Instituto Nacional de Matemática Pura e Aplicada-IMPA

Competing Growth, Interfaces and Geodesics in First-Passage Percolation Models on Voronoi Tilings

Phd Thesis

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Contents

"Mesmo que eu fale em línguas,

a dos homens e a dos anjos,

se me falta o amor,

sou um metal que ressoa,

um címbalo retumbante.

Mesmo que tenha o dom da profecia,

o saber de todos os mistérios e de todo o conhecimento,

mesmo que tenha a fé mais total,

a que transporta montanhas,

se me falta o amor,

nada sou."

Paulo [Primeira Epístola aos Coríntios, versículo 13].

Aos meus pais,

José e Jussara.

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Chapter 1

Introduction

"In its efforts to learn as much as possible about nature, modern physics has found that certain things can never be known with certainty. Much of our knowledge must always remain uncertain. The most we can know is in terms of probabilities."

Richard P. Feynman [The Feynman Lectures on Physics].

Since its inception, first-passage percolation theory has become a major tool to describe the growth of an interface in many physical and biological situations (see [20]). In this work we consider two-dimensional stochastic growth processes that are defined in terms of first-passage percolation models on random graphs. We begin by introducing the single spatial growth process based on the notion of first-passage time and answering some questions that are known for first-passage percolation models on some periodic graphs but were unanswered for the random graph case. These questions are related to the compactness of the limit set (Theorem 1.3 and Corollary 1.1) and to the speed of convergence to the time constant (Theorems 1.4 and 1.5). The proof of the first result relies on some renormalization ideas while the proof of the second one is based on the method of bounded increments, considered by Kesten (see [18]) to study the fluctuations in the square lattice context.

After that, we consider a competing spatial growth model by permitting more than one cluster to develop over the same environment, where each Voronoi tile is acquired by the cluster which first arrives there. In the literature, Häggstrom and Pemantle (see [10])) considered the $\mathbf{Z}^2$ lattice case with exponential passage times and proved that when $k = 2$ coexistence (mutual unbounded growth between the clusters) occurs with positive probability for all initial configuration of seeds. Later Garet and Marchand (see [8]) extended this result for ergodic $\mathbf{Z}^d$ lattice FPP models. Derrida and Dickman (see [7]) performed numerical simulations to predict the value of the roughening exponent ζ of the "competition interface", namely the one-dimensional boundary between the clusters, in the two-dimensional square =lattice starting from a deterministic corner of angle $\alpha \in (0, 2\pi)$. They found exponents $\zeta = 1/3$, $\zeta = 2/3$ and $\zeta = 1$ for angles $\alpha \in (0, \pi)$, $\alpha = \pi$ and $\alpha \in (\pi, 2\pi)$ respectively. The values 1/3 and 2/3 are in concordance with others random interfaces, but $\zeta = 1$ looks incompatible with the observed intrinsic roughness of the competition interface. Indeed for $\alpha \in (0, \pi]$ the competition interface showed a deterministic orientation and the roughening exponent was measured around this orientation, but for $\alpha \in (\pi, 2\pi)$ they claimed that the measurable value might not reflect the intrinsic roughness of the competition interface because it seemed to have a random orientation. Here we show that the probability of coexistence goes to one as the distance between the initial seeds goes to infinity, and that every branch of the competition interface has a random orientation (Theorems 1.6, 1.7 and 1.8).

To study the structure of this interface we use the notion of geodesic in first-passage percolation. Existence and coalescence of semi-infinite geodesics with asymptotic directions are obtained by following the same method due to Licea, Piza and Newman (see [25],[21],[26]) to obtain the same result for first-passage percolation models on the two dimensional square lattice (Theorems 1.9, 1.10 and 1.11). We study the coalescence function related to the coalescence property, and obtain same asymptotic results that were conjectured by Howard and Newman (see [14]) in the Euclidean first-passage percolation case (Theorem 1.12).

Finally, we introduce some exponents that are related to the roughness (fluctuations) of the single growing cluster, of geodesics and of the "competition interface", and derive some scaling relations between them (Theorem 1.13).

1.1 Voronoi first-passage percolation

The Voronoi first-passage percolation model first appear in the work of Vahidi-Asl and Wierman in 1987 (see [29]). In order to introduce this model, let $\mathcal{V} \in \mathbf{R}^2$ be the set of points realized in a two-dimensional homogeneous Poisson point process with intensity 1 (see [6]). In such a process, the number of points in a Borel set $B \subseteq \mathbf{R}^2$ is a Poisson random variable with mean $l(B)$, where $l(B)$ denotes the Lebesgue measure (area) of B. $\mathcal{V}$ is then a countable set of points, such that each compact set in $\mathbf{R}^2$ contains only a finite subset of $\mathcal{V}$.

For each $v \in \mathcal{V}$ there is a corresponding polygonal region C_v defined by

$$C_v \equiv \{x \in \mathbf{R}^2;\ |x - v| \leq |x - \tilde{v}|\ \forall\, \tilde{v} \in \mathcal{V}\}.$$

The family $\{C_v;\ v \in \mathcal{V}\}$ is called the Voronoi tiling of the plane based on $\mathcal{V}$, and each polygonal region C_v is called the Voronoi cell (or tile) located at v. On this family of cells we define two graph structures. In the first one, denoted by

$$\mathcal{G} \equiv (\mathcal{V}, \mathcal{E}),$$

the vertex set $\mathcal{V}$ is exactly the set $\mathcal{V}$ given before and the edge set $\mathcal{E}$ is determined by pairs of vertices $(v, \tilde{v})$ such that C_v and $C_{\tilde{v}}$ share a common edge in their boundaries. With probability one, all faces of $\mathcal{G}$ are triangles. The second one, denoted by

$$\mathcal{G}^* \equiv (\mathcal{V}^*, \mathcal{E}^*),$$

consist of the vertices and edges of the polygonal regions C_v. Notice that for each $v^* \in \mathcal{V}^*$ we can also define the Voronoi cell located at v^* by using the same procedure we have done for $v \in \mathcal{V}$, and that the vertices in $\mathcal{V}^*$ are the circuncenters of the triangles in $\mathcal{G}$ (see [24]). Moreover, the edges of $\mathcal{G}^*$ may be seen as line segments of the perpendicular bisectors of the edges of $\mathcal{G}$ and vice-versa. This establishes a one-to-one correspondence,between the edges of $\mathcal{G}$ and $\mathcal{G}^*$

$$* : e \in \mathcal{E} \to e^* \in \mathcal{E}^*,$$

which express the duality between $\mathcal{G}$ and $\mathcal{G}^*$ as planar graphs. The random graphs $\mathcal{G}$ and $\mathcal{G}^*$ are called the Delaunay triangulation and the Voronoi tessellation respectively (see Figure 1.1).

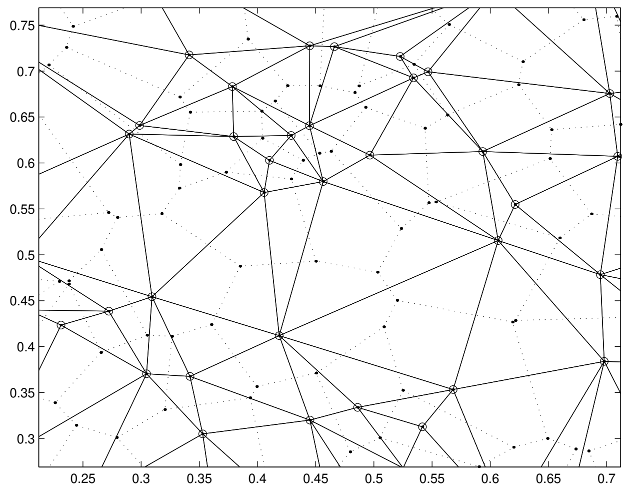

Figure 1.1: The Voronoi tessellation and the Delaunay triangulation.

Given an arbitrary graph $G \equiv (V, E)$, we say that γ is a path in G if γ is an ordered sequence of vertices, $\gamma \equiv (v_n)_I$, where I is either equal to $\{1, ..., m\}$, for some $m \geq 1$, or $\mathbf{N} \equiv \{1, 2, ...\}$ (the set of natural numbers) or $\mathbf{Z}$ (the set of integer numbers), so that for all $n \in I$, $(v_n, v_{n+1}) \in E$ whenever $n+1 \in I$. Define that $\gamma \equiv (v_n)_I$ is a path connecting v to $\tilde{v}$ if $I = \{1, ..., m\}$ and $v = v_1, v_m = \tilde{v}$. A path $\gamma \equiv (v_n)_I$ is a self avoiding path if $v_n \neq v_{\tilde{n}}$ whenever $n \neq \tilde{n}$, and is a circuit if $I = \{1, ...m\}$, $(v_2, ..., v_{m-1})$ is a self avoiding path and $v_1 = v_m$. We denote by $v \in \gamma$ or by $e \in \gamma$ if there is $n \in I$ so that $v_n = v$ in the first case, or if there is $n \in I$ so that $n+1 \in I$ and $e = (v_n, v_{n+1})$ in the second case. We also denote by $\tilde{\gamma} \subseteq \gamma$ if $v, e \in \gamma$ for all $v, e \in \tilde{\gamma}$. If G is a planar graph embedded in $\mathbf{R}^2$ and $B, \tilde{B} \subseteq \mathbf{R}^2$, define that a path $\gamma \equiv (v_n)_I$ in G is a path connecting B to $\tilde{B}$, if $I = \{1, ..., m\}$, $C_{v_1} \cap B \neq \emptyset$ and $C_{v_m} \cap \tilde{B} \neq \emptyset$.

To introduce the first-passage percolation model on $\mathcal{G}$, let $\{\tau_e;\ e \in \mathcal{E}\}$ be a set of i.i.d. non negative random variables with common distribution F and independent of the Poisson point process which determines $\mathcal{G}$. For each finite path γ in $\mathcal{G}$ define,

$$t(\gamma) \equiv \sum_{e \in \gamma} \tau_e.$$

The pair $(\mathcal{G}, (\tau_e)_{\mathcal{E}})$ can be thought of as a model for the spread of some fluid through a porous random medium. The medium is modeled by the edges of $\mathcal{G}$ and τ_e represent the passage time along the edge e. With this physical interpretation in mind, $t(\gamma)$ is called the passage time along the path γ and F is called the passage time distribution. Given $v, \tilde{v} \in \mathcal{V}$, the first-passage time from v to $\tilde{v}$ is defined by

$$T(v, \tilde{v}) \equiv \inf\{t(\gamma);\ \gamma \in \mathcal{C}(v, \tilde{v})\},$$

where $\mathcal{C}(v, \tilde{v})$ is the set of all self avoiding paths connecting v to $\tilde{v}$. Notice that if $\tau_e \equiv 1$ for all $e \in \mathcal{E}$, then T represents the graph metric over $\mathcal{G}$. Generally, T is a pseudo-metric over $\mathcal{G}$. It is symmetric,

$$T(v, \tilde{v}) = T(v, \tilde{v}),$$

and satisfies the triangle inequality,

$$T(v, \tilde{v}) \leq T(v, \hat{v}) + T(\hat{v}, \tilde{v}).$$

It is called a pseudo-metric because there may exist vertices with $T(v, \tilde{v}) = 0$ but $v \neq \tilde{v}$.

The symmetry can be deduced directly from the definition of T. The triangle inequality is a consequence of the following. Let $v, \hat{v}, \tilde{v} \in \mathcal{V}$ and $\epsilon > 0$. Then there exists $\gamma^1 \in \mathcal{C}(v, \hat{v})$

and $\gamma^2 \in \mathcal{C}(\hat{v}, \tilde{v})$ so that

$$t(\gamma^1) + t(\gamma^2) \leq T(v, \hat{v}) + T(\hat{v}, \tilde{v}) + \epsilon.$$

However, for $i = 1, 2,$

$$\gamma^i = (v_1^i, v_2^i, ..., v_{m_i}^i)$$

where $v_1^1 = v, v_{m_1}^1 = \hat{v} = v_1^2$ and $v_{m_2}^2 = \tilde{v}$. This implies that

$$\gamma = (v_1^1, ..., v_{m_1}^1, v_2^2, ..., v_{m_2}^2) \in \mathcal{C}(v, \tilde{v}).$$

By the definition of T and the last inequality,

$$T(v, \tilde{v}) \leq t(\gamma) = t(\gamma^1) + t(\gamma^2) \leq T(v, \hat{v}) + T(\hat{v}, \tilde{v}) + \epsilon.$$

Since this is valid for all $\epsilon > 0$, we obtain the triangle inequality.

To extend the definition of T for all pair of points in the plane notice that, for all fixed $x \in \mathbf{R}^2$, almost surely there exists a unique $v \in \mathcal{V}$ so that x belongs to the interior of C_v. Denote this vertex by v_x and for $x, y \in \mathbf{R}^2$ define the first-passage time from x to y by

$$T(x, y) \equiv T(v_x, v_y).$$

An important property of the Voronoi first-passage percolation model is the Euclidean invariance of the law of T: for every rigid motion

$$S : x \to S(x) = A(x) + z,$$

where A is a rotation of the plane and $z \in \mathbf{R}^2$, the law of $T(x, y)$ is equal to the law of $T(S(x), S(y))$.

From now on we denote the probability space which defines the Voronoi first-passage percolation model by $(\Omega, \mathcal{F}, P)$ and the pair

$$(\mathcal{G}, (\tau_e)_{\mathcal{E}}) \equiv (\mathcal{G}(\omega), (\tau_e(\omega))_{\mathcal{E}(\omega)})$$

is a function of $\omega \in \Omega$.

Consider the following construction of $(\Omega, \mathcal{F}, P)$ (except in Chapter 3). Let $z_1, z_2, ...$ be a spiral ordering of $\mathbf{Z}^2$, and for each $k \geq 1$ define

$$B_k \equiv z_k + [-1/2, 1/2]^2.$$

Consider

$$\{N_k;\ k \geq 1\},$$

a collection of i.i.d. Poisson random variables with intensity 1,

$$\{\tau_{k,l}^{m,n};\ k \geq 1, l \geq 1, m \geq k, n \geq 1 \text{ and } n > l \text{ whenever } k = m\},$$

a collection of i.i.d. non negative random variables with common distribution F (the passage time distribution), and

$$\{U_{k,l};\ k \geq 1, l \geq 1\},$$

a collection of independent random points in the plane so that $U_{k,l}$ has an uniform distribution in the square box B_k. We also impose that all these collections are independent of each other.

To determine the point configuration $\mathcal{V}$, at each square box B_k we put N_k points given by $U_{k,1}, ..., U_{k,N_k}$. This procedure determines a Poisson point process from the collection $\{N_k, U_{k,l};\ k, l \geq 1\}$. Given $\mathcal{V}$, for each $e \in \mathcal{E}$ we know that there exist an unique pair $(U_{k,l}, U_{m,n})$, where either $m > k$ or $m = k$ and $n > l$, so that

$$e = (U_{k,l}, U_{m,n}).$$

Therefore we define

$$\tau_e = \tau_{k,l}^{m,n}.$$

Notice that for each $k \geq 1$ we can see the probability space where $N_k, \{U_{k,l};\ l \geq 1\}$ and $\{\tau_{k,l}^{m,n};\ l \geq 1, m \geq k, n \geq 1 \text{ and } n > l \text{ whenever } k = m\}$ are defined as

$$(\Omega_k, \mathcal{F}_k, P_k),$$

where

$$\Omega_k \equiv \mathbf{N} \times (B_k)^\infty \times \mathbf{R}^\infty.$$

The probability space $(\Omega, \mathcal{F}, P)$ is defined by the product space of $(\Omega_k, \mathcal{F}, P_k)$ for $k \geq 1$.

For each $B \subseteq \mathbf{R}^2$, we will denote the Delaunay triangulation of the plane based on $\mathcal{V}(B) \equiv \mathcal{V} \cap B$ by

$$\mathcal{G}(B) \equiv (\mathcal{V}(B), \mathcal{E}(B)).$$

This means that to each $v \in \mathcal{V}(B)$ we define the Voronoi cell restrict to the knowledge of vertices inside B by

$$C_v(B) \equiv \{x \in \mathbf{R}^2;\ |x - v| \leq |x - \tilde{v}|\ \forall\, \tilde{v} \in \mathcal{V}(B)\},$$

and the edge set $\mathcal{E}(B)$ is determined by pair of vertices $(v, \tilde{v})$ in $\mathcal{V}(B)$ so that $C_v(B)$ and $C_{\tilde{v}}(B)$ share a common edge in their boundaries. The passage times $\{\tau_e;\ e \in \mathcal{E}(B)\}$ is determined by the same procedure as we have done before. Notice that $\mathcal{V}(B)$ is always a subset of $\mathcal{V}$ and for all $v \in \mathcal{V}(B)$,

$$C_v \subseteq C_v(B).$$

Although, $\mathcal{E}(B)$ may have edges which are not in $\mathcal{E}$. By the above construction, if B and $\tilde{B}$ are two disjoint regions in the plane, then the law of the configuration $(\mathcal{G}(B), \{\tau_e\}_{e \in \mathcal{E}(B)})$ is independent of the law of the configuration $(\mathcal{G}(\tilde{B}), \{\tau_e\}_{e \in \mathcal{E}(\tilde{B})})$. Nevertheless, the model determined by $(\mathcal{G}, \{\tau_e\}_{e \in \mathcal{E}})$ has long-range dependence due to arbitrarily large Voronoi cells.

In the course of the text we will consider several times two bond percolation models. The first one, known as the bond percolation model on the Delaunay triangulation, is defined as follows: for each realization of the Poisson point process that determines the graph $\mathcal{G} \equiv (\mathcal{V}, \mathcal{E})$ we consider a collection of i.i.d. Bernoulli random variables with parameter $p \in [0, 1]$ given by $\{X_e;\ e \in \mathcal{E}\}$. We impose that this collection is independent of the Poisson point process which determines the graph structure and denote the law of this model by P_p. The second one, known as the bond percolation model on the Voronoi tessellation, is defined similarly:

for each realization of the Poisson point process that determines the graph $\mathcal{G}^* \equiv (\mathcal{V}^*, \mathcal{E}^*)$ we consider a collection of i.i.d. Bernoulli random variables with parameter $p \in [0,1]$ given by $\{X_e^*;\ e^* \in \mathcal{E}^*\}$. We also impose that this collection is independent of the Poisson point process which determines the graph structure and denote the law of this model by P_p^*. In both models it is a common sense to declare that an edge is open, if the random variable which is indexed to this edge takes the value one, and closed otherwise. The probability law that we are considering in both cases refers to the randomness of the graph structure and also to the randomness of the state of an edge (open or closed).

Denote the set of vertices in $\mathcal{G}$ that can be connected to the cell containing the origin by an open path (a path which only pass through open edges) by W_0, and define

$$p_{cb} \equiv \inf\{p > 0;\ P_p(\#(W_0) = +\infty) > 0\}. \tag{1.1}$$

We say that a path γ^* in $\mathcal{G}^*$ is a LR^* (left-right) open crossing of the rectangle $B = [k,l] \times [\tilde{k}, \tilde{l}]$ if $\gamma^* \equiv (v_n^*)_I$ ($I = \{1, ..., m\}$) is an open self avoiding path connecting $\{k\} \times [\tilde{k}, \tilde{l}]$ to $\{l\} \times [\tilde{k}, \tilde{l}]$ and $v_n^* \in B$ for all $n = 2, ..., m-1$ (see Figure 1.2). For each $n \geq 1$ define the

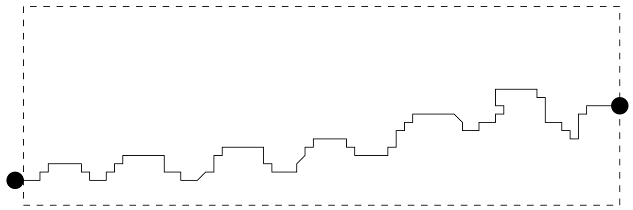

Figure 1.2: A LR^* open crossing.

event

$$A_n^* \equiv [\exists \text{ a } LR^* \text{ open crossing of } [0,3n] \times [0,n]],$$

and the critical probability

$$\hat{p}_{cb}^* \equiv \inf\{p > 0;\ \liminf_{n \to +\infty} P_p^*(A_n^*) = 1\}. \tag{1.2}$$

It is known that $p_{cb}, \hat{p}^*_{cb} \in (0,1)$ (see [2], [32]) and by the duality relation between $\mathcal{G}$ and $\mathcal{G}^*$ it is conjectured that

$$p_{cb} + \hat{p}^*_{cb} = 1.$$

In Section 2.3 we prove that $1 \leq p_{cb} + \hat{p}^*_{cb}$ by relating these critical probabilities to the compactness of the limit set (see Proposition 2.3).

1.2 Single spatial growth

Based on the idea of a fluid which is spreading through a random medium, let us imagine the following picture. Suppose that the cell which contains the origin is a source that is injecting a black fluid in the medium. Consider the set of cells (vertices) that can be reached by the fluid until time t defined by

$$\tilde{G}_0(t) \equiv \{v \in \mathcal{V};\ T(v_0, v) < t\},$$

and its fattened version (see Figure 1.3),

$$G_0(t) \equiv \cup_{v \in \tilde{G}_0(t)} C_v.$$

The process $\{G_0(t)\}_{t \geq 0}$ defines a two dimensional stochastic growth model which satisfies the following properties almost surely:

- $G_0(s) \subseteq G_0(t)\ \forall s \leq t$;
- $G_0(t)$ is a connect region in the plane;
- $\lim_{t \to +\infty} G_0(t) = \mathbf{R}^2$.

Indeed, the first and the second properties are immediate consequences of the definition of $G_0(t)$. The third one is a consequence of the fact that, almost surely, $\mathcal{V}$ is locally finite, and this implies that for all $v \in \mathcal{V}$, $T(v_0, v) < +\infty$.

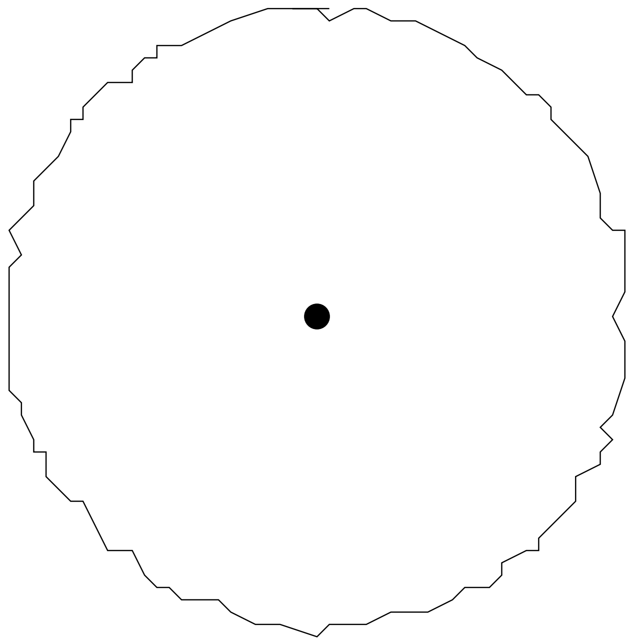

Figure 1.3: A single growing cluster.

The first aim when we have a stochastic growth model $\{G_0(t)\}_{t>0}$ is to find conditions that ensure the convergence of the rescaled set $t^{-1}G_0(t)$. In order to do so, we shall first look carefully to the variable T involved in the definition of $G_0(t)$. The key property of T that allow us to deduce something about the long time behavior of $G_0(t)$ is the sub-additivity of its expected value. In fact, let $\vec{e}_1 \equiv (1,0)$ and $\vec{x} \in S^1 \equiv \{x; |x| = 1\}$. By the Euclidean invariance, $E_n \equiv ET(0, n\vec{x})$ does not depend on $\vec{x}$ for all $n \geq 1$. By the triangle inequality, if $ET(0, \vec{e}_1) < +\infty$, $(E_n)_{n\geq 1}$ defines a sub-additive sequence of non-negative real numbers so that

$$\lim_{n\to+\infty} \frac{E_n}{n} = \mu(F) \equiv \inf_{n\geq 1}\{\frac{E_n}{n}\} \leq E_1 < +\infty. \tag{1.3}$$

Moreover, it is not hard to prove that for all $x \in \mathbf{R}^2$

$$\frac{ET(0, nx)}{n} \to \mu|x|, \text{ as } n \to +\infty. \tag{1.4}$$

Motivated by equation 1.4 we would expect that, under some conditions on F, almost surely the rescaled first-passage time $r^{-1}T(0, rx)$ will also converge to $\mu|x|$ and $t^{-1}G_0(t)$, will converge to

$$B_0 \equiv \{x \in \mathbf{R}^2;\ |x| \leq 1/\mu\}. \tag{1.5}$$

This is given by the following two theorems proved by Vahidi-Asl and Wierman (see [29],[30]). The proof of the first theorem is based on the sub-additive ergodic theorem of Kingman (see [19]). For the second one, they used the same general structure present in the proof of the shape theorem for the lattice first-passage percolation model by Cox and Durret (see [5]). Let τ_j, for $j = 1, 2, 3$, be three independent passage times with common distribution F.

Theorem 1.1 (Ergodic Theorem) *$E(\min\{\tau_1, \tau_2, \tau_3\}) < +\infty$ is a sufficient and necessary condition to ensure that there exist $\mu(F) \in [0, \infty)$, given by 1.3, such that almost surely*

$$\lim_{r\to\infty} \frac{T(0, rx)}{r} = \mu|x|.$$

Theorem 1.2 (Shape Theorem) *Assume that $E(\min\{\tau_1, \tau_2, \tau_3\}) < +\infty$. Then:*

- *if $\mu(F) > 0$, $E(\min\{\tau_1, \tau_2, \tau_3\}^2) < +\infty$ is a sufficient and necessary condition to ensure that there exists an Euclidean ball B_0, given by 1.5, such that for all $\epsilon > 0$ almost surely*

$$(1 - \epsilon)B_0 \subseteq t^{-1}G_0(t) \subseteq B_0(1 + \epsilon) \text{ for all } t \text{ sufficiently large }.$$

- *if $\mu = 0$, then for every compact set K,*

$$P(K \subseteq t^{-1}G_0(t) \text{ for all } t \text{ sufficiently large }) = 1.$$

Although, Theorems 1.1 and 1.2 do not give a complete answer for the morphology of $G_0(t)$ because they do not establish when $\mu > 0$, or in the shape theorem, when B_0 is compact. The first result of this work is to set a sufficient condition on F to ensure that $\mu > 0$. Recall that we have defined in Section 1.1 the critical probability $\hat{p}^*_{cb}$ given by 1.2.

Theorem 1.3 *Suppose that $F(0) < 1 - \hat{p}^*_{cb}$, then there exist $C_1, C_2, C_3 > 0$ such that*

$$P(T(0, n\vec{e}_1) < C_1 n) \leq C_2 \exp(-C_3 n).$$

Moreover, we have the following corollary of Theorem 1.3.

Corollary 1.1 *If $F(0) < 1 - \hat{p}^*_{cb}$, then under the assumption of Theorem 1.1*

$$\mu(F) > 0.$$

Proof of Corollary 1.1. By Theorem 1.3 and the Borel-Cantelli lemma, if $F(0) < 1 - \hat{p}^*_{cb}$ then there exist $C_1 > 0$ so that almost surely

$$\liminf_{n\to\infty} \frac{T(0, n\vec{e}_1)}{n} \geq C_1.$$

But under the assumption of Theorem 1.1, almost surely the sequence $n^{-1}T(0, n\vec{e}_1)$ converge to $\mu(F)$ when n goes to infinity. Therefore,

$$0 < C_1 \leq \mu(F).$$

□

The proof of Theorem 1.3 is performed in Chapter 2, Section 2.2. To establish the exponential bound of short passage times, the Voronoi first-passage percolation model is compared with an auxiliary site (l-dependent) percolation model in $\mathbf{Z}^2$ via the renormalization technique. The difficulty with an approach of this type is that there necessarily exist arbitrarily large Voronoi cells and hence the model has long-range dependence. We also make use of the duality between $\mathcal{G}$ and $\mathcal{G}^*$ as planar graphs.

1.3 Fluctuations around the time constant

A natural question, which has received much attention from statistical physicists (see [20], [15]), is how rough is the boundary of the set $G_0(t)$? Or equivalently, how fast is the convergence $n^{-1}T(0, n\vec{e}_1) \to \mu$? Our result concerning these questions are the following.

Theorem 1.4 *There exists $q_0 \in (0,1)$ such that: if $F(0) < q_0$ and $\int e^{ax} F(dx) < +\infty$ then $\mu(F) > 0$ and: for all $\epsilon > 0$ there exists $n_0 > 0$ such that for all $n \geq n_0$,*

$$E(T(0,n) - ET(0,n))^2 \leq n^{1+\epsilon}; \tag{1.6}$$

for all $\kappa \in (1/2, 1)$ there exists $\nu_1, \nu_2 \in (0,1)$ and $C_i > 0$, such that for all $n \geq 1$ and for all $x \leq C_1 n^{\nu_1}$

$$P(|T(0,n) - ET(0,n)| \geq x n^{\kappa}) \geq C_3 \exp(-C_4 x^{\nu_2}), \tag{1.7}$$

and

$$\mu n \leq ET(0,n) \leq \mu n + C_5 n^{\kappa} (\log n)^{1/\nu_2}. \tag{1.8}$$

Moreover for any $\nu \in (0, \nu_1/2)$ there exists constants C_6, C_7 (depending on ν) such that for all $x \in [n^{\nu}, n^{\nu_1 - \nu}]$

$$P(|T(0,n) - \mu n| \geq x n^{\kappa}) \leq C_6 \exp(-C_7 x^{\nu_2}) \tag{1.9}$$

The proof of 1.6 and 1.7 follows the method of bounded differences that represents $T(0,n) - ET(0,n)$ as a sum of martingales differences. 1.8 and 1.9 are derived from 1.6 and 1.7. As a consequence of 1.9 we have the following improved version of Theorem 1.2.

Theorem 1.5 *If $F(0) < q_0$ and for some $a > 0$, $\int e^{ax} F(dx) < +\infty$ then for all $\kappa \in (1/2, 1)$, almost surely there exists $t_0 > 0$ such that for all $t > t_0$*

$$(t - t^{\kappa}) B_0 \subseteq G_0(t) \subseteq (t + t^{\kappa}) B_0. \tag{1.10}$$

1.4 Competing spatial growth and competition interfaces

Now let us imagine that we pick two different cells (vertices) representing two different sources, say v_1 and v_2, and at each source we inject a different fluid. Imagine that the fluid injected from v_1 is black and the fluid injected from v_2 is gray. We impose also that each cell can be occupied by only one fluid and it gets occupied in a finite time. When times goes to infinity a "competition interface" is formed by faces which are shared by cells with different colors. The shape theorem suggest us that we would expect some kind of regularity for the shape of this "competition interface".

To be more precise, assume that passage time distribution F is a continuous function to ensure that almost surely for all different vertices v and $\tilde{v}$ we have that $T(v,\hat{v}) \neq T(\tilde{v},\hat{v})$ for all $\hat{v}$. Denote by $v_{1,r} \equiv v_{r\vec{e}_1}$ and by $v_{2,r} \equiv v_{-r\vec{e}_1}$ (recall that $x \in C_{v_x}$). Let Ω_r^2 be the set of all configurations ω such that $v_{1,r} \neq v_{2,r}$. For each $v \in \mathcal{V}$, let $M(v) \equiv \min_{j=1,2}\{T(v_{j,r},v)\}$. For each $\omega \in \Omega_r^2$ and $j = 1,2$ define the growing cluster

$$\tilde{G}_{j,r}(t) \equiv \{v \in \mathcal{V};\ T(v_{j,r},v) < t \text{ and } M(v) = T(v_{j,r},v)\}.$$

For $\omega \in (\Omega_r^2)^c$ we define $\tilde{G}_{1,r}(t)$ as before and $\tilde{G}_{2,r}(t) \equiv \emptyset$ for all $t > 0$.

