset $\{f^{-n}(0)\}_{n\geq 1}$ of pre-images of zero is not dense in $\mathbb{R}$. Hence, there exists an open interval $U = (a,b)$ such that U does not contain pre-images of zero and due to the fact that f is increasing (in each connected component of its domain) we have that $\forall n \geq 0$, $f^n(U) = (f^n a, f^n b)$. Then,

(1) $\forall n \geq 0$, $f^n(U)$ does not contain pre-images of zero.

(2) $\forall n \geq 0$, $f^n(U)$ is in the same atom of the partition $\mathbb{P}_f$ induced by f on $\mathbb{R}$.

Using that f is transitive there exists K_0 such that $U \cap f^{K_0}U \neq \emptyset$. Therefore, the subset

$$J = \bigcup_{n \geq 0} f^{nK_0}U$$

is an interval such that $f^{K_0}J \subset J$. From (1) and (2) there exists $A \in \mathbb{P}_f$ (the partition induced by f on $\mathbb{R}$) such that $J \in A$. In particular, J is a bounded interval. Then, there exists a bounded subset J such that J has not empty interior and $f^{K_0}J \subset J$. This is a contradiction with the fact f is transitive.

Definition 1.2.2 *Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be an increasing alternating system. We say that f is affine if f is a segment of straight line in each interval of the partition $\mathbb{P}_f$ induced by f in $\mathbb{R}$. We say that f is symmetric if $f(x) = -f(-x)$ for every $x \neq 0$.*

Corollary 1.2.1 *If f is a symmetric affine alternating system then f is transitive.*

<u>Proof</u>: Let $\{x_n^\pm\}_{n\geq 0}$, $\{u_n^\pm\}_{n\geq 0}$ be the sequences induced by f, $I = (x_0^-, x_0^+)$ the distributor interval of f and $\{I_{n+1}^- = (u_n^-, u_{n+1}^-)\}_{n\geq 0}$, $\{I_{n+1}^+ = (u_{n+1}^+, u_n^+)\}_{n\geq 0}$ the partition induced by f on I. In the linear case it is very easy to understand the transformation $f_I : I \hookleftarrow$ induced by f on the distributor interval I. In fact remembering the alternating dynamics of sequences induced by f:

$$\begin{array}{l} u_{n+1}^- \longrightarrow x_n^+ \longrightarrow x_{n-1}^+ \longrightarrow \cdots \longrightarrow x_1^+ \longrightarrow x_0 \\ u_{n+1}^+ \longrightarrow x_n^- \longrightarrow x_{n-1}^- \longrightarrow \cdots \longrightarrow x_1^- \longrightarrow u_0 \end{array}$$

and due to linearity and symmetry f_I is just an affine expansive transformation of I, that is, f_I is a segment of straight line in each atom $I_n^\pm$ such that $f_I(I_n^-) = (0, x_0^+)$, $f_I(I_n^+) = (x_0^-, 0)$ and $f_I'(x) > 1$, $\forall x \in I$ where it makes sense. An equivalent way of expansivity is the following: For Lebesgue a.e $x \in I$, the numerical representation of x with respect to f_I is unique (numerical representation means a sequence $(a_0 a_1 \cdots)$ of infinite symbols such that $a_m = \pm n$, if $f_I^m(x) \in I_n^{\pm}$, $m, n \geq 0$). This fact implies that for Lebesgue a.e $x \in \mathbb{R}$ the numerical representation of x with respect to f (this means a sequence $(a_0 a_1 \cdots)$ of two symbols such that $a_m = 1$ if $f^m(x) > 0$ and $a_m = -1$ if $f^m(x) < 0$, $m \geq 0$) is unique. Therefore f is expansive in the sense of Definition 1.2.1 and then f is transitive.

Define $f_a : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ in the following way: take $x_0, a \in \mathbb{R}$ such that x_0 is greater than 1 and $1 - \epsilon < a < 1$ (Think $\epsilon > 0$ and small). Put $u_1^- = -\frac{1}{x_0}$ and $x_1^+ = \frac{x_0}{a}$.

$$f_a(x) = \begin{cases} \textit{segment joining } (-x_0, 0) \textit{ with } (u_1^-, x_0) \;, & \textit{if} \;\; u_1^- \geq x \geq -x_0 \\ -\frac{1}{x} & \textit{if} \;\; 0 > x \geq u_1^- \\ \textit{segment joining } (x_0, 0) \textit{ with } (x_1^+, x_0) \;, & \textit{if} \;\; x_1^+ \geq x \geq x_0 \\ ax & \textit{if} \;\; x \geq x_1^+ \end{cases}$$

Define f_a in $(-\infty, u_0] \cup (0, x_0]$ so that f_a is symmetric $\;(f_a(x) = -f_a(-x))$

Corollary 1.2.2 *f_a is a transitive alternating system.*

Proof: On the one hand f_a is an increasing alternating system, this follows from Lemma 1.1.1. On the other hand, f_a is expansive. Indeed, observe that $\{x_n^{\pm}\}_{n\geq 1} = \{\pm\frac{x_0}{a^n}\}_{n\geq 1}$ and $\{u_n^{\pm}\}_{n\geq 1} = \{\pm\frac{a^n}{x_0}\}_{n\geq 1}$. Fix $n \geq 2$ and take $x \in I_n^-$, then $x \in (\frac{-a^{n-1}}{x_0}, \frac{-a^n}{x_0})$, so

$$(f_a)_I'(x) = f'(x)f'(f(x)) \cdots f(f^{n-1}x) = \frac{1}{x^2}a^{n-1} > \frac{x_0^2 a^{n-1}}{a^{2n-2}} = \frac{x_0^2}{a^{n-1}} > 1$$

where $(f_a)_I : I \hookleftarrow$ is the transformation induced by f_a on the distributor interval I. The rest of the proof follows like the affine case.

1.3 Robustly Transitive Alternating Systems.

Definition 1.3.1 *Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a C^∞ alternating system. We say that f is C^1 robustly transitive if $\exists\ \epsilon > 0$ such that $\forall g : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ ϵ-C^1 close to f (i.e, $|f(x) - g(x)| < \epsilon$ and $|f'(x) - g'(x)| < \epsilon\ \forall x \in \mathbb{R} - \{0\}$) we have that g is transitive.*

Remark: The Boole transformation $B(x) = x - \frac{1}{x}$ is not C^1 robustly transitive. Indeed, the diagonal line $\{y = x\}$ is an oblique asymtote, therefore with an arbitrarily small perturbation we obtain a function g C^1 close to f such that g is increasing and g has a fixed point p_g, so $g[p_g, +\infty) = [p_g, +\infty)$ and g is not transitive.

Definition 1.3.2 *Let f be a C^1 alternating system. We say that f is expanding (with respect to the transformation f_I induced by f on the distributor interval I) if f is an expansive alternating system such that there exist constants $C > 0$ and $\sigma > 1$ such that:*

$$|f_I'(x)| \geq C\sigma^{n(x)},\ \text{for } x \in I \text{ such that } f^n(x) \neq 0\ \forall n \geq 1,$$

where $n(x) = min\{n \geq 1 : f^n x \in I\}$.

Remark: The Boole transformation $B(x) = x - \frac{1}{x}$ is an expansive alternating system but it is not expanding.

Lemma 1.3.1 *Let f be an increasing C^1 alternating system. Suppose that:*

(1) There exists $\delta > 0$ such that $f(x) < x - \delta$, $\forall x > 0$ and $f(x) > x + \delta$, $\forall x < 0$.
(2) There exists $a > 0$ such that $f'(x) > a\ \forall x \neq 0$.
(3) There exists $L > 0$ (think L large) such that $M = sup\{f'(x) : |x| > L\} < 1$.

Then there exists $\epsilon > 0$ with the following property: $\forall g$ ϵ-C^1 close to f there exists $K \geq 1$ such that $n_f(x) - K < n_g(x) < n_f(x) + K$, $\forall x \in I_g$ (and such that $g^n x \neq 0\ \forall n \geq 1$), where I_g is the distributor interval of g and n_g, n_f

are the return time functions with respect to g and f.

<u>**Proof**</u>: Observe that hypothesis (1) and (2) guarantee that any g, sufficiently C^1 close to f, is again an alternating system, this follows easily from Lemma 1.1.1. Fix g an ϵ-C^1 perturbation of f such that g is an increasing alternating system.

Denote by $\{x_n^{\pm}\}_{n\geq 0}$, $\{u_n^{\pm}\}_{n\geq 0}$ the sequences induced by f and $\{z_n^{\pm}\}_{n\geq 0}$, $\{w_n^{\pm}\}_{n\geq 0}$ the sequences induced by g. We will do the proof using the sequences $\{x_n^{+}\}$, $\{u_n^{-}\}$ and $\{z_n^{+}\}$, $\{w_n^{-}\}$, i.e we will perturb f in $[x_0^-, 0) \cup [x_0^+, +\infty)$ (the same proof works perturbing f in $(-\infty, x_0^-] \cup (0, x_0^+])$. Also, we will omit the signs $\pm$.

Take $N, P \in \mathbb{Z}^+$ such that $x_N > L$ and $x_N \geq z_P \geq x_{N+1}$ (this is possible because $x_n \to +\infty$, $z_n \to +\infty$). Observe that if $z_{P+n} = x_{N+n}$ $\forall n \geq 0$ then the Lemma follows, suppose that $x_{N+1} \geq z_P > x_N$. Take $K \in \mathbb{Z}^+$ such that:

$$n_f(x) - K < n_g(x) < n_f(x) + K,$$

$\forall x \in [z_0^-, w_{P+1}]$ such that $g^n(x) \neq 0$, $f^n(x) \neq 0$, $\forall n \geq 0$.

