Stably ergodic diffeomorphisms which are not partially hyperbolic

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Abstract

We show stable ergodicity of a class of conservative diffeomorphisms of $\mathbb{T}^n$ which do not have any hyperbolic invariant subbundle. Moreover, the uniqueness of SRB (Sinai-Ruelle-Bowen) measure for non-conservative C^1 perturbations of such diffeomorphisms is verified. This class strictly contains non-partially hyperbolic robustly transitive diffeomorphisms constructed by Bonatti-Viana [4] and so we answer the question posed there on stable ergodicity of such systems.

1 Introduction

One of the main aims of dynamical systems is to answer the following questions:

1. Are the important topological or metric properties satisfied by majority of dynamical systems?
2. Under which conditions such properties persist after small perturbation of the system?

Ergodicity is a basic feature of conservative dynamical systems that yields the description of the average time spent by typical orbits in different regions of the phase space. For non-conservative systems the existence of SRB measures is a natural candidate for the same purpose and they are defined as follows.

Let M be a compact manifold and $f : M \to M$. Given an $f-$invariant Borel probability measure μ, we call basin of μ the set $B(\mu)$ of $x \in M$ such that:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\phi(f^i(x)) = \int \phi d\mu \quad \text{for every} \qquad \phi \in C^0(M)$$

and say that μ is a physical or SRB (Sinai-Ruelle-Bowen) measure for f if $B(\mu)$ has positive Lebesgue measure .

A program proposed a few years ago by Palis [11] contains a conjecture related to the first question:

Every system can be C^r approximated, any $r \geq 1$, by one having finitely many SRB measures with their basins covering a full Lebesgue measure of the phase space.

By the above conjecture, we expect that for a"majority" of diffeomorphisms, the average time spent by typical orbits in different regions of the phase space is described by at most a finite number of measures.

In the same direction, in [1] the authors show the existence of finitely many SRB measures with basins covering a full Lebesgue measure of the ambient manifold, for a large class of partially hyperbolic systems and more generally for systems displaying dominated splitting.

Let M be a closed, compact riemannian manifold with volume form ω. A C^2-volume preserving diffeomorphism $f : M \to M$ is stably ergodic if there is a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^2_\omega(M)$, the space of C^2-volume preserving diffeomorphisms of M, such that every $g \in \mathcal{U}$ is ergodic.

Considering the question (2), we want to know the necessary conditions to get stable ergodicity. First, Anosov in [2] proved ergodicity of the Lebesgue measure for globally hyperbolic systems. Later, Pugh and Shub proved stable ergodicity of a large class of partially hyperbolic systems. The main condition to get ergodicity in these results is "accessibility": any two points of the phase space can be joined by a C^1-path consisting of consecutive segments, which are part of stable or unstable foliations (see [5] for a recent result in stable ergodicity).

Recently F. Rodriguez [7] showed the stable ergodicity of partially hyperbolic automorphisms of $\mathbb{T}^n$ for which the accessibility is not satisfied.

In this paper we show the stable ergodicity of an open set in $\mathrm{Diff}^1_\omega(\mathbb{T}^n)$ admitting no invariant hyperbolic subbundle. In particular, we answer the question posed in [4] about stable ergodicity of the constructed robustly transitive example there. The novelty of our work can be explained as follows.

The existence of invariant foliations tangent to the hyperbolic subbundles of partially hyperbolic systems ([8]) is the main tool for proving the ergodicity of such systems. In the present work, no invariant hyperbolic subbundle is available. We have a dominated splitting and a non-uniform hyperbolicity property for this splitting which is explained in the Preliminary section. Moreover, we show the uniqueness of the SRB measures constructed in [1], for non-conservative perturbations.

The class $\mathcal{V} \subset \mathrm{Diff}^1(\mathbb{T}^n)$ under consideration consists of diffeomorphisms which are deformation of an Anosov diffeomorphism. To define $\mathcal{V}$, let f_0 be a linear Anosov diffeomorphism of n-dimensional torus $\mathbb{T}^n$ (in fact, we need f_0 only to be an Anosov diffeomorphism on $M = \mathbb{T}^n$ whose foliations lifted to $\mathbb{R}^n$ are global graphs of C^1 functions). Denote by $TM = E^s \oplus E^u$ the hyperbolic splitting for f_0 with dim $(E^s) = s$, dim $(E^u) = u$ and let $V = \bigcup V_i$ be a finite union of small balls. We suppose that f_0 has at least a fixed point outside V. and say that $f \in \mathcal{V}$, if it satisfies the following open conditions in C^1 topology :

1. TM admits a dominated decomposition and there exists small continuous cone fields C^{cu}, C^{cs} invariant for Df and Df^{-1} containing respectively E^u and E^s

2. f is C^1 close to f_0 in the complement of V, i.e for $x \notin V$ there is $\sigma < 1$:

$$\|(Df|T_xD^{cu})^{-1}\| < \sigma \quad \text{and} \quad \|Df|T_xD^{cs}\| < \sigma$$

3. There exists some small $\delta_0 > 0$ such that for $x \in V$:

$$\|(Df|T_xD^{cu})^{-1}\| < 1 + \delta_0 \quad \text{and} \quad \|(Df|T_xD^{cs}\| < 1 + \delta_0$$

   where D^{cu}, D^{cs} are disks tangent to C^{cu} and C^{cs}

Theorem 1. *Every $f \in \mathcal{V} \cap \mathrm{Diff}^2_\omega(\mathbb{T}^n)$ is stably ergodic.*

For non-conservative perturbations we prove the uniqueness of SRB measures. An important property required in this case, called "volume hyperbolicity", is defined as follows.

Definition 1.1. *Let $f : M \to M$ be a C^1 diffeomorphism and $TM = E^1 \oplus E^2$; we say that this decomposition has volume hyperbolicity property, if for some $C > 0$ and $\lambda < 1$:*

$$|det(Df^n(x)|E^1)| \leq C\lambda^n, |det(Df^{-n}(x)|E^2)| \leq C\lambda^n$$

Theorem 2. *Any $f \in \mathcal{V} \cap \mathrm{Diff}^2(\mathbb{T}^n)$ having volume hyperbolicity property for $TM = E^{cs} \oplus E^{cu}$ has a unique SRB measure with a full Lebesgue measure basin.*

In Theorem 2, the volume hyperbolicity has the main role for proving non-uniform hyperbolicity. Roughly speaking, by means of this property and a good geometry of the invariant leaves of f_0, typical orbits do not stay a long time in V.

In Section 3, we give an example of non-partially hyperbolic diffeomorphisms which satisfy the hypothesis of Theorem 1 and 2. Observe that although our class is not partially hyperbolic, some weak form of hyperbolicity called "dominated splitting" exists. To justify this dominated splitting condition we make the following comments:

1. Having a unique SRB measure with full Lebesgue measure basin, in a C^1 robust way is more difficult than stable ergodicity in conservative setting.
   It is good to mention that if for a C^1 open set $\mathcal{U}$ of diffeomorphisms there is just one SRB measure μ (depending on f) with supp $(\mu) = M$, then it is easy to see that all $f \in \mathcal{U}$ are topologically transitive and by a result in [3], TM admits a dominated decomposition.

2. The persistence of positive measure sets of invariant tori due to Kolmogrov, Arnold, Moser, Herman and others shows that of course some form of hyperbolicity is needed to get ergodicity. On the other hand, one hopes that stable ergodicity implies dominated splitting.

In Section 2, we give some definitions which will be used in the rest of the paper and in Section 3 the example of robusrtly transitive and non-partially hyperbolic diffeomorphisms of $\mathbb{T}^n$ is constructed.

In Sections 4 and 5, we analyse the geometry of the basins of the SRB measures constructed in [1] for systems with dominated splitting. It is shown that for each such measure there exists some disk almost contained in the basin of it and the radius of the disk is large enough to intersect the stable manifold of a fixed point q outside V. As the intersection of $W^s(q)$ with the mentioned disk in the basin of each SRB measure is transversal, we can C^1 approximate these disks by the λ-lemma.

After approximating the basins of two SRB measures the idea now, is to apply some local accessibility argument. In Sections 6 and 7, we prove the existence of local stable manifolds and absolute continuity of their holonomy for a positive measure subset of unstable manifold of q. By this we prove that $B(\mu_i) \cap B(\mu_j) \neq \emptyset$, for any two SRB measures. Then, the definition of the basin of SRB measures implies that $\mu_i = \mu_j$.

It is worthwile to emphasize that, in Pesin's theory for construction of local stable manifolds the set of regular points in the sense of Lyapunov plays a crucial role. By Oseledets' theorem they occupy a total probability subset of the ambient manifold, but in Theorem 1 we need to use these results for non-regular points. In Sections 6 and 7, we show that the coexistence of non-uniform hyperbolicity and a good control of the angle between the subbundles corresponding to non-uniform contraction and non-uniform expansion, enable us to construct stable manifolds and prove the absolute continuity of their holonomy.

