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Infinite Horizon Incomplete Markets: Equilibrium, Default and Bubbles

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TESE APRESENTADA PARA A OBTENÇÃO DO GRAU DE DOUTOR EM CIÊNCIA.

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À Gricelda.

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Friedrich Nietzsche

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Resumo

Estudaremos uma economia com mercados sequencias, e horizonte de tempo infinito, na qual a estrutura financeira e composta por ativos sujeitos a inadimplência.

Modelamos a inadimplência nas promessas através de requerimentos de colateral, os quais são exógenamente fixados e são exigidos de cada agente ao momento de fazer uma promessa futura. Esta estrutura esta baseada no modelo de colateral, em economias com dois periodos, introducido por Dubey, Geanakoplos, and Zame (1995).

No caso de serem negociados na economia só ativos reais que vivem um periodo, mostraremos que sempre existe um equilíbrio competitivo. Para garantir isto não precisaremos de introducir restrições exógenas, na forma de condições de transversalidade ou restrições à divida, para evitar os assim chamados esquemas de Ponzi.

Logo, extenderemos o resultado anterior ao caso de ativos multiperiodo e estudaremos a existência de equilíbrio em um contexto bastante geral: permitiremos a participação incompleta dos agentes ao longo do tempo e os requerimentos de colateral, que poderão depender do nivel de preços, serão constituidos tanto por bens duraveis (colateral físico) quanto por ativos reais (colateral financeiro),

Neste último cenário estudaremos a ocorrencia de especulação nos preços dos bens duraveis e dos ativos.

O fato dos ativos sujeitos a inadimplência poderem ser vistos como derivativos de aqueles ativos que são usados como colateral , nos permitira mostrar que as bolhas especulativas na economia são causadas por distorções ou nos preços dos bens duraveis ou nos preços dos ativos livres de default em relação ao seus valores fundamentais. A partir disto, obteremos condições suficientes para garantir a inexistência de especulação nos mercados.

PALAVRAS CHAVE : Equilibrio, Colateral, Condições de Transversalidade, Ativos Multiperiodo, Bolhas Especulativas.

Summary

In an infinite-horizon sequential economy, we study a general equilibrium model with financial assets which are subject to default.

We model default through exogenous collateral requirements, which are demanded from the borrowers for each unit sold by them. We base this model on the two-period economy with default introduced by Dubey, Geanakoplos, and Zame (1995).

In the case of markets which allow only short-lived assets, we show that equilibrium always exists, without introducing either transversality conditions or debts constraints to avoid the possibility of Ponzi schemes.

We extend the previous result to the case of long-lived assets and address existence of equilibrium in a very general context, where we allow incomplete participation of the agents and both physical and financial collateral requirements.

Within this last scenario we study the occurrence of speculation in the prices of durable goods and assets. The fact that assets subject to default may be regarded as derivatives, whose underlying assets are those securities that are used as collateral requirements, allows us to show that asset price bubbles are caused by bubbles in the prices of durable goods and of default-free assets. We also obtain sufficient conditions for the non-existence of speculation in the markets.

KEYWORDS : Equilibrium, Collateral, Multiperiod Assets, Transversality Conditions, Asset Pricing Bubbles.

Contents

Introduction

The study of economies where agents may default on their promises has been a key area of interest in the recent general equilibrium literature. Both default itself and the necessity for its anticipation play an important role in the economy, particularly in the incomplete financial markets case, since there is the possibility that the presence of default might improve efficiency relative to economies whose agents must make good on their promises.

In fact, as was shown in a seminal work by Zame (1993), in the context of a two-period incomplete markets economy, if default is allowed, then the agents may negotiate a bigger volume of contracts and thus more efficiently allocate their resources along time. Zame (1993) wrote

> *"If markets are incomplete and traders are only able to enter into contracts that they will be able to execute regardless of future events, contingent contracting may be severely restricted. Moreover, opening new markets may not relieve these restrictions. Default promotes efficiency in a way that opening new markets does not, by making it possible for traders to enter into contracts that they will be able to execute with high probability, but not with certainty."*

Thus much research has been done on equilibrium in economies with default and various mechanisms for restricting the default level.

Utility penalties for default, which are proportional to the default level, have been studied by Dubey, Geanakoplos, and Shubik (1990), in the finite-horizon case; and by Araujo, Monteiro, and Páscoa (1996) in the infinite-horizon case.

The modelling of default via exogenous collateral requirements, which must be constituted by the borrowers for each unit they sell, is based on the pioneering work of Dubey, Geanakoplos, and Zame (1995) in the two-period economy case. This framework was recently extended, in the finite-horizon context, to the case of endogenous collateral requirements; see Araujo, Orillo, and Páscoa (2000) or Araujo, Fajardo, and Páscoa (2001).

On the other hand, in the model of infinite-horizon incomplete markets without default, it has so far been necessary to impose exogenous conditions that prevent indefinite debt accumulation (the so-called Ponzi Schemes) in order to guarantee equilibrium.

Along these lines, in economies with short-lived assets the existence of equilibrium has been established via either debt constraints or transversality conditions (see Magill and Quinzii (1994, 1996), Hernandez and Santos (1996), Levine and Zame (1996), Florenzano and Gourdel (1996) for trees with finitely many branches at each node, and Araujo, Monteiro, and Páscoa (1996) for trees with a continuum of branches).

These debt constraints or transversality conditions are imposed on the budget set, but do not follow from budgetary considerations or individual rationality. Furthermore, the state prices that are used as a present value process in the transversality conditions, although personalized, are chosen jointly with the other equilibrium variables, without a clear objective criterium among the continuum of possibilities compatible with the absence of arbitrage.

In the case of economies with long-lived assets (i.e., with assets which last more than a single period) the debt restrictions, given by transversality conditions or debts constraints, are only sufficient to guarantee the *generic* existence of equilibrium relative to the aggregate inicial endowments of the economy.

It would therefore be interesting to address the existence of equilibrium in a model where Ponzi schemes are not a priori ruled out from the agents' choice sets, but instead would presumably not occurr in equilibrium.

Hence, we will study an incomplete markets model with assets protected by collateral, where default causes the seizure of the collateral by the lender. The collateral obligation coefficients are given exogenously and are the same for all traders in the economy. The resulting collateral bundles of commodities may be stored or consumed by the borrowers or by the lenders. Our model is therefore based on the seminal work of Dubey, Geanakoplos, and Zame(1995).

In this context we show that, when there is no enforcement other than the seizure of collateral by the lenders, then equilibrium always exists (without the need of any additional hypotheses such as debts constraints or transversality conditions) in infinite-horizon economies with sequential trades.

Moreover we show that, in the case where there are utility penalties and where the collateral requirements may consist both of durable goods and of financial assets, an a priori implicit debt restriction guarantees the existence of equilibrium. This contrasts with the results on generic existence of equilibrium in multiperiod-asset economies without default, as is the case in Magill and Quinzii (1996) and in Levine and Zame (1996).

In this last framework it is natural to study the occurrence of speculation in equilibrium prices.

This question has already been studied for infinite-horizon economies by Santos and Woodford (1997) and Magill and Quinzii (1996), who show that finite-lived assets exhibit no bubbles in their equilibrium prices; and that if the aggregate endowment is bounded by a portfolio trading plan, at every state of nature, then those infinite-lived assets that have positive net supply will also exhibit bubble-free non-arbitrage prices.

In our context we show that there are no bubbles in the prices both of finite-lived assets and of assets that go to default in finite horizon. In the case of infinite-lived assets, we obtain conditions on the prices of durable goods and of default-free assets that guarantee the non-existence of speculation in the economy.

We prove that distortions relative to the fundamental value, in prices of assets backed by collateral, are always caused by bubbles in the prices of those assets that constitute the collateral.

Insertion in the Literature and Contribution

The analysis of infinite-horizon economies has been developed within two classes of models: those of overlapping generations and those of consumers with infinite life. This project extends the second class, although it is also possible to explore its implications for the first class via the second chapter's results (which allow for incomplete agent participation). When markets are incomplete and consumers heterogeneous, new difficulties appear that did not occur in models with representative infinite-life consumers, as in Lucas (1978).

As in models with infinite horizon and complete markets, the problem of the consumer does not have a solution if successive postponement of the debt payment is allowed. In fact, the consumer would have all the interest in running increasingly into debt, throughout time, using new credit to pay debt interests. These Ponzi schemes were avoided in the literature by the introduction of debt constraints or transversality conditions (see Blanchard and Fischer (1989, chapter 2)). Debt constraints place a uniform bound on the debt in all periods. Transversality conditions, in the deterministic case, require that the debt increases asymptotically slower than the interest rate (see Kehoe (1989), for instance).

When markets are incomplete, the choice of which deflator to use in the transversality condition becomes troublesome. In fact, there is no longer one unique vector of present values of one unit of returns in the future. The equivalent martingale measure, which serves as a deflator of asset prices in the absence of arbitrage, is unique under the complete markets hypothesis, but turns out to be indeterminate in the case of incomplete markets. Magill and Quinzii (1994, 1996) proposed a solution that consists of a transversality condition on the budget restriction, requiring that the present value of an agent's debt tend asymptotically to zero, where the deflator would be personalized but determined in equilibrium. Hernandez and Santos (1996) proposed the use of the most punishing deflator among the continuum of possibilities compatible with the absence of arbitrage (or, in others words, that the budget set be defined as the intersection of all the sets that satisfy one condition of transversality for a certain equivalent martingale measure). In this way, the arbitrary imposition of a transversality condition is strengthened by the arbitrary choice of the deflator.

Magill and Quinzii (1994) show that there is a one-to-one correspondence between equilibria of economies with a priori transversality conditions and equilibria of economies whose budget sets include an implicit debt constraint (requiring the value of the portfolio to be a bounded sequence along the event-tree). Actually, given an equilibrium with implicit debt constraint, it is possible to replace this constraint by a node-by-node explicit debt constraint which is non-binding at this equilibrium (by picking an explicit upper bound greater than the supremum of the sequence of equilibrium debt values). Any of these three equilibrium concepts assumes that agents' choices are restricted to portfolios that do not allow for Ponzi schemes. Even in the case of the non-binding explicit debt constraint equilibrium, it should be noted that without this constraint equilibria might not exist, since portfolios whose debt-value sequences are not bounded could now be chosen as well.

Moreover, the existence of equilibrium is only guaranteed by such restrictions if all of the financial assets are short-lived. Magill and Quinzii (1996) and Levine and Zame (1996) have shown that if the financial markets include long-lived assets, then equilibrium exists only for a dense set of endowments.

In the first chapter of this work, we explore the structure of a model with default and exogenous collateral and show the existence of equilibrium without a priori imposing either restrictions on debt or exogenous transversality conditions (see Theorem 1.2). The obligation of constituting collateral in terms of durable goods whenever an asset is sold will bound the asymptotic explosion of the debt, since the collateral is required to be different from zero for all assets negotiated in the economy, and the aggregate endowment of these durable goods is bounded. Hence, in this model it is the feasibility together with the collateral requirements that rule out Ponzi schemes (as explained in Remark 1.4 below). A recursive argument shows that feasible short-sale allocations are bounded, at each node, by the sum of aggregate depreciated endowments along the relevant path divided by the collateral coefficient at this node. However, we do not require collateral coefficients to be uniformly bounded away from zero along the entire infinite tree, and therefore there is no uniform upper bound on asset short sales that could be derived from the feasibility equations.

In Chapter 2 we explore a very general framework that extends the previously discussed model. We now allow multiperiod assets in the financial markets; such assets might be subject to default or default-proof. We also assume that the collateral requirements may consist of durable goods (physical collateral) or of assets (financial collateral), and that there are utility penalties for agents who default (as in Dubey, Geanakoplos and Zame (1995)). We show the existence of equilibrium in a context that includes incomplete agent participation, which paves the way for the study of these questions in overlapping generations models (see Theorem 2.1 and Corollary 2.2)

Now, since there are enforcement mechanisms other than the seizure of the collateral bundles by the lenders, the value of the collateral requirements may be smaller than the amount that the borrowers receive from the lenders once a promise is made. In fact, due to the presence of the utility penalties, the agents might have to pay more than the depreciated value of the collateral if the latter is smaller than the asset's promises. Thus, in equilibrium, the cost of selling one unit of a asset might be negative (that is, the borrower might obtain gains today, at the moment he short sells) since, in contrast to the situation studied in chapter 1, in future states of nature the asset's vendor might have to pay dividends whose value exceeds the depreciated value of the collateral.

It is due to this last fact that, in a more general context, we have to impose exogenous bounds on debt growth, in the form of debt constraints, in order to guarantee the existence of equilibrium. Moreover, we show that there always exist prices which centralize the agents' decisions, in contrast to what occurs in the current literature on models with default (such models only guarantee the *generic* existence of equilibrium when there are multiperiod assets).

In the special case where the only assets in the economy are those subject to default and where the only enforcement is the seizure of collateral by the lenders, we show that equilibrium always exists, without the need for a priori transversality conditions ou debt constraints; this is a extension of the Theorem 1.2 to the case of multi-period assets.

Another much studied question in the infinite-horizon general equilibrium literature has been the existence of speculation in non-arbitrage prices. In a seminal work, Santos and Woodford (1997) gave a very broad characterization of the existence of bubbles in an incomplete markets model with incomplete agent participation. They showed that, if it is possible to replicate the economy's aggregate endowment at finite cost then – given an finite lived asset or, a infinite-lived asset with positive net supply – there is always a state-price process such that the asset's price has no distortions relative to the fundamental value.

In order to guarantee the absence (in the strong sense) of speculation[1] in assets with positive

[1] That is, that independently of the chosen state-price process, the asset's price has no distortions relative to the fundamental value.

net supply, the aforementioned authors impose an assumption that assures a sufficient degree of impatience in the agents' preferences. But in the case of infinite-lived assets with zero net supply the results of Santos and Woodford are not conclusive.

Magill e Quinzii (1996), however, use arguments based on the previous results to show that, in a model where the debt growth is restricted by transversality conditions, the equilibrium prices of assets with zero net supply exhibit bubbles, with real efects, only if the markets are incomplete.

That is, if the markets are sequentially complete and the prices of the assets with zero net supply exhibit bubbles, then there are other prices which support the same allocation and are free of speculation. If the markets are incomplete, however, there might be equilibrium prices with bubbles which cannot be replaced by other prices without the agents change their consumption allocations.

These facts show the importance of studying speculation in the case where the assets are subject to default; the model in chapter 2 is a natural framework for this research. Thus, since collateral might consist of both durable physical goods and of financial assets, we show that if we impose upper bounds on the amounts required for collateral, then bubbles in the prices of collateral-backed assets are caused by distortions in the prices of durable goods or of default-proof assets relative to their fundamental values.

Furthermore, this last result allows us to obtain conditions, in the form of transversality conditions, on the prices of commodities and of default-proof assets that guarantee the absence of bubbles in equilibrium prices.

Although in this paper we do not study hypotheses on the primitives of the economy that might assure the validity of this transversality conditions, it is important to note that these conditions are sufficient regardless of whether the relevant asset has positive net supply or not.

Methodology and Results

In the first chapter, we construct a model of a tree with a countable set of periods. The assets traded at each node are real and, for simplicity, deliver returns only at the nodes which immediately follow the one where the trade took place. We suppose that the initial endowments at each node are uniformly bounded from above and that the exogenous collateral requirements are positive for all assets traded in the event-tree. Preferences are described by utilities that are additive in time and in states of nature.

Agents have the possibility of defaulting on their promises. Since the omission of the payments gives rise to collateral seizure by the lender, and since no other enforcement exists in case of default (such as utility loss or credit restrictions), each agent, whether an asset seller or buyer, turns out to have as payoff coefficient, at each state, the minimum between the value of real return and the value of the collateral that is established at the moment of the negotiation.

We actually start by considering finite-horizon economies, where it is relatively easy to prove the existence of equilibrium, since the equations of feasibility imply that aggregate asset sales are bounded from above at each node. The technique of proof follows the generalized game method, introducing an artificial auctioneer at each period and state. In this way, we extend the Dubey, Geanakoplos, and Zame (1995) results to the multi-periods case (see Theorem 1.1).

Then, in order to study the infinite horizon case, we analyze the equilibrium sequence associated with the truncated economies with increasing terminal periods. We prove that the sequence of marginal utilities of endowments income, evaluated in this equilibrium, of each agent at each node, is uniformly bounded. This crucial step will allow us to obtain cluster points for the Lagrange multipliers. We extract cluster points, node by node, from all the equilibrium variables, using the fact that feasible short-sales have an upper bound, node by node, due to the required purchase of durable collateral. There might not exist a uniform upper bound on short sales along the infinite tree, but the countability of nodes allows us to use a diagonalization argument to extract the desired cluster points. We prove the feasibility of the vector of cluster points and then it remains only to verify individual optimality, which will follow from an argument by contradiction that uses the Kuhn-Tucker conditions of the truncated problems.

Although we did not impose any a priori transversality conditions it is interesting to examine whether such conditions hold in equilibrium. Taking as deflator the vector of cluster points of Lagrange multipliers in truncated economy equilibria, we show that no agent wants to be a lender at infinity and that collateral constraints are binding at infinity. Nevertheless, in the incomplete markets model without default, transversality conditions require that no agent be a borrower at infinity. We show that this condition holds if endowments are assumed to be uniformly bounded from below, which we do not require in our existence result (see Remark 1.5 below).

Finally, we make a remark about the welfare properties of equilibrium in the exogenous collateral economy: we prove that a constrained efficiency property is achieved, which is a result that had already been obtained by Dubey, Geanakoplos, and Zame (1995) in the case of two-period models.

In the second chapter, we extend the model to the case of multiperiod assets. The structure of uncertainty and the characterization of the durable goods are essentially the same as those already described.

Incomplete agent participation is allowed, and the financial structure comprises two types of securities:

-Default-free assets, whose sale is restricted via exogenous short-sales constraint or via margin requirements, which must constituted by the borrowers for each unit of asset that they trade, and

-Assets subject to default which are protected both by collateral requirements (which might vary with the price level, in contrast to what happened in the previous chapter) and by utility penalties which are proportional to the volume of default.

Hence, in order to show that equilibrium always exists, we assume that the agents' total debt level is uniformly bounded along time through the imposition of an implicit debt constraint on the budget set of each participant in the economy. We prove existence by using essentially the same technique as the one in the proof of Theorem 1.2: we first show that there is equilibrium in a truncated economy with a finite number of periods. We then show that the Lagrange multipliers of these finite-dimensional problems are uniformly bounded at each node of the event-tree. We show next, using our hypotheses, that the allocations are also uniformly bounded, and we thereby obtain cluster points for the multipliers as well as for consumption allocations, portfolios, and prices.

We then use the implicit debt constraint to show that such limits constitute an equilibrium in the original economy.

If all of the negotiated assets are subject to default and there are no utility penalties for default, then the total cost of selling an asset, in equilibrium, is positive (because the agents will act as lenders, since the future returns will be non-negative. This is analogous to the non-arbitrage condition given by Proposition 1.1) and therefore the implicit debt constraint becomes unnecessary for guaranteeing equilibrium.

Finally, in order to obtain sufficient conditions for the non-existence of bubbles, we use the conditions which guarantee that the commodity prices and asset prices generate *finite* optimum for the agents' problems; we also characterize the functional form of this prices in terms of the returns given by the assets.

Chapter 1

Collateral Avoids Ponzi Schemes

In this chapter we study an infinite-horizon sequential economy in which, at every period and in every state of nature, short-lived assets subject to default are negotiated. We assume that the asset sellers are required to constitute exogenously given bundles of physical collateral, in order to protect the lenders in case of default. We suppose that the only form of enforcement in case of default is the seizure of the collateral by the lenders.

We show that in such economies there is always an equilibrium, even without imposing a priori transversality conditions or debt constraints.

Thus, the collateral requirements, which constitute more plausible restrictions than transversality conditions or debt constraints (especially in a context where the agents might not fulfill their promises), guarantee that the agents cannot indefinitely accumulate growing debts, by using new capital to pay past debts. In this sense, the collateral requirements avoid the so-called Ponzi Schemes.

1.1 The Economy

1.1.1 Model

Uncertainty: The model of uncertainty is essentially the one developed by Magill and Quinzii (1994, 1996). We consider an economy in which $\tau \in \mathbb{N} \cup \{\infty\}$ denotes the length of the time horizon. The set of periods is $\mathcal{T} = \{0, 1, \ldots, \tau - 1\}$ when τ is finite and $\mathcal{T} = \mathbb{N}$ when τ is infinite. Let S denote the set of states of the nature. In the case of finite horizon, we suppose that the cardinality of S is finite.

The available information at period t in $\mathcal{T}$ is the same for every agent (symmetric information) and is given by a partition $\mathcal{F}_t$ of S, where the state of nature lies. We suppose that there is no information at $t = 0$, that is, $\mathcal{F}_0 = S$. When $\tau < \infty$, we consider $\mathcal{F}_{\tau-1} = \{\{s\} : s \in S\}$. So the information structure in the economy is given by a family: $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{\tau-1}\}$.

Our model reveals information along time, so if $t < t'$ then $\mathcal{F}_{t'}$ is finer than $\mathcal{F}_t$. The number of sets in $\mathcal{F}_t$ is finite for all t in $\mathcal{T}$.

Every pair $\xi = (t, \sigma)$, with t in $\mathcal{T}$ and σ in $\mathcal{F}_t$, is called a node of our economy. The set of all nodes, $\mathcal{D}^\tau$, is the *event-tree* induced by $\mathcal{F} : \mathcal{D}^\tau = \{(t, \sigma) \ : \ t \in T \ , \sigma \in \mathcal{F}_t\}$.