Consider its fattened version given by

$$G_{j,r}(t) \equiv \cup_{v \in \tilde{G}_{j,r}(t)} C_v.$$

As before, we have the following properties:

- $G_{j,r}(s) \subseteq G_{j,r}(t)$, $\forall s \leq t$ for $j = 1,2$;
- $G_{j,r}(t)$ is a connected region in the plane for $j = 1,2$;
- $\lim_{t\to+\infty} G_{1,r}(t) \cup G_{2,r}(t) = \mathbf{R}^2$.

Notice that if $\omega \in \Omega_r^2$ then for each fixed $t > 0$, $G_{1,r}(t) \cap G_{2,r}(t)$ is empty or is a finite union of polygonal curves determined by faces which are shared by cells with different colors. Furthermore, if we denote by

$$G_{j,r} \equiv \lim_{t\to+\infty} G_{j,r}(t),$$

then

$$\psi_r \equiv G_{1,r} \cap G_{2,r}$$

is a polygonal curve that we will call the competition interface between $G_{1,r}$ and $G_{2,r}$. Note that ψ_r may be seen as a path in $\mathcal{G}^*$. With this picture in mind we construct the following representation of ψ_r. For $x, y \in \mathbf{R}^2$, let denote the line segment connecting x and y as $[x, y]$. Let $e_0^* = (v_0^*, \tilde{v}_0^*)$ be the unique (almost surely) edge in $\mathcal{G}^*$ which is the closest edge in ψ_r to $r\vec{e}_1$ so that $[v_0^*, \tilde{v}_0^*]$ intersect $[-r\vec{e}_1, r\vec{e}_1]$ in exactly one point. We can assume that v_0^* lies above the line segment $[-r\vec{e}_1, r\vec{e}_1]$ and that $\tilde{v}_0^*$ lies below the line segment $[-r\vec{e}_1, r\vec{e}_1]$. Then define $x_1 \equiv v_0^*$ and $x_0 \equiv \tilde{v}_0^*$. For each $n \geq 0$, given $x_{-n}, x_{-(n-1)}, ..., x_0, x_1, ..., x_{n+1}$, we know that there exists an edge $e_n^* = (x_{n+1}, v_n^*)$ in $\mathcal{G}^*$ such that $[x_{n+1}, v_n^*] \subseteq \psi_r$ and $v_n^* \neq x_n$, and there exists an edge $e_{-n}^* = (x_{-n}, \tilde{v}_{-n}^*)$ in $\mathcal{G}^*$ such that $[x_{-n}, \tilde{v}_{-n}^*] \subseteq \psi_r$ and $\tilde{v}_{-n}^* \neq x_{-(n-1)}$. If $v_n^* \neq \tilde{v}_{-n}^*$ and $v_n^* \neq x_{-n}$, then define $x_{n+2} \equiv v_n^*$ and $x_{-(n+1)} \equiv \tilde{v}_{-n}^*$. Otherwise, re-define the index by keeping $x_1, x_2, ..., x_{n+1}$ as before and putting $x_{n+2} = \tilde{v}_{-n}^*, x_{n+3} = \tilde{v}_{-(n-1)}^*, ..., x_m = \tilde{v}_0^*$. Therefore, by induction we construct a representation of the interface, given by $\psi_r \equiv (x_i)_{n\in I}$, where the set I is either finite, if the interface is a closed polygonal curve, or equal to $\mathbf{Z}$, if the interface is an infinite polygonal curve. Note that $\{x_n;\ n \in I\}$ is exactly the set of vertices in the polygonal curve ψ_r and that $[x_n, x_{n+1}] \subseteq \psi_r$ whenever $n, n+1 \in I$. Observe also that if we move along the interface by following $x_n, x_{n+1}, ...$ then on the right hand side we always see the black color and on the left hand side we always see the gray color. We say that this representation of the interface is the black-gray representation. From now on we will always see the interface as a polygonal path with the black-gray representation. If $\omega \in (\Omega_r^2)^c$ we define that the interface is empty.

For each $A \subseteq \mathbf{R}^2$, define $\#_{\mathcal{V}}(A) \equiv \#(A \cap \mathcal{V})$, where $\#(B)$ is the number of points (cardinality) inside the set $B \subseteq \mathbf{R}^2$. It is easy to see that ψ_r is either empty or a closed polygonal curve if and only if $\min_{j=1,2}\{\#_{\mathcal{V}}(G_{j,r})\} < +\infty$. In fact, this possibility may happen if one of these two growing clusters completely surround the other one which then is prevented to grow indefinitely. We say that mutual unbounded growth occurs if

$$\min_{j=1,2}\{\#_{\mathcal{V}}(G_{j,r})\} = +\infty. \tag{1.11}$$

The process $\{G_{1,r}(t), G_{2,r}(t)\}_{t>0}$ may be seen as a competing spatial growth model. In order to understand the question of mutual unbounded growth and the geometrical structure of the competition interface it is convenient to introduce new objects related to polygonal paths. We consider only polygonal paths ψ which can be represent as an ordered sequence $(x_n)_I$, with $I \subseteq \mathbf{Z}$, of vertices in ψ with the property that the line segment connecting consecutive vertices is contained in ψ, and for every $x \in \psi$ there exist $n \in I$ so that $x \in [x_n, x_{n+1}]$. We also impose that I is either equal to $\{1, ..., m\}$, for some $m \geq 1$, or $\mathbf{N}$ or $\mathbf{Z}$. We refer to $(x_n)_I$ as a representation of ψ if it satisfies the above properties and also identify $\psi = (x_n)_I$. If $\#(I) < +\infty$, we say that ψ is a finite path, if $I = \mathbf{N}$ we say that ψ is a uni-path starting from x_1, and if $I = \mathbf{Z}$ we say that ψ is a bi-path. Note that in our context we can always see a path in $\mathcal{G}$ or in $\mathcal{G}^*$ as a polygonal path.

Let $\vec{x}, \vec{y} \in S^1$. We say that a uni-path $\psi = (x_n)_{\mathbf{N}}$ is a $\vec{x}$-path if

$$\frac{x_n}{|x_n|} \to_{n\to+\infty} \vec{x}.$$

We also say that a bi-path $\psi \equiv (x_n)_{\mathbf{Z}}$ is a $(\vec{x},\vec{y})$-path if $\psi^+ \equiv (x_n)_{\mathbf{N}}$ is a $\vec{x}$-path, and $\psi^- \equiv (y_n)_{\mathbf{N}}$ is a $\vec{y}$-path, where $y_n = x_{-n}$. The following questions are relevant. If mutual unbounded growth occurs would we expect that the competition interface is a $(\vec{x},\vec{y})$-path for some $\vec{x}$ and $\vec{y}$? What is the behavior of this interface when the distance between the initial vertices $v_{1,r}, v_{2,r}$ goes to infinity?

Theorem 1.6 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Then almost surely, or the competition interface ψ_r is a closed curve or there exist $\vec{x}_1, \vec{x}_2 \in S^1$ such that ψ_r is a $(\vec{x}_1, \vec{x}_2)$-path.*

With Theorem 1.6 in hands we are able to define the following random variables for almost all configurations. Recall that we had fixed the representation of the competition interface as the black-gray representation. Consider the parameterization of S^1 given by $\{\exp(i\theta);\ \theta \in (0, 2\pi]\}$, where i is the imaginary number. Let $\theta_{1,r}$ and $\theta_{2,r}$ be defined as follows:

- if ψ_r is either empty or a closed curve, we define $\theta_{1,r} = \theta_{2,r} = 0$;
- if ψ_r is a $(\vec{x}_1, \vec{x}_2)$-path, we define $\theta_{1,r} \in (0, 2\pi]$ so that $\vec{x}_1 = e^{i\theta_{1,r}}$ and $\theta_{2,r} \in (0, 2\pi]$ so that $\vec{x}_2 = e^{i\theta_{2,r}}$.

Theorem 1.7 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Then*

$$\lim_{r\to\infty} \theta_{1,r} = \frac{\pi}{2} \text{ in probability}$$

and

$$\lim_{r\to\infty} \theta_{2,r} = \frac{3\pi}{2} \text{ in probability.}$$

Theorem 1.6 is proved in Chapter 5, Section 5.2. Theorem 1.7 is proved in Chapter 5, Section 5.3. It is a corollary of Theorem 1.7 that the probability of the event that mutual unbounded growth occurs goes to 1 as the distance between the initial sources $v_{1,r}$ and $v_{2,r}$ goes to infinity.

Corollary 1.2 *Under the same assumption of Theorem 1.7,*

$$P(\min_{j=1,2}\{\#\mathcal{V}(G_{j,r})\} = +\infty) \to_{r\to+\infty} 1.$$

Proof of Corollary 1.2. Fix $\epsilon_0 \in (0, \pi/2)$ and notice that

$$|\theta_{1,r} - \pi/2| \le \epsilon_0 \Rightarrow \theta_{1,r} \ne 0.$$

But $\theta_{1,r} \ne 0$ implies that the competition interface is infinite and in particular, mutual unbounded growth will occur. Therefore

$$P(|\theta_{1,r} - \pi/2| \le \epsilon_0) \le P(\min_{j=1,2}\{\#\mathcal{V}(G_{j,r})\} = +\infty).$$

Thus Theorem 1.7 and the last inequality conclude the proof of Corollary 1.2.

$\square$

Now we introduce the competing spatial growth model for arbitrary $k \geq 2$. For each $j = 1, ..k$ define,

$$\vec{e}_j^k \equiv \exp(2i\pi(j-1)/k) \text{ and } v_{j,r} \equiv v_{r\vec{e}_j^k}.$$

Let Ω_r^k be the set of all configurations ω such that $v_{j,r} \neq v_{l,r}$ for all $j \neq l$. For each $v \in \mathcal{V}$, let $M(v) \equiv \min_{j=1,...,k}\{T(v_{j,r}, v)\}$. For each $\omega \in \Omega_r^k$ and $j = 1, ..., k$ define the growing cluster

$$\tilde{G}_{j,r}(t) \equiv \{v \in \mathcal{V}; T(v_{j,r}, v) < t \text{ and } M(v) = T(v_{j,r}, v)\}.$$

If $\omega \in (\Omega_r^k)^c$ and there is $j < l$ so that $v_{j,r} = v_{l,r}$ we will define $\tilde{G}_{j,r}(t)$ as before and $\tilde{G}_{l,r}(t) = \emptyset$ for all $t > 0$. Consider its fattened version,

$$G_{j,r}(t) \equiv \cup_{v \in \tilde{G}_{j,r}(t)} C_v.$$

Let

$$G_{j,r} \equiv \lim_{t \to +\infty} G_{j,r}(t)$$

and denote by J the set of j so that $G_{j,r} \neq \emptyset$. If $\#(J) \geq 2$, define the competition interface by

$$\psi_r \equiv \cap_{j \in J} G_{j,r}.$$

If $\#(J) = 1$, define that the competition interface is empty (see Figure 1.4).

Now the competition interface will have a different structure and we need to introduce a new object. We say that a polygonal path ψ is an interface branch if $\psi = (x_n)_{\mathbf{N}}$ is a uni-path and

$$\psi \subseteq \cap_{j \in J} G_{j,r}.$$

In Chapter 5 we will see that, if each $j \in \{1, ..., k\}$ represents a different color, then for every interface branch $\psi = (x_n)_{\mathbf{N}}$ there exist $j_1 \neq j_2$ and $n_0 \geq 1$ so that for all $n \geq n_0$, if we move along ψ by following $x_n, x_{n+1}, ...$ then in the right hand side we always see the color j_1 and

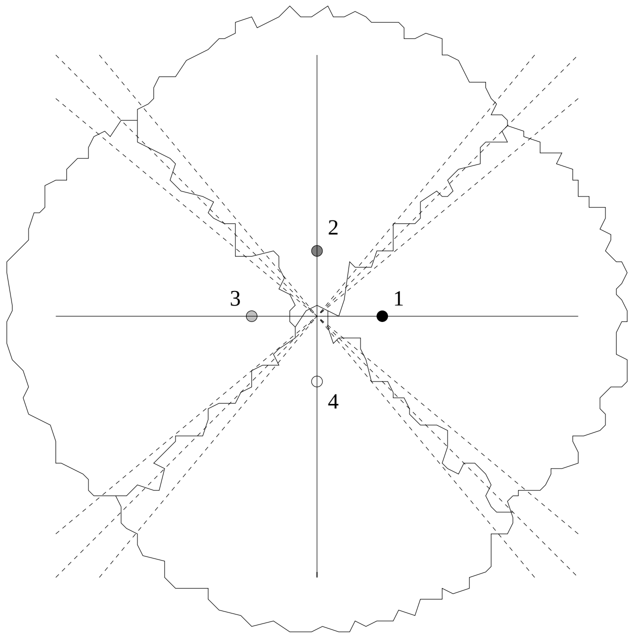

Figure 1.4: Four growing clusters.

in the left hand side the color j_2. As in the case of two growing clusters, would we expect that every interface branch of the competition interface is a $\vec{x}$-path for some $\vec{x}$?

We say that mutual unbounded growth occurs if

$$\min_{j=1,\dots,k}\{\#_{\mathcal{V}}(G_{j,r})\} = +\infty.$$

Would we expect that the event that mutual unbounded growth occurs has high probability for sufficiently distant initial vertices?

For each $s > 0$ and $\vec{x} \in S^1$, let

$$l^s_{\vec{x}} \equiv \{t\vec{x}; s \leq t\},$$

and for each $j = 1, \dots, k$ define

$$S^1_{j,r} \equiv \{\vec{x} \in S^1;\ \exists\, s > 0 \text{ so that } l^s_{\vec{x}} \subseteq G_{j,r}\}.$$

Given $\epsilon \in (0, \pi/k)$ let

$$S_1^1(\epsilon) \equiv \{e^{i\theta}; \theta \in ((2\pi - \pi/k) + \epsilon, 2\pi] \cup [0, \pi/k - \epsilon)\},$$

and for each $j = 2, ..., k$, let

$$S_j^1(\epsilon) \equiv \{e^{i\theta}; \theta \in ((2j\pi/k - \pi/k) + \epsilon, (2j\pi/k + \pi/k) - \epsilon)\}.$$

Theorem 1.8 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Then there exist $l_0 \in \mathbf{N}$ and $\vec{x}_j \in S^1$ for $j = 1, ..., l_0$ such that every interface branch of the competition interface is a $\vec{x}_j$-path for some $j \in \{1, ..., l_0\}$. Furthermore,*

$$P(\cap_{j=1}^k [S_j^1(\epsilon) \subseteq S_{j,r}^1]) \to_{r\to\infty} 1.$$

Theorem 1.8 is proved in Chapter 5, Section 5.4. It is a corollary of Theorem 1.8 that the probability of the event that mutual unbounded growth occurs converges to 1 as the distance between the initial vertices goes to infinity.

Corollary 1.3 *Under the same assumption of Theorem 1.8,*

$$P(\min_{j=1,...,k}\{\#\nu(G_{j,r})\} = +\infty) \to_{r\to+\infty} 1.$$

Proof of Corollary 1.3. Indeed, to prove this corollary is sufficient to observe that

$$[S_{j,r}^1 \neq \emptyset \text{ for all } j = 1, ..., k] \subseteq [\min_{j=1,...,k}\{\#\nu(G_{j,r})\} = +\infty],$$

and then apply Theorem 1.8.

□

Moreover, this theorem tell us something about the shape of these k growing clusters. In fact, for small $\epsilon > 0$ and $j = 1, ..., k$, consider the cone C_ϵ^j given by

$$C_j^\epsilon \equiv \{s\vec{x}; s > 0 \text{ and } \vec{x} \in S_j^1(\epsilon)\}.$$

We say that the predominant color of the cone C_j^ϵ is the color n if for all $l \neq n$, $G_{l,r} \cap C_j^\epsilon$ is a bounded region and $G_{n,r} \cap C_j^\epsilon$ is an unbounded region. Then for all large $r > 0$, with probability close to 1, the predominant color of the cone C_j^ϵ is the color j. This permit us to conclude that for all large $r > 0$, with probability close to 1, for all large $t > 0$, the the union of the k colored growing clusters will looks like an Euclidean ball cut into k colored slices (see Figure 1.4). Observe that for $k = 2$, Theorem 1.8 is equivalent to Theorems 1.6 together with 1.7.

1.5 Geodesics

Given $v, \tilde{v} \in \mathcal{V}$ recall that we have denoted the set of all self avoiding paths connecting v to $\tilde{v}$ by $\mathcal{C}(v, \tilde{v})$. We say that a self avoiding path $\rho(v, \tilde{v}) \in \mathcal{C}(v, \tilde{v})$ is a geodesic connecting v to $\tilde{v}$ if

$$T(v, \tilde{v}) = t(\rho(v, \tilde{v})),$$

and that a self avoiding path ρ in $\mathcal{G}$ is a geodesic if for all $v, \tilde{v} \in \rho$, there exist a geodesic $\rho(v, \tilde{v})$ connecting v and $\tilde{v}$ so that $\rho(v, \tilde{v}) \subseteq \rho$. Generally, geodesics we will be denoted by the symbol ρ. Recall that we have defined in Section 1.1 the critical probability p_{cb} given by 1.1.

Proposition 1.1 *Suppose that $F(0) < p_{cb}$. Then almost surely for all $v, \tilde{v} \in \mathcal{V}$ the set of geodesics between v and $\tilde{v}$ is non empty and finite. Furthermore, if F is continuous then there exists exactly one geodesic between v and $\tilde{v}$.*

Proof of proposition 1.1. To prove the existence, for each $v \in \mathcal{V}$ and $n \geq 1$ define

$$d_n(v) \equiv \inf\{T(v, \hat{v});\ |\hat{v}| \geq n\}.$$

We claim that, almost surely, for all $v \in \mathcal{V}$,

$$d_n(v) \to +\infty \text{ as } n \to +\infty. \tag{1.12}$$

To prove 1.12 fix $\epsilon > 0$ in order that $F(\epsilon) < p_{cb}$ and define

$$X_e^\epsilon \equiv 1_{[\tau_e \le \epsilon]},$$

Then $\{X_e^\epsilon;\ e \in \mathcal{E}\}$ defines a bond percolation model with parameter

$$p^\epsilon = P(X_e^\epsilon = 1) = F(\epsilon) < p_{cb}.$$

Since $p^\epsilon < p_{cb}$, then percolation of open edges does not occur. This permit us to conclude that, for all $x \in \mathbf{R}^2$, there is infinitely many disjoint blocking circuits in the dual graph $\mathcal{G}^*$,

$$\sigma_1^*, \sigma_2^*, ...$$

which are surrounding x and

$$\forall\, e^* \in \sigma_n^*,\ X_e^\epsilon = 0$$

(or equivalently $\tau_e > \epsilon$). Call each circuit σ_n^* by an ϵ-blocking circuit surrounding x. Notice that every path γ in $\mathcal{G}$ that cross at least k disjoint ϵ-blocking circuit satisfies

$$t(\gamma) \ge \epsilon k.$$

Given $K > 0$ there is $n_K > 0$ such that inside the ball with radius n_K there is at least K/ϵ disjoint ϵ-blocking circuits surrounding v. Thus for every $n \ge n_K$, every path γ in $\mathcal{G}$, connecting v to a vertex in $\hat{v} \in \{x;\ |x| > n\} \cap \mathcal{V}$, will cross at least K/ϵ disjoint ϵ-circuit and then

$$d_n(v) \ge K \text{ for all } n \ge n_K.$$

To finish the proof of the existence, let $v, \tilde{v} \in \mathcal{V}$. By equation 1.12, there exists n large enough so that

$$d_n(v) > T(v, \tilde{v}) + 1.$$

But every path γ in $\mathcal{G}$ connecting v to $\tilde{v}$ and which contains a vertex in $\hat{v} \in \{x;\ |x| > n\}$ must have the passage time greater than $d_n(v)$. So,

$$t(\gamma) \ge d_n(v),$$

and consequently,

$$t(\gamma) > T(v, \tilde{v}) + 1.$$

Therefore, the passage time between v and $\tilde{v}$ will be attained by a path in the finite collection of paths that connect v to $\tilde{v}$ and lie inside the ball with radius n.

If F is a continuous function, almost surely, for any pair of different finite paths γ and $\tilde{\gamma}$ we have that $t(\gamma) \neq t(\tilde{\gamma})$. Therefore, as $F(0) = 0 < p_{cb}$, then almost surely for all $v, \tilde{v} \in \mathcal{V}$ there will be exactly one path that attains the minimum passage time.

$\square$

Now that we know conditions for the existence of finite geodesics we will study the existence of semi-infinite geodesics. We say that a path ρ in $\mathcal{G}$ is a uni-geodesic if ρ is a geodesic and if the polygonal version of ρ is a uni-path. For $\vec{x} \in S^1$ we say that ρ is a $\vec{x}$-geodesic if ρ is a geodesic and the polygonal version of ρ is a $\vec{x}$-path. Generally, uni-geodesics starting from v will be denoted by ρ_v and $\vec{x}$-geodesics starting from v will be denoted by $\rho_v(\vec{x})$. By using the same methods due to Licea, Newman and Piza for lattice first-passage percolation models (see [25],[26],[21]), and due to Howard and Newman for Euclidean first-passage percolation (see [13],[14]), to study the existence of semi-infinite geodesics with asymptotic directions we obtain the following results.

Theorem 1.9 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Then almost surely, for every $v \in \mathcal{V}$ the following is true:*

- *for every uni-geodesic ρ_v starting from v there exists $\vec{x} \in S^1$ such that ρ_v is a $\vec{x}$-geodesic;*
- *for every $\vec{x} \in S^1$ there exists at least one $\vec{x}$-geodesic starting from v.*

Theorem 1.10 *Assume that F is continuous and fix $\vec{x} \in S^1$. Then, a.s. for all $v \in \mathcal{V}$ there exists at most one $\vec{x}$-geodesic starting from v. Moreover, for all $v, \tilde{v} \in \mathcal{V}$, if $\rho_v(\vec{x})$ and $\rho_{\tilde{v}}(\vec{x})$ do exist, then they must coalesce. This means that there exists $c \equiv c(v, \tilde{v}, \vec{x}) \in \mathcal{V}$ such that*

$$\rho_v(\vec{x}) = \rho(v, c) \cup \rho_c(\vec{x}) \text{ and } \rho_{\tilde{v}}(\vec{x}) = \rho(\tilde{v}, c) \cup \rho_c(\vec{x}).$$

Theorem 1.11 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. For each fixed $\vec{x} \in S^1$ and for all $v, \tilde{v} \in \mathcal{V}$, let $c(\vec{x}, v, \tilde{v})$ be the coalescence point given by Theorem 1.9 and Theorem 1.10. Define*

$$H^{\vec{x}}(v, \tilde{v}) = T(v, c) - T(\tilde{v}, c).$$

Then almost surely for all $v, \tilde{v} \in \mathcal{V}$, for all sequence $(x_n)_{n \geq 1}$ so that

$$|x_n| \to_{n \to \infty} +\infty \text{ and } \frac{x_n}{|x_n|} \to_{n \to \infty} \vec{x}$$

we have that

$$\lim_{n \to \infty} T(v, x_n) - T(\tilde{v}, x_n) = H^{\vec{x}}(v, \tilde{v}).$$

Moreover, $H^{\vec{x}}(v, \tilde{v}) \neq 0$ for all $v \neq \tilde{v}$.

For $x, y \in \mathbf{R}^2$ define

$$H^{\vec{x}}(x, y) \equiv H^{\vec{x}}(v_x, v_y).$$

We call $H^{\vec{x}}(v, \tilde{v})$ the coalescence function. This result was firstly proved by Newman in the lattice first-passage percolation context to study the micro-structure of the boundary of $G_0(t)$ when $t \to +\infty$ (see [25]). It was conjecture by Howard and Newman that the asymptotic behavior of $H^{\vec{x}}(y, 0)$, as $|y| \to +\infty$, is $-\mu(F)\vec{x} \cdot y$, where $\mu(F)$ is the time constant which appear in Theorem 1.1 and $x \cdot y$ is the standard Euclidean inner product between x and y (see [14]). The following theorem is related to this conjecture. Define

$$S^1_+ \equiv \{\vec{x} \in S^1;\ \vec{x} \cdot \vec{e}_1 > 0\}$$

and

$$S^1_- \equiv \{\vec{x} \in S^1;\ \vec{x} \cdot \vec{e}_1 < 0\}.$$

Denote the set of all weak limits of $\{r^{-1} H^{\vec{x}}(r\vec{e}_1, 0)\}_{r>0}$ by $\mathbf{H}^{\vec{x}}$.

Theorem 1.12 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Then for all $\vec{x} \in S^1$, $\mathbf{H}^{\vec{x}}$ is non empty and every $h \in \mathbf{H}^{\vec{x}}$ is almost surely a constant. Furthermore*

- $\forall \vec{x} \in S^1_+,\ h \in \mathbf{H}^{\vec{x}} \Rightarrow h \leq 0$;
- $\forall \vec{x} \in S^1_-,\ h \in \mathbf{H}^{\vec{x}} \Rightarrow h \geq 0$;
- $\lim_{r\to\infty} \frac{H^{\vec{e}_1}(r\vec{e}_1, 0)}{r} = -\mu(F)$ *in probability;*
- $\lim_{r\to\infty} \frac{H^{\vec{e}_2}(r\vec{e}_1, 0)}{r} = 0$ *in probability.*

1.6 Critical exponents and scaling relations

A subject that has attracted considerable attention in recent years is the nature of the fluctuation of growing surfaces and geodesics. For instance, the surface between $G_0(t)$ and its complement grows linearly in t with a deterministic shape. The magnitude of the fluctuations of this surface about its asymptotic shape is believed to be typically of order t^χ for some $\chi \in (0,1)$. Another exponent $\xi \in (0,1)$ is such that r^ξ is the order of deviations of a finite geodesic, connecting two points x and y with $|x-y| = r$, about the line segment $[x, y]$.

It is conjectured that these exponents satisfy the scaling relations

$$\chi = 2\xi - 1 \text{ and } \xi = 2\chi, \tag{1.13}$$

which leads to the exact values

$$\chi = \frac{1}{3} \text{ and } \xi = \frac{2}{3}.$$

This conjecture is based on the assumption that the growth model introduced in terms of first-passage percolation lies in the universality class governed by the KPZ partial differential equation (see [20], [15], [16]). By definition universality means that the large scale properties of the fluctuations are essentially independent of the growth rule and of the microscopic details, within the given class.

Here, we will consider several exponents that are possible formal definitions of χ and ξ and present some rigorous results that point out in the direction of 1.13. We also will consider another type of exponent that measures the magnitude of the deviations of the interface, defined in the competing spatial growth model, about its asymptotic directions.

We say that $\kappa \in [0,1]$ satisfies **A1** if there exists constants ν_1, ν_2 such that for all $\nu \in (0, \nu_1/2)$ there exist $C_i > 0$ (depending on ν) such that for all $x \in [n^{\nu}, n^{\nu_1 - \nu}]$

$$P(|T(0,n) - \mu n| \geq x n^{\kappa}) \leq C_6 \exp(-C_7 x^{\nu_2}), \tag{1.14}$$

where μ is the time constant given by 1.3. Let

$$\chi_1 \equiv \inf\{\kappa \in [0,1];\ \kappa \text{ satisfies } \mathbf{A1}\}.$$

This exponent measure the magnitude of the fluctuations of $T(x,y)$ from its asymptotic value $\mu|x-y|$. Notice that Theorem 1.4 implies $\chi_1 \leq 1/2$.

Now we say that $\kappa \in [0,1]$ satisfies **A2** if almost surely there exist $t_0 > 0$ so that for all $t > t_0$,

$$(t - t^{\kappa})B_0 \subseteq G_0(t) \subseteq (t + t^{\kappa})B_0,$$

where B_0 is the limit shape given by 1.5. Let

$$\chi_2 \equiv \inf\{\kappa \in [0,1];\ \kappa \text{ satisfies } \mathbf{A2}\}.$$

This exponent measure the roughness of the boundary of $G_0(t)$ about its asymptotic shape and it is believed that $\chi_1 = \chi_2$. Note that Theorem 1.5 implies that $\chi_2 \leq 1/2$.

Considering now the fluctuations of finite geodesics, we say that $\kappa \in [0,1]$ satisfies **A3** if the probability of the event that

$$\sup_{v \in \rho(0, r\vec{e}_1)} d(v, [0, r\vec{e}_1]) > r^{\kappa},$$

has at least a sub-exponential decay with r, where $d(x,B)$ is the Euclidean distance between the point x and the set B. Let

$$\xi_1 \equiv \inf\{\kappa \in [0,1];\ \kappa \text{ satisfies } \mathbf{A3}\}.$$

We will also consider the fluctuations of semi-infinite geodesics about its asymptotic direction as follows. For each $x \in \mathbf{R}^2$ and $\vec{x} \in S^1$, let $H_x(\vec{x})$ be the hyperplane that is perpendicular to the direction $\vec{x}$ and contains the point x. Given a polygonal path $\gamma \equiv (x_1, x_2, ...)$ that is a $\vec{x}$-path, let $x_r \in \gamma \cap H_{y_r^{\vec{x}}}(\vec{x})$ so that

$$|x_r - y_r^{\vec{x}}| = \max_{x \in \gamma \cap H_{y_r^{\vec{x}}}(\vec{x})} \{|x - y_r^{\vec{x}}|\},$$

where $y_r^{\vec{x}}$ is the point in the semi-infinite line segment $l_{x_1}^{\vec{x}}$ (starting from x_1 and with direction $\vec{x}$) at distance r from x_1. Let $\kappa \in [0,1]$ and say that γ is κ-straight about $l_{x_1}^{\vec{x}}$ if there exists $r_0 > 0$ so that for all $r > r_0$

$$|x_r - y_r^{\vec{x}}| \leq r^{\kappa}.$$

We say that κ satisfies **A4** if ,almost surely, every $\vec{x}$-geodesic $\rho_v(\vec{x})$ is κ-straight about $l_v^{\vec{x}}$. Let

$$\xi_2 \equiv \inf\{\kappa \in [0,1];\ \kappa \text{ satisfies } \mathbf{A4}\}.$$

Related to the interface we will define the following exponent. Consider the interface $\psi_r \equiv (x_n)_{\mathbf{Z}}$ defined in the competing spatial growth model. Let $\vec{x}_1$ and $\vec{x}_2$ be the interface directions given by Theorem 1.6 (see Section 1.4). We say that $\kappa \in [0,1]$ satisfies **A5** if, almost surely, whenever ψ_r is a bi-infinite path, $\psi_r^+ \equiv (x_0, x_1, ...)$ is κ-straight about $l_{x_0}^{\vec{x}_1}$ and $\psi_r^- \equiv (..., x_{-1}, x_0)$ is κ-straight about $l_{x_0}^{\vec{x}_2}$. Let

$$\zeta \equiv \inf\{\kappa \in [0,1];\ \kappa \text{ satisfies } \mathbf{A5}\}.$$

Theorem 1.13 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Then we have the following scaling relations;*

$$\chi_2 \leq \chi_1 \leq \frac{1}{2} \textit{ and } \zeta \leq \xi_2 \leq \xi_1 \leq \frac{\chi_1 + 1}{2} \leq \frac{3}{4}.$$

We believe that

$$\chi_1 = \chi_2 = \chi,\ \xi_1 = \xi_2 = \xi,$$

and that these exponents satisfies the scaling relations 1.13. We also conjecture that

$$\zeta = \xi.$$

There is another exponent related to the interface and which measures the rate of convergence of $\theta_{j,r}$ to its asymptotic value (see Theorem 1.7). We say that $\kappa \in [0,1]$ satisfies **A6** if

$$P(|\theta_{1,r} - \frac{\pi}{2}| > r^{-\kappa}) \to_{r\to\infty},$$

and

$$P(|\theta_{2,r} - \frac{3\pi}{2}| > r^{-\kappa}) \to_{r\to\infty},$$

and let

$$\upsilon \equiv \inf\{\kappa \in [0,1];\ \kappa \text{ satisfies } \mathbf{A6}\}.$$

At this moment we do not know if there exist any relation of υ with the others exponents or even what should be its value.