(1) Assume that N is sufficiently large such that $x_{(N+1+n)} - x_{(N+n)} > C\epsilon M$, $\forall n \geq 0$, where C is a very large positive constant (this is possible because we are assuming that $M < 1$, so the difference $x_{(N+n+1)} - x_{(N+n)}$ grows exponentially fast with n for N large).

<u>Claim 1.</u> If $z \in [x_{(N+n)} + \epsilon M, x_{(N+1+n)} - \epsilon M]$, for some $n \geq 0$ then:

$$x_{(N+1+n)} < g^{-1}(z) < x_{(N+2+n)}.$$

<u>Proof of Claim 1:</u> Fix $z \in [x_{(N+n)} + \epsilon M, x_{(N+1+n)} - \epsilon M]$. If f_+ and g_+ denote the right branches de f and g, let us denote by f^{-1} and g^{-1} their inverses. We know that:

(2) $\frac{1}{f'(y)} > \frac{1}{M}$ if $|y| > L$.
(3) $f^{-1}(y) - \epsilon < g^{-1}(y) < f^{-1}(y) + \epsilon$, $\forall y$.

Using (2), the mean value theorem and that f is increasing we have:

$$|f^{-1}(z) - f^{-1}(x_{(N+n)})| = f^{-1}(z) - x_{(N+1+n)} \geq \frac{1}{M}(z - x_{(N+n)}) > \frac{M\epsilon}{M} = \epsilon,$$

from (3) we obtain $x_{(N+1+n)} < g^{-1}(z)$. Similarly, $g^{-1}(z) < x_{(N+2+n)}$ and this completes the proof of Claim 1.

<u>Claim 2.</u>
If $0 < x_{(N+n+1)} - z < M\epsilon$ for some $n \geq 1$ then $g^{-1}(z) - x_{(N+n+1)} > M\epsilon$.

<u>Proof of Claim 2:</u> If N is sufficiently large then, using that f is increasing, we have that $x_{N+n+1} < f^{-1}(z) - \epsilon$ (see (1)). From (3) we have:

$$x_{N+n+1} < g^{-1}(z)$$

Take N sufficiently large (see (1)) such that:

$$(x_{N+1} - M\epsilon - x_N) > \epsilon(M^2 + 1).$$

Therefore, from our hypothesis and the fact $\{x_{N+n}\}$ is monotone and increasing:

$$(4)\ (z - x_{N+n}) > (x_{(N+n+1)} - M\epsilon - x_{(N+n)} > \epsilon(M^2 + 1),\ \forall n \geq 0.$$

We will do the proof by contradiction. Suppose that:

$$(5)\ g^{-1}(z) - x_{(N+n+1)} < M\epsilon.$$

$$A = |g(g^{-1}z) - f(g^{-1}z)| = |z - f(x_{(N+n+1)}) - (f(g^{-1}z) - f(x_{(N+n+1)}))|$$

from (2), (4), (5) and the fact $x_{N+n} = f(x_{N+n+1})$ we obtain:

$$A \geq |z - x_{(N+n)}| - |f(g^{-1}z) - f(x_{(N+n+1)})|$$

$$> |z - x_{(N+n)}| - M|g^{-1}(z) - x_{N+n+1}| > \epsilon(M^2 + 1) - M^2\epsilon = \epsilon.$$

This is a contradiction with (3) and the proof of Claim 2 is complete.

Fix a positive constant d (for example $d = \frac{1}{2}$) and take $\epsilon > 0$ such that:

(6) $\epsilon < d(\frac{1}{M} - 1)$ and $M\epsilon < d$.

Take N sufficiently large such that $x_{(N+1)} - x_N$ is large enough (see (1)).

Claim 3. If $z \in (x_M, x_{(M+1)})$ and $z - x_M > d$ then $g^{-1}(z) - x_{(M+1)} > d$ for every $M \geq N$.

Proof of Claim 3: We know that $x_{(M+1)} = f^{-1}(x_M)$. Then, using (2), (3), (6) and the mean value theorem we get:

$$g^{-1}(z) - x_{(M+1)} > f^{-1}(z) - f^{-1}(x_M) - \epsilon > \frac{1}{M}(z - x_M) - \epsilon$$

$$> \frac{1}{M}d - d(\frac{1}{M} - 1) = d.$$

This completes the proof of Claim 3.

Claims 1,2 and 3 say that the return time function of g is bounded to the right with respect to the return time function of f, i.e $\exists \tilde{K}$ such that $n_g < n_f + \tilde{K}$ (In fact $\tilde{K} = K \pm 1$, where K is the possible growth of n_g in a compact part of the real line).

If g is a ϵ-C^1 perturbation of f such that for some $a > 0$

$$a < \tilde{m} < g' < \tilde{M} < 1$$

where $\tilde{m}$ =inf$\{g'(x) : x \neq 0\}$ and $\tilde{M}$ =sup$\{g'(x) : |x| > \tilde{L}\}$, for some positive large $\tilde{L}$, then the same proof works cosidering f as a ϵ-C^1 perturbation of g, i.e $\exists \tilde{\tilde{K}}$ such that $n_f - \tilde{\tilde{K}} < n_g$. This completes the proof of Lemma 1.3.1.

Theorem 1.3.1 *Let f be an increasing alternating system. Suppose that:*

(1) f is expanding.
(2) There exists $\delta > 0$ such that $f(x) < x - \delta$, $\forall x > 0$ and $f(x) > x + \delta$, $\forall x < 0$.
(3) There exists $a > 0$ such that $f'(x) \geq a$ $\forall x \neq 0$.
(4) There exists $L > 0$ such that $\sigma \frac{m}{M} > 1$ where σ is given by Definition 1.3.2, m =min$\{f'(x) : |x| \geq L\}$ and M =max$\{f'(x) : |x| \geq L\}$.
(5) $M < 1$.

Then f is C^1 robustly transitive.

Proof: We will prove that any C^1 perturbation sufficiently close to f is an increasing expansive alternating system, then due to Theorem 1.2.1, the result follows. This is done showing that the transformations, induced by C^1 perturbations of f on the respectives distributor intervals, are expansive. Hence the respective perturbation is expansive. Observe that hypotheses (2) and (3) guarantee that any g, sufficiently C^1 close to f, is again an alternating system, this follows easily from Lemma 1.1.1.

From (4) we can take $\epsilon_1 > 0$ such that $\sigma \frac{(m-\epsilon_1)}{M} > 1$.

From (3) we can take $\epsilon_2 > 0$ such that $f'(x) - \epsilon_2 > 0\ \forall x \neq 0$.

(2), (3) and (5) allow us to use Lemma 1.3.1. So, take $\epsilon_3 > 0$ such that if g is a ϵ_3-C^1 perturbation of f we have that:

$$n_f - \tilde{K} < n_g < n_f + \tilde{K}$$

for some $\tilde{K} \geq 1$, where I_g is the distributor interval of g and n_f, n_g are the return time function with respect to f and g.

Take $\epsilon_4 = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. Fix g an ϵ_4-C^1 perturbation of f and fix $x \in I_g$ such that $g^n x \neq 0$ and $f^n x \neq 0\ \forall n \geq 1$.

Let $x_0^- < 0 < x_0^+$ be the two roots of f and $y_0^- < 0 < y_0^+$ be the two roots of g.

Denote,

$$\begin{aligned}
p_1 =& \#\{g^j(x) \in [-L, y_0^-] \cup [y_0^+, L] : j = 1, \cdots, n_g(x) - 1\} \\
p_2 =& \#\{f^j(x) \in [-L, x_0^-] \cup [x_0^+, L] : j = 1, \cdots, n_f(x) - 1\} \\
\tilde{m} =& \min\{f'z : z \in [-L, y_0^-] \cup [y_0^+, L]\} \\
\tilde{M} =& \max\{f'z : z \in [-L, x_0^-] \cup [x_0^+, L]\}.
\end{aligned}$$

Hence, from the definition of g_{I_g}, the fact $|g' - f'| < \epsilon_4$, from (1) and the previous arguments we have:

$$
\begin{aligned}
(g)'_{I_g}(x) &= g'(x)g'(gx)\cdots g'(g^{n_g(x)-1}x) \\
&> (f'(x)-\epsilon_4)(f'(gx)-\epsilon_4)\cdots(f'(g^{n_g(x)-1}x)-\epsilon_4)) \\
&= f'(x)f'(fx)\cdots f'(f^{n_f(x)-1}x)\Big(1-\tfrac{\epsilon_4}{f'(x)}\Big)\tfrac{(f'(gx)-\epsilon_4)\cdots(f'(g^{n_g(x)-1}x)-\epsilon_4))}{f'(fx)\cdots f'(f^{n_f(x)-1}x)} \\
&> C\sigma^{n_f(x)}\Big(1-\tfrac{\epsilon_4}{f'(x)}\Big)\tfrac{(\tilde{m}-\epsilon_4)^{p_1}}{\tilde{M}^{p_2}}\Big(\tfrac{(m-\epsilon_4)^{n_g(x)-1-p_1}}{M^{n_f(x)-1-p_2}}\Big) \\
&> \tilde{C}\Big(\sigma\tfrac{(m-\epsilon_4)}{M}\Big)^{n_f(x)}
\end{aligned}
$$

where $\tilde{C}=\tilde{C}(\epsilon_4)=C\frac{(\tilde{m}-\epsilon_4)^{p_1}}{\tilde{M}^{p_2}}\frac{M^{p_2+1}}{(m-\epsilon_4)^{p_1+1-\tilde{K}}}\inf\{1-\frac{\epsilon_4}{f'(x)}:x\neq 0\}$.
Thus, if $\{x_n^{\pm}\}_{n\geq 0}$, $\{u_n^{\pm}\}_{n\geq 0}$ are the sequences induced by f, then there exists $N\geq 1$ such that:

$$\forall x\in(u_N^-,0)\cup(0,u_N^+) \text{ we have } (g)'_{I_g}(x)>1.$$

Take $\epsilon_5>0$ such that if g is a ϵ_5-C^1 perturbation of f in the compact set $[x_{N-1}^-,u_N^-]\cup[u_N^+,x_{N-1}^+]$ then:

$$(g)'_{I_g}(x)>1\ \forall x\in(y_0^-,u_N^-)\cup(u_N^+,y_0^+).$$

To finish take $\epsilon=\min\{\epsilon_4,\epsilon_5\}$. Observe that if ϵ_5 is sufficiently small then the numbers p_1 and p_2 do not change and the previous estimations are still valid. This completes the proof.