If $\xi = (t, \sigma)$ then $\tilde{t}(\xi) = t$ is the period associated with the node. We say that $\xi' = (t', \sigma')$ is a *successor* of $\xi = (t, \sigma)$ if $t' \geq t$ and $\sigma' \subseteq \sigma$; we use the notation $\xi' \geq \xi$. The set $\xi^+ = \{\xi' \in \mathcal{D}^\tau \ : \ \xi' \geq \xi \ , \tilde{t}(\xi') = \tilde{t}(\xi) + 1\}$ is the set of *immediate successors* of ξ in $\mathcal{D}^\tau$. We denote by b_ξ^τ the cardinality of ξ^+ and suppose that it is *finite* for all ξ in $\mathcal{D}^\tau$. Because $\mathcal{F}_t$ is finer than $\mathcal{F}_{t-1}$ for all t in $\mathcal{T}$, there is only one predecessor for each $\xi \in \mathcal{D}^\tau$. We denote this node by ξ^-. We denote by ξ_0 the node at $t = 0$.

Agents and Commodities: There exists a finite set of commodities, $\mathcal{L}$, at each node of the event-tree $\mathcal{D}^\tau$. So the set of goods in the economy is given by $\mathcal{D}^\tau \times \mathcal{L} = \{(\xi, l) \ : \xi \in \mathcal{D}^\tau \ , l \in \mathcal{L}\}$ and we suppose that they suffer a partial depreciation at the node branches.

The structure of depreciation in the event-tree is given by a family of $\mathcal{L} \times \mathcal{L}$-matrix with positive entries: $\{Y_\xi^c; Y_\xi^s\}_{\xi \in \mathcal{D}^\tau}$, where $(Y_\xi^c)_{l,l'}$ denotes the amount of the good l that is obtained at the node ξ if one unit of the good l' was *consumed* at the node ξ^-. Thus, for example, if the commodity l' is perishable then $(Y_\xi^c)_{l,l'} = 0$ for all pairs (ξ, l). Otherwise, there exist commodities such that $(Y_\xi^c)_{l,l'} \neq 0$. This is, for instance, the case of perfectly durable goods, simply one period older goods or goods whose consumption causes their partition into many pieces. Analogously, $(Y_\xi^s)_{l,l'}$ denotes the amount of commodity l that is obtained at the node ξ if one unit of the good l' was *stored* at the node ξ^-. It is clearly important to differentiate the depreciation in the event-tree in the case of a commodity such as tobacco, which is perfectly storable but not durable.

At each node there are spot markets for the trading of commodities. Let $p = (p_{\xi,l})$ in $\mathbb{R}_+^{\mathcal{D}^\tau \times \mathcal{L}}$ be the spot price process and $p_\xi = (p_{\xi,l} \ : l \in \mathcal{L})$ be the spot price vector at the node $\xi \in \mathcal{D}^\tau$.

There exists a finite set, $\mathcal{I}$, of infinite-life agents in the economy. They demand commodities (for consumption and storage) in spot markets and negotiate assets at every node of $\mathcal{D}^\tau$.

We characterize each agent i in $\mathcal{I}$ by an endowment process $w^i = (w^i(\xi, l) \ : (\xi, l) \in \mathcal{D}^\tau \times \mathcal{L})$

that belongs to the non-negative orthant of $\mathbb{R}^{\mathcal{D}^\tau \times \mathcal{L}}$, which we denote by X^i. So the endowment of the agent i at the node $\xi \in \mathcal{D}^\tau$ is $w^i(\xi) = \left(w^i(\xi, l) \ : l \in \mathcal{L}\right) \in \mathbb{R}^{\mathcal{L}}_+$.

Each agent in the economy chooses a consumption plan free of collateral $x^i = \left(x^i(\xi, l) \ : (\xi, l) \in \mathcal{D}^\tau \times \mathcal{L}\right) \in X^i$ in the event-tree[1] and a storage plan free of collateral y^i in $\tilde{X}^i = \{y \in X^i : y(\xi) = 0 \ \forall \xi \in \mathcal{D}^\tau, \ b^\tau_\xi = 0\}$.

The utility function $U^i : X^i \to \mathbb{R}_+$ represents the preferences of the agent i.

Assets and Collateral: We work with a structure of real assets that live for only one period.

Let $\mathcal{J}$ denote the set of securities in the economy and $\mathcal{J}(\xi) \subset \mathcal{J}$ the set of real assets negotiated at the node ξ. So the set of assets in the event-tree is given by $\mathcal{D}^\tau(\mathcal{J}) = \{(\xi, j) \ : \xi \in \mathcal{D}^\tau \ , j \in \mathcal{J}(\xi)\}$. We suppose that the cardinality of $\mathcal{J}(\xi)$ is finite for all $\xi \in \mathcal{D}^\tau$ and $\mathcal{J}(\xi) = \emptyset$ for all *terminal node* in $\mathcal{D}^\tau$ - that is, the nodes in $\mathcal{D}^\tau$ such that $b^\tau_\xi = 0$.

The function $A : \mathcal{D}^\tau \times \mathcal{J} \to \mathbb{R}^{\mathcal{L}}_+$ characterizes the promises of the asset's structure, so $A(\xi, j)$ describes the bundles yielded by asset j at the immediate successors nodes of ξ^-. We suppose that $A(\xi, j) = 0$ if $j \notin \mathcal{J}(\xi^-)$ and, for each asset j in $\mathcal{J}(\xi)$, that there exists a node μ in ξ^+ such that $A(\mu, j)$ is different from zero, that is, there are no trivial securities in the economy.

If $j \in \mathcal{J}(\xi)$ then denote by $q_{\xi,j}$ the unit price of the asset j at the node ξ. At each node ξ of the event-tree, denote by $\theta^i(\xi, j)$ the number of units of the asset $j \in \mathcal{J}(\xi)$ bought by the agent i at the node and by $\varphi^i(\xi, j)$ the number of units of the asset sold. Let $Z^i(\xi) = \theta^i(\xi) - \varphi^i(\xi)$ be the portfolio of agent i.

In this model we suppose that, for every unit of the asset j in $\mathcal{J}(\xi)$ sold by agent i at the node ξ, he should establish a collateral $C^\xi_j \in \mathbb{R}^{\mathcal{L}}_+$, which is given *exogenously* and has the purpose of protecting the buyer when the sellers do not honor their commitments.

The collateral established by the seller of the asset j at the node ξ, C^ξ_j, can be decomposed, as in Dubey, Geanakoplos, and Zame (1995), as $C^\xi_j = C^{W,\xi}_j + C^{B,\xi}_j + C^{L,\xi}_j$, where $C^{W,\xi}_j$ denotes the part that is stored, such as commodities that are fragile, or that can be easily stolen; $C^{B,\xi}_j$ is the part of the collateral that is held by the borrower, such as a house or a car; and $C^{L,\xi}_j$ is the part that is held by the lender, such as a painting.

At each immediate successor ξ' of ξ, the collateral established at ξ is depreciated. In this way, the seller of asset j at the node ξ delivers at each immediate successor ξ' of ξ the amount $D^{\xi'}_j \equiv \min\left\{p_{\xi'} A(\xi', j) \ , p_{\xi'}\left[Y^c_{\xi'}(C^{B,\xi}_j + C^{L,\xi}_j) + Y^s_{\xi'} C^{W,\xi}_j\right]\right\}$ to the asset's buyer. This means

[1] It is not necessary to restrict the consumption bundles of the agent because the collateral structure guarantees the uniform limitation of the consumption allocations in equilibrium (see Lemma 1 and Remark 4).

that since the only enforcement mechanism in case of default is the seizure of the collateral, each debtor decides to deliver the minimum between his debt and the depreciated value of the collateral in this state. Similarly, each lender expects to receive only the minimum between the claim and the market value of the collateral.

For convenience of notation, let us define $C^{\xi} \equiv \left(C_j^{\xi}\right)_{j \in \mathcal{J}(\xi)}$, $D^{\xi} \equiv \left(D_j^{\xi}\right)_{j \in \mathcal{J}(\xi^-)}$, $Y_{\xi} C^{\xi^-} \equiv Y_{\xi}^{c}\left(C^{B,\xi^-} + C^{L,\xi^-}\right) + Y_{\xi}^{s} C^{W,\xi^-}$, and for all terminal nodes in the event-tree, $C^{\xi} \equiv 0$. Moreover, given a vector $z = (z_1, z_2, \ldots, z_n)$ in the Euclidean space $\mathbb{R}^n$, we denote by $||z||_{\Sigma}$ the norm of the sum of z, that is, the value of $\sum_{k=1}^{n} |z_k|$.

Therefore, the economy with *exogenous collateral*, $\mathcal{E}_{ex}^{\tau}$, is characterized by the event-tree $\mathcal{D}^{\tau}$, the utility functions that represent the agents' preferences $\mathcal{U} = (U^i)_{i \in \mathcal{I}}$, the agents' endowment processes $\mathcal{W} = (w^i)_{i \in \mathcal{I}}$, the assets structure $\mathcal{A} = \left(\mathcal{J}, \left(\mathcal{J}(\xi)\right)_{\xi \in \mathcal{D}^{\tau}}, \left(A(\xi, j)\right)_{(\xi, j) \in \mathcal{D}^{\tau} \times \mathcal{J}}, C^{\xi}_{\{\xi \in \mathcal{D}^{\tau}\}}\right)$ and the depreciation $\mathcal{Y} = \left(Y_{\xi}^{s}; Y_{\xi}^{c}\right)_{\xi \in \mathcal{D}^{\tau}}$.

1.1.2 Equilibrium in $\mathcal{E}_{ex}^{\tau}\left(\mathcal{D}^{\tau}, \mathcal{U}, \mathcal{W}, \mathcal{A}, \mathcal{Y}\right)$

Given $p \in \mathbb{R}_{+}^{\mathcal{D}^{\tau} \times \mathcal{L}}$ a spot price process and $q \in \mathbb{R}_{+}^{\mathcal{D}^{\tau}(\mathcal{J})}$ an asset price process, the agent i in $\mathcal{I}$ can choose an allocation $\left(x^i, y^i, \theta^i, \varphi^i\right)$ in the state space $I\!E^{\tau} = X^i \times \tilde{X}^i \times \mathbb{R}_{+}^{\mathcal{D}^{\tau}(\mathcal{J})} \times \mathbb{R}_{+}^{\mathcal{D}^{\tau}(\mathcal{J})}$, subject to the budgetary restrictions :

$$p_{\xi_0}\left[x^i(\xi_0) + y^i(\xi_0)\right] + p_{\xi_0} C^{\xi_0} \varphi^i(\xi_0) + q_{\xi_0} Z^i(\xi_0) \leq p_{\xi_0} w^i(\xi_0); \tag{1.1}$$

$$p_{\xi}\left[x^i(\xi) + y^i(\xi)\right] + p_{\xi} C^{\xi} \varphi^i(\xi) + q_{\xi} Z^i(\xi)$$
$$\leq p_{\xi} w^i(\xi) + p_{\xi}\left[Y_{\xi}^{c} x^i(\xi^-) + Y_{\xi} C^{\xi^-} \varphi^i(\xi^-) + Y_{\xi}^{s} y^i(\xi^-)\right] + D^{\xi} Z^i(\xi^-), \tag{1.2}$$

$\forall \xi \in \mathcal{D}^{\tau} : \xi > \xi_0$.

The agent's *process of consumption* in the event-tree is $\left(x^i(\xi) + C^{B,\xi}\varphi^i(\xi) + C^{L,\xi}\theta^i(\xi)\right)_{\xi \in \mathcal{D}^{\tau}}$ and $(\theta^i(\xi), \varphi^i(\xi))_{(\xi) \in \mathcal{D}^{\tau}} \equiv (\theta^i(\xi, j), \varphi^i(\xi, j))_{(\xi, j) \in \mathcal{D}^{\tau}(\mathcal{J})}$ are the asset's buying and selling processes of the agent i.

So the budget set for agent i is $\mathcal{B}_{ex}^{\tau, i}(p, q) = \left\{(x, y, \theta, \varphi) \in I\!E^{\tau} \text{ s.t. equations (1.1) and (1.2) hold}\right\}$.

The homogeneity of equations (1) and (2) in (p_{ξ}, q_{ξ}) implies that this pair can be normalized. So we consider the pair in $\Delta_{+}^{\mathcal{L}+\mathcal{J}(\xi)-1} = \{v \in \mathbb{R}_{+}^{\mathcal{L}+\mathcal{J}(\xi)} : \sum_k v_k = 1\}$. Therefore, the space of prices will be $\mathcal{P}^\tau = \{(p,q) = (p_\xi, q_\xi)_{\xi \in \mathcal{D}^\tau} \ : \ (p_\xi, q_\xi) \in \Delta_+^{\mathcal{L}+\mathcal{J}(\xi)-1}\}$.

Let us define $V^i\big(x,y,\theta,\varphi\big) \equiv U^i\Big[\big(x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\big)_{b_\xi^\tau > 0}; \big(x(\xi)\big)_{b_\xi^\tau = 0}\Big]$.

Definition: An *equilibrium* for the economy $\mathcal{E}_{ex}^\tau$ is a vector $[(\overline{x},\overline{y},\overline{\theta},\overline{\varphi});(\overline{p},\overline{q})]$ in $\big(\mathbb{E}^\tau\big)^{\mathcal{I}} \times \mathcal{P}^\tau$ with $\big(\overline{x},\overline{y},\overline{\theta},\overline{\varphi}\big) = \big(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\big)_{i\in\mathcal{I}}$, such that:

- The allocation $(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i)$ solves:

(1.3)
$$\max_{(x,y,\theta,\varphi)} \quad V^i\big(x,y,\theta,\varphi\big),$$
$$\text{subject to} \quad (x,y,\theta,\varphi) \in \mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q}).$$

- The following feasibility conditions are satisfied:

(1.4)
$$\sum_{i\in\mathcal{I}} \big[\overline{x}^i(\xi_0) + \overline{y}^i(\xi_0) + C^{\xi_0}\overline{\varphi}^i(\xi_0)\big] = \sum_{i\in\mathcal{I}} w^i(\xi_0);$$

(1.5)
$$\sum_{i\in\mathcal{I}} \Big[\overline{x}^i(\xi) + \overline{y}^i(\xi) + C^\xi\overline{\varphi}^i(\xi)\Big] = \sum_{i\in\mathcal{I}} \Big[w^i(\xi) + Y_\xi^c\overline{x}^i(\xi^-) + Y_\xi C^{\xi^-}\overline{\varphi}^i(\xi^-) + Y_\xi^s\overline{y}^i(\xi^-)\Big],$$

for all $\xi \in \mathcal{D}^\tau : \xi > \xi_0$.

- The pair $(\overline{\theta},\overline{\varphi})$ satisfies:

(1.6)
$$\sum_{i\in\mathcal{I}} \overline{\theta}^i = \sum_{i\in\mathcal{I}} \overline{\varphi}^i.$$

Remark 1.1. If the structure of depreciation were the same for commodities consumed or stored, that is $Y_\xi^s = Y_\xi^c$ for all ξ in $\mathcal{D}^\tau$, then the agents would not be interested in storing commodities free of collateral. In this way, equations (1.4) and (1.5) in the equilibrium definition would be equivalent to the condition that, for each node in the economy, the total demand of commodities is equal to the total endowment accumulated until this date. That is,

(1.7)
$$\sum_{i\in\mathcal{I}} \Big[\overline{x}^i(\xi) + C^\xi\overline{\varphi}^i(\xi)\Big] = \sum_{i\in\mathcal{I}} \sum_{k=0}^{\tilde{t}(\xi)} Y\big(\xi,\xi^{-k}\big) w^i(\xi^{-k}),$$

where ξ^{-k} denotes the k-times predecessor of the node ξ, and $Y\big(\xi,\xi^{-(k+1)}\big) = Y_\xi^c Y\big(\xi^-,(\xi^-)^{-k}\big)$, with $Y(\xi,(\xi^-)^0) \equiv I$, is the accumulated depreciation factor.

Therefore, in the finite horizon case, this equivalence implies that when securities markets are cleared (equation (1.6)), the consumption allocations are uniformly bounded in equilibrium. The proof of this fact, in the general case ($Y^c \neq Y^s$), is contained in Lemma 1.

In the following section, we shall give sufficient conditions that guarantee the existence of equilibrium in the case of finite horizon. Then, we will show the existence of equilibrium in infinite horizon economies by using the existence for the finite case and a non-arbitrage condition (Proposition 1.1) satisfied by prices in equilibrium.

1.2 Equilibrium Existence in the Finite Horizon Case

The aim of this section is to prove the following theorem that characterizes the existence of equilibrium in finite horizon ($\tau < \infty$).

Theorem 1.1 *For an economy $\mathcal{E}_{ex}^{\tau}\left(\mathcal{D}^{\tau}, \mathcal{U}, \mathcal{W}, \mathcal{A}, \mathcal{Y}\right)$ in which*

a. *For all agents $i \in \mathcal{I}$, w^i belongs to $\mathbb{R}_{++}^{\mathcal{D}^{\tau} \times \mathcal{L}}$;*

b. *The utility functions $U^i : X^i \rightarrow \mathbb{R}_{+}$ are continuous , strictly increasing, strictly quasi-concave and $U^i(0) = 0$;*

c. *The collateral vector C_j^{ξ} is different from zero, for all (ξ, j) in $\mathcal{D}^{\tau}(\mathcal{J})$,*

there exists an equilibrium.

Some comments about the hypotheses: Conditions a. and b. are classical in finite horizon incomplete markets models.[2] Condition c. guarantees that the requirements of collateral have nontrivial implications for the wealth of the agents.

The existence of an equilibrium where assets backed by no collateral are not traded can be established without the third hypothesis. In fact, if $C_j^{\xi} \equiv 0$ for a node ξ and an asset $j \in \mathcal{J}(\xi)$, then the security does not deliver any return at the immediate successor nodes $\mu \in \xi^{+}$, and the lenders (or the borrowers) cannot improve their utility through the consumption of the collateral

[2] We say that a function $F : X^i \rightarrow \mathbb{R}_{+}$ is *strictly quasi-concave* if it is quasi-concave and given $x, y \in X^i$ such that $F(x) \neq F(y)$ we have that $F(\lambda x + (1 - \lambda)y) > \min\{F(x), F(y)\}$, for all $\lambda \in (0, 1)$.

bundle. Moreover, it follows from the non-arbitrage condition (see Proposition 1.1, at the end of this section) that, in equilibrium, the price of asset j is zero.

Therefore agents are indifferent to the amount of asset j traded and, to show existence of equilibrium, we may suppose that the asset j is not traded.

For the proof of Theorem 1.1 we need some previous results.

Lemma 1.1 *Under hypotheses a. and c. in Theorem 1.1, an allocation* (x, y, θ, φ) *in* $\left(\mathbb{E}^{\tau}\right)^{\mathcal{I}}$ *that satisfies feasibility conditions of the equilibrium definition in finite horizon is bounded.*

Proof: Let (x, y, θ, φ) be an allocation satisfying the feasibility conditions, then

$$\sum_{i\in\mathcal{I}} \left[x^i(\xi_0) + y^i(\xi_0) + \sum_{j\in\mathcal{J}(\xi_0)} C_j^{\xi_0} \varphi^i(\xi_0, j)\right] = \sum_{i\in\mathcal{I}} w^i(\xi_0). \tag{1.8}$$

Therefore,

$$\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \left[x^i(\xi_0, l) + y^i(\xi_0, l) + \sum_{j\in\mathcal{J}(\xi_0)} C_{j,l}^{\xi_0} \varphi^i(\xi_0, j)\right] = \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} w^i(\xi_0, l) \leq W\mathcal{I}, \tag{1.9}$$

where in the last inequality $W = \max_{\{\xi\in\mathcal{D}^{\tau}, i\in\mathcal{I}\}} ||w^i(\xi)||_{\Sigma}$.

Let $\overline{Y} = \max\{(Y_{\xi}^{\alpha})_{l,l'} : (\alpha, \xi, l, l') \in \{c, s\} \times \mathcal{D}^{\tau} \times \mathcal{L} \times \mathcal{L}\}$. Then it follows from the feasibility conditions that, for all $\xi \in \mathcal{D}^{\tau}$such that $\tilde{t}(\xi) > 0$, the inequality below holds:

$$\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \left[x^i(\xi, l) + y^i(\xi, l) + \sum_{j\in\mathcal{J}(\xi)} C_{j,l}^{\xi} \varphi^i(\xi, j)\right] \tag{1.10}$$

$$\leq W\mathcal{I} + \overline{Y}\mathcal{L} \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \left[x^i(\xi^-, l) + y^i(\xi^-, l) + \sum_{j\in\mathcal{J}(\xi^-)} C_{j,l}^{\xi^-} \varphi^i(\xi^-, j)\right].$$

From the expression above and equation (1.9), we have that for ξ in $\mathcal{D}^{\tau}$ such that $\tilde{t}(\xi) = t$

$$\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \left[x^i(\xi, l) + y^i(\xi, l) + \sum_{j\in\mathcal{J}(\xi)} C_{j,l}^{\xi} \varphi^i(\xi, j)\right] \leq W\mathcal{I} \sum_{k=0}^{t} (\overline{Y}\mathcal{L})^k. \tag{1.11}$$

Now, due to hypothesis c. in Theorem 1.1, we have that $m_\xi = \min_{j\in\mathcal{J}(\xi)} ||C_j^\xi||_\Sigma > 0$, therefore

$$\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} x^i(\xi,l) \leq W\mathcal{I}\sum_{k=0}^{\tau}(\overline{Y}\mathcal{L})^k = \chi < \infty, \tag{1.12}$$

$$\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} y^i(\xi,l) \leq \chi \tag{1.13}$$

$$\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(\xi)} \varphi^i(\xi,j) \leq \frac{\chi}{m_\xi} = \Psi_\xi < \infty. \tag{1.14}$$

The feasibility condition in the portfolios implies that

$$\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(\xi)} \theta^i(\xi,j) \leq \Psi_\xi,\ \forall \xi \in \mathcal{D}^\tau. \tag{1.15}$$

This concludes the proof because $x^i, y^i, \theta^i, \varphi^i$ are positive streams. □

Remark 1.2

i. In this model we do not have a uniform bounded short sales constraint in the event-tree. Nevertheless, there is a uniform borrowing constraint in equilibrium, that is, for each agent i in $\mathcal{I}$, the value of the short sales, $\overline{q}_\xi\overline{\varphi}^i(\xi)$, is uniformly bounded (See the discussion in Section 1.4).

ii. When the securities markets feasibility conditions hold, it follows from the proof above that the aggregate consumption is uniformly bounded along the event-tree. That is,

$$\sum_{i\in\mathcal{I}}\left[x^i(\xi,l) + \sum_{j\in\mathcal{J}(\xi)} C_{j,l}^{B,\xi}\varphi^i(\xi,j) + \sum_{j\in\mathcal{J}(\xi)} C_{j,l}^{L,\xi}\theta^i(\xi,j)\right] \leq \chi. \tag{1.16}$$

for all (ξ,l) in $\mathcal{D}^\tau \times \mathcal{L}$. This is a direct consequence of equation (2.49).