Remarks

The first-passage percolation model was originally defined in the context of the $\mathbf{Z}^d$ lattice by Hammersley and Welsh in 1965 (see [12]). This means that the underline graph is $(\mathbf{Z}^d, \mathbf{E}^d)$, where $\mathbf{E}^d \equiv \{(z,\tilde{z});\ |z-\tilde{z}| = 1\}$, and $\{\tau_e;\ e \in \mathbf{E}^d\}$ is a set of i.i.d. non-negative random variables with common distribution F. The first passage time is defined by following the same procedure we have done for the Voronoi case.

There is another type of first-passage percolation model introduced by Howard and Newman in 1997 (see [13]) (Euclidean first-passage percolation). In such a model the graph structure is given by (V,E) where V is the realization of a homogeneous (two-dimensional)

Poisson point process and E is determined by all pairs $(v, \tilde{v})$ so that $v, \tilde{v} \in V$ and $v \neq \tilde{v}$ (the complete graph over V). The passage time through the edge $e \in E$ is a function of the point process:

$$\tau_e \equiv |v - \tilde{v}|^{\alpha},$$

where $e = (v, \tilde{v})$ and $\alpha > 1$ is a fixed parameter. The passage time along a path γ is defined by

$$t(\gamma) \equiv (\sum_{e \in \gamma} \tau_e)^{1/\alpha}.$$

The first-passage time from v to $\tilde{v}$ is the infimum of $t(\gamma)$, where γ runs over all self avoiding path connecting v to $\tilde{v}$.

For these models there are ergodic and shape theorems (see [5] for the lattice case and [13] for the Euclidean case). By isotropy of the Euclidean model, the limit shape is also an Euclidean ball, although for the lattice case the exact limit shape is unknown. An improved version of Theorem 1.4 was proved for the lattice case, when $F(0) < 1/2$ and there exists $a > 0$ so that $\int e^{ax} dF(x) < +\infty$, by Kesten (see [18]) and by Alexander (see [1]). In the Euclidean case, an improved version of Theorem 1.4 was proved, when $\alpha > 1$, by Howard and Newman (see [14]).

We believe that all theorems presented here are also valid for two-dimensional Euclidean first-passage percolation models with $\alpha \geq 2$. This is so because all argumentation used in our proofs are based on the existence of the coalescence function (see Section 5.1) and on the Euclidean invariance, that are known for this case (see [14]). For the the two-dimensional lattice case we need to be more careful, but adding an assumption on the curvature of the limit shape (see [25]), the existence of the coalescence function for many directions is valid. This encourage us to believe that all theorems presented here concerning the interface for $k = 2$ are also valid under this assumption. By using the same methods, we would believe that Theorem 1.12 about the asymptotic of the coalescence function is also valid. Although, it is not clear how to prove Theorem 1.8 without isotropy.

Overview

Chapter 2 is dedicated to the study of the relation between $F(0)$, the atom at the origin of the passage time distribution, and the time constant $\mu(F)$. In Section 2.1 is determined a sufficient condition on $F(0)$ to ensure that $\mu(F) = 0$ (see Proposition 2.1). In Section 2.2, we prove Theorem 1.3. Section 2.3 is dedicated to prove some inequalities involving critical probabilities of bond and site Voronoi percolation models via the relation between F and $\mu(F)$ (see Propositions 2.3 and 2.4).

In Chapter 3 we prove Theorem 1.4 by adapting the method of bounded differences for the Voronoi context. Theorem 1.5 is deduced form Theorem 1.4 as in [1].

In Chapter 4 we develop the notion of geodesics and by using the same methods due to Licea, Newman and Piza for lattice first-passage percolation (see [25],[26],[21]), and due to Howard and Newman for Euclidean first-passage percolation (see [13],[14]), we prove Theorem 1.9 and Theorem 1.10.

The competing spatial growth model is studied in Chapter 5. Firstly we prove Theorem 1.11 and part of Theorem 1.12 (see Theorem 5.1). In Section 5.2 we explore the relation between the coalescence function and the geometrical structure of $G_{j,r}$, to prove Theorem 1.6. After that we prove Theorem 1.7 in Section 5.3. In Section 5.4, we finish the proof of Theorem 1.12 and use this result to prove Theorem 1.8. Finally, in Section 5.5 we prove the scaling inequalities in Theorem 1.13.

Chapter 2

Compactness of the limit shape

In the previous chapter we have defined the Voronoi first-passage percolation model. We considered the Deulanay triangulation $\mathcal{G} \equiv (\mathcal{V}, \mathcal{E})$ based on a realization of a two-dimensional homogeneous Poisson point process, and to each edge $e \in \mathcal{E}$ we have associated τ_e, the passage time along e, where the random variables τ_e are i.i.d. and independent of the Poisson point process that determines $\mathcal{G}$. In this chapter we will study the relation between the time constant $\mu(F)$, given in Theorem 1.1, and the passage time distribution F. Recall that Theorem 1.3 allows us to prove that

$$F(0) < 1 - \hat{p}^*_{cb} \Rightarrow \mu(F) > 0$$

(see Corollary 1.1).

2.1 The atom at zero passage time

In this section we will go in the opposite direction and see a situation where the time constant is zero. Recall that in Section 1.1 we have considered the bond percolation model $\{X_e;\ e \in \mathcal{E}\}$, with parameter p, on the random graph $\mathcal{G} \equiv (\mathcal{V}, \mathcal{E})$ and defined the critical probability p_{cb} given by 1.1.

Proposition 2.1 *If $F(0) > p_{cb}$, then with positive probability, for all $t > 0$, $G_0(t)$ is an unbounded subset of the plane.*

Proof of Proposition 2.1. Given $\{\tau_e;\, e \in \mathcal{E}\}$ we can define the following auxiliary process. Consider $\{X_e;\, e \in \mathcal{E}\}$ where $X_e \equiv 1_{[\tau_e=0]}$. Then this process defines a bond percolation model on the random graph $\mathcal{G}$ with parameter $p = F(0) > p_{cb}$. Recall that we had denoted the open cluster that contains the origin by W_0. Thus, with positive probability $\#(W_0) = +\infty$. But for all $t > 0$

$$W_0 \subseteq \tilde{G}_0(t),$$

and consequently, with positive probability, for all $t > 0$, $G_0(t)$ is an unbounded subset of the plane.

$\square$

A corollary of this simple proposition is that a necessary condition to ensure that $G_0(t)$ has a compact limit shape is that $F(0) \leq p_{cb}$. For the square lattice first-passage percolation model it was proved by Kesten that the limit shape is compact if and only if $F(0) < 1/2$ (see [17]).

2.2 On the decay of short passage times

In this section we will prove Theorem 1.3. The basic tool that is handled in the course of the proof is static renormalization. We also explore the duality between $\mathcal{G}$ and $\mathcal{G}^*$ as planar graphs.

Proof of Theorem 1.3. First let consider the particular case where the passage time distribution is a Bernoulli distribution with parameter p, i.e.: $P(\tau_e = 1) = p = 1 - P(\tau_e = 0)$. Let $L \geq 1$ be a parameter which value will be specified later. Define for $z = (z^1, z^2) \in \mathbf{Z}^2$, $|z|_0 \equiv \max_{j=1,2}\{|z^j|\}$. For each $A \subseteq \mathbf{R}^2$, we denote the boundary of A by $\partial(A)$. For each

$z = (z^1, z^2) \in \mathbf{Z}^2$ and $r > 0$ consider the box

$$B_z^{rL} \equiv Lz + [-rL, rL]^2,$$

and the circuit, $\Upsilon_z \equiv \{\tilde{z};\ |z - \tilde{z}|_0 = 2\}$. Paths and circuits in $\mathbf{Z}^2$ will be denoted by Γ and Υ respectively. Divide $B_z^{L/2}$ into thirty six sub-box with the same size and declare that $B_z^{L/2}$ is a full box if all these thirty six sub-boxes contain at least one point of $\mathcal{V}$. Let

$$H_z^L \equiv [\forall\, \bar{z} \in \Upsilon_z,\ B_{\bar{z}}^{L/2} \text{ is a full box }].$$

Recall that, for $B, \tilde{B} \subseteq \mathbf{R}^2$, $\mathcal{C}(B, \tilde{B})$ denote the set of all self avoiding path connecting B to $\tilde{B}$, i.e., the set of self avoiding paths $\gamma = (v_1, ..., v_m)$ such that $C_{v_1} \cap B \neq \emptyset$ and $C_{v_m} \cap \tilde{B} \neq \emptyset$. Let

$$\mathcal{C}_z^L \equiv \{\gamma \in \mathcal{C}(\partial(B_z^{3L/2}), B_z^{L/2});\ \forall v \in \gamma,\ C_v \cap B_z^{3L/2} \neq \emptyset\}. \tag{2.1}$$

We say that $B_z^{L/2}$ is a good box if for every $\tilde{z} \in \Gamma_z$, $B_{\tilde{z}}^{L/2}$ is a full box and every path $\gamma \in \mathcal{C}_z^L$ satisfies that $t(\gamma) \geq 1$ (see Figure 2.1). Let

$$G_z^L \equiv [\forall\, \gamma \in \mathcal{C}_z^L,\ t(\gamma) \geq 1].$$

Then

$$[B_z^{L/2} \text{ is a good box }] = H_z^L \cap G_z^L.$$

Given a square box $B \subseteq \mathbf{R}^2$ we had denoted the Delaunay triangulation of the plane based on $\mathcal{V}(B) \equiv \mathcal{V} \cap B$ by $\mathcal{G}(B) \equiv (\mathcal{V}(B), \mathcal{E}(B))$ (see Section 1.1). To each $v \in \mathcal{V}(B)$ we had defined the Voronoi cell restrict to the configuration of vertices in B by

$$C_v(B) \equiv \{x \in \mathbf{R}^2;\ |x - v| \leq |x - \tilde{v}|\ \forall\, \tilde{v} \in \mathcal{V}(B)\},$$

and the edge set $\mathcal{E}(B)$, by pair of vertices $(v, \tilde{v})$ in $\mathcal{V}(B)$ so that $C_v(B)$ and $C_{\tilde{v}}(B)$ share a common face. Let us denote the σ-algebra generated by the events that just depend on

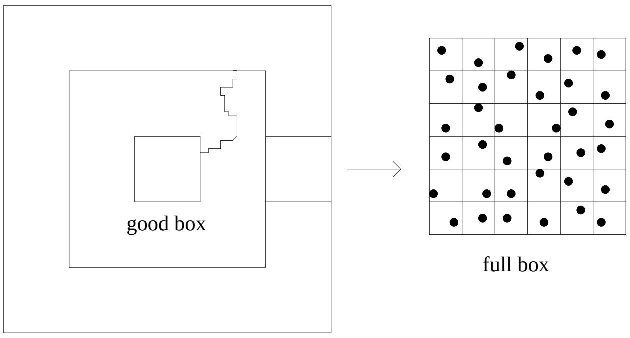

Figure 2.1: Renormalization.

the knowledge of $(\mathcal{G}(B), (\tau_e)_{\mathcal{E}(B)})$ by $\mathcal{F}(B)$ (see Section 1.1). Then it is easy to see that $H_z^L \in \mathcal{F}(B_z^{5L/2})$, but it is not true that $G_z^L \in \mathcal{F}(B_z^{5L/2})$. Nevertheless,

$$H_z^L \cap G_z^L \in \mathcal{F}(B_z^{5L/2}). \tag{2.2}$$

To prove 2.2 let $\mathcal{C}_z^L(B_z^{5L/2})$ be the set of all self avoiding paths γ in $\mathcal{G}(B_z^{5L/2})$ which connect $\partial(B_z^{3L/2})$ to $B_z^{L/2}$ and such that for all $v \in \gamma$, $C_v \cap B_z^{3L/2} \neq \emptyset$. Also denote by $G_z^L(B_z^{5L/2})$, the event that every path $\gamma \in \mathcal{C}_z^L(B_z^{5L/2})$ satisfies that $t(\gamma) \geq 1$ by. Then

$$G_z^L(B_z^{5L/2}) \in \mathcal{F}(B_z^{5L/2}).$$

Now we claim that,

$$H_z^L \cap G_z^L = H_z^L \cap G_z^L(B_z^{5L/2}). \tag{2.3}$$

To prove 2.3, notice that as $\mathcal{V}(B^{5L/2})_z \subseteq \mathcal{V}$, each path γ that belongs to $\mathcal{C}_z^L(\omega)$ or to $\mathcal{C}_z^L(B_z^{5L/2})(\omega)$ is an ordered sequence of vertices in $\mathcal{V}$. Thus, 2.3 follows from that on the set H_z^L we have that

$$\mathcal{C}_z^L = \mathcal{C}_z^L(B_z^{5L/2}). \tag{2.4}$$

The proof of 2.4 is based on the next statements. On the set H_z^L the following holds:

1. $\{v \in \mathcal{V};\ C_v \cap B_z^{3L/2} \neq \emptyset\} = \{v \in \mathcal{V}(B_z^{5L/2});\ C_v(B_z^{5L/2}) \cap B_z^{3L/2} \neq \emptyset\}$;

2. if $v \in \mathcal{V}(B_z^{5L/2})$ and $C_v(B_z^{5L/2}) \cap B_z^{3L/2} \neq \emptyset$ then $C_v(B_z^{5L/2}) = C_v$.

Indeed, let us assume that we are only considering configurations in H_z^L. Firstly let us prove that, for all $v \in \mathcal{V}$,

$$C_v \cap B_z^{3L/2} \neq \emptyset \Rightarrow C_v \subseteq B_z^{2L}. \tag{2.5}$$

Suppose that 2.5 is not true. If we can find $x \in C_v \cap B_z^{3L/2}$ and $y \in C_v \cap (B_z^{2L})^c$, by convexity of C_v, we can assume that $x \in \partial(B_z^{3L/2})$. Then

$$d(y, B_z^{2L}) + \frac{L}{2} \leq |x - y| \leq |x - v| + |y - v|.$$

As every box $B_{\tilde{z}}^{L/2}$ with $\tilde{z} \in \Gamma_z$ is a full box, there exist $v(x), v(y) \in \mathcal{V}$ so that

$$|v(x) - x| \leq \frac{\sqrt[2]{2}L}{6} \text{ and } |v(y) - y| \leq d(y, B_z^{2L}) + \frac{\sqrt[2]{2}L}{6}.$$

Although $x, y \in C_v$, which implies that

$$|v - x| \leq |v(x) - x| \text{ and } |v - y| \leq |v(y) - y|.$$

Thus,

$$d(y, B_z^{2L}) + \frac{L}{2} \leq d(y, B_z^{2L}) + \frac{\sqrt[2]{2}L}{3},$$

which leads to a contradiction because $\frac{\sqrt[2]{2}}{3} < \frac{1}{2}$. Similarly, one can prove that on H_z^L, for all $v \in \mathcal{V}(B_z^{5L/2})$,

$$C_v(B_z^{5L/2}) \cap B_z^{3L/2} \neq \emptyset \Rightarrow C_v(B_z^{5L/2}) \subseteq B_z^{2L}, \tag{2.6}$$

and that for all $v \in \mathcal{V}$,

$$C_v \cap (B_z^{5L/2})^c \neq \emptyset \Rightarrow C_v \subseteq (B_z^{2L})^c. \tag{2.7}$$

Note that for all $v \in \mathcal{V}(B_z^{5L/2})$, $C_v \subseteq C_v(B_z^{5L/2})$. Therefore, statements 1 and 2 follow from 2.5, 2.6 and 2.7. These last statements ensure that, on H_z^L, the piece of the graph determined by the cells C_v with $v \in \{v \in \mathcal{V};\ C_v \cap B_z^{3L/2} \neq \emptyset\}$ coincides with the piece of the graph structure determined by the cells $C_v(B_z^{5L/2})$ with $v \in \{v \in \mathcal{V}(B^{5L}2);\ C_v(B^{5L}2) \cap B_z^{3L/2} \neq \emptyset\}$. In particular, every path in $\mathcal{C}_z^L$ is also a path in $\mathcal{C}_z^L(B_z^{5L/2})$ and vice-versa.

Now 2.2 implies that for all $k \geq 1$, the events

$$[B_{z_1}^{L/2} \text{ is a good box}], ..., [B_{z_k}^{L/2} \text{ is a good box}]$$

are independent whenever $|z_j - z_n|_0 > 5$ (see Section 1.1).

For $l \geq 1$, let us consider $\{Y_z;\ z \in \mathbf{Z}^2\}$, the l-dependent site percolation model on the square lattice with parameter p. This means that for all $k \geq 1$, the random variables $Y_{z_1}, ..., Y_{z_k}$ are independent whenever $|z_j - z_n|_0 > l$ and identically distributed with $P(Y_z = 1) = p = 1 - P(Y_z = 0)$. For each $L > 0$, we can define the following auxiliary process $Y^L \equiv \{Y_z^L\}_{z \in \mathbf{Z}^2}$, where

$$Y_z^L \equiv 1_{[B_z^{L/2} \text{ is a good box}]}.$$

Note that Y^L is a 5-dependent site percolation model on the square lattice with parameter

$$p^L = P(B_z^L \text{ is a good box}).$$

The key step to apply the renormalization technique is the following proposition. Recall that we had defined in Section 1.2 the critical probability $\hat{p}_{cb}^*$ (see 1.2).

Proposition 2.2 *If* $F(0) = 1 - p < 1 - \hat{p}_{cb}^*$ *then*

$$P(Y_z^L = 1) = P(Y_0^L = 1) \to_{L \to \infty} 1.$$

Proof of Proposition 2.2. We show that $P(Y_0^L = 0) \to 0$ as $L \to +\infty$. First note that

$$P(Y_0^L = 0) \leq P((H_0^L)^c) + P((G_0^L)^c). \tag{2.8}$$

By the definition of a two-dimensional homogeneous Poisson point process with intensity 1,

$$P([-L/6, L/6]^2 \cap \mathcal{V} = \emptyset) = \exp(-L^2/36),$$

and by translation invariance,

$$P((H_0^L)^c) \leq 16 P(B_0^{L/2} \text{ is not a full box }) \leq 576 \exp(-L^2/36). \tag{2.9}$$

To prove that the second term in the right hand side of 2.8 also converges to zero when L goes to infinity, let us define $X_{e^*} \equiv \tau_e$, where e^* is determined by the duality relation expressed in Section 1.1. Then $\{X_{e^*};\, e^* \in \mathcal{E}^*\}$ defines a bond percolation model on the random graph $\mathcal{G}^*$. If there exists an open circuit σ^* in $\mathcal{G}^*$ which surrounds $B_0^{L/2}$ and lies inside $B_0^{3L/2}$, then every path γ in $\mathcal{G}$ connecting $\partial(B_0^{3L/2})$ to $B_0^{L/2}$ have and edge crossing with σ^*. Denote by F_L^* the event that an open circuit σ^* in $\mathcal{G}^*$ which surrounds $B_0^{L/2}$ and lies inside $B_0^{3L/2}$ does not exist. Thus the previous crossing argument implies that,

$$P((G_0^L)^c) \leq P_p^*(F_L^*) \leq 4(1 - P_p^*(A_L^*)). \tag{2.10}$$

The last inequality in 2.10 is a consequence of the following. In the same as way we had defined a left-right open crossing of a rectangle, we can define a TB^* (top-bottom) open crossing of a rectangle $[k, l] \times [\tilde{k}, \tilde{l}]$ by the existence of an open path in $\mathcal{G}^*$ that lies inside $[k, l] \times [\tilde{k}, \tilde{l}]$ and connects $[k, l] \times \{\tilde{l}\}$ to $[k, l] \times \{\tilde{k}\}$. Consider the events

$$A_L^{1*} \equiv \exists \text{ a } LR^* \text{ open crossing of } [-3L/2, 3L/2] \times [L/2, 3L/2],$$

$$A_L^{2*} \equiv \exists \text{ a } TB^* \text{ open crossing of } [-3L/2, -L/2] \times [-3L/2, 3L/2],$$

$$A_L^{3*} \equiv \exists \text{ a } LR^* \text{ open crossing of } [-3L/2, 3L/2] \times [-3L/2, -L/2],$$

$$A_L^{4*} \equiv \exists \text{ a } TB^* \text{ open crossing of } [L/2, 3L/2] \times [-3L/2, 3L/2].$$

By the Euclidean invariance of the model, $P_p^*(A_L^{j*}) = P^*(A_L^*)$ for all $j = 1, 2, 3, 4$. One can easily see that (see Figure 1.2),

$$\cap_{1 \leq j \leq 4} A_L^{j*} \subseteq (F_L^*)^c.$$

As $p > \hat{p}_{cb}^*$, by using 2.8, 2.9, 2.10 and the definition of $\hat{p}_{cb}^*$, we obtain Proposition 2.2.

$\square$

To express the connection between the process Y^L and the passage time T, we shall make another definition. For each $r \in \mathbf{R}$ let us denote the integer part of r by $[r]$. Given a collection $Y \equiv \{Y_z\}_{z\in\mathbf{Z}^2}$ of Bernoulli random variables, we define the random variable M_m^Y as the maximum number of disjoint open circuits Υ in $\mathbf{Z}^2$ surrounding the origin and lying inside $\{z \in \mathbf{Z}^2;\ |z|_0 \leq m\}$. We say that two circuits Υ_1 and Υ_2 in $\mathbf{Z}^2$ are l-distant if for every $z_1 \in \Upsilon_1$ and $z_2 \in \Upsilon_2$, $|z_1 - z_2|_0 > l$. Then notice that if there exist M_m^Y open circuits surrounding the origin and lying inside $\{z \in \mathbf{Z}^2;\ |z|_0 \leq m\}$, then there exist $[M_m^Y(l+1)^{-1}]$ l-distant open circuits surrounding the origin and lying inside $\{z \in \mathbf{Z}^2;\ |z|_0 \leq m\}$. We will say that $(B_{z_j}^{L/2})_I$ is a circuit of boxes if $(z_j)_I$ is a circuit in $\mathbf{Z}^2$, and that two circuits of boxes $(B_{z_j}^{L/2})_I, (B_{\tilde{z}_j}^{L/2})_{\tilde{I}}$ are l-distant if $(z_j)_I, (\tilde{z}_j)_{\tilde{I}}$ are l-distant. Now consider the auxiliary process Y^L and notice that each open (with respect to Y^L) circuit Υ in $\mathbf{Z}^2$ determines a circuit of good boxes in $\mathbf{R}^2$. Therefore, we can find $[M_{[nL^{-1}]}^{Y^L}(l+1)^{-1}]$ l-distant circuits of good boxes surrounding the origin and lying inside $[-n, n]^2$.

Now suppose that we have an integer number M of 5-distant circuits of good boxes surrounding the origin and lying inside $[-n, n]^2$. Let us order this sequence of circuits

$$\Upsilon_1^L, \Upsilon_2^L, ..., \Upsilon_M^L$$

so that Υ_j lies in interior of the region bounded by Υ_{j+1}. Given a self-avoiding path $\gamma \in \mathcal{C}(v_0, v_{n\vec{e}_1})$ (connecting C_{v_0} to $C_{v_{n\vec{e}_1}}$) let us define the following sequence of good boxes. Let $B_1(\gamma)$ be the box where occurs the first intersection between Υ_1^L and γ, moving along γ from v_0 to $v_{n\vec{e}_1}$, and let $\tilde{B}_1(\gamma)$ be the box where occurs the last intersection between Υ_1^L and γ. For $j = 1, ..., M-1$, given $B_1(\gamma), \tilde{B}_1(\gamma), ..., B_j(\gamma), \tilde{B}_j(\gamma)$, let $B_{j+1}(\gamma)$ be the box where occurs the first intersection between Υ_{j+1}^L and γ coming from $\tilde{B}_j(\gamma)$ and let $\tilde{B}_{j+1}(\gamma)$ be the box where occurs the last intersection between Υ_j^L and γ coming from $\tilde{B}_j(\gamma)$. Denote by $z_j^L = Lz_j$ the circuncenter of $B_j(\gamma)$ and notice that for each good box $B_j(\gamma)$ we can find an edge $e_j = (v_j, \tilde{v}_j) \in \gamma$ so that

$$v_j, \tilde{v}_j \in B_{z_j}^{5L/2} \text{ and } \tau_{e_j} = 1. \tag{2.11}$$

Indeed, for every path $\gamma \in \mathcal{C}(v_0, v_{n\vec{e}_1})$ define

$$\mathcal{C}(\gamma, j) \equiv \{\tilde{\gamma} \subseteq \gamma;\ \tilde{\gamma} \in \mathcal{C}^L_{z_j}\}$$

(see 2.1). Since $B_j(\gamma)$ is a good box, for all $\tilde{\gamma} \in \mathcal{C}(\gamma, j)$, $t(\tilde{\gamma}) \geq 1$. This allows us to ensure that we can find $e_j = (v_j, \tilde{v}_j) \in \tilde{\gamma} \subseteq \gamma$ so that $\tau_{e_j} = 1$. Moreover, since every box $B^{L/2}_{\tilde{z}}$ with $\tilde{z} \in \Gamma_{L^{-1}z^L_j}$ is a full box, by 2.3 we can deduce that every $\tilde{\gamma} \in \mathcal{C}(\gamma, j)$ lies inside $B^{5L/2}_{z_j}$. This ensures that $v_j, \tilde{v}_j \in B^{5L/2}_{z_j}$. However, $L^{-1}|z^L_j - z^L_l|_0 > 5$ for all $j \neq l$, which implies that

$$(B^{5L/2}_{z_j}) \cap (B^{5L/2}_{z_l}) = \emptyset,$$

and consequently,

$$e_j \neq e_l \,\forall\, j \neq l \tag{2.12}$$

By 2.11 and 2.12, if we have M 5-distant circuits of good boxes that surround the origin and lie inside $[-n, n]^2$, then every path γ in $\mathcal{G}$ connecting C_{v_0} to $C_{v_{n\vec{e}_1}}$ has at least M edges e so that $\tau_e = 1$. Thus $M \leq t(\gamma)$ and consequently

$$[M^{Y^L}_{[nL^{-1}]}/6] \leq T(0, n\vec{e}_1). \tag{2.13}$$

2.13 allows us to deduce that

$$\begin{aligned} P(T(0, n\vec{e}_1) < Cn) &\leq P([M^{Y^L}_{[nL^{-1}]}/6] < Cn) \\ &\leq P(M^{Y^L}_{[nL^{-1}]} < 6CL([nL^{-1}] + 1)). \end{aligned} \tag{2.14}$$

If $Y \equiv \{Y_z\}_{z \in \mathbf{Z}^2}$ is a collection of i.i.d. Bernoulli random variables with parameter p sufficiently close to 1, then there exist $C > 0$ so that the probability of the event $[M^Y_m < Cm]$ decays exponentially fast with m (see [9]). Since Y^L is 5-dependent, by using Proposition 2.2 and Theorem ?? of Ligget, Schonmann and Stacey (see [23]), we can choose L_0 large enough to ensure the existence of $C_1, C_2, C_3 > 0$ so that

$$P(M^{Y^{L_0}}_m < C_1 m) \leq C_2 \exp(-C_3 m). \tag{2.15}$$

Choose $C_4 > 0$ small enough such that $6C_4L_0(m+1) \leq C_1m$ and apply 2.14 and 2.15 to obtain that

$$P(T(0, n\vec{e}_1) < C_4n) \leq C_2 \exp(-\frac{C_3n}{L_0}).$$

For the general case, if $F(0) = P(\tau_e = 0) < 1 - \hat{p}^*_{cb}$, we can fix $\epsilon > 0$ such that $F(\epsilon) < 1 - \hat{p}^*_{cb}$ and define the auxiliary process $\{\tau^\epsilon_e;\ e \in \mathcal{E}\}$, where

$$\tau^\epsilon_e \equiv 1_{[\tau_e > \epsilon]}.$$

So, τ^ϵ_e has a Bernoulli distribution with parameter

$$p^\epsilon = P(\tau_e > \epsilon) > \hat{p}^*_{cb},$$

and consequently,

$$P(\tau^\epsilon_e = 0) < 1 - \hat{p}^*_{cb}. \tag{2.16}$$

If we denote by T^ϵ the first-passage time associated to the collection passage times $\{\tau^\epsilon_e\}_{e\in\mathcal{E}}$, it follows that

$$T^\epsilon(0, n\vec{e}_1) \leq \frac{T(0, n\vec{e}_1)}{\epsilon}. \tag{2.17}$$

From the case of Bernoulli passage times and 2.16, we know that there exists $C_1, C_2, C_3 > 0$ such that

$$P(T^\epsilon(0, n\vec{e}_1) < C_1n) \leq C_2 \exp(-C_3n). \tag{2.18}$$

Thus 2.17 and 2.18 imply that we can choose $C_5 > 0$ such that $C_5 < \epsilon C_1$ to obtain that

$$P(T(0, n\vec{e}_1) < C_5n) \leq P(T^\epsilon(0, n\vec{e}_1) < C_1n) \leq C_2 \exp(-C_3n).$$

□

2.3 Inequalities for critical probabilities

We conclude Chapter 2 by proving several inequalities between critical probabilities in percolation models. Consider the Voronoi first-passage percolation model with Bernoulli passage times with parameter $P(\tau_e = 0) = q$. Note that it satisfies the condition of Theorem 1.1 (the ergodic theorem), and so we can define the time constant $\mu(q)$. Corollary 1.1 and Proposition 2.1 will allow us to derive some inequalities involving critical probabilities by exploring the relation between $F(0) = P(\tau_e = 0)$ and $\mu(F)$. By duality, it is conjectured that for a wide class of bond percolation models the equality, $1 = p_{cb} + \hat{p}^*_{cb}$ holds (see [9]).

Proposition 2.3 $1 \leq \hat{p}^*_{cb} + p_{cb}$.

Proof of proposition 2.3. We know by Proposition 2.1 that,

$$q > p_{cb} \Rightarrow \mu(q) = 0. \tag{2.19}$$

By Corollary 1.1,

$$q < 1 - \hat{p}^*_{cb} \Rightarrow \mu(q) > 0. \tag{2.20}$$

Therefore, if we assume that $\hat{p}^*_{cb} + p_{cb} < 1$ then we can find $q \in (p_{cb}, 1 - \hat{p}^*_{cb})$. Nevertheless, it leads to a contradiction because by 2.19, $\mu(q) = 0$, and by 2.20, $\mu(q) > 0$.

□

Now, let us consider a site version of the first-passage percolation model. As before, we have a random point configuration $\mathcal{V}$ given by the Poisson point process and then we construct the random graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as before. But now, to each vertex $v \in \mathcal{V}$, we associate one non-negative random variable τ_v so that $\{\tau_v\}_{v \in \mathcal{V}}$ is i.i.d., with common distribution F, and independent of the Poisson process which determines $\mathcal{G}$. We restrict our self to the case of Bernoulli variables with $P(\tau_v = 0) = q = 1 - p$. Define

$$t_s(\gamma) \equiv \sum_{v \in \gamma} \tau_v$$

and $T_s(v, \tilde{v}) \equiv \inf\{t(\gamma); \gamma \in \mathcal{C}(v, \tilde{v})\}$. The subindex indicates that these quantities refer to the *site* first-passage percolation model.

The proof of the shape theorem by Vahidi-Asl and Wierman (see [30]) can be extended for this class of models. Again, by isotropy, if the limit shape is compact then it is an Euclidean ball. It remains to see under which conditions the limit shape is compact. Let us denote the time constant in the site first-passage percolation model by $\mu_s(F)$.