Now the important point is that the union of the basin of SRB measures constructed in [1] contains a full Lebesgue measure subset of the phase space. So, by our uniqueness result, for Lebesgue almost all $x \in M$:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \phi(f^i(x)) = \int \phi d\mu \quad \text{for every} \qquad \phi \in C^0(M).$$

This is equivalent to the ergodicity of the Lebesgue measure for conservative diffeomorphisms.

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2 Preliminary

We may consider some ways of relaxing uniform hyperbolicity, like:

- non-uniform hyperbolicity
- partial hyperbolicity
- dominated splitting

Non-uniform hyperbolicity:
This approach is due to Pesin [13] and it refers to diffeomorphisms with nonzero Lyapunov exponents in a full measure subset of phase space. Recall that λ is a Lyapunov exponent at x if $\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda$ for some vector $v \in T_x M$. By Oseledets' theorem Lyapunov exponents exist for a total probability subset of M.

Dominated splitting:
This approach is due to Mañé and refers to diffeomorphisms with a continuous decomposition of tangent bundle of the phase space : $TM = E^{cs} \oplus E^{cu}$, with the following property:

$$\|Df|E^{cs}_x\|.\|Df^{-1}|E^{cu}_{f(x)}\| \leq \lambda < 1 \quad \text{for all} \quad x \in M$$

From here on just to emphasize the domination we write $TM = E^{cs} \prec E^{cu}$. Whenever we have a dominated splitting on TM, there are two cone fields C^{cu}, C^{cs} with the following properties:

$$C^{cu}_a(x) = \{v_1 + v_2 \in E^{cs} \oplus E^{cu}; \|v_1\| \leq a\|v_2\|\}, \qquad Df(C^{cu}_a(x)) \subset C^{cu}_{\lambda a}(f(x))$$

$$C^{cs}_a(x) = \{v_1 + v_2 \in E^{cs} \oplus E^{cu}; \|v_2\| \leq a\|v_1\|\}, \; Df^{-1}(C^{cs}_a(x)) \subset C^{cs}_{\lambda a}(f^{-1}(x))$$

A system for which $TM = E^s \prec E^c \prec E^u$ is a dominated splitting and E^s, E^u are respectively uniformly contracting and expanding (at least one of them is nontrivial) is called partially hyperbolic. If both uniform contracting and expanding subbundles exist, we call the diffeomorphism as "*strongly partially hyperbolic*".

Key property: "Non-uniformly hyperbolic" dominated splitting
To construct SRB measures for systems under consideration in this paper or more generally with a dominated splitting, by the methods in [1] we need to verify "non-uniform hyperbolicity" in a total Lebesgue measure set in the following sense. There is some $c_0 > 0$ such that

- There exists a full Lebesgue measure set H such that for $x \in H$:

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df|E^{cu}_{f^j(x)})^{-1}\| \leq -c_0 \tag{1}$$

and also :

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df|E^{cs}_{f^j(x)}\|\leq -c_0 \qquad (2)$$

We mention that the above conditions imply nonzero Lyapunov exponents. Let us just mention that in the Pesin theory, some invariant measure is fixed and non-uniformly hyperbolic systems refers to ones without zero Lyapunov exponent in a total measure set. But, we are working with the Lebesgue measure which is not invariant for non-conservative diffeomorphisms of $\mathcal{V}$. In this paper by non-uniform hyperbolicity we refer to the above conditions.

To verify non-uniform hyperbolicity for the diffeomorophisms in Theorems 1 and 2, we use the volume hyperbolicity property defined in the Introduction (Definition 1.1).

For non trivial examples of diffeomorphisms with volume hyperbolicity property we mention the following (see [3]).

- **Conservative systems**: Any C^1 conservative diffeomorphism with a dominated splitting $TM = E^1 \prec E^2$ has the volume hyperbolicity property.

From this and the continuity of $detDf$, we conclude the following corollary:

Corollary 2.1. *For any $f\in\mathcal{V}\cap\mathrm{Diff}^2_\omega(\mathbb{T}^n)$, there exists $\sigma_1>1$ and $C>0$ such that*

$$|det(Df^n(x)|T_x(D^{cs}))|\leq C\sigma_1^{-n},\quad |det(Df^{-n}(x)|T_x(D^{cu}))|\leq C\sigma_1^{-n},$$

where D^{cs}, D^{cu} are disks tangent to C^{cs}, C^{cu}.

3 An example for Theorem 1

Here we give an example of systems that satisfy the hypothesis of Theorem 1. The first non-partially hyperbolic and robustly transitive example is constructed in [4] on $\mathbb{T}^4$. we apply their method and show that it works in larger dimensions. We start with a linear Anosov diffeomorphism f_0 on the $\mathbb{T}^n$ for which

$$T_x(\mathbb{T}^n)=\mathbb{R}^n=E^s_1\prec E^s_2\cdots\prec E^s_{n-2}\prec E^u$$

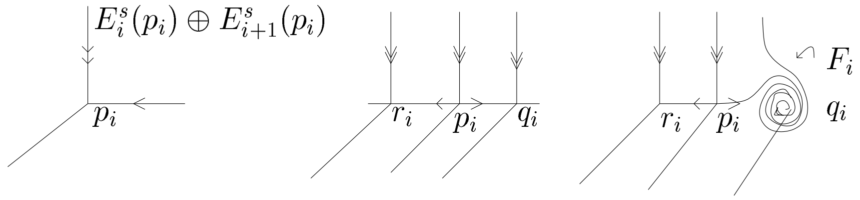


Figure 1: Deformation of Anosov

where dim $(E^u) = 2$ and $\dim(E^s_i) = 1$.

We may suppose that f_0 has fixed points $p_1, p_2, ..., p_{n-2}$. Let $V = \bigcup B(p_i, \delta)$ be a union of small balls centered at p_i and radius small $\delta > 0$. The idea is to deform the Anosov diffeomorphism inside V, passing first through a flip bifurcation along $E^s_i \oplus E^s_{i+1}$ inside $B_i = B(p_i, \delta)$ and then other deformation (see fig 1), always composing with discrete time map of Hamiltonian vector fields to get volume preserving diffeomorphism. (For an example of such vector fields see [4])

More precisely, first we modify along stable direction $E^s_i(p_i) \oplus E^s_{i+1}(p_i)$ for $1 \leq i \leq n-3$ until the index of p_i drops one and two fixed points q_i, r_i are created. These new fixed points are of index $n-2$. In the next step composing with another Hamiltonian (two dimensional), we mix the two contracting subbundles of $T_{q_i}M$ corresponding to $E^s_i(q_i)$ and $E^s_{i+1}(q_i)$. After these deformations we have:

$$T_{q_i}M = E_1 \prec \cdots \prec E_{i-1} \prec F_i \prec \cdots \prec E^u$$

where F_i is two dimensional and corresponds to the complex eigenvalue. Finally we do the same for p_{n-2}, but in the unstable direction of it.

In this way we get an open set $\tilde{\mathcal{V}}$ in C^1 topology, of diffeomorphisms the conditions 1-3 mentioned in the Introduction and:

- There exist a hyperbolic fixed point q with stable index $s =$ dimension of E^s of the Anosov one (in the example is $n-2$), such that its stable manifold intersect any disk tangent to C^{cu} with radius more than ϵ_0, for some small $\epsilon_0 > 0$. The similar thing for the unstable manifold and disks tangent to C^{cs} happens. This is just because of the denseness of invariant leaves of the Anosov diffeomorphism f_0: Take a compact part of $W^s(q, f_0)$ to be ϵ_0 dense and taking V small enough to garantee permanence of this part during the deformations.

Remark 3.1. *Clearly the last item above is satisfied for $f \in \mathcal{V}$ of Theorems 1 and 2, as V is small enough.*

In what follows, we see that f is robustly transitive but it is not partially hyperbolic.

Lemma 3.2. *$f \in \tilde{\mathcal{V}}$ is robustly transitive.*

Proof. The proof goes as in $\mathbb{T}^4$ case in [4, Lemma 6.8] and we just remember the steps. The main idea to prove robust transitivity is to show the robust density of the stable and unstable manifold of an hyperbolic fixed point. We show the density of invariant manifolds of q defined in Remark 3.1 (see Proposition 5.1).

Let U and V be to open subsets. Using λ-Lemma and the ϵ_0 density of invariant manifolds of q we intersect some iterate of U with V and get transitivity for f. $\square$

Lemma 3.3. *$f \in \tilde{\mathcal{V}}$ is not partially hyperbolic.*

Proof. This is just because of the definition of partially hyperbolic systems : f is partially hyperbolic if $TM = E^s \oplus E^c \oplus E^u$ is a decomposition into continuous subbundles where at least two of them are nonzero and E^s and E^u are respectively uniformly contracting and expanding. Suppose that f is partially hyperbolic. First of all note that by continuity of subbundles and the existence of a dense orbit by lemma 3.2, the dimension of E^s is constant.