The lemma proven above allows us to bound the consumption and the portfolios in the economy with the aim of proving Theorem 1.1

We achieve this by establishing the existence of equilibrium in a generalized game with a finite set of utility maximizing consumers and auctioneers at each node maximizing the value of the excess demand in the markets.

Therefore, we define the generalized game $\mathcal{G}_{ex}^\tau$ in the following way:

- Given (p,q) in $\mathcal{P}^\tau$, each agent i maximizes V^i in the truncated budget set $\mathcal{B}_{ex}^{\tau,i}(p,q,2\Psi,2\chi)$, where

$$\mathcal{B}_{ex}^{\tau,i}(p,q,2\Psi,2\chi) = \left\{ \begin{array}{rl} (x,y,\theta,\varphi) \in \mathcal{B}_{ex}^{\tau,i}(p,q) & : x(\xi,l) \leq 2\chi,\ y(\xi,l) \leq 2\chi,\ \theta(\xi,j) \leq 2\Psi_\xi \\ & \varphi(\xi,j) \leq 2\Psi_\xi \quad \forall (l,\xi,j) \in \mathcal{L} \times \mathcal{D}^\tau \times \mathcal{J}. \end{array} \right\}.$$

- Given an allocation $\left(x^i, y^i, \theta^i, \varphi^i\right)_{i\in\mathcal{I}}$, the auctioneer at the node ξ_0 chooses (p_{ξ_0}, q_{ξ_0}) in $\Delta_+^{\mathcal{L}+\mathcal{J}(\xi_0)-1}$ in order to maximize

$$p_{\xi_0} \sum_{i\in\mathcal{I}} \left[x^i(\xi_0) + y^i(\xi_0) + C^{\xi_0}\varphi^i(\xi_0) - w^i(\xi_0)\right] + q_{\xi_0} \sum_{i\in\mathcal{I}} Z^i(\xi_0).$$

- Given an allocation $\left(x^i, y^i, \theta^i, \varphi^i\right)_{i\in\mathcal{I}}$, the auctioneer at the node ξ, with $\xi > \xi_0$ and $b_\xi^\tau > 0$, chooses $(p_\xi, q_\xi) \in \Delta_+^{\mathcal{L}+\mathcal{J}(\xi)-1}$ in order to maximize

$$p_\xi \sum_{i\in\mathcal{I}} \left[x^i(\xi) + y^i(\xi) + C^\xi\varphi^i(\xi) - w^i(\xi) - Y_\xi^c x^i(\xi^-) - Y_\xi C^{\xi^-}\varphi^i(\xi^-) - Y_\xi^s y^i(\xi^-)\right] + q_\xi \sum_{i\in\mathcal{I}} Z^i(\xi).$$

- There is one auctioneer for each terminal node of $\mathcal{D}^\tau$ such that, given $\left(x^i, y^i, \theta^i, \varphi^i\right)_{i\in\mathcal{I}}$, his objective is to maximize

$$p_\xi \sum_{i\in\mathcal{I}} \left[x^i(\xi) - w^i(\xi) - Y_\xi^c x^i(\xi^-) - Y_\xi C^{\xi^-}\varphi^i(\xi^-) - Y_\xi^s y^i(\xi^-)\right].$$

So, an equilibrium for the generalized game $\mathcal{G}_{ex}^\tau$ is a vector $\left[(\overline{x}, \overline{y}, \overline{\theta}, \overline{\varphi}), (\overline{p}, \overline{q})\right]$ in $\left(\mathbb{E}^\tau\right)^{\mathcal{I}} \times \mathcal{P}^\tau$ which solves the four items stated above.

Lemma 1.2 *If there exists an equilibrium for the generalized game $\mathcal{G}_{ex}^\tau$ then there exists an equilibrium for the economy $\mathcal{E}_{ex}^\tau$.*

Proof: See the Appendix.

In order to show that there is equilibrium in $\mathcal{E}_{ex}^\tau$ it is enough to guarantee that the generalized game $\mathcal{G}_{ex}^\tau$ has equilibrium.

Lemma 1.3 *Under the hypothesis of Theorem 1 there exists a pure strategies equilibrium for the generalized game $\mathcal{G}_{ex}^\tau$.*

Proof: This result follows from the equilibrium existence theorem in a generalized game of Debreu (1952). In fact, the objective functions of the agents are continuous and quasi-concave in their strategies. Furthermore, the objective functions of the auctioneers are continuous and linear in their own strategies, and therefore quasi-concave.

The correspondence of admissible strategies, for the agents and for the auctioneers, has compact domain and compact-, convex- and nonempty-values. Such correspondences are upper semi-continuous, because it has compact values and a closed graph. The lower semi-continuity of interior correspondences follows from hypothesis a. in Theorem 1.1 (see Hildenbrand (1974, ch. II.1.2)). Because the closure of a lower semi-continuity correspondence is also lower semi-continuous, the continuity of these set functions is guaranteed. We can apply Kakutani's fixed point theorem to the correspondence of optimal strategies in order to find the equilibrium. □

We have shown the equilibrium existence theorem for the economy with finite horizon $\mathcal{E}_{ex}^{\tau}$. Now, we shall give a *non-arbitrage condition* that holds when the commodities and assets prices are in equilibrium. This condition has already appeared in the literature in Dubey, Geanakoplos, and Zame (1995) and is very important for our proof of the equilibrium existence in the infinite horizon case.

Proposition 1.1 *Let $(\overline{p}, \overline{q}) \in \mathcal{P}^{\tau}$ be an equilibrium price vector, then for all pairs (ξ, j) in the set $\mathcal{D}^{\tau}(\mathcal{J})$*

$$\overline{p}_{\xi} C_j^{\xi} - \overline{q}_{\xi,j} \geq 0.$$

Moreover, if $C_j^{B,\xi} \neq 0$ then the inequality above is strict.

Proof: See the Appendix.

Proposition 1.1 states that in equilibrium agents cannot borrow more resources, by selling an asset, than the value of the associated collateral bundle . In the case that $C_j^{B,\xi} \neq 0$, this avoids the possibility of an agent making unbounded short sales at a node. Nevertheless, the non-arbitrage condition is not enough to guarantee that the value of the loan is a bounded sequence (that is, that $\left(\overline{q}_{\xi} \overline{\varphi}_{\xi}^{i}\right) \in l^{\infty}\left(\mathcal{D}^{\tau}\right)$). Feasibility, in durable goods markets, is also required for this purpose (see Remark 1.4 below).

Moreover, the non-arbitrage condition implies that the vector of equilibrium commodity prices $(\overline{p}_\xi)$ is bounded from below, node by node. In fact, since $(\overline{p}_\xi, \overline{q}_\xi) \in \Delta_+^{\mathcal{L}+\mathcal{J}(\xi)+1}$ we have that

$$||\overline{p}_\xi||_\Sigma \geq \left(1 + \max_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}(\xi)} C_{j,l}^{\xi}\right)^{-1}. \tag{1.17}$$

1.3 Existence of Equilibria in $\mathcal{E}_{ex}^{\infty}$

In this section we establish the existence of equilibrium in infinite horizon economies by truncating $\mathcal{E}_{ex}^{\infty}$ to finite horizon and using the results already obtained.

Given an economy $\mathcal{E}_{ex}^{\infty}(\mathcal{D}^\infty, \mathcal{U}, \mathcal{W}, \mathcal{A}, \mathcal{Y})$, we define the *truncated economy* $\mathcal{E}_{ex}^{T}$ as a finite horizon economy with T+1 periods such that:

$$\begin{aligned} &\mathcal{D}^T = \left\{\xi \in \mathcal{D}^\infty \,:\, \tilde{t}(\xi) \leq T\right\}, &&U^{T,i}\left(\left(x(\xi)\right)_{\{\xi \in \mathcal{D}^T\}}\right) = U^i\left(\left(x(\xi)\right)_{\xi \in \mathcal{D}^T}; 0\right),\\ &w^{T,i} = \left(w^i(\xi) \,:\, \xi \in \mathcal{D}^T\right), &&\mathcal{A}^T = \mathcal{A}|_{\mathcal{D}^{T-1}},\\ &\mathcal{Y}^T = \mathcal{Y}|_{\mathcal{D}^T}. \end{aligned}$$

Now, suppose that the economy $\mathcal{E}_{ex}^{\infty}$ satisfies

A. For any agent i in $\mathcal{I}$, w^i belongs to $X_+^i = \mathbb{R}_{++}^{\mathcal{D}^\infty \times \mathcal{L}}$ and there exists $\overline{w}$ in $\mathbb{R}_{++}$, such that $||w^i(\xi)||_\Sigma \leq \overline{w}$ for all $\xi \in \mathcal{D}^\infty$;

B. The utility functions $U^i : X^i \to \mathbb{R}_+ \cup \{\infty\}$ are time- and state-separable in the setting

$$U^i(x) = \sum_{\xi \in \mathcal{D}^\infty} u_i\big(\xi, x(\xi)\big),$$

and are *finite* for all x in $l_+^\infty(\mathcal{D}^\infty \times \mathcal{L}) \subset X^i$. The function $u_i(\xi, \cdot) : \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}_+$ satisfies $u_i(\xi, 0) = 0$, is continuous, strictly increasing and concave for all ξ in $\mathcal{D}^\infty$;

C. The collateral vector C_j^ξ is different from zero for all (ξ, j) in $\mathcal{D}^\infty(\mathcal{J})$;

D. The structure of depreciation in the event-tree is $[Y_\xi^c, Y_\xi^s] \equiv \big[diag\big[a_l(\xi)\big], diag\big[b_l(\xi)\big]\big]_{l \in \mathcal{L}}$, and there is a scalar $\kappa \in (0,1)$ such that $\max_{l \in \mathcal{L}} \{a_l(\xi), b_l(\xi)\} < \kappa$ for all ξ in $\mathcal{D}^\infty$.

From the former section, we know that there exists an equilibrium $\big[(x^T, y^T, \theta^T, \varphi^T); (p^T, q^T)\big]$ for the truncated economy $\mathcal{E}_{ex}^T$ for all $T \in \mathbb{N}$.

Remark 1.3 Conditions A, B and C are analogous to the finite horizon hypotheses. Condition D guarantees a strong depreciation along the event-tree. This implies that the total endowment accumulated until each node is uniformly bounded in the event-tree.

In the subsection below we prove, using a sequence of finite horizon equilibria with increasing terminal periods: $\left[(x^T, y^T, \theta^T, \varphi^T); (p^T, q^T)\right]_{T\in \mathbb{N}}$, the main result of this work that guarantees the equilibrium existence in $\mathcal{E}^{\infty}_{ex}$. That is:

Theorem 1.2 *For an economy $\mathcal{E}^{\infty}_{ex}$ satisfying hypotheses A, B, C and D there exists an equilibrium.*

Remark 1.4 On Bounds on Short Sales. As mentioned above in Remark 1.2, we do not have a uniformly bounded short sales constraint in the economy $\mathcal{E}^{\infty}_{ex}$. Thus, given equilibrium prices, agents have the possibility of allocating their wealth in portfolios that are not uniformly bounded along the event-tree.

In fact, consider for instance an economy $\mathcal{E}^{\infty}_{ex}$ characterized by an event-tree $\mathcal{D}^{\infty}$ in which there are only two branches at each node, that is, $\xi^+ = \{\xi_u, \xi_d\}$ for any node ξ; only one commodity $\mathcal{L} = \{l\}$ is negotiated at each node; and having two agents $\mathcal{I} = \{A, B\}$ with endowments $w^A(\xi_0) = w^B(\xi_0) = w^A(\xi_d) = w^B(\xi_d) = 1$; $w^A(\xi_u) = w^B(\xi_u) = 1 - 2^{-(\tilde{t}(\xi)+2)}$ and utility functions $\{U^A, U^B\}$ satisfying the hypotheses of Theorem 1.2. Given ξ in $\mathcal{D}^{\infty}$, the depreciation structure is characterized by $Y_{\xi_u} = \left(Y^s_{\xi_u}, Y^c_{\xi_u}\right) = (0, 0.5)$ and $Y_{\xi_d} = \left(Y^s_{\xi_d}, Y^c_{\xi_d}\right) = (0, 0)$.
At each node ξ there is only one asset j_ξ for trading, with returns given by $A(\xi_u, j_\xi) = 2^{-(\tilde{t}(\xi)+2)}$, $A(\xi_d, j_\xi) = 1$. The collateral requirements for such an asset are $C^{\xi}_j = C^{B,\xi}_j = 2^{-\tilde{t}(\xi)}$.

By Theorem 1.2, there exists an equilibrium. Therefore, given equilibrium prices $(\overline{p}, \overline{q}) \in \mathcal{P}^{\infty}$, both agents can choose the portfolio $(\tilde{\theta}, \tilde{\varphi}) = (0, \tilde{\varphi})$, where

$$\tilde{\varphi}(\xi) = \frac{\overline{p}_\xi}{\overline{p}_\xi C^{\xi}_j - \overline{q}_\xi} = \frac{1}{(1 + \frac{1}{2^{\tilde{t}(\xi)}}) - \frac{1}{\overline{p}_\xi}}. \tag{1.18}$$

Now, because $(\overline{p}_\xi, \overline{q}_\xi)$ belongs to $\Delta^{\mathcal{L}+\mathcal{J}(\xi)-1}_{+}$, Proposition 1.1 guarantees that $\overline{p}_\xi$ tends to one as $\tilde{t}(\xi)$ tends to infinity. This implies that

$$\tilde{\varphi}(\xi) \longrightarrow \infty \quad \text{as} \quad \tilde{t}(\xi) \to \infty. \tag{1.19}$$

Thus, we have established an example in which, at equilibrium prices $(\overline{p}, \overline{q})$, the agents can choose a portfolio allocation that is not uniformly bounded in the event-tree.

However, under hypotheses A, C and D, equation (1.9) implies that the *feasible allocations* (x, y, θ, φ) satisfy

$$\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}}\Big[x^i(\xi,l)+y^i(\xi,l)+\sum_{j\in\mathcal{J}(\xi)}C^{\xi}_{j,l}\varphi^i(\xi,j)\Big] \tag{1.20}$$

$$\leq \overline{w}\mathcal{I}+\kappa\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}}\Big[x^i(\xi^-,l)+y^i(\xi^-,l)+\sum_{j\in\mathcal{J}(\xi^-)}C^{\xi^-}_{j,l}\varphi^i(\xi^-,j)\Big],$$

for all ξ in $\mathcal{D}^{\infty}$. Therefore, analogous to the finite horizon case, the aggregate demand of commodities is uniformly bounded in the event-tree. That is, for any node ξ in the economy, we have

$$\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}}\Big[x^i(\xi,l)+y^i(\xi,l)+\sum_{j\in\mathcal{J}(\xi)}C^{\xi}_{j,l}\varphi^i(\xi,j)\Big]\leq\overline{w}\mathcal{I}\sum_{s=0}^{\infty}\kappa^s=\frac{\overline{w}\mathcal{I}}{1-\kappa}<\infty. \tag{1.21}$$

It follows from the equation above that, node by node, there is a bounded short sales constraint

$$\sum_{i\in\mathcal{I}}\varphi^i(\xi,j)\leq\frac{1}{||C^{\xi}_j||_{\Sigma}}\frac{\overline{w}\mathcal{I}}{(1-\kappa)}<\infty. \tag{1.22}$$

If collateral coefficients were uniformly bounded from below by a strictly positive scalar, we could find a uniform upper bound on feasible allocations. But we have no need to impose such an assumption.

Finally, note that in equilibrium, the value of short sales, $\overline{q}_{\xi}\overline{\varphi}^i(\xi)$, is uniformly bounded. In fact, it follows from Proposition 1.1 that

$$\overline{q}_{\xi}\overline{\varphi}^i(\xi)\leq\overline{p}_{\xi}C^{\xi}\overline{\varphi}^i(\xi)\leq||C^{\xi}\overline{\varphi}^i(\xi)||_{\Sigma}\leq\frac{\overline{w}\mathcal{I}}{(1-\kappa)}, \tag{1.23}$$

where the last inequality is a consequence of equation (1.21).

In a seminal paper, Magill and Quinzii (1994) established the existence of equilibria for an economy with transversality conditions imposed a priori on agents' budget sets. These equilibria were also shown to be in one to one correspondence with equilibria of an economy whose budget sets included an implicit debt constraint of the form $(qZ)\in l^{\infty}(\mathcal{D}^{\infty}(\mathcal{J}))$. An equilibrium with implicit debt constraint is also an equilibrium with an explicit constraint of the form $q_{\xi}Z_{\xi}\geq -M$. Moreover, it is possible to choose M so that the explicit constraint is not binding at this equilibrium (in fact, given an equilibrium with implicit debt constraint with debt value $(\overline{q}\overline{Z})$, it is sufficient to take $M>||\overline{q}\overline{Z}||_{\infty}$).

In our context, the collateral structure dispenses with a priori transversality conditions or debt constraints, but there is, nevertheless, a debt constraint (given by inequality (1.23)) which any feasible allocation must satisfy, at non-arbitrage prices.

1.3.1 Proof of Theorem 1.2

Consider a sequence of equilibria with increasing terminal periods: $\left[(x^T,y^T,\theta^T,\varphi^T);(p^T,q^T)\right]_{T\in I\!N}$. To shorten the inequalities below, define

$$M_\xi^T(x,y,\theta,\varphi) \equiv p_\xi^T x(\xi)+p_\xi^T y(\xi)+p_\xi^T C^\xi\varphi(\xi)+q_\xi^T Z(\xi), \tag{1.24}$$

$$L_\xi^{T,i}(x,y,\theta,\varphi) \equiv M_\xi^T(x^{T,i},y^{T,i},\theta^{T,i},\varphi^{T,i})-M_\xi^T(x,y,\theta,\varphi), \tag{1.25}$$

$$L_\xi^{T,i} \equiv L_\xi^{T,i}(0,0,0,0), \tag{1.26}$$

$$S_\xi^T(x,y,\theta,\varphi) \equiv p_\xi^T\left\{Y_\xi^c x(\xi^-)+Y_\xi C^{\xi^-}\varphi(\xi^-)+Y_\xi^s y(\xi^-)\right\}+D^\xi Z(\xi^-), \tag{1.27}$$

$$R_\xi^{T,i}(x,y,\theta,\varphi) \equiv S_\xi^T(x^{T,i},y^{T,i},\theta^{T,i},\varphi^{T,i})-S_\xi^T(x,y,\theta,\varphi), \tag{1.28}$$

$$R_\xi^{T,i} \equiv R_\xi^{T,i}(0,0,0,0), \tag{1.29}$$

$$V^{T,i}(x,y,\theta,\varphi) \equiv U^{T,i}\left[\left(x(\xi)+C^{B,\xi}\varphi(\xi)+C^{L,\xi}\theta(\xi)\right)_{b_\xi^T>0},\left(x(\xi)\right)_{b_\xi^T=0}\right], \tag{1.30}$$

$$V^{i}(x,y,\theta,\varphi) \equiv U^{i}\left[\left(x(\xi)+C^{B,\xi}\varphi(\xi)+C^{L,\xi}\theta(\xi)\right)_{\xi\in\mathcal{D}^\infty}\right]. \tag{1.31}$$

The next result follows from Kuhn-Tucker's theorem and Slater's Condition for the agent's maximization problem (see Avriel (1976)):

Lemma 1.4 *Given an equilibrium* $\left[(x^T,y^T,\theta^T,\varphi^T),(p^T,q^T)\right]$ *for the truncated economy* $\mathcal{E}_{ex}^T$, *there exist Lagrange multipliers* $\left(\mu_\xi^{Ti}\right)_{\{\xi\in\mathcal{D}^T\}}\in\mathbb{R}_+^{\mathcal{D}^T}$ *for each* i *in* $\mathcal{I}$ *such that*

$$\mu_{\xi_0}^{Ti}L_{\xi_0}^{T,i}=\mu_{\xi_0}^{Ti}p_{\xi_0}^T w^i(\xi_0)\ ; \tag{1.32}$$

$$\mu_\xi^{Ti}\{L_\xi^{T,i}-R_\xi^{T,i}\}=\mu_\xi^{Ti}p_\xi^T w^i(\xi)\quad\forall\xi\in\mathcal{D}^T:\xi>\xi_0. \tag{1.33}$$

Moreover, for every (x,y,θ,φ) *in the state-space* $I\!E^T$, *we have:*

$$V^{T,i}(x,y,\theta,\varphi)-V^{T,i}(x^{T,i},y^{T,i},\theta^{T,i},\varphi^{T,i})\le-\sum_{\xi\ge\xi_0}\mu_\xi^{Ti}L_\xi^{T,i}(x,y,\theta,\varphi)+\sum_{\xi>\xi_0}\mu_\xi^{Ti}R_\xi^{T,i}(x,y,\theta,\varphi).$$

Observation: In a strict sense, the Kuhn-Tucker conditions for the consumer problem in the economy $\mathcal{E}_{ex}^T$ include the Lagrange multipliers associated to sign restrictions ($x(\xi,l)\ge0$, $y(\xi,l)\ge0$, $\theta(\xi,j)\ge0$, $\varphi(\xi,j)\ge0$) in the budget set.