We will say that a path γ is an open path (for the site percolation case) if for all $v \in \gamma$, $\tau_v = 1$. Let us define

$$A_n \equiv [\exists \text{ a } LR \text{ open crossing of } [0,3n] \times [0,n]],$$

where a LR open crossing of the rectangle $B = [k,l] \times [n,m]$ is defined by the existence of a self avoiding open (site) path $\gamma \equiv (v_n)_I$ in $\mathcal{G}$ with $I = \{1,...,m\}$, connecting $\{k\} \times [n,m]$ to $\{l\} \times [n,m]$ and such that, $v_n \in B$ for all $n = 2,...,m-1$. As before, we define the following critical probability.

$$\hat{p}_{cs} \equiv \inf\{p > 0;\ \liminf_{n \to +\infty} P_p(A_n) = 1\}.$$

Note that we also have that $\hat{p}_{cs} \in (0,1)$ (see [2]).

Theorem 2.1 *Suppose that $F(0) < 1 - \hat{p}_{cs}$, then there exist $C_1, C_2, C_3 > 0$ such that*

$$P(T_s(0, n\vec{e}_1) < C_1 n) \leq C_2 \exp(-C_3 n).$$

Proof of Theorem 2.1. This proof follows exactly the same strategy that was used in the proof of Theorem 1.3. We can define a box B_z^L being good if for every $\tilde{z} \in \Gamma_z$, $B_{\tilde{z}}^L$ is a full box and every path $\gamma \in \mathcal{C}_z^L$ (see 2.1) satisfies that $t_s(\gamma) \geq 1$. In the same way $\{1_{[B_z^{L/2}]};\ z \in \mathbf{Z}^2\}$ will be a 5-dependent site percolation model. The difference is that,

$$P(B_z^{L/2} \text{ is not a good box}) \leq 576 \exp(-L^2/36) + 4(1 - P_p(A_L)).$$

Since we assumed that $p > \hat{p}_{cs}$ then we obtain that

$$P(B_z^{L/2} \text{ is a good box}) \to 1 \text{ as } L \to +\infty.$$

It is simple to see that the rest of the proof follows the same argumentation as in the proof of Theorem 1.3.

$\square$

As in Corollary 1.1, Theorem 2.1 allows us to prove that

$$F(0) < 1 - \hat{p}_{cs} \Rightarrow \mu_s(q) > 0.$$

Define,

$$p_{cs} \equiv \inf\{p > 0;\ P_p(\#(W_0) = +\infty) > 0)\},$$

where P_p is the law of Voronoi site percolation model with parameter p. It was proved by Zvavitch that $p_{cs} \in [1/2, 1)$ (see [32]). As in the proof of Proposition 2.1,

$$F(0) > p_{cs} \Rightarrow \mu_s(q) = 0.$$

By using the same argumentation which was presented in the proof of Proposition 2.3 we can obtain the following proposition.

Proposition 2.4 $1 \leq \hat{p}_{cs} + p_{cs}$.

It was conjectured that $\hat{p}_{cs} = p_{cs}$ and, by the self matching property of the Voronoi site percolation, that $\hat{p}_{cs} = p_{cs} = 1/2$ (see [9] and [29]).

Chapter 3

Speed of convergence to the time constant

In this chapter we will study the fluctuations of $T(0, n\vec{e}_1)$ around its asymptotic value μn and prove Theorem 1.4.

3.1 Martingales with bounded increments

The basic tool that we will used here is called the method of bounded differences. This method represents $T(0,n)-ET(0,n)$ as a sum of martingale differences and, after estimating the sum of squares of these differences, applies standard exponential bounds for martingales with bounded differences. Here we follow the structure presented by Kesten (see [18]) that used this method to study the fluctuations around the time constant in the lattice case. We formulate the abstract martingale estimate used by Howard and Newman (see [14]).

Theorem 3.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_k\}_{k\geq 0}$ be an increasing family of σ-algebras of measurable sets. Let $\{M_k\}_{k\geq 0}$, $M_0 = 0$, be a martingale with respect to the filtration $\{\mathcal{F}_k\}_{k\geq 0}$ and let $\{U_k\}_{k\geq 0}$ be a collection of positive random variables that are*

$\mathcal{F}$-measurable. Assume that the increments $\Delta_k = M_k - M_{k-1}$ satisfy

$$|\Delta_k| \leq c, \text{ for some constant } c \geq 0, \tag{3.1}$$

and

$$E(\Delta_k^2 \mid \mathcal{F}_{k-1}) \leq E(U_k \mid \mathcal{F}_{k-1}). \tag{3.2}$$

Assume further that for some finite constants $0 < C_1, \gamma$ and x_0 with $x_0 \geq c^2$, we have that for all $x \geq x_0$,

$$P(\sum_{k\geq 1} U_k > x) \leq C_1 \exp(x^\gamma). \tag{3.3}$$

Then almost surely $M = \lim_{k\to\infty} M_m$ exists. Moreover, there exists finite constants $0 < C_2, C_3$, that do not depend on c, x_0 and such that for all $x \leq x_0^\gamma$,

$$P(|M| > x\sqrt{x_0}) \leq C_2 \exp(-C_3 x). \tag{3.4}$$

In order to use Theorem 3.1 we will have to consider the following construction of $(\Omega, \mathcal{F}, P)$. Let $\delta \in (0,1)$ and $n \geq 1$. Divide the plane into square boxes $B_k^{\delta,n}$ with (side) length n^δ and so that the origin is the center of $B_1^{\delta,n}$. Let $B_1^{\delta,n}, B_2^{\delta,n}, ...$ be a spiral ordering of $\{B_k^{\delta,n}\}_{k\geq 1}$. Consider

$$\{N_k^{\delta,n};\ k \geq 1\},$$

a collection of i.i.d. Poisson random variables with intensity $n^{2\delta}$,

$$\{\tau_{k,l}^{m,p};\ k \geq 1, l \geq 1, m \geq k, p \geq 1 \text{ and } p > l \text{ whenever } k = m\},$$

a collection of i.i.d. non negative random variables with common distribution F (the passage time distribution), and

$$\{V_{k,l}^{\delta,n};\ k \geq 1, l \geq 1\},$$

a collection of independent random points in the plane so that $V_{k,l}^{\delta,n}$ has an uniform distribution in the square box $B_k^{\delta,n}$. We also impose that all these collections are independent of each other. To determine the point configuration $\mathcal{V}$, at each square box $B_k^{\delta,n}$ we put $N_k^{\delta,n}$

points given by $V_{k,1}^{\delta,n}, ..., V_{k,N_k^{\delta,n}}^{\delta,n}$. This procedure determines a Poisson point process $\mathcal{V}$ from the collection $\{N_k^{\delta,n}, V_{k,l}^{\delta,n};\ k,l \geq 1\}$. Given $\mathcal{V}$, for each $e \in \mathcal{E}$ we know that there exist an unique pair $(V_{k,l}^{\delta,n}, V_{m,p}^{\delta,n})$, where either $m > k$ or $m = k$ and $p > l$, so that $e = (V_{k,l}^{\delta,n}, V_{m,p}^{\delta,n})$. Therefore we define $\tau_e = \tau_{k,l}^{m,p}$. Notice that for each $k \geq 1$ we can see the probability space where $N_k^{\delta,n}$, $\{V_{k,l}^{\delta,n};\ l \geq 1\}$ and $\{\tau_{k,l}^{m,p};\ l \geq 1, m \geq k, p \geq 1$ and $p > l$ whenever $k = m\}$ are defined as

$$(\Omega_k, \mathcal{A}_k, P_k),$$

where

$$\Omega_k \equiv \mathbf{N} \times (B_k^{\delta,n})^{\infty} \times \mathbf{R}^{\infty}.$$

The probability space $(\Omega, \mathcal{F}, P)$ is defined by the product space of $(\Omega_k, \mathcal{A}_k, P_k)$ over $k \geq 1$. We denote by $\mathcal{F}_k$ the σ-algebra generated by the collections $N_j^{\delta,n}$, $\{V_{j,l}^{\delta,n}; l \geq 1\}$ and $\{\tau_{j,l}^{m,p}; l \geq 1, m \geq j, p \geq 1$ and $p > l$ whenever $j = m\}$ for $j = 1, ..., k$.

3.2 Main estimative

With the above construction of the filtration $\{\mathcal{F}_k\}_{k\geq 0}$, given a random variable X on Ω we can define the martingale $M_k = E(X \mid \mathcal{F}_k) - EX$ and in this case

$$E(X \mid \mathcal{F}_k) = \int X(\omega_1, ..., \omega_k, \sigma_{k+1}, ...) \prod_{j \geq k+1} P_j(d\sigma_j). \tag{3.5}$$

Let $\sigma = (\sigma_j)_{j\geq 1} \in \Omega$ and $\omega = (\omega_j)_{j \geq 1} \in \Omega$ and define $[\omega, \sigma]_k = (\omega_1, ..., \omega_k, \sigma_{k+1}, ...) \in \Omega$. Notice that, if we integrate 3.5 with respect to $P_k(d\sigma_k)$ then nothing changes. This implies that $\Delta_k = M_k - M_{k-1}$ will be equal to

$$\int X[\omega, \sigma]_k - X[\omega, \sigma]_{k-1} \prod_{j \geq k} P_j(d\sigma_j). \tag{3.6}$$

Proof of Theorem 1.4. We will divide the proof of Theorem 1.4 in four parts.

Part 1: proof of 1.6. Our aim is to study $X[\omega, \sigma]_k - X[\omega, \sigma]_{k-1}$ when $X = T_n = T(0, n)$. For each box $B_k = B_k^{\delta,n}$ there exist (almost surely) a circuit Γ_k, which only depend on the location of points outside B_k, surrounding B_k and so that any change in the point location inside B_k does not change the graph structure (the Delaunay triangulation) outside Γ_k. Notice that, by the above definition, $\Gamma_k[\omega,\sigma]_k = \Gamma_k[\omega,\sigma]_{k-1}$. Let F_k be the event that a geodesic between 0 and $n\vec{e}_1$ touches Γ_k and consider the random variable $I_k = I_{F_k}$(the indicator function of the set F_k). Then we claim that

$$I_k[\omega,\sigma]_k = 0 \text{ and } I_k[\omega,\sigma]_{k-1} = 0 \Rightarrow T_n[\omega,\sigma]_k - T_n[\omega,\sigma]_{k-1} = 0. \tag{3.7}$$

Indeed, if $I_k[\omega,\sigma]_k = 0$ then a geodesic $\rho_n[\omega,\sigma]_k$, for $T_n[\omega,\sigma]_k$, is also a path in the graph $\mathcal{G}[\omega,\sigma]_{k-1}$ because it does not intersect Γ_k. Thus $T_n[\omega,\sigma]_k = t(\rho_n[\omega,\sigma]_k) \geq T_n[\omega,\sigma]_{k-1}$. Analogously, if $I_k[\omega,\sigma]_{k-1} = 0$ then $T_n[\omega,\sigma]_{k-1} = t(\rho_n[\omega,\sigma]_{k-1}) \geq T_n[\omega,\sigma]_k$, which completes the proof of 3.7.

Now to deal with the case

$$T_n[\omega,\sigma]_{k-1} < T_n[\omega,\sigma]_k \tag{3.8}$$

(and after the case $T_n[\omega,\sigma]_k < T_n[\omega,\sigma]_{k-1}$) we will introduce the following modification of the geodesics ρ_n. Notice that $T_n[\omega,\sigma]_{k-1} < T_n[\omega,\sigma]_k$ implies that $I_k[\omega,\sigma]_{k-1} = 1$(by the same reason we prove 3.7). Suppose that a geodesic ρ_n, for the configurations $[\omega,\sigma_{k-1}]$ intersects Γ_k. Denote by $\bar{\Gamma}_k$ the region bounded by Γ_k. Firstly let us assume that the points 0 and $n\vec{e}_1$ do not belong to $\bar{\Gamma}_k$. Then ρ_n must enter into $\bar{\Gamma}_k$ for the first time through point $v_n^1 \in \Gamma_k$ and exit from $\bar{\Gamma}_k$ for the last time through a point $v_n^2 \in \Gamma_k$. Let γ_n be the path that connect v_n^1 to v_n^2 by only using vertices and edges of Γ_k(we have two of them) and follows the counter clockwise orientation from v^1 to v_n^2. Define $\rho_n' = \rho(0, v_n^1 \cup \gamma_n \cup \rho(v_n^2, n\vec{e}_1)$. Then ρ_n' is also a path in $\mathcal{G}[\omega,\sigma]_k$ and so

$$T_n[\omega,\sigma]_{k-1} + t(\gamma_n) = T(v_n^1, v_n^2)[\omega,\sigma]_{k-1} + t(\rho_n')$$

$$\geq T_n[\omega,\sigma]_k + T(v_n^1, v_n^2)[\omega,\sigma]_{k-1}.$$

Notice that $t(\gamma_n) \leq t(\Gamma_k)$ and that $t(\Gamma_k)[\omega,\sigma]_{k-1} = t(\Gamma_k)[\omega,\sigma]_k$. This with the last inequality and 3.8 imply that

$$0 < T_n[\omega,\sigma]_k - T_n[\omega,\sigma]_{k-1} \le t(\Gamma_k)[\omega,\sigma]_k. \tag{3.9}$$

Secondly, without lost of generality, let us assume that $0 \in \bar{\Gamma}_k$. If $n\vec{e}_1$ do not belongs to $\bar{\Gamma}_k$, let v_n^1 be last exit from Γ_k, otherwise set $v_n^1 = n\vec{e}_1$. In this case

$$T_n[\omega,\sigma]_{k-1} + T(0,v_n^1)[\omega,\sigma]_k = T(0,v_n^1)[\omega,\sigma]_{k-1} + T(v_n^1,n\vec{e}_1)[\omega,\sigma]_{k-1} + T(0,v_n^1)[\omega,\sigma]_k.$$

Notice that $T(v_n^1,n\vec{e}_1)[\omega,\sigma]_{k-1} = T(v_n^1,n\vec{e}_1)[\omega,\sigma]_k$ and that $T(v_n^1,n\vec{e}_1)[\omega,\sigma]_k + T(0,v_n^1)[\omega,\sigma]_k \ge T_n[\omega,\sigma]_k$. This with the last inequality and 3.8 imply that

$$0 < T_n[\omega,\sigma]_k - T_n[\omega,\sigma]_{k-1} \le \max_{x,y\in\bar{\Gamma}_k} T(x,y)[\omega,\sigma]_k. \tag{3.10}$$

Thus if 3.8 holds then

$$0 < T_n[\omega,\sigma]_k - T_n[\omega,\sigma]_{k-1} \le \max\{t(\Gamma_k)[\omega,\sigma]_k, \max_{x,y\in\bar{\Gamma}_k} T(x,y)[\omega,\sigma]_k\}. \tag{3.11}$$

Analogously, in the case where $T_n[\omega,\sigma]_k < T_n[\omega,\sigma]_{k-1}$ we obtain that

$$0 < T_n[\omega,\sigma]_{k-1} - T_n[\omega,\sigma]_k \le \max\{t(\Gamma_k)[\omega,\sigma]_{k-1}, \max_{x,y\in\bar{\Gamma}_k} T(x,y)[\omega,\sigma]_{k-1}\}. \tag{3.12}$$

Now we have to bound the right hand side of 3.11 and 3.12. To do this,we will have to construct from $\omega = (\omega_k)_{k\ge1}$ a truncated configuration $\hat{\omega} = (\hat{\omega}_k)_{k\ge1}$ as follows: let $\delta \in (0,1)$ and $C, B > 0$ be constants which the values will be specified later. Define $\hat{N}_{\delta,n} = \min\{\max\{N^{\delta,n},1\}, Cn^{2\delta}\}$ and $\hat{\tau}_{k,l}^{m,p} = \min\{\tau_{k,l}^{m,p}, B\log n\}$. We construct a new point process $\hat{\mathcal{V}}$ by the following procedure: in each box $B_k^{\delta,n}$ we put $\hat{N}_k^{\delta,n}$ points given by $V_{k,1}^{\delta,n}, ..., V_{k,\hat{N}_k^{\delta,n}}^{\delta,n}$. From the point location $\hat{\mathcal{V}}$ we can define as before the graph structure $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ (the Delaunay triangulation based on $\hat{\mathcal{V}}$) and attach to each edge $\hat{e} = (V_{k,l}^{\delta,n}, V_{m,p}^{\delta,n})$ the random variable $\hat{\tau}_{k,l}^{m,p}$. We define the auxiliary random variable by $\hat{T}_n = T(0,n\vec{e}_1)(\hat{\mathcal{V}}, (\hat{\tau}_{\hat{e}})_{\hat{\mathcal{E}}})$, i.e., $\hat{T}_n$ is the first-passage time from 0 to $n\vec{e}_1$ with respect to the configuration $\hat{\omega} = (\hat{\mathcal{V}}, (\hat{\tau}_{\hat{e}})_{\hat{\mathcal{E}}})$.

Notice that

$$t(\Gamma_k)(\hat{\omega}) \le 49BCn^{2\delta}\log n \text{ and that } \max_{x,y\in\bar{\Gamma}_k} T(x,y)(\hat{\omega}) \le 49BCn^{2\delta}\log n. \tag{3.13}$$

Passage times and related quantities, when defined in terms of $\hat{\omega}$ instead of ω, will be decorated with a hat. For instance, we put $\hat{T}_n(\omega) = T_n(\hat{\omega})$ and $\rho_n(\hat{\omega}) = \hat{\rho}_n(\omega)$. With this convention, 3.7, 3.11, 3.12 and 3.13 imply that

$$|\hat{T}_n[\omega,\sigma]_k - \hat{T}_n[\omega,\sigma]_{k-1}| \leq 49BCn^{2\delta}\log n \times \max\{\hat{I}_k[\omega,\sigma]_k, \hat{I}_k[\omega,\sigma]_{k-1}\}, \tag{3.14}$$

where $\hat{I}_k$ is the event that $\hat{\rho}_n$ intersects $\bar{\Gamma}_k$.

Lemma 3.1 *If $F(0) < q_0$ and for some $a > 0$, $\int e^{ax}F(dx) < +\infty$ then for all $\delta \in (0, 1/2)$ we can find constants $A_0, C_0, B_0 > 0$, related to the construction of $\hat{\omega}$, and also $C_i > 0$, that depend on $\delta \in (0, 1/2)$ and $A_0, C_0, B_0 > 0$, so that*

$$E|\hat{\rho}_n| \leq C_1 n \tag{3.15}$$

and for all $x \geq 0$

$$P(|T_n - \hat{T}_n| > x) \leq C_1\exp(-C_2n^{2\delta}) + C_3\exp(-C_4x) \tag{3.16}$$

and

$$E(T_n - \hat{T}_n)^2 \leq C_5. \tag{3.17}$$

The next lemma together with 3.14 will allow us to use the method of bounded differences for $\hat{T}_n$. Let $U_k = n^{6\delta}\hat{I}_k$.

Lemma 3.2 *If $F(0) < q_0$ and for some $a > 0$, $\int e^{ax}F(dx) < +\infty$ then for all $\delta \in (0, 1/8)$ there exist constants C_i such that for all $y \geq n^{1+8\delta}$*

$$P(\sum_{k\geq 1} U_k > y) \leq C_1\exp(-C_2y^{1/2}). \tag{3.18}$$

To estimate $E(\hat{\Delta}_k^2 \mid \mathcal{F}_{k-1})$ we notice first that by 3.14

$$\hat{\Delta}_k^2 = (\int \hat{T}_n[\omega,\sigma]_k - \hat{T}_n[\omega,\sigma]_{k-1}\prod_{j\geq k} P_j(d\sigma_j))^2$$

$$\le C' n^{4\delta} (\log n)^2 \int \max\{\hat{I}_k[\omega,\sigma]_k, \hat{I}_k[\omega,\sigma]_{k-1}\} \prod_{j\ge k} P_j(d\sigma_j),$$

where $C' = (49B_0C_0)^2$. By integrating the last inequality with respect to $P_k(d\omega_k)$ we obtain that

$$E(\hat{\Delta}_k^2 \mid \mathcal{F}_{k-1}) \le 2C' n^{4\delta} (\log n)^2 E(\hat{I}_k \mid \mathcal{F}_{k-1}) \le E(U_k \mid \mathcal{F}_{k-1}). \tag{3.19}$$

By the definition of U_k,

$$E(\sum_k U_k) \le n^{6\delta} E(|\hat{\rho}_n|).$$

Thus, by Lemma 3.1, for all n large enough,

$$E \sum_k U_k \le C n^{1+6\delta}.$$

The above inequality together with 3.19 yields that for all n large enough,

$$E(\hat{T}_n - E\hat{T}_n)^2 = \sum_k E\hat{\Delta}_k^2 \le E \sum_k U_k \le C n^{1+6\delta}. \tag{3.20}$$

By Schwarz's inequality,

$$E(T_n - ET_n)^2 - E(\hat{T}_n - E\hat{T}_n)^2 = E[(T_n - \hat{T}_n)(T_n + \hat{T}_n)] + E(T_n - \hat{T}_n)E(T_n + \hat{T}_n)$$

$$\le [E(T_n - \hat{T}_n)^2 E(T_n + \hat{T}_n)^2]^{1/2} + E(T_n - \hat{T}_n)E(T_n + \hat{T}_n). \tag{3.21}$$

Note that ET_n and $E\hat{T}_n$ are of order n and that ET_n^2 and $E\hat{T}_n^2$ are of order n^2 (see [29]). Thus, 3.21 together with Lemma 3.1 implies that for some constant $B > 0$

$$E(T_n - ET_n)^2 - E(\hat{T}_n - E\hat{T}_n)^2 \le Bn. \tag{3.22}$$

Therefore, 3.20 and 3.22 yield 1.6.

Part 2: proof of 1.7. If we take $c = 49B_0C_0n^{2\delta}\log n$, where B_0, C_0 are giving by Lemma 3.1) and $x_0 = n^{1+8\delta}$, then $c^2 \le x_0$ and by Lemma 3.2,

$$P(\sum_k U_k > x) \le C_1 \exp(-C_2 x^{1/2}) \text{ whenever } x \ge x_0.$$

Therefore, the martingale $\hat{M}_k = E(\hat{T}_n \mid \mathcal{F}_k) - E(\hat{T}_n)$ satisfies 3.13.2 and 3.3 with the above constants c, x_0 and with the collection of random variables $\{U_k\}_k$ as in 3.19. By Theorem 3.1 there exists C_3, C_4(that do not depend on n) such that

$$P(|\hat{T}_n - E\hat{T}_n| > xn^{\frac{1+8\delta}{2}}) \leq C_3 \exp(-C_4 x) \text{ whenever } x \leq n^{\frac{1+8\delta}{2}}. \tag{3.23}$$

By Lemma 3.1 and 3.23, we can find constants $C_i > 0$ so that for all $n \geq 1$, for all $x \leq n^{\frac{1+8\delta}{2}}$,

$$P(|T_n - ET_n| > xn^{\frac{1+8\delta}{2}}) \leq$$

$$P(|\hat{T}_n - T_n| > \frac{x}{3} n^{\frac{1+8\delta}{2}}) + P(|\hat{T}_n - E\hat{T}_n| > \frac{x}{3} n^{\frac{1+8\delta}{2}}) \leq$$
$$C_1 \exp(-C_2 n^{2\delta}) + C_3 \exp(-C_4 \frac{x}{3} n^{\frac{1+8\delta}{2}}) + C_5 \exp(-C_6 \frac{x}{3}).$$

But for $x \leq n^{\frac{1+8\delta}{2}}$ we can bound the last term by $C_7 \exp(-C_8 x^{\frac{4\delta}{1+8\delta}})$. For $\kappa \in (1/2, 1)$, if we write $\kappa = 1/2 + 4\delta$ then $\delta = (\kappa - 1/2)/4 < 1/8$ and we obtain 1.7 with $\nu_1 = \kappa$ and $\nu_2 = 1/2 - 1/4\kappa$.

Part 3: proof of 1.8. Firstly we claim that by applying the same general argument in the prof of Lemma 4.1 in [14] one can show that 1.7 yields that for some constant $C > 0$

$$ET_{2n} \geq 2T_n - Cn^{\kappa} (\log n)^{1/\nu_1}. \tag{3.24}$$

Now assume the functions $a : [0, +\infty) \to \mathbf{R}$ and $g : [0, +\infty) \to [0, +\infty)$ satisfy the following conditions:

$$a(n)/n \to_{n\to\infty} \mu \in \mathbf{R},\ g(n)/n \to_{n\to\infty} 0,$$

and

$$\psi = \limsup_{n\to\infty} g(2n)/g(n) < 2.$$

Then, for any $c > 1/(2 - \psi)$ for all n sufficiently large,

$$a(n) \leq \mu n + cg(n)$$

(see Lemma 4.2 in [14]). By taking $a(n) = ET_n$ and $g(n) = C(\log n)^{1/\nu_1}$, this last claim together with 3.24 yields 1.8.

Part 4: proof of 1.9. To obtain 1.9, notice that 1.7, 1.8 yield that for some constants C_i, for all $x \in [C_1(\log n)^{1/\nu_2}, n^{\nu_1}]$

$$P(|T_n - \mu n| > 2xn^{\kappa}) \leq C_2 \exp(-C_3 x^{\nu_1})$$

which implies 1.9.

$\square$

Proof of Lemma 3.1. Recall for each $n \geq 1$ we have defined a the new point location $\hat{\mathcal{V}} = \hat{\mathcal{V}}(n)$ by imposing that in each box $B_k^{\delta,n}$ (with radius n^{δ}) we have at least one point and at most $Cn^{2\delta}$ points. Also recall that objects when defined in terms of $\hat{\mathcal{V}}$ are decorated with a hat. Notice that if $\mathcal{V} = \hat{\mathcal{V}}$ inside the box $[-Cn, Cn]^2$ and $\hat{\rho}_n \subseteq [-Cn, Cn]^2$ then $\hat{\rho}_n$ is also a path in $\mathcal{G}$, and thus

$$T_n - \hat{T}_n = t(\rho_n) - \hat{t}(\hat{\rho_n}) \leq t(\hat{\rho_n}) - \hat{t}(\hat{\rho_n}) = \sum_{e\hat{\rho}_n} \tau_e - \hat{\tau}_e \leq \sum_{e \in \hat{\rho}_n} \tau_e I_{[\tau_e > B \log n]}.$$

Therefore, for all $C > 0$

$$P(|T_n - \hat{T}_n| > x) \leq P(\mathcal{V} \neq \hat{\mathcal{V}} \text{ inside the box } [-Cn, Cn]^2)$$

$$+P(\hat{\rho}_n \not\subseteq [-Cn, Cn]^2) + P(\sum_{e \in [-Cn, Cn]^2} \tau_e I_{[\tau_e > B \log n]} > x). \tag{3.25}$$

By the definition of the Poisson point process, given $C > 0$ one can find $A_1, A_2 > 0$ such that

$$P(\mathcal{V} \neq \hat{\mathcal{V}} \text{ inside the box } [-Cn, Cn]^2) \leq A_1 \exp -(A_2 n^{2\delta}). \tag{3.26}$$

Let $\mathcal{C}_0(m)$ be the set of all self avoiding paths starting at the origin and with at least m edges. In order to obtain an upper bound for the second term in the right hand side of

3.25 notice that follows from the renormalization developed to prove Theorem 1.3 that there exists $\alpha_i > 0$ so that for all $m \geq 1$

$$P(\exists\, \gamma \in \hat{\mathcal{C}}_0 \text{ exiting } [-m,m]^2 \text{ and with } |\gamma| < \alpha_1 m) \leq \alpha_2 \exp(-\alpha_3 m).$$

Thus,

$$P(\hat{\rho}_n \not\subseteq [-Cn, Cn]^2) \leq P(|\hat{\rho}_n| \geq \alpha_1 Cn) + \alpha_2 \exp(-\alpha_3 n/C). \tag{3.27}$$

To control the lenght of $|\rho_n|$ we use the following lemma.

Lemma 3.3 *There exists constants $q_0 \in (0,1)$, $\alpha_0 > 0$ and $C_i > 0$, that does not depend on n, sucht that, if $F(0) < q_0$ then*

$$P(\exists\, \gamma \in \hat{\mathcal{C}}_0(m) \textit{ with } t(\gamma) < \alpha m) \leq C_1 \exp(-C_2 m).$$

Now

$$P(|\hat{\rho}_n| \geq \alpha_1 Cn) \leq P(\hat{T}_n \geq \beta\alpha Cn) + P(\exists\, \gamma \in \mathcal{C}_0(\alpha Cn) \text{ with } t(\gamma) < \beta\alpha Cn). \tag{3.28}$$

By lemma 3.3 we can find β_i so that

$$P(\exists\, \gamma \in \mathcal{C}_0(\alpha_1 Cn) \text{ with } t(\gamma) < \beta_1\alpha_1 Cn) \leq \beta_2 \exp(-\beta_3 n/C). \tag{3.29}$$

To bound the probability of the event that $\hat{T}_n \geq \beta\alpha Cn$ we use the following lemma.

Lemma 3.4 *There exists a path $\hat{\gamma}_n = \hat{\gamma}_n(\hat{\mathcal{V}})$ between 0 and $n\vec{e}_1$, and constants $A_i > 0$ that does not depend on n so that*

$$P(|\hat{\gamma}_n| \geq A_3 n) \leq A_3 \exp(-A_4 n).$$

Therefore, by Lemma 3.4,

$$P(\hat{T}_n \geq \beta\alpha_1 Cn) \leq \sum_{i=1}^{A_3 n} P(\sum_{e \in \hat{\gamma}_n} \hat{\tau}_e \geq \beta\alpha_1 Cn \mid |\hat{\gamma}_n| = i) P(|\hat{\gamma}_n| = i) + A_3 \exp(-A_4 n).$$

But $\hat{\gamma}_n$ is independent of $\hat{\tau}_e$ and then

$$P(\sum_{e\in\hat{\gamma}_n} \hat{\tau}_e \geq \beta\alpha Cn \mid |\hat{\gamma}_n| = i) \leq e^{-a\beta\alpha Cn}(Ee^{a\tau})^i \leq (e^{-a\beta\alpha C}Ee^{a\tau})^n.$$

If we choose $C > 0$ so that $e^{-a\beta\alpha C}Ee^{a\tau} < e^{-1}$ then we obtain that

$$P(\hat{T}_n \geq \beta\alpha Cn) \leq e^{-n} + A_3e^{-A_4n}. \tag{3.30}$$

By using 3.27, 3.28, 3.29 and 3.30 we obtain that for some constants A_i

$$P(\hat{\rho}_n \not\subseteq [-A_5n, A_5n]^2) \leq A_6\exp(-A_7n). \tag{3.31}$$

To bound the third term in the right hand side of 3.25 notice that given $C > 0$ then there exists constants A_i so that

$$P(\#\{\, e \in [-Cn, Cn]^2\} \geq A_8n^2) \leq A_9\exp(-A_{10}n^2). \tag{3.32}$$

To see that 3.32 holds, denote by V, E and F the number of vertices, edges and faces of the Delaunay triangulation inside the box $[-Cn, Cn]^2$. Then $3E = 2F$ and, by the Euler formula, $V - E + F = 2$. This implies that the number of edges is comparable with the number of vertices which yields 3.32.

