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Equilibrium States for Non-uniformly Expanding Maps

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Abstract

We construct equilibrium states, including measures of maximal entropy, for a large (open) class of non-uniformly expanding maps on compact manifolds. Moreover, we study uniqueness of these equilibrium states, as well as some of their ergodic properties.

1 Introduction

The theory of equilibrium states originates from statistical mechanics and was thoroughly developed, in the classical setting of uniformly hyperbolic dynamical systems, in the seventies and eighties, especially by Sinai, Ruelle, Bowen, Parry and Walters.

In general, given a continuous transformation $f : M \to M$ on a compact metric space, and given a continuous function ϕ, we call **equilibrium state** for (f, ϕ) a Borel probability measure μ_ϕ such that

$$h_{\mu_\phi}(f) + \int \phi d\mu_\phi = \sup_{\mu \in \mathcal{I}} \{h_\mu(f) + \int \phi d\mu\},$$

where the supremum is taken over the set $\mathcal{I}$ of f-invariant probabilities. That is, an equilibrium state is a maximum of the function $F_\phi : \mathcal{I} \to \mathbb{R}$ defined by

$$F_\phi(\mu) = h_\mu(f) + \int \phi d\mu.$$

It is now classical that for uniformly hyperbolic diffeomorphisms, as well as for uniformly expanding maps, equilibrium states always exist and they are unique if the potential is Hölder continuous, assuming that the transformation f is transitive. See [Pa64, Si72, Bow75, Rue69]. Moreover, the equilibrium states

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coincide with the Gibbs measures, that is, the invariant probability measures satisfying

$$\mu(\mathcal{B}_\epsilon(n,x)) \cong \exp\big(\sum_{i=0}^{n-1}\phi(f^i(x)) - nP\big) \tag{1}$$

for some $P \in \mathbb{R}$, called the *pressure* of ϕ, where $\mathcal{B}_\epsilon(n,x)$ is the dynamical ball of length n and size ϵ around x,

$$\mathcal{B}_\epsilon(n,x) = \{y \in M; d(f^i(y), f^i(x)) \le \epsilon, \text{ for every } 0 \le i \le n-1\},$$

and $\cong$ means equality up to a uniform factor, independent of x and n. In this setting, the pressure P is given by

$$P = \sup_{\mu\in\mathcal{I}} \big\{h_\mu(f) + \int \phi d\mu\big\}.$$

On the other hand, little is know concerning equilibrium states for dynamical systems outside the uniformly hyperbolic category. The goal of this work is to initiate such a theory for a robust (open) class of non-uniformly expanding maps. This class will be defined precisely in the next section. Here we just mention one of its main features:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\|Df(f^i(x))^{-1}\| \le -2c < 0 \tag{2}$$

for "most" points, including a full measure set relative to the equilibrium states μ_ϕ that we construct. We prove that equilibrium states do exist for every potential ϕ with small variation, that is, such that

$$\sup\phi - \inf\phi < K \tag{3}$$

for some convenient constant K. As a consequence of (2), these measures are (non-uniformly) expanding, that is, μ_ϕ-almost every point has only positive Lyapunov exponents. Moreover, μ_ϕ has a weak Gibbs property like (1) with $\cong$ meaning equality up to a factor with subexponential growth.

The basic strategy for the construction is to find a subset $\mathcal{K}$ of invariant probability measures which are expanding and such that μ-almost every point has infinitely many hyperbolic times, in the sense of [Al97, ABV00]. We prove that $\nu \mapsto F_\phi(\nu)$ is upper-semicontinuous on $\mathcal{K}$ and there exist maximum on $\mathcal{K}$ of F_ϕ. Using (3) we check that the maximum obtained on $\mathcal{K}$ is really a maximum of F_ϕ over all invariant probabilities.

These arguments apply, in particular, when the potential ϕ is constant, in which case μ_ϕ maximizes the entropy:

$$h_{\mu_\phi}(f) = h_{top}(f).$$

However, in this case we can go much further. Using a different approach, via semi-conjugation to a one-sided subshift of finite type, we are able to prove that the maximal entropy measure is unique and a Markov measure. If f is topologically mixing, this measure is Bernoulli.

Closing this introduction, we mention some questions that are naturally raised by our results. The first one is to prove the uniqueness, or at least the finiteness of the equilibrium states for Hölder potentials in the general situation of (3), under topological transitivity. The other one concerns the need of the hypothesis (3) itself. Ongoing work indicates that equilibrium states do exist also for potentials with large variation, but they may have both positive and negative Lyapunov exponents: from the viewpoint of these measures, the dynamics looks hyperbolic, rather than expanding.

2 Setting and Statements

We always consider a $C^{1+\alpha}$ local diffeomorphism $f : M^l \to M^l$, defined on a compact Riemannian manifold with dimension l. Let m be normalized Lebesgue measure on M. We suppose that f satisfies, for positive constants δ_0, β, δ_1, σ_1, and p, $q \in \mathbb{N}$,

(H1) There exists a covering $B_1, \ldots, B_p, \ldots, B_{p+q}$ of M such that every $f|B_i$ is injective and

- f is uniformly expanding at every $x \in B_1 \cup \cdots \cup B_p$:
$$\|Df(x)^{-1}\| \leq (1+\delta_1)^{-1}.$$
- f is never too contracting: $\|Df(x)^{-1}\| \leq (1+\delta_0)$ for every $x \in M$.

(H2) f is everywhere volume-expanding: $|\det Df(x)| \geq \sigma_1$ with $\sigma_1 > q$.

Define
$$V = \{x \in M; \|Df(x)^{-1}\| > (1+\delta_1)^{-1}\}.$$

(H3) There exists a set $W \subset B_{p+1} \cup \cdots \cup B_{p+q}$ containing V such that
$$M_1 > m_2 \quad \text{and} \quad m_2 - m_1 < \beta$$
where m_1 and m_2 are the infimum and the supremum of $|\det Df|$ on V, respectively, and M_1 and M_2 are the infimum and the supremum of $|\det Df|$ on W^c, respectively.

Definition 2.1. The supremum of F_ϕ over the set all invariant probability is called the *pressure of* ϕ and will be denoted by $P(\phi)$

Definition 2.2. Given $\phi : M \to \mathbb{R}$ continuous, we say that ϕ has *ρ-low variation* if
$$\max_{x \in M} \phi(x) < P(\phi) - \rho h_{top}(f).$$

Our first main result is

Theorem A. *Assume hypotheses (H1), (H2), (H3) hold, with δ_0 and β sufficiently small. Then, there exists ρ such that if ϕ is a continuous potential with ρ-low variation then ϕ has some equilibrium state. Moreover, these equilibrium states are hyperbolic measures, with all Lyapunov exponents bigger than some $c(\delta_1, \sigma_1, p, q) > 0$, and Jacobian bigger than 1.*

For maximal entropy mesures we are able to say a lot more, under the following additional hypothesis:

(H4) There exists a Markov partition $\mathcal{R} = \{R_1, \ldots, R_d\}$ for f such that

- $\mathcal{R}$ is transitive: for any i, j there exists a k such that $f^k(R_i) \cap R_j \neq \emptyset$; For simplicity in the proves we also assume that $W \subset R_1$.

Definition 2.3. The *dynamical ball* of length n and size ϵ around x is the set

$$\mathcal{B}_\epsilon(n, x) = \{y \in M; d(f^i(y), f^i(x)) \leq \epsilon, \text{ for every } 0 \leq i \leq n\}$$

We say that a system (f, μ) is Bernoulli (respectively Markov), if it is ergodically equivalent to a subshift of finite type endowed with a Bernoulli (respectively Markov) measure. See e.g. [Mañ87] for definitions.

Theorem B. *Assume hypotheses (H1), (H2), (H3), (H4) hold with δ_0 and β sufficiently small. Then there exists a unique invariant measure μ_{max} with $h_{\mu_{max}}(f) = h_{top}(f)$. This measure also satisfies,*

1. *all Lyapunov exponents of μ_{max} are larger than some $c(\delta_1, \sigma_1, p, q) > 0$;*
2. *μ_{max} a weak Gibbs measure for $\phi = 0$, in the sense that for μ-almost every $x \in M$ and any $n \geq 1$ there exist constants $K_n(x)$, such that*

$$K_n(x)^{-1} \leq \frac{\mu_{max}(B_\epsilon(n, x))}{e^{(-nh_{top}(f))}} \leq K_n(x)$$

   and $\lim_{n \to \infty} \frac{1}{n} \log K_n(x) = 0$.
3. *(f, μ_{max}) is a Markov system and, if f is topologically mixing, a Bernoulli system.*

Remark 2.4. Since the pressure $P(\phi)$ is continuous in C^0 topology, the low-variation is a open condition in the C^0 topology. This provide us robustness in the class of the low variation continuous potentials.