We claim that $\dim(E^s) = n - 2$ and this gives a contradiction, because in $T_{p_i}M$ there does not exist $n-2$ contracting invariant directions. To prove the claim observe that if we suppose that $\dim\,(E^s) = j < n-2$, then by the decomposition of $T_{q_j}M$:

$$T_{q_j}M = E_1 \prec \cdots \prec E_{j-1} \prec F_j \cdots \prec E^u.$$

By definition, $E^s(q_j)$ must contain $E_1 \oplus \cdots \oplus E_{j-1}$ and then as F_j does not have any invariant subbundle we conclude that $\dim\,(E^s(q_j)) \geq j+1$ and this is a contradiction, because $\dim(E^s) = j$.
By investigating $T_{p_{n-2}}M$, it is obvious that f also can not have any continuous unstable subbundle. $\square$

4 cu-Gibbs measures

Gibbs measures in partially hyperbolic dynamical systems, as measures absolutely continuous along unstable foliation were constructed by Sinai and Pesin. ([12])

For systems with only dominated splitting, in some cases we may call a probability measure as cu-Gibbs, when its conditional measures with respect to a measurable family of center-unstable disks (tangent to C^{cu}) is absolutely continuous with respect to the Lebesgue measure of disks.

In [1], SRB measures for the systems with dominated decomposition having non-uniform hyperbolicity property and a technical condition, are constructed. The technical condition so called simultaneous hyperbolic times is satisfied for diffeomorphisms in $\mathcal{V}$ by techniques introduced in [1]. However in the Appendix we show that, it is not necessary to verify such technical condition for constructing SRB measures of diffeomorphisms in $\mathcal{V}$. The constructed SRB measures are in fact cu-Gibbs measures. Let us recall briefly the construction of cu-Gibbs measures:

Fix a C^2 disk tangent to C^{cu} at every point of it and intersecting H (the set of points having non-uniformly hyperbolic behaviour) in a positive Lebesgue measure where by measure we refer the Lebesgue measure of the disk. Now consider the sequence μ_n of averages of forward iterate of Lebesgue measure restricted to such disk and then prove that a definite fraction of each average corresponds to a measure ν_n which is absolutely continuous with respect to Lebesgue measure along the iterate of disk with uniformly bounded densities. Finally, show that absolute continuity passes to ν, the limit of ν_n. More precisely:

Proposition 4.1 ([1]). *There exists a cylinder $\mathcal{C}$ (a diffeomorphic image of product of two balls B^u, B^s of dimensions $dim(E^{cu})$ and $dim(E^{cs})$ in M) and a family $\mathcal{K}_\infty$ of disjoint disks contained in $\mathcal{C}$ which are graph over B^u such that*

1. *The union of all the disks in $\mathcal{K}_\infty$ has positive ν measure .*
2. *The resteriction of ν to that union has absolutely continuous conditional measure along the disks in $\mathcal{K}_\infty$*

So we have $\mu = \nu + \eta$ where ν is absolutely continuous with a bounded away from zero Radon-Nikodym derivative along a cu-disks family. In this way we conclude that there exists disks γ where Leb_γ-almost every point in γ is regular and by absolute continuity of the stable manifolds "for regular points", one gets a μ positive measure set in the same ergodic component. Normalizing the restriction of μ to the ergodic component above, we get an ergodic invariant probability measure μ^*.

As the conditional measure of μ with respect to $\mathcal{K}_\infty$ is the sum of the conditional measures of ν and η we conclude the following:

Lemma 4.2. *There exists a disk D^∞ in $\mathcal{K}_\infty$, such that Leb_{D^∞}-almost every point of D^∞ belong to the basin of μ^*.*

By Proposition 6.4 in [1], $M = \bigcup B(\mu_i)$ forgetting a negligible set, where μ_i's are cu-Gibbs ergodic and SRB measures. By the above lemma we get the following corollary.

Corollary 4.3. *Let f be as in Theorem 2, then $M = \bigcup B(\mu_i)$ (mod 0) where μ_i are ergodic SRB measures and for each μ_i there exists a disk D_i^∞ tangent to center-unstable cone field such that $D_i^\infty \subset B(\mu_i)$ ($Leb_{(D_i^\infty)}$-mod 0).*

In the next Section we prove that for $f \in \mathcal{V}$, $B(\mu_i) \cap B(\mu_j) \neq \emptyset$ for all $i \neq j$, but as μ_i's are ergodic so they are the same one.

5 Uniqueness of cu-Gibbs measures

Sketch of the proof of Theorems 1 and 2

In order to show the uniqueness of the cu-Gibbs measures, we prove that their basins have non empty intersection. For this, we use Remark 3.1 for the diffeomorphisms in $\mathcal{V}$ and Proposition 5.1 below to approximate the basins of SRB measures. Then by means of local stable manifolds intersect the basins corresponding to the different measures. Observe that any two points in some local stable manifold belong to the basin of the same measure. Let q be as in Remark 3.1.

Proposition 5.1. *The stable manifold of $q, W^s(q(f), f)$ is dense and intersects transversally each D_i^∞.*

Remark 5.2. *This intersection is a crucial part of the proof of ergodicity. Let's mention that just denseness does not imply intersection with D_i^∞.*

Proof. To prove the Proposition 5.1 we claim that some iterate of D_i^∞ contains a disk tangent to center-unstable conefield with radius more than ϵ_0 which also almost every point in it belong to $B(\mu_i)$. This proves the Proposition because of ϵ_0 denseness of the $W^s(q)$, see Remark 3.1. we prove the claim as following:

Consider a lift $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ of f, and π_u as the projection along stable foliation of f_0 (the Anosov one) from $\mathbb{R}^n$ to $\mathbb{R}^u$. As D_i^∞ is tangent to conefield C^{cu} at each point of it, we consider a global graph Γ (the graph of a C^1 function $\gamma : \mathbb{R}^u \to \mathbb{R}^s$, $\|D\gamma\| \leq \epsilon$ (angle of conefield), which contains D_i^∞. Now consider the iterates $\Gamma_n := \tilde{f}^n(\Gamma)$ which all of them are graph of C^1 functions with small derivative, this is because $f^n(\Gamma)$ is a proper embedding of $\mathbb{R}^u$ in $\mathbb{R}^n$ whose tangent space at every point is in C^{cu} and C^{cu} is forward invariant.

Now as Df expands area of disks on center unstable direction by arguments of [4, Lemma 6.8] there exists some point x_0 in $f^{n_0}(D_i^\infty)$ such that its positive orbit never intersects V, so any small disk in Γ_{n_0} around x_0 will have some iterate containing a disk with radius at least ϵ_0. (See Remark 3.1 for ϵ_0). In this way we can show the denseness of $W^s(q)$ and $W^u(q)$, too.

Now observe that because of the invariance of continuous cone field C^{cs}, the global stable manifold of q is tangent to C^{cs} at any point and consequently the intersection of $W^s(q)$ and D_i^∞ is transversal. □

Using λ-lemma, for n large enough $f^n(D_i^\infty)$ and $W^u(q)$ are C^1 near enough. On the other hand, in Section 6 we show that almost every point of $W^u(q)$ have a local stable manifold. This implies that there exists $S \subset W^u(q)$ with $Leb(S) > 0$ such that for all $x \in S$ the size of $W^s_{loc}(x)$ is uniformly bounded away from zero and $W^s_{loc}(x)$ intersects $f^n(\mathcal{D}_i^\infty)$ for n large enough. We need an absolute continuity property proved in Section 7 to conclude the following:

$$Leb_{f^n(\mathcal{D}_i^\infty)}\Big(\bigcup_{x \in S} W^s_{loc}(x) \cap B(\mu_i) \cap f^n(\mathcal{D}_i^\infty)\Big) > 0$$

We would get the same thing for μ_j and this enables us to find at least two points x, y respectively in $B(\mu_i)$ and $B(\mu_j)$ such that they are in the local stable manifold of the same point in S (see fig 2). This means

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^i(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^i(y) \quad \text{for every} \quad \phi \in C^0(M),$$

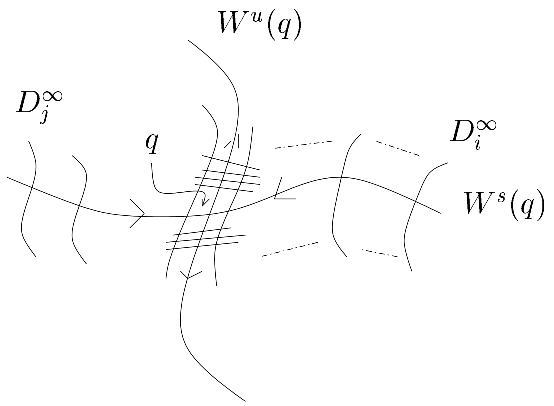


Figure 2: Intersecting basins via local stable manifolds

and consequently $B(\mu_i) \cap B(\mu_j) \neq \emptyset$ which implies $\mu_i = \mu_j$. We have proved that the decomposition of $\mathbb{T}^n$ (mod 0) by the basin of SRB measures contains a unique element (mod 0) or there exists just one SRB measure whose basin has full Lebesgue measure.
If f preserves the Lebesgue measure, by dominated splitting the volume hyperbolicity is satisfied (see Preliminary). So by Theorem 2 for almost all points

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^i(x) = \int \phi d\mu \quad \text{for every} \qquad \phi \in C^0(M)$$

and immediately we have ergodicity of Lebesgue measure, completing the proof of the Theorem 1.