Nevertheless, this does not affect the validity of the equations above because we work with vectors in the state-space $I\!E^T$.

Note that Lemma 1.4 holds without any hypothesis about *differentiability* of the utility functions (See Theorem 4.41 in Avriel(1976)).

The following result is a direct consequence of the above lemma and its proof is in the Appendix.

Lemma 1.5 *For all ξ in $\mathcal{D}^T$ such that $b_\xi^T = 0$, we have*

$$u^i(\xi, x(\xi)) - u^i(\xi, x^{T,i}(\xi)) \leq \mu_\xi^{Ti} p_\xi^T (x(\xi) - x^{T,i}(\xi)). \tag{1.34}$$

Moreover, if $b_\xi^T > 0$ then

$$\mu_\xi^{Ti} L_\xi^{T,i} \leq u^i\big(\xi, x^{T,i}(\xi) + C^{B,\xi}\varphi^{T,i}(\xi) + C^{L,\xi}\theta^{T,i}(\xi)\big) + \sum_{\eta \in \xi^+} \mu_\eta^{Ti} \{L_\eta^{T,i} - p_\eta^T w^i(\eta)\}. \tag{1.35}$$

Because $(x^{T,i}, y^{T,i}, \theta^{T,i}, \varphi^{T,i})$ is an equilibrium allocation for the agent i in $\mathcal{E}_{ex}^T$, it is bounded. Then there exists β in $\mathbb{R}_{++}^{\mathcal{L}}$ such that $x^{T,i}(\xi) + C^{B,\xi}\varphi^{T,i}(\xi) + C^{L,\xi}\theta^{T,i}(\xi) \leq \beta$, for all ξ in $\mathcal{D}^T$. This bound does not depend on T or i because of the hypotheses A and D.[3] So it follows from Lemma 1.5 and hypothesis B that, for all ξ in $\mathcal{D}^T$,

$$\mu_\xi^{Ti} L_\xi^{T,i} \leq \sum_{\{\xi' \in \mathcal{D}^T; \xi' \geq \xi\}} u^i(\xi', \beta). \tag{1.36}$$

From Lemma 1.4 it follows that

$$\mu_\xi^{Ti} L_\xi^{T,i} = \mu_\xi^{Ti} p_\xi^T w^i(\xi) + \mu_\xi^{Ti} R_\xi^{T,i},$$

where $\mu_\xi^{Ti} R_\xi^{T,i} \geq 0$. Thus, from hypothesis A and hypothesis B, we conclude that

$$\mu_\xi^{Ti} \|p_\xi^T\|_\Sigma \min_{(l,i) \in \mathcal{L} \times \mathcal{I}} w^i(\xi, l) \leq \sum_{\{\xi' \in \mathcal{D}^\infty : \xi' \geq \xi\}} u^i(\xi', \beta). \tag{1.37}$$

So, follows from equation (1.17) that, for all ξ in $\mathcal{D}^T$, we have

$$\mu_\xi^{Ti} \leq \frac{\left(1 + \max_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}(\xi)} C_{j,l}^\xi\right) \max_{i \in \mathcal{I}} \sum_{\{\xi' \in \mathcal{D}^\infty : \xi' \geq \xi\}} u^i(\xi', \beta)}{\min_{(l,i) \in \mathcal{L} \times \mathcal{I}} w^i(\xi, l)} < \infty. \tag{1.38}$$

Observe that the latter bound depends neither on T nor on $i \in \mathcal{I}$. This proves the following:

Lemma 1.6 *The sequence $\left\{\mu_\xi^{Ti},\ T \in I\!N,\ T \geq \tilde{t}(\xi),\ i \in \mathcal{I}\right\}$ of marginal utilities of endowments income at node ξ of the event-tree is uniformly bounded.*

[3]This fact is a consequence of equation (1.21) and market clear conditions.

Define the set $I\!F(\xi) = \mathbb{R}_+^{\mathcal{L}\times\mathcal{I}} \times \mathbb{R}_+^{\mathcal{L}\times\mathcal{I}} \times \mathbb{R}_+^{\mathcal{J}(\xi)\times\mathcal{I}} \times \mathbb{R}_+^{\mathcal{J}(\xi)\times\mathcal{I}} \times \Delta_+^{\mathcal{L}+\mathcal{J}(\xi)-1} \times \mathbb{R}_+^{\mathcal{I}}$.
It follows from equation (1.21), hypotheses C and D and the lemma above that the sequence

$$\left\{\left(x^T(\xi), y^T(\xi), \theta^T(\xi), \varphi^T(\xi), p_\xi^T, q_\xi^T, \mu_\xi^T\right)\right\}_{T>\tilde{t}(\xi)} \subset I\!F(\xi)$$

is uniformly bounded for each ξ in $\mathcal{D}^\infty$. Countability of $\mathcal{D}^\infty$ implies that there exists an order in its nodes $\{\xi_1, \xi_2, \dots\}$. So we know that there is a subsequence $\{T_k^1\}_{k\in I\!N} \subset I\!N$ such that

$$\lim_{k\to\infty} \left\{\left(x^{T_k^1}(\xi_1), y^{T_k^1}(\xi_1), \theta^{T_k^1}(\xi_1), \varphi^{T_k^1}(\xi_1), p_{\xi_1}^{T_k^1}, q_{\xi_1}^{T_k^1}, \mu_{\xi_1}^{T_k^1}\right)\right\}$$

exists in $I\!F(\xi_1)$. In the same way, there exists a subsequence $\{T_k^2\}_{k\in I\!N} \subset \{T_k^1\}_{k\in I\!N}$ such that the sequence is convergent in ξ_2. Repeating this process throughout $\mathcal{D}^\infty = \{\xi_1, \xi_2, \dots\}$ we obtain a sequence of families $\{T_k^1\}_{k\in I\!N} \supseteq \{T_k^2\}_{k\in I\!N} \supseteq \dots$ such that

$$\lim_{k\to\infty} \left\{\left(x^{T_k^s}(\xi_s), y^{T_k^s}(\xi_s), \theta^{T_k^s}(\xi_s), \varphi^{T_k^s}(\xi_s), p_{\xi_s}^{T_k^s}, q_{\xi_s}^{T_k^s}, \mu_{\xi_s}^{T_k^s}\right)\right\}.$$

exists in $I\!F(\xi_s)$.

Define the sequence $\{T_k\}_{k\in I\!N}$ as $T_k = T_k^k$. Then, for a natural number s, we have $\{T_k\}_{k\geq s} \subseteq \{T_k^s\}_{k\in I\!N}$ and it is an infinite set. We obtain that

$$\lim_{k\to\infty} \left\{\left(x^{T_k}(\xi_s), y^{T_k}(\xi_s), \theta^{T_k}(\xi_s), \varphi^{T_k}(\xi_s), p_{\xi_s}^{T_k}, q_{\xi_s}^{T_k}, \mu_{\xi_s}^{T_k}\right)\right\}$$

exists for all s $\in I\!N$. So we have found a sequence $\{T_k\}_{k\in I\!N}$ such that

$$\left\{\left(x^{T_k}, y^{T_k}, \theta^{T_k}, \varphi^{T_k}, p^{T_k}, q^{T_k}, \mu^{T_k}\right)\right\} \in I\!F,$$

where $I\!F = l_+^\infty(\mathcal{D}^\infty \times \mathcal{L} \times \mathcal{I}) \times l_+^\infty(\mathcal{D}^\infty \times \mathcal{L} \times \mathcal{I}) \times \mathbb{R}_+^{\mathcal{D}^\infty(\mathcal{J})\times\mathcal{I}} \times \mathbb{R}_+^{\mathcal{D}^\infty(\mathcal{J})\times\mathcal{I}} \times \mathcal{P}^\infty \times \mathbb{R}_+^{\mathcal{D}^\infty\times\mathcal{I}}$, converges when k goes to infinity to some allocation $(\overline{x}, \overline{y}, \overline{\theta}, \overline{\varphi}, \overline{p}, \overline{q}, \overline{\mu}) \in I\!F$.

Lemma 1.7 *The allocation* $\left[(\overline{x}, \overline{y}, \overline{\theta}, \overline{\varphi}), (\overline{p}, \overline{q})\right]$ *is an equilibrium for* $\mathcal{E}_{ex}^\infty$.

Proof: The feasibility conditions follow directly from the fact that $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ is the limit of $\left(x^{T_k,i}, y^{T_k,i}, \theta^{T_k,i}, \varphi^{T_k,i}\right)$ as k goes to infinity. Because $\left(x^{T_k,i}, y^{T_k,i}, \theta^{T_k,i}, \varphi^{T_k,i}\right)$ is a $\mathcal{E}_{ex}^{T_k}$-equilibrium it satisfies the feasibility conditions, and so does its limit. We have to show the optimality of $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ in $\mathcal{B}_{ex}^{\infty,i}(\overline{p}, \overline{q})$.

Suppose, by contradiction, that there is $\left(x, y, \theta, \varphi\right)$ in $\mathcal{B}_{ex}^{\infty,i}(\overline{p}, \overline{q})$ and $\delta \in \mathbb{R}_{++}$ such that

$$V^i\left(x, y, \theta, \varphi\right) - V^i\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right) \geq \delta > 0. \tag{1.39}$$

We claim that for all $\xi \in \mathcal{D}^{\infty}$ we have:

$$u^i\Big(\xi, x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\Big) - u^i\Big(\xi, \overline{x}^i(\xi) + C^{B,\xi}\overline{\varphi}^i(\xi) + C^{L,\xi}\overline{\theta}^i(\xi)\Big)$$

$$\leq \lim_{k\to\infty}\Big\{ - \mu_\xi^{T_k,i}L_\xi^{T_k,i}(x,y,\theta,\varphi) + \sum_{\eta\in\xi^+}\mu_\eta^{T_k,i}R_\eta^{T_k,i}(x,y,\theta,\varphi)\Big\}.$$

In fact, given $\xi \in \mathcal{D}^{\infty}$ there exists a natural number k such that $\tilde{t}(\xi) < T_k$, then applying Lemma 1.4 to $\mathcal{E}_{ex}^{T_k}$ the claim follows from taking the pointwise limit as k tends to infinity. From this inequality, given $N \in I\!N$, we have

$$\sum_{\{\xi, 0\leq\tilde{t}(\xi)\leq N\}}\left(u^i\Big(\xi, x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\Big) - u^i\Big(\xi, \overline{x}^i(\xi) + C^{B,\xi}\overline{\varphi}^i(\xi) + C^{L,\xi}\overline{\theta}^i(\xi)\Big)\right)$$

$$\leq \lim_{k\to\infty}\left\{ - \sum_{\{\xi, 0\leq\tilde{t}(\xi)\leq N\}}\mu_\xi^{T_k,i}L_\xi^{T_k,i}(x,y,\theta,\varphi) + \sum_{\{\xi, 1\leq\tilde{t}(\xi)\leq N+1\}}\mu_\xi^{T_k,i}R_\xi^{T_k,i}(x,y,\theta,\varphi)\right\}.$$

By taking the pointwise limit in equations (1.32) and (1.33) of Lemma 1.4, we obtain:

$$\lim_{k\to\infty}\mu_{\xi_0}^{T_k,i}L_{\xi_0}^{T_k,i} = \overline{\mu}_0^i\overline{p}_{\xi_0}w^i(\xi_0), \tag{1.40}$$

$$\lim_{k\to\infty}\mu_\xi^{T_k,i}\{L_\xi^{T_k,i} - R_\xi^{T_k,i}\} = \overline{\mu}_\xi^i\overline{p}_\xi w^i(\xi). \tag{1.41}$$

So from $(x,y,\theta,\varphi) \in \mathcal{B}_{ex}^{\infty,i}(\overline{p},\overline{q})$ it follows that:

$$\sum_{\{\xi, 0\leq\tilde{t}(\xi)\leq N\}}\left(u^i\Big(\xi, x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\Big) - u^i\Big(\xi, \overline{x}^i(\xi) + C^{B,\xi}\overline{\varphi}^i(\xi) + C^{L,\xi}\overline{\theta}^i(\xi)\Big)\right)$$

$$\begin{aligned}
&\leq \lim_{k\to\infty}\sum_{\{\xi,\tilde{t}(\xi)=N+1\}}\mu_\xi^{T_k,i}R_\xi^{T_k,i}(x,y,\theta,\varphi)\\
&\leq \lim_{k\to\infty}\sum_{\{\xi,\tilde{t}(\xi)=N+1\}}\mu_\xi^{T_k,i}R_\xi^{T_k,i}\\
&\leq \lim_{k\to\infty}\sum_{\{\xi,\tilde{t}(\xi)=N+1\}}\mu_\xi^{T_k,i}L_\xi^{T_k,i}\\
&\leq \sum_{\{\xi,\tilde{t}(\xi)\geq N+1\}}u^i(\xi,\beta),
\end{aligned}$$

where the last inequality follows by taking the pointwise limit in the inequality (1.36).

Because $V^i(x,y,\theta,\varphi) = \sum_{\xi \in \mathcal{D}^\infty} u^i\Big(\xi, x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\Big)$ there is a N^* in $I\!N$ such that

$$\sum_{\{\xi, \tilde{t}(\xi) \le N\}} u^i\Big(\xi, x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\Big) - V^i(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i) \ge \frac{\delta}{2}, \qquad \text{for all } N \ge N^*.$$

Therefore,

$$\frac{\delta}{2} \le \sum_{\{\xi, \tilde{t}(\xi) \ge N+1\}} u^i(\xi, \beta), \qquad \text{for all } N \ge N^*. \tag{1.42}$$

We obtain a contradiction with equation (1.39), because hypothesis B guarantees that

$$\lim_{N \to \infty} \sum_{\{\xi, \tilde{t}(\xi) \ge N+1\}} u^i(\xi, \beta) = 0. \tag{1.43}$$

We conclude that $\left(\overline{x}^i, \overline{y}^i, \overline{\varphi}^i, \overline{\theta}^i\right)$ is optimal in $\mathcal{B}_{ex}^{\infty,i}(\overline{p}, \overline{q})$. □

Remark 1.5 On Transversality Conditions. In the incomplete markets literature, the existence of equilibrium in infinite horizon economies has been guaranteed through debt constraints or transversality conditions in the agent's budget set, in order to limit the value of short sales and hence to prevent Ponzi schemes. In our model, it is not necessary to restrict the agent's budget set because the collateral structure together with feasibility limits the asymptotic explosion of the debt.

Nevertheless, at equilibria whose existence was established in Theorem 1.2 we can guarantee the validity of some transversality conditions. In fact, we have

- The discounted value at a node ξ of the total loans given by the agent $i \in \mathcal{I}$ at the period T converges to zero as T goes to infinity:

$$\lim_{T \to \infty} \sum_{\{\xi' \ge \xi\, ;\, \tilde{t}(\xi') = T\}} \overline{\mu}^i_{\xi'} \overline{q}_{\xi'} \overline{\theta}^i(\xi') = 0. \tag{1.44}$$

- Although for each period T the discounted value, at a node ξ, of the total collateral established by the agent $i \in \mathcal{I}$ is greater than or equal to the discounted value of the total resources borrowed by the agent, when T goes to infinity these amounts become equal:

$$\lim_{T \to \infty} \sum_{\{\xi' \ge \xi\, ;\, \tilde{t}(\xi') = T\}} \overline{\mu}^i_{\xi'} \left(\overline{p}_{\xi'} C^{\xi'} - \overline{q}_{\xi'}\right) \overline{\varphi}^i(\xi') = 0. \tag{1.45}$$

Condition (1.44) says that no agent wants to be a lender at infinity. As Magill and Quinzii (1994) remarked, this is an uncontroversial condition: if this condition were not satisfied, agent i could find a preferred consumption stream by decreasing his lending. The analogous condition for borrowing is more controversial, particularly when imposed a priori in agents' budget sets, as Magill and Quinzii (1994) recognized: to justify that agent i prevents himself from not being a borrower at infinity, we would have to argue that he does not expect to find counterparts wishing to be lenders at infinity, for the very same present value vectors $(\overline{\mu}_{\xi}^{i})$ that i uses in condition (1.44). In our context, the transversality condition on borrowing holds under an assumption on endowments which is not necessary for the existence of equilibria.

In fact, if we had assumed that there is a positive scalar $\underline{w}$ such that $w^i(\xi,l) \geq \underline{w}$, for all (ξ,l,i) in $\mathcal{D}^{\infty} \times \mathcal{L} \times \mathcal{I}$., then the discounted value at ξ of the total resources borrowed by agent $i \in \mathcal{I}$ at period T goes to zero as T tends to infinity:

$$\lim_{T\to\infty} \sum_{\{\xi' \geq \xi\,;\, \tilde{t}(\xi')=T\}} \overline{\mu}_{\xi'}^{i} \overline{q}_{\xi'} \overline{\varphi}^{i}(\xi') = 0. \tag{1.46}$$

Recall that, in equilibrium, the sequence of debt values $(\overline{q}\overline{Z})$ is bounded, but the personalized state prices $(\overline{\mu}_{\xi}^{i})$ (which can be interpreted as marginal utilities of endowment income) may explode if agent i's endowments went to zero fast enough. By assumption B, an agent's utility evaluated at aggregate endowment is finite. Therefore, Lagrange multipliers would explode if agent i's endowments were to go to zero at a higher rate than the rate of convergence of his utilities evaluated at aggregate endowments.

Notice that equations (1.44) and (1.46) guarantee that the discounted value of agent i's portfolio at period t goes to zero as time goes to infinity, which was the Magill and Quinzii (1994) transversality condition. These facts follow from equations (1.21), (1.36) and the non-arbitrage condition (see Proposition 1.1). The proofs are contained in the Appendix.

Remark 1.6 A Social Welfare Property. We now present a social welfare property fulfilled by equilibrium allocations in $\mathcal{E}_{ex}^{\tau}$. Dubey, Geanakoplos, and Zame (1995) showed that in a two period model with collateral it is not possible to improve the social welfare by means of a tax and subsidy intervention in the initial period, keeping spot markets cleared, at the equilibrium prices. It is not difficult to prove an extension of this result for the economy $\mathcal{E}_{ex}^{\tau}$. In our model the equilibrium allocations cannot be dominated in the Pareto sense by feasible allocations that satisfy agents' budget restrictions, at equilibrium prices, at all nodes of the economy except at a node ξ where there may be transferences subsidizing or taxing agents.

More precisely, given a node ξ in D^τ, the equilibrium allocation $\left(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\right)_{i\in\mathcal{I}}$ for the economy $\mathcal{E}^\tau_{ex}$ dominates, in the Pareto sense, any feasible allocation $\left(x^i,y^i,\theta^i,\varphi^i\right)_{i\in\mathcal{I}}$ that satisfies the budget restrictions of the agents at each node $\mu\neq\xi$ at the original equilibrium prices (this fact is proven in the Appendix).

1.4 Concluding Remarks

In this chapter we have shown that it is not necessary to impose exogenous conditions, in the form of debt constraints or transversality conditions, to avoid Ponzi schemes in incomplete markets, provided that there is a structure of collateral that protects agents in case of default.

As mentioned in the introduction, the analysis of infinite horizon economies may be carried out not just in the context of infinite-life consumers, but also in the context of overlapping generations. In the latter and, in particular, for the case of the simplest models where money is not introduced, it is possible to find a natural extension of our results. In fact, one only needs a structure of incomplete participation of agents in the event-tree. Theorem 1.1 would still hold since the finite horizon economy would still have a finite number of agents, as long as at each node the set of agents allowed to trade remains finite. Therefore, it is not hard to establish a version of Theorem 1.2 in this context, using the fact that both truncated equilibrium allocations and Lagrange multipliers are still uniformly bounded at each node.

However, in order to address the existence of equilibria in overlapping generations models with money we would have to introduce infinitely lived assets, such as fiat money, as in the incomplete markets model without default by Santos and Woodford (1997). Our approach seems to be extendable to multiperiod assets and actually two interesting situations that were absent in the model without default would now occur and deserve special attention. One is the case where assets die as default occurs and the other is the case where there is the possibility of automatic renegotiation when assets default. In addition, it becomes particularly important to identify the secondary market as only those agents that issue assets should constitute collateral.

We did not contemplate enforcement mechanisms aside from the seizure of the collateral. Nevertheless it would be possible to introduce penalties on the utility functions as in Dubey, Geanakoplos, and Zame (1995) or credit restrictions dependent on past default (see Kehoe and Levine (1993) or Alvarez and Jermann (2000)). In the presence of these mechanisms, defaulters may choose to surrender more than the collateral value, and the non-arbitrage condition in prices stated in the chapter would therefore no longer hold. The existence argument would have to be carefully redone. Failure of honoring debts can also be modelled in the context of a bankruptcy model where agents' endowments may be confiscated by creditors, as in the two-period model by Araujo and Páscoa (2001), but the infinite horizon version of this model has not yet been studied. Another line of research deals with endogenizing the collateral structure. Araujo, Orrillo, and Páscoa (2000) allowed agents to choose the collateral coefficients backing their short-sales, provided that these sellers purchase a default insurance at the same time (or equivalently, as long as a spread penalizing defaulters is deducted from the asset sale price).