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3 Examples

In this section, we sketch the construction of a non-hyperbolic map f_0 that satisfies the conditions in theorems A and B above. It will be clear from this construction that these conditions hold, in fact, for every map f C^1-close to f_0.

We start by considering any Riemann manifold that supports an expanding map $g : M \to M$. For simplicity, choose $M = \mathbb{T}^n$ the n-dimensional torus, and g an endomorphism induced from a linear map with eigenvalues $\lambda_n > \cdots > \lambda_1 > 1$. Denote by $E_i(x)$ the eigenspace associated to the eigenvalue λ_i in $T_x M$.

Since g is an expanding map, g admits a transitive Markov partition $R_1, \ldots, R_d$ with arbitrary small diameter. We may suppose that $g|R_i$ is injective for every $i = 1, \ldots, d$. Replacing by a iterate if necessary, we may suppose that there exists a fixed point p_0 of g and, renumbering if necessary, this point is contained in the interior of the rectangle R_d of the Markov partition.

Considering a small neigbourhood $W \subset R_d$ of p_0 we deform g inside W along the direction E_1. This deformation consists essentially in rescaling the expansion along the invariant manifold associated to E_1 by a real function α. Let us be more precise:

Considering W small, we may identify W with a neighbourhood of 0 in $\mathbb{R}^n$ and p_0 with 0. Without loss of generality, suppose that $W = (-2\epsilon, 2\epsilon) \times B_{3r}(0)$, where $B_{3r}(0)$ is the ball or radius $3r$ and center 0 in $\mathbb{R}^{n-1}$. Consider a function $\alpha : (-2\epsilon, 2\epsilon) \to \mathbb{R}$ such $\alpha(x) = \lambda_1 x$ for every $|x| \geq \epsilon$ and for small constants γ_1, γ_2:

1. $(1 + \gamma_1)^{-1} < \alpha'(x) < \lambda_1 + \gamma_2$
2. $\alpha'(x) < 1$ for every $x \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$;
3. α is C^0-close to λ_1: $\sup\limits_{x \in (-\epsilon, \epsilon)} |\alpha(x) - \lambda_1 x| < \gamma_2$,

Also, we consider a bump function $\theta : B_{3r}(0) \to \mathbb{R}$ such $\theta(x) = 0$ for every $2r \leq |x| \leq 3r$ and $\theta(x) = 1$ for every $0 \leq |x| \leq r$. Suppose that $\|\theta'(x)\| \leq C$ for every $x \in B_{3r}(0)$. Considering coordinates $(x_1, \ldots, x_n)$ such that $\partial_{x_i} \in E_i$, define f_0 by:

$$f_0(x_1, \ldots, x_n) = (\lambda_1 x_1 + \theta(x_2, \ldots, x_n)(\alpha(x_1) - \lambda_1 x_1), \lambda_2 x_2, \ldots, \lambda_n x_n)$$

Observe that by the definition of θ and α we can extend f_0 smoothly to $\mathbb{T}^n$ as $f_0 = g$ outside W. Now, is not difficult to prove that f_0 satisfies the conditions (H1), (H2), (H3) and (H4) above.

First, we have that $\|Df_0(x)^{-1}\|^{-1} \geq \min\limits_{i=1,\ldots,n} \|\partial_{x_i} f_0\|$. Observe that:

$$\partial_{x_1} f_0(x_1, \ldots, x_n) = (\alpha'(x_1)\theta(x_2, \ldots, x_n) + (1 - \theta(x_2, \ldots, x_n))\lambda_1, 0, \ldots, 0)$$

$$\partial_{x_i} f_0(x_1, \ldots, x_n) = ((\alpha(x_1) - \lambda_1)\partial_{x_i}\theta(x_2, \ldots, x_n), 0, \ldots, \lambda_i, 0, \ldots, 0), \text{ for } i \geq 2.$$

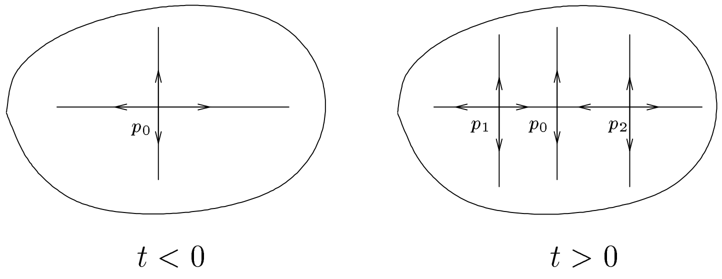

Figure 1: Pitchfork bifurcation

Then, since $\|\partial_{x_i}\theta(x)\| \leq C$ for every $x \in B_{3r}(0)$, and $\alpha(x_1) - \lambda_1 x_1 \leq \gamma_2$ we have that $\|\partial_{x_i} f_0\| > (\lambda_i - \gamma_2 C)$ for every $i = 2, \ldots, n$. Moreover, by condition 1, $\|\partial_{x_1} f_0\| \leq \max\{\alpha'(x_1), \lambda_1\} \leq \lambda_1 + \gamma_2$, if we choose γ_2 small in such way that $\lambda_2 - \gamma_2 C > \lambda_1 + \gamma_2$ then:

$$\|\partial_{x_i} f_0\| > \|\partial_{x_1} f_0\|, \text{ for every } i \geq 2.$$

Notice also that $\|\partial_{x_1} f_0\| \geq \min\{\alpha'(x_1), \lambda_1\} \geq (1+\gamma_1)^{-1}$. This prove that:

$$\|Df_0(x)^{-1}\|^{-1} \geq \min_{i=1,\ldots,n} \|\partial_{x_i} f_0\| \; (1+\gamma_1)^{-1}.$$

Since f coincides with g outside W, we have $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1}$ for every $x \in W^c$. Together with the above inequality, this proves condition (H1), with $\delta_0 = \gamma_1$.

Choosing γ_1 small and $p = d-1$, $q = 1$, $B_i = R_i$ for every $i = 1, \ldots, d$, condition (H2) is imediate. Indeed, observe that the Jacobian of f_0 is given by the formula:

$$\det Df_0(x) = (\alpha'(x_1)\theta(x_2, \ldots, x_n) + (1 - \theta(x_2, \ldots, x_n))\lambda_1) \prod_{i=2}^{n} \lambda_i.$$

Then, if we choose $\gamma_1 < \prod_{i=2}^{n} \lambda_i - 1$:

$$\det Df_0(x) > (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1.$$

Therefore, we may take $\sigma_1 = (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1$.

To verify property (H3) for f_0, observe that if we denote by

$$V = \{x \in M; \|Df_0(x)^{-1}\| > (1+\delta_1)^{-1}\},$$

with $\delta_1 < \lambda_1 - 1$, then $V \subset W$. Indeed, since $\alpha(x_1)$ is constant equal to $\lambda_1 x_1$ outside W we have that $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1} < (1+\delta_1)^{-1}$, for every $x \in W^c$.

Given γ_3 close to 0, we may choose δ_1 close to 0 and α satisfying the conditions above in such way that,

$$\sup_{x,y\in V} \alpha'(x_1) - \alpha'(y_1) < \gamma_3.$$

If m_1 and m_2 are the infimum and the supremum of $|\det Df_0|$ on V, respectively,

$$m_2 - m_1 \leq C(\sup_{x,y\in V} \alpha'(x_1) - \alpha'(y_1)) < \gamma_3 C,$$

where $C = \prod_{i=2}^{n} \lambda_i$. Then, we may take $\beta = \gamma_3 C$ in (H3). If M_1 is the infimum of $|\det Df_0|$ on W^c, $M_1 > m_2$, since $\lambda_1 > (1+\delta_1) \geq \sup_{x\in V} \alpha'(x)$. Condition (H4) is clear from the construction, since $f_0 = g$ outside $W \subset R_d$, so the Markov property of $\{R_1, \ldots, R_d\}$ is not affected by the pertubation.