Remark: In Theorem 2 we prove the uniqueness of the SRB measures. The unique SRB measure is absolutely continuous along disks which are unstable manifolds corresponding to positive Lyapunov exponents. By [9] one has the following:

$$h_\mu(f) = \sum \lambda_i^+ \quad \text{where} \quad \lambda_i^+ = max\{0, \lambda_i\},$$

where λ_i are the Lyapunov exponents of the ergodic measure μ. In fact, as the basin of the physical measure construced in Theorem 2 occupies a total Lebesgue measure set of manifold, it will be the unique measure among the ergodic measures with nonzero Lyapunov exponents which satisfy the Pesin's formula.

we observe that with the same method with which we have proved the uniqueness of SRB measures, one also can show that μ is the unique ergodic measure satisfying Pesin's equality and having u(=dim E^u) positive Lyapunov exponents. Then by ergodic decomposition theorem it is the unique invariant probability with the mentioned properties. so, the following question is interesting:

Question 1. *Does any f as in Theorem 2 have only one measure satisfying the Pesin's equality?*

6 Non-uniform hyperbolicity

In this chapter we show a non-uniformly hyperbolic behaviour for a full Lebesgue measure subset on the unstable manifold of the persistent hyperbolic fixed point q. Then we construct local stable manifold for the point of this subset .

To prove a non-uniform hyperbolic behavior, we will "follow the orbit of points" and observe that they spend a definitive part of their time, out of the perturbation region and conclude that they "remember hyperbolicity of the Anosov one". More precisely let W be a $u-$dimensional submanifold of $\mathbb{T}^n$ and π the natural projection from $\mathbb{R}^n$ to $\mathbb{T}^n$. We call W dynamically flat according to the following definition.

Definition 6.1. *W is dynamically flat if for $\widetilde{W_n}$, any lift of $f^n(W)$ to $\mathbb{R}^n$, $Leb(\widetilde{W_n} \cap K) \leq C$ where K is any unit cube in $\mathbb{R}^n$ and C is a constant depending only on f.*

Lemma 6.2. *$W^u(q)$ is dynamically flat.*

Proof. Consider $\mathcal{F}_0(q)$ the leaf of unstable foliation of f_0 which passes through q and let $\mathcal{F}_n = f^n(\mathcal{F}_0(q))$. As $\mathcal{F}_0$ is a leaf of a linear Anosov diffeomorphism, any lift of it to $\mathbb{R}^n$ will be a u-affine subspace and is a proper image of $\mathbb{R}^u$ to $\mathbb{R}^n$. By invariance of the thin conefield C^{cu}, we conclude that the tangent space of any lift of $\mathcal{F}_n$, which we call also $\mathcal{F}_n$, at every point is in C^{cu} and it is also proper image of $\mathbb{R}^u$. In this way for any unit cube K, $\mathcal{F}_n \cap K$ can be seen as the graph of a C^1 function with u-dimensional base of the cube as its domain. This function has an small norm of derivative which is independent of cube K and n, this is because its graph is tangent to C^{cu}. So $\mathcal{F}_n \cap K$ has a uniformly bounded area (with respect to Lebesgue measure of $\mathcal{F}_n$) and this is what we want, because the intersection of the unstable manifold of q with

K is contained in the limit of $\mathcal{F}_n \cap K$.

□

Proposition 6.3. *Let W be a dynamically flat submanifold and f satisfying the hypothesis of Theorem 2, then every small disk in W contains a Lebesgue total measure (Lebesgue measure of W) subset for which:*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|(Df|E^{cs}_{f^j(x)})\| \leq -c_0 \quad c_0 > 0$$

Proof. Here we use the same argument of [1] and show that:

Lemma 6.4. *There exists $\epsilon > 0$ and a total Lebesgue measure subset of any small disk D in W, such that $\#\{0 \leq j < n : f^j(x) \notin V\} \geq \epsilon n$ for every large n .*

Proof. we choose a partition in domains $B_1, B_2 \cdots B_{p+1} = V$ of $\mathbb{T}^n$ such that there exists K_i, L_i with $B_i \in \pi(K_i)$ and $f(B_i) \in \pi(L_i)$ where K_i , L_i are a finite open cubes in $\mathbb{R}^n$) and estimate the Lebesgue measure of the sets $[\underline{i}]$'s where $\underline{i}$ is an array $(i_0, i_1, ..., i_{n-1})$ and $[\underline{i}]$ is defined as points in D such that $f^j(x) \in B_{i_j}$ for $0 \leq j < n$. In fact, we prove the following lemma. Let σ_1 be as in Corollary 2.1 then:

Lemma 6.5. *$Leb([\underline{i}]) \leq C\sigma_1^{-n}$ (where C is a constant depending only to f)*

Proof. By the choice of B_i and induction we have that $f^j([\underline{i}]) \in \pi(\widetilde{W}_n \cap L_{i_{j-1}})$, where $\widetilde{W}_n$ is a lift of $f^n(W)$ to $\mathbb{R}^n$.
To conclude lemma we use area expanding (Corollary2.1 property along disks tangent to center unstable conefield and the fact that intersection of $\widetilde{W}_n$with a unit cube has a uniformly bounded volume. By induction

$$Leb([\underline{i}]) \leq \sigma_1^{-n} Leb(f^n([\underline{i}]) \leq \sigma_1^{-n} Leb(\widetilde{W} \cap L_{i_{n-1}}) \leq C\sigma_1^{-n}$$

□

Now we show how to conclude Lemma 6.4. Let $g(\underline{i})$ be the number of values $0 \leq j \leq n-1$ for which $i_j \leq p$. We note that the total number of arrays with $g(\underline{i}) \leq \epsilon n$ is bounded by

$$\sum_{k \leq \epsilon n} \binom{n}{k} p^k \leq \sum_{k \leq \epsilon n} \binom{n}{k} p^{\epsilon n}$$

and applying Stirling's formula gives that it is bounded by $e^{\beta_0 n} p^{\epsilon n}$ (β_0 goes to zero as ϵ goes to zero). So, the union of the sets $[\underline{i}]$ for which $g(\underline{i}) \leq \epsilon n$ has Lebesgue measure less than $Ce^{\beta_0 n} p^{\epsilon n} \sigma_1^{-n}$. Choosing ϵ small enough such that $e^{\beta_0} p^{\epsilon} < \sigma_1$, we are in the setting of Borel-Cantelli lemma and conclude Lemma 6.4. $\square$

By this lemma the Proposition 6.3 is proved just taking $c_0 = -\log(\sigma^{\epsilon}(1+\delta_0)^{1-\epsilon})$ and δ_0 small enough. $\square$

Corollary 6.6. *Almost all points of local unstable manifold of q satisfy non-uniform hyperbolicity property.*

For any x satisfying the conclusion of Proposition 6.3, there exists $N(x)$ such that for $n \geq N$

$$\prod_{i=0}^{n-1} \|Df_{|E^{cs}(f^i(x))}\| \leq \lambda^n$$

we remember that $\lambda = \sigma^{\epsilon}(1+\delta_0)^{1-\epsilon}$ which is less than one if δ_0 is small enough.

Corollary 6.7. *There exists a positive Lebesgue measure subset $S \subset W^u(q)$, $N \in \mathbb{N}$ and $\lambda < 1$ such that $\forall x \in S$:*

$$\forall n > N \qquad \prod_{i=0}^{n-1} \|Df_{|E^{cs}(f^i(x))}\| \leq \lambda^n$$

The points of S are not necessarily regular in the sense of Lyapunov. We can not use Pesin theory directly for the existence of invariant manifolds and absolute continuity of their holonomy. By dominated splitting and non-uniform hyperbolicity as above we can construct local stable manifolds.

Proposition 6.8. *Every point of S has an stable manifold, whose size is bounded away from zero.*

Proof. We can construct local invariant disks using only domination property but in general case we do not know whether these disks are stable manifolds or not. For $f \in \mathcal{V}$ by Corollary 6.7, we are able to prove that the disks passing through the point of S are stable manifolds, i.e $d(f^n(x), f^n(y)) \to 0$ exponentially fast, for $y \in W^{cs}_{loc}(x)$.
Denote $Emb(D^u, M)$ the space of C^1 embeddings from D^u to M endowed with the C^1 topology where D^u is the u-dimensional ball of radius one.