Corollary 1.3.1 *Define $f_a:\mathbb{R}-\{0\}\longrightarrow\mathbb{R}$ by $f_a(x)=ax-\frac{1}{x}$ $(0<a<1)$. If a is sufficiently close to 1 then f_a is robustly transitive.*

<u>Proof</u>: Let us check the hypotheses of Theorem 1.3.1. Indeed, f_a satisfies the hypoteses of Lemma 1.1.1, so f_a is an increasing C^∞ alternating system. Let us see that f_a is expanding with respect to the transformation induced by f on the distributor interval $I=[-\frac{1}{\sqrt{\lambda}},0)\cup(0,\frac{1}{\sqrt{\lambda}}]$. Take $x<0$ (think x very close to 0) then:

$$
\begin{aligned}
&x\to f_a(x)=ax-\tfrac{1}{x}<-\tfrac{1}{x}\to f_a^2(x)<f_a(-\tfrac{1}{x})=-\tfrac{a}{x}+x<-\tfrac{a}{x}\to \\
&\to f_a^3(x)<f_a(-\tfrac{a}{x})=-\tfrac{a^2}{x}+\tfrac{x}{a}<-\tfrac{a^2}{x}\to\cdots\to f_a^{n(x)-1}(x)<-\tfrac{a^{n(x)-2}}{x}
\end{aligned}
$$

on another hand $f_a^{n(x)-1}(x) \in (x_0^+, x_1^+) = (\frac{1}{\sqrt{a}}, x_1^+)$, then $\frac{1}{x^2} > \frac{1}{a^{2n(x)-2}}$. It follows that:

$$f_I'(x) = f'(x)f'(fx)\cdots f'(f^{n(x)-1}x) > \frac{1}{x^2}a^{n(x)-1} > \frac{1}{a}\left(\frac{1}{a}\right)^{n(x)}.$$

Take a sufficiently close to 1 so that $f'(x) > 1 \ \forall x \in [x_0^-, u_1^-]$. Hypothesis (2), (3), (4) and (5) of Theorem 1.3.1 follow easily and this completes the proof.

1.4 Ergodic Alternating Systems.

We begin the study of the ergodicity (with respect to Lebesgue measure) of alternating systems. Our goal is to look for sufficient conditions in order to guarantee the existence of ergodic absolutely continuous probabilities with respect to transformations induced, by expansive alternating systems, on compact intervals. As before, if f is an increasing alternating system, consider the distributor interval I of f, the sequences $\{x_n^-\}_{n\geq 0}$, $\{x_n^+\}_{n\geq 0}$, $\{u_n^-\}_{n\geq 0}$, $\{u_n^+\}_{n\geq 0}$ induced by f and the partition $\mathbb{P}_f$ induced by f on $\mathbb{R}$. Remember that the atoms of $\mathbb{P}_f$ are defined as:

$$\{I_{n+1}^- = (u_n^-, u_{n+1}^-)\}_{n\geq 0}, \ \ \{I_{n+1}^+ = (u_{n+1}^+, u_n^+)\}_{n\geq 0}.$$

$$\{B_{n+1}^- = (x_{n+1}^-, x_n^-)\}_{n\geq 0}, \ \ \{B_{n+1}^+ = (x_n^+, x_{n+1}^+)\}_{n\geq 0}.$$

Theorem 1.4.1 *Let f be a C^2 increasing alternating system. Suppose that:*

(1) f is expanding with respect to the transformation f_I induced by f on I.

(2) $sup_{n\geq 1} sup\{\frac{f'x_0}{f'y_0}\frac{f'x_1}{f'y_1}\cdots\frac{f'x_n}{f'y_n} : x_0, y_0 \in \xi^{(n)} \in \xi \vee f^{-1}\xi \vee \cdots \vee f^{-n+1}\xi\} < \infty$, where ξ is the partition induced by f on I and

$$x_j = f^{n(x_{j-1})}(x_{j-1}),\ y_j = f^{n(y_{j-1})}(y_{j-1}),\ j \geq 1.$$

(3) The sequences

$$M_n = max\{f'(x) : x \in B_n^- \cup B_n^+\}$$

$$m_n = min\{f'(y) : y \in B_n^- \cup B_n^+\}$$

have the property $\frac{M_n}{m_n} < 1 + \epsilon_n$*, where* $\{\epsilon_n\}$ *is a sequence of positive numbers such that* $\forall x_0$

$$\sum_{j=0}^{\infty} (\epsilon_1 + \cdots + \epsilon_{n(x_j-1)}) < \infty.$$

Then f *is ergodic with respect to Lebesgue measure.*

Proof: Observe that the transformation f_I induced by f on the distributor interval I is an expanding Markov map (see [3],[8],[11] for definitions and details) the only thing that we have to prove is that f_I has bounded distortion with respect to the partition $\xi = \{I_n^-, I_n^+ : n \geq 1\}$ induced by f on I. i.e there exist a constant $M > 0$ such that

$$\sup\{\frac{|(f_I^n)'x|}{|(f_I^n)'y|} : x, y \in \xi^{(n)}, n \geq 1\} < M$$

where $\xi^{(n)}$ is a generic element of the partition $\xi \vee f^{-1}(\xi) \vee \cdots \vee f^{-n+1}(\xi)$.

In this case, due to the Folklore theorem (see [3],[8],[11]), there exists an ergodic f_I-invariant probability equivalent to Lebesgue measure. Hence, using a clasical argument in infinite ergodic theory (see for example [9]), we have that f is ergodic with respect to Lebesgue measure. Indeed, fix $m \geq 1$ and take $x_0, y_0 \in \xi^{(m)}$. Using the definition of f_I we have:

$$\frac{(f_I^m)'(x_0)}{(f_I^m)'(y_0)} = \prod_{i=0}^{m-1} \prod_{j=0}^{n(x_i)-1} \frac{f'(f^j x_i)}{f'(f^j y_i)}$$

From (2) $\frac{f'(x_0)f'(x_1)\cdots f'(x_{m-1})}{f'(y_0)f'(y_1)\cdots f'(y_{m-1})}$ is uniformly bounded by a constant $C > 0$.

On the other hand, from (3) we obtain:

$$\frac{(f_I^m)'(x_0)}{(f_I^m)'(y_0)} < C\exp(\sum_{j=1}^{\infty}(\epsilon_1 + \cdots \epsilon_{n(x_j)-1}))$$

In the last statement we used the fact $\log(1+\epsilon) < \epsilon$ for $\epsilon > 0$. This completes the proof.

Corollary 1.4.1 *If f is an affine symmetric alternating system then f is ergodic (with respect to Lebesgue measure).*

Proof: It follows observing that the hypothesis of Theorem 1.4.1 are satisfied with the sequence $\epsilon_n = 0 \ \forall n \geq 1$.

Remark: In [2] is shown that the Boole transformation is ergodic, this is done taking control on the quantities (2) and (3) of theorem 1.4.1. In [8] the same result is shown using a differentiable compactification of $\mathbb{R}$. If a differentiable compactification is possible for an alternating system f (think in an alternating systems with oblique asymptotes) then the ergodicity of f follows if $h \circ f \circ h^{-1}$ is ergodic for some differentiable function $h : \mathbb{R} \to I$ and I is some bounded interval.

Chapter 2

Bidimensional Alternating Systems.

In this chapter we begin the study of diffeomorphisms on the plane having a curve of discontinuities. We will characterize the transitivity in the set of alternating system. Also, to understand the problem of robust transitivity, we will do a definition of Anosov diffeomorphisms for diffeomorphisms on the plane with a curve of discontinuities. Finally we will show conditions that guarante the existence of physical measures with respect to transformations induced by alternating systems on bounded subsets of the plane.

2.1 Alternating Systems.

Let $\gamma, \delta : \mathbb{R} \to \mathbb{R}^2$ be transversal smooth curves such that $\gamma = (\gamma_1, \gamma_2)$ is horizontal (i.e, $\frac{d\gamma_1}{dt}(t_0) \neq 0 \ \forall t_0 \in \mathbb{R}$), $\delta = (\delta_1, \delta_2)$ is vertical ($\frac{d\delta_2}{dt}(t_0) \neq 0$ $\forall t_0 \in \mathbb{R}$) and γ intersects δ in a single point. Suppose that $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ is a homeomorphism. We say that F is transitive if there exists a point $p \in \mathbb{R}^2$ such that $o(p) = \{F^n p : n \in \mathbb{Z}\}$ is dense in $\mathbb{R}^2$. Let $A = \{\alpha_1, \cdots, \alpha_n\}$ be a collection of n smooth horizontal (or vertical) curves ($\alpha_i : \mathbb{R} \to \mathbb{R}^2$) without intersections, if $B = \{\beta_1, \cdots, \beta_{n+1}\}$ is a collection of $n+1$ smooth curves, we say that B is between A if $\forall$ element E in the partition induced by A (on $\mathbb{R}^2$) $\exists \ ! \ i \in \{1, \cdots, n+1\}$ such that $\beta_i \in E$. For every $p \in \mathbb{R}^2 - \{\gamma\}$ define $\mathrm{sign}_\gamma(p) = 1$ if $p \in R^+$ and $\mathrm{sign}_\gamma(p) = -1$ if $p \in R^-$, where $R^\pm$ are the respective connected components of $\mathbb{R}^2 - \{\gamma\}$.