1.5 Appendix

Proof of Lemma 1.2 *Feasibility.* Assuming that $\left[(\overline{x},\overline{y},\overline{\theta},\overline{\varphi}),(\overline{p},\overline{q})\right]$ is an equilibrium for the generalized game, we have that

$$\overline{p}_{\xi_0}\left[\overline{x}^i(\xi_0)+\overline{y}^i(\xi_0)\right]+\overline{p}_{\xi_0}C^{\xi_0}\overline{\varphi}^i(\xi_0)+\overline{q}_{\xi_0}\left[\overline{\theta}^i(\xi_0)-\overline{\varphi}^i(\xi_0)\right]\leq\overline{p}_{\xi_0}w^i(\xi_0), \tag{1.47}$$

$$\begin{aligned}\overline{p}_{\xi}\left[\overline{x}^i(\xi)+\overline{y}^i(\xi)\right]+\overline{p}_{\xi}C^{\xi}\overline{\varphi}^i(\xi)+\overline{q}_{\xi}\overline{Z}^i(\xi)\\ \leq\overline{p}_{\xi}w^i(\xi)+\overline{p}_{\xi}\left[Y^c_{\xi}\overline{x}^i(\xi^-)+Y_{\xi}C^{\xi^-}\overline{\varphi}^i(\xi^-)+Y^s_{\xi}\overline{y}^i(\xi^-)\right]+D^{\xi}\overline{Z}^i(\xi^-),\end{aligned} \tag{1.48}$$

$\forall\xi\in\mathcal{D}^{\tau}:\xi>\xi_0.$

Adding in i gives:

$$\overline{p}_{\xi_0}\Big[\sum_{i\in\mathcal{I}}\left(\overline{x}^i(\xi_0)+\overline{y}^i(\xi_0)-w^i(\xi_0)+C^{\xi_0}\overline{\varphi}^i(\xi_0)\right)\Big]+\overline{q}_{\xi_0}\sum_{i\in\mathcal{I}}\left(\overline{\theta}^i(\xi_0)-\overline{\varphi}^i(\xi_0)\right)\leq 0, \tag{1.49}$$

$$\begin{aligned}\overline{p}_{\xi}\Big[\sum_{i\in\mathcal{I}}\left(\overline{x}^i(\xi)+\overline{y}^i(\xi)+C^{\xi}\overline{\varphi}^i(\xi)\right)\Big]+\overline{q}_{\xi}\sum_{i\in\mathcal{I}}\overline{Z}^i(\xi)\\ \leq\overline{p}_{\xi}\left[\sum_{i\in\mathcal{I}}\left[w^i(\xi)+Y^c_{\xi}\overline{x}^i(\xi^-)+Y_{\xi}C^{\xi^-}\overline{\varphi}^i(\xi^-)+Y^s_{\xi}\overline{y}^i(\xi^-)\right]\right]+D^{\xi}\sum_{i\in\mathcal{I}}\overline{Z}^i(\xi^-),\end{aligned} \tag{1.50}$$

$\forall\xi\in\mathcal{D}^{\tau}:\xi>\xi_0.$

From the fact that $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})$ solves the auctioneer's problem, we have

$$\sum_{i\in\mathcal{I}}\left(\overline{x}^i(\xi_0)+\overline{y}^i(\xi_0)-w^i(\xi_0)+\sum_{j\in\mathcal{J}(\xi_0)}C^{\xi_0}_j\overline{\varphi}^i(\xi_0,j)\right)\quad\leq\quad 0, \tag{1.51}$$

$$\sum_{i\in\mathcal{I}}\left(\overline{\theta}^i(\xi_0)-\overline{\varphi}^i(\xi_0)\right)\quad\leq\quad 0. \tag{1.52}$$

Given $\xi\in(\xi_0)^+$, equations (1.50), (1.52) and the fact that $(\overline{p}_{\xi},\overline{q}_{\xi})$ solves the auctioneer's problem at the node ξ it follows that

$$\sum_{i\in\mathcal{I}}\left(\overline{x}^i(\xi)+\overline{y}^i(\xi)+C^{\xi}\overline{\varphi}^i(\xi)\right)\quad\leq\quad\sum_{i\in\mathcal{I}}\left(w^i(\xi)+Y^c_{\xi}\overline{x}^i(\xi^-)+Y_{\xi}C^{\xi_0}\overline{\varphi}^i(\xi^-)+Y^s_{\xi}\overline{y}^i(\xi^-)\right), \tag{1.53}$$

$$\sum_{i\in\mathcal{I}}\left(\overline{\theta}^i(\xi)-\overline{\varphi}^i(\xi)\right)\quad\leq\quad 0. \tag{1.54}$$

Repeating the arguments made at the nodes $\xi \in (\xi_0)^+$, we have that inequalities (1.53) and (1.54) hold for the nodes with period $t = 2$. In fact, the same argument across time shows that inequalities (1.53) and (1.54) hold for all successors ξ from ξ_0 in $\mathcal{D}^\tau$.

Equations (1.51), (1.52), (1.53) and (1.54) imply that $\overline{x}^i(\xi, l) \leq \chi$, $\overline{y}^i(\xi, l) \leq \chi$, $\overline{\varphi}^i(\xi, j) \leq \Psi_\xi$ and $\overline{\theta}^i(\xi, j) \leq \Psi_\xi$ for all nodes ξ in $\mathcal{D}^\tau$.

Then, it follows from the fact that the allocation $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ belongs to $\mathcal{B}^{\tau,i}_{ex}(\overline{p}, \overline{q}, 2\Psi, 2\chi)$ and the monotonicity of the utility functions that the inequalities (1.47), (1.48), (1.49) and (1.50) are, in fact, equalities.

We claim that inequality (1.51) is, in fact, an equality too. To prove this, suppose that there exists $l \in \mathcal{L}$ such that

$$\sum_{i \in \mathcal{I}} \left[\overline{x}^i(\xi_0, l) + \overline{y}^i(\xi_0, l) - w^i(\xi_0, l) + \sum_{j \in \mathcal{J}(\xi_0)} C^{\xi_0}_{j,l} \overline{\varphi}^i(\xi_0, j) \right] < 0, \tag{1.55}$$

then $\overline{p}(\xi_0, l) = 0$, implying that at the node ξ_0 the agents' consumption of the commodity should be the maximum available, $\overline{x}^i(\xi_0, l) = 2\chi$, which contradicts the consumption bounds already obtained. So, for all $l \in \mathcal{L}$

$$\sum_{i \in \mathcal{I}} \left[\overline{x}^i(\xi_0, l) + \overline{y}^i(\xi_0, l) - w^i(\xi_0, l) + \sum_{j \in \mathcal{J}(\xi_0)} C^{\xi_0}_{j,l} \overline{\varphi}^i(\xi_0, j) \right] = 0. \tag{1.56}$$

For convenience of notation denotes by $\mathcal{J}^{nt}(\xi)$ the set of *non-trivial securities* negotiated at the node ξ, that is,

$$\mathcal{J}^{nt}(\xi) = \left\{ j \in \mathcal{J}(\xi) \,:\, \left(C^{L,\xi}_j \neq 0 \right) \,\vee\, \left(\exists \mu \in \xi^+ : A(\mu, j) > 0 \wedge Y_\mu C^\xi_j > 0 \;, \right) \right\}.$$

Now suppose that there exists an asset $j \in \mathcal{J}^{nt}(\xi_0)$ such that equation (1.52) is a strict inequality. Then the price of this asset is zero, i.e $\overline{q}(\xi_0, j) = 0$. Therefore the agents are motivated to buy the greatest amount possible of units of this asset, so $\overline{\theta}^i(\xi_0, j) = 2\Psi_{\xi_0}$, which contradicts the bounds already obtained.

We have shown that inequality (1.51) is in fact an equality and for all asset $j \in \mathcal{J}^{nt}(\xi_0)$ we have

$$\sum_{i \in \mathcal{I}} \overline{\theta}^i(\xi_0, j) = \sum_{i \in \mathcal{I}} \overline{\varphi}^i(\xi_0, j). \tag{1.57}$$

Let us consider a node ξ in $(\xi_0)^+$. By arguments analogous to the one made for the node ξ_0, we show that equality holds in inequality (1.53) and for all asset $j \in \mathcal{J}^{nt}(\xi)$ inequality (1.54) is in fact an equality.. By applying these results to the trees's nodes with period $t = 2$ and repeating the process along time, we have that inequalities (1.51) and (1.53) are in fact equalities. Moreover, inequalities (1.52) and (1.54) are equalities for all non-trivial asset in the event-tree $\mathcal{D}^\tau$. Therefore, the feasibility conditions hold for the allocation $\left[(\tilde{x},\tilde{y},\tilde{\theta},\tilde{\varphi}),(\overline{p},\overline{q})\right]$ given by

$$\tilde{x}^i(\xi) = \overline{x}^i(\xi) + \sum_{j\in\mathcal{J}(\xi)-\mathcal{J}^{nt}(\xi)} C_j^{B,\xi}\overline{\varphi}^i(\xi,j), \tag{1.58}$$

$$\tilde{y}^i(\xi) = \overline{y}^i(\xi) + \sum_{j\in\mathcal{J}(\xi)-\mathcal{J}^{nt}(\xi)} C_j^{W,\xi}\overline{\varphi}^i(\xi,j),, \tag{1.59}$$

$$\tilde{\theta}^i(\xi,j) = \begin{cases} \overline{\theta}^i(\xi,j) & j\in\mathcal{J}^{nt}(\xi), \\ 0 & j\in\mathcal{J}(\xi)-\mathcal{J}^{nt}(\xi), \end{cases} \tag{1.60}$$

$$\tilde{\varphi}^i(\xi,j) = \begin{cases} \overline{\varphi}^i(\xi,j) & j\in\mathcal{J}^{nt}(\xi), \\ 0 & j\in\mathcal{J}(\xi)-\mathcal{J}^{nt}(\xi). \end{cases} \tag{1.61}$$

Optimality.

We want to prove that $\left(\tilde{x},\tilde{y},\tilde{\theta},\tilde{\varphi}\right)$ solves the consumer's problem in $\mathcal{E}_{ex}^\tau$ at prices $(\overline{p},\overline{q})$. This means $\left(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i\right)$ is solution of:

$$\begin{array}{ll} \max_{(x,y,\theta,\varphi)} & V^i(x,y,\theta,\varphi) \\ \text{subject to} & (x,y,\theta,\varphi)\in\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q}). \end{array} \tag{1.62}$$

We know that $\left(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i\right)$ solves:

$$\begin{array}{ll} \max_{(x,y,\theta,\varphi)} & V^i(x,y,\theta,\varphi) \\ \text{subject to} & (x,y,\theta,\varphi)\in\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q},2\Psi,2\chi)\subset\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q}). \end{array} \tag{1.63}$$

This follows since allocation $\left(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i\right)$ belongs at $\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q},2\Psi,2\chi)$ and gives the same level of consumption for agent i than as allocation $\left(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\right)$.

So suppose that $\left(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i\right)$ does not solve (1.62). Then there exists $(x,y,\theta,\varphi)\in\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q})$ such that:

$$V^i(x,y,\theta,\varphi) > V^i(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i). \tag{1.64}$$

Because $\left(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i\right)$ satisfies the feasibility conditions, $\tilde{x}^i(\xi,l)\le\chi$, $\tilde{y}^i(\xi,l)\le\chi$, $\tilde{\varphi}^i(\xi,j)\le\Psi_\xi$ and $\tilde{\theta}^i(\xi,j)\le\Psi_\xi$ for all (l,ξ,j) in $\mathcal{L}\times\mathcal{D}^\tau\times\mathcal{J}$, it is an interior point of $\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q},2\chi,2\Psi)$. Therefore, due to the finite number of nodes in our tree, there is $\lambda\in(0,1)$, λ near zero, such that:

$$(1-\lambda)(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i)+\lambda(x,y,\theta,\varphi)\in\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q},2\chi,2\Psi), \tag{1.65}$$

and from U^i strict quasi-concavity:

$$V^i\left[(1-\lambda)(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i)+\lambda(x,y,\theta,\varphi)\right] > V^i\left[\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i\right]. \tag{1.66}$$

This contradicts the fact that $(\tilde{x}^i,\tilde{y}^i,\tilde{\theta}^i,\tilde{\varphi}^i)$ solves the problem (1.63).

So, the allocation $\left[(\tilde{x},\tilde{y},\tilde{\theta},\tilde{\varphi}),(\overline{p},\overline{q})\right]$ is an equilibrium for the economy $\mathcal{E}^{\tau}_{ex}$. □

Proof of Proposition 1.1 Let $(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i)$ be the equilibrium's allocation associated to equilibrium prices $(\overline{p},\overline{q})$ of the agent i. Suppose, by contradiction, that there is a node ξ' and $j'\in\mathcal{J}(\xi')$ such that $\overline{p}_{\xi'}C^{\xi'}_{j'}-\overline{q}_{\xi',j'}<0$.

Let $l'\in\mathcal{L}$, $y=\overline{y}^i$, $\theta=\overline{\theta}^i$, $\varphi(\xi,j)=\overline{\varphi}^i(\xi,j)$, for all $(\xi,j)\neq(\xi',j')$, and $x(\xi,l)=\overline{x}^i(\xi,l)$ for all $(\xi,l)\neq(\xi',l')$.

Because the equilibrium price $\overline{p}_{\xi}$ is strictly positive, we can take

$$\varphi(\xi',j') > \overline{\varphi}^i(\xi',j'), \tag{1.67}$$

$$x(\xi',l') = \overline{x}^i(\xi',l')+\frac{\overline{p}_{\xi'}C^{\xi'}_{j'}-\overline{q}_{\xi',j'}}{\overline{p}_{\xi',l'}}\left(\overline{\varphi}^i(\xi',j')-\varphi(\xi',j')\right) > \overline{x}^i(\xi',l'). \tag{1.68}$$

Then because of the strict monotonicity of the utility function of the agent i, we can find an allocation (x,y,θ,φ) that improves utility at prices $(\overline{p},\overline{q})$ and is in $\mathcal{B}^{\tau,i}_{ex}(\overline{p},\overline{q})$. This contradicts the fact that $(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i)$ is optimal.

In the case where $C^{B,\xi'}_{j'}\neq 0$, if $\overline{p}_{\xi'}C^{\xi'}_{j'}-\overline{q}_{\xi',j'}=0$ then we define $\varphi(\xi',j')>\overline{\varphi}^i(\xi',j')$ and keep the other allocations. Therefore, we can find an allocation (x,y,θ,φ) in $\mathcal{B}^{\tau,i}_{ex}(\overline{p},\overline{q})$ that improves utility. □

Proof of Lemma 1.5 Let ξ' be such that $b^T_{\xi'}=0$. Define $\theta(\xi)=\theta^{T,i}(\xi)$, $\varphi(\xi)=\varphi^{T,i}(\xi)$, $y(\xi)=y^{T,i}(\xi)$ for all $\xi\in\mathcal{D}^T$ and $x(\xi)=x^{T,i}(\xi)$ for all $\xi\in\mathcal{D}^T$, $\xi\neq\xi'$. It follows from item ii. in Lemma 4 that

$$u^i(\xi',x(\xi'))-u^i(\xi',x^{T,i}(\xi'))\leq\mu^{Ti}_{\xi'}p^T_{\xi'}(x(\xi')-x^{T,i}(\xi')). \tag{1.69}$$

In the case where $b^T_{\xi}>0$, remember that $q^T_{\mu}\equiv 0$ and $C^{\mu}\equiv 0$ for all $\mu\in\mathcal{D}^T$ such that $b^T_{\mu}=0$, and consider the allocation (x, y, θ, φ) in $I\!E^T$ given by

$$x(\xi) = \begin{cases} x^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{1.70}$$

$$y(\xi) = \begin{cases} y^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{1.71}$$

$$\theta(\xi) = \begin{cases} \theta^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{1.72}$$

$$\varphi(\xi) = \begin{cases} \varphi^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{1.73}$$

It follows from Lemma 1.4 and hypothesis B that

$$-u^i\Big(\xi', x^{T,i}(\xi') + C^{B,\xi'}\varphi^{T,i}(\xi') + C^{L,\xi'}\theta^{T,i}(\xi')\Big) \leq -\mu_{\xi'}^{Ti} L_{\xi'}^{T,i} + \sum_{\eta \in (\xi')^+} \mu_\eta^{Ti} R_\eta^{T,i}.$$

Applying equations (1.32) and (1.33) we obtain the result. □

Proof of Claims in Remark 1.5 Given ξ in $\mathcal{D}^\infty$ and $K \in I\!N$, it follows from equation (1.36) that

$$\sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \overline{\mu}_{\xi'}^i \left[\left(\overline{p}_{\xi'} C^{\xi'} - \overline{q}_{\xi'}\right)\overline{\varphi}^i(\xi') + \overline{q}_{\xi'}\overline{\theta}^i(\xi')\right] \leq \lim_{T \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \mu_{\xi'}^{Ti} L_{\xi'}^{T,i}, \tag{1.74}$$

$$\leq \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') \geq K\}} u^i(\xi', \beta). \tag{1.75}$$

Taking the limit as K goes to infinity in the last inequalities, we conclude that

$$\lim_{K \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \overline{\mu}_{\xi'}^i \left(\overline{p}_{\xi'} C^{\xi'} - \overline{q}_{\xi'}\right)\overline{\varphi}^i(\xi') + \lim_{K \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \overline{\mu}_{\xi'}^i \overline{q}_{\xi'} \overline{\theta}^i(\xi') \leq 0. \tag{1.76}$$

Equations (1.44) and (1.45) follow since the non-arbitrage condition (Proposition 1.1) guarantees that both of the terms in the left side of (1.77) are non-negative.

Now, if we suppose that $w^i(\xi, l) \geq \underline{w}$, it follows from equations (1.21), (1.37) and (1.45) that

$$0 \leq \lim_{K \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \overline{\mu}_{\xi'}^i \overline{q}_{\xi'} \overline{\varphi}^i(\xi') = \lim_{K \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \overline{\mu}_{\xi'}^i \overline{p}_{\xi'} C^{\xi'} \overline{\varphi}^i(\xi') \tag{1.77}$$

$$\leq \lim_{K \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \overline{\mu}_{\xi'}^i \|\overline{p}_{\xi'}\|_\Sigma \max_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}(\xi')} C_{j,l}^{\xi'} \overline{\varphi}^i(\xi', j), \tag{1.78}$$

$$\leq \frac{\overline{w}\mathcal{I}}{(1-\kappa)} \lim_{K \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') = K\}} \overline{\mu}_{\xi'}^i \|\overline{p}_{\xi'}\|_\Sigma, \tag{1.79}$$

$$\leq \frac{\overline{w}\mathcal{I}}{(1-\kappa)} \frac{1}{\underline{w}} \lim_{K \to \infty} \sum_{\{\xi' \geq \xi : \tilde{t}(\xi') \geq K\}} u^i(\xi', \beta) = 0. \tag{1.80}$$

Proof of Claim in Remark 1.6 Suppose by contradiction that there is an allocation $\left(x^i, y^i, \theta^i, \varphi^i\right)_{i\in\mathcal{I}}$ that satisfies the markets clear conditions, belongs in the agent's budget set for all nodes $\mu \neq \xi$ and $V^i(x^i, y^i, \theta^i, \varphi^i) > V^i(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i)$ for all i in I. Then it follows from the individual optimality of the equilibrium allocation that at the node ξ,

$$p_\xi\left(x^i(\xi) + y^i(\xi) + C^\xi \varphi^i(\xi) - w^i(\xi)\right) + q_\xi Z^i(\xi) > p_\xi\left[Y^c_\xi x^i(\xi^-) + Y_\xi C^{\xi^-} \varphi^i(\xi^-) + Y^s_\xi y^i(\xi^-)\right] + D^\xi Z^i(\xi^-),$$

Adding in $i \in I$ we contradict the fact that $\left(x^i, y^i, \theta^i, \varphi^i\right)_{i\in\mathcal{I}}$ is feasible. □

Chapter 2

Multiperiod Assets and Bubbles in Markets subject to Default

Now we extend the results from the previous chapter to the case of long-lived assets and especially of infinite-lived assets.

We allow also incomplete participation in the markets by the agents, as well as the existence of physical and financial collateral requirements, which might vary with the price levels. Such requirements might consist both of assets which are subject to default and of assets which are default-free.

We study the existence of equilibrium and obtain sufficient conditions for the non-existence of speculative bubbles.

We prove that bubbles in the economy are caused by speculation in the prices of default-free assets, thus reducing the characterization of bubbles to the validity of transversality conditions, that depends on the prices of durable goods and of assets not subject to default.

2.1 The Economy

The model of uncertainty and the characterization of commodities used here is essentially the one developed in Araujo, Páscoa, and Torres-Martínez (2000). Along the following lines we recall the main features of this model; introduce the possibility of incomplete household participation, as in Santos and Woodford (1996); and define the structure of multiperiod securities in the economy.

2.1.1 Uncertainty

We work in an economy with infinite horizon and discrete time. The set of trade periods is $T = \{0, 1, \dots\}$. We suppose that there is no uncertainty at $t = 0$. There are S_1 states of nature at $t = 1$. In general, given a history of realization of the states of nature for the first $t-1$ periods: $\overline{s}_t = (s_0, s_1, \dots, s_{t-1})$, there exist $S(\overline{s}_t)$ states of nature at period t. We suppose that these sets are *finite*.

An information set $\xi = (t, \overline{s}_t, s)$, where $t \in T$ and $s \in S(\overline{s}_t)$, is called a *node* of the economy. The period associated to node ξ is denoted by $\tilde{t}(\xi)$. The (unique) predecessor of node ξ is denoted by ξ^- and the only information set at $t = 0$ is ξ_0. Given a node $\xi = (t, \overline{s}_t, s)$, we denote by ξ^+ the set of immediate successors of ξ, that is, the set of nodes $\mu = (t+1, \overline{s}_{t+1}, s')$, where $\overline{s}_{t+1} = (\overline{s}_t, s)$. There exists a natural order in the information structure: given nodes $\xi = (t, \overline{s}_t, s)$ and $\mu = (t', \overline{s}_{t'}, s')$, we say that μ is a *successor* of the node ξ, and write $\mu \geq \xi$, if $t' \geq t$ and $\overline{s}_{t'} = (\overline{s}_t, \dots)$.

The set of nodes or information sets in the economy, called the *event-tree*, is denoted by $\mathcal{D}$. The set of nodes with period t is denoted by $\mathcal{D}_t$, and, given an information set ξ, $\mathcal{D}(\xi)$ denotes the sub-tree whose root is ξ (that is, the set of successors of the node ξ).