Using notation of [8], M is an immediate relative pseudo hyperbolic set for f if there exists a continuous function ρ such that:

$$\|Df_{|E^{cs}(x)}\| < \rho(x) < m(Df_{|E^{cu}(x)}) \tag{3}$$

where $m(T) = \|T^{-1}\|^{-1}$

In our case, dominated splitting and compactness of $\mathbb{T}^n$ imply relative pseudo hyperbolicity. We deduce that, there exist continuou sections

- $\phi^u : M \to Emb(D^u, M)$
- $\phi^s : M \to Emb(D^s, M)$

such that $W^{cs}_\epsilon(x) := \phi^s(x)D^s_\epsilon, W^{cu}_\epsilon(x) := \phi^s(u)D^u_\epsilon$, have the following properties:

1.
   - $T_x W^{cs}_1(x) = E^{cs}(x)$
   - $T_x W^{cu}_1(x) = E^{cu}(x)$
2. Local invariance property . for all $0 < \epsilon_1 < 1$ there is $0 < \epsilon_2 < 1$
   - $f(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f(x))$
   - $f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-1}(x))$

Given any c we can take ϵ_1 such that :

$$1 - c < \frac{\|Df_{|T_y W^{cs}(x)}\|}{\|Df_{|E^{cs}(x)}\|} < 1 + c \qquad \text{when} \qquad d(x,y) < \epsilon_1\,, y \in W^{cs}_{\epsilon_1}(x) \tag{4}$$

We can take this ϵ_1 uniformly in x as M is compact and the section is continuous with image in embeddings endowed with C^1 topology. Choosing ϵ_2 such that $f^i(W^{cs}_{\epsilon_2}(x)) \in W^{cs}_{\epsilon_1}(f^i(x))$ for all $0 \le i \le N$ we show that $\forall n \in \mathbb{N},\ d(f^n(x), f^n(y)) \le \epsilon_1$. In fact, $d(f^n(x), f^n(y))$ goes to zero as n goes to infinity. We prove it by induction; let us define:

$$\overline{\lambda} := (1 + c)\lambda \tag{5}$$

and c is adjusted such that $\overline{\lambda} < 1$. As $d(f^i(x), f^i(y)) \le \epsilon_1$ for $0 \le i \le N+k-1$ we have:

$$d(f^{N+k}(x), f^{N+k}(y)) \leq (1+c)\|Df_{|E^{cs}(f^{N+k-1}(x))}\|d(f^{N+k-1}(x), f^{N+k-1}(y)) \leq$$

$$\leq \prod_{i=0}^{N+k-2} \|Df_{|T_{z_i}W^s_{\epsilon_1}(f^i(x))}\| \, \|Df_{|E^{cs}(f^{N+k-1}(x))}\| \, d(x,y)$$

$$\leq (1+c)^{n+k} \prod_{i=0}^{N+k-1} \|Df_{|E^{cs}(f^i(x))}\|d(x,y) \leq \overline{\lambda}^{n+k} d(x,y)$$

where $z_i \in W^{cs}_{\epsilon_1}(f^i(x))$; this is all by Mean Value Theorem. □

7 Absolute continuity

[1] In this Section we prove that the holonomy map by the local stable manifolds constructed on S is absolutely continuous:

Theorem 3. *For large n, holonomy map from $S \subset W^u_{local}(q(f), f)$ to $f^n(\mathcal{D}^i_\infty)$ is absolutely continuou i.e it sends the nonzero measure subset of S to a nonzero measure subset of $f^n(\mathcal{D}^i_\infty)$.*

Let us remember that holonomy map h is defined on whole S for large n. From now on we call its inverse by π which is holonomy along stable manifolds from $f^n(\mathcal{D}^i_\infty)$ to $W^u(q)$. We are going to prove that if B is a measure zero set in $h(S) \subset f^n(\mathcal{D}^i_\infty)$ then $Leb(\pi(B)) = 0$ and then conclude that $Leb(h(S)) \neq 0$. For this, it is enough to show that for every disk $D \subset f^n(\mathcal{D}^i_\infty)$ with center in $h(S)$, the holonomy π from D to $W^u(q)$ does not increase measures, more than a constant which is uniform for all such disks :

$$Leb(\pi(D)) < KLeb(D)$$

because for any measurable set B with zero measure, we can cover it by a family of disks $\mathcal{D}$ such that $\sum_{D\in\mathcal{D}} m(D)$ is arbitrary close to zero. As $Leb(\pi(D)) \leq KLeb(D)$, we conclude that $Leb(\pi(B)) = 0$. From now on S' represents $h(S)$.
The proof of this absolute continuity result goes in the same spirit of [14]. The difference is that here the points for which we construct stable manifolds

[1] I thank Krerley Irraciel for usefull discussions on this section.

are not necessarily regular. We see that an asymptotic hyperbolicity and a good control on the angles of two invariant subbundles is enough to get an absolute continuity result. A short sketch of the proof is as follows.

To compare $Leb(D)$ and $Leb(\pi(D))$, we iterate sufficiently such that $f^n(D)$ and $f^n(\pi(D))$ "become near enough". But after such iteration, $f^n(D)$ may have an strange figure, so in 7.2 we consider a covering of $f^n(S') \cap f^n(D)$ by $B_i := B(a_n, f^n(x_i))$ (ball of radius a_n with center $f^n(x_i)$) where x_i is in S' and a_n is much larger than $d(x_i, \pi_n(x_i))$ where π_n is defined naturally by $\pi_n = f^n \circ \pi \circ f^{-n}$.

By the specific choice of a_n, in 7.3 we show that $Leb(B_i) \approx Leb(\pi_n(B_i))$. Indeed, the dominated splitting of the tangent bundle allows us to choose them in such a good way. Finally, in 7.4 we prove some distortion results and come back to compare the volume of D and $\pi(D)$.

7.1 Some general statements

Let us fix some notations and definitions:

- d_1(resp.d_2) := restriction of the Riemannian metric of manifold to $f^n(D)$ (resp. $W^u(q)$)
- d_s := Intrinsic metric of stable manifolds
- d := Riemannian metric of the manifold M
- $a \preceq b$ means $a \leq kb$ for a uniform $k > 0$ $a, b \in \mathbb{R}$

Definition 7.1. *If E, F are two subspaces of the same dimension in $\mathbb{R}^n$ the angle between them, $\measuredangle(E, F)$, is the norm of following linear operator :*

$$L : E \to E^{\perp} \quad \text{such that} \quad Graph(L) := \{(v, L(v)), v \in E\} = F$$

Definition 7.2. *A thin cone $C_\epsilon(E)$with angle ϵ around E is defined as subspaces S s.t $\measuredangle(S, E) \leq \epsilon$*

By the definition of cones, it is easy to see that:

Lemma 7.3. *If $C^{cu}_\epsilon, C^{cs}_\epsilon$ are two conefield which contain E^{cu}, E^{cs} (the subbundles of dominated splitting) then $Df_x C^{cu}_\epsilon(x) \subset C^{cu}_{\lambda\epsilon}(f(x))$, for some $0 < \lambda < 1$ or in other words the angle will decrease exponentially.*

Proof. Take $S \in C^{cu}(x)$ and $v \in S$. By definition $v = v_1 \oplus v_2$ where $v_1 \in E^{cu}, v_2 \in E^{cs}$ and by dominated splitting (see Preliminary Section):

$$\frac{\|Df_x(v_2)\|}{\|Df_x(v_1)\|} \leq \lambda \frac{\|v_2\|}{\|v_1\|}$$

and this means that $\measuredangle(Df(S), E^{cu}(f(x))) \leq \lambda \measuredangle(S, E^{cu}(x))$ by definition 7.1. ☐

Let us state an easy lemma that gives us some good relations between d_1, d_2 and d.

Lemma 7.4. *If $\mathbb{R}^n = S \oplus U (U = S^{\perp})$ and h is a C^1 function from $B(0,\delta) \subset S$ (ball of radius δ) to F where F is in a small cone $C_\epsilon(U)$. Suppose that $T_x(graph(h)) \subset C_\epsilon(S), \forall x \in graph(h)$ then:*

- $d_h(z,0) \leq C(\epsilon) d(z,0), \quad d_h$ *:distance on* $graph(h), C(\epsilon) \approx 1$
- $Leb(graph(h)) \approx Leb(B(0,\delta))$

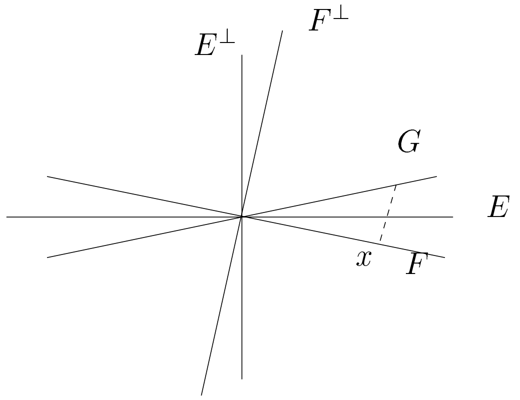
Proof. By the hypothesis on the $graph(h)$ and the definition of angle, we conclude that $\|D_x h\| \leq \epsilon$ and the proof of the first item goes just by the Mean Value Theorem. The second item is also easy to prove just by the formula of volume for graph of a function (see Chapter 1 of [6] for the formulas). ☐

In what follows we show some general results about the derivative of diffeomorphisms and conefields.