The arguments above show that the hypotheses $(H1), (H2), (H3)$ and $(H4)$ are satisfied by f_0. Moreover, if we one takes $\alpha(0) = 0$, then p_0 is fixed point for f_0, which is not a reppeler, since $\alpha'(0) < 1$. Therefore, f_0 is not a uniformly expanding map.

It is not difficult to see that this construction may be carried out in such way that f_0 does not satisfy the expansiveness property: there is a fixed hyperbolic saddle point p_0 such that the stable manifold of p_0 is contained in the unstable manifold of two other fixed points (see figure 1).

For a discussion of related examples see [Ca93] and [BV00].

4 Expanding measures and hyperbolic times

The proof of Theorem A occupies this section and the next one. Let us begin by detailing a bit more our strategy to prove the existence of equilibrium states:

- To exhibit a subset $\mathcal{K}$ of invariant measures such that all their Lyapunov exponents are positive, and almost every point has infinitely many hyperbolic times.
- To show that there exists a common generating partition for all the measures in $\mathcal{K}$. This allows us to prove that the function $\mu \to h_\mu(f) + \int \phi d\mu$ is upper-semicontinuous on $\mathcal{K}$. Using this, we get that the maximum of $h_\mu(f) + \int \phi d\mu$ for measures μ in $\mathcal{K}$ is attained.
- To prove that if the potential has low variation, the maximum obtained over $\mathcal{K}$ is, in fact, a global maximum for the function $h_\mu(f) + \int \phi d\mu$ over all invariant measures.

Let us begin by stating our precise conditions on the constants δ_0 and β in the theorem. According to [ABV00, Appendix], if f satisfies (H1) then there exists $\gamma_0 < 1$ depending only on (σ_1, p, q) such that Lebesgue almost every point spends at most a fraction γ_0 of time inside $B_{p+1} \cup \cdots \cup B_{p+q}$.

Reducing δ_0 if necessary, we may find constants $\alpha > 0$, as close to 1 as we want, and $c > 0$ such that

$$(1+\delta_0)^\alpha (1+\delta_1)^{-(1-\alpha)} < e^{-2c} < 1. \tag{4}$$

We take $\alpha > \gamma_0$. By hypothesis (H3), $m_2 < M_1$ and $m_2 - m_1 < \beta$. So, taking β and δ_0 small, and α close enough to 1, we ensure that

$$\alpha m_2 + (1-\alpha) M_2 < \gamma_0 m_1 + (1-\gamma_0) M_1 - l \log(1+\delta_0). \tag{5}$$

Preparing the definition of $\mathcal{K}$, we introduce the convex set $K_\alpha \subset \mathcal{I}$ given by

$$K_\alpha = \{\mu \in \mathcal{I}; \mu(V) \leq \alpha\}.$$

Since Lebesgue almost every point spends at most a fraction $\gamma_0 < \alpha$ of time inside $V \subset B_{p+1} \cup \cdots \cup B_{p+q}$, all the ergodic absolutely continuous invariant measures constructed in [ABV00] belong to K_α. In particular, K_α is non-empty. We will see that K_α contains the equilibrium states of potentials with low variation.

Let us recall the ergodic decomposition theorem, as it is proven in [Mañ87]:

Theorem 4.1. *Given any invariant measure, there are ergodic invariant measures $\{\mu_x : x \in M\}$ depending measurably on the point x and such that*

$$\int f d\mu = \int (\int f d\mu_x) d\mu \quad \textit{for every } f \in L^1(d\mu).$$

Moreover, this decomposition $\{\mu_x : x \in M\}$ is essentially unique and, in fact, $\mu_x = \lim n^{-1} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ at μ-almost every point.

Our distinguished set of invariant measures is the non-empty set $\mathcal{K} \subset K_\alpha$ defined by

$$\mathcal{K} = \{\mu \in \mathcal{I}; \mu_x \in K_\alpha \text{ for } \mu\text{-a.e. } x\}$$

Definition 4.2. We say that a measure ν is *f-expanding with exponent c* if for ν-almost every $x \in M$ we have:

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -2c < 0.$$

Remark 4.3. If ν is an ergodic invariant measure, by the ergodic theorem the above definition is equivalent to $\int \log |Df^{-1}| d\nu \leq -2c$.

Remark 4.4. Given any ergodic invariant measure ν, the Lyapunov exponents of (f, ν) are all positive if and only if ν is f^N-expanding for some iterate N. See [ABV00]. Also, if every invariant measure ν is f-expanding, then f is a uniformly expanding map. See [AAS01].

In general, N depends on ν. So the following question remains open:

Question 1. If f has positive Lyapunov exponents for every invariant measure, is f necessarily a uniformly expanding map?

The next statement proves that all measures in $\mathcal{K}$ are f-expanding, with uniform exponent. Let $c > 0$ be as in (4).

Lemma 4.5. *Every measure in $\mu \in \mathcal{K}$ is f-expanding with exponent c:*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le -2c < 0 \quad \textit{for } \mu\textit{-almost every } x \in M.$$

Proof. Suppose that $\mu \in \mathcal{K}$ is ergodic. Since $\mu \in K_\alpha$ and $\mu(V) \le \alpha$, by the ergodic theorem almost every point spends a fraction less than α inside V: there exist an invariant set A with $\mu(A) = 1$ such that for every $x \in A$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_V(f^i(x)) \le \alpha.$$

By hypothesis (H1), we have that $\|Df(y)^{-1}\| \le (1+\delta_0)$ for every $y \in V$. Moreover, $\|Df(y)^{-1}\| \le (1+\delta_1)^{-1}$ for $y \in V^c$. This implies that

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| &\le \frac{1}{n} \log (1+\delta_0)^{\alpha n} (1+\delta_1)^{(1-\alpha)n} \\ &\le \log (1+\delta_0)^{\alpha} (1+\delta_1)^{(1-\alpha)} < -2c < 0 \end{aligned}$$

for all $x \in A$.

To finish , let H be the set of x that satisfy the condition in the conclusion of the lemma. Since every $\mu \in \mathcal{K}$ is $\mu = \{\mu_x\}$ a convex combination of ergodic measures μ_x in K_α, by the previous case, $\mu_a(H) = 1$ for μ almost every a, and this implies that $\mu(H) = \int \mu_x(H) d\mu = 1$.

□

We need the key notion of hyperbolic time, first introduced by Alves [Al97].

Definition 4.6. We say that n is a *hyperbolic time* for x with exponent c, if for every $0 \le j \le n$:

$$\prod_{k=0}^{j-1} \|Df(f^{n-k}(x))^{-1}\| \le e^{-cj}.$$

To prove that for an f-expanding measure almost every point admits infinitely many hyperbolic times, we need the following lemma due to Pliss. See for instance [ABV00] for a proof.

Lemma 4.7. *Given $A \geq c_2 > c_1 > 0$, let $d_0 = \frac{c_2 - c_1}{A - c_1}$. If $a_1, \ldots, a_n$ are real numbers such that $a_i \leq A$ and*

$$\sum_{i=1}^{n} a_i \geq c_2 n$$

then there are integer numbers $l > d_0 n$ and $1 < n_1 < \cdots < n_l \leq n$ so that, for every $0 \leq k \leq n_i$ and $i = 1, \ldots, l$:

$$\sum_{j=k+1}^{n_i} a_j \geq c_1 (n_i - n)$$

Using this lemma, we get

Lemma 4.8. *For every invariant measure ν which is f-expanding with exponent c, there exists a full ν-measure set $H \subset M$ such that every $x \in H$ has infinitely many hyperbolic times $n_i = n_i(x)$ with exponent c and, in fact, the density of hyperbolic times at infinity is larger than some $d_0 = d_0(c) > 0$:*

1. $\displaystyle\prod_{k=0}^{j-1} \|Df^{-1}(f^{n_i - k}(x))\| \leq e^{-cj}$ *for every* $0 \leq j \leq n_i$
2. $\displaystyle\liminf_{n \to \infty} \frac{\sharp\{0 \leq n_i \leq n\}}{n} \geq d_0 > 0.$

Proof. By the definition of f-expanding measures, there exists a set H with $\nu(H) = 1$ such that given any $x \in H$ we have

$$\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| \leq -\frac{3c}{2} n$$

for every n large enough. Then, it suffices to take $A = \sup\limits_{x \in M} -\log \|Df(x)^{-1}\|$, $c_1 = c$, $c_2 = \frac{3c}{2}$ and $a_i = -\log \|Df(f^{i-1}(x))^{-1}\|$ in the previous lemma. □

The next lemma asserts that points at hyperbolic times have unstable manifolds with size uniformly bounded from below.