Similarly, we define the sign of a point with respect to the vertical curve δ

Definition 2.1.1 *Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be a homeomorphism. we say that F is an alternating system if:*

(AS1) For every $n \geq 1$, $F^{-n}(\gamma)$ consists of 2^n smooth horizontal curves between the previous $F^{-n+1}(\{\gamma\}) \cup \cdots \cup F^{-1}(\{\gamma\}) \cup \{\gamma\}$ and they meet δ in a single point. For every $n \geq 1$, $F^n(\delta)$, consists of 2^n smooth vertical curves between the previous $F^{n-1}(\{\delta\}) \cup \cdots \cup F(\{\delta\}) \cup \{\delta\}$ and they meet γ in a single point.

(AS2) For every $p \in \mathbb{R}^2$ such that $F^n p$ exists $\forall n \in \mathbb{Z}$ we have: $\forall M \geq 1$ $\exists N \geq 1$ such that $sign_\gamma(F^M p) = -sign_\gamma(F^N p)$ (i.e $F^n p$ is over and below the curve γ indefinitely) and $\forall M \geq 1$ $\exists N \geq 1$ such that $sign_\delta(F^{-M} p) = -sign_\delta(F^{-N} p)$ (i.e $F^{-n} p$ is on the right and on the left of the the curve δ infinitely many times).

Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be an alternating system. If $\alpha, \beta : \mathbb{R} \to \mathbb{R}^2$ are horizontal smooth curves, we say that α *is over* β ($\alpha > \beta$) if $\forall$ vertical straight line l we have that any intersection of l with $\alpha(\mathbb{R})$ is greater than any intersection of l with $\beta(\mathbb{R})$ (Make a similar definition to order vertical smooth curves). We say that F is increasing if F is increasing in each connected component of the partition induced by γ on $\mathbb{R}^2$ and each connected component of the partition induced by δ on $\mathbb{R}^2$.

Theorem 2.1.1 *Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be a diffeomorphism. Suppose that:*

(1) F is transitive.
(2) $\mathbb{R}^2 - \{\gamma\}$ admits transversal F-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that:
(i) Every leaf of $\mathcal{F}^s$ intersects every leaf of $\mathcal{F}^u$ in a single point.
(ii) $\mathcal{F}^s$ is horizontal and $\mathcal{F}^u$ is vertical.
(iii) $\mathcal{F}^s$ and $\mathcal{F}^u$ are continuous (i.e, the holonomy maps are continuous).
(3) F and F^{-1} are increasing in each connected component of their domains.

Then F is an alternating system.

Proof: Observe that the proof of Theorem 1.1.1 is still valid for any transitive unidimensional transformation, say f, such that f is continuos and increasing in each connected component of its domain (i.e, the proof does not depend on differentiability). Consider the projection $f = P_\delta : \delta \to \delta$ on δ along $\mathcal{F}^s$ with respect to F (i.e, for $p \in \delta$, $f(p)$ is the intersection of the leaf of $\mathcal{F}^s$ containing $F(p)$ with δ) and $g = P_\gamma : \gamma \to \gamma$ the projection on γ along $\mathcal{F}^u$ with respect to F^{-1} (i.e, for $p \in \gamma$, $g(p)$ is the intersection of the leaf of $\mathcal{F}^u$ containing $F^{-1}(p)$ with γ). The proof follows from Theorem 1.1.1, observing that f and g are unidimensional, increasing (increasing in each connected component of theirs domains) and transitive. (The important thing here is that, necessarily, the discontinuities curves γ and δ of an increasing transitive diffeomorphism are explosive, and so they behave like unidimensional alternating systems).

2.2 Transitive alternating systems

Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be an increasing alternating system. A directive curve of F is any connected component of $F^{-n}(\gamma)$ or any connected component of $F^n(\delta)$, for any natural number n. For each $n \geq 0$ consider the extreme directive curves of $F^{-n}(\gamma)$ and $F^n(\delta)$ (extreme means the greater one and smaller one) and their images and inverse images. This subset of extreme directive curves will be called just directive curves. Like unidimensional alternating systems, any increasing (bidimensional) alternating system F induces a structure, which will be called the Markov structure induced by F, say

$$(F, \{x_n^{\pm}\}_{n\geq 0}, \{u_n^{\pm}\}_{n\geq 0}, \{y_n^{\pm}\}_{n\geq 0}, \{w_n^{\pm}\}_{n\geq 0}, \mathbb{P}_F, R, F_R : R \to R)$$

where,

(1) $\{x_n^{\pm}\}_{n\geq 0}, \{u_n^{\pm}\}_{n\geq 0}$ are the horizontal directive curves and $\{y_n^{\pm}\}_{n\geq 0}, \{w_n^{\pm}\}_{n\geq 0}$ are the vertical directive curves. They have the following property with respect to F (i.e considering future images of F):

$$\begin{array}{l} u_{n+1}^- \longrightarrow x_n^+ \longrightarrow x_{n-1}^+ \longrightarrow \cdots \longrightarrow x_1^+ \longrightarrow x_0^+ \longrightarrow \gamma \\ u_{n+1}^+ \longrightarrow x_n^- \longrightarrow x_{n-1}^- \longrightarrow \cdots \longrightarrow x_1^- \longrightarrow x_0^- \longrightarrow \gamma. \end{array}$$

Similarly, for vertical directive curves with respect to F^{-1}.

(2) The directive curves induce a partition $\mathbb{P}_F$ on $\mathbb{R}^2$. $\mathbb{P}_F$ will be called the partition induced by F on $\mathbb{R}^2$.

(3) R is the subset bounded by the directive curves $F^{-1}(\{\gamma\})$ and $F(\{\delta\})$. The subset R will be called the distributor subset of F.

(4) $F_R : R \to R$ is the transformation defined by $F_R(p) = F^{n(p)}p$, where

$n(p)$ =min$\{n \geq 1 : F^n p \in R\}$ for every p such that $F^n(p) \notin \{\gamma\} \cup \{\delta\}$ $\forall n \in \mathbb{Z}$.

Observe that $F_R : R \to R$ is well defined, for points in the plane outside a denumerable subset of vertical and horizontal curves. We will call F_R the transformation induced by F on the distributor subset R.

Definition 2.2.1 *An alternating system F is expansive if $\forall p \neq q$ (and such that $F^n(p)$, $F^n(q)$ are defined $\forall n \in \mathbb{Z}$) there exists $A \neq B \in \mathbb{P}_F$ (the partition induced by F on $\mathbb{R}^2$) and $N \in \mathbb{Z}$ such that $F^N(p) \in A$ and $F^N(q) \in B$.*

Theorem 2.2.1 *Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be an increasing alternating system. then,*

F is expansive $\Longleftrightarrow$ F is transitive.

Proof: ($\Rightarrow$) Let Σ be the space of bilateral sequences of two symbols $\{1, -1\}$ with its usual topology and Bernoulli probability. Σ is compact and the shift

$$\sigma : \Sigma \longrightarrow \Sigma,\ \sigma(\cdots a_{-1}\tilde{a_0}a_1 \cdots) = (\cdots a_{-1}a_0\tilde{a_1} \cdots)$$

is a homeomorphism. We construct a homeomorphism $h : A \longrightarrow \Sigma'$ such that $h \circ F = \sigma \circ h$, where $A = \mathbb{R}^2$ (Mod 0) and $\Sigma' = \Sigma$ (Mod 0). Take $A = \mathbb{R}^2 - \{(x_0, y_0) : y_n \in \gamma \text{ or } y_{-n} \in \delta, \text{ for some } n \geq 0\}$. A is the complement of the subset of points that meet the curve γ or δ. Σ' the subset of sequences in Σ except those which terminate (to the right and/or to the left) with an infinite string of 1s or -1s. Define $h : A \longrightarrow \Sigma'$, $h(x_0, y_0) = (\cdots a_{-1}a_0a_1 \cdots)$, where

$$a_n = \begin{cases} 1 & if\ y_n\ \ is\ \ over\ \ \gamma \\ -1 & if\ y_n\ \ is\ \ below\ \ \delta \end{cases}$$

For $p_0 = (x_0, y_0) \in \mathbb{R}^2$ and $n \in \mathbb{Z}$ write $p_n = F^n(p_0) = (x_n, y_n)$ in what follows. Using (AS2) given $p_0 \in A$ there exists $a \in \Sigma'$ such that $h(p_0) = a$. Observing the geometry of an alternating system it is clear that if p_0 and p_0' are in different elements of the partition $\mathbb{P}_F$ induced by F in $\mathbb{R}^2$ then there exists $n \geq 1$ such that y_n is over γ and y_n' is below γ. So by hypotheses $p_0 \neq p_0' \Longrightarrow h(p_0) \neq h(p_0')$. Therefore, h is well defined and is one to one.
Now we show that h is onto Σ'. For that take $a = (\cdots a_{-1}a_0a_1\cdots) \in \Sigma'$ and denote
$Q_{+1} = \{p \in \mathbb{R}^2 : p \text{ is over } \gamma\}$, $Q_{-1} = \{p \in \mathbb{R}^2 : p \text{ is below } \gamma\}$ Define:

$$A_1 = \overline{F^{-1}(Q_{a_{-1}}) \cap Q_{a_0}}$$

$$A_n = \overline{F^{-n}(Q_{a_{-n}}) \cap F^{-n+1}(Q_{a_{-n+1}}) \cap \cdots \cap Q_{a_0}}\ ,\ \ n \geq 1$$

$$B_1 = \overline{F(Q_{a_1})}$$

$$B_n = \overline{F^n(Q_{a_n}) \cap F^{n-1}(Q_{a_{n-1}}) \cap \cdots F(Q_{a_1})}\ , n \geq 1$$

If $a \in \Sigma'$ then there exist $n, m \geq 1$ such that
$a_1, a_2, \cdots, a_{n-1} = a_{-1}, a_{-2}, \cdots, a_{-m+1} = a_0$, but $a_n = a_{-m} = -a_0$.
So A_m and B_n are bounded by two directive curves. Therefore $A_m \cap B_n$ is a nested sequence of compact sets. Take $p \in \bigcap_{n\geq 1}(A_n \cap B_n)$, observe that $p \in A$ and so we have $h(p) = a$. The continuity of h is a consequence of the continuity of F.