Therefore,

$$P(\sum_{e\in[-Cn,Cn]^2} \tau_e I_{[\tau_e > B\log n]} > x) \leq A_9\exp(-A_{10}n^2)+$$

$$\sum_{i=1}^{A_8n^2} P(\sum_{e\in[-Cn,Cn]^2} \tau_e I_{[\tau_e > B\log n]} > x \mid \#\{\, e \in [-Cn, Cn]^2\} = i)P(\#\{\, e \in [-Cn, Cn]^2\} = i).$$

If we choose $B = 8/a$ and again use the independence between the point location and the time configuration then we can find A_{11} (see Kesten [18]) so that

$$P(\sum_{e\in[-Cn,Cn]^2} \tau_e I_{[\tau_e > B\log n]} > x \mid \#\{\, e \in [-Cn, Cn]^2\} = i) \leq A_{11}\exp(-ax/2) \tag{3.33}$$

To prove 3.17, by 3.16 for all $y \geq 0$

$$P(|T_n - \hat{T}_n| > ny) \leq C_1 e^{-C_2 n^{2\delta}} + C_3 e^{-C_4 ny}. \tag{3.34}$$

Since $|T_n - \hat{T}_n| \leq \max\{T_n, \hat{T}_n\}$ then we also have that

$$P(|T_n - \hat{T}_n| > ny) \leq P(T_n > ny/2) + P(\hat{T}_n > ny/2) \leq C_5 e^{-C_6 yn}, \tag{3.35}$$

for $y \geq y_0$ (as in the proof of 3.30). So 3.34 and 3.35 yields that

$$E(T_n - \hat{T}_n)^2 = 2n^2[\int_0^{y_0} P(|T_n - \hat{T}_n| > ny) y dy + \int_{y_0}^{\infty} P(|T_n - \hat{T}_n| > ny) y dy]. \tag{3.36}$$

By using 3.34 and 3.35, one can see that the right hand side of 3.36 is bounded by

$$2n^2[C_1 e^{-C_2 n^{2\delta}} \int_0^{y_0} y dy + C_3 \int_0^{y_0} y e^{-C_4 ny} dy + C_5 \int_{y_0}^{\infty} y e^{-C_6 ny} dy]$$

$$= 2n^2[C_1 y_0^2 e^{-C_2 n^{2\delta}} + \frac{C_3}{C_4^2 n^2} - C_3 e^{-C_4 n y_0}(\frac{y_0}{C_4 n} + \frac{1}{C_4^2 n^2}) + C_5 e^{-C_6 n y_0}(\frac{y_0}{C_6 n} + \frac{1}{C_6^2 n^2})],$$

which is bounded by a constant independent of n, and this completes the proof of 3.17.

□

Proof of Lemma 3.2. For each vertex $v \in \hat{\mathcal{V}}$ there exists at most 9 circuits Γ_k such that $v \in \bar{\Gamma}_k$. Indeed, by noticing that $\Gamma_k \subseteq \cup_{j \in K_k} B_j$, where K_k is set of index of the boxes B_j which intersect B_k in their boundaries. Then

$$\sum_k U_k = n^{6\delta} \sum_k \hat{I}_k \leq 9n^{6\delta} |\hat{\rho}_n|.$$

which yields

$$P(\sum_k \hat{U}_k > y) \leq P(|\hat{\rho}_n| > y/9n^{6\delta}). \tag{3.37}$$

Now we claim that for some constants C_i, for all $z \geq C_0 n^{1+\delta}$,

$$P(|\hat{\rho}_n| > z) \leq C_1 \exp(-C_2 z) \tag{3.38}$$

Indeed, for $C > 0$

$$P(|\hat{\rho}_n| > z) \leq P(\sum_{e\in\hat{\gamma}_n} \hat{\tau}_e > Cz) + P(\exists\, \gamma \in \mathcal{C}_0(z) \text{ with } \hat{t}(\gamma) < Cz). \tag{3.39}$$

By Lemma 3.4, we can find a constant $A_i > 0$ so that

$$P(\sum_{e\in\hat{\gamma}_n} \hat{\tau}_e > Cz) \leq \sum_{i=1}^{A_0 n} P(\sum_{e\in\hat{\gamma}_n} \hat{\tau}_e > z \mid |\hat{\gamma}_n| = i) P(|\hat{\gamma}_n| = i) + A_1 \exp(-A_2 n)$$

$$\leq \sum_{i=1}^{A_0 n} e^{-Caz} (Ee^{a\tau})^i P(|\hat{\gamma}_n| = i) + A_1 \exp(-A_2 n) \leq e^{-Caz} (Ee^{a\tau})^{A_0 n^{1+\delta}} + A_1 \exp(-A_2 n).$$

Therefore if we take α_i given by Lemma 3.3 and set $C = \alpha_1$, then this last inequality together with 3.39 yields 3.38.

By 3.37 and 3.38 for all $y \geq 9C_0 n^{1+7\delta}$(put $z = y/9n^{6\delta}$),

$$P(\sum_k U_k > y) \leq C_1 \exp(-C_2 y/9n^{6\delta}) \leq \exp(-C_2 y^{1/2}/9). \tag{3.40}$$

To justify the last inequality in 3.40, notice that $y^{1/2} \leq y/n^{6\delta}$(take the square in both sides) if and only if $n^{12} \leq y$, which is true for $\delta \in (0, 1/8)$ and $y \geq n^{1+8\delta}$.

□

Proof of Lemma 3.3. Let $F(y, \alpha)$ be the event that there exists a self-avoiding path ρ starting at 0 with at least y edges but with at most αy open edges. For each $z \geq y$, let $F(y, z, \alpha)$ be the event that there exists a self-avoiding path ρ starting at 0 with exactly z edges but with at most αy open edges. Then

$$P(F(y, \alpha)) = \sum_{z\geq y} P(F(y, z, \alpha)). \tag{3.41}$$

Since each edge is open or closed according to a Bernoulli distribution with parameter $q = P(e \text{ is closed })$, that is independent of the distribution of $\hat{\mathcal{V}}$, and the expected value of the number of self-avoiding paths starting at 0 with exactly z edges grows at most exponentialy fast (uniformily in n), we can find $B > 1$ so that

$$P(F(y,z,\alpha)) \leq B^z q^{z-\alpha y} = \exp(\alpha \log(1/q) y) \exp((\log(B) + \log(q)) z). \tag{3.42}$$

Therefore, if we take $q_0 = 1/B$ and $q < q_0$ then $-C_0 = \log(B) + \log(q) < 0$. This together with 3.41 3.42 implies that

$$P(F(y,\alpha)) \leq C_1 \exp((\alpha \log(1/q) - C_0) y), \tag{3.43}$$

where $C_1 = 1/(1 - e^{-C_0})$.

Thus if we take $\alpha < C_0/\log(1/q)$ then $-C_2 = \alpha \log(1/q) - C_0 < 0$ and so

$$P(F(y,\alpha)) \leq C_1 \exp(-C_2 y). \tag{3.44}$$

Now choose $\epsilon > 0$ such that $F(\epsilon) \in (F(0), q_0)$ and define the bond percolation model by $X_e = I_{[\tau_e > \epsilon]}$. Then

$$P(e \text{ is closed }) = P(X_e = 0) = F(\epsilon) < q_0. \tag{3.45}$$

By 3.45 and Proposition 3.44, we can find α_0 and $C_i > 0$ so that

$$P(F(y,\alpha_0)) \leq C_1 \exp(-C_2 y). \tag{3.46}$$

Although, $\epsilon I_{[X_e=1]} < \tau_e$, and so if we choose $\alpha < \alpha_0$ then

$$P(\exists\, \gamma \in \mathcal{C}_0(m) \text{ with } t(\gamma) < \alpha m) \leq P(F(m,\alpha_0)),$$

and this together with 3.46 finishes the proof of lemma 3.3.

□

Proof of Lemma 3.4. Here we give a sheme of the proof that follows the ideas of Vahidi-Asl and Wierman (see [29]). For each , $L > 0$ and $z \in (L\mathbf{Z}^2)$ let $B_z^L = z + [-L/2, L/2]$.

Construct four triangles by inserting the diagonals in B_z^L. We say that the box B_z^L has property T if there is at least one point $v \in \hat{\mathcal{V}}$ in each of the four triangles in B_z^L. A box B_z^{4L} has property 4T if each of its sixteen subsquares $B_{z'}^L$ has property T. A circuit of squares which all have property T is called a 4T-circuit.

Then if there exists a 4T-circuit with 0 and $n\vec{e}_1$ in its interior we can construct a self-avoiding path γ_n between 0 and $n\vec{e}_1$ which lies completely inside the T-circuit and its interior.

Define a site percolation model by declaring vertex z open if B_z^{4L} has property 4T. Notice that, by the definition of $\hat{\mathcal{V}}$

$$\lim_{L\to\infty} P(z \text{ is open }) = 1 \tag{3.47}$$

uniformily in n.

Denote by C the smalest open circuit around $[0, n\vec{e}_1]$ and by H its interior. Let N_z denote the cardinality of $B_z^{4L} \cap \hat{\mathcal{V}}$ and consider $N = \sum_{z \in C \cup H} N_z$. Therefore

$$|\gamma_n| \leq N. \tag{3.48}$$

By 3.47 and some standard results concerning site percolation in the supercritical regime (see [9]), we can choose L_0 sufficiently large in order to ensure that there exist constants $C_j > 0$ such that

$$P(N \geq C_1 n) \leq C_2 \exp(-C_3 n),$$

and this together with 3.48 yields Lemma 3.4.

□

3.3 Roughness at the boundary

Here we prove Theorem 1.5. This proof parallels the arguments of Alexander (see [1]) where the analogous result for the lattice case are presented.

Proof of Theorem 1.5 Let $\kappa > 1/2$ and κ_1, κ_2 so that $1/2 < \kappa_2 < \kappa_1 < \kappa$. Suppose that

$$\exists\, t > 0 \text{ and } x \in G_0(t) \text{ but } x \notin (t + t^\kappa)B_0. \tag{3.49}$$

Then

$$T(0,x) \leq t \text{ and } (t + t^\kappa) < \mu|x|,$$

and consequently,

$$t^\kappa \leq \mu|x| - t \leq \mu|x| - T(0,x). \tag{3.50}$$

If $\mu|x| \leq 2t$, then 3.50 implies that, for $|x|$ large enough,

$$|x|^{\kappa_1} < \frac{\mu^\kappa}{2^\kappa}|x|^\kappa \leq t^\kappa \leq \mu|x| - T(0,x).$$

If $\mu|x| \geq 2t$, then 3.50 implies that, for $|x|$ large enough,

$$|x|^{\kappa_1} < \frac{\mu}{2}|x| \leq \mu|x| - t \leq \mu|x| - T(0,x).$$

Now suppose that,

$$\exists\, x \notin G_0(t) \text{ but } x \in (t - t^\kappa)B_0. \tag{3.51}$$

Then

$$t < T(0,x) \text{ but } \mu|x| \leq t - t^\kappa,$$

and consequently, for $|x|$ large enough

$$|x|^{\kappa_1} < t^\kappa \leq t - \mu|x| \leq T(0,x) - \mu|x|.$$

Thus if 3.49 or 3.51 occurs for arbitrarily large t, we can find a sequence $(x_n)_{n\geq 1}$ so that

$$|x_n| \to_{n\to\infty} +\infty$$

and for all $n \geq 1$,

$$|T(0,x_n) - \mu|x_n|| > |x_n|^{\kappa_1}.$$

For each $x \in \mathbf{R}^2$ there is $z \in \mathbf{Z}$ so that $x \in z + [-1/2, 1/2]^2 \equiv B_z$ and

$$|T(0,x) - \mu|x|| \leq |T(0,z) - \mu|z|| + \max_{y \in B_z}\{T(z,y)\} + \mu\sqrt[2]{2}.$$

Therefore, if 3.49 or 3.51 occurs for arbitrarily large t, we can find a sequence $(z_n)_{n\geq 1}$, $z_n \in \mathbf{Z}^2$, so that

$$|z_n| \rightarrow_{n \rightarrow +\infty} +\infty$$

and for all $n \geq 1$,

$$|T(0,z_n) - \mu|z_n|| > |z_n|^{\kappa_2} \text{ or } \max_{y \in B_{z_n}}\{T(z_n,y)\} > |z_n|^{\kappa_2}.$$

By Theorem 1.4

$$\sum_{z \in \mathbf{Z}^2} P(|T(0,z) - \mu|z|| > |z|^{\kappa_2}) < +\infty$$

and

$$\sum_{z \in \mathbf{Z}^2} P(\max_{y \in B_z}\{T(y,z)\} > |z|^{\kappa_2}) = \sum_{z \in \mathbf{Z}^2} P(\max_{y \in B_0}\{T(y,0)\} > |z|^{\kappa_2}) < +\infty.$$

Thus, by the Borel-Cantelli lemma the probability of the event that 3.49 or 3.51 occurs for arbitrarily large t has zero probability, and consequently, if $\kappa > \chi_1$ then almost surely, for all t large enough

$$(t - t^{\kappa})B_0 \subseteq G_0(t) \subseteq (t + t^{\kappa})B_0.$$

$\square$

Chapter 4

Geodesics

For $v, \tilde{v} \in \mathcal{V}$ we had defined (see section 1.5) that a self avoiding path $\rho(v, \tilde{v})$ in $\mathcal{G}$ is a geodesic connecting v to $\tilde{v}$ if

$$T(v, \tilde{v}) = t(\rho(v, \tilde{v})),$$

and proved that $F(0) < p_{cb}$ is a sufficient condition to ensure the existence of geodesics connecting two arbitrary vertices (see Proposition 1.1). We also had defined that a self avoiding path ρ in $\mathcal{G}$ is a geodesic if for all $v, \tilde{v} \in \rho$, there exist a geodesic $\rho(v, \tilde{v})$ connecting v and $\tilde{v}$ so that $\rho(v, \tilde{v}) \subseteq \rho$. In this chapter we will study the existence and geometric properties of semi-infinite geodesics.

4.1 Deviations of finite geodesics

One of the basic ingredients in study of the existence of $\vec{x}$-geodesics is a control of the fluctuations of finite geodesics, and it is given in the following theorem. For $v, \tilde{v} \in \mathcal{V}$, denote by $g(v, \tilde{v})$ the set of all vertices that belongs to some geodesic connecting v to $\tilde{v}$.

Theorem 4.1 *Assume that $F(0) < p_{cb}$ and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Let $\chi \in (1/2, 1)$. Then for all $\xi \in (\frac{\chi+1}{2}, 1)$, there exists $\delta_0 > 0$, $C_1, C_2 > 0$ such that* *for all $r > 0$ large enough*

$$P(\sup_{v \in g_r} d(v, [0, r\vec{e}_1]) \geq r^{\xi}) \leq C_1 \exp(-C_2 r^{\delta_0}),$$

where $g_r \equiv g(v_0, r\vec{e}_1)$.

*Proof of theorem 4.1.*The proof of this theorem uses the same methods due to Howard and Newman to prove the analogue result for the Euclidean first-passage percolation model (see [14]). Let $\tilde{\chi} \in (\chi, 1)$, $\delta \equiv \tilde{\chi} - \chi$ and $\xi \equiv \frac{\tilde{\chi}+1}{2} \in (0, 1)$. For each $r \geq 1$ define the following subsets of the plane:

$$A_r^0 \equiv \{x \in \mathbf{R}^2;\ d(x, [0, r\vec{e}_1]) < r^{\xi}\},$$

$$A_r^1 \equiv \{x \in \mathbf{R}^2 \backslash A_r^0;\ d(x, A_r^0) < r^{\xi}\},$$

and

$$A_r^2 \equiv \{x \in \mathbf{R}^2 \backslash A_r^0;\ d(x, A_r^0) < r^{\xi} + \sqrt{2}\}.$$

Denote by F_r the event defined by the following properties:

- $v_0, v_{r\vec{e}_1} \in A_r^0$
- for all $e = (v, \tilde{v}) \in \mathcal{E}$ with $|v| \leq 2r$ or $|\tilde{v}| \leq 2r$, then $|v - \tilde{v}| \leq r^{\xi}$.

Consider the random variable

$$T_z \equiv \max_{v \in B_z \cap \mathcal{V}} \{T(z, v)\},$$

where B_z is the square box centered at z and with side length 1. Thus we have the following inclusions:

$$[\sup_{v \in g_r} d(v, [0, r\vec{e}_1]) \geq r^{\xi}] \cap F_r \subseteq$$

$$[\exists v \in \mathcal{V} \cap A_r^1;\ T(0, v) + T(v, r\vec{e}_1) = T(0, r\vec{e}_1)] \subseteq D(r) \tag{4.1}$$

where

$$D(r) \equiv [\exists z \in \mathbf{Z}^2 \cap A_r^2;\ T(0, z) + T(z, r\vec{e}_1) \leq T(0, r\vec{e}_1) + 2T_z].$$

Define

$$\Delta(z, r\vec{e}_1) \equiv \mu|z - r\vec{e}_1| + \mu|z| - \mu|r\vec{e}_1|$$

and notice that

$$T(0, z) + T(z, r\vec{e}_1) \leq T(0, r\vec{e}_1) + 2T_z$$

if and only if,

$$\Delta(z, r\vec{e}_1) \leq (T(0, r\vec{e}_1) - \mu|r\vec{e}_1|) + (\mu|z| - T(0, z)) +$$

$$(\mu|z - r\vec{e}_1| - T(z, r\vec{e}_1)) + 2T_z.$$

This implies that

$$D(r) \subseteq \cup_{j=0}^{3} D_j(r) \tag{4.2}$$

where

$$D_0(r) \equiv [\exists z \in \mathbf{Z}^2 \cap A_r^2;\ T_z \geq \frac{\Delta(z, r\vec{e}_1)}{8}]$$

$$D_1(r) \equiv [\exists z \in \mathbf{Z}^2 \cap A_r^2;\ |T(z, r\vec{e}_1) - \mu|z - r\vec{e}_1|| \geq \frac{\Delta(z, r\vec{e}_1)}{4}]$$

$$D_2(r) \equiv [\exists z \in \mathbf{Z}^2 \cap A_r^2;\ |T(0, z) - \mu|z|| \geq \frac{\Delta(z, r\vec{e}_1)}{4}]$$

$$D_3(r) \equiv [\exists z \in \mathbf{Z}^2 \cap A_r^2;\ |T(0, r\vec{e}_1) - \mu|r\vec{e}_1|| \geq \frac{\Delta(z, r\vec{e}_1)}{4}].$$

4.1 and 4.2 provide us the following inequality:

$$P(\sup_{v \in g_r} d(v, [0, r\vec{e}_1]) \geq r^{\xi}) \leq P(F_r^c) + P(r), \tag{4.3}$$

where

$$P(r) \equiv \sum_{j=0}^{3} P(D_j(r)).$$

By the Pythagorean theorem, we can find constants B_1, B_2 such that for all $r > 0$ large and $z \in \mathbf{Z}^2 \cap A_r^2$ we have that

$$B_1 r^{\tilde{\chi}} = B_1 r^{2\xi - 1} \leq \Delta(z, r\vec{e}_1) \leq B_2 r^{\xi} = B_2 r^{\frac{\tilde{\chi}+1}{2}},$$

$$r^{\xi} \leq |z|, |z - r\vec{e}_1| \leq 2r. \tag{4.4}$$

In order to bound $P(r)$ we shall first see that there exist $\epsilon \in (0, \epsilon_0)$, where ϵ_0 is given in Theorem 1.4, such that for every $\lambda = \Delta(z, r\vec{e}_1)/4$ and every $y = |z|, |z - r\vec{e}_1|, r$, with $z \in A_r^2$ and $r > 0$ large,

$$\lambda \in (|y|^{\chi+\epsilon}, |y|^{\chi+(1-\epsilon)}).$$

To do so, choose $\delta_0 \in (0, \min\{\chi, \delta\})$ and $\epsilon < \min\{\delta_0, \epsilon_0\}$. By 4.4, for every large $r > 0$

$$|y|^{\chi+\epsilon} \leq (2r)^{\chi+\epsilon} < B_1 r^{\tilde{\chi}} \leq \lambda$$

$$\leq B_2 r^{\frac{\tilde{\chi}+1}{2}} < r^{\xi(\chi+(1-\epsilon))} \leq |y|^{\chi+(1-\epsilon)}.$$

Thus, by Theorem 1.4 and the Euclidean invariance,

$$P(D_1(r)) \leq \sum_{z \in \mathbf{Z}^2 \cap A_r^2} C_1 \exp(-C_2 \frac{\lambda}{|z - r\vec{e}_1|^{\chi}}),$$

$$P(D_2(r)) \leq \sum_{z \in \mathbf{Z}^2 \cap A_r^2} C_1 \exp(-C_2 \frac{\lambda}{|z|^{\chi}}),$$

$$P(D_3(r)) \leq \sum_{z \in \mathbf{Z}^2 \cap A_r^2} C_1 \exp(-C_2 \frac{\lambda}{r^{\chi}}). \tag{4.5}$$

Notice also that, by 4.4 and 4.5, for $j = 1, 2, 3$

$$P(D_j(r) \leq C_1 \exp(-C_2 r^{\delta_0}).$$

Therefore, by 4.3 and the last inequality we have that

$$P(\sup_{v \in g_r} d(v, [0, r\vec{e}_1]) \geq r^{\xi}) \leq P(F_r^c) + P(D_0(r)) +$$

$$\#(\mathbf{Z}^2 \cap A_r^2)\{3C_1 \exp(\frac{-C_2 r^{\delta}}{4})\}. \tag{4.6}$$

As $\int e^{ax} dF(x) < +\infty$ we have that for some $B > 0$, $E(e^{BT_0}) < +\infty$ (see [29]). Thus for every $\kappa \in (0,1)$,

$$P(T_0 \geq r^{\kappa}) \leq E(e^{BT_0}) e^{-Br^{\kappa}}. \tag{4.7}$$

But, $\#(\mathbf{Z}^2 \cap A_r^2)$ is at most of order $r^{2\xi}$, and thus Theorem 4.1 follows from 4.6, 4.7 together with Lemma 4.1 below.

Lemma 4.1 *For all $\kappa > 0$, there exists $C_3, C_4 > 0$ such that*

$$P(\exists\, x \in \mathbf{R}^2;\ |x| \leq 2r, |x - v_x| \geq r^\kappa) \leq C_3 \exp(-C_4 r^{2\kappa}).$$

Proof of Lemma 4.1. We follow the same proof due to Howard and Newman (see [14]). In fact, notice that for large $r > 0$,

$$[\exists\, x \in \mathbf{R}^2;\ |x| \leq 2r, |x - v_x| \geq r^\kappa] \subseteq$$

$$[\exists\, z \in \mathbf{Z}^2;\ |z| \leq 2r, |z - v_z| \geq \frac{r^\kappa}{2}].$$

By the definition of the Poisson point process, this larger event has probability bounded $C_3 \exp(-C_4 r^{2\kappa})$, which, for smaller C_4 is bounded by (for large r) $C_3 \exp(-C_4)$.

□

4.2 Uni-geodesics

To prove the existence of uni-geodesics with asymptotic directions we will use some results related to trees in $\mathbf{R}^2$. Let us assume that F is continuous and define $\mathcal{T}_v$ as the union taken over all $\tilde{v} \in \mathcal{V}$ of the geodesic between v and $\tilde{v}$. Since almost surely, for all $v, \tilde{v}$ the geodesic connecting v to $\tilde{v}$ exists and is unique (by Proposition 1.1), $\mathcal{T}_v$ is a tree spanning all $v \in \mathcal{V}$. For each $x = |x|e^{i\alpha}, y = |y|e^{i\beta} \in \mathbf{R}^2 \backslash \{0\}$, where $\alpha, \beta \in [0, 2\pi)$, let $ang(x, y) \equiv \min\{|\beta - \alpha|, 2\pi - |\beta - \alpha|\}$. Given $\epsilon \in [0, \pi)$ and $x \in \mathbf{R}^2 \backslash \{0\}$ let

$$C(x, \epsilon) \equiv \{y \in \mathbf{R}^2 \backslash \{0\};\ ang(y, x) \leq \epsilon\},$$

If $\mathcal{T}$ is a tree embedded in $\mathbf{R}^2$, for each pair $v, \tilde{v} \in \mathcal{T}$ let $R_{out}(v, \tilde{v})$ be the set of all $\hat{v} \in \mathcal{T}$ such that the unique path in $\mathcal{T}$ connecting v to $\hat{v}$ touches $\tilde{v}$. For each $f : [0, +\infty) \to [0, +\infty)$

so that $\lim_{r\to\infty} f(r) = 0$, we say that $\mathcal{T}$ is f-straight at v if, for all but finitely many $\tilde{v} \in \mathcal{T}$,

$$R_{out}(v, \tilde{v}) \subseteq\ v + C(\tilde{v} - v, f(|\tilde{v} - v|)).$$

Theorem 4.2 *Assume that $F(0) < p_{cb}$ and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Let $\chi \in (1/2, 1)$. Then for all $\xi \in (\frac{\chi+1}{2}, 1)$, almost surely for all $v \in \mathcal{V}$ $\mathcal{T}_v$ is f-straight at v with $f(r) \equiv r^{\xi-1}$.*

Proof of Theorem 4.2. Without loss of generality, we restrict attention to $v = v_0$. From Lemma 4.1, Theorem 4.1 and the Borel-Cantelli lemma, we have that for all $\xi \in (\frac{\chi+1}{2}, 1)$, almost surely there exists a large $M_0 > 0$ so that for any $v \in \mathcal{V}$ with $|v| > M_0$, and any edge $e = (\hat{v}, \tilde{v}) \in \mathcal{E}$ with $|\hat{v}| > M_0$, the following holds:

$$|\hat{v} - \tilde{v}| \leq \hat{v}^{\xi} \text{ and } d(\hat{v}, [v_0, v]) \leq |v|^{\xi}, \forall\, \hat{v} \in g(v_0, v).$$

Thus Theorem 4.2 follows from Lemma 4.2.

Lemma 4.2 *Let $\xi \in (0, 1)$ and $\epsilon \in (0, 1 - \xi)$. Suppose that $(v_1, v_2, ...)$ is any sequence of distinct points in $\mathbf{R}^2$ with $\lim_{n\to+\infty} |v_n| = +\infty$, and such that for all large n,*

$$|v_{n+1} - v_n| \leq |v_n|^{\xi} \text{ and } d(v_m, [v_1, v_n]) \leq |v_n|^{\xi} \text{ for } 1 \leq m \leq n.$$

Then there exist $C_1 > 0$ and $m^* > 0$ such that

$$ang(v_m, v_n) \leq C_1 |v_m|^{-\epsilon} \text{ whenever } m^* \leq m < n.$$

Proof of Lemma 4.2. We will follow the proof due to Howard and Newman in [14]. Choose L large enough that $L^{\xi} < L^{1-\epsilon} < L/3$, and then choose m^* so that for every $n \geq m^*$:

1. $|v_n| > L$;

2. $|v_{n+1} - v_n| \leq v_n^{\xi}$;

3. $d(v_m, [v_1, v_n]) \leq |v_n|^\xi \forall m \in \{1, ..., n\}$.

Now suppose $m^* \leq m \leq n$. First notice that, by item 3,

$$ang(v_m, v_n) < \frac{\pi}{2}.$$

If $|v_n| \leq 3|v_m|$, then

$$ang(v_n, v_m) \leq \frac{\pi}{2} \sin(ang(v_n, v_m)) \leq \frac{\pi |v_n|^{1-\epsilon}}{2|v_m|} \leq C|v_m|^{-\epsilon}, \tag{4.8}$$

where $C = 3^{1-\epsilon}\pi/2$. If $|v_n| > 3|v_m|$, define the following subsequence of $v_m, v_{m+1}, ..., v_n$. Let $v_{n_1} = v_m$ and suppose that $v_{n_1}, ..., v_{n_k}$ have been selected; if $v_{n_k} = v_n$, we stop. Otherwise, let

$$n_{k+1} \equiv \max\{l;\ n_k < l \leq n,\ |v_l| \leq 3|v_{n_k}|\}.$$

By construction, the $|v_{n_k}|$ are increasing with $|v_{n_{k+1}}| \leq 3|v_{n_k}|$. By item 2, for each $n_{k+1} < n$, we must have $2|v_{n_k}| \leq |v_{n_{k+1}}|$, and consequently,

$$2^{k-1}|v_m| \leq |v_{n_k}|.$$

From this and a repeated application of the previous case, it follows that

$$ang(v_m, v_n) = ang(v_{n_1}, v_n) \leq \sum_k ang(v_{n_k}, v_{n_{k+1}}) \leq C(\sum_k 2^{-(k-1)\epsilon})|v_m|^{-\epsilon}. \tag{4.9}$$

Choosing $C_1 \geq \max\{C, C(\sum_k 2^{-(k-1)\epsilon})\}$ and using 4.8 and 4.9 we finish the proof of Lemma 4.2.

$\square$

We say that $U \subseteq \mathbf{R}^2$, is asymptotically omni-directional if for all $L > 0$,

$$\{\frac{u}{|u|};\ u \in U, |u| > L\}$$

is dense in S^1. One can see that for a homogeneous Poisson process, with probability one, $\mathcal{V}$ is omni-directional.

Lemma 4.3 *Suppose that $\mathcal{T}$ is a tree whose vertex set $\mathcal{V} \subseteq \mathbf{R}^2$ is locally finite and asymptotically omni-directional and such that every vertex has finite degree. Suppose further that $\mathcal{T}$ is f-straight at v. Then $\mathcal{T}$ satisfies the following properties:*

- *every uni-path in $\mathcal{T}$ starting from v is a $\vec{x}$-path for some $\vec{x} \in S^1$;*
- *for every $\vec{x} \in S^1$, there is at least one $\vec{x}$-path in $\mathcal{T}$ starting from v.*

Proof of Lemma 4.3. We will follow the proof due to Howard and Newman (see [14]). Let $(v_n)_{n\geq 1}$ be a uni-path in T. Then f-straightness implies that for large m,

$$ang(v_n - v, v_m - v) \leq f(|v - v_m|), \text{ for } n \geq m.$$

Since, $|v_m| \to_{m\to\infty} +\infty$ ($\mathcal{V}$ is locally finite), it follows that $(v_n|v_n|^{-1})$ is a Cauchy sequence, and consequently converges. Since $\mathcal{V}$ is omni-directional and each vertex has finite degree, it follows that, for a given $\vec{x} \in S^1$, starting from $v_1 = v$, one can construct a uni-path $v_1, v_2, ...$ in $\mathcal{T}$ such that for each n, $R_{out}(v, v_n)$ contains a sequence $v_1^n, v_2^n, ...$ with

$$\frac{v_l^n}{|v_l^n|} \to_{l\to\infty} \vec{x}.$$

From the first part, it follows that the uni-path $(v_n)_{n\geq 1}$ must have an asymptotic direction, say $\vec{y}$. But f-straightness implies that

$$ang(\vec{x}, v_n - v) \leq f(|v_n - v|),$$

which yields $\vec{x} = \vec{y}$.

$\square$

Proof of Theorem 1.9. It follows from Theorem 4.2 and Lemma 4.3.

$\square$

To complete the first part concerning the description of uni-geodesics we shall ask about uniqueness. As Newman points out in [25], we can not ask that almost surely for all directions $\vec{x} \in S^1$ there exist at most one $\vec{x}$-geodesic starting from $v \in \mathcal{V}$. In fact, there will be infinitely many (random) directions $\vec{x}$ with more than one $\vec{x}$-geodesic with the same starting point (see also [21] and [14]). However, if we fix one direction and ask for the almost surely uniqueness with respect to this direction, then the answer is affirmative and is given by the next proposition.