Lemma 4.9. *There exists $\epsilon_0 > 0$ such that for every x and n_i a hyperbolic time of x, if $z \in M$ satisfies $f^{n_i}(z) \in B_{\epsilon_0}(f^{n_i}(x))$ then*

$$d(f^{n_i - j}(x), f^{n_i - j}(z)) \leq e^{\frac{-c}{2} j} d(f^{n_i}(z), f^{n_i}(x)) \quad \textit{for every } 0 \leq j \leq n_i(x).$$

Proof. Since Df is uniformly continuous and a local diffeomorphism, there exists ϵ_0 such that for every $\xi, \eta \in M$ with $\xi \in B_{\epsilon_0}(\eta)$ then

$$\frac{\|Df(\xi)^{-1}\|}{\|Df(\eta)^{-1}\|} \leq e^{\frac{c}{2}}.$$

By definition, if n_i is hyperbolic time for x then $\prod_{k=0}^{j-1}\|Df(f^{n_i-k}(x))^{-1}\| \le e^{-cj}$ for every $0 \le j \le n_i$. Observe that $d(f^{n_i-1}(z), f^{n_i-1}(x)) \le \epsilon_0$. This is because $f^{n_i}(z) \in B_{\epsilon_0}(f^{n_i}(x))$ and, by the previous observation, the norm of the derivarive of the inverse branch of f that sends $f^{n_i}(x)$ to $f^{n_i-1}(x)$ is less than $e^{-\frac{c}{2}}$ restricted to $B_{\epsilon_0}(f^{n_i}(x))$. Arguing by induction,

$$\prod_{k=0}^{j-1}\|Df(f^{n_i-k}(z))^{-1}\| \le e^{\frac{-c}{2}j} \quad \text{for all} \quad 0 \le j \le n_i.$$

We conclude that $d(f^{n_i-j}(x), f^{n_i-j}(z)) \le e^{\frac{-c}{2}j} d(f^{n_i}(x), f^{n_i}(z))$, proving the lemma. □

Since c is fixed, we will write simply ν-expanding to mean ν-expanding with exponent c.

5 Existence of equilibrium states for low variation potentials

Our main aim in this section is to establish the existence of equilibrium states for low variation potentials, in order to prove theorem A. In particular, this applies to $\phi = 0$, which always has low variation. Thus, our construction also yields maximal entropy measures for these transformations.

Beforehand, we use some results of the previous section to establish expansiveness for measures in $\mathcal{K}$.

Definition 5.1. Given $\epsilon > 0$ we define the set $A_\epsilon(x)$ by:

$$A_\epsilon(x) = \{y \in M; d(f^n(x), f^n(y)) \le \epsilon \text{ for every } n \ge 0\}.$$

By definition, f is an **expansive map** with expansiveness constant $\tilde{\epsilon}$ if and only if $A_\epsilon(x) = \{x\}$ for every $x \in M$ and $\epsilon < \tilde{\epsilon}$.

Lemma 5.2. *Suppose that $\mu \in \mathcal{K}$ is ergodic and let ϵ_0 be as constructed in lemma 4.9. Then for μ-almost every $x \in M$ and any $\epsilon < \epsilon_0$,*

$$A_\epsilon(x) = \{x\}.$$

Proof. By corollary 4.8 we have that almost every $x \in M$ has infinitely many hyperbolic times $n_i(x)$. By lemma 4.9, if $z \in A_\epsilon(x)$ with $\epsilon < \epsilon_0$ then for any n_i we have

$$d(x, z) \le e^{-\frac{c}{2}n_i} d(f^{n_i}(x), f^{n_i}(z)) \le e^{-\frac{c}{2}n_i}\epsilon.$$

Making $n_i \to \infty$ we deduce that $x = z$. □

Let $\mathcal{P} = \{P_1, \ldots, P_d\}$ be any partition of M in measurable sets with diameter less than ϵ_0. From the above lemma we get

Corollary 5.3. *$\mathcal{P}$ is a generating partition for every $\mu \in \mathcal{K}$.*

Proof. Define

$$\mathcal{P}^{(n)} = \{C^{(n)} = P_{i_0} \cap \cdots \cap f^{-n+1}(P_{i_{n-1}})\}, \quad \text{for each } n \geq 1.$$

We need to prove that given any measurable set A and given $\delta > 0$ there exist elements $C_1^{(n)}, \ldots, C_m^{(n)}$ of $\mathcal{P}(n)$ such that

$$\mu(\bigcup C_i^{(n)} \Delta A) \leq \delta.$$

Consider $K_1 \subset A$ and $K_2 \subset A^c$ compact sets such that $\mu(K_1 \Delta A) \leq \delta$ and $K_2 \Delta A^c \leq \delta$. Let $r = d(K_1, K_2) > 0$. Lemma 5.2 gives that if n is big enough then diam$\mathcal{P}^{(n)}(x) \leq \frac{r}{2}$, for x in a set with μ-measure bigger than $1-\delta$. Consider the sets $C_1^{(n)}, \ldots, C_m^{(n)}$ that intersect K_1. Then

$$\begin{aligned} \mu(\bigcup C_i^{(n)} \Delta A) &= \mu(\bigcup C_i^{(n)} - A) + \mu(A - \bigcup C_i^{(n)}) \\ &\leq \mu(A - K_1) + \mu(A^c - K_2) + \delta \leq 3\delta. \end{aligned}$$

This proves the claim. □

Remark 5.4. Recalling the definition of entropy with respect to a partition $\mathcal{Q}$,

$$H_\mu(\mathcal{Q}) = \sum_{Q \in \mathcal{Q}} -\mu(Q) \log \mu(Q),$$

we have that for any partition $\mathcal{Q}$ such $\mu_0(\partial Q) = 0$, for each $Q \in \mathcal{Q}$, then the function $\mu \to H_\mu(\mathcal{Q})$ is continuous in μ_0. This imply that

$$\mu \mapsto h_\mu(f, \mathcal{P}) = \inf_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{P}^{(n)}).$$

is upper-semicontinuous in μ_0.

Observe that, by corollary 5.3, $\mathcal{P}$ is a generating partition for every $\mu \in \mathcal{K}$. So, as a consequence of Kolmogorov-Sinai's theorem(see e.g. [Mañ87]),

Corollary 5.5. *For every measure $\mu \in \mathcal{K}$ we have $h_\mu(f) = h_\mu(f, \mathcal{P})$.*

The next lemma will allow us prove that if ϕ has ρ-low variation, the function F_ϕ restrict to $\mathcal{K}$ has a maximum μ_ϕ and this maximum is in fact an equilibrium state for ϕ.

Lemma 5.6. *All ergodic measures η outside $\mathcal{K}$ have small entropy: there exists $\rho < 1$ such that if $\eta \in \mathcal{K}^c$ is ergodic then*

$$h_\eta(f) \leq \rho h_{top}(f).$$

Proof. As we are supposing that η is ergodic, we have that $\eta(V) > \alpha$, since every ergodic measure μ such $\mu(V) \leq \alpha$ lies in $\mathcal{K}$.