Finally, a set $\mathcal{D}' \subset \mathcal{D}$ will be called a *sub-tree* of the event-tree if there is a node $\xi \in \mathcal{D}'$, called the *root* of the sub-tree, such that $\mathcal{D}' \subset \mathcal{D}(\xi)$ and $\mathcal{D}' \cap \mathcal{D}(\mu) = \phi$, for all nodes $\mu \in \mathcal{D}(\xi) - \mathcal{D}'$.

2.1.2 Commodities

We suppose that at each node of the event-tree $\mathcal{D}$ the economy has a finite set $\mathcal{L}$ of commodities which can be consumed or stored. Besides, these commodities may suffer partial depreciation at the node branches; this depreciation varies according to whether the commodity is for consumption or storage. Note that if the depreciation factors for consumption and storage are the same, then the agents have no interest in storing commodities. Therefore, if one unit of the good $l \in \mathcal{L}$ is consumed at the node ξ, then at each node $\mu \in \xi^+$ we obtain an amount $(Y^c_\mu)_{l,l'}$ of the good l'. Analogously, $(Y^s_\xi)_{l,l'}$ denotes the amount of commodity l' that is obtained at the node ξ if one unit of the good l was stored at the node ξ^-. These structures are very general, allowing for instance for goods that are perishable, perfectly durable, simply one period older or perfectly storable but non-durable.

Spot markets for commodity negotiations are available at each node. We denote by $p_\xi = \big(p_{\xi,l} \, : \, l \in \mathcal{L}\big)$ the vector of spot prices at the node $\xi \in \mathcal{D}$.

2.1.3 Agents

Incomplete participation of the agents is allowed in the economy, and therefore at each node ξ of the event-tree there exists a *finite* set $\mathcal{H}(\xi)$ of agents able to trade in the spot markets.

We suppose that $\mathcal{H}(\xi)$ is a subset of the agents set $\mathcal{H}$, which is countable. We denote by $\mathcal{D}^h$ the sub-tree of nodes at which the agent $h \in \mathcal{H}$ can trade in spot markets. The root of the sub-tree $\mathcal{D}^h$ is denoted by ξ^h. We say that the agent h is *infinite-lived* if for all $t \in \mathbb{N}$ there exists a node $\xi \in \mathcal{D}^h$ such that $\tilde{t}(\xi) = t$. Otherwise we say that the agent h is *finite-lived*.

The set $\delta\mathcal{D}^h$ denotes the *terminal nodes* for the agent h. That is, the set of nodes $\xi \in \mathcal{D}^h$ for which $\mathcal{D}(\xi) \cap \mathcal{D}^h = \{\xi\}$ (if such nodes exist; otherwise we suppose that $\delta\mathcal{D}^h$ is empty).

Two technical restrictions that appear in the literature in Santos and Woodford (1997) are also needed:

a. For each agent $h \in \mathcal{H}$, if $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$ then $\xi^+ \subset \mathcal{D}^h$,

b. For each node $\xi \in \mathcal{D}$ there exists at least one agent $h \in \mathcal{H}$ for which $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$.

We denote by $\tilde{\mathcal{H}}(\xi)$ the set of agents able to negotiate assets at the node ξ. Hence, we have that $h \in \mathcal{H}(\xi)$ if and only if $\xi \in \mathcal{D}^h$, and $h \in \tilde{\mathcal{H}}(\xi)$ if and only if $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$.

At each node $\xi \in \mathcal{D}^h$, the agent h can choose a *collateral-free consumption allocation* $x^h(\xi) \in \mathbb{R}^{\mathcal{L}}_+$. We denote by $x^h = \left(x^h(\xi)\right)_{\xi \in \mathcal{D}^h}$ the collateral-free consumption plan of the agent h. At the nodes $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$ the agent h is also able to choose a *collateral-free storage allocation* $y^h(\xi) \in \mathbb{R}^{\mathcal{L}}_+$. A family of such allocations, $y^h = \left(y^h(\xi)\right)_{\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)}$, is called a collateral-free storage plan of the agent h.

We use the notation $X^h = \mathbb{R}^{\mathcal{D}^h \times \mathcal{L}}_+$ and $Y^h = \mathbb{R}^{(\mathcal{D}^h - \delta\mathcal{D}^h)\times\mathcal{L}}_+$ to denote respectively the consumption- and the storage-spaces of the agent h in the economy.

Each agent $h \in \mathcal{H}$ is characterized by an endowment process $w^h \in X^h$ and a utility function $U^h : X^h \to \mathbb{R}_+$ that represents his preferences in the consumption space X^h.

2.1.4 Assets

The set $\mathcal{J}$ of assets, including long-lived or infinite-lived real securities, that can be negotiated in the economy is the union of four disjoint sets:[1]

- The set $\mathcal{J}^b$ of assets that, not being subject to default, have exogenous bounds on short sales at each node where they are traded;

[1] The structure of real securities also allows the study of models with nominal securities, since Geanakoplos and Mas-Collel (1989) show that a model with securities that deliver nominal returns can be converted into a family of models with real securities.

- The set $\mathcal{J}^m$ of assets which are not subject to default but whose short sales require margin deposits. Such margin requirements may consist of either consumption goods or assets in $\mathcal{J}^b$;

- The set $\mathcal{J}^d$ of assets which might suffer default but which are *individually* protected by collateral requirements. These collateral requirements are exogenously imposed on the agents; they may consist of consumption goods (Physical Collateral) as well as of assets (Financial Collateral);

- The set $\mathcal{J}^t$ of families of real short-lived assets (tranches) that are subject to default but *collectively* protected by exogenous collateral requirements. Within each family $k \in \mathcal{J}^t$ there is an priority as to the payment of the returns. We assume for the sake of simplicity that the collateral requirements for each $k \in \mathcal{J}^t$ consist solely of consumption goods (Physical Collateral) or of assets in $\mathcal{J}^b \cup \mathcal{J}^m$.

We now explain the characteristics of and the rules for trading in each of the aforementioned asset types:

The set $\mathcal{J}^b$: Assets with Bounded Short-Sales

Each asset $j \in \mathcal{J}^b$ is characterized by

1. A node $\xi_j \in \mathcal{D}$ where it is issued and a set of *terminal nodes*, $T_j \subset \mathcal{D}(\xi_j)$.
   We suppose that if $T_j \neq \phi$ then, given nodes μ_1, $\mu_2 \in T_j$, we have $\mathcal{D}(\mu_1) \cap \{\mu_2\} = \phi$.[2]

2. The sub-tree where the asset is negotiated
$$\mathcal{N}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi < \kappa \quad, \kappa \in T_j\},$$

3. The processes $A(\xi, j)$ of bundle promises, which are defined in the sub-tree of nodes where the asset gives returns, $\mathcal{R}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi_j < \xi \leq \kappa \quad, \kappa \in T_j\}$, and,

4. Real numbers $\left(b_\xi^j\right)_{\xi \in \mathcal{N}(j)} \in \mathbb{R}_{++}^{\mathcal{N}(j)}$ that bound the number of units of the asset j that each agent can sell at node ξ.

Thus, given an asset $j \in \mathcal{J}^b$ and a node $\xi \in \mathcal{N}(j)$, each agent $h \in \mathcal{H}(\xi)$ chooses a portfolio $Z^h(\xi, j) = \left(\theta^h(\xi, j) - \varphi^h(\xi, j)\right) \in \mathbb{R}$ in the asset, subject to the condition $\varphi^h(\xi, j) \leq b_\xi^j$; here we denote by $\theta^h(\xi, j) = \max\{Z^h(\xi, j), 0\}$ the number of units of the asset j in ξ bought, and by $\varphi^h(\xi, j) = -\min\{Z^h(\xi, j), 0\}$ the number of units sold.

[2] This technical condition aims to avoid ambiguity when dealing with the elements of T_j.

Such a portfolio costs

$$\mathrm{PV}_{\xi,j}(\theta^h,\varphi^h) \equiv q_{\xi,j}\left(\theta^h(\xi,j)-\varphi^h(\xi,j)\right), \tag{2.1}$$

(where $q_{\xi,j}$ denotes the asset's unit price at node ξ), and delivers, at each node $\mu \in \xi^+$, returns equal to

$$\mathrm{R}_{\mu,j}(\theta^h,\varphi^h) \equiv (p_\mu A(\mu,j)+q_{\mu,j})\left(\theta^h(\xi,j)-\varphi^h(\xi,j)\right). \tag{2.2}$$

The next condition, which guarantees that the real returns of assets in $\mathcal{J}^b$ do not grow unboundedly along the tree, will be necessary for existence of equilibrium; this condition is commonplace in the literature of sequential equilibrium in infinite-horizon economies with multi-period asset trading (see Magill and Quinzii (1996)):

ASSUMPTION A: For each asset $j \in \mathcal{J}^b$, the real promises $(A(\xi,j))_{\xi\in\mathcal{R}(j)}$ belong to $l^\infty(\mathcal{L}\times\mathcal{R}(j))$.

We denote by $\mathcal{J}^b(\xi)$ the (finite) set of assets $j \in \mathcal{J}^b$ which are traded at node ξ.

The set $\mathcal{J}^m$: Assets with Margin Requirements

Just like the assets in $\mathcal{J}^b$, each asset $j \in \mathcal{J}^m$ is characterized by the node at which it is issued, ξ_j, and by the set T_j of terminal nodes. We denote by $\mathcal{N}(j)$ the sub-tree of nodes where the asset is traded. The real promises, as in the previous cases, are denoted by $[A(\xi,j)]_{\xi\in\mathcal{R}(j)}$.

We denote by $q_{\xi,j}$ the price of asset $j \in \mathcal{J}^m$ at node $\xi \in \mathcal{N}(j)$. Each asset in the set $\mathcal{J}^m$ will have margin requirements which will be purchased for each units of the asset that are sold, at the various nodes where the asset is traded.

We assume that such requirements at node ξ depend on the prices of the consumption goods and of the assets in $\mathcal{J}^b$, as well as on the price of the asset itself at this node. Furthermore, such requirements might consist of physical goods or of assets in $\mathcal{J}^b$.

Hence, given an asset $j \in \mathcal{J}^m$ and a node $\xi \in \mathcal{N}(j)$, the margin requirements $M_{\xi,j} = [M^1_{\xi,j}, M^2_{\xi,j}]$ are given by the functions

$$M^1_{\xi,j}: \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{J}^b(\xi)}_+ \times \mathbb{R}_+ \to \mathbb{R}^{\mathcal{L}}_+, \tag{2.3}$$

$$M^2_{\xi,j}: \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{J}^b(\xi)}_+ \times \mathbb{R}_+ \to \mathbb{R}^{\mathcal{J}^b(\xi)}_+, \tag{2.4}$$

where $M^1_{\xi,j}(p,q,q_{\xi,j})$ (resp. $M^2_{\xi,j}(p,q,q_{\xi,j})$) denotes the physical (resp. financial) margin requirements, where consumption prices are given by $p \in \mathbb{R}^{\mathcal{L}}_+$, the prices of assets with bounded short-sales are given by $q \in \mathbb{R}^{\mathcal{J}^b(\xi)}_+$, and the price of the asset itself is given by $q_{\xi,j}$.[3]

We assume that the following hypothesis holds:

ASSUMPTION B: For each asset $j \in \mathcal{J}^m$, the real promises $(A(\xi,j))_{\xi\in\mathcal{R}(j)}$ belong to $l^\infty(\mathcal{L}\times\mathcal{R}(j))$, and the functions $M^1_{\xi,j}$, $M^2_{\xi,j}$ are continuous in $\left(p_\xi, (q_{\xi,j'})_{j'\in\mathcal{J}^b(\xi)\cup\{j\}}\right)$ and different from zero, for every node $\xi \in \mathcal{N}(j)$.

We suppose that the commodities and the assets in $\mathcal{J}^b$ that can be used as margin requirements for the asset $j \in \mathcal{J}^m$ at node ξ are the same along the sub-tree $\mathcal{N}(j)$ and are independent of the prices. That is, given a commodity $l \in \mathcal{L}$, if $\left(M^1_{\xi,j}(p,q,q_{\xi,j}\right)_l \neq 0$ then $\left(M^1_{\mu,j}(p',q',q'_{\mu,j}\right)_l \neq 0$ for every node $\mu \in \mathcal{N}(j)$. Analogously, given an asset j' in $\mathcal{J}^b(\xi)$, if $\left(M^2_{\xi,j}(p,q,q_{\xi,j}\right)_{j'} \neq 0$ then $\left(M^2_{\mu,j}(p',q',q'_{\mu,j}\right)_{j'} \neq 0$ for every node $\mu \in \mathcal{N}(j)$.[4]

Moreover, we suppose that given prices for the commodities and for the assets in $\mathcal{J}^b \cup \{j\}$, the requirements $M_{\xi,j} = [M^1_{\xi,j}, M^2_{\xi,j}]$ are uniformly bounded from above by a positive vector $\overline{M}_j \in \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{J}^b(\xi_j)}_+$, that not depends on the node $\xi \in \mathcal{N}(j)$.

The bundle of goods that is used as the margin, will be stored. Therefore, an agent who sells a unit of asset $j \in \mathcal{J}^m$, at node $\xi \in \mathcal{N}(j)$, will have at each node $\mu \in \xi^+$ his margin deposit transformed into $Y^w_\mu M^1_{\xi,j}\left(p_\xi, (q_{\xi,j'})_{j'\in\mathcal{J}^b(\xi)\cup\{j\}}\right)$.

As was mentioned previously, the agents are not allowed to default on assets that belong to the set $\mathcal{J}^m$. Thus the purpose of the margin requirements is to bound the number of short sales at each node. The dependence of the margin requirements on the prices of the goods and assets in $\mathcal{J}^b$ will allow these requirements to be adjusted along the sub-tree $\mathcal{N}(j)$, thus preventing the agents from indefinitely accumulating higher and higher shorted positions in asset j, in case they could benefit from the successive depreciations in the value of M^j.

In order to simplify the notation we denote by $\mathrm{MV}_{\xi,j}$ the value of the margin that is required for each unit of the asset j sold at ξ, and by $\mathrm{DMV}_{\mu,j}$ the depreciated value of this margin at the

[3]In order to simplify the notation, when there is no ambiguity we will omit the explicit dependence of the margin requirements on the prices (p,q).

[4]This assumption implies that if $\left(M^2_{\xi,j}\right)_{j'} \neq 0$ then $T_{j'} \subset \bigcup_{\kappa\in T_j\cap\mathcal{D}(\xi)} \mathcal{D}(\kappa)$. Therefore, the assets $j \in \mathcal{J}^m$ which are infinite-lived can only be protected by infinite-lived assets in $\mathcal{J}^b$.

node $\mu \in \xi^+$,

$$\text{MV}_{\xi,j} \equiv p_\xi M^1_{\xi,j} + \sum_{j' \in \mathcal{J}^b(\xi)\cup\{j\}} q_{\xi,j'} (M^2_{\xi,j})_{j'}, \tag{2.5}$$

$$\text{DMV}_{\mu,j} \equiv p_\mu Y^w_\mu M^1_{\xi,j} + \sum_{j' \in \mathcal{J}^b(\xi)\cup\{j\}} (p_\mu A(\mu, j') + q_{\mu,j'}) (M^2_{\xi,j})_{j'}, \tag{2.6}$$

Now, if an agent sells a unit of asset $j \in \mathcal{J}^m$ at node ξ, but wishes to keep his short position at a node $\mu \in \xi^+$, then he must adjust his margin deposit. Thus he must increase his margin deposit if the value of the requirements at μ, $\text{MV}_{\mu,j}$ is greater than the depreciated value of the margin at ξ, $\text{DMV}_{\mu,j}$; and decrease his margin deposit otherwise.

Hence, given an asset $j \in \mathcal{J}^m$ and a node $\xi \in \mathcal{N}(j)$, each agent $h \in \tilde{\mathcal{H}}(\xi)$ will choose a portfolio $(\theta^h(\xi, j); \varphi^h(\xi, j))$ in the asset. Here $\theta^h(\xi, j)$ denotes the number of units bought by the agent and $\varphi^h(\xi, j)$ denotes the number of units sold. Since the agents must constitute a margin for each unit sold at node ξ, the total cost for trading the allocation will be

$$\text{PV}_{\xi,j}(\theta^h; \varphi^h) \equiv \text{MV}_{\xi,j}\varphi^h(\xi, j) + q_{\xi,j} \left(\theta^h(\xi, j) - \varphi^h(\xi, j)\right); \tag{2.7}$$

and the value of the returns delivered at each node $\mu \in \xi^+$ will be

$$\text{R}_{\mu,j}(\theta^h; \varphi^h) \equiv \text{DMV}_{\mu,j}\varphi^h(\xi, j) + (p_\mu A(\mu, j) + q_{\mu,j}) \left(\theta^h(\xi, j) - \varphi^h(\xi, j)\right). \tag{2.8}$$

Just as with assets in $\mathcal{J}^b$, we denote by $\mathcal{J}^m(\xi)$ the finite set of assets that require margin deposits and are negotiated at ξ.

The set $\mathcal{J}^d$: Assets protected by Exogenous Collateral

Each asset $j \in \mathcal{J}^d$ is characterized by the node $\xi_j \in \mathcal{D}$ where it is issued; the set of *terminal nodes*, $T_j \subset \mathcal{D}(\xi_j)$; the sub-tree where the asset is *potentially* negotiated, [5]

$$\mathcal{N}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi < \kappa \quad, \kappa \in T_j\};$$

and the processes $A(\xi, j)$ of bundle promises, which are defined in the set of nodes where the asset delivers returns, $\mathcal{R}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi_j < \xi \leq \kappa \quad, \kappa \in T_j\}$.

If the asset j in $\mathcal{J}^d$ is negotiated at the node $\xi \in \mathcal{D}$ then we denote by $q_{\xi,j}$ the unit price of the asset at this node, associated with the process $[A(\mu, j)]_{\mu \geq \xi}$. Such prices may vary along the tree $\mathcal{N}(j)$ according to the rules for renegotiation that will be set out in the next section.

[5]Since we allow the possibility of default along the event-tree, the assets might foreclosed before the original terminal dates $\kappa \in T_j$.

Moreover, there exist, at each node $\xi \in \mathcal{N}(j)$, collateral requirements $C_{\xi,j} = [C^1_{\xi,j}, C^2_{\xi,j}] \in \mathbb{R}^L_+ \times \mathbb{R}_+^{\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)\cup\mathcal{J}^d(\xi)}$, that depend on the price level and are given by the functions

$$C^1_{\xi,j} : \mathbb{R}^L_+ \times \mathbb{R}_+^{\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)\cup\mathcal{J}^d(\xi)} \times [0,1]_+^{\mathcal{J}^d(\xi)} \rightarrow \mathbb{R}^{\mathcal{L}}_+, \tag{2.9}$$

$$C^1_{\xi,j} : \mathbb{R}^L_+ \times \mathbb{R}_+^{\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)\cup\mathcal{J}^d(\xi)} \times [0,1]_+^{\mathcal{J}^d(\xi)} \rightarrow \mathbb{R}_+^{\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)\cup\mathcal{J}^d(\xi)}, \tag{2.10}$$

where $C^1_{\xi,j}(p_\xi, q_\xi; \beta_\xi)$ (resp. $C^2_{\xi,j}(p_\xi, q_\xi; \beta_\xi)$) represents the physical (resp. financial) collateral requirements for the asset $j \in \mathcal{J}^d$ at the node $\xi \in \mathcal{N}(j)$, and depends on the commodity price p_ξ at this node, and on the asset prices $q_{\xi,j'}$ and anonymous renegotiation rules $\beta_{\xi,j'}$ of the assets $j' \in \mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi) \cup \mathcal{J}^d(\xi)$. These rules, which are taken as given by the agents, are explained in the next section.

As usual, for each node ξ the set $\mathcal{J}^d(\xi)$ denotes the finite set of assets in $\mathcal{J}^d$ that are negotiated in the economy at node ξ.

For notational convenience, we refer to the collateral requirements at the node ξ for the asset $j \in \mathcal{J}^d$ as $C_{\xi,j} = [C^1_{\xi,j}, C^2_{\xi,j}]$.

As in Dubey, Geanakoplos, and Zame (1995) we allow the existence of penalties $\lambda^h_{\xi,j}$ for each node $\xi \in (\mathcal{D}^h - \{\xi^h\})$ and asset $j \in \mathcal{J}^d(\xi^-)$; such penalties correspond to the loss of utility for each unit of default incurred by agent h at the various nodes ξ and assets j.

It is worth noting that in this model the collateral requirements for the assets $j \in \mathcal{J}^d$ will consist of either commodities (*physical collateral*) or assets (*financial collateral*), or both. The physical collateral for the asset $j \in \mathcal{J}^d$ at the node $\xi \in \mathcal{N}(j)$ can be divided, as in Dubey, Geanakoplos, and Zame (1995), among $C^1_{\xi,j} = C^{1,W}_{\xi,j} + C^{1,B}_{\xi,j} + C^{1,L}_{\xi,j}$, where $C^{1,W}_{\xi,j}$, $C^{1,B}_{\xi,j}$, $C^{1,L}_{\xi,j}$ denote respectively the part that is stored, the part that is held by the borrower and the part that is held by the lender.

The physical collateral is subject to depreciation and we denote by $Y_\mu \diamond C^1_{\xi,j} = Y^c_\mu\big(C^{1,B}_{\xi,j} + C^{1,L}_{\xi,j}\big) + Y^w_\mu C^{1,W}_{\xi,j}$ the depreciated bundle of collateral $C^1_{\xi,j}$ at the node $\mu \in \xi^+$.