($\Leftarrow$) Similar to the unidimensional case.

Definition 2.2.2 *Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be an alternating system. F is affine if F is affine in every atom of the partition $\mathbb{P}_F$ induced by F on $\mathbb{R}^2$, that is, given $A \in \mathbb{P}_F$ there exist real numbers $\lambda(A) = \lambda$, $\sigma(A) = \sigma$, $x_0(A) = x_0$, $y_0(A) = y_0$ such that $F_{|_A} = (\lambda x + x_0, \sigma y + y_0)$.*

To construct an affine alternating system we proceed as follows:

Fix $b, x_0^-, x_0^+, c, y_0^-, y_0^+ \in \mathbb{R}$ such that $x_0^-, y_0^- < 0 < x_0^+, y_0^+$ and $1 \geq b, c > 0$.

Define a function $F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{x = 0\}$ as follows:

(1) for every $n \geq 1$ take $x_n^+, x_n^-, y_n^+, y_n^- \in \mathbb{R}$ such that:

$$x_{n-1}^+ < bx_n^+,\ bx_n^- < x_{n-1}^-,\ y_{n-1}^+ < cy_n^+,\ cy_n^- < y_{n-1}^-.$$

(2) Take any sequences $\{u_n^-\}_{n\geq1}$, $\{u_n^+\}_{n\geq1}$, $\{v_n^-\}_{n\geq1}$, $\{v_n^+\}_{n\geq1}$ such that

$$\begin{array}{l} x_0^- < u_n^- < 0 < u_n^+ < x_0^+ \text{ and } u_n^- \nearrow 0,\ u_n^+ \searrow 0 \\ y_0^- < v_n^- < 0 < v_n^+ < y_0^+ \text{ and } v_n^- \nearrow 0,\ v_n^+ \searrow 0. \end{array}$$

(3) Denote $l(x_0) = \{x = x_0, y > 0\}$ (superior verticals). Define F affine so that:

$$\begin{array}{l} l(0) \longrightarrow \{x = x_0^+\}, \\ l(x_n^-) \longrightarrow \{x = u_{n+1}^+\}\ \forall n \geq 0, \\ l(x_n^+) \longrightarrow \{x = x_{n+1}^+\}\ \forall n \geq 0, \end{array}$$

and such that $\forall x_0 \in \mathbb{R}$

$$\begin{array}{l} \{x = x_0, y_n^+ < y < y_{n+1}^+\} \longrightarrow \{*, y_{n-1}^+ < y < y_n^+\}\ \forall n \geq 1, \\ \{x = x_0, y_0^+ < y < y_1^+\} \longrightarrow \{*, 0 < y < y_0^+\}, \\ \{x = x_0, v_1^+ < y < y_0^+\} \longrightarrow \{*, y_0^- < y < 0\}, \\ \{x = x_0, v_{n+1}^+ < y < v_n^+\} \longrightarrow \{*, y_n^- < y < y_{n-1}^-\}\ \forall n \geq 1, \end{array}$$

where $*$ is the first coordinate of the respective $F(x_0, y)$. If $y < 0$ Define $F(x, y) = -F(-x, -y)$.

Observe that $F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{x = 0\}$ is a homeomorphism, in fact F is differentiable except in the horizontal and vertical lines

$$\{y = y_n^\pm\},\ \{y = v_n^\pm\},\ \{x = x_n^\pm\},\ \{x = u_n^\pm\}.$$

(Observe that these lines are the directive curves of F). F is affine in the following sense: $\forall A \in \mathbb{P}_F$ we have that $F|_A(x, y) = (\lambda x + x_0, \sigma y + y_0)$, for some real numbers $\lambda = \lambda(A), \sigma = \sigma(A), x_0 = x_0(A), y_0 = y_0(A)$.

Corollary 2.2.1 *If $F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{x = 0\}$ is an affine symmetric (i.e, $F(p) = -F(-p)\ \forall p$) alternating system then F is transitive.*

Proof: If F is an affine symmetric alternating system then the projection on $\{x = 0\}$ along horizontal lines (with respect to F) and the projection on $\{y = 0\}$ along vertical lines (with respect to F^{-1}) are unidimensional affine symmetric alternating systems. This implies that the following subsets:

$$\{F^{-n}(\{y = 0\}) : n \geq 0\} \text{ and } \{F^n(\{x = 0\}) : n \geq 0\}$$

are dense on $\mathbb{R}^2$ and therefore F is expansive.

We will study another kind of transitive diffeomorphisms. They are motivated by the following area preserving diffeomorphism:

$$F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{y = -x\},\ F(x, y) = (x + \tfrac{1}{y}, y - \tfrac{1}{y} - x)$$

which was introduced by Hénon as an asymptotic form of the equations of motion of the restricted three body problem [12]. In [13] it was shown that F is transitive. In our context F is an increasing expansive alternating system .

Theorem 2.2.2 *Given $\epsilon > 0$ (small) the map $F_a : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{y = -ax\}$, $F_a(x, y) = (x + \frac{1}{y}, ay - ax - \frac{a}{y})$ with $1 - \epsilon < a < 1$ is a transitive alternating system.*

Theorem 2.2.2 follows from several Lemmas.

Lemma 2.2.1 *Given $y_0 > 0$, $x_0 \geq 0$ there exists $n \geq 1$ such that $0 \geq y_n$. (that is, every point in the first quadrant changes its second coordinate signal or meets the x-axis after an iterate sufficiently large)*

Proof: Consider the following extreme directive curves:

$\gamma_1 = F_a^{-1}(\{(x, y) : y = 0, x > 0\})$

.

.

$\gamma_n = F_a^{-1}(\{(x, y) \in \gamma_{n-1} : y > -ax\})$

.

and their intersections with the positive y-axis ($\{(x,y) : x = 0, y > 0\}$) say $\{z(1), z(2), \cdots, z(n), \cdots\}$. These points exist and they are unique because directive curves are monotone smooth real functions.

Claim: $z(n) \to +\infty$ when $n \to +\infty$. Indeed, keeping in mind the formula of F_a and definition of $z(n)$ we have:

$(x_0, y_0) = (0, z(n)) \longrightarrow (x_1, y_1) = (\frac{1}{z(n)}\ ,\ az(n) - ax_1) \longrightarrow (x_2, y_2)$

$(x_2, y_2) = (\frac{1}{z(n)} + \frac{1}{y_1}\ ,\ a^2 z(n) - a^2 x_1 - ax_2) \longrightarrow (x_3, y_3) \cdots \longrightarrow (x_n, y_n) = (*, 0)$

$(x_n, y_n) = (\frac{1}{z(n)} + \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_{n-1}}\ ,\ a^n z(n) - a^n x_1 - a^{n-1} x_2 - \cdots - ax_n)$

$$z(n) = x_1 + \frac{1}{a}x_2 + \cdots + \frac{1}{a^{n-1}}x_n$$

$$= \frac{1}{z(n)} + \frac{1}{a}(\frac{1}{z(n)} + \frac{1}{y_1}) + \cdots + \frac{1}{a^{n-1}}(\frac{1}{z(n)} + \frac{1}{y_1} + \cdots + \frac{1}{y_{n-1}}).$$

Observe that the definition of $\{\gamma_n\}$ implies $y_0, y_1, \cdots, y_{n-1} > 0$. Then

$$z(n) > \frac{1}{z(n)} \sum_{j=1}^{n} (\frac{1}{a})^j.$$

Since $a < 1$ the series $\sum(\frac{1}{a})^n$ diverges and thus the claim follows. To finish the proof of the Lemma, given $y_0 > 0$, $x_0 \geq 0$ take $z(n)$ sufficiently large so that the point (x_0, y_0) is below the curve γ_n. Following the dynamics of directive curves we have the conclusion of Lemma 2.2.1.

Lemma 2.2.2 *Given $x_0 < 0$, $y_0 > 0$ there exists $n \geq 1$ such that either $0 \geq y_n$ or else $x_n \geq 0$.*

Proof: Suppose $y_n > 0$ and $x_n < 0$ $\forall n \geq 1$. Then $x_1 = x_0 + \frac{1}{y_0} > x_0$, $x_2 = x_1 + \frac{1}{y_1} > x_1$, $\cdots$ $x_n > x_{n-1} > \cdots > x_1 > x_0$

$y_1 = ay_0 - ax_1 < y_0 - ax_0 < y_0 - x_0$

$y_2 = ay_1 - ax_2 < y_1 - x_2 < y_0 - x_0 - x_0$.....By induction, $y_n < y_0 - nx_0$.

therefore
$x_2 = x_1 + \frac{1}{y_1} > x_0 + \frac{1}{y_0} + \frac{1}{y_0 - x_0}, \cdots\cdots, x_{n+1} > x_0 + \frac{1}{y_0} + \frac{1}{y_0 - x_0} + \cdots + \frac{1}{y_0 - nx_0}$.

The series $\sum \frac{1}{y_0 - nx_0}$ diverges to $+\infty$, thus for some n, $a_n > 0$ and from this we get a contradiction.

Similarly we prove the corresponding results for the case $y < 0$ because F_a is symmetric, i.e, $F_a(p) = -F_a(-p)$, $\forall p \in \mathbb{R}^2$. We will use the notation $\triangle x_n = x'_n - x_n$ and $\triangle y_n = y'_n - y_n$, $n \in \mathbb{Z}$.