If we denote by $\lambda_1(x) \geq \lambda_2(x) \geq \dots \lambda_s \geq 0 > \lambda_{s+1} \cdots \geq \lambda_l(x)$ the Lyapunov exponents in x, we know that $\lambda_i = \lambda_i(x)$ are constant in η almost everywhere. By the theorem of Oseledets [Ose68],

$$\int \log|\det Df| d\eta = \sum_{i=1}^{l} \lambda_i. \tag{6}$$

On the other hand, we have that $\lambda_l > -\log(1+\delta_0)$, since by hyphotesis, $\|Df(x)^{-1} \leq (1+\delta_0)\|$. By Ruelle's inequality (see [Rue78]) we have that

$$h_\eta(f) \leq \sum_{i=1}^{s} \lambda_i = \int \log|\det Df(x)| d\eta - \sum_{i=s+1}^{l} \lambda_i. \tag{7}$$

Since $m_2 = \sup_{x \in V} \log|\det Df(x)| < M_2 = \sup_{x \in V^c} \log|\det Df(x)|$ and $\eta(V) > \alpha$ we have:

$$h_\eta(f) \leq \int \log|\det Df| d\eta \leq \eta(V)m_2 + (1-\eta(V))M_2 + (l-s)\log(1+\delta_0) \leq \alpha m_2 + (1-\alpha)M_2 + l\log(1+\delta_0)$$

Let μ_0 be any ergodic absolutely continuous invariant measure as constructed in [ABV00]. Since Lebesgue almost every point spends at most a fraction γ_0 of time inside $W \subset B_{p+1} \cup \cdots \cup B_{p+q}$, we have that $\mu_0(W) < \gamma_0$. As f is $C^{1+\alpha}$ and μ_0 is absolutely continuous, we may use Pesin's entropy formula (see [Pes77]):

$$h_{\mu_0}(f) = \int \log|\det Df| d\mu_0 \geq \mu_0(W)m_1 + (1-\mu_0(W))M_1$$

As $\mu_0(W) \leq \gamma_0$ and $m_1 < M_1$, we conclude that

$$\gamma_0 m_1 + (1-\gamma_0)M_1 \leq h_{\mu_0}(f).$$

On the other hand, using $\eta(V) > \alpha$ and $m_2 < M_2$, we have
By condition (5),

$$\alpha m_2 + (1-\alpha)M_2 < \gamma_0 m_1 + (1-\gamma_0)M_1 - l\log(1+\delta_0).$$

So, we can choose $\rho < 1$ such that

$$h_\eta(f) \leq \alpha m_2 + (1-\alpha)M_2 + l\log(1+\delta_0) < \rho(\gamma_0 m_1 + (1-\gamma_0)M_1) < \rho h_{\mu_0}(f) \leq \rho h_{top}(f).$$

This proves lemma 5.6. $\square$

Observe that follows from the lemma 5.6 and the variational principle:

Corollary 5.7 (Variational Principle for expanding measures). *If ϕ is a ρ-low variation potential, then:*

$$\sup_{\nu\in\mathcal{K}} h_\nu(f) + \int \phi d\nu = P(\phi)$$

In particular,

$$\sup_{\nu\in\mathcal{K}} h_\nu(f) = h_{top}(f)$$

Proof. Denote by E the set of all ergodic invariant probabilities, to prove the lemma, we just need to prove that:

$$\sup_{\nu\in\mathcal{K}} F_\phi(\nu) = \sup_{\nu\in E} F_\phi(\nu) \tag{8}$$

Since

$$P(\phi) = \sup_{\nu\in E} F_\phi(\nu)$$

To prove 8, note that by lemma 5.6 we have that for $\nu \in \mathcal{K}^c$ ergodic, then $h_\nu(f) \le \rho h_{top}(f)$. This imply that:

$$F_\phi(\nu) = h_\nu(f) + \int \phi d\nu \le \rho h_{top}(f) + \max_{x\in M} \phi(x) d\nu < P(\phi)$$

and this prove the corollary.

□

6 Proof of theorem A

First of all, observe that the previous corollary asserts that $\sup_{\nu\in\mathcal{K}} F_\phi(\nu) = P(\phi)$. To prove that there exists some equilibrium states, consider a sequence of measures μ_k such $F_\phi(\mu_k)$ converge to $P(\phi)$.

Without loss of generality, we suppose that $\mu_k \to \mu$ weakly. We prove that μ is an equilibrium state for ϕ and belongs to $\mathcal{K}$.

First, we claim that $F_\phi(\mu) = P(\phi)$. In fact, fix $\mathcal{P}$ a partition with diameter less than ϵ_0, such $\mu(\partial P) = 0$ for any $P \in \mathcal{P}$. Observe that, since $\mu_k \in \mathcal{K}$, we have that $h_{\mu_k}(f) = h_{\mu_k}(f, \mathcal{P})$, by corollary 5.5. Then:

$$P(\phi) = \sup_{\nu\in\mathcal{K}} F_\phi(\nu) = \limsup F_\phi(\mu_k) = \limsup h_{\mu_k}(f) + \int \phi d\mu_k$$

As, by remark 5.4, the function $\nu \to h_\nu(f, \mathcal{P})$ is upper-semicontinuous in μ, thus:

$$\limsup h_{\mu_k}(f,\mathcal{P}) + \int \phi d\mu_k \leq h_\mu(f,\mathcal{P}) + \int \phi d\mu \leq h_\mu(f) + \int \phi d\mu = F_\phi(\mu)$$

Then, we have

$$P(\phi) = \sup_{\nu\in\mathcal{K}} F_\phi(\nu) \leq F_\phi(\mu) \leq P(\phi),$$

which imply that μ is equilibrium state for ϕ.

Now, we prove that any measure η such $F_\phi(\eta) = \sup_{\nu\in\mathcal{K}} F_\phi(\nu)$ belongs to $\mathcal{K}$, proving that all equilibrium states belong to $\mathcal{K}$.

In fact, if $\eta = \{\eta_x\}$ is the ergodic decomposition of η, we should prove that the set $S = \{x \in M; \eta_x \in K_\alpha\}$ is a η-full measure set.

Using the fact that

$$h_\eta(f) = \int h_{\eta_x}(f)\, d\eta(x),$$

see [Ro67], we have $F_\phi(\eta) = \int F_\phi(\eta_x)\, d\eta(x)$

Suppose by contradiction that $\eta(S^c) > 0$. Observe that if $y \in S^c$, then η_y is in the hyphoteses of lemma 5.6 and thus:

$$F_\phi(\eta_y) = h_{\eta_y} + \int \phi\, d\eta_y < \rho h_{top}(f) + \max_{x\in M} \phi(x) d\mu_y < P(\phi), \tag{9}$$

since ϕ is a ρ-low variation potential.

As for every $x \in S$ we have that $F_\phi(\mu_x) \leq P(\phi)$, the inequality 9 implies that $F_\phi(\eta) = \int F_\phi(\eta_x)\, d\eta(x) < P(\phi)$, which is a contradiction. Then, $\eta(S) = 1$ which imply, by the definition of $\mathcal{K}$ hat $\eta \in \mathcal{K}$ and that all equilibrium states are in $\mathcal{K}$, completing he prove of theorem A.

7 Measures of Maximal Entropy

In the next two sections, we give another proof of the existence of a measure with maximum entropy. Moreover, we prove that this measure is unique, if f is topologically transitive and we derive the other properties in the statement of theorem B. Throughout, we assume the additional hypothesis (H4): existence of a transitive Markov partition. Observe that we do not require this partition to be generating.

Our strategy is to transfer our problem to a subshift of finite type $\sigma^+ : \Sigma^+ \to \Sigma^+$, using the semiconjugacy between f and σ^+ provided by hypothesis (H4). This transference is not complete, in the sense that some measures can not be transported to the shift. But we will show that any f-invariant measure with big entropy is ergodically equivalent to some measure in the shift. Then we use the classical fact that transitive subshifts of finite type have exactly one measure of maximal entropy.

If we denote $\partial R = \bigcup_{n=1}^{d} \partial R_i$ and $\tilde{M} = M - \bigcup_{n\geq 0} f^{-n}(\partial R)$, we can define a map $\Pi : \tilde{M} \to \Sigma^+$ over a subshift of finite type Σ^+ associate to some transition matrix A, by

$$\Pi(x) = (i_0, \ldots, i_n, \ldots) \text{ such that } f^n(x) \in P_{i_n}\}.$$

Observe that this map is possibly only a semiconjugacy between f and σ^+, where σ^+ is one-sided shift over Σ^+. We denote $\tilde{\Sigma}^+ = \Pi(\tilde{M})$. Using this semiconjugation and the invariance of $\partial\mathcal{R}$, we will prove that any ergodic measure with big entropy gives zero measure to $\partial\mathcal{R}$.