Lemma 7.5. *If h is a C^1 function from $B(0,r) \subset E$ to F such that $\|Dh(x)\| \leq a$ (small) where F is a linear subspace with $\measuredangle(E^{\perp}, F) \leq b$ (also small) then $graph(h)$ will be graphic of a new function $\tilde{h} : Dom(\tilde{h}) \subset E \to E^{\perp}$ and $\|D\tilde{h}\| \leq Ka$ (where K is constant converging to 1 when b goes to zero)*

Lemma 7.6. *If h is a C^1 function from $B(0,r) \subset E$ to $E^{\perp}$ such that $\|Dh(x)\| \leq a$ (small) and F is a linear subspace with the same dimension of E , with $\measuredangle(E,F) \leq b$ (also small) then $graph(h)$ will be graphic of a new function $\tilde{h} : Dom(\tilde{h}) \subset F \to F^{\perp}$ and $\|D\tilde{h}\| \leq 2(a+b)$*

The proof of the Lemma 7.5 comes out just by the definition of angle and the derivative of a function. We prove Lemma 7.6 as follows.



Proof. First observe that by the definition of angle in Definition 7.1, $\|Dh(x)\|$ is equal to the angel between E and $T_{(x,h(x))}graph(h)$. So, to prove the Lemma suppose that $\measuredangle(E,F) = b$ and $\measuredangle(E,G) = a$ with a, b small. Let f be linear maps from F to $F^\perp$ whose graph is E a nd $\tilde{g} : E \to F^\perp$ and $g : F \to F^\perp$ be the maps with G as their graph. We are going to show that $\|g\| \leq 2(a+b)$. By the definition of $\measuredangle(F,E)$ and using Lemma 7.5 we have (see figure 7.1):

$$\|g(x)\| \leq \|f(x)\| + \|\tilde{g}(x+f(x))\| \leq \|f(x)\| + Ka\|x+f(x)\|$$

where K is near to one and is obtained by Lemma 7.5. so we get :

$$\frac{\|g(x)\|}{\|x\|} \leq b + Ka(\sqrt{1+b^2}) \leq 2(a+b)$$

and the proof of Lemma 7.6 is complete just by taking $G = T_{(x,h(x))}graph(h)$. $\square$

The dependence of invariant subbundles E^{cu}, E^{cs} to the base point is very important for proving the Theorem 3. The following control of the angles is a product of dominated decomposition.

Lemma 7.7. *There exist constants: $0 < \alpha < 1$ and $0 < \theta < 1$ with following poperty:*

if $d(f^i(x), f^i(y)$ is small for $i = 0, ..., n$ then for any two subspaces S_1, S_2 respectively in $C^{cu}(x), C^{cu}(y)$ (small cones):

$$\measuredangle(Df^n_x(S_1), Df^n_y(S_2)) \preceq \theta^n + dist(f^n(x), f^n(y))^\alpha.$$

Proof. See [15] □

Remark 7.8. *In the above Lemma, $\theta < 1$ comes from dominated splitting and we can take $\theta^2 = \lambda$ where λ is as Lemma 7.3.*

7.2 Covering $f^n(S')$ by graph of C^1 functions

In what follows we will show that for every point x in $S' \cap D$, $f^n(D)$ locally can be seen as graph of a C^1 function from $E^{cu}(f^n(x))$ to $E^{cs}(f^n(x))$ with norm of derivative converging to zero uniformly as n goes to infinity. By this we intend to consider flat disks in the high iterates of D covering $f^n(S')$. Let us call $y_n := f^n(x), y'_n := \pi_n(y_n)$.

We mention that, for all n, $f^n(D)$ is tangent to a thin cone which varies continuously. In the following Lemma we show that locally, they are graphs over apropriately disks whose radius decays as n grows up.

Lemma 7.9. *There exists $\delta > 0$ such that for $\delta_1 < \delta$ and any $x \in M$ If*

$$h : B^{cu}_{\delta_1}(0) \subset E^{cu}(x) \to E^{cs}(x),\ h(0) = 0,\ \|Dh(\xi)\| \leq k,\ \forall \xi \in B^{cu}_{\delta_1}$$

and graph $(h) \subset B^{cu}_{\delta} \times B^{s}_{\delta}$, then

$W = f(graph(h)) \cap (B^{cu}_{\gamma\delta_1} \times B^{s}_{\delta})$ will be also graph of some $\tilde{h}$ with the following properties:

1. *Its domain contains $B^{cu}_{\gamma\delta_1}$ and $\tilde{h}(0) = 0$;*
2. *$\|D\tilde{h}(\tilde{\xi})\| \leq k\theta$, $\forall \tilde{\xi} \in B^{cu}_{\gamma\delta_1} \subset E^{cu}(f(x))$;*
3. *$\bar{\lambda} < \gamma$ where $\bar{\lambda}$ is defined as (5) in Section 6.*

Proof. As f is C^2 , there exists δ such that for all $x \in M$, f can be written as : $f(\xi, \eta) = (A^{cu}(\xi) + \phi^{cu}(\xi, \eta), A^{cs}(\eta) + \phi^{cs}(\xi, \eta)$, where $(\xi, \eta) \in B^{cu}_{\delta} \times B^{cs}_{\delta}$ and $\|D(\phi^{cu}, \phi^{cs})\| \leq \epsilon$. Just to reduce the notations suppose that x is a fixed point. We define :

- $\alpha(\xi) = \tilde{\xi} := A^{cu}(\xi) + \phi^{cu}(\xi, h(\xi) = A^{cu}(\xi + (A^{cu})^{-1}(\xi)\phi^{cu}(\xi, h(\xi));$
- $\beta(\xi) = A^{cs}(h(\xi)) + \phi^{cs}(\xi, h(\xi));\qquad \xi \in B^{cu}_{\delta}(0);$

Now as $\|(A^{cu})^{-1}\| \leq 1+\delta_0$ choosing ϵ small enough we deduce that:

$$\|(A^{cu})^{-1}\| Lip(\phi^{cu}(\xi, h(\xi))) < 1,$$

and this shows that $\alpha = A^{cu}(.)(Id + (A^{cu})^{-1}(.)\phi^{cu}(., h(.))$ is invertible. So, it is enough to determine the domain of α^{-1} and defining $\tilde{h} = \beta \circ \alpha^{-1}$ for proving the first part of the lemma.
Observe that

$$\|\alpha(\xi)\| \geq \|A^{cu}(\xi)\| - \|\phi^{cu}(\xi, h(\xi)\| \geq (\frac{1}{1+\delta_0} - 2\epsilon)\|\xi\| > \gamma \|\xi\| \ ,$$

where γ is near to one as δ_0 is small enough. Now by the aid of the proof of the inverse function theorem α^{-1} is defined on $B^{cu}_{\gamma\delta_1}$ and

$$\tilde{h} = \beta \circ \alpha^{-1} : B^{cu}_{\gamma\delta_1} \to B^{cs}$$

is what we want. Observe that as $\bar{\lambda} < 1$, the third part of the Lemma also turns out.