Lemma 2.2.3 *Given p_0, p'_0 different points such that $\triangle x_0 < 0$, $\triangle y_0 > 0$, then there exist A, B different atoms of $\mathbb{P}_{F_a}$ (The Partition induced by F_a in $\mathbb{R}^2$) and $N \geq 0$ such that $p_N \in A$ and $p'_N \in B$.*

Proof: Suppose that p_n, p'_n are in the same atom of the partition $\mathbb{P}_{F_a}$ $\forall n \geq 0$. In particular, y_n, y'_n have the same sign $\forall n \geq 0$. Therefore
$\triangle x_1 = x'_1 - x_1 = x'_0 + \frac{1}{y'_0} - (x_0 + \frac{1}{y_0}) = (x'_0 - x_0) + \frac{(y_0 - y'_0)}{y'_0 y_0} =$

$$= \triangle x_0 + \frac{(-\triangle y_0)}{y'_0 y_0} < \triangle x_0 < 0$$

$$\triangle y_1 = y'_1 - y_1 = (ay'_0 - ax'_1) - (ay_0 - ax_1) = a\triangle y_0 - a\triangle x_1 > 0 \ (\Rightarrow y'_1 > y_1)$$

$$\triangle x_2 = \triangle x_1 + \frac{(-\triangle y_1)}{y'_1 y_1} = \triangle x_0 + \frac{(-\triangle y_0)}{y'_0 y_0} + \frac{(-\triangle y_1)}{y'_1 y_1} < \triangle x_1 < \triangle x_0 < 0$$

$$\triangle y_2 = y'_2 - y_2 = a\triangle y_1 - a\triangle x_2 = a^2\triangle y_0 - a^2\triangle x_1 - a\triangle x_2 > 0 \ (\Rightarrow y'_2 > y_2)$$

by induction we have:

. $\triangle x_n < \triangle x_{n-1} < \cdots < \triangle x_1 < \triangle x_0 < 0$

. $\triangle x_n = \triangle x_0 + \frac{(-\triangle y_0)}{y'_0 y_0} + \cdots + \frac{(-\triangle y_{n-1})}{y'_{n-1} y_{n-1}}$ (*)

. $\triangle y_n = a^n\triangle y_0 - a^n\triangle x_1 - \cdots - a^2\triangle x_{n-1} - a\triangle x_n > a(-\triangle x_n)$ (**)

. $y'_n > y_n \ \forall n \geq 1$ (***)

Claim: $\triangle x_n \longrightarrow -\infty$. Suppose this is not true. From (**) $-\triangle x_n \nearrow \tilde{c} > 0$ and from (*) $\frac{\triangle y_n}{y'_n y_n} \to 0$. So, from (**) $\triangle y_n > a(-\triangle x_0) > 0 \ \forall n \geq 1$, then $\frac{1}{y'_n y_n} \to 0$. Using (AS2) set in a subsequence $\{n_k\}$ all natural numbers such that $y_{n_k} > 0 \ \forall k$ (this implies $y'_{n_k} > 0 \ \forall k$). Observe that $\{y'_{n_k}\}$ does not converge neither to a real number nor to ∞.
From (***) $(y'_{n_k})^2 > y'_{n_k} y_{n_k}$, i.e $y'_{n_k} \to +\infty$ when $k \to +\infty$ and from this a contradiction.

To finish the proof of Lemma 2.2.3, the claim says that points satisfying $\triangle x_0 < 0$, $\triangle y_0 > 0$, after many positive iterates separate so much that they finish being in different atoms of the partition $\mathbb{P}_{F_a}$.

Lemma 2.2.4 *Given p_0, p'_0 different points such that $\triangle x_0 > 0, \;\; \triangle y_0 > 0$, then there exist A, B different atoms of $\mathbb{P}_{F_a}$ (Partition induced by F_a in $\mathbb{R}^2$) and $N \geq 0$ such that $p_{-N} \in A$ and $p'_{-N} \in B$.*

__**Proof**__: Suppose p_{-n} and p'_{-n} are in the same atom of $\mathbb{P}_{F_a}$, $\forall n \geq 1$. In particular y_{-n} and y'_{-n} have the same sign $\forall n \geq 0$. Then we have:

$$\triangle y_{-1} = y'_{-1} - y_{-1} = x'_0 + \tfrac{y'_0}{a} - (x_0 + \tfrac{y_0}{a}) = (x'_0 - x_0) + \tfrac{1}{a}(y'_0 - y_0) >$$

$$> \triangle x_0 + \triangle y_0 > \triangle y_0 > 0$$

$$\triangle x_{-1} = x'_{-1} - x_{-1} = (x'_0 - \tfrac{1}{y'_{-1}}) - (x_0 - \tfrac{1}{y_{-1}}) = (x'_0 - x_0) + (\tfrac{1}{y_{-1}} - \tfrac{1}{y'_{-1}}) =$$

$$= \triangle x_0 + \tfrac{\triangle y_{-1}}{y_{-1}y'_{-1}} > \triangle x_0 > 0$$

$$\triangle y_{-2} = y'_{-2} - y_{-2} = x'_{-1} + \tfrac{y'_{-1}}{a} - (x_{-1} + \tfrac{y_{-1}}{a}) = \triangle x_{-1} + \tfrac{1}{a}(y'_{-1} - y_{-1}) >$$

$$> \triangle x_{-1} + \triangle y_{-1} > 2\triangle x_0 + \triangle y_0.$$

By induction,

. $\triangle x_{-n} > \triangle x_{-n+1} > \cdots > \triangle x_{-1} > \triangle x_0 > 0$

. $\triangle y_{-n} > \triangle y_{-n+1} > \cdots > \triangle y_{-1} > \triangle y_0 > 0$

. $\triangle y_{-n} > n\triangle x_0 + \triangle y_0 \longrightarrow +\infty$

To finish the proof of Lemma 2.2.4, note as before that points satisfying $\triangle x_0 > 0$, $\triangle y_0 > 0$ after many negative iterates separate so much that they finish being in different atoms of partition $\mathbb{P}_{F_a}$.

__Proof of Theorem 2.2.2__:
F_a is C^∞ inversible with inverse $F_a^{-1}(x,y) = (x - \frac{a}{ax+y}, \frac{ax+y}{a})$. It is easy to check property (AS1) of definition 2.1.1. Lemmas 2.2.1 and 2.2.2 prove property (AS2) of Definition 2.1.1. On another hand F is expansive, by Lemmas 2.2.3 and 2.2.4. Indeed, given p_0, p'_0 different points in $\mathbb{R}^2$ (such that all future and past iterates exist) their coordinates either satisfy $\triangle x_0 > 0$, $\triangle y_0 > 0$ or $\triangle x_0 < 0$, $\triangle y_0 > 0$ (if $\triangle y_0 < 0$ change the role of p_0 and p'_0) If $\triangle x_0 = 0$ or $\triangle y_0 = 0$ (vertical or horizontal position) then after one iteration

the images of p_0, p_0' are again in one of the previous situations. Then from Theorem 2.2.1 F_a is transitive.

2.3 Robustly Transitive Alternating Systems.

Definition 2.3.1 *Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be a C^∞ alternating system. We say that F is C^1 robustly transitive if there exists $\epsilon > 0$ such that $\forall\ G : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ ϵ-C^1 close to F, i.e*

$$\|G(p) - F(p)\| < \epsilon \textit{ and } \|D_pG - D_pF\| < \epsilon\ \forall p \in \mathbb{R}^2 - \{\gamma\}$$

we have that G is transitive.

Remark: Denote by F the Henon map, then F is not C^1 robustly transitive. Indeed, in [13] it is shown that F is transitive. The important observation is that the projection on the line $\{x = 0\}$ along the (non-uniform) unstable foliation (see [14]) with respect to F, say f, is a unidimensional alternating system such that f is continuous, transitive, increasing and the diagonal line is an oblique asymptote for f. Then, with any small C^0 perturbation we destroy the transitivity of f (and hence the transitivity of F), therefore F is not C^0-robustly transitive.

Definition 2.3.2 *Let $F : \mathbb{R}^2 - \{\gamma\} \longrightarrow \mathbb{R}^2 - \{\delta\}$ be a C^∞ alternating system. We say that F is Anosov if F is an expansive increasing alternating system such that the transformation $F_R : R \hookleftarrow$, induced by F on the distributor subset R, admits a continuous DF_R-splitting $E^s \oplus E^u$ such that for some constants $C > 0$ and $\sigma > 1$ we have:*

$$\|D_pF_R(v)\| \geq C\sigma^{n(p)}\|v\|,\ \forall v \in E^u \textit{ and}$$
$$\|D_p(F_R)^{-1}(v)\| \geq C\sigma^{n(p)}\|v\|,\ \forall v \in E^s$$

$\forall p \in R$ such that $F^n(p) \neq 0\ \forall n \in \mathbb{Z}$, where $n(p) =$min$\{n \geq 1 : F^n p \in R\}$.

We think that this is the right definition of Anosov diffeomorphisms for diffeomorphims on the plane with a curve of discontinuities. The main problem is to obtain conditions such that Anosov alternating systems become C^1

robustly transitives. Like the unidimensional case, it is hoped that the previous definition is not enough to prove C^1 robust transitivity of alternating systems. Certainly, if F is an Anosov alternating system then to obtain that F is C^1 robustly transitive, the projection maps of F (with respect to the continuous foliations induced by the topological conjugation between F and the bilateral shifts, see Theorem 2.2.1) must be C^0 far from the diagonal line, and some condition about the distortion of F at the infinite has to be imposed. We just formulate some questions:

(A) Let F be an Anosov alternating system. Suppose that:

(i) Any C^1 perturbation of F sufficiently close to F is an increasing alternating system.
(ii) There exists $L > 0$ such that $\sigma\frac{m}{M} > 1$, where σ is the constant given by Definition 2.3.2, $M = \sup\{\|D_pF\| : \|p\| \geq L\}$ and $m = \inf\{\|D_pF\| : \|p\| \geq L\}$.