To start our construction we introduce the Ruelle-Perron-Frobenius transfer operator $\mathcal{L}_\phi : C^0(M) \to C^0(M)$ associated to a potential ϕ, defined by

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$

Observe that this operator is bounded, since ϕ is a continuous function and M is compact. In the case $\phi = 0$ we denote $\mathcal{L}_\phi$ by $\mathcal{L}$. If $\mathcal{M}$ is the space of finite measures over M, the dual operator $\mathcal{L}^\star : \mathcal{M} \to \mathcal{M}$ is defined by the formula

$$\int g d\mathcal{L}^\star \mu = \int \mathcal{L} g d\mu.$$

Since f is a local diffeomorphism, and M is connected, the number $p > 1$ of preimages of each point is constant. Thus, we have imediatelly that $\mathcal{L}1 = \sum_{f(y)=x} 1 = p$.

Lemma 7.1. *The spectral radii of $\mathcal{L}$ and $\mathcal{L}^\star$ are equal to p. Moreover, there exists a eigenmeasure ν of $\mathcal{L}^\star$ associated to p.*

Proof. Denote by r the spectral radius of $\mathcal{L}$. To prove that $\mathcal{L}^\star$ has some engein-measure associate to p, we consider $G : \mathcal{M}_1 \to \mathcal{M}_1$ over the set of probabilities measures $\mathcal{M}_1$ defined by:

$$G(\mu) = \frac{\mathcal{L}^\star \mu}{\int \mathcal{L}1 d\mu}.$$

Since G is continuous and $\mathcal{M}_1$ is a compact and convex set, by the Tychonoff-Schauder fixed point theorem, there exists a probability measure ν such that $G(\nu) = \nu$, that is, $\mathcal{L}^\star \nu = (\int \mathcal{L}1 d\nu)\nu = p\nu$.

Now, we prove that the spectral radius of $\mathcal{L}$ r is equal to p. In fact, we have $r \geq p$, since ν is a eigenmeasure associate to p. For the converse inequality, observe that $\|\mathcal{L}g\| = \|\sum_{f(y)=x} g(y)\| \leq p\|g\|$, which implies that $\|\mathcal{L}\| \leq p$. Since $r = \lim_{n\to\infty} \|\mathcal{L}^n\|^{\frac{1}{n}}$ and $\|\mathcal{L}^n\| \leq \|\mathcal{L}\|^n$, we have that $r \leq p$. ☐

Lemma 7.2. *Let ν be any Borel probability such that $\mathcal{L}_\phi^\star \nu = \lambda\nu$ and h a ν-integrable function such $\mathcal{L}_\phi h = \lambda h$. Then, the measure $\mu = h\nu$ is f-invariant.*

Proof. Let u be any continuous function on M. Observe that:

$$\int u \circ f d\mu = \int u \circ f h d\nu = \lambda^{-1} \int \mathcal{L}_\phi(h.u \circ f) d\nu = \lambda^{-1} \int \mathcal{L}_\phi h u d\nu = \int h u d\nu = \int u d\mu$$

Thus, μ is f invariant.

□

Observe that $\mathcal{L}1 = p$, since every point admits exactly p preimages. As direct consequence of the last lemma :

Corollary 7.3. *Any probability such $\mathcal{L}\nu = p\nu$ is f-invariant probability.*

Definition 7.4. The *Jacobian* of a measure μ with respect to f is the (essentially unique) function $J_\mu f$ satisfying

$$\mu(f(A)) = \int_A J_\mu f d\mu.$$

for any measurable set A such that $f|_A$ is injective.

In other words, $J_\mu f$ is defined by the formula $J_\mu f = \frac{d\mu \circ f}{d\mu}$. The Jacobian of a measure does not necessarily exist. But, if f is countable-to-one then $J_\mu f$ does exist for every f-invariant measure.

Lemma 7.5. *Let ν be any Borel probability such that $\mathcal{L}_\phi^\star \nu = p\nu$. Then, the Jacobian of ν is $J_\nu f = pe^{-\phi}$.*

Proof. Let A be any measurable set such that $f|_A$ is injective. Take a sequence $\{g_n\} \in C^0(M)$ such that $g_n(x) \to \chi_A$ for ν-a.e. and $\|g_n\|_{C^0} \leq 2$. Then,

$$\mathcal{L}_\phi(e^{-\phi} g_n)(x) = \sum_{f(y)=x} e^{\phi(y)}.e^{-\phi(y)} g_n(y) = \sum_{f(y)=x} g_n(y).$$

Observe that the last expression converges to $\chi_{f(A)}(x)$ almost everywhere ν. Then, by the dominated convergence theorem,

$$\int pe^{-\phi} g_n d\nu = \int \mathcal{L}_\phi(e^{-\phi} g_n) d_\nu \to \nu(f(A)).$$

Then,

$$\nu(f(A)) = \int_A pe^{-\phi} d\nu,$$

which proves the lemma. □

In what follows we consider $\phi = 0$ and let ν be any Borel probability, satisfying $\mathcal{L}^\star \nu = p\nu$. The first observation is that ν has a constant Jacobian $J_\nu f = p > 1$. As a consequence,

Proposition 7.6. *ν-almost every point spends a definite fraction of the time outside V: there exists $0 < \gamma = \gamma(p) < 1$ such that*

$$\nu(\{x \in M; \limsup \frac{\sharp\{0 \leq i \leq n; f^i(x) \in V\}}{n} < \gamma\}) = 1.$$

Proof. Consider the Markov partition $R_1, \ldots, R_d$. By hypothesis, f is one-to-one in each R_i and V is contained in R_1. We are going to prove a bit more than we stated: almost every point spends a definite fraction of the time outside R_1.

Denote by $[i] = [i_0, \ldots, i_n] = \{x \in M; 0 \leq j \leq n, f^j(x) \in R_{i_j}\}$. We claim that $\nu([i]) \leq Kp^{-n}$ where C is a constant that depends only on $R_1, \ldots, R_d$. In fact, since $f^j([i]) \subset R_{i_j}$, we have that f is one-to-one on $f^j([i])$. Therefore, by the definition of Jacobian,

$$\nu(f^{j+1}([i])) = \int_{f^j([i])} J_\nu f \, d\nu \geq p\nu(f^j([i])).$$

It follows, by induction, that

$$\nu([i]) \leq p^{-n}\nu(R_{i_0}).$$

So, choosing $C = \max\{\nu(R_1), ..., \nu(R_d)\}$, we establish our claim.

Let us now use this claim to prove proposition 7.6. Let

$$A_\gamma(n) = \{x \in M; \frac{\sharp\{0 \leq i \leq n; f^i(x) \in V\}}{n} \geq \gamma\}.$$

and

$$A_\gamma = \{x \in M; \limsup \frac{\sharp\{0 \leq i \leq n; f^i(x) \in V\}}{n} \geq \gamma\}$$

We will prove that there exists $\gamma(p)$ and $0 < \eta = \eta(p) < 1$ such that, for some constant $K = K(\eta)$:

$$\nu(A_\gamma(n)) \leq K\eta^n. \tag{10}$$

Notice that the proposition is a direct consequence of 10. In fact, observe that :

$$A_\gamma = \bigcap_{i \geq 1} \bigcup_{n \geq i} A_\gamma(n)$$

and assuming 10, A has zero measure by the Borel-Cantelli lemma. So, we only have to prove (10).

Define $I_n = \{[i]; \sharp\{0 \leq j \leq n; i_j = 1\} \geq \gamma n\}$. Then

$$A_\gamma(n) \subset \bigcup_{[i] \in I_n} [i].$$

With a simple computation we get

$$\sharp I_n \leq \sum_{r \geq (\gamma)n} \binom{n}{r} d^{n-r}.$$

The following is an elementary consequence of Stirling's formula, see [Tah02] or [BV00]: there exists a universal constant $B > 0$, such that

$$r > \frac{kn}{k+1} \quad \Rightarrow \quad \binom{n}{r} \leq B((1+\frac{1}{k})(1+k)^{\frac{1}{k}})^r.$$

Then, since $\nu([i]) \leq Cp^{-n}$, we have that $\nu(A_\gamma(n)) \leq \sharp I_n C p^{-n}$. For $\gamma \geq \frac{k}{k+1}$:

$$\nu(A_\gamma(n)) \leq \frac{n}{k} BC((1+\frac{1}{k})(1+k)^{\frac{1}{k+1}})^n d^{\frac{n}{k+1}} p^{-n}.$$

Taking $k_0(p)$ big enough such for any $k \geq k_0$ holds $(1+\frac{1}{k})(1+k)^{\frac{1}{k+1}} d^{\frac{1}{k+1}} < p$, if $\gamma > \frac{k_0}{k_0+1}$ we have that any $1 > \eta > (1+\frac{1}{k})(1+k)^{\frac{1}{k+1}} d^{\frac{1}{k+1}} p^{-1}$, satisfies the claim, for $K(\eta)$ bigger enough.