We suppose that the following assumption holds:

ASSUMPTION C: For each asset $j \in \mathcal{J}^d$, the real promises $(A(\xi, j))_{\xi\in\mathcal{R}(j)}$ belong to $l^\infty(\mathcal{L}\times\mathcal{R}(j))$, and the functions $C^1_{\xi,j}$, $C^2_{\xi,j}$ are continuous in $\big(p_\xi, (q_{\xi,j'})_{j'\in\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)\cup\mathcal{J}^d(\xi)\cup\{j\}}, \beta\big)$ and different from zero, for every node $\xi \in \mathcal{N}(j)$.

We suppose that the commodities and the assets in $\mathcal{J}^b\cup\mathcal{J}^m\cup\mathcal{J}^d$ that can be used as collateral for the asset $j \in \mathcal{J}^d$ at node ξ, are the same along the sub-tree $\mathcal{N}(j)$ and are independent of the price level. That is, given a commodity $l \in \mathcal{L}$, if $\Big(C^1_{\xi,j}(p, q, \beta)\Big)_l \neq 0$ then $\big(C^1_{\mu,j}(p', q', \beta')\big)_l \neq 0$ for all node $\mu \in \mathcal{N}(j)$ and prices (p', q', β'). Analogously, given an asset $j' \neq j$ in $\mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi) \cup \mathcal{J}^d(\xi)$, if $\Big(C^2_{\xi,j}(p, q, \beta)\Big)_{j'} \neq 0$ then $\big(C^2_{\mu,j}(p', q', \beta')\big)_{j'} \neq 0$ for all price vectors (p', q', β')

and nodes $\mu \in \mathcal{N}(j)$.

Moreover, we suppose that given prices for the commodities and for the assets, the collateral requirements $C_{\xi,j} = [C^1_{\xi,j}, C^2_{\xi,j}]$ are uniformly bounded from above by a positive vector $\overline{C}_j \in \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{J}^b(\xi_j)\cup\mathcal{J}^m(\xi_j)\cup\mathcal{J}^d(\xi_j)}_+$ that is independent of the node.

Now, in order to prevent two assets j_1 and j_2 from mutually protecting each other via collateral requirements, we impose a *pyramiding* structure on the assets in $\mathcal{J}^d$.[6]

ASSUMPTION D: The set $\mathcal{J}^d$ is a disjoint union of sets $(\mathcal{A}_k; k \geq 0)$, which are defined by the recursive rule:

$$\mathcal{A}_0 = \{j \in \mathcal{J}^d : \left(C^2_{\xi,j}\right)_{j'} \neq 0 \Rightarrow j' \in \mathcal{J}^m \cup \mathcal{J}^b\},$$

$$\mathcal{A}_k = \left\{ j \in \mathcal{J}^d : \left(C^2_{\xi,j}\right)_{j'} \neq 0 \Rightarrow j' \in \left(\bigcup_{r=0}^{k-1} \mathcal{A}_r\right) \cup \mathcal{J}^b \cup \mathcal{J}^m \right\} - \bigcup_{r=0}^{k-1} \mathcal{A}_r.$$

Note that Assumption C guarantees that the sets $\mathcal{A}_k$ are independent of the price level and of the nodes.

Renegotiation Rules for the Assets in $\mathcal{J}^d$.

The agents in the economy take as given "*anonymous renegotiation rules*" $\beta_{\xi,j} \in [0,1]$, for every asset $j \in \mathcal{J}^d$ and node $\xi \in \mathcal{R}(j)$.

So, given an asset $j \in \mathcal{A}_0$ and a node $\xi \in \xi_j^+$,

- If $\beta_{\xi,j} = 1$ then the agents expect to receive, for each unit of asset j bought by them at the node ξ_j, the bundle $A(\xi,j)$ at this node, and to negotiate their long-position at prices $q_{\xi,j}$, with collateral requirements $C_{\xi,j}$.

- If $\beta_{\xi,j} = 0$, then the agents suppose that the asset $j \in \mathcal{A}_0$ goes to default at this node, delivering the depreciated value of the collateral

$$\text{DCV}_{\xi,j} \equiv p_\xi Y_\xi \diamond C^1_{\xi_j,j} + \sum_{j' \in \mathcal{J}^b(\xi_j)\cup\mathcal{J}^m(\xi_j)} \text{R}_{\xi,j'}\left(\left(C^2_{\xi_j,j}\right)_{j'}, 0\right), \tag{2.11}$$

  as well as the foreclosure.

- Finally, if $\beta_{\xi,j} \in (0,1)$, then the agents expect to receive, for each unit of asset j bought at ξ_j, a percentage of the original promises, $\beta_{\xi,j} A(\xi,j)$, and a collateral proportional to

[6]For more details on this concept see Dubey, Geanakoplos and Zame (1996).

the non-delivered promises, that is, $(1-\beta_{\xi,j})\text{DCV}_{\xi,j}$. Moreover, the asset suffers an automatic renegotiation and is transformed into a new asset with the promises process $[\beta_{\xi,j}A(\mu,j)]_{\{\mu>\xi\ ,\ \mu\in\mathbb{R}(j)\}}$ and new collateral requirements $[\beta_{\xi,j}C^1_{\xi,j},\beta_{\xi,j}C^2_{\xi,j}]$. The agents therefore expect to sell their long-positions at prices $\beta_{\xi,j}q_{\xi,j}$.

For the sake of simplicity, we will keep the notation j for such new assets in $\mathcal{A}_0$ and modify their collateral and price processes, in the sub-tree with root ξ, to $\left[\beta_{\xi,j}C_{\mu,j};\beta_{\xi,j}q_{\mu,j}\right)_{\{\mu>\xi\ ,\ \mu\in\mathcal{N}(j)\}}$.

The interpretation of $\beta_{\xi,j}$ is similar at the nodes $\xi\in\mathcal{N}(j)$ with $\tilde{t}(\xi)>\tilde{t}(\xi_j)+1$. In fact,

- If $\beta_{\xi,j}=1$ then the agents expect that the asset $j\in\mathcal{A}_0$, *that was negotiated at the node* $\xi-$, delivers the bundle associated to its promises. That is, the agents receive the commodities bundle $\left(\prod_{\xi_j<\nu<\xi}\beta_{\nu,j}\right)A(\xi,j)$. If the node ξ is not a terminal node of j, $\xi\notin T_j$, then the agents also expect to sell their long-positions at prices $\left(\prod_{\xi_j<\nu<\xi}\beta_{\nu,j}\right)q_{\xi,j}$.
- In the case where $\beta_{\xi,j}=0$ all lenders expect to receive the depreciated value of the collateral associated to the asset j,[7]

  (2.12)
  $$\text{DCV}_{\xi,j}\equiv\left(\prod_{\xi_j<\nu<\xi}\beta_{\nu,j}\right)\left[p_\xi Y_\xi\diamond C^1_{\xi-,j}+\sum_{j'\in\mathcal{J}^b(\xi-)\cup\mathcal{J}^m(\xi-)}\text{R}_{\xi,j'}\left(\left(C^2_{\xi-,j}\right)_{j'},0\right)\right],$$
  as well as the asset's foreclosure in the market.
- If $\beta_{\xi,j}\in(0,1)$ then the asset j delivers $\beta_{\xi,j}$ percent of the promises made at the node ξ^-, $\beta_{\xi,j}\left[\left(\prod_{\xi_j<\nu<\xi}\beta_{\nu,j}\right)A(\xi,j)\right]$, and delivers a proportional part of the depreciated value of the collateral constituted at the node ξ^-, $(1-\beta_{\xi,j})\text{DCV}_{\xi,j}$.

  In the case where ξ is not a terminal node of j, then j is renegotiated with prices $\left(\prod_{\xi_j<\nu\leq\xi}\beta_{\nu,j}\right)q_{\xi,j}$, new collateral requirements $\left(\prod_{\xi_j<\nu\leq\xi}\beta_{\nu,j}\right)[C^1_{\xi,j},C^2_{\xi,j}]$, and a new promises process given by $\left[\left(\prod_{\xi_j<\nu\leq\xi}\beta_{\nu,j}\right)A(\mu,j)\right]_{\{\mu>\xi\ ,\ \mu\in\mathbb{R}(j)\}}$.

In sum, each unit of the asset $j\in\mathcal{A}_0$ bought by an agent $h\in\tilde{\mathcal{H}}(\xi)$ has value $\left(\prod_{\xi_j<\nu\leq\xi}\beta_{\nu,j}\right)q_{\xi,j}$, at the node $\xi\in\mathcal{N}(j)$, and delivers, at each node $\mu\in\xi^+$, an amount

$$L_{\mu,j}\equiv\left[\beta_{\mu,j}\left(p_\mu\tilde{\beta}_{\xi,j}A(\mu,j)\right)+(1-\beta_{\mu,j})\text{DCV}_{\mu,j}\right]\tag{2.13}$$

and is sold at prices $\tilde{\beta}_{\mu,j}q_{\mu,j}$, where $\tilde{\beta}_{\xi,j}$ is the renegotiation rule at the node ξ with respect to the node ξ_j

$$\tilde{\beta}_{\xi,j}=\prod_{\xi_j<\nu\leq\xi}\beta_{\nu,j}.\tag{2.14}$$

[7] If the asset j suffers default before the node ξ then the collateral clearly is equal to zero.

On the other hand, at each node $\xi \in \mathcal{D}$, agents h in $\tilde{\mathcal{H}}(\xi)$ are able to determine *endogenous renegotiation rules* $\left[\alpha^h_{\xi,j}\right]_{j\in\mathcal{J}(\xi^-)}$.

That is, for each unit of the asset $j \in \mathcal{A}_0$ sold by an agent $h \in \tilde{\mathcal{H}}(\xi)$, he borrow the value $\tilde{\beta}_{\xi,j}q_{\xi,j}$ for his short positions, and he must establish a collateral $\tilde{\beta}_{\xi,j}[C^1_{\xi,j}, C^2_{\xi,j}]$, whose value is

$$\mathrm{CV}_{\xi,j} \equiv \tilde{\beta}_{\xi,j}\left[p_\xi C^1_{\xi,j} + \sum_{j'\in\mathcal{J}^b\cup\mathcal{J}^m} q_{\xi,j'}\left(C^2_{\xi,j}\right)_{j'}\right]. \tag{2.15}$$

Moreover, he delivers, at each node $\mu \in \xi^+$, the amount

$$B^h_{\mu,j} \equiv \left[\alpha^h_{\mu,j}\left(p_\mu\tilde{\beta}_{\xi,j}A(\mu,j)\right) + (1-\alpha^h_{\mu,j})\mathrm{DCV}_{\mu,j}\right] \tag{2.16}$$

where $\alpha^h_{\mu,j}$ belongs to the interval [0,1] and represents the endogenous renegotiation rule of the agent h for the asset $j \in \mathcal{A}_0$, at the node μ. The interpretation of the different values of $\alpha^h_{\mu,j}$ is analogous to that made for the rules $\beta_{\mu,j}$, but now from the borrowers point of view:

- If $\alpha^h_{\mu,j} = 1$ then the agent h delivers, for each unit of asset $j \in \mathcal{J}^d$ sold by him at the node ξ (remember that $\mu \in \xi^+$), the bundle $\tilde{\beta}_{\xi,j}A(\mu,j)$ at this node, and renegotiates his short position at prices $\alpha^h_{\xi,j}\tilde{\beta}_{\xi,j}q_{\mu,j}$.
- If $\alpha^h_{\mu,j} = 0$ then the agent h goes to default at this node in the asset $j \in \mathcal{J}^d$: he delivers, for each unit of the asset sold at the node ξ, the depreciated value of the collateral $\mathrm{DCV}_{\mu,j}$, and forecloses his short position in asset j.
- If $\alpha^h_{\mu,j} \in (0,1)$ then the agent h delivers, for each unit of asset j sold at the node ξ, a percentage of the promises, $\alpha^h_{\mu,j}\tilde{\beta}_{\xi,j}A(\mu,j)$, he also expects that the lenders seize a proportional part of the collateral constituted at the node ξ,

$$(1-\alpha^h_{\mu,j})\tilde{\beta}_{\xi,j}[Y_\mu \diamond C^1_{\xi,j} + \sum_{j'\in\mathcal{J}^b\cup\mathcal{J}^m} A(\mu,j')\left(C^2_{\xi,j}\right)_{j'}, C^2_{\xi,j}],$$

  and, in the case where μ is not a terminal node of the asset, he sells his short position with unit price $\alpha^h_{\mu,j}\tilde{\beta}_{\xi,j}q_{\mu,j}$.

Therefore, the portfolio $(\theta^h(\xi,j), \varphi^h(\xi,j))$ for the agent $h \in \tilde{\mathcal{H}}(\xi)$ in the asset $j \in \mathcal{A}_0$, at node ξ, is worth

$$\mathrm{PV}_{\xi,j}(\theta^h,\varphi^h) \equiv \mathrm{CV}_{\xi,j}\varphi^h(\xi,j) + \tilde{\beta}_{\xi,j}q_{\xi,j}\left(\theta^h(\xi,j) - \varphi^h(\xi,j)\right), \tag{2.17}$$

and delivers, at each node $\mu \in \xi^+$,

$$\mathrm{R}_{\mu,j}(\theta^h,\varphi^h) \equiv \left(L_{\mu,j} + \tilde{\beta}_{\mu,j}q_{\mu,j}\right)\theta^h(\xi,j) - \left(B^h_{\mu,j} + \alpha^h_{\mu,j}\tilde{\beta}_{\xi,j}q_{\mu,j} - \mathrm{DCV}_{\mu,j}\right)\varphi^h(\xi,j). \tag{2.18}$$

Now, for the assets in $j \in \bigcup_{k \neq 0} \mathcal{A}_k \subset \mathcal{J}^d$, the lenders also take as given anonymous renegotiation rules $\beta_{\xi,j}$, and the borrowers also choose endogenous renegotiation rules $\alpha^h_{\xi,j}$, at the nodes $\xi \in \mathcal{N}(j)$. For these assets, however, the depreciated value of collateral constituted at ξ^- depends on the returns of other assets in $\mathcal{J}^d$ as

$$\text{(2.19)} \qquad \mathrm{DCV}_{\xi,j} \equiv \tilde{\beta}_{\xi^-,j} \left[p_\xi Y_\xi \diamond C^1_{\xi^-,j} + \sum_{j' \in \mathcal{J}^b(\xi^-) \cup \mathcal{J}^m(\xi^-) \cup \mathcal{J}^d(\xi^-)} \mathrm{R}_{\xi,j'} \left(\left(C^2_{\xi^-,j} \right)_{j'}, 0 \right) \right],$$

Note that Assumption D guarantees that equation (2.19) is well-defined. Therefore, analogously to the assets in $\mathcal{A}_0$, a portfolio $(\theta(\xi,j), \varphi(\xi,j))$ in the asset $j \in \mathcal{J}^d - \mathcal{A}_0$ is worth $\mathrm{PV}_{\xi,j}(\theta,\varphi)$, at the node ξ, and delivers, at each node μ, which is the immediate successor of the node ξ, a quantity $\mathrm{R}_{\mu,j}(\theta,\varphi)$, where the functions $\mathrm{PV}_{\xi,j}$ and $\mathrm{R}_{\mu,j}$ are defined by equations (2.17) and (2.18), using equation (2.19).

It is important to note that it is not necessary to identify secondary markets in the economy. That is, once every agent expects to receive returns as a function of the anonymous renegotiation rules $\beta_{\xi,j}$, which are given and are the same for all participants of the markets, then they are indifferent to the identity of the borrower.

Moreover, in some cases the endogenous renegotiation rules are the same for every agent. In fact, if we consider an asset whose returns[8] are less than its associated depreciated collateral at a node, then the optimal strategy for the agents able to negotiate this asset at this node is to take $\alpha^h_{\xi,j}$ equal to one, since other choices would diminish their wealth at this node.

Now, if there are no penalties for default, that is $\lambda^h \equiv 0$ for every agent h in $\mathcal{H}$, then the only enforcement in case of default is the collateral seizure by the lenders. So if the promises of the asset are greater than the associated collateral at a node ξ, then the optimal strategy is to take the value of $\alpha^h_{\xi,j}$ equal to zero, that is, going to default. Therefore, in the absence of these penalties, the only case in which the agents have different endogenous renegotiation rules is when the value of the collateral is the same as the face value.

We assume that in equilibrium there is a compatibility between the anonymous renegotiation rules $\beta_{\mu,j}$ and the endogenous rules $\alpha^h_{\mu,j}$, for each node $\mu \in \xi^+$ and asset j in $\mathcal{J}^d(\xi)$.

In fact, we require that the anonymous rule $\beta_{\mu,j}$ be a mean of the endogenous rules $\alpha^h_{\mu,j}$

$$\text{(2.20)} \qquad \beta_{\mu,j} = \sum_{h \in \tilde{\mathcal{H}}(\xi)} \alpha^h_{\mu,j} p^h(\xi,j).$$

where $p^h(\xi,j)$ is the percentage of the total short-position on asset j owned at the node ξ by the agent h.

[8] At a node ξ the returns of the asset $j \in \mathcal{J}^d(\xi^-)$ are given by the value of real promises $\tilde{\beta}_{\xi^-,j} A(\xi,j)$ plus the spot price $\tilde{\beta}_{\xi^-,j} q_{\xi,j}$. If ξ is a terminal node for j, then the spot price is taken as zero.

This implies that, in equilibrium, the total return delivered by an asset j will be equal to the total expected return, that is [9]

$$(2.21) \qquad \sum_{h\in\tilde{\mathcal{H}}(\xi)} \left(L_{\mu,j} + \tilde{\beta}_{\mu,j} q_{\mu,j}\right) \overline{\theta}^h(\xi, j) = \sum_{h\in\tilde{\mathcal{H}}(\xi)} \left(B^h_{\mu,j} + \alpha^h_{\mu,j} \tilde{\beta}_{\xi,j} q_{\mu,j}\right) \varphi^h(\xi, j),$$

where $\overline{\theta}^h(\xi, j)$ denotes the total long position of the agent h, in the asset $j \in \mathcal{J}^d$, at the node ξ,

$$(2.22) \qquad \overline{\theta}^h(\xi, j) \equiv \theta^h(\xi, j) + \sum_{j'\in\mathcal{J}^d(\xi)} \tilde{\beta}_{\xi,j'} \left(C^2_{\xi,j'}\right)_j \varphi^h(\xi, j'),$$

and $\theta^h(\xi, j)$ (resp. $\varphi^h(\xi, j)$) is the *collateral-free* long-position (resp. short-position) in the asset j [10] of the agent h, at the node ξ.

The set $\mathcal{J}^t$: Short-Lived Tranches protected by Collateral

There are other financial structures in the markets which are protected by collateral requirements, such as the assets $j \in \mathcal{J}^d$, but which do not form "pyramids" with the latter. In fact, the assets $j \in \mathcal{J}^d$ may cause a chain of defaults if their promises are not honored: that is, default on an asset $j \in \mathcal{A}_k$ might cause defaults on all those assets in $\mathcal{A}_{k+1}$ that use it as collateral, and in their turn these might cause defaults on assets in $j \in \mathcal{A}_{k+2}$, and so forth.

One way of preventing this, while at the same time having a small amount of physical goods and assets protect (at least indirectly) all of the economy's assets which are subject to default, is to allow families of assets to be protected by collective collateral requirements. In this way the agents avoid having to constitute excessive collateral requirements, and they are also able to protect several promises with a single collateral requirement.

Consider for instance two assets j_1 and j_2 in $\mathcal{J}^d$, both of which are individually protected, at the node ξ, by the same collateral requirement $C_\xi \in \mathbb{R}^{\mathcal{L}}_+$, which is independent of the price level. Thus, if a given agent wishes to short sell a unit of j_1 and a unit of j_2, then he will have to twice constitute the bundle of goods C_ξ; this would entail very high costs. In this section we will study the scenario where the agent can protect both promises *jointly*, with the same collateral bundle C_ξ, but giving priority to the asset j_1 over the asset j_2 when it comes to paying the returns. In doing so, the agent may receive a smaller amount, since the buyers of asset j_2 will be more exposed to default. The agent, however, might improve her situation relative to the scenario where he sells either j_1 or j_2 but not both.

[9] Note that equation (2.21) is a consequence of both the usual market clearing condition in equilibrium, $\sum_{h\in\tilde{\mathcal{H}}(\xi^-)} \overline{\theta}^h(\xi^-, j) = \sum_{h\in\tilde{\mathcal{H}}(\xi^-)} \varphi^h(\xi^-, j)$, and the fact that the rule $\beta_{\xi,j}$ is not a variable chosen by the agents.

[10] In the sense that it does not involve assets backed as collateral.

More formally, we suppose for simplicity that there exist only *short-lived tranches* along the event-tree $\mathcal{D}$. The set of tranches issued at the node ξ is given by a set $\mathcal{J}^t(\xi)$, where each $k \in \mathcal{J}^t(\xi)$ is characterized by a family of short-lived real assets $\{j_k^1, j_k^2, \dots, j_k^{n_k}\}$ which make individual promises $A(\mu, j_k^m)_{m\in\{1,\dots,n_k\}}$ at the immediate successor nodes $\mu \in \xi^+$, and are jointly protected by collateral requirements $C^k = [C_k^1, C_k^2]$. Such collateral requirements depend on the price level, and are given by the functions

$$C_k^1 : \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)} \times \mathbb{R} \to \mathbb{R}_+^{\mathcal{L}}, \tag{2.23}$$

$$C_k^2 : \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)} \times \mathbb{R} \to \mathbb{R}_+^{\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)}, \tag{2.24}$$

where, analogously to the margin requirements for assets in $\mathcal{J}^m$ and to the collateral requirements for assets in $\mathcal{J}^d$, $C_k^1(p_\xi, q_\xi, q_{\xi,k})$ (resp. $C_k^2(p_\xi, q_\xi, q_{\xi,k})$) denotes the amount of physical collateral (resp. financial collateral) that the borrowers of the tranche $k \in \mathcal{J}^t(\xi)$ must constitute for each unit that they sell. These amounts depend on the prices of the consumption goods at the node ξ, on the prices of the assets in $\mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi)$, and on the sale price of tranche k itself, $q_{\xi,k}$.