Proposition 4.1 *Suppose that F is continuous. Then for any fixed direction $\vec{x} \in S^1$, a.s. for all $v \in \mathcal{V}$ there exists at most one $\vec{x}$-geodesic starting from v.*

Proof of proposition 4.1. Let $v, \hat{v} \in \mathcal{V}$ and assume that $\hat{v}$ is a vertex where occurs a bifurcation of two geodesics starting from v: i.e., there exist $e_i = (\hat{v}, \hat{v}_i)$ and $\rho_{\hat{v}_i}$ such that after $\hat{v}$ one uni-geodesic starting from v coincides with $\rho_{\hat{v}_1}$, and the other one coincides with $\rho_{\hat{v}_2}$. Define the uni-geodesic $\rho^+(v, \hat{v}) = (v_n)_n$, starting from $\hat{v}$ as follows. Let $k(\hat{v})$ be the number of of vertices of the polygon $C_{\hat{v}}$ and denote by $(\hat{v}^*_1, ..., \hat{v}^*_k)$, the set of the vertices of $C_{\hat{v}}$ with the counter clock-wise orientation so that the angle in $(0, 2\pi)$ between $\hat{v}$ and $\hat{v}_1$ is strictly less than the angle in $(0, 2\pi)$ between $\hat{v}$ and $\hat{v}_j$ for all $j = 2, ..., k$. Without loss of generality let us suppose that there exist $j_1 < j_2$ such that the edge $(\hat{v}^*_{j_1}, \hat{v}^*_{j_1+1})$ in $\mathcal{G}^*$ corresponds to the edge e_1 and that the edge $(\hat{v}^*_{j_2}, \hat{v}^*_{j_2+1})$ in $\mathcal{G}^*$ correspond to the edge e_2. Thus, set $v_1 = \hat{v}$ and $v_2 = \hat{v}_2$. From v_2 we follow ρ_{v_2} until we find a new bifurcation vertex $\tilde{v}$. Define the analogous objects we have defined for $\hat{v}$ such as $\hat{v}_i$ and $\rho_{\hat{v}_i}$. Then from $\tilde{v}$ we follow $\rho_{\tilde{v}_1}$ and apply the same rule have defined for $\tilde{v}$ every time we see a new bifurcation vertex. Since we are in the plane, if $\rho_{\hat{v}_1}$ and $\rho_{\hat{v}_2}$ are $\vec{x}$-geodesics, then $\rho^+(v, \hat{v})$ is a $\vec{x}$-geodesic (starting from $\hat{v}$) because any uni-geodesic contained in between $\rho_{\hat{v}_1}$ and $\rho_{\hat{v}_2}$ must be a $\vec{x}$-geodesic. This permits us to deduce that, if we denote by $D_{\vec{x}}$ the event that for all $v \in \mathcal{V}$ there exist at most one $\vec{x}$-geodesic starting from v, then $D_{\vec{x}}$ must occur unless the event $B_{\vec{x}} := [\rho^+(v, \hat{v})$ is an $\vec{x}$-geodesic for some $v, \hat{v} \in \mathcal{V}]$ occurs. Since $\mathcal{V}$ is enumerable and for each $v, \hat{v} \in \mathcal{V}$,

$\rho^+(v, \hat{v})$ cannot be an $\vec{x}$-geodesic for more than one $\vec{x}$, for each configuration ω the set of $\vec{x} \in S^1$ so that $I_{B_{\vec{x}}}(\omega) = 1$ is enumerable. Therefore

$$\int P(B_{\vec{x}}) d\vec{x} = \int [\int I_{B_{\vec{x}}} d\vec{x}] dP = 0, \tag{4.10}$$

where $d\vec{x}$ is the Lebesgue measure on S^1. Thus, 4.10 implies that there exists $S \subseteq S^1$, with total Lebesgue measure, such that for all $\vec{x} \in S$ we have that $P(D_{\vec{x}}) = 1$. By the Euclidean invariance, $P(D_{\vec{x}})$ does not depend on $\vec{x}$, which yields Proposition 4.1.

□

4.3 Coalescence

Let $\vec{x} \in S^1$. Proposition 4.1 ensures that almost surely, for all $v \in \mathcal{V}$ there exists at most one $\vec{x}$-geodesic starting from v. Recall that we had denoted this almost surely unique object by $\rho_v(\vec{x})$. The second part of Theorem 1.10 deal with the following question: suppose we fix $\vec{x} \in S^1$ and choose two different vertices v and $\tilde{v}$ in $\mathcal{V}$. Assume also that there exist a unique $\vec{x}$-geodesic starting from v and a unique $\vec{x}$-geodesic starting from $\tilde{v}$. Would we expect that they coalesce at some vertex $c \in \mathcal{V}$?

This question looks like uniqueness of infinite cluster in percolation problems because, in some sense, we are asking for a unique road to reached the infinity by following the direction $\vec{x}$. Licea and Newman in [21] gave a proof of this coalescence behavior for the lattice case (see also [13] for the Euclidean case) by following the same method due to Burton and Keane (see [4]) to prove the uniqueness of infinite clusters for two-dimensional square lattice percolation models. Here we also take advantage of this and make the necessary adjustments for the Voronoi case to prove the following theorem. Let $\mathcal{S}(\vec{x})$ denote the union over all $v \in \mathcal{V}$ of $\rho_v(\vec{x})$. Then $\mathcal{S}(\vec{x})$ is a forest with say $N(\vec{x})$ disjoint trees.

Theorem 4.3 *Assume that F is continuous. Then for all $\vec{x} \in S^1$,*

$$P(N(\vec{x}) \geq 2) = 0.$$

Proof of theorem 4.3. By isotropy it is sufficient to prove that

$$P(N(\vec{e}_1) \geq 2) = 0.$$

Firstly we will show that

$$P(N(\vec{e}_1) \geq 2) > 0 \Rightarrow P(N(\vec{e}_1) \geq 3) > 0. \tag{4.11}$$

To do so, let $\delta > 0$ and $x_i, \tilde{x}_i \in \mathbf{R}^2, i = 1, ..., j$ and denote by $A_\delta(x_1, ..., x_j, \tilde{x}_1, ..., \tilde{x}_j)$ the event determined by the following:

1. at each $B_\delta(x_i)$ and $B_\delta(\tilde{x}_i)$ there is one unique vertex v_i and $\tilde{v}_i$ respectively;
2. $e_i = (v_i, \tilde{v}_i)$ is one edge in $\mathcal{E}$ and $e_i \in \rho_{v_i}(\vec{e}_1)$;
3. after v_i, $\rho_{v_i}(\vec{e}_1)$ has vertices only with strictly positive coordinates;
4. the second coordinate of x_i and $\tilde{x}_i$ are increasing with the index i, the first coordinate of x_i is strictly less than $-\delta$ and the first coordinate of $\tilde{x}_i$ is strictly greater than δ;
5. $\rho_{v_i}(\vec{e}_1)$ are disjoint.

If $P(N(\vec{e}_1) \geq 2) > 0$ then there exist $\delta > 0$, $x_1, x_2, \tilde{x}_1, \tilde{x}_2 \in \mathbf{R}^2$ such that

$$P(A_\delta(x_1, x_2, \tilde{x}_1, \tilde{x}_2)) > 0.$$

Now define c_2 as the maximum between the second coordinate of x_2 and $\tilde{x}_2$ and c_1 as the minimum between the second coordinate of x_1 and x_2, and consider the rectangle $R_0 \equiv [-\delta, \delta] \times (c_1 - \delta, c_2 + \delta)$. Let z_0 be the circumcenter of the rectangle R_0, let M_0 be the vertical length of R_0 and define $z_l \equiv z_0 + l\vec{e}_2$, where $\vec{e}_2 \equiv (0, 1)$. For $l \in \mathbf{Z}$, define $R_l \equiv z_l + R_0$

and $A(l) \equiv A_\delta(x_1^l, x_2^l, \tilde{x}_1^l, \tilde{x}_2^l)$ where $x_j^l = x_j + z_l \in R(l)$ and $\tilde{x}_j^l = \tilde{x}_j + z_l$. By translation invariance, $P(A(l)) = P(A(0))$ and, by Fatou lemma,

$$0 < P(A(0)) \leq P(\limsup_l A(l)) \leq P(\cup_{l_1 \neq l_2} A(l_1) \cap A(l_2)).$$

Thus, there are $l_1 < l_2$ such that $P(A(l_1) \cap A(l_2)) > 0$. But the geodesic starting from $v_1^{l_1}$ can not intersect the geodesic starting from $v_1^{l_2}$ or the geodesic starting from $v_2^{l_2}$ because otherwise, by planarity and the definition of $A(l_1)$, the geodesic starting at $v_1^{l_1}$ will intersect the geodesic starting from $v_2^{l_1}$ which is a contradiction in the definition of $A(l_1)$. So,

$$A(l_1) \cap A(l_2) \subseteq A_\delta(x_1^{l_1}, x_1^{l_2}, x_2^{l_2}, \tilde{x}_1^{l_1}, \tilde{x}_1^{l_2}, \tilde{x}_2^{l_2}).$$

Then

$$0 < P(A_\delta(x_1^{l_1}, x_1^{l_2}, x_2^{l_2}, \tilde{x}_1^{l_1}, \tilde{x}_1^{l_2}, \tilde{x}_2^{l_2})),$$

and consequently, $P(N(\vec{e}_1) \geq 3) > 0$.

For $m, k \geq 0$, let

$$F_{m,k} \equiv [\text{some geodesic intersect } R_{m,k} \text{ but no other vertex in } Q_m]$$

where

$$R_{m,k} \equiv \{x = (x_1, x_2) \in \mathbf{R}^2;\ 0 \leq x_1 \leq m, |x_2| \leq k\}$$

and

$$Q_m \equiv \{x = (x_1, x_2) \in \mathbf{R}^2;\ x_1 \leq m\}.$$

Secondly we will show that

$$P(N(\vec{e}_1) \geq 3) > 0 \Rightarrow P(F_{m,k}) > 0 \tag{4.12}$$

for some $m, k \geq 0$. To verify 4.12 we shall divide the proof into two parts. The first one is when the passage time distribution has unbounded support, and the second one is when the passage time distribution has bounded support. Although, in both situation we will use a local modification argument which is formalized by the following lemma. Let K be the set of all sequences

$$I = ((k_j, l_j, n_j, m_j))_{j=1,\dots,q} \in (\mathbf{N}^4)^q \tag{4.13}$$

where $1 \leq q$, for $j \neq i$ $(k_j, l_j, n_j, m_j) \neq (k_i, l_i, n_i, m_i)$, $k_1 \leq \dots \leq k_q$, and either $k_j < n_j$ or $l_j < m_j$. Each $I \in K$ will correspond to a random vector given by

$$(\tau_{k_j,l_j}^{n_j,m_j})_{j=1,\dots,q},$$

and which live in the probability space

$$(\Omega_I, \mathcal{F}_I, P_I)$$

where

$$\Omega_I \equiv \mathbf{R}^q.$$

Let

$$\hat{\Omega}_I \equiv \{\hat{\omega}_I;\ \exists\, \omega_I \in \Omega_I \text{ with } (\hat{\omega}_I, \omega_I) \in \Omega\}.$$

Recall that the probability space (Ω, P) is a product space (see Section 1.1), that

$$\Omega = \hat{\Omega}_I \times \Omega_I$$

and that,

$$P = \hat{P}_I \times P_I.$$

Lemma 4.4 *For $I \subseteq K$, $A \subseteq \Omega$ and $\omega_1 \in \hat{\Omega}_I$ define*

$$A_{I,\omega_1} \equiv \{\omega_2 \in \Omega_I; \omega = (\omega_1, \omega_2) \in A\}.$$

Let $\{R_I; I \in K\}$ be a family of events $R_I \in \mathcal{F}_I$ such that $P_I(R_I) > 0$ for all I. Then define the map on $\mathcal{F}$ by

$$\Phi_I(A) \equiv \{\omega_1 \in \hat{\Omega}_I; P_I(A_{I,\omega_1}) > 0\} \times R_I.$$

Suppose that $W(\omega)$ is a random element of K and $\tilde{\Phi}(A)$, for $A \subseteq \Omega$, is defined as

$$\tilde{\Phi}(A) \equiv \cup_{I \in K}[\{\omega_1 \in \hat{\Omega}_I; A(I)_{I,\omega_1} \neq \emptyset\} \times R_I],$$

where $A(I) \equiv A \cap [W = I]$. Therefore, for $A \in \mathcal{F}$, $\tilde{\Phi}(A)$ contains $\Phi(A) \in \mathcal{F}$ defined as the following union

$$\Phi(A) \equiv \cup_{I \in K} \Phi_I(A(I)).$$

Furthermore, $P(A) > 0$ implies $P(\Phi(A)) > 0$.

Proof of Lemma 4.4. By noting that

$$P_I(A(I)_{I,w_1}) > 0 \Rightarrow A(I)_{I,w_1} \neq \emptyset,$$

is easy to see that $\Phi(A) \subseteq \tilde{\Phi(A)}$. Since K is countable, if $P(A) > 0$ then there exists $I \in K$ such that $P(A(I)) > 0$ (note that $A = \cup_{I \in K} A(I)$). For this I, by Fubini theorem

$$0 < P(A(I)) = \int_{\hat{\Omega}_I} P_I(A(I)_{I,w_1}) \hat{P}_I(dw_1).$$

Define $\hat{A}_I \equiv \{w_1; P_I(A(I)_{I,w_1}) > 0\}$ and so by the last equation, $\hat{P}_I(\hat{A}_I) > 0$. But, according to the definition of Φ_I,

$$P(\Phi_I(A(I))) = \hat{P}_I(\hat{A}_I) P_I(R_I) > 0.$$

Since $\Phi(A)$ contains $\Phi_I(A(I))$, we conclude that $P(\Phi(A)) > 0$.

□

Let us suppose that F has unbounded support, i.e.: $\{r > 0; F(r) = 1\} = \emptyset$. As it was noticed above,

$$P(N(\vec{e}_1) \geq 3) > 0 \Rightarrow P(A_\delta(x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)) > 0,$$

for some $\delta > 0$ and $x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ in the plane. So, without loss of generality suppose that the second coordinate of these points increase with the index, and consider the rectangle

$R \equiv [-\delta, \delta] \times [C_S, C_N]$, where $C_S \equiv \min\{x_1^2, \tilde{x}_1^2\}$ and $C_N \equiv \max\{x_3^2, \tilde{x}_3^2\}$ (the index 2 indicate the second coordinate). Denote by Ξ the set of edges which cross the rectangle R and the vertical coordinate axis, and observe that for all $w \in A_\delta(x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3), e_i \equiv (v_i, \tilde{v}_i) \in \Xi$ (remember that $x_i \in C_{v_i}, \tilde{x}_i \in C_{\tilde{v}_i}$). Define the event B_λ by those configurations that for all $e = (v_1, v_2) \in \Xi$ there exists γ connecting v_1 to v_2 but not using edges in Ξ such that $t(\gamma) < \lambda$. As

$$\lim_{\lambda \to +\infty} P(B_\lambda) = 1,$$

we can choose $\tilde{\lambda} > 0$ in order that

$$P(A_\delta(x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \cap B_{\tilde{\lambda}}) > 0. \tag{4.14}$$

Now we will define $W(\omega)$ by the following procedure: given $\omega \in \Omega$ we define

$$W(\omega) \equiv ((k_j, l_j, n_j, m_j))_{j=1,\ldots,q}.$$

by ordering all (k, l, n, m) so that

$$e(\omega) = (U_{k,l}(\omega), U_{n,m}(\omega) \in \Xi(\omega),$$

according to 4.13. Heuristically, W represents the indexes of the edges $e \in \Xi$. For each $I \in K$, let

$$R_I \equiv (\tilde{\lambda}, +\infty)^q \subseteq \Omega_I.$$

Let

$$A \equiv A_\delta(x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \cap B_{\tilde{\lambda}}.$$

Consider $\tilde{\Phi}$ defined in Lemma 4.4 with W and R_I as above. These objects satisfy the assumption of Lemma 4.4 (note that $P_I(R_I) > 0$ because F has unbounded support) and thus there exist

$$\Phi(A) \subseteq \tilde{\Phi}(A) \text{ with } P(\Phi(A)) > 0.$$

Now take $\tilde{\omega} \in \tilde{\Phi}(A)$. The path $\rho_{\tilde{v}_i}(\vec{e}_1)(\omega)$ remains a $\vec{e}_1$-geodesic for the configuration $\tilde{\omega}$ because the only difference between ω and $\tilde{\omega}$ is in the passage times τ_e with $e \in \Xi$ (note that by the definition of A, $\rho_{\tilde{v}_i}(\vec{e}_1)$ do not intersect Ξ). Notice that $\tilde{\omega} \in B_{\tilde{\lambda}}$ and that for every edge $e \in \Xi$, $\tau_e(\tilde{w}) > \tilde{\lambda}$. This implies that no geodesic can have an edge in Ξ. Therefore

$$\tilde{\Phi}(A) \subseteq F_{m,k},$$

where $m \equiv \delta + \max\{\tilde{x}_1^1, \tilde{x}_2^1, \tilde{x}_3^1\}$ and $k \equiv \max\{C_N, C_S\}$. This permit us to conclude that $P(F_{m,k}) > 0$.

For the case where F has bounded support we shall be more careful in the modification of the passages times. In fact we will need to use a fine control on the length of paths in the random graph $\mathcal{G}$. Let g denote the graph metric associated to the random graph $\mathcal{G}$, i.e., for $A, B \subseteq \mathcal{V}$, $g(A, B)$ is the minimum number of edges that one path should pass to go from A to B. Given $R \subseteq \mathbf{R}^2$, let define $I(R)$ as the number of Voronoi cells intersecting R.

Proposition 4.2 *There exist $\nu, \sigma \in (0, +\infty)$ such that almost surely,*

$$\lim_{n\to\infty} \frac{g(v_0, v_{n\vec{e}_1})}{n} = \nu \ \textit{and} \ \lim_{n\to+\infty} \frac{I([0, n\vec{e}_1])}{n} = \sigma.$$

Proof of proposition 4.2. Consider the first-passage percolation model where, the passage time distribution F is concentrated at the value 1, i.e.,$F(x) = 1_{[1,+\infty]}$. Then, if we denote the first-passage time with passage distribution F by T_F it is easy to see that $T_F = g$.

But such a F satisfies the assumption of Theorems 1.2 and 1.3. The second convergence follows from subadditivity and ergodicity.

□

Denote by $\lambda(F)$ the supremum of the support of F and recall that we had denoted by $\mu(F)$ the time constant given by Theorem 1.1. We claim that

$$\mu(F) < \lambda(F)\nu. \tag{4.15}$$

Indeed, notice that by choosing a path γ_n that minimizes the graph distance between v_0 and $v_{n\vec{e}_1}$, we obtain that

$$T(0, n\vec{e}_1) \leq \sum_{e \in \gamma_n} \tau_e.$$

But given $\mathcal{V}$, $\{\tau_e\}_{e\in\mathcal{E}}$ is independent of γ_n. This implies that

$$\begin{aligned} E(T(0, n\vec{x})) &\leq E(\sum_{e\in\gamma_n} \tau_e) = E(E(\sum_{e\in\gamma_n} \tau_e \mid \mathcal{V})) \\ &= E(g(v_0, v_{n\vec{x}})E(\tau_e)). \end{aligned}$$

Thus,

$$\mu(F) \leq \nu E(\tau_e),$$

and by noticing that $E(\tau_e) = \int x dF(x) < \lambda(F)$ we obtain 4.15.

For x_1, x_2, x_3 in the definition of $A_\delta(x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ let denote by $C_m^{\tilde{\epsilon}}$ the event that $\rho_{v_i}(\vec{e}_1)$ touches the line $\{(x^1, x^2); x^1 = m\}$, for $i = 1, 2, 3$, for the first time (coming from v_i) within the vertical segment $Q_{m,\tilde{\epsilon}}$ of length $2\tilde{\epsilon}m$ centered at the point $m\vec{e}_1$. By the definition of $\vec{e}_1$-geodesic,

$$\lim_{m\to+\infty} P(C_m^{\tilde{\epsilon}}) = 1 \tag{4.16}$$

for all $\tilde{\epsilon} > 0$.

For $m, k > 0$ let $C_{m,k}$ denote the event that $\rho_{v_i}(\vec{e}_1)$, for $i = 1, 2, 3$, do not intersect $\{(x^1, x^2); x^1 \in [0, m], |x^2| > k\}$. Then again, by the definition of $\vec{e}_1$-geodesics, for any fixed $m > 0$,

$$\lim_{k\to+\infty} P(C_{m,k}) = 1. \tag{4.17}$$

For $\epsilon, \tilde{\epsilon} > 0$, denote as $B_m^{\epsilon,\tilde{\epsilon}}$ the event that for every z with first coordinate between $-\delta$ and δ and second coordinate between $\max\{x_3^2, \tilde{x}_3^2\}$ and $\min\{x_1^2, \tilde{x}_1^2\}$ and every $u \in Q_{\tilde{\epsilon},m}$,

$$T(z, u) < (\mu + \epsilon)m. \tag{4.18}$$

From the shape theorem it follows that, for any $\epsilon > 0$ and any $\tilde{\epsilon}$ small enough,

$$\lim_{m\to+\infty} P(B_m^{\epsilon,\tilde{\epsilon}}) = 1. \tag{4.19}$$

Lemma 4.5 *For all λ such that $\mu < \lambda\nu$, there exists $\epsilon_0, \tilde{\epsilon}_0 > 0$ such that for all $\epsilon < \epsilon_0$, $\tilde{\epsilon} < \tilde{\epsilon}_0$ and for any rectangle R,*

$$\lim_{m \to +\infty} P(\lambda I(R) + (\mu+\epsilon)m + \lambda I(Q_{m,\tilde{\epsilon}}) \leq (\lambda-\epsilon) g(R, Q_{m,\tilde{\epsilon}})) = 1$$

Proof of lemma 4.5. Fix $\eta > 0$ such that

$$\mu + \eta < \lambda\nu.$$

Note that,

$$\lambda I(R) + (\mu+\epsilon)m + \lambda I(Q_{m,\tilde{\epsilon}}) \leq (\lambda-\epsilon) g(R, Q_{m,\tilde{\epsilon}})$$

if and only if

$$\lambda I(R) + \lambda I(Q_{m,\tilde{\epsilon}}) + \epsilon g(R, Q_{m,\tilde{\epsilon}}) + \epsilon m \leq \lambda g(R, Q_{m,\tilde{\epsilon}}) - \mu m.$$

By Proposition 4.2

$$P(\lambda g(R, Q_{m,\tilde{\epsilon}}) < (\mu+\eta)m) \to_{m\to\infty} 0.$$

Therefore, in order to obtain Lemma 4.5, we need to prove that

$$P(\lambda I(R) + \lambda I(Q_{m,\tilde{\epsilon}}) + \epsilon g(R, Q_{m,\tilde{\epsilon}}) + \epsilon m > \eta m) \to_{m\to\infty} 0. \tag{4.20}$$

However, $g(R, Q_{m,\tilde{\epsilon}}) \leq g(R, m\vec{e}_1)$ and by Proposition 4.2, we can find $C_1, C_2 > 0$ so that

$$P(g(R, m\vec{e}_1) > C_2 m) \to_{m\to\infty} 0,$$

and

$$P(I(Q_{l,1}) > C_1 l) \to_{l\to\infty} 0.$$

This implies that

$$P(\lambda I(R) + \lambda I(Q_{m,\tilde{\epsilon}}) + \epsilon g(R, Q_{m,\tilde{\epsilon}}) + \epsilon m > \lambda I(R) + O(\epsilon, \tilde{\epsilon})m) \to_{m\to+\infty} 0, \tag{4.21}$$

where

$$O(\epsilon, \tilde{\epsilon}) \equiv \tilde{\epsilon}_0 \lambda C_1 + \epsilon_0 (C_2 + 1).$$

If we choose ϵ_0 and $\tilde{\epsilon}_0$ such that for all $\epsilon < \epsilon_0$ and for all $\tilde{\epsilon} < \tilde{\epsilon}_0$ (notice that $\lambda I(R)$ does not depend on m)

$$P(O(\epsilon, \tilde{\epsilon})m + I(R) > \eta m) \to_{m\to\infty} 1, \tag{4.22}$$

we obtain that for all $\epsilon < \epsilon_0$ and $\tilde{\epsilon} < \tilde{\epsilon}_0$ the probability in the LRS of 4.20 is less than

$$P_1(\epsilon, \tilde{\epsilon}, m) + P_2(\epsilon, \tilde{\epsilon}, m), \tag{4.23}$$

where

$$P_1(\epsilon, \tilde{\epsilon}, m) \equiv P(\lambda I(R) + \lambda I(Q_{m,\tilde{\epsilon}}) + \epsilon g(R, Q_{m,\tilde{\epsilon}}) + \epsilon m > \lambda I(R) + O(\epsilon, \tilde{\epsilon})m),$$

and

$$P_2(m, \epsilon, \tilde{\epsilon}) \equiv P(O(\epsilon, \tilde{\epsilon})m + I(R) > \eta m).$$

Thus 4.21, 4.22 and 4.23 imply 4.20.

$\square$

For each $\lambda > 0$ and $\epsilon, \tilde{\epsilon} > 0$ define,

$$D \equiv [\lambda I(R) + (\mu + \epsilon)m + \lambda I(Q_{m,\tilde{\epsilon}}) \leq (\lambda - \epsilon)g(R, Q_{m,\tilde{\epsilon}}]$$

and

$$A \equiv A_\delta(x_1, x_2, x_3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \cap C_m^{\tilde{\epsilon}} \cap C_{m,k} \cap B_m^{\epsilon,\tilde{\epsilon}} \cap D$$

(see Figure 4.1). By 4.15, 4.16, 4.17, 4.19 and Lemma 4.5, if we fix $\epsilon > 0$ such that

$$\mu + \epsilon < (\lambda(F) - \epsilon)\nu,$$

then we can find $\tilde{\epsilon}$ small enough, m and k large enough such that

$$P(A) > 0.$$

For $j = 1, 2, 3$, let ρ_j the piece of $\rho_{v_j}(\vec{e}_1)$ between $\tilde{v}_j$ and the first time it intersect $Q_{\tilde{\epsilon},m}$). Notice that, for any configuration $\omega \in A$, and any points z and u as in the definition of

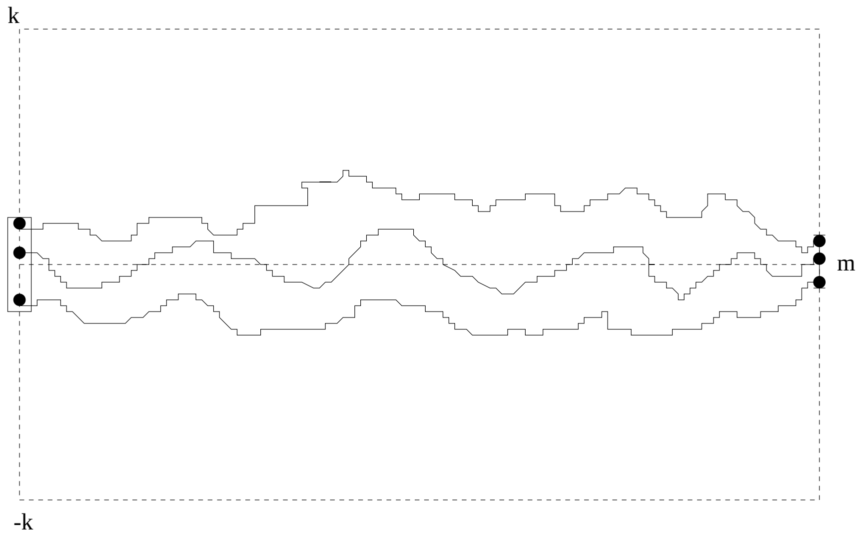

Figure 4.1: The event A.

$B_m^{\epsilon,\tilde{\epsilon}}$, we can find a path $\gamma(z,u)$, from z to u, which first moves vertically down and uses cells which intersect the rectangle R (recall that $R \equiv [-\delta,\delta] \times [C_S, C_N]$), then follows ρ_1, then moves vertically again, and

$$t(\gamma(z,u)) < \lambda I(R) + (\mu+\epsilon)m + \lambda I(Q_{m\tilde{\epsilon}}) \leq (\lambda-\epsilon) g(R, Q_{m,\tilde{\epsilon}}). \tag{4.24}$$

Define Ξ as the set of edges in the interior of the region bounded by ρ_1, ρ_3, $Q_{m,\tilde{\epsilon}}$ and R. Now we will define $W(\omega)$, as follows: given $\omega \in \Omega$ we define

$$W(\omega) \equiv ((k_j, l_j, n_j, m_j))_{j=1,\ldots,q}.$$

by ordering all (k,l,n,m) so that

$$e(\omega) = (U_{k,l}(\omega), U_{n,m}(\omega) \in \Xi(\omega),$$

according to 4.13. Heuristically, W represents the indexes of the edges $e \in \Xi$. For each $I \in K$, let

$$R_I \equiv (\lambda-\epsilon, \lambda)^q \subseteq \Omega_I.$$

Consider $\tilde{\Phi}$ defined in Lemma 4.4 with $W(\omega)$ and R_I as above. These objects satisfy the assumption of Lemma 4.4 (note that $P_I(R_I) > 0$ since $F(\lambda - \epsilon) < 1$) and thus there exists

$$\Phi(A) \subseteq \tilde{\Phi}(A) \text{ with } P(\Phi(A)) > 0.$$

The point is that, for $\omega \in \tilde{\Phi}(A)$ any geodesic ρ originated in $\{(x^1, x^2); x^1 \leq m\}$ other than in the rectangle $R_{m,k}$ could not touch the middle geodesic $\rho_{v_2}(\vec{e}_1)$ without either touching ρ_1 or ρ_3 or else having a segment $\tilde{\rho}$ entirely using edges of Ξ. In the first case it would coalesce with $\rho_{v_1}(\vec{e}_1)$ or with $\rho_{v_3}(\vec{e}_1)$. In the second case, as for all $e \in \Xi$, $\tau_e > \lambda - \epsilon$ then

$$t(\tilde{\rho}) > (\lambda - \epsilon) g(R, Q_{m,\tilde{\epsilon}}),$$

which can not happen for $\omega \in \tilde{\Phi}$ by 4.24. As in the previous case this permit us to conclude that we can find $m, k \geq 0$ so that $P(F_{m,k}) > 0$.

Finally we will show that for all $m, k \geq 0$,

$$P(F_{m,k}) = 0. \tag{4.25}$$

Indeed, let consider a rectangular array of non-intersecting translates $R^u_{m,k}$ of the basic rectangle $R_{m,k}$ indexed by $z \in \mathbf{Z}^2$, and also consider the corresponding event $F^z_{m,k}$. If $F^z_{m,k}$ and $F^{\tilde{z}}_{m,k}$ occur, then the corresponding trees in $\mathcal{S}(\vec{x})$ must be disjoint. So, if N_L is the number of $F^z_{m,k}$ which occurs with $z \in [0, L]^2$, then

$$N_L \leq \#(\{ \text{ edges crossing the boundary of } [0, L]^2\}).$$

But the expected value of the number of edges crossing the boundary of $[0, L]^2$ is of order L. Using translation invariance,

$$E(N_L) = n_L P(F_{M,k}),$$

where n_L is the number of rectangles $R^z_{m,k}$ in $[0, L]^2$. If $P(F_{m,k}) > 0$ we are lead to a contradiction because n_L is of order L^2.

Therefore, 4.11, 4.12, 4.25 allow us to prove that

$$P(N(\vec{e}_1) \geq 2) = 0.$$

$\square$

Proof of Theorem 1.10. By Theorem 1.9 and Theorem 4.3, $S(\vec{x})$ is a tree spanning all $\mathcal{V}$. Thus, for all $v, \tilde{v} \in \mathcal{V}$ the $\vec{x}$-geodesics $\rho_v(\vec{x})$ and $\rho_{\tilde{v}}(\vec{x})$ have a non empty intersection. However, by Proposition 4.1 if $\rho_v(\vec{x})$ and $\rho_v(\vec{x})$ ever meet then they must coalesce.