$\square$

Corollary 7.7. *The measure ν is f-expanding with exponent $c_1 = c_1(p)$, and the measures of cylinders decay exponentially fast.*

Proof. By proposition 7.6 we have that for ν almost every point spends a definite fraction $\gamma(p)$ of the time outside of R_1. By hypothesis, we have that $\log|\det Df(x)^{-1}|$ is less than $(1+\delta_1)^{-1}$ for every x outside V. Then

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq \frac{1}{n} \log(1+\delta_1)^{-(1-\gamma)}(1+\delta_0)^{\gamma})^n \leq -2c_1(p) < 0.$$

The second part is a direct consequence of the proof of proposition 7.6, since there exists a constant C such for every cylinder $[i]$, $\nu([i]) \leq Cp^{-n}$. $\square$

Definition 7.8. Let (i_n) be the itinerary of x, defined by $f^n(x) \in R_{i_n}$, for each $n \geq 0$. We define $[x]$ to be the set

$$[x] = [i_0, ..., i_n, ...] = \{y \in M; f^n(y) \in R_{i_n}\}.$$

In the uniformly expanding case the set $[x]$ always consists of the single point x. In our present context we prove that this occurs almost everywhere, relative to the measures that interest us:

Corollary 7.9. *For ν-almost every x we have $[x] = \{x\}$.*

Proof. By proposition 7.7 we have that ν is a f-expanding measure. Thus, there exists a set H with total ν-measure such every $x \in H$ have infinitely many hyperbolic times. Given $x \in H$ and n a hyperbolic time for x:

$$\operatorname{diam}([i_0, \ldots, i_n]) \leq Ce^{-cn}.$$

As $[x] = \bigcap_{n\geq 0} [i_0, \ldots, i_n]$, and $\operatorname{diam}([x]) \leq Ce^{-cn}$ for every $n \geq 0$ hyperbolic time for x, we have that $[x] = \{x\}$ for every $x \in H$.

$\square$

Lemma 7.10. *There exists a set H with $\nu(\tilde{H}) = 1$ such that $\Pi|_{\tilde{H}}$ is injective and a subset $H \subset \tilde{H}$ such that $\nu(H) = 1$ and for every $x \in H$, $f^{-1}(x) \subset \tilde{H}$.*

Proof. By corollary 7.9 we can choose a mensurable set $\tilde{H}$ such that $\nu(\tilde{H}) = 1$ and $\Pi|_{\tilde{H}}$ is injective. Observe that $\nu(f^{-1}(\tilde{H})) = 1$. In fact, if it is not true, there exists a set A with positive measure $\nu(A) > 0$ such that $f|A$ is injective and $f(A) \cap \tilde{H} = \emptyset$. It is a absurd, since $\nu(f(A)) = p\nu(A) > 0$ and $\nu(\tilde{H}) = 1$. Then, if we define

$$H = \bigcap_{n=0}^{\infty} f^{-n}(\tilde{H})$$

we have that $\nu(H) = 1$ and for every $x \in H$, $f^{-1}(x) \subset \tilde{H}$ ☐

8 Proof of theorem B

Proof. First, observe that ν is invariant , by corollary 7.3. Also, as the jacobian of ν is constant and equal to p, the entropy of ν is $h_\nu(f) = \int J_\nu f = \log p$. To prove that ν has maximal entropy, we use the lemmas below:

Lemma 8.1. *There exists a number $\theta < 1$ such if η is a ergodic measure giving full measure to $\partial\mathcal{R}$ then:*

$$h_\eta(f) \leq \theta \log p$$

Proof. Consider $K = f(\partial R)$ and $f|_K$. Observe that, since $f : K \to K$ is a expanding map and $\partial\mathcal{R}$ is connected compact set, by classical theorems (e.g. [Bow75]) there exists some measure β such $h_\beta(f) = h_{top}(f|_K)$ and supp$\beta = \partial\mathcal{R}$. Then, we just need prove the inequality above for β.

Observe that the set $A = \{x \in K/\sharp\{f^{-1}(x) \cap \partial\mathcal{R}\} \leq p - 1\}$ is a non-empty. In fact, since $p > 1$, for some i there exists $R_j, R_l \subset f(R_i)$. Choosing $x \in R_l$ and $y \in R_j$ and γ a path connecting x and y, we have that there exists some $z \in \gamma \cap K \neq \emptyset$, which imply that $f^{-1}(z) \cap R_i =\neq \emptyset$. Then, $z \in A$. Also, we have that A is open, since K is compact. As supp$\beta = K$, we have that $\beta(A) > 0$. If $k(y) = \sharp f^{-1}(x) \cap \partial\mathcal{R}$, these imply that:

$$h_\beta(f) \leq \int_K \log k(y) d\beta = \int_A k(y) d\beta + \int_{A^c} k(y) d\beta < \log p$$

☐

Lemma 8.2. *If μ is an f invariant ergodic measure, then*

$$\int \log J_\mu f d\mu \leq \log p.$$

The equality holds, iff $J_\mu f = p$.

Proof. By previous lemma, we can suppose that $\mu(\partial\mathcal{R}) = 0$. If we consider the cover $\{f(R_j)\}$ of M, each element R_i intersects exact p elements $f(R_j)$ of this cover, since each point of M has exact p preimages. Thus,

$$\int J_\mu f d\mu = \sum_{j=1}^{d} \int_{R_j} J_\mu f d\mu = \sum_{j=1}^{d} \mu(f(R_j)) = p \sum_{j=1}^{d} \mu(R_j) = p$$

Then, by Jensen's inequality for the logarithmic function:

$$\int \log J_\mu f d\mu = \int \log J_\mu f d\mu \leq \log p.$$

and the equality is attained iff $J_\mu f$ is constant and equal to $J_\mu f = p$. ☐

Corollary 8.3. *The topological entropy of f is $\log p$. In particular, for every measure μ with maximal entropy, the Jacobian is constant equal to p.*

Proof. As f is finite-to-one, by Rohlin's equality we have that $h_\mu(f) = \int \log J_\mu f d\mu$. By lemma 8.2 we have that $h_{top}(f) \leq \log p$. As f is semiconjugate with σ^+, we have that $h_{top}(f) \geq h_{top}(\sigma^+)$. Then:

$$\log p \geq h_{top}(f) \geq h_{top}(\sigma^+) = \log p.$$

If μ is a maximal measure, then $h_\mu(f) = \int \log J\mu f d\mu = \log p$, then the Jacobian is constant by lemma 8.2. ☐

From the above lemma, we have that $h_\nu(f) = \log p$ is maximal. Assume that R is transitive, and let us conclude the uniqueness of this measure.

We will use the semiconjugacy $\Pi : \tilde{M} \to \Sigma^+$ between $f : \tilde{M} \to \tilde{M}$ and $\sigma^+ : \Sigma^+ \to \Sigma^+$, defined above. Define a σ^+-invariant measure ν^+ over Σ^+ by:

$$\nu^+(A) = \nu(\Pi^{-1}(A))$$

Observe that, by corollary 7.9, there exists a set H such $\nu(H) = 1$ and Π restrict to H is injective. Thus, ν^+ is ergodic equivalent to ν and its entropy is $\log p$.

Given η a ergodic invariant measure $\eta \in \mathcal{K}$ with $h_\eta(f) = h_{top}(f)$. By lemma 8.2 we have that η has Jacobian constant equal to $p > 1$. Then, by proposition 7.6 and hypothesis [H4], η is f-expanding. By the proof of corollary 7.9, we have that there exists a set H with $\eta(H) = 1$ such that the semiconjugacy Π is injective on H . It allow us consider the push-forward η^+ of η by Π. As η and η^+ are ergodic equivalent, we have $\log p = h_\eta(f) = h_{\eta^+}(\sigma) = h_{top}(\sigma^+)$. Thus, η^+ is a maximal entropy measure for σ^+. As Σ^+ is a subshift of finite type topologically transitive, there exists a unique maximal measure and this measure is a Markov measure(see [Pa64]). Then, $\eta^+ = \nu^+$ and this implies that :

$$\eta = \Pi^\star \eta^+ = \Pi^\star \nu^+ = \nu.$$

If Σ^+ is topologically mixing, then ν^+ is mixing ([Bow75]). Since, by Ornstein's Theorem ([Mañ87]), every mixing Markov measure is Bernoulli, we have that ν^+ is Bernoulli, which implies that ν is Bernoulli also.