Now we will verify the claim about derivative of $\tilde{h}$. By dominated splitting we have $0 < \theta < 1$ such that $\|(A^{cu})^{-1}(f(x))\| \, \|A^{cs}(x)\| \leq \theta^2$ (θ^2 is just the λ in Lemma 7.3). By choosing ϵ small enough such that $\|D\beta\| \leq \dfrac{k}{\sqrt{\theta}}$ we get

$$\|D\tilde{h}(\tilde{\xi})\| \leq \|D\beta(\xi)\| \, \|D\alpha^{-1}(\tilde{\xi})\| \leq \frac{k}{\sqrt{\theta}} \|A^{cs}\| \, \|(A^{cu})^{-1}\| \, \|D(I+T)^{-1}\|$$

where $T = (A^{cu})^{-1}\phi^{cu}(\xi, h(\xi))$, on the other hand we have

$$\|D(I+T)^{-1}\| = \|(I+DT)^{-1}\| \leq \sum_{i=0}^{\infty} \|(DT)^i\| = \frac{1}{1-\|DT\|} \leq \frac{1}{\sqrt{\theta}}$$

for ϵ is small enough. so $\|D\tilde{h}(x)\| \leq k\theta$ □

Let us see how to cover $f^n(S') \cap f^n(D)$ by disks:

Take $x \in S \cap D$ and let $y_n := f^n(x), y'_n := \pi_n(y_n)$. There exists $\delta > 0$ and C^1 functions h_x such that, $h_x : E^{cu}_\delta(x) \to E^{cs}(x)$ and the graph of h_x is a ball around x. (δ is chosen uniformly in D). Now by Lemma 7.9 there exists $h_{f^n(x)} : B^{cu}_{\gamma^n\delta}(x) \to E^{cs}(f^n(x))$ such that $h_{f^n(x)}(B^{cu}_{\gamma^n\delta}(x))$ is a ball around y_n and also we have a good control on the derivative of them : $\|Dh_n(x)\| \leq k\,\theta^n$. Applying Lemma 7.4 we get:

$$d(z, y_n) \leq d_1(z, y_n) \leq k_n d(z, y_n) \quad \forall z \in \text{graph}(h_n) \text{ and } k_n \to 1 \quad ,$$

and this gives that $h_{f^n(x)}(B^{cu}_{\gamma\delta}(x))$ is a ball of radius arbitrary near to $2a_n := \gamma^n\delta$ by taking n large enough. we call this ball $\bar{B}_n$ (around y_n) and B_n the ball with radius a_n around y_n.

we mention here that $\bar{B}_n$ is also graph of a function from E^{cu} to $(E^{cu})^{\perp}$ over $P(\bar{B}_n)$ where P is the orthogonal projection along $(E^{cu})^{\perp}$.

Remark 7.10. *By the estimate of the derivative of h_n, $P(\bar{B}_n)$ is contained in the ball of radius $2a_n(1+C\theta^n)$ and contains the ball of radius $2a_n(1-C\theta^n)$ where C depends on the angle of $(E^{cu})^{\perp}$ and E^{cs}.*

In what follows we are working with $\bar{B}_n$ as the graph of the mentioned new C^1 function which we call it also h_n and it is easy to see that $\|Dh_n\| \leq K\theta^n$ (Lemma 7.5).

Now we define a new transformation from $\bar{B}_n$ to $W^u(q)$ which is very near to holonomy π_n. Let's $z \in \bar{B}_n$ and define $\mathcal{P}(z)$ by translation along $E^{cu}(y_n)^{\perp}$ which is orthogonal to the tangent space of all points of $\bar{B}_n$. One important property of $\mathcal{P}$ is that $d(z, \mathcal{P}(z))$ is exponentially small. Indeed, we choose a_n small enough for $d(z, \mathcal{P}(z))$ be comparable to the $d(y_n, y'_n) = \bar{\lambda}^n$.

7.3 Comparing measures of B_i and $\pi_n(B_i)$:

In the previous Section we saw how to cover $f^n(S') \cap f^n(D)$ by balls $\bar{B}_i$. In what follows it is proved that the volume of these disks does not increase "a lot" by holonomy. Indeed, we have to take a_n in a good way to have this property. The most important property for a_n is:

$$\frac{\bar{\lambda}^n}{a_n} \to 0 \tag{6}$$

and the main proposition to prove is the following.

Proposition 7.11. *There is a constant $I > 0$, independent of n, such that $Leb(\pi_n(B_i)) \leq I\,Leb(B_i)$*

To prove the above Proposition, we start with some lemmas.

Lemma 7.12. *There is a choice of a_n satisfying (6) such that for every $z \in \bar{B}_i, d(z, \mathcal{P}(z)) \preceq \bar{\lambda}^n$.*

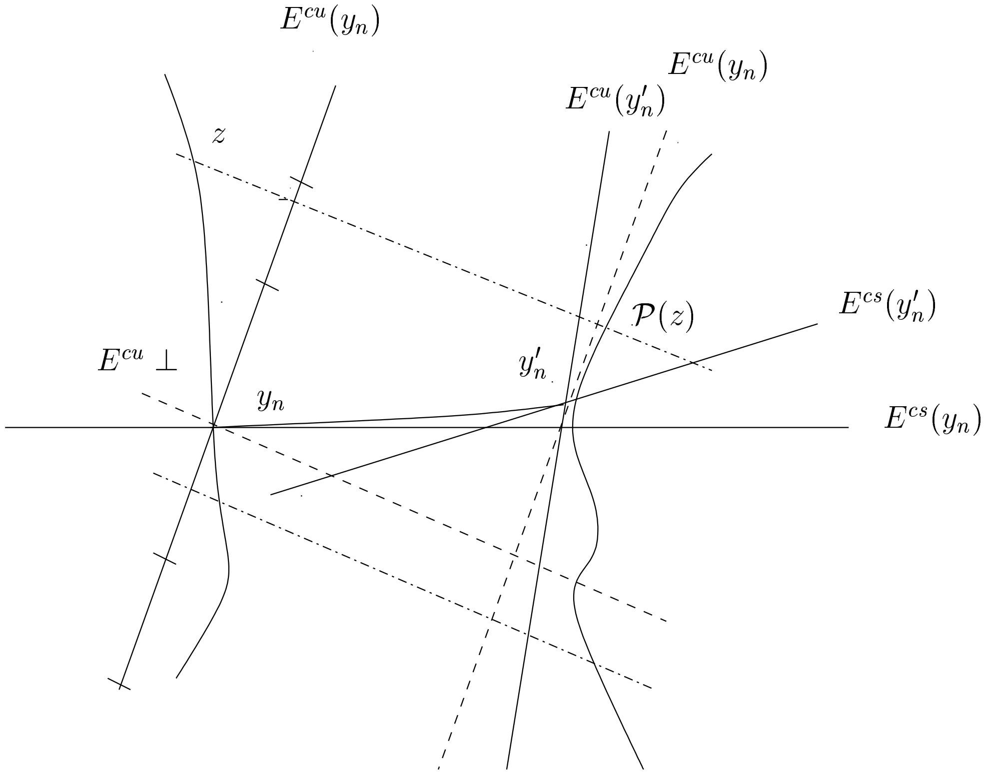


Figure 3: Image of graphics

Proof. When n is large enough we can consider $\mathcal{P}(\bar{B_n})$ also as a graph over $E^{cu}(y_n)$ to $E^{cu}(y_n)^{\perp}$, but we have to consider the angle between $E^{cu}(y_n)$ and $E^{cu}(y'_n)$ to calculate norm of derivative of the new function. To estimate the norm of the derivative of the C^1 functions whose graphs are $\bar{B_n}$ and $\mathcal{P}(\bar{B_n})$ we use lemmas 7.6 and 7.5. Using Mean Value Theorem and Remark 7.10 we have (see figure 3) :

$$d(z,\mathcal{P}(z)) \leq k(2a_n + 2Ca_n\theta^n)\theta^n +$$

$$+(2a_n + 2Ca_n\theta^n)(k\theta^n + \measuredangle(E^{cu}(y_n), E^{cu}(y'_n))) + \bar{\lambda}^n$$

Note that the first term containing angles, in the above relations is because of the deviation of $E^{cu}(y_n)$ from $E^{cu}(y'_n)$ and applying lemma 7.6 :

$$\measuredangle(E^{cu}(y_n), E^{cu}(y'_n)) \preceq \theta^n + d(y_n, y'_n)^{\alpha},$$

and so:

$$d(z, \mathcal{P}(z)) \preceq a_n\theta^n + a_n d(y_n, y_n')^\alpha + \bar{\lambda}^n.$$

So, to finish the proof of Lemma 7.12 it is enough to have the following two conditions :

- $a_n \approx \bar{\lambda}^n\theta^{-n}$
- $a_n \approx \bar{\lambda}^{n(1-\alpha)}$

Remember that by Lemma 7.9, we need another restriction on a_n to have graph of functions to use Mean Value Theorem.