$\square$

Chapter 5

Competition interfaces

In this chapter we will focus on the morphology of k growing clusters which are competing for space. Firstly, let us restrict the attention for the case $k = 2$. From now on we will suppose that F is continuous. In Section 1.4, we had considered the competing spatial growth process

$$\{G_{1,r}(t), G_{2,r}(t)\}_{t>0},$$

which describes the growth of two clusters which are competing for space. Recall that we had defined

$$\Omega_r^2 \equiv [v_{1,r} \neq v_{2,r}],$$

where $(-1)^{j+1} r\vec{e}_1 \in C_{v_{j,r}}$, for $j = 1, 2$. Given $\omega \in \Omega_r^2$, a cell $C_v(\omega)$ is contained in $G_{i,r}(t)$ if and only if

$$T(v_{i,r}, v) < t \text{ and } T(v_{i,r}, v) < T(v_{j,r}, v),$$

where $i, j = 1, 2$ and $i \neq j$. We had considered the the regions

$$G_{i,r} \equiv \lim_{t\to+\infty} G_{i,r}(t),$$

and defined the interface path by

$$\psi_r \equiv G_{1,r} \cap G_{2,r}.$$

For $\omega \in (\Omega_r^2)^c$, we assumed that the interface is empty.

The heuristic idea in the analysis of the shape of ψ_r is the following: suppose we paint the region $G_{1,r}$ with the black color and the region $G_{2,r}$ with the gray color. Fix a direction $\vec{x}$ and denote the semi-infinite line starting from the origin and with direction $\vec{x}$ by $l_{\vec{x}}^0$. Say that $l_{\vec{x}}^0$ becomes of the color $j \in \{1,2\}$ if there exists $s > 0$ so that $l_{\vec{x}}^s \subseteq G_{j,r}$. Then we have the following possibilities:

1. $l_{\vec{x}}^0$ becomes eventually black;

2. $l_{\vec{x}}^0$ becomes eventually gray;

3. the color of $l_{\vec{x}}^0$ alternates infinitely often.

Denote the sequence of cells which intersect in the line $l_{\vec{x}}^0$ by $(v_{\vec{x}}^n)_{n\geq 1}$. Then we would expect that, almost surely, the sequences of geodesics $(\rho(v_{1,r}, v_{\vec{x}}^n))_{n\geq 1}$ and $(\rho(v_{2,r}, v_{\vec{x}}^n))_{n\geq 1}$ will converge to the $\vec{x}$-geodesics $\rho_{v_{1,r}}(\vec{x})$ and $\rho_{v_{2,r}}(\vec{x})$ respectively. By Theorem 1.10, these two $\vec{x}$-geodesics must coalesce and this permits us to see that the predominant color of the line $l_{\vec{x}}^0$ is the color of who first arrived in the coalescence point.

5.1 Coalescence function

In this Section we will prove Theorem 1.11 and part of Theorem 1.12 (see Theorem 5.1 below). Recall that we defined the coalescence function (under the assumption of Theorem 1.11) by

$$H^{\vec{x}}(v, \tilde{v}) \equiv T(v, c) - T(\tilde{v}, c),$$

where $c \equiv c(v, \tilde{v}, \vec{x})$ is the coalescence point given by Theorem 1.10.

Proof of Theorem 1.11. First notice that given $\vec{x} \in S^1$, almost surely for all $v \in \mathcal{V}$ and all sequence of vertices $v_1, v_2, ... \in \mathcal{V}$ such that

$$\frac{v_n}{|v_n|} \to_{n\to+\infty} \vec{x},$$

the sequence of finite geodesics $\rho(v, v_n)$ converges to the unique $\vec{x}$-geodesic $\rho_v(\vec{x})$. Indeed, for a small $\epsilon > 0$ let $\vec{x}_+(\epsilon)$ (resp. $\vec{x}_-(\epsilon)$) be the direction obtained by rotating $\vec{x}$ by an angle ϵ in the clockwise (resp. counter-clockwise) direction. By Theorem 1.9 and Proposition 4.1, almost surely there exist a unique $\rho_v(\vec{x}_+(\epsilon))$ and $\rho_v(\vec{x}_-(\epsilon))$. These paths bifurcate at some $\tilde{v}$. Let us see these paths as polygonal paths in $\mathbf{R}^2$. Then

$$\mathbf{R}^2 \backslash \{\rho_v(\vec{x}_+(\epsilon)) \cup \rho_v(\vec{x}_-(\epsilon))\}$$

consists of two connected components that we will denoted by $R_v^{in}(\vec{x}, \epsilon)$ and $R_v^{out}(\vec{x}, \epsilon)$. The component $R_v^{in}(\vec{x}, \epsilon)$ is characterized by containing sequences $x_1, x_2, ... \in \mathbf{R}^2$ such that $|x_n| \to \infty$ while the angle between $x_n/|x_n|$ and $\vec{x}$ converges to a point in $(-\epsilon, \epsilon)$. By uniqueness of finite geodesics, once n is large enough that $v_n \in R_v^{in}(\vec{x}, \epsilon)$, $\rho(v, v_n)$, except for its initial portion from v to $\tilde{v}$, must be entirely within the closure of $R_v^{in}(\vec{x}, \epsilon)$ and thus the same must be true for any subsequence limit γ of $\rho(v, v_n)$. Since this is true for every $\epsilon > 0$, it follows that such γ is automatically a $\vec{x}$-geodesic starting from v. Then by Proposition 4.1, such path should be $\rho_v(\vec{x})$, as claimed.

By the previous paragraph, almost surely, for any $v, \tilde{v}, \in \mathcal{V}$

$$\rho(v, x_n) \to_{n \to +\infty} \rho_v(\vec{x}) \text{ and } \rho(\tilde{v}, x_n) \to_{n \to +\infty} \rho_{\tilde{v}}(\vec{x}).$$

Therefore, by Theorem 1.10, almost surely, for all large n,

$$T(v, x_n) - T(\tilde{v}, x_n) = T(v, c) - T(\tilde{v}, c) \equiv H^{\vec{x}}(v, \tilde{v}).$$

□

For $\vec{x} \in S^1$ and $x, y \in \mathbf{R}^2$, we had defined

$$H^{\vec{x}}(x, y) \equiv H^{\vec{x}}(v_x, v_y).$$

The following theorem proves part of Theorem 1.12 and will be useful in the study of the asymptotic behavior of the interface directions when the distance between the initial sources goes to infinity.

Theorem 5.1 *Assume that F is continuous and that there exists $a > 0$ such that $\int e^{ax} dF(x) < +\infty$. Then*

$$\lim_{r\to\infty} \frac{H^{\vec{e}_1}(r\vec{e}_1, 0)}{r} = -\mu(F) \text{ in probability.}$$

Proof of Theorem 5.1. Let

$$H_r \equiv \{(x^1, x^2) \in \mathbf{R}^2;\ x^1 = r\vec{e}_1\},$$

and and let y_r be the last crossing of $\rho_0(\vec{e}_1)$ and H_r, moving along $\rho_0(\vec{e}_1)$ from the origin. We claim that

$$-T(r\vec{e}_1, 0) \le H^{\vec{e}_1}(r\vec{e}_1, 0) \le T(r\vec{e}_1, y_r) - T(y_r, 0). \tag{5.1}$$

Indeed, denote the coalescence point between $\rho_{r\vec{e}_1}(\vec{e}_1)$ and $\rho_0(\vec{e}_1)$ by c. The left hand side of inequality 5.1 is easily deduced from the definition of $H^{\vec{e}_1}(r\vec{e}_1, 0)$ and from sub-additivity. To show the right hand side of inequality 5.1, suppose firstly that $y_r \notin \rho(0, c)$. Then $c \in \rho(0, y_r)$ and, as c is the coalescence point, this implies that $c \in \rho(r\vec{e}_1, y_r)$. Thus

$$H^{\vec{e}_1}(r\vec{e}_1, 0) = T(r\vec{e}_1, c) - T(0, c) = T(r\vec{e}_1, y_r) - T(y_r, 0).$$

If $y_r \in \rho(0, c)$, then

$$T(0, c) = T(0, y_r) + T(y_r, c)$$

and consequently,

$$H^{\vec{e}_1}(r\vec{e}_1, 0) = T(r\vec{e}_1, c) - T(0, c) = (T(r\vec{e}_1, c) - T(c, y_r)) - T(0, y_r).$$

By sub-additivity,

$$T(r\vec{e}_1, c) - T(c, y_r) \le T(r\vec{e}_1, y_r),$$

which implies that

$$H^{\vec{e}_1}(r\vec{e}_1, 0) \le T(r\vec{e}_1, y_r) - T(y_r, 0).$$

Although,

$$T(r\vec{e}_1, y_r) - T(y_r, 0) \le \max_{z\in[-y_r,y_r]}\{T(r\vec{e}_1, z)\} - T(H_r, 0). \tag{5.2}$$

By Theorem 1.2, almost surely

$$\lim_{r\to\infty} \frac{-T(r\vec{e}_1, 0)}{r} = -\mu(F) \text{ and } \lim_{r\to\infty} \frac{-T(H_r, 0)}{r} = -\mu(F). \tag{5.3}$$

Thus 5.1, 5.2, 5.3, allow us to conclude the proof of Theorem 5.1 since we prove that

$$\lim_{r\to\infty} \frac{\max_{z\in[-y_r,y_r]}\{T(r\vec{e}_1, z)\}}{r} = 0 \text{ in probability.} \tag{5.4}$$

Let $\delta > 0$, $\epsilon > 0$ and $I_s \equiv [-s\vec{e}_2, s\vec{e}_2]$, where $\vec{e}_2 \equiv (0,1)$. By translation invariance,

$$P(\max_{z\in[-y_r,y_r]}\{T(r\vec{e}_1, z)\} \ge \delta r) \le$$

$$P(\max_{z\in I_{\epsilon r}}\{T(0,z)\} \ge \delta r) + P(|y_r - r\vec{e}_1| \ge \epsilon r). \tag{5.5}$$

We claim that we can find $C_1 > 0$ so that

$$P(\max_{z\in I_s}\{T(0,z)\} > C_1 s) \to_{s\to\infty} 0. \tag{5.6}$$

In fact, it is a corollary of Theorem 1.1: take $C_1 > \mu(F)$ and note that by Fatou lemma,

$$\limsup_s P(\max_{z\in I_s}\{T(0,x)\} > C_1 s) \le P(\limsup_s [\max_{z\in I_s}\{T(0,z)\} > C_1 s]).$$

But, for all $\omega \in \limsup_s [\max_{z\in I_s}\{T(0,z)\} > C_1 s]$ we can find sequences $(r_k)_{k\ge 1}$ and $(s_k)_{k\ge 1}$ with $|r_k| \to_{k\to+\infty} +\infty$ and $r_k \le s_k$, such that

$$C_1 r_k \le C_1 s_k < T(0, r_k\vec{e}_2).$$

Since $C_1 > \mu$, by Theorem 1.1,

$$P(\limsup_s [\max_{z\in I_s}\{T(0,z)\} > C_1 s]) = 0.$$

Given $\delta > 0$ we choose $\epsilon > 0$ such that $\delta > C_1\epsilon$ and let $s = \epsilon r$. By 5.5

$$P(\max_{z\in[-y_r,y_r]}\{T(r\vec{e}_1, z)\} \ge \delta r) \le$$

$$P(\max_{z\in I_s}\{T(0,z)\} \geq C_1 s) + P(|y_r - r\vec{e}_1| \geq \epsilon r). \tag{5.7}$$

However, $\rho_0(\vec{e}_1)$ is a $\vec{e}_1$-geodesic and thus,

$$\frac{y_r}{|y_r|} \to_{r\to+\infty} \vec{e}_1,$$

which implies that for all $\epsilon > 0$

$$P(|y_r - r\vec{e}_1| > \epsilon r) \to_{r\to+\infty} 0. \tag{5.8}$$

Therefore 5.7, 5.6 and 5.8 permit us to deduce 5.4 and finish the proof of Theorem 5.1.

□

5.2 Geometry of the competition interface

In Section 1.4 we have introduced what a $(\vec{x},\vec{y})$-path it is. By using the coalescence function we are able to prove Theorem 1.6.

Proof of theorem 1.6. For $s \geq 0$ and $\vec{x} \in S^1$ let

$$l^s_{\vec{x}} \equiv \{t\vec{x};\, t \geq s\}.$$

Define the following decomposition of S^1:

$$S^1_{j,r} \equiv \{\vec{x}; \exists\, s_0 > 0 \text{ with } l^s_{\vec{x}} \subseteq G_{j,r}\ \forall\, s > s_0\};$$

for $j = 1, 2$, and

$$S^1_{0,r} \equiv (S^1_{1,r} \cup S^1_{2,r})^c.$$

Notice that for every $\vec{x} \in S^1_{0,r}$, the line segment $l^0_{\vec{x}}$ intersects infinitely many times the region of $G_{1,r}$ and the region of $G_{2,r}$. This implies that for every $\vec{x} \in S^1_{0,r}$

$$\liminf_{s\to+\infty}(T(-r\vec{e}_1, s\vec{x}) - T(r\vec{e}_1, s\vec{x})) \leq 0 \leq \limsup_{s\to+\infty}(T(-r\vec{e}_1, s\vec{x}) - T(r\vec{e}_1, s\vec{x})). \tag{5.9}$$

Although, we know that for every $\vec{x} \in S^1$ almost surely if $v_{1,r} \neq v_{2,r}$, then

$$\lim_{s \to +\infty} (T(-r\vec{e}_1, s\vec{x}) - T(r\vec{e}_1, s\vec{x})) = H^{\vec{x}}(v_{2,r}, v_{1,r}) \neq 0. \tag{5.10}$$

Let

$$D \equiv \{\exp(i\theta);\ \theta = \frac{2k\pi}{2^n} \text{ with } n \geq 1, 1 \leq k \leq 2^n\}.$$

Then D is an enumerable and dense subset of S^1. By Theorem 1.11, almost surely if $v_{1,r} \neq v_{2,r}$, for all $\vec{x} \in D$ there exists $H^{\vec{x}}(v_{2,r}, v_{1,r}) \neq 0$. This together with 5.9 and 5.10 allow us to conclude that

$$P(D \cap S^1_{0,r} = \emptyset) = 1. \tag{5.11}$$

Now we will use a geometrical lemma and to introduce this lemma we will consider a general situation (non-random). Fix $l \geq 2$, and suppose that

$$\mathbf{R}^2 = \cup_{j=1}^{l} G_j,$$

where for all $j = 1, ..., l$, G_j is a connected region in the plane, and consider also that $\cap_{j=1}^{l} G_j$ is a finite union of polygonal uni-paths. Recall that we had defined that ψ is an interface branch if ψ is a uni-path and $\psi \subseteq \cap_{j=1}^{l} G_j$. Assume that each j represent a different color. For each $j = 1, ..., l$, define

$$S^1_j \equiv \{\vec{x} \in S^1;\ \exists\, s > 0 \text{ so that } l^s_{\vec{x}} \subseteq G_j\},$$

and

$$S^1_0 \equiv (\cup_{j=1}^{l} S^1_j)^c).$$

Lemma 5.1 *Assume that $D \cap S^1_0 = \emptyset$. Then every interface branch is a $\vec{x}$-path for some $\vec{x} \in S^1$. Furthermore, the set of $\vec{x} \in S^1$ so that there is an interface branch which is a $\vec{x}$-path is finite.*

Thus Theorem 1.6 follows from 5.11 and Lemma 5.1.

$\square$

Proof of Lemma 5.1. To prove the first part of Lemma 5.1 notice that

$$D = \lim_{n \to \infty} D_n,$$

where

$$D_n \equiv \{\exp(i\theta_k);\ \theta_k = \frac{2k\pi}{2^n} \text{ with } 1 \leq k \leq 2^n\}.$$

Let $\vec{x}, \vec{y} \in S^1$ and denote by $S(\vec{x}, \vec{y})$ the interval of S^1 which consist of all directions $\vec{z}$ so that, if one moves along S^1, in the counter-clockwise orientation, from $\vec{x}$ to $\vec{y}$ then it must pass through $\vec{z}$. For each fixed $n \geq 1$, D_n defines a partition of the plane into cones

$$\mathbf{R}^2 = \cup_{k=1}^{2^n} C_k^n,$$

where

$$C_k^n \equiv \{r\vec{x};\ \vec{x} \in S(e^{i\theta_k}, e^{i\theta_{k+1}}\}.$$

Since we assumed that $S_0^1 \cap D = \emptyset$, for every $\vec{y} \in D$, every interface branch can not intersect infinitely many times the semi-infinite line segment $l_0^{\vec{y}}$. This implies that given an interface branch ψ, we can find a sequence of cones $(C_{k_n}^n)_{n \geq 1}$ (that depend on the interface branch) so that for all $n \geq 1$ the interface branch ψ is eventually inside $C_{k_n}^n$ and,

$$C_{k_{n+1}}^{n+1} \subseteq C_{k_n}^n.$$

However there must be a unique $\vec{x} \in S^1$ so that

$$l_0^{\vec{x}} = \cap_{n \geq 1} C_{k_n}^n,$$

and in this case it is easy to see that ψ must be a $\vec{x}$-path.

Let $j_1, j_2 \in \{1, ..., l\}$, $j_1 \neq j_2$, we say that an interface branch $\psi \equiv (x_n)_{\mathbf{N}}$ has the (j_1, j_2) representation if there exists n_0 so that moving along ψ by following $x_{n_0}, ..., x_n, x_{n+1}, ...$ then on the right hand side one always sees the color j_1 and on the left hand side one always sees the color j_2. Then the second part of Lemma 5.1 is a consequence of the following statements:

1. for each interface branch ψ there is a unique pair (j_1, j_2) so that ψ have the (j_1, j_2) representation;

2. for every pair of interface branches $\psi, \tilde{\psi}$ with the same (j_1, j_2) representation we must have that $\psi \subseteq \tilde{\psi}$ or $\tilde{\psi} \subseteq \psi$.

To prove the first statement, let us suppose that there are two different colors, say j and l, so that if we move along the uni-path $\psi = (x_1, x_2, ...)$, then on the right hand side we see infinitely many times the color j and the color l (the argument for the left hand side case is similar). Without loss of generality, let us suppose that on the right hand side of the edge (x_1, x_2) we see a Voronoi cell with the color j. Let $x_{n_1}(l)$ be the first time we see on the right hand side of the edge (x_n, x_{n+1}) a Voronoi cell with the color l, $x_{n_1}(j)$ be the first time after $x_{n_1}(l)$ that we see on the right hand side of the edge (x_n, x_{n+1}) a Voronoi cell with the color j, and $x_{n_2}(l)$ be the first time after $x_{n_1}(j)$ that we see on the right hand side of the edge (x_n, x_{n+1}) a Voronoi cell with the color l. Let $\psi(x_1, x_{n_1}(j))$ be the part of the polygonal path ψ between x_1 and $x_{n_1}(j)$. As $G_{j,r}$ is a connected region, we can find two polygonal paths, say $\gamma_1(j)$ and $\gamma_2(j)$, with the color j and connecting x_1 and $x_{n_1}(j)$ to $v_{j,r}$ respectively. Since these two paths can not intersect the interface branch ψ, the closed curve determined by concatenating $\psi(x_1, x_{n_1}(j))$, $\gamma_1(j)$ and $\gamma_2(j)$, divides the plane into two regions, one R_1 that is bounded by this curve, and another one R_2 that is unbounded. Note that the point $x_{n_1}(l)$ is necessarily inside R_1 and the point $x_{n_2}(l)$ is necessarily inside R_2. However, $G_{l,r}$ is a connected region which makes possible to connect $x_{n_1}(l)$ to $x_{n_2}(l)$ by a polygonal path with the color l. But it leads to a contradiction because in such a case this polygonal path must cross the boundary of R_1.

To prove the second statement, let ψ and $\tilde{\psi}$ be two interface branches with the same (j_1, j_2) representation. The interface branch ψ determines two paths of cells: $\gamma_1(\psi)$ composed by cells of the color j_1 and lying in the right hand side of ψ, $\gamma_2(\psi)$ composed by cells of the color j_2 and lying in the left hand side of ψ. Analogously, $\tilde{\psi}$ determines path of cells: $\gamma_1(\tilde{\psi})$

composed by cells of the color j_1 and lying in the right hand side of $\tilde{\psi}$ and $\gamma_2(\tilde{\psi})$ composed by cells of the color j_2 and lying in the left hand side of $\tilde{\psi}$. But we can connect $\gamma_1(\psi)$ to $\gamma_1(\tilde{\psi})$ by a path composed by cells of the color j_1. Denote such a path by $\tilde{\gamma}_1$. The path of cells $\gamma_1(\psi) \cup \tilde{\gamma}_1 \cup \gamma_1(\tilde{\psi})$ divide the plane into two regions, say E_1 and E_2. As this path is composed by only cells of the color j_1 we obtain that either $G_{j_2,r} \subseteq E_1$ or $G_{j_2,r} \subseteq E_2$. However, it leads to a contradiction because $\gamma_2(\psi)$ and $\gamma_2(\tilde{\psi})$ must be contained in different regions.

Now, with statement 1 and 2 in hands it is easy to conclude that there exist at most a finite number of possible directions for interface branches.

$\square$

There is a conjecture related to the topology of $S^1_{1,r}$ and $S^1_{2,r}$, which is the following. In the proof of Lemma 5.1 it is not difficult to see that if $l = 2$ then S^1_1 and S^1_2 are intervals in S^1. This together with 5.11 allows us to conclude that almost surely

$$\#(S^1_r) \leq 2.$$

Conjecture 5.1 *Almost surely, the interface ψ is closed if and only if $S^1_r = \emptyset$. Moreover, if $S^1_r \neq \emptyset$ and ψ is $(\vec{x}, \vec{y})$-path, then $S^1_r = \{\vec{x}, \vec{y}\}$.*

5.3 Mutual growth of two clusters

The second step in the description of the interface is given by Theorem 1.7. We strongly use the coalescence function and the ergodicity of the model. Recall that the probability space $(\Omega, \mathcal{F}, P)$ is the product space of $(\Omega_k, \mathcal{F}_k, P_k)$ over $k \geq 1$, where for each $k \geq 1$, $(\Omega_k, \mathcal{F}_k, P_k)$ describes the model restricted to the square box B_k (see Section 4.3). Denote by $(\Omega^l_k, \mathcal{F}^l_k, P^l_k)$ the product space of $(\Omega_i, \mathcal{F}_i, P_i)$ over $i = k, k+1, ..., l$. The tail σ-algebra is denoted by $\mathcal{F}^\infty$ and is defined as the biggest σ-algebra contained in $\mathcal{F}^\infty_n$ for all $n \geq 1$. Note that $\mathcal{F}^\infty$ contains the collection of events which are independent of $\mathcal{F}_1^m$ for all $m \geq 1$. Furthermore, by the Kolmogorov $0-1$ law, every random variable which is measurable with respect to $\mathcal{F}^\infty$ is almost surely a constant equal to its expected value (see [28]).

Proof of Theorem 1.7. The proof of this theorem is based on the following statements. For $i = 1, 2$ denote by $\mathbf{W}_i$ the set of all subsequencial weak limits of $\{\theta_{i,r};\ r > 0\}$. Then:

1. $\mathbf{W}_i \neq \emptyset$ for $i = 1, 2$;

2. for all $\theta_1 \in \mathbf{W}_1$ we have that $E(\theta_1) = \frac{\pi}{2}$ and for all $\theta_2 \in \mathbf{W}_2$ we have that $E(\theta_2) = \frac{3\pi}{2}$;

3. for $i = 1, 2$ every random variable in $\mathbf{W}_i$ is measurable with respect to $\mathcal{F}^\infty$.

To prove the first statement notice that $\theta_{i,r} \in [0, 2\pi]$ and thus $\{\theta_{i,r};\ r > 0\}$ is tight. By Prohorov theorem the set of weak limits is non empty (see [3]).

Statement 2 follows from Theorem 5.1. Indeed, if

$$M_r \equiv [\vec{e}_1 \in S^1_{1,r}, -\vec{e}_1 \in S^1_{2,r}],$$

then

$$[H^{\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) > 0, H^{-\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) < 0] \subseteq M_r, \tag{5.12}$$

and by symmetry,

$$P(H^{\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) > 0) = P(H^{-\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) < 0). \tag{5.13}$$

By Theorem 5.1, 5.12 and 5.13,

$$P(M_r) \rightarrow_{r\rightarrow\infty} 1. \tag{5.14}$$

Notice that the random variable $\theta_{1,r} \mid M_r$ (conditioned on the event M_r) takes its value in the interval $[0, \pi]$ and is symmetric with respect to $\pi/2$. Similarly, $\theta_{2,r} \mid M_r$ takes its value in the interval $[\pi, 2\pi]$ and is symmetric with respect to $3\pi/2$. Therefore

$$E(\theta_{1,r} \mid M_r) = \frac{\pi}{2} \text{ and } E(\theta_{2,r} \mid M_r) = \frac{3\pi}{2}. \tag{5.15}$$

By 5.14 and 5.15

$$E(\theta_{1,r}) \to_{r\to\infty} \frac{\pi}{2} \text{ and } E(\theta_{2,r}) \to_{r\to\infty} \frac{3\pi}{2}. \tag{5.16}$$

As $\theta_{1,r}$ and $\theta_{2,r}$ takes their values in a compact set, 5.16 implies statement 2.

To prove statement 3 we will use the following proposition which is proved after we complete the proof of Theorem 1.7.

Proposition 5.1 *For $i = 1, 2$, for all $k \geq 1$ there exists $\theta^k_{i,r}$, measurable with respect to $\mathcal{F}^\infty_{k+1}$, such that*

$$|\theta_{i,r} - \theta^k_{i,r}| \to_{r\to\infty} 0$$

in probability.

In fact, notice that if

$$Y_n \to_{n\to\infty} Y \text{ in distribution}$$

and

$$|X_n - Y_n| \to_{n\to\infty} 0 \text{ in probability}$$

then

$$X_n \to_{n\to\infty} Y \text{ in distribution}$$

(see [3]). By Proposition 5.1, for all $k \geq 1$ every $\theta_i \in \mathbf{W}_i$ is also a subsequence weak limit of $\{\theta^k_{i,r};\ r > 0\}$. This implies that θ_i is $\mathcal{F}^\infty_{k+1}$ measurable. Since $k \geq 1$ is arbitrary, θ_i is $\mathcal{F}^\infty$.

Statement 1, 2 and 3 together with the Kolmogorov $0 - 1$ law imply that $\mathbf{W}_1 = \{\pi/2\}$ and $\mathbf{W}_2 = \{3\pi/2\}$. Thus

$$\theta_{1,r} \to_{r\to\infty} \frac{\pi}{2} \text{ and } \theta_{2,r} \to_{r\to\infty} \frac{3\pi}{2},$$

in distribution. However, convergence in distribution to a delta implies convergence in probability and so we finish the proof of Theorem 1.7.

□

Proof of proposition 5.1. To prove Proposition 5.1 we will use the following lemma and also we will introduce some new notation. Firstly, recall that each configuration $\omega \in \Omega$ is a sequence $\omega = (\omega_k)_{k\geq 1}$, and each ω_k represent the model restricted to the box $B_k = z_k + [-1/2, 1/2]^2$. Indeed,

$$\omega_k = (N_k, (U_{kl})k, l, (\tau_{kl}^{mn})_{k,l,m,n}),$$

where N_k is number of vertices in the box B_k, $U_{k,1}, ..., U_{k,N_k}$ are the positions of the vertices inside the box B_k, and $\tau_{k,l}^{mn}$ are the travel times attached to the edge $(U_{k,l}, U_{m,n})$. For each $k \geq 1$ and $1 \leq j \leq k$ define

$$\phi_k^j(\omega_k) = (0, (U_{kl})_{k,l}, (\tau_{kl}^{mn})_{k,l,m,n}),$$

and let

$$\phi_k(\omega) \equiv (\phi_k^1(\omega_1), ..., \phi_k^k, \omega_{k+1}, \omega_{k+2}, ...) \equiv \omega^k.$$

Lemma 5.2 *Assume that $\{A_r\}_{r>0}$ is a collection of measurable subsets of Ω. Then $A_r^k \equiv \phi_k^{-1}(A_r)$ is $\mathcal{F}_{k+1}^{\infty}$ measurable. Furthermore;*

- $P(A_r) \to_{r\to\infty} 1 \Rightarrow P(A_r^k) \to_{r\to\infty} 1;$
- $P(A_r) = 1 \Rightarrow P(A_r^k) = 1.$

Proof of lemma 5.2. To prove that A_r^k is $\mathcal{F}_{k+1}^{\infty}$ measurable, recall that $\Omega = \prod_{j\geq 1} \Omega_j$ and take $\tilde{\omega} = (\tilde{\omega}_j)_{j\geq 1}, \bar{\omega} = (\bar{\omega}_j)_{j\geq 1} \in \Omega$. Suppose that $\tilde{\omega}_j = \bar{\omega}_j$ for all $j \geq k+1$. The transformation $\omega \to \omega^k$ deletes all vertices of $\mathcal{V}(\omega)$ inside $\cup_{i=1}^k B_i$ and this implies that the only part of the configuration $\omega = (\omega_j)_{j\geq 1}$ that will determine the graph structure $\mathcal{G}(\omega^k) \equiv (\mathcal{V}(\omega^k), \mathcal{E}(\omega^k))$ and the passage time configuration $(\tau_e(\omega^k))_{e\in\mathcal{E}(\omega^k)}$ corresponds to ω_j, with $j \geq k+1$. Thus

$$1_{A_r^k}(\tilde{\omega}) = 1_{A_r}(\phi_k(\tilde{\omega})) = 1_{A_r}(\phi_k(\bar{\omega})) = 1_{A_r^k}(\bar{\omega}),$$

which implies that $1_{A_r^k}$ is $\mathcal{F}_{k+1}^{\infty}$ measurable.

For convenience, let us write $\omega = (\hat{l}, \omega(k))$ where $\hat{l} = (l_1, .., l_k) \in \mathbf{N}^k$ and l_j represents the number of vertices contained in the box B_j, and $\omega(k)$ corresponds to the part of the configuration of ω that is independent of $l_1, ..., l_k$. Let $\hat{0} = (0, ..., 0)$. Then for all $\omega = (\hat{l}, \omega(k))$, $\phi(\omega) = (\hat{0}, \omega(k))$. Notice that, by the construction of the probability space (see Section 4.3), we can write the probability measure P as the product of P_k with $\mathbf{P}_k$ where P_k is the law of the random vector $N_1, ..., N_k$, where N_j is i.i.d random variables with a Poisson distribution, and $\hat{P}_k$ is the law which correspond to the part of the configuration of ω that is independent of $N_1, ..., N_k$. By the Fubini theorem

$$P(A_r) = \sum_{\hat{l} \in \mathbf{N}^k} \mathbf{P}_k((\hat{l}, \omega(k)) \in A_r \mid (N_j)_{j=1,...,k} = \hat{l}) P_k(N(k) = \hat{l}). \tag{5.17}$$

However, for all $\hat{l} \in \mathbf{N}^k$,

$$P_k(N(k) = \hat{l}) > 0, \tag{5.18}$$

and

$$\sum_{\hat{l} \in \mathbf{N}^k} P_k(N(k) = \hat{l}) = 1. \tag{5.19}$$

By 5.17, 5.18 and 5.19, for all $\hat{l} \in \mathbf{N}^k$,

$$P(A_r) \to_{r \to \infty} 1 \Rightarrow \mathbf{P}_k((\hat{l}, \omega(k)) \in A_r) \to_{r \to \infty} 1,$$

and

$$P(A_r) = 1 \Rightarrow \mathbf{P}_k((\hat{l}, \omega(k)) \in A_r) = 1.$$

By observing that

$$P(A_r^k) = \hat{P}_k((\hat{0}, \hat{\omega}) \in A_r),$$

one can finish the proof of Lemma 5.2.