Then, is F C^1 robustly transitive?

(B) Let F be an Anosov alternating system. Suppose that the projection maps of F, with respect to the continuous foliations induced by the topological conjugation between F and the bilateral shifts (see Theorem 2.2.1), are unidimensional alternating systems satisfying the hypotheses of Theorem 1.3.1. Then, is F C^1 robustly transitive?

(C) Is there an example of a C^1 robustly transitive Anosov alternating system?

2.4 Ergodic Alternating Systems.

Let $\epsilon > 0$ be small, $1-\epsilon < a < 1$ and $F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{y = -ax\}$ be an increasing symmetric alternating system. Suppose that F is expansive and $\mathbb{R}^2$ admits (mod 0) transversal F-invariant stable and unstable foliations, say $\mathcal{F}^s$ and $\mathcal{F}^u$ respectively. Let $f = P_\delta : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be the projection on the discontinuity line $\delta = \{y = -ax\}$ along $\mathcal{F}^s$. Observe that f is an unidimensional increasing symmetric expansive alternating system and its differentiability depends on the differentiability of the foliation $\mathcal{F}^s$.

In the next Lemma we construct physical measures (or S.R.B measures) for F in bounded regions of the plane. We will use R to represent the region bounded by the directive curves $F^{-1}(\{y=0\})$ and $F(\{y=-ax\})$ and $f_I : I \hookleftarrow$ will denote the transformation induced by f on the distributor interval I (see Chapter 1).

Lemma 2.4.1 *Let F be an alternating system in the previous conditions and suppose that $\mathcal{F}^s$ is a C^1 foliation. If f_I admits an ergodic f_I-invariant probability μ_I absolutely continuous (with respect to Lebesgue measure) then the transformation $F_R : R \hookleftarrow$ induced by F on R admits a F_R-invariant probability μ on the Borel σ-algebra of R such that the basin of μ has full Lebesgue measure.*

Proof: The construction of the S.R.B measure for F is very similar to that of transitive $C^{1+\epsilon}$ Anosov diffeomorphisms. The main ingredients are the uniform contraction due to $\mathcal{F}^s$ and its absolutely continuity. So we just do a sketch of proof and refer to [10] for details. In our case the projection f is an unidimensional increasing symmetric expansive alternating system. Hence $f_I : I \hookleftarrow$ is just an expansive transformation of I (see Chapter 1 for details) and we are asuming that f_I admits an ergodic absolutely continuous f_I-invariant measure μ_I. The basic idea is to induce from μ_I a probability on the Borel σ-algebra of R using the projection $\pi : R \to I$ on I along $\mathcal{F}^s$. It is important to note that F-stable foliation and F_R-stable foliation coincide and that π semi-conjugate the dynamics of F_R with the dynamics of f_I, that is $\pi : R \to I$ is continuous onto I and $f_I \circ \pi = \pi \circ F_I$. Denote by $\mathbb{B}$ the Borel σ-algebra of I. We will define a sequence of probabilities $\{\mu_n\}$ as follows: given $B \in \mathbb{B}$ define $\mu_0(E) = \mu_I(B)$, where $E = \pi^{-1}(B)$

$$\mu_1(E) = \mu_0(F_R^{-1}E),\ E \in \mathbb{A}_1 = F_R(\pi^{-1}\mathbb{B})$$

$$\mu_2(E) = \mu_0(F_R^{-2}E),\ E \in \mathbb{A}_2 = F_R^2(\pi^{-1}\mathbb{B})$$

.

.

1) $\{\mathbb{A}_n\}_{n\geq 0}$ is a nested sequence of σ-algebras.
2) For each $n \geq 0$, μ_n is a F_R-invariant probability on $\mathbb{A}_n$ and $\mu_{n+1}|_{\mathbb{A}_n} = \mu_n$.
3) Define $\mu_\infty(E) = \mu_n(E)$ if $E \in \mathbb{A}_n$, μ_∞ is a finitely additive probability on the algebra $\mathbb{A} = \bigcup_{n\geq 0} \mathbb{A}_n$. It is a delicate problem to see that μ_∞ really is additive. Extend μ_∞ to a probability μ on $\tilde{\mathbb{A}}$ the smallest σ-algebra that

contains $\mathbb{A}$. To finish the construction of μ it has to be proved that $\tilde{\mathbb{A}}$ is (Mod 0) the Borel σ-algebra of R.
μ is a physical measure for F_R such that the basin of μ has full Lebesgue measure, in fact, F_R is Bernoulli with respect to this measure. This is basically the ergodic theory of Anosov diffeomorphisms, in our case we take advantage of good understanding of projection along $\mathcal{F}^s$ and the possibility of define an uniformly hyperbolic diffeomorphism in a bounded region of $\mathbb{R}^2$. The fundamental hypothesis are: (1) existence of an unidimensional ergodic abolutely continuous probability with respect to an unidimensional alternating systems and (2) absolute continuity of the stable foliation.

Corollary 2.4.1 *Let $F : \mathbb{R}^2 - \{y = 0\} \longrightarrow \mathbb{R}^2 - \{x = 0\}$ be a symmetric affine alternating system (symmetric means $F(p) = -F(-p)\ \forall p \in \mathbb{R}^2$). If R is the rectangle bounded by the two horizontal lines $F^{-1}(\{y = 0\})$ and the two vertical lines $F(\{x = 0\})$ then the transformation $F_R : R \hookleftarrow$ induced by F on R admits a physical measure on the Borel σ-algebra of R.*

Proof: F is an expansive alternating system. Indeed, denote $l = \{x = 0\}$ (the y-axis). Consider $P_l : l \hookleftarrow$ the projection on l along horizontal lines, that is, $P_l = \pi_2 \circ F$ where $\pi_2(x, y) = (0, y)$. Observe that P_l is an unidimensional symmetric affine alternating system. By the unidimensional theory (Chapter 1) the subset of P_l-inverse images of zero is dense in l, in other words the subset of F-inverse images of x-axis is dense in $\mathbb{R}^2$. Similarly, projecting along vertical lines we obtain that F-images of y-axis is a dense subset of $\mathbb{R}^2$. Both things imply that the subset of directive lines is dense in $\mathbb{R}^2$ and so F is expansive. On another hand, F satisfies the hypothesis of Lemma 2.4.1, in fact the stable foliation $\mathcal{F}^s$ is the subset of horizontal lines and the projection P_l on l along $\mathcal{F}^s$ is an unidimensional symmetric affine alternating system. Therefore (due to the unidimensional theory) the Lebesgue measure is an ergodic P_l-invariant probability. This completes the proof.

Chapter 3

Questions and further work

(1) Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a C^2 expansive alternating system such that f is convex in $(-\infty, 0)$ and concave in $(0, +\infty)$. Is f ergodic with respect to Lebesgue measure ? (A way to answer this question is to show that the transformation f_I, induced by f on the distributor interval I, has bounded distortion with respect to the partition $\mathbb{P}_{f_I}$ induced by f on I)

(2) If the answer of (1) is negative then add a hypothesis such that the statement becomes useful in practice.

Definition: Let $f : \mathbb{R} - \{P_1, P_2\} \longrightarrow \mathbb{R}$ be a non-singular transformation. We say that f is an alternating system if:
(AS1) $f^{-1}(0)$ consist of 3 numbers $\{x_0^-, r_0, x_0^+\}$ such that $x_0 < P_1 < r_0 < P_2 < x_0$.
$f^{-2}(0)$ consist of 3^2 numbers $\{x_1^-, r_1^1, u_1^-, P, Q, v_1^-, v_1^+, r_1^2, x_1^+\}$ such that:

$x_1^- < x_0^- < r_1^1 < u_1^- < P_1 < P < r_0 < Q < v_1^- < P_2 < v_1^+ < x_0^+ < r_1^2 < x_1^+$.

(Let us say that the 3^2 numbers of $f^{-2}(0)$ are monotonely between the previous numbers $\{x_0^-, P_1, r_0, P_2, x_0^+\}$) Successively, $f^{-n}(0)$ consist of 3^n numbers monotonely between the previous ones.
(AS2) The subset $\bigwedge_f = \{x \in \mathbb{R} : f^n(x), n \geq 1$, is smaller and greater than P_1 infinitely many times$\}$ has full Lebesgue measure.

(3) Let $C^\infty_{T,+}$ be the subset of smooth functions $f : \mathbb{R} - \{P_1, P_2\} \longrightarrow \mathbb{R}$ such that: (a) $f'(x) > 0\ \forall x$ (b) f is transitive.

If $f \in C^{\infty}_{T,+}$ then, is f an alternating system ?

Let $f : \mathbb{R} - \{P_1, P_2\} \longrightarrow \mathbb{R}$ be an alternating system. For every $n \geq 1$, take extreme numbers of $f^{-n}(0)$ (the smaller one and greater one) and their respective pre-images. This proceeding provides monotone sequences $\{x_n^{\pm}\}$, $\{u_n^{\pm}\}$, $\{v_n^{\pm}\}$ such that:
(a) $\forall n \geq 1$, $x_n^- < x_0^- < u_n^- < P_1 < u_{n+1}^+ < P < Q < v_n^- < P_2 < v_n^+ < x_0^+ < x_n^+$.
(b) $x_n^- \searrow -\infty$, $u_n^- \nearrow P_1$, $u_n^+ \searrow P_1$, $v_n^- \nearrow P_2$, $v_n^+ \searrow P_2$, $x_n^+ \nearrow +\infty$.
Item (a) gives a partition $\mathbb{P}_f$ of $\mathbb{R}$ (partition induced by f on $\mathbb{R}$). Say that an alternating system is affine if f is a segment of straight line in each atom of the partition $\mathbb{P}_f$ induced by f on $\mathbb{R}$.