For part 2, we are going to prove a kind of Gibbs property for ν first. In the following, ν is the unique maximal measure obtained in the last theorem.

Lemma 8.4. *For ν-a.e. $x \in M$ there exist a sequence $n_i = n_i(x)$ with $n_i \to \infty$ and a number K such that:*

$$K^{-1} \leq \frac{\nu(\mathcal{B}_\epsilon(n_i, x))}{p^{n_i}} \leq K.$$

Proof. Observe that if n_i is a hyperbolic time of x, we have that $f^{-n_i}(B_\epsilon(f^{n_i}(x)) = \cap_{j=0}^{n_i} f^{-j}(B_\epsilon(f^k(x)))$, because f^{-j} restrict to $B_\epsilon(f(x))$ is a contraction for every $j = 1, \ldots, n_i$ then,

$$\nu(B_\epsilon(f^{n_i}(x)) == \int_{\nu(\mathcal{B}(n_i,x))} d\nu \circ f^{n_i} = \int_{\nu(\mathcal{B}(n_i,x))} p^{n_i} d\nu = p^{n_i}\nu(\mathcal{B}(n_i, x)).$$

If $m = \inf\{\nu(B_\epsilon(y); y \in \operatorname{supp}\nu\}$ then we have that $m > 0$ because if $m = 0$, as $\operatorname{supp}\nu$ is compact, taking a sequence $x_i \to x, x \in \operatorname{supp}\nu$ such that $\nu(B_\epsilon(x_i)) < \frac{1}{i}$ we have that for any $\delta < 1$, $\nu(B_{\delta\epsilon}(x)) = 0$ because $B_{\delta\epsilon}(x) \subset B_\epsilon(x_i)$, if i is bigger enough. This is in contradiction with $x \in \operatorname{supp}\nu$.

Then we have that

$$\frac{\nu(\mathcal{B}(n_i, x))}{p^{-n_i}} \geq m.$$

Analogously,

$$K \geq \frac{\nu(\mathcal{B}(n_i, x))}{p^{-n_i}}.$$

$\square$

We will study now the distribution of hyperbolic times, in order to prove the subexponential measure control on the dynamical balls. We consider the function $n_1 : M \to \mathbb{N}$ the *first hyperbolic time function.* If x has not hyperbolic time, we define $n_1(x) = \infty$.

Lemma 8.5. *Let ν be a maximal entropy measure, as obtained in item 1. Then*

$$\int n_1(y) d\nu < +\infty.$$

Proof. Let $L(n) = \{x \in M; n_1(x) \geq n\}$ and

$$R(n) = \{x \in M; \frac{\sharp\{0 \leq i \leq n; f^i(x) \in V\}}{n} \geq \gamma\}.$$

By lemma 7.6 we have that if $\gamma > \gamma_0$ then:

$$\nu(R(n)) \leq Kb^n$$

for some $b < 1$. As $L(n) \subset R(n)$,

$$\nu(L(n)) \leq \nu(R(n)) \leq Kb^n.$$

This implies that $\sum_{i=0}^{+\infty} \nu(L(i)) < \infty$. Then,

$$\int n_1(y) d\nu = \sum_{i=0}^{+\infty} \nu(L(i)) < \infty.$$

$\square$

Lemma 8.6. *Let D be defined by $D = \{x \in M; \limsup \frac{n_{i+1}(x)}{n_i(x)} = 1\}$. Then, $\nu(D) = 1$.*

Proof. Define $L_\theta(n) = \{x \in M; n_1(x) \geq \theta n\}$. As we know

$$\sum_{n=1}^{+\infty} \nu(L_\theta(n)) < +\infty.$$

Then, if $x \in D^c$ we have

$$\lim \frac{n_{i+1}}{n_i} = \theta$$

for some subsequence (n_i).Then, x belongs to the set L defined by:

$$L = \bigcup_{\theta \in \mathbb{Q} \cap [1,+\infty)} \bigcap_{i=0}^{+\infty} \bigcup_{n \geq i} f^{-n}(L_\theta(n)).$$

As ν is f-invariant, $\nu(f^{-n}(L_\theta(n)) = \nu(L_\theta(n))$ and $\sum_{n=1}^{+\infty} \nu(L_\theta(n)) < +\infty$. Then, by Borel-Cantelli lemma,

$$\nu(L) = 0.$$

As $D^c \subset L$, we get $\nu(D) = 1$. $\square$

In order to prove part 2 of theorem B, we show:

Lemma 8.7. *For ν almost every $x \in M$ if $n_i = n_i(x)$ is the sequence of hyperbolic times of x then $K_{n_i} = K_{n_i}(x)$ such that,*

$$\frac{\nu(\mathcal{B}_\epsilon(n_i, x))}{\nu(\mathcal{B}_\epsilon(n_{i+1}, x))} \leq K_{n_i},$$

with $\limsup \frac{1}{n_i} \log K_{n_i} = 0$.

Proof. By lemma 8.6 there exists a set D of total probability $\nu(D) = 1$ such that, for any $x \in D$,

$$\lim \frac{n_{i+1} - n_i}{n_i} = 0.$$

Consider $x \in D \cap \mathrm{supp}\nu$. Then, as for any hyperbolic time n_i of x:

$$\nu(B_\epsilon(f^{n_i}(x)) = \nu(f^{n_i}(\mathcal{B}_\epsilon(n_i, x))) = p^{n_i}\nu(\mathcal{B}_\epsilon(n_i, x))$$

if $M = \max\limits_{z,y \in M} \frac{\nu(B_\epsilon(y))}{\nu(B_\epsilon(z)}$ then:

$$\frac{\nu(B_\epsilon(f^{n_i}(x))}{\nu(B_\epsilon(f^{n_{i+1}}(x))} = p^{n_i - n_{i+1}} \frac{\nu(\mathcal{B}_\epsilon(n_i, x))}{\nu(\mathcal{B}_\epsilon(n_{i+1}, x))} \leq M.$$

Then, if we define $K_{n_i}(x) = Mp^{n_{i+1} - n_i}$ we have:

$$\frac{\nu(\mathcal{B}_\epsilon(n_i, x))}{\nu(\mathcal{B}_\epsilon(n_{i+1}, x))} \leq K_{n_i},$$

and

$$\limsup \frac{1}{n_i} \log K_{n_i} = \limsup \frac{n_{i+1} - n_i}{n_i} + \frac{\log MP}{n_i} = 0.$$

□

Proof of item 2: Observe that if $n_i \leq n \leq n_{i+1}$, by the definition:

$$\mathcal{B}_\epsilon(n_{i+1}, x) \subset \mathcal{B}_\epsilon(n, x) \subset \mathcal{B}_\epsilon(n_i, x).$$

Then, by lemma 8.7,

$$\frac{\nu(\mathcal{B}_\epsilon(n, x))}{\nu(\mathcal{B}_\epsilon(n_{i+1}, x))} \leq \frac{\nu(\mathcal{B}_\epsilon(n_i, x))}{\nu(\mathcal{B}_\epsilon(n_{i+1}, x))} \leq K_{n_i}.$$

As, by lemma 8.4

$$\frac{\nu(\mathcal{B}_\epsilon(n_{i+1}, x))}{p^{-n_{i+1}}} < m.$$

Then,

$$\frac{\nu(\mathcal{B}_\epsilon(n, x))}{p^- n} \leq mK_{n_i}p^{n - n_{i+1}}$$

and

$$\limsup \frac{1}{n_i} \log mK_{n_i}p^{n - n_{i+1}} \leq \limsup 2\frac{n_{i+1} - n_i}{n_i} + \frac{\log mp}{n_i} = 0.$$

The converse inequality is analogous. This proves part 2. □

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