□

By Theorem 1.6 and Lemma 5.2, for $i = 1, 2$ we can define a $\mathcal{F}_{k+1}^{\infty}$ measurable function $\theta_{i,r}^{k}$ by

$$\theta_{i,r}^{k}(\omega) \equiv \theta_{i,r}(\omega^{k}).$$

Analogously, by Theorem 1.11 and Lemma 5.2 for all $\vec{x} \in S^1$ we can define a $\mathcal{F}_{k+1}^{\infty}$ measurable function $H_k^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1)$ by

$$H_k^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1)(\omega) \equiv H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1)(\omega^k).$$

By Theorem 5.1 and Lemma 5.2,

$$P(H_k^{\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) > 0) \to_{r\to\infty} 1, \tag{5.20}$$

and

$$P(H_k^{-\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) < 0) \to_{r\to\infty} 1. \tag{5.21}$$

To continue the proof of this proposition we will use the following proposition.

Proposition 5.2 *For all $\vec{x} \in S^1 \backslash \{\vec{e}_1, -\vec{e}_1\}$,*

$$P(H_k^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) = H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) \neq 0) \to_{r\to\infty} 1.$$

Proof of Proposition 5.2. To prove Proposition 5.2 we will need to control the discrepancy between the graph structure determined by $\mathcal{V}(\omega)$ and the graph structure determined by $\mathcal{V}(\omega^k)$. To do so we will use a renormalization idea similar to one used to prove Theorem 1.3.

Let $L > 0$ and $z \in \mathbf{Z}^2$. Divide the square box

$$B_z^{L/2} \equiv Lz + [-L/2, L/2]^2$$

into 36 equal square boxes and define that this box is a good box if all these thirty six sub square boxes have at least one vertex of $\mathcal{V}$. Then $Y^L \equiv \{Y_z^L;\ z \in \mathbf{Z}^2\}$, where

$$Y_z^L \equiv 1_{[B_z^{L/2} \text{ is a good box }]},$$

defines an independent site percolation model in $\mathbf{Z}^2$ with parameter

$$p^L = P(B_0^{L/2} \text{ is a good box }).$$

Let Υ_L^1 be the first open circuit surrounding the origin. By the definition of the Poisson point process,

$$p^L \to_{L\to\infty} 1.$$

This allows us to conclude that we can fix $L > 0$ so that $\cup_{i=1}^k B_i \subseteq B_0^{L/2}$, and for all $\epsilon > 0$ there exist $C_1, C_2 > 0$ such that

$$P(\Upsilon_L^1 \not\subseteq [-\epsilon n, \epsilon n]^2) \leq C_1 \exp(-C_2 n) \tag{5.22}$$

(see [9]). With each open circuit $\Upsilon = (z_1, ..., z_m)$ in $\mathbf{Z}^2$ we associate a circuit of boxes in $\mathbf{R}^2$ by $B(\Upsilon) = (B_{z_1}^{L/2}, ..., B_{z_m}^{L/2})$. Note that

$$\mathbf{B}(\Upsilon) \not\subseteq [-\epsilon L n, \epsilon L n]^2 \Rightarrow \Upsilon \not\subseteq [-\epsilon n, \epsilon n]^2.$$

By Lemma 5.2,

$$\rho_x^k(\vec{x})(\omega) \equiv \rho_x(\vec{x})(\omega^k),$$

is well defined. Using the same method presented in the proof of Theorem 1.3 one can prove that the graph structure determined by $\mathcal{V}(\omega)$ outside $B(\Upsilon_L^1)(\omega)$ is equal to the graph structure determined by $\mathcal{V}(\omega^k)$ outside $B(\Upsilon_L^1)(\omega)$. Therefore, if $\rho_x(\vec{x}) \cap B(\Upsilon_L^1) = \emptyset$ and $\rho_x^k(\vec{x}) \cap B(\Upsilon_L^1) = \emptyset$ then we have that

$$\rho_x(\vec{x}) = \rho_x^k(\vec{x}).$$

Denote by $O_2^\epsilon(\vec{x})$ the event that there exists at least one of the geodesics $\rho_{r\vec{e}_1}(\vec{x}), \rho_{-r\vec{e}_1}(\vec{x}), \rho_{r\vec{e}_1}^k(\vec{x}), \rho_{-r\vec{e}_1}^k(\vec{x})$ which intersects $[-\epsilon L r, \epsilon L r]^2$, and by O_1^ϵ the event that $\Upsilon_L^1 \not\subseteq [-\epsilon r, \epsilon r]^2$. By the argument in the previous paragraph, the event that

$$H_k^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) \neq H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1).$$

is contained in the union $O_1^\epsilon(\vec{x}) \cup O_2^\epsilon(\vec{x})$, and consequently,

$$P(H_k^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) \neq H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) \text{ or } v_{r\vec{e}_1} = v_{-r\vec{e}_1}) \leq P(\Omega_2^r) + P(O_1^\epsilon) + P(O_2^\epsilon(\vec{x})). \quad (5.23)$$

By Euclidean invariance

$$P(O_2^\epsilon(\vec{x})) \leq P(\rho_{v_0}(\vec{x}) \cap (-r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset) +$$

$$P(\rho_{v_0}(\vec{x}) \cap (r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset) + P(\rho_{v_0}^k(\vec{x}) \cap (-r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset)$$

$$+P(\rho_{v_0}^k(\vec{x}) \cap (r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset). \quad (5.24)$$

By Lemma 5.2 and 5.24, to prove that for some $\epsilon > 0$,

$$P(O_2^\epsilon(\vec{x})) \to_{r\to\infty} 0, \quad (5.25)$$

we only need to prove that for some $\epsilon > 0$,

$$P(\rho_{v_0}(\vec{x}) \cap (-r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset) \to_{r\to\infty} 0 \quad (5.26)$$

and

$$P(\rho_{v_0}(\vec{x}) \cap (r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset) \to_{r\to\infty} 0. \quad (5.27)$$

Recall that for $\epsilon > 0$ and for non zero $x \in \mathbf{R}^2$ we had defined the cone

$$C(x, \epsilon) \equiv \{y \in \mathbf{R}^2;\ ang(x, y) < \epsilon\},$$

and that $\vec{x} \neq \vec{e}_1, -\vec{e}_1$. Thus we can find $\epsilon > 0$ such that for all $r > 0$ large enough,

$$C(\vec{x}, \epsilon) \cap (-r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) = \emptyset,$$

and

$$C(\vec{x}, \epsilon) \cap (r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) = \emptyset.$$

By Fatou lemma, for all $\epsilon > 0$,

$$\limsup_r P(\rho_{v_0}(\vec{x}) \cap (-r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset) \leq$$

$$P(\limsup_r [\rho_{v_0}(\vec{x}) \cap (-r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset]) \leq$$

$$P(\rho_{v_0}(\vec{x}) \text{ intersect i.o. } (C(\vec{x}, \epsilon))^c),$$

and

$$\limsup_r P(\rho_{v_0}(\vec{x}) \cap (r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset) \leq$$

$$P(\limsup_r [\rho_{v_0}(\vec{x}) \cap (r\vec{e}_1 + [-\epsilon Lr, \epsilon Lr]^2) \neq \emptyset]) \leq$$

$$P(\rho_{v_0}(\vec{x}) \text{ intersect i.o. } (C(\vec{x}, \epsilon))^c).$$

However, $\rho_{v_0}(\vec{x})$ has the asymptotic direction $\vec{x}$. This implies that for all $\epsilon > 0$,

$$P(\rho_{v_0}(\vec{x}) \text{ intersect i.o. } (C(\vec{x}, \epsilon))^c) = 0,$$

which implies 5.26 and 5.27, and consequently we obtain 5.25 for some $\epsilon > 0$. By the definition of the Poisson point process,

$$P(\Omega_2^r) \to_{r \to \infty} 1. \tag{5.28}$$

By 5.25, 5.23, 5.22 and 5.28, we obtain Proposition 5.2.

$\square$

Continuing the proof of Proposition 5.1, to each $n \geq 1$ define

$$D_n \equiv \{\exp(i\alpha_j);\ \alpha_j = \frac{2\pi j}{2^n} \text{ and } j = 0, 1, ..., 2^n - 1\} \subseteq S^1,$$

and let

$$A_r^n \equiv [H^{\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) > 0,\ H^{-\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) < 0]$$

$$\cap [H_k^{\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) > 0,\ H_k^{-\vec{e}_1}(-r\vec{e}_1, r\vec{e}_1) < 0]$$

$$\cap [\text{ for all } \vec{x} \in D_n,\ H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) = H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1)].$$

By 5.20, 5.21 and Proposition 5.2 for all $n \geq 1$,

$$P(A_r^n) \to_{r \to \infty} 1. \tag{5.29}$$

Let $\epsilon > 0$ and $n \geq 1$ so that $2^{-n} < \epsilon$. We claim that for $j = 1, 2$, for all $r > 0$

$$A_r^n \subseteq [|\theta_{j,r} - \theta_{j,r}^k| \leq \epsilon]. \tag{5.30}$$

Let us prove 5.30 for $j = 1$. The proof for $j = 2$ is similar. Indeed, first notice that for all configuration in A_r^n we have that

$$\theta_{1,r}, \theta_{1,r}^k \in [0, \pi].$$

If $|\theta_{1,r} - \theta_{1,r}^k| > \epsilon$, without loss of generality, let us assume that

$$0 \leq \theta_{1,r} < \theta_{1,r} + \epsilon \leq \theta_{1,r}^k \leq \pi.$$

Since $2^{-n} < \epsilon$, we can find $l \geq 0$ so that

$$0 \leq \alpha_l \leq \theta_{1,r} < \alpha_{l+1} \leq \theta_{1,r} + 2^{-n} < \theta_{1,r}^k.$$

If $\vec{x} = \exp(i\theta)$ and $\theta \in [0, \theta_{1,r}^k] \cap \{\alpha_j;\ 0 \leq j \leq 2^n - 1\}$ then

$$H_k^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) > 0.$$

If $\vec{x} = \exp(i\theta)$ and $\theta \in [\theta_{1,r}^k, \pi] \cap \{\alpha_j;\ 0 \leq j \leq 2^n - 1\}$ then

$$H_k^{\vec{x}}(-r\vec{e}, r\vec{e}_1) < 0.$$

This implies that for $\vec{x} = \exp(i\alpha_{l+1})$,

$$H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) \neq H_k^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1),$$

and consequently,

$$[|\theta_{1,r} - \theta_{1,r}^k| > \epsilon] \subseteq (A_n^r)^c.$$

With 5.29 and 5.30 we can conclude the proof of Proposition 5.1.

$\square$

Given $\epsilon > 0$ let $C(\vec{e}_2, \epsilon)$ be the cone with direction $\vec{e}_2 = (0, 1)$, and angle ϵ, and let $C(-\vec{e}_2, \epsilon)$ the cone with direction $-\vec{e}_2$ and angle ϵ (see Section 4.2). We will say that a bi-path ψ is eventually inside $C(\vec{e}_2, \epsilon) \cup C(-\vec{e}_2, \epsilon)$ if there exists $i_0 \geq 0$ so that $x_i \in C(\vec{e}_2, \epsilon) \cup C(-\vec{e}_2, \epsilon)$ for all i with $|i| > i_0$. Thus we have the following corollary of Theorem 1.7.

Corollary 5.1 *Under the assumption of Theorem 1.7, for all $\epsilon > 0$,*

$$P(\psi_r \text{ is eventually inside } C(\vec{e}_2, \epsilon) \cup C(-\vec{e}_2, \epsilon)) \to_{r \to +\infty} 1.$$

Proof of Corollary 5.1. Indeed, note that

$$|\theta_{1,r} - \frac{\pi}{2}| < \epsilon \text{ and } |\theta_{2,r} - \frac{3\pi}{2}| < \epsilon$$

implies that ψ_r is eventually inside $C(\vec{e}_2, \epsilon) \cup C(-\vec{e}_2, \epsilon)$.

$\square$

5.4 The competition interface between k clusters

Before focusing on the morphology of the interface between k clusters let us prove Theorem 1.12.

Proof of Theorem 1.12. We had considered the subsets of S^1,

$$S^1_+ \equiv \{\vec{x} \in S^1;\ \vec{x} \cdot \vec{e}_1 > 0\}$$

and

$$S^1_- \equiv \{\vec{x} \in S^1;\ \vec{x} \cdot \vec{e}_1 < 0\}.$$

We also had denoted by $\mathbf{H}^{\vec{x}}$ the set of all weak limits of $\{\frac{H^{\vec{x}}(r\vec{e}_1, 0)}{r};\ r > 0\}$. In Theorem 5.1 we had seen that

$$\lim_{r \to \infty} \frac{H^{\vec{e}_1}(r\vec{e}_1, 0)}{r} = -\mu(F) \text{ in probability.}$$

By symmetry and Theorem 5.1 we can assume that $\vec{x} \neq \vec{e}_1, -\vec{e}_1$. By sub-additivity

$$\frac{|H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1)|}{r} \leq \frac{T(-r\vec{e}_1, r\vec{e}_1)}{r}.$$

However, by the shape theorem,

$$\frac{T(-r\vec{e}_1, r\vec{e}_1)}{r} \to_{r\to\infty} 2\mu(F) \text{ in probability}$$

and this with the last inequality implies tightness of $\{\frac{H^{\vec{x}}(r\vec{e}_1,0)}{r};\ r > 0\}$. Proposition 5.2 implies that every weak limit of $\{\frac{H^{\vec{x}}(r\vec{e}_1,0)}{r};\ r > 0\}$ is $\mathcal{F}^\infty$ measurable and then, by the Kolmogorov $0-1$ law, is almost surely a constant. This proves the first part of Theorem 1.12. It remains to prove that

$$\forall\, \vec{x} \in S^1_+,\ h \in \mathbf{H}^{\vec{x}} \Rightarrow h \leq 0, \tag{5.31}$$

$$\forall\, \vec{x} \in S^1_-,\ h \in \mathbf{H}^{\vec{x}} \Rightarrow h \geq 0, \tag{5.32}$$

and that

$$\lim_{r\to\infty} \frac{H^{\vec{e}_2}(r\vec{e}_1, 0)}{r} = 0 \text{ in probability.} \tag{5.33}$$

5.31 and 5.32 follows from Proposition 5.3.

Proposition 5.3 *For all $\vec{x} \in S^1_+$,*

$$P(H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) > 0) \to 1 \text{ as } r \to +\infty,$$

and for all $\vec{x} \in S^1_-$,

$$P(H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) < 0) \to 1 \text{ as } r \to +\infty.$$

Proof of Proposition 5.3. By symmetry, we only need to prove the second statement. Consider $\vec{x} = \exp(i\theta) \in S^1_-$ with $\theta \in (\pi/2, 3\pi/2)$. So, choose ϵ small enough such that $\theta \in (\pi/2 + \epsilon, 3\pi/2 - \epsilon)$. Thus,

$$P(\psi_r \text{ is eventually inside } C^\epsilon_{north} \cup C^\epsilon_{south}) \leq P(H^{\vec{x}}(-r\vec{e}_1, r\vec{e}_1) < 0),$$

since

$$[\psi_r \text{ is eventually inside } C^{\epsilon}_{north} \cup C^{\epsilon}_{south}] \subseteq [\vec{x} \in S^1_{2,r}].$$

By Corollary 5.1 we obtain Proposition 5.3.

□

Turning back to the proof of Theorem 1.12, notice that by symmetry

$$E(H^{\vec{e}_2}(-r\vec{e}_1, r\vec{e}_1)) = 0,$$

and this with the first part of Theorem 1.12 implies 5.33.

□

Reminding what was done in Section 1.4, we defined the competing spatial growth process

$$\{G_{1,r}(t), ..., G_{k,r}(t)\}_{t>0},$$

which describes the growth of k clusters which are competing for space. We also considered the the regions

$$G_{j,r} \equiv \lim_{t \to +\infty} G_{j,r}(t),$$

and denoted the set of j so that $G_{j,r} \neq \emptyset$ by J. If $\#(J) \geq 2$, we have defined the interface by

$$\psi_r \equiv \cap_{j \in J} G_{j,r}.$$

If $\#(J) = 1$, we assumed that the interface is empty.

Proof of theorem 1.8. To prove the first part of Theorem 1.8 recall we had defined that ψ is an interface branch if ψ is a uni-path and

$$\psi \subseteq \cap_{j \in J} G_{j,r},$$

where

$$J \equiv \{j \in \{1, ..., k\};\ G_{j,r} \neq \emptyset\}.$$

We also had defined

$$S^1_{j,r} \equiv \{\vec{x} \in S^1;\ \exists\, s > 0 \text{ so that } l^s_{\vec{x}} \subseteq G_{j,r}\},$$

and

$$S^1_{0,r} \equiv (\cup_{j \in J} S^1_{j,r})^c.$$

To use Lemma 5.1, let

$$D \equiv \{\exp(i\theta);\ \theta = \frac{2k\pi}{2^n} \text{ with } n \geq 1, 1 \leq k \leq 2^n\}.$$

By Theorem 1.11, almost surely if $l, \tilde{l} \in J$, $l \neq \tilde{l}$, for all $\vec{x} \in D$ there exists

$$H^{\vec{x}}(v_{\tilde{l},r}, v_{l,r}) \neq 0.$$

Although, if $\vec{x} \in S^1_{0,r}$ then there is $l, \tilde{l} \in J$ so that the line segment $l^0_{\vec{x}}$ intersects infinitely many times the regions $G_{l,r}$ and $G_{\tilde{l},r}$. This implies that

$$\liminf_s (T(v_{\tilde{l},r}, s\vec{x}) - T(v_{l,r}, s\vec{x})) \leq 0 \leq \limsup_s (T(v_{\tilde{l},r}, s\vec{x}) - T(v_{l,r}, s\vec{x})),$$

and consequently,

$$P(D \cap S^1_{0,r} = \emptyset) = 1.$$

Therefore, by Lemma 5.1, there exist $\vec{x}_1, ..., \vec{x}_{l_0}$ so that every interface branch is a $\vec{x}_j$-path for some $j \in \{1, ..., l_0\}$.

To prove the second part, let $\epsilon \in (0, \pi/k)$ and define

$$S^1_1(\epsilon) \equiv \{e^{i\theta}; \theta \in ((2\pi - \pi/k) + \epsilon, 2\pi] \cup [0, \pi/k - \epsilon)\},$$

and

$$S^1_j(\epsilon) \equiv \{e^{i\theta}; \theta \in ((2j\pi/k - \pi/k) + \epsilon, (2j\pi/k + \pi/k) - \epsilon)\}.$$

for $j = 2, ..., k$. By the Euclidean invariance

$$P(S^1_j(\epsilon) \subseteq S^1_{j,r}) = P(S^1_1(\epsilon) \subseteq S^1_{1,r})$$

for all $j = 1, ..., k$. Consequently, we only need to prove that

$$P(S_1^1(\epsilon) \subseteq S_{1,r}^1) \to_{r\to\infty} 1. \tag{5.34}$$

Indeed, let

$$A_r \equiv \cap_{j=1}^k A_r^j$$

where

$$A_r^j \equiv [H^{\vec{e}_j^k}(v_{l,r}, v_{j,r}) > 0, \, \forall \, l \neq j].$$

By the Euclidean invariance and Proposition 5.3, for all $j = 1, ..., k$

$$P(A_r^j) \to_{r\to\infty} 1,$$

and consequently,

$$P(A_r) \to_{r\to\infty} 1. \tag{5.35}$$

Now let

$$\vec{y}_k^{+\epsilon} \equiv \exp(i\theta_k^{+\epsilon}), \text{ and } \vec{y}_k^{-\epsilon} \equiv \exp(i\theta_k^{-\epsilon}),$$

where

$$\theta_k^{+\epsilon} \equiv \frac{\pi}{k} - \epsilon \text{ and } \theta_k^{-\epsilon} \equiv (2\pi - \frac{\pi}{k}) + \epsilon.$$

Define

$$B_r(\epsilon) \equiv [\forall \, j \neq 1, \, H^{\vec{y}_k^{+\epsilon}}(v_{j,r}, v_{1,r}) > 0, H^{\vec{y}_k^{-\epsilon}}(v_{j,r}, v_{1,r}) > 0].$$

Again by Euclidean invariance and Proposition 5.3,

$$P(B_r(\epsilon)) \to_{r\to\infty} 1.$$

This with 5.35 implies that

$$P(A_r \cap B_r(\epsilon)) \to_{r\to\infty} 1. \tag{5.36}$$

Now we claim that

$$A_r \cap B_r(\epsilon) \subseteq [S_1^1(\epsilon) \subseteq S_{1,r}^1]. \tag{5.37}$$

In fact, we had denoted by $S(\vec{x},\vec{y})$ the interval in S^1, consisting of all directions $\vec{z}\in S^1$ so that if one moves from $\vec{x}$ to $\vec{y}$ in the counter clockwise orientation then it must pass through $\vec{z}$. Thus

$$S^1_1(\epsilon)=S(\vec{y_k}^{-\epsilon},\vec{y_k}^{+\epsilon}).$$

However, $\vec{y_k}^{+\epsilon},\vec{y_k}^{-\epsilon}\in S^1_{1,r}$, and then there exist $s>0$ so that

$$l_s^{\vec{y_k}^{-\epsilon}},l_s^{\vec{y_k}^{+\epsilon}}\subseteq G_{1,r}.$$

Since $G_{1,r}$ is a connected region, we can find two finite polygonal paths, say γ^+ and γ^-, that are containing in the region $G_{1,r}$ and connect $s\vec{y_k}^{+\epsilon}$ and $s\vec{y_k}^{-\epsilon}$ to $v_{1,r}$, respectively. Notice that the bi-infinite curve $l_s^{\vec{y_k}^{-\epsilon}}\cup l_s^{\vec{y_k}^{+\epsilon}}\cup\gamma^+\cup\gamma^-$ divide the plane into two regions, say R_1 and R_2, such that

$$S(\vec{y_k}^{-\epsilon},\vec{y_k}^{+\epsilon})=\{\vec{z}\in S^1;\ \exists s>0\text{ with }l_s^{\vec{z}}\subseteq R_1\},$$
$$S(\vec{y_k}^{+\epsilon},\vec{y_k}^{-\epsilon})=\{\vec{z}\in S^1;\ \exists s>0\text{ with }l_s^{\vec{z}}\subseteq R_2\}.$$

Now, suppose that there exists $\vec{x}\in S(\vec{y_k}^{-\epsilon},\vec{y_k}^{+\epsilon})$ and a color $j\neq 1$ such that the semi-infinite line segment $l_0^{\vec{x}}$ intersects infinitely many times the region $G_{j,r}$. In particular, $l_0^{\vec{x}}$ is eventually inside R_1. But $l_0^{\vec{x}}$ intersects infinitely many times the region $G_{j,r}$ and this implies that there should exist at least one point, say x_j, of $G_{j,r}$ inside the region R_1. The point x_j can be connected to the vertex $v_{j,r}$ by using a polygonal path, say γ_j, with the color j (or equivalently, lying inside $G_{j,r}$). As the boundary of R_1 has the color 1, the path γ_j can not touch any point inside R_2 and consequently $v_{j,r}\in R_1$. However, we are considering configurations in A_r, and thus

$$\vec{e_j^k}\in S^1_{j,r}\Rightarrow\exists s_0>0;\ \forall\, s>s_0,\ l_s^{\vec{e_j^k}}\in G_{j,r}.$$

This leads to a contradiction because for $s>0$, large enough, R_1 and $l^{\vec{e_j^k}}$ has empty intersection and every path from $v_{j,r}$ to $l^{\vec{e_j^k}}$ must cross the boundary of R_1 that has the color 1. This proves 5.37, and 5.36, 5.37 proves 5.34.

□

5.5 Inequalities for critical exponents

*Proof of Theorem 1.13.*Recall the definition of χ_1, ξ_1 in Section 1.6, Chapter 1 . We divide the proof in four parts.

Part1: the inequality $\chi_1 \leq 1/2$ is follows from Theorem 1.4 and from the definition of χ_1. To prove that $\chi_2 \leq \chi_1$ we simple mimic the proof of Theorem 1.5.

Part 2: to prove that $\xi_1 \leq \frac{\chi_1+1}{2}$ notice that it follows from the definitions of ξ_1 and χ_1 together with Theorem 4.1.

Part 3: we prove that $\xi_2 \leq \xi_1$. Let $\xi_1 < \kappa$ and $\xi_1 < \kappa_1 < \kappa$. As in the proof of Theorem 4.2, by using the definition of χ_1 and Lemma 4.2, if $\rho_v(\vec{x}) = (v_1, v_2, ...)$ is $\vec{x}$-geodesic, then there exists $C > 0$ and $m^* > 0$ so that $n, m \geq m^*$

$$ang(v_m, v_n) < C|v_m|^{\kappa_1 - 1}.$$

Then the proof of the third part follows from the next lemma (see Section 1.6).

Lemma 5.3 *Let $(x_n)_{n\geq 1}$ be a sequence of points in $\mathbf{R}^2$ so that:*

$$\frac{x_n}{|x_n|} \rightarrow_{n\rightarrow\infty} \vec{x},\ |x_n| \rightarrow_{n\rightarrow\infty} +\infty, \tag{5.38}$$

and for all $m \geq m^*$,

$$ang(x_m, x_n) \leq C|v_m|^{\kappa_1 - 1}. \tag{5.39}$$

Then for all $\kappa > \kappa_1$, for all $r > 0$ large enough,

$$|y_r^{\vec{x}} - x_r| \leq r^{\kappa}.$$

Proof of lemma 5.3. Without loss of generality, let us assume that $x_1 = 0$ and that $\vec{x} = \vec{e}_1$, witch implies that $y_r^{\vec{e}_1} = r\vec{e}_1$. By 5.38 and 5.39, there exists $r_0 > 0$ so that, for all $r > r_0$

$$ang(x_r, \vec{e}_1) \leq C|x_r|^{\kappa_1 - 1}.$$

However we can find $B > 0$ and $r_1 > 0$, where B does not depend on $r > 0$, so that for all $r > r_1$

$$||x_r|\vec{e}_1 - x_r| \le ang(x_r, \vec{e}_1)B|x_r|, \text{ and } |x_r| \le 2|r|.$$

Consequently, for all $r > \max\{r_0, r_1\}$,

$$|y_r^{\vec{e}_1} - x_r| \le ||x_r|\vec{e}_1 - x_r| \le 2BCr^{\kappa_1},$$

which implies that for all $r > 0$ large enough,

$$|y_r^{\vec{e}_1} - x_r| \le r^{\kappa}.$$

□

Part 4: we prove that $\zeta \le \xi_2$. Let $\kappa \in (\xi_2, 1)$ and $\kappa_1 \in (\xi_2, \kappa)$. Let ψ_r be the interface and consider the black-gray representation, $\psi_r \equiv (x_n)_{\mathbf{Z}}$ (see Section 1.4). Let us restrict our attention to the interface branch $\psi^+ \equiv (x_0, x_1, ...)$. By Theorem 1.6, this interface branch has the direction $\vec{x}_1$. For $j = 1, 2$, let $(v_n^j)_{n\ge1}$ be the sequence of vertices in the region $G_{j,r}$ so that the cell $C_{v_n^j}$ has an edge boundary that belongs to ψ^+ and define the following tree:

$$T_j \equiv \cup_{n\ge1}\rho(v_{j,r}, v_n^j) \subseteq G_{j,r}.$$

This tree is determined by all vertices and edges that belongs to some geodesic connecting $v_{j,r}$ to v_n^j (see Figure 5.1). We claim that:

- T_j contains at least one uni-path, say ρ_j;
- ρ_j is a $\vec{y}_j$-geodesic for some $\vec{y}_j \in S^1$;
- $\vec{y}_j = \vec{x}_1$.

Since T_j is locally finite and $\#(T_j) = +\infty$, it must contain at least one uni-path starting from $v_{j,r}$, say ρ_j. By the definition of T_j, ρ_j must be an uni-geodesic, and by Theorem 1.9

it must be a $\vec{y}_j$-geodesic for some $\vec{y}_j \in S^1$. Notice that ρ_j is contained in $G_{j,r}$ and there may not exist any uni-geodesic starting from $v_{j,r}$ and caught between the interface branch ψ^+ and the uni-geodesic ρ_j. This with Theorem 1.9 implies that $\vec{y}_j = \vec{x}_1$.

Let us denote $\rho_j = (v_j^1, v_j^2, ...,)$, where $v_j^1 = v_{j,r}$ (r fixed). Consider the line segments $l_{v_1^1}^{\vec{x}_1}$, $l_{x_1}^{\vec{x}_1}$, $l_{v_2^1}^{\vec{x}_1}$. Recall that $H_x(\vec{x})$ is the hyperplane that is perpendicular to the direction $\vec{x}$ and contains the point x, and also consider the points $z_j(s)$, that belongs to $l_{v_j^1}^{\vec{x}_1}$ and $|z_j(s) - v_j^1| = s$, and the point $z_+(s)$, that belongs to $l_{x_1}^{\vec{x}_1}$ and $|z_+ - x_1| = s$. Let, $y_j(s)$ be the point that belongs to the intersection between ρ_j and $H_{z_j}^{\vec{x}_1}$, and that maximizes the distance from z_j, and let $y_+(s)$ be the point that belongs to the intersection between ψ^+ and $H_{z+}^{\vec{x}_1}$, and that maximizes the distance from z_+. Since ψ^+ is caught between ρ_1 and ρ_2,

$$|z_+(s) - y_+(s)| \leq \max_{j=1,2}\{|z_+(s) - y_j(s)|\}.$$

However,

$$|z_+(s) - y_j(s)| \leq |v_1 - v_2| + |z_j(s) - y_j(s)|,$$

and consequently, by the definition of ξ_2, for all $s > 0$ lager enough,

$$|z_+(s) - y_+(s)| \leq Cr + s^{\kappa_1} \leq s^{\kappa}.$$

This implies that $\zeta \leq \kappa$. Since $\kappa > \xi_2$ is arbitrary, we obtain that $\zeta \leq \xi_2$.

$\square$

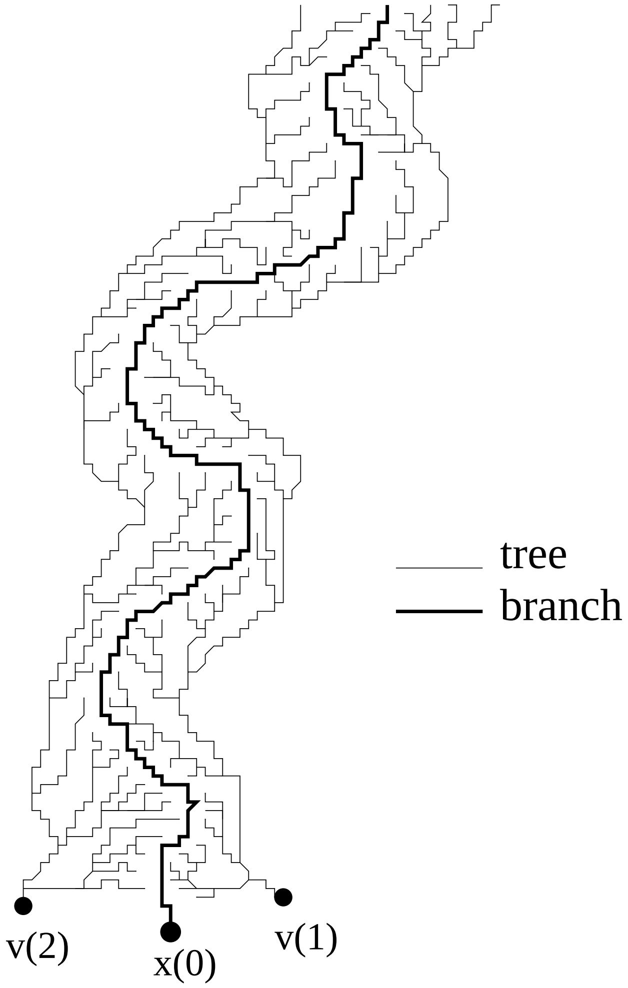

Figure 5.1: The interface and the trees T_1 and T_2.

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