(4) If f is an affine alternating system then, is f transitive, ergodic ? What is the dynamics of the induced transformation by f on $I = [x_0^-, P_1) \cup (P_1, P] \cup [P, Q] \cup [Q, P_2) \cup (P_2, x_0^+]$? In the non-linear case, look for conditions such that the transformation $f_I : I \hookleftarrow$ induced by f on I is expansive and ergodic.

(5) For each $n \geq 1$ denote by $AS(n)$ the subset of increasing alternating systems with n explosive discontinuities. Which elements in $C(n)$ do admit a σ-finite absolutely continuous invariant measure ?

Let $(X, \mathbb{B}, m)$ be a measure space such that m is a infinite, σ-finite measure and $T : X \hookleftarrow$ a non-singular transformation. In the context of [9], a set $W \subset X$ is called a wandering set (for T) if the sets $\{T^{-n}W\}_{n=0}^{\infty}$ are disjoint. The dissipative part of T, $\mathcal{D}(T) \subset X$, is the measurable union of wandering sets. The non-singular transformation T is called conservative if $m(\mathcal{D}(T)) = 0$.
Definition: A conservative, ergodic, measure preserving transformation T of $(X, \mathbb{B}, m)$ is called *rationally ergodic* if there is a set $A \in \mathbb{B}$, $0 < m(A) < \infty$, satisfying a *Renyi inequality*: $\exists M > 0$ such that

$$\int_A (\sum_{j=0}^{n-1} 1_A \circ T^j)^2 dm < M \left(\int_A \sum_{j=0}^{n-1} 1_A \circ T^j dm \right)^2, \forall n \geq 1.$$

If $T : (X, \mathbb{B}, m) \hookleftarrow$ is rationally ergodic then there is a sequence $\{a_n\} \nearrow \infty$ such that whenever $A \in \mathbb{B}$ satisfies a Renyi inequality,

(i) $\frac{1}{a_n} \sum_{k=0}^{n-1} m(B \cap T^{-k}C) \longrightarrow m(B)m(C)$ as $n \to \infty$ $\forall B, C \in \mathbb{B} \cap A$

and there exists $\{a_{n_k}\}$ a subsequence of $\{a_n\}$ such that

(ii) $\frac{1}{N}\sum_{k=1}^{N}\frac{1}{a_{n_k}}\sum_{j=0}^{n_k-1} f\circ T^j \longrightarrow \int_X f dm$ a.e as $N\to\infty$ $\forall f\in L^1(m)$.

(See [9] for details). The sequence $\{a_n\}$ defined by (i) is called a *return sequence* for T and sometimes denoted $a_n(T)$. A conservative, ergodic, measure preserving transformation satisfying (ii) is called *weakly homogeneous.*

(6) Let $f:\mathbb{R}-\{P_1,\cdots,P_n\}\longrightarrow\mathbb{R}$ be an <u>affine</u> alternating system with n explosive discontinuities. Is f a conservative, ergodic, measure preserving transformation with respect to some absolutely continuos σ-finite measure ? Does $I=[x_0^-,P_1)\cup(P_1,P_2)\cup\cdots\cup(P_n,x_0^+]$ (the distributor interval of f) satisfy a Renyi inequality ? If these questions have afirmative answers, compute a return sequence of f. Show non-linear examples of rationally ergodic alternating systems with n explosive discontinuities (Note that the Boole's transformation is rationally ergodic [2],[9]).

<u>Bidimensional alternating systems with n curves of discontinuities:</u>

(7) Let F_a be a bidimensional alternating system like in Theorem 2.2.3. Due to uniform contraction, is the stable foliation absolutely continuous ? C^1 ? C^2 ? Let $f:\mathbb{R}-\{0\}\longrightarrow\mathbb{R}$ be the projection on the discontinuity line $\{y=-ax\}$ along the stable foliation $\mathcal{F}^s$. Does the transformation f_I, induced by f on the distributor interval I (see Chapter 1), have bounded distortion ?

Denote by R_a the region bounded by the directive curves $F_a^{-1}(\{y=0\})$ and $F_a(\{y=-ax\})$. Due to the unidimensional theory and Lemma 2.4.1, an affirmative answer to this question says that there exists μ_{R_a} an ergodic physical measure for the transformation $F_{R_a}:R_a\hookleftarrow$ induced by F_a on R_a.

(8) Does $\mu_{R_a}\to\lambda$ when $a\to 1$, where λ is the Lebesgue measure on R? Here R is the region bounded by the directive curves $F^{-1}(\{y=0\})$ and $F(\{y=-x\})$ and F is the Hénon-Devaney map $F(x,y)=(x+\frac{1}{y},y-\frac{1}{y}-x)$. Is the limit λ ergodic ?

Let $\gamma_1,\gamma_2,\cdots,\gamma_n:\mathbb{R}\to\mathbb{R}^2$ be horizontal curves and $\delta_1,\delta_2,\cdots,\delta_n:\mathbb{R}\to\mathbb{R}^2$ vertical curves such that γ_i intersects δ_i in a single point, $i=1,\cdots,n$.
<u>Definition</u>: Let $F:\mathbb{R}^2-\{\gamma_1,\gamma_2,\cdots,\gamma_n\}\longrightarrow\mathbb{R}^2-\{\delta_1,\delta_2,\cdots,\delta_n\}$ be continuous, inversible, C^∞ except possibly in a set of zero Lebesgue measure. Say

that F is an alternating system if:
(AS1) For each $n \geq 1$, $F^{-n}(\{\gamma_1, \cdots, \gamma_n\})$ are horizontal curves monotonely between the previous $F^{-n+1}(\{\gamma_1, \cdots, \gamma_n\})$. Similarly, $F^n(\{\delta_1, \cdots, \delta_n\})$ are vertical curves monotonely between the previous $F^{n-1}(\{\delta_1, \cdots, \delta_n\})$.
(AS2) The subset $\bigwedge_F = \{p \in \mathbb{R}^2 : F^n p$ exists $\forall n \in \mathbb{Z}$ and $F^n p$ is over and below the curve γ_1 infinitely many times$\}$ has full Lebesgue measure.

(9) Let $F : \mathbb{R}^2 - \{\gamma_1, \cdots, \gamma_n\} \longrightarrow \mathbb{R}^2 - \{\delta_1, \cdots, \delta_n\}$ be an increasing diffeomorphism such that $\mathbb{R}^2$ admits transversal F-invariant foliations. If F is transitive then, is F an alternating system ?

Let us construct an affine alternating system with two lines of discontinuities. Take two straight lines, say $\gamma_i = \{y = a_i\}$, $\delta_i = \{x = b_i\}$, where a_i, b_i $(i = 1, 2)$ are real numbers such that $a_1 < a_2$ and $b_1 < b_2$. Denote:

$A_1 = \{(x, y) : y < a_1\}$, $B_1 = \{(x, y) : x < b_1\}$
$A_2 = \{(x, y) : a_1 < y < a_2\}$, $B_2 = \{(x, y) : b_1 < x < b_2\}$
$A_3 = \{(x, y) : y > a_2\}$, $B_3 = \{(x, y) : x > b_2\}$

Define $F : \mathbb{R}^2 - \{\gamma_1, \gamma_2\} \longrightarrow \mathbb{R}^2 - \{\delta_1, \delta_2\}$ affinely such that:

$A_1 \to B_1$, $A_2 \to B_2$, $A_3 \to B_3$

(See the construction of an affine bidimensional alternating system with a line of discontinuity).

(10) Are affine bidimensional alternating systems with n lines of discontinuities transitive? Let R be the rectangle bounded by the discontinuity lines $\{y = a_1\}$, $\{y = a_n\}$ and any two vertical lines, then is the Lebesgue measure with respect to the Borel σ-algebra of R ergodic ? Show a non-linear example of a bidimensional expansive alternating system with n curves of discontinuities.
Is $(x, y) \to (x + \frac{1}{y-a_1} + \cdots + \frac{1}{y-a_n}, y - \frac{1}{y-a_1} - \cdots - \frac{1}{y-a_n} - x)$ an expansive altenating system with n lines of dicontinuities ?

Fractal analysis of alternating systems:
Let $EAS(n, m)$ be the set of n-dimensional expansive alternating systems $(n = 1, 2)$ with m discontinuities. Take $f \in EAS(n, m)$ and a small open neighbourhood $U \in$Dom(f). Due to expansivity, f carries a symbolic dynamics, so $\bigcup_{n=0}^{\infty} f^{-n}U$ is an open and dense set. Denote by $\Lambda(f)$ the complement of this set.

(11) Construct an alternating system such that $\Lambda(f)$ is not bounded and has positive Lebesgue measure.

Let us define the Hausdorff dimension of a set $F \subset \mathbb{R}^n$ and the Hausdorff dimension of a Borel measure on $\mathbb{R}^n$. The diameter of a set $U \subset \mathbb{R}^n$ is denoted by $|U|$. If $\{U_i\}$ is a countable collection of sets of diameter at most $\delta > 0$ that cover F, i.e $F \subset \bigcup_{i=1}^{\infty} U_i$ with $|U_i| < \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F. Given $t \geq 0$, define the t-dimensional Hausdorff measure of F as

$$H^t(F) = \liminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} |U_i|^t : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

It is not difficult to see that there is a unique value t_0 such that $H^t(F) = \infty$ if $t < t_0$ and $H^t(F) = 0$ if $t > t_0$. Define the Hausdorff dimension of F, written $\dim_H F$, as being this value t_0. Let μ be a Borel σ-finite measure on $\mathbb{R}^n$. Define the Hausdorff dimension of μ as:

$$\dim_H \mu = \inf \left\{ \dim_H F : \mu(F^c) = 0 \right\}.$$

(12) Due to expansivity f is topologically conjugate to a shift. For which $f \in EAS(n,m)$ there exists μ, a measure induced by this conjugation, such that $\dim_H \mu = n$?

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