- $a_n \leq \gamma^n\delta$

So choose $a_n = min(\bar{\lambda}^n\theta^{-n}, \bar{\lambda}^{n(1-\alpha)}, \gamma^n\delta)$. As $\bar{\lambda} < \gamma$, already $\frac{\bar{\lambda}^n}{a_n} \to 0$. ☐

Lemma 7.13. *$\pi_n(B_i)$ is contained in a ball around y_n' of radius near enough to $\frac{3}{2}a_n$ as n is large enough.*

Proof. For $z \in B_n$, $\pi_n(z)$ lies in the $W^u(q)$ which is contained in the graph of a function defined globally and the graph is tangent to a thin cone field. So, by Lemma 7.4 we deduce that for $z \in B_n$, $d_2(\pi_n(z), y_n') \leq \frac{3}{2}d(\pi_n(z), y_n')$ so we get :

$$d_2(\pi_n(z), y_n') \leq \frac{3}{2}(d(\pi_n(z), y_n')) \leq \frac{3}{2}\left(d(\pi_n(z), z) + d(z, y_n) + d(y_n, y_n')\right)$$

$$\leq \frac{3}{2}(d_s(\pi_n(z), z) + d_1(z, y_n) + d_s(y_n, y_n') \leq \frac{3}{2}(\bar{\lambda}^n + k_n a_n + \bar{\lambda}^n)$$

$$= \frac{3}{2}a_n(k_n + \frac{2\bar{\lambda}^n}{a_n})$$

So, choosing a_n as in the Lemma 7.12 the proof is complete. ☐

Lemma 7.14. *$\mathcal{P}(\bar{B}_i)$ contains $\pi_n(B_i)$.*

Proof. For every $z \in \bar{B}_i$ by triangular inequality for distances on the ambient manifold :

$$d_2(\mathcal{P}(z), y_n') \geq d(\mathcal{P}(z), y_n') \geq d(z, y_n) - d(y_n', y_n) - d(z, \mathcal{P}(z))$$

$$\geq \frac{1}{k_n} d_1(z, y_n) - d_s(y'_n, y_n) - d(z, \mathcal{P}(z)) \geq \frac{1}{k_n} d_1(z, y_n) - 2\bar{\lambda}^n.$$

Here we have used Lemma 7.12, as $k_n \to 1$ and $\frac{\bar{\lambda}^n}{a_n} \to 0$ we conclude that $\mathcal{P}(\bar{B}_i)$ contains a ball around with radius near to $2a_n$ and by Lemma 7.13 it contains $\pi_n(B_i)$. □

With this specific choice of a_n in the above Lemmas, we are able to prove the Proposition 7.11.

Proof. As $Leb(\bar{B}_i) \leq I_1 Leb(B_i)$, for a constant I_1 not depending to n and just depends to dimension ofB_i, we have :

$$Leb(\pi_n(B_i)) \leq Leb(\mathcal{P}(\bar{B}_i)) \approx Leb(\bar{B_i})) \leq I_1 Leb(B_i)$$

and the Proposition is proved. □

Up to now we have covered $S_n := f^n(S') \cap f^n(D)$ by a family of disks such that the volume of whose images under holonomy is comparable to their volume. By Besicovich covering theorem [10] we can cover S_n with a countable locally finite subfamily $\{B_i\}_i$ that is, there is a constant C only depending to the dimension of D such that, the intersection of any $C+1$ disk of such subfamily is empty set.

7.4 Distortion estimates

Now we state the distortion controls statements. In what follows, by $Jf(x, A)$ we mean $det(Df_x|A)$

Lemma 7.15 (Bounded Distortion). *There are $P_1, M > 0$ such that for any $z \in B_i$ the following are satisfied:*

- $\frac{1}{M} \leq \frac{Jf^{-n}(y_n, T_{y_n} B_i)}{Jf^{-n}(y'_n, T_{y'_n} \mathcal{P}(B_i))} \leq M$
- $\frac{1}{P_1} \leq \frac{Jf^{-n}(z, T_z(B_i))}{Jf^{-n}(y_n, E^{cu}_{y_n})} \leq P_1$

Proof. The problem is that in general we do not have Hölder control of the centre unstable fibers. But in the case of the dominated decomposition or in other words when we have hyperbolicity property for the angles, one can show statements near to Hölder continuity .

As f is C^2 function, we conclude that there exist constants $R_1, R_2 > 0$ such that if $z_1, z_2 \in M, d(z_1, z_2) \leq 1$ and S_1, S_2 are subspaces of $\mathbb{R}^n$ with dimension u (dimension of E^{cu}) then:

$$|\log Jf^{-1}(z_1, A_1) - \log Jf^{-1}(z_2, A_2)| \leq R_1 d(z_1, z_2) + R_2 \measuredangle(A_1, A_2). \tag{7}$$

Now using the above inequality(7) and Lemma 7.7 we have :

$$|\log Jf^{-n}(y_n, E^{cu}(y_n)) - \log Jf^{-n}(y'_n, E^{cu}(y'_n)) \leq$$

$$\leq R_1(\sum_{i=0}^{n-1} dist(f^{-i}(y_n), f^{-i}(y'_n)) + R_2(\sum_{i=0}^{n-1} \measuredangle(E^{cu}(f^{-i}(y_n), E^{cu}(f^{-i}(y'_n))))$$

$$\leq \frac{CR_2}{1-\theta} + (KR_2 + R_1) \sum_{i=0}^{n-1} dist(f^{-i}(y_n), f^{-i}(y'_n)^{\alpha}.$$

So using another time (7) we conclude :

$$|\log Jf^{-n}(y_n, T_{y_n} B_i) - \log Jf^{-n}(y'_n, T_{y'_n} \mathcal{P}(B_i)| \leq$$

$$|\log Jf^{-n}(y_n, T_{y_n} B_i) - \log Jf^{-n}(y_n, (E^{cu}(y_n))|+$$

$$|\log Jf^{-n}(y_n), E^{cu}(y_n)) - \log Jf^{-n}(y'_n), E^{cu}(y'_n)|+$$

$$|\log Jf^{-n}(y'_n), E^{cu}(y'_n)) - \log Jf^{-n}(y'_n), T_{y'_n} \mathcal{P}(B_i)| \leq$$

$$\frac{R_2}{1-\theta} + (KR_2 + R_1) \sum_{i=0}^{n-1} dist(f^{-i}(y_n), f^{-i}(y'_n)^{\alpha} + 2R_2 \sum_{i=0}^{n-1} \theta^{n-i} \tag{8}$$

As y_n, y'_n are on the same strong stable manifold all of the terms appeared in (8) are summable and the proof of the first item of the lemma is complete. In fact, our argument show that we can substitute y_n, y'_n respectively by any point $w_n \in B_i \cap f^n(S')$ and $\pi_n(w_n)$. The second item of the lemma follows by the same arguments noting that the size of $B_i \subset f^n(D)$, a_n, is exponentially small.

Now we apply distortion estimates of jacobians to get

$$Leb(\pi(D)) \leq \sum_i Leb(f^{-n}(\pi_n(B_i)) \leq MP_1^2 \sum_i Leb(f^{-n}(B_i)) \frac{Leb(\pi_n(B_i))}{Leb(B_i)}$$

$$\leq IMP \sum_i Leb(f^{-n}(B_i))$$

But as $\{B_i\}_i$ is a locally finite family covering S_n and by f^{-n} the areas of disks tangent to C^{cu} decreases, taking n sufficiently large we see that $\sum_i Leb(f^{-n}(B_i)) \leq ALeb(D)$. So taking $P_1^2 = P$ we conclude

$$\frac{Leb(\pi(D))}{Leb(D)} \leq IMPA(universal)$$

□

8 Appendix: Simultaneous hyperbolic times

In [1, Theorem 6.3] ergodic $cu-$Gibbs measures for diffeomorphisms with dominated splitting and the non-uniformly hyperbolic property like in the Preliminary section, are constructed. This measures are absolutely continuous along a family of disks which are tangent to center-unstable conefield.

Proposition 8.1. *For $f \in \mathcal{V}$, the $cu-$Gibbs measures as above are SRB i.e their basin has positive Lebesgue measure.*

To prove that these measures are SRB, one need to show that for points in the support of these measures, all the Lyapunov exponents (in the E^{cs} direction) are negative. To provide negative Lyapunov exponents, in [1], the authors add the condition of "simultaneous hyperbolic times". We show that for $f \in \mathcal{V}$ it is not necessary to verify this condition and see that the $cu-$Gibbs measure constructed there, are indeed SRB measure.

For any $y \in Supp(\mu)$ where μ is one of such cu-Gibbls measures, there exists x such that $y \in D^\infty(x)$ where $D^\infty(x)$ is tangent E^{cu} at any point of it and moreover it is the local strong unstable manifold of x (see [1, Lemma 3.7]).

Lemma 8.2. *If $f \in \mathcal{V}$ then for Lebesgue almost all point of $D^\infty(x)$ the Lyapunov exponents in the E^{cs} direction are negative.*

Proof. By the above observations about $D^\infty(x)$ we may consider the lift of $D^\infty(x)$ to $\mathbb{R}^n$ included in the graph of a global C^1 function $\gamma : \mathbb{R}^u \to \mathbb{R}^s$ with $T_{(z,\gamma(z))} graph(\gamma) \in C^{cu}(z, \gamma(z))$. So, by the definition of dynamically flat submanifols in Section 6, $D^\infty(x)$ is contained in a dynamically flat submanifold and by Proposition 6.3 for almost all points in $D^\infty(x)$ all the Lyapunov exponents in the E^{cs} direction are negative. $\square$

For proving that the $cu-$Gibbs measures are really SRB, or the basin of them has positive volume, we repeat the same argument of [1, Proposition 6.4]:

Proof. Let μ be such a Gibbs ergodic measure. There exists some disk D^∞ such that almost every point in D^∞ is in the basin of μ. By absolute continuity of stable lamination of the points in $D^\infty \cap B(\mu)$ and the fact that these stable manifolds are contained in $B(\mu)$, we conclude that the basin of μ must have positive Lebesgue measure. $\square$

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