That is, even though the assets j_k^m that constitute the family $k \in \mathcal{J}^t(\xi)$ are short-lived, they may be protected both by physical goods and by multiperiod assets in $\mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi)$.

Again for the sake of simplicity we denote such requirements by $C^k = [C_k^1, C_k^2]$, recalling nevertheless that such baskets may vary with price changes.

We assume also that the physical collateral requirements can be broken down into $C_k^1 = C_k^{1,B} + C_k^{1,W}$, where $C_k^{1,B}$ denotes the portion of the collateral that is consumed by the borrower and $C_k^{1,W}$ denotes the portion that must be stored. We assume that no part of the collateral may be consumed by the lender; since each asset j_k^m may be separately negotiated by the lenders in the market, this assumption avoids defining a collateral assignment rule among the various buyers of the assets in k.

We impose the following conditions, which guarantee that the collateral requirements for each tranche vary continuously with the price levels:

ASSUMPTION E: For each tranche $j \in \mathcal{J}^t(\xi)$, the functions C_k^1, C_k^2 are continuous in $\left(p_\xi, (q_{\xi,a})_{a\in\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)\cup\{k\}}\right)$ and different from zero.

We denote by $Y_\mu \diamond C_k^1 \equiv Y_\mu^c C_k^{1,B} + Y_\mu^w C_k^{1,W}$ the depreciated value at the node $\mu \in \xi^+$ of the physical collateral constituted at ξ. As usual, we assume that the financial collateral C_2^k suffers no depreciation.

Now, at the immediate successor nodes $\mu \in \xi^+$ the asset j_k^m has priority over the assets $(j_k^r)_{r>m}$ as regards payments of the returns. *We suppose, for simplicity, that there do not exist utility penalties for agents who default.*

Therefore, each borrower of one unit of the tranche k delivers, at a node $\mu \in \xi^+$, the minimum between the depreciated value of the collateral

$$\mathrm{DCV}_{\mu,k} \equiv p_\mu Y_\mu \diamond C_k^1 + \sum_{j\in\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)} \mathrm{R}_{\mu,j}\left(\left(C_k^2\right)_j, 0\right) \tag{2.25}$$

and the total value of the promises made by the assets $\{j_k^1, j_k^2, \ldots, j_k^{n_k}\}$. That is,

$$B_{\mu,k} \equiv \min\left\{\mathrm{DCV}_{\mu,k}\,;\, \sum_{m=1}^{n_k} p_\mu A(\mu, j_k^m)\right\}. \tag{2.26}$$

On the other hand, the lenders are able to trade each asset $\{j_k^m\}_{\{k\in\mathcal{K},\ m\in\{1,\ldots,n_k\}\}}$. So for each unit of the asset j_k^m bought by a lender, he expects to receive, at each node $\mu \in \xi^+$, the amount

$$L_{\mu,k}^{j_k^1} \equiv \min\left\{p_\mu A(\mu, j_k^1)\,;\, \mathrm{DCV}_{\mu,k}\right\} \tag{2.27}$$

$$L_{\mu,k}^{j_k^m} \equiv \left[\min\left\{\sum_{i=1}^{m} p_\mu A(\mu, j_k^i)\,;\, \mathrm{DCV}_{\mu,k}\right\} - \sum_{i=1}^{m-1} p_\mu A(\mu, j_k^i)\right]^+, \quad m > 1. \tag{2.28}$$

We denote by q_{ξ,j_k^m} the price of asset $j_k^m \in k$ at the node ξ, and we assume that the sale price of a tranche k at node ξ is given by $q_{\xi,k} \equiv \sum_{m=1}^{n_k} q_{\xi,j_k^m}$.[11]

Hence, each agent $h \in \tilde{\mathcal{H}}(\xi)$ gets to choose a portfolio $\left[\left(\theta^h(\xi, j_k^m)\right)_{\{m=1,2,\ldots,n_k\}}; \varphi^h(\xi,k)\right]$ in tranche $k \in \mathcal{J}^t(\xi)$, paying an amount equal to

(2.29)

$$\mathrm{PV}_{\xi,k}(\theta^h, \varphi^h) \equiv \left[p_\xi C_k^1 + \sum_{j\in\mathcal{J}^b(\xi)\cap\mathcal{J}^m(\xi)} q_{\xi,j}\left(C_k^2\right)_j - q_{\xi,k}\right]\varphi^h(\xi,k) + \sum_{m=1}^{n_k} q_{\xi,j_k^m}\theta^h(\xi, j_k^m).$$

At each node $\mu \in \xi^+$ this portfolio yields returns equal to

$$\mathrm{R}_{\mu,k}(\theta^h, \varphi^h) \equiv \mathrm{DCV}_{\mu,k}\varphi^h(\xi,k) + \sum_{m=1}^{n_k} L_{\mu,k}^{j_k^m}\theta^h(\xi, j_k^m) - B_{k,\mu}\varphi^h(\xi,k). \tag{2.30}$$

We thus characterize the economy $\mathcal{E}\,(\mathcal{D}, \mathcal{C}, \mathcal{H}, \mathcal{J})$ by specifying the structure of uncertainty $\mathcal{D}$, the set of durable goods $\mathcal{C} = \left(\mathcal{L}, Y_\xi^c, Y_\xi^w\right)$, the characteristics of the agents $\mathcal{H} = \left(\mathcal{H}(\xi), \tilde{\mathcal{H}}(\xi), w^h, U^h\right)$, and the financial asset structure $\mathcal{J} = \left(\mathcal{J}^b, \mathcal{J}^m, \mathcal{J}^d, \mathcal{J}^t\right)$.

[11] We are therefore interested in the existence of equilibrium in the case where the prices of the assets $j_m \in k$ are not perturbed relative to the tranche price $q_{\xi,k}$. This hypothesis is a good approximation of what occurs in markets where there are no significant transaction costs for buying each asset j_k^m separately.

2.1.5 Equilibrium

For notational convenience let us define the following sets:

- Assets that are bought at node $\xi \in \mathcal{D}$:

$$\mathcal{J}_B(\xi) = \mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi) \cup \mathcal{J}^d(\xi) \cup \left(\bigcup_{k \in \mathcal{J}^t(\xi)} \{j_m : j_m \in k\} \right),$$

- Assets and tranches that are sold at node $\xi \in \mathcal{D}$: $\mathcal{J}_S(\xi) \equiv \mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi) \cup \mathcal{J}^d(\xi) \cup \mathcal{J}^t(\xi)$,
- Assets (indexed by nodes) that can be bought by the agent $h \in \mathcal{H}$:

$$\mathcal{D}^h_B(\mathcal{J}) \equiv \left\{ (\xi, a) : \xi \in \mathcal{D}^h \ , a \in \mathcal{J}_B(\xi) \right\},$$

- Assets and tranches (indexed by nodes) that can be sold by the agent $h \in \mathcal{H}$:

$$\mathcal{D}^h_S(\mathcal{J}) \equiv \left\{ (\xi, a) : \xi \in \mathcal{D}^h \ , a \in \mathcal{J}_S(\xi) \right\},$$

- Assets in $\mathcal{J}^d$ that can be negotiated by the agent $h \in \mathcal{H}$:

$$\mathcal{D}^h(\mathcal{J}^d) \equiv \left\{ (\xi, a) : \xi \in \mathcal{D}^h \ , a \in \mathcal{J}^d(\xi) \right\}.$$

Given a spot price process $(p_\xi)_{\{\xi \in \mathcal{D}\}}$, an asset price process $\left[(q_{\xi,j})_{\{\xi \in \mathcal{D}, j \in \mathcal{J}_B(\xi)\}} \right]$, and anonymous renegotiation rules $(\beta_{\xi,j})_{\{j \in \mathcal{J}^d, \xi \in \mathcal{R}(j)\}}$, each agent h in $\mathcal{H}$, able to trade in the sub-tree $\mathcal{D}^h$, can choose an allocation $(x, y, \theta, \varphi, \alpha)$ in the state-space
$\mathbb{E}^h = X^h \times Y^h \times \mathbb{R}_+^{\mathcal{D}^h_S(\mathcal{J})} \times \mathbb{R}_+^{\mathcal{D}^h_B(\mathcal{J})} \times [0,1]_+^{\mathcal{D}^h(\mathcal{J}^d)}$, subject to:

$$p_{\xi^h} \left[x(\xi^h) + y(\xi^h) \right] + \sum_{a \in \mathcal{J}_S(\xi^h)} \mathrm{PV}_{\xi^h, a}(\theta, \varphi) \leq p_{\xi^h} w^h(\xi^h), \tag{2.31}$$

and, for all $\xi > \xi^h \in \mathcal{D}^h - \delta\mathcal{D}^h$,

$$\begin{aligned} p_\xi \left[x(\xi) + y(\xi) \right] + \sum_{a \in \mathcal{J}_S(\xi)} \mathrm{PV}_{\xi,a}(\theta, \varphi) \\ \leq p_\xi w^h(\xi) + p_\xi \left[Y^c_\xi x(\xi^-) + Y^w_\xi y(\xi^-) \right] + \sum_{a \in \mathcal{J}_S(\xi^-)} \mathrm{R}_{\xi,a}(\theta, \varphi), \end{aligned} \tag{2.32}$$

and, for every node $\xi \in \delta\mathcal{D}^h$,

$$p_\xi x(\xi) \leq p_\xi w^h(\xi) + p_\xi \left[Y^c_\xi x(\xi^-) + Y^w_\xi y(\xi^-) \right] + \sum_{a \in \mathcal{J}_S(\xi^-)} \mathrm{R}_{\xi,a}(\theta, \varphi). \tag{2.33}$$

Moreover, the short positions in the assets $j \in \mathcal{J}^b(\xi)$ are bounded from above by means of $\varphi(\xi, j) \leq b_j^{\xi}$.

Note that given an asset $j \in \mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi)$, an allocation $(x, y, \theta(\xi, j), \varphi(\xi, j))$ that satisfies the aforementioned restrictions is collateral-free, in the sense that such goods and assets are not used to protect other assets that are traded by the agent. We therefore proceed just as we did with assets $j \in \mathcal{J}^d$ (see equation (2.22)), and denote the total amount of asset j bought by agent h at node ξ by

$$\overline{\theta}(\xi, j) \equiv \theta(\xi, j) + \sum_{j' \in \mathcal{J}^m(\xi)} \left(M^2_{\xi, j'}\right)_j \varphi(\xi, j') + \sum_{j' \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi, j'} \left(C^2_{\xi, j'}\right)_j \varphi(\xi, j') + \sum_{k \in \mathcal{J}^t(\xi)} \left(C^2_k\right)_j \varphi(\xi, k), \quad \forall j \in \mathcal{J}^b(\xi), \tag{2.34}$$

$$\overline{\theta}(\xi, j) \equiv \theta(\xi, j) + \sum_{j' \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi, j'} \left(C^2_{\xi, j'}\right)_j \varphi(\xi, j') + \sum_{k \in \mathcal{J}^t(\xi)} \left(C^2_k\right)_j \varphi(\xi, k), \quad \forall j \in \mathcal{J}^m(\xi), \tag{2.35}$$

The total amount of goods demanded for consumption by agent $h \in \mathcal{H}$ at node ξ is given by

$$\overline{x}(\xi) \equiv x(\xi) + \sum_{j \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi, j} \left(C^{1,B}_{\xi, j} + C^{1,L}_{\xi, j}\right) \varphi(\xi, j) + \sum_{k \in \mathcal{J}^t(\xi)} C^{1,B}_k \varphi(\xi, k). \tag{2.36}$$

And the total amount of goods demanded for storage is given by

$$\overline{y}(\xi) \equiv y(\xi) + \sum_{j \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi, j} C^{1,W}_{\xi, j} \varphi(\xi, j) + \sum_{k \in \mathcal{J}^t(\xi)} C^{1,W}_k \varphi(\xi, k) + \sum_{j \in \mathcal{J}^m(\xi)} M^1_{\xi, j} \varphi(\xi, j). \tag{2.37}$$

Now, given prices and renegotiation rules (p, q, β), we define the budget set of agent h as

$$\mathcal{B}^h(p, q, \beta) = \left\{(x, y, \theta, \varphi, \alpha) \in I\!\!E^h : \text{equations (2.31), (2.32) and (2.33) hold}\right\}. \tag{2.38}$$

Nevertheless, depending on the generality of the economy's financial structure, it might be necessary to restrict agent h's possible portfolio choices. Thus, as is already usual in the infinite-horizon economy literature (see Magill and Quinzii (1994, 1996), Levine and Zame (1996), and Florenzano and Gourdel (1996)) we shall also work with two alternative definitions for the budget-set:

- The budget set, of the agent h, with explicit debts constraints M^h:

$$\mathcal{B}^{M^h, h}(p, q, \beta) = \left\{(x, y, \theta, \varphi, \alpha) \in \mathcal{B}^h(p, q, \beta) : \sum_{a \in \mathcal{J}_S(\xi)} \text{PV}_{\xi, a}(\theta, \varphi) \geq -M^h, \ \forall \xi \in \mathcal{D}^h - \delta\mathcal{D}^h\right\}.$$

- The budget set with implicit debt constraint:

$$\mathcal{B}^{DC,h}(p,q,\beta)=\left\{(x,y,\theta,\varphi,\alpha)\in\mathcal{B}^h(p,q,\beta):\left(\sum_{a\in\mathcal{J}_S(\xi)}\mathrm{PV}_{\xi,a}(\theta,\varphi)\right)\in l^\infty(\mathcal{D}^h-\delta\mathcal{D}^h)\right\}$$

We generally denote by $\mathcal{B}^{*,h}(p,q,\beta)$ any of the budget-sets defined above.

Because our objective is to show the existence of equilibrium, we restrict the space of prices to $\mathcal{P}\equiv\{(p_\xi,q_\xi)_{\xi\in\mathcal{D}}:(p_\xi,q_\xi)\in\Delta_+^{\mathcal{L}+\mathcal{J}_B(\xi)-1}\}$, where Δ_+^{n-1} denotes the n-dimensional simplex.

Note that given an allocation $(x^h,y^h,\theta^h,\varphi^h,\alpha^h)\in\mathcal{B}^{*,h}(p,q,\beta)$, the associated consumption at a node $\xi\in\mathcal{D}^h$ is [12]

$$c_\xi(x^h,y^h,\theta^h,\varphi^h)\equiv\overline{x}^h(\xi)-\sum_{j\in\mathcal{J}^d(\xi)}\tilde{\beta}_{\xi,j}\left[C^{1,L}_{\xi,j}\varphi^h(\xi,j)-C^{1,L}_{\xi,j}\overline{\theta}^h(\xi,j)\right],\tag{2.39}$$

and the *penalty for default* at ξ is

$$\begin{aligned}P^h_\xi(\varphi^h,\alpha^h)\;&\equiv\;\sum_{j\in\mathcal{J}^d(\xi^-)}\lambda^h_{\xi,j}\left[p_\xi\tilde{\beta}_{\xi^-,j}A(\xi,j)+\tilde{\beta}_{\xi^-,j}q_{\xi,j}-\left(\alpha^h_{\xi,j}\tilde{\beta}_{\xi^-,j}q_{\xi,j}+B^h_{\xi,j}\right)\right]^+\varphi^h(\xi^-,j)\\&\equiv\;\sum_{j\in\mathcal{J}^d(\xi^-)}\lambda^h_{\xi,j}\left(1-\alpha^h_{\xi,j}\right)\left[p_\xi\tilde{\beta}_{\xi^-,j}A(\xi,j)+\tilde{\beta}_{\xi^-,j}q_{\xi,j}-\mathrm{DCV}_{\xi,j}\right]^+\varphi^h(\xi^-,j)\end{aligned}$$

It is useful, to shorten the notations, define the objective function of the agent h, evaluated in an allocation $(x,y,\theta,\varphi,\alpha)$ in the budget set $\mathcal{B}^{*,h}(p,q,\beta)$, by

$$V^h(x,y,\theta,\varphi,\alpha)\equiv U^h\left[(c_\xi(x,y,\theta,\varphi))_{\xi\in\mathcal{D}^h}\right]-\sum_{\xi\in\mathcal{D}^h}P^h_\xi(\varphi,\alpha).\tag{2.40}$$

It follows that given prices and anonymous renegotiation rules $(p,q,\beta)\in\mathcal{P}\times[0,1]^{\mathcal{D}^h(\mathcal{J}^d)}$, then consumer h's problem in the economy $\mathcal{E}$ is

$$\begin{aligned}&\max && V^h(x,y,\theta,\varphi,\alpha),\\&\text{subject to} && (x,y,\theta,\varphi,\alpha)\in\mathcal{B}^{*,h}(p,q,\beta).\end{aligned}\tag{2.41}$$

We now define the three possible notions of equilibrium that arise from the alternative definitions for budget-set:

[12] Throughout the chapter we use the convention that given a node ξ, then $\varphi^h(\xi)=\overline{\theta}^h(\xi)=0$ for the agents $h\notin\tilde{\mathcal{H}}(\xi)$.

Definition 1 *An Equilibrium (respectively, an Equilibrium with Explicit Debts Constraints, or an Equilibrium with Implicit Debt Constraint) for the Infinite Horizon Exogenous Collateral Economy* $\mathcal{E}(\mathcal{D},\mathcal{C},\mathcal{H},\mathcal{J})$ *is a vector of prices and anonymous renegotiation rules* $(p,q,\beta) \in \mathcal{P} \times [0,1]^{\mathcal{D}(\mathcal{J})}$ *such that there exist allocations* $\left[x^h, y^h, \theta^h, \varphi^h, \alpha^h\right]_{h\in\mathcal{H}} \in \Pi_{h\in\mathcal{H}} \mathbb{E}^h$ *that satisfy*

I. For each agent $h \in \mathcal{H}$*, the allocation* $(x^h, y^h, \theta^h, \varphi^h, \alpha^h)$ *solves the problem (2.41) in* $\mathcal{B}^h(p,q,\beta)$ *(respectively* $\mathcal{B}^{M^h,h}(p,q,\beta)$*, or* $\mathcal{B}^{DC,h}(p,q,\beta)$*).*

II. Asset markets are cleared, that is, for each node ξ *if* $j \in \mathcal{J}^m(\xi) \cup \mathcal{J}^b(\xi)$ *or* $j \in \mathcal{J}^d(\xi)$ *(with* $\tilde{\beta}_{\xi,j} \neq 0$*) we have*

$$\sum_{h\in\tilde{\mathcal{H}}(\xi)} \overline{\theta}^h(\xi,j) = \sum_{h\in\tilde{\mathcal{H}}(\xi)} \varphi^h(\xi,j). \tag{2.42}$$

III. Tranche markets are cleared, that is, for each node ξ *and tranche* $k \in \mathcal{J}^t(\xi)$ *we have*

$$\sum_{h\in\tilde{\mathcal{H}}(\xi)} \theta^h(\xi,j_m) = \sum_{h\in\tilde{\mathcal{H}}(\xi)} \varphi^h(\xi,k), \quad \forall j_k^m \in k. \tag{2.43}$$

IV. The total demand for commodities is equal to the total wealth at each node of the economy

$$\sum_{h\in\mathcal{H}(\xi_0)} \overline{x}^h(\xi_0) + \sum_{h\in\mathcal{H}(\xi_0)} \overline{y}^h(\xi_0) = \sum_{h\in\mathcal{H}(\xi)} w^h(\xi_0), \tag{2.44}$$

$$\sum_{h\in\mathcal{H}(\xi)} \overline{x}^h(\xi) + \sum_{h\in\tilde{\mathcal{H}}(\xi)} \overline{y}^h(\xi) = \sum_{h\in\mathcal{H}(\xi)} w^h(\xi) + \sum_{h\in\tilde{\mathcal{H}}(\xi^-)} \left[Y_\xi^c \overline{x}^h(\xi^-) + Y_\xi^s \overline{y}^h(\xi^-)\right]. \tag{2.45}$$

V. The endogenous renegotiation rules perfect-foresight the anonymous rules. In other words, for each node ξ *and asset* $j \in \mathcal{J}^d(\xi^-)$ *such that* $\tilde{\beta}_{\xi^-,j} \neq 0$*, we have*

$$\sum_{h\in\tilde{\mathcal{H}}(\xi^-)} \beta_{\xi,j}\overline{\theta}^h(\xi^-,j) = \sum_{h\in\tilde{\mathcal{H}}(\xi^-)} \alpha^h_{\xi,j}\varphi^h(\xi^-,j). \tag{2.46}$$

2.2 Existence of Equilibrium

In this section we state the main result of the paper, that is, the theorem that guarantees the existence of equilibrium in the Exogenous Collateral Economy.

Theorem 2.1: *Suppose that Assumptions A to E are satisfied, and*

a. *For each agent h in $\mathcal{H}$, $w^h(\xi) \in \mathbb{R}^{\mathcal{L}}_{++}$ for all node ξ in $\mathcal{D}^h$. Moreover, there exists a upper bound in the aggregated initial endowments at each node of the economy, that is, $\sum_{h\in\mathcal{H}(\xi)} w^h(\xi, l) \leq \overline{W}$ for all (ξ, l) in $\mathcal{D} \times \mathcal{L}$.*

b. *The utility function for the agent h, U^h, is separable in the time and in the states of nature, in the sense that*

$$U^h(c) = \sum_{\xi\in\mathcal{D}^h} u^h(\xi, c(\xi)),$$

where the functions $u^h(\xi, \ )$ are strict concave, continuous and nondecreasing, with $u^h(\xi, 0) = 0$.

c. *For each agent h, the penalties for default $\lambda^h_{\xi,j}$, are nonnegative for all assets $j \in \mathcal{J}^d(\xi^-)$ and nodes ξ in $\mathcal{D}^h - \{\xi_j\}$.*

d. *There exists $\kappa \in (0, 1)$ such that the structure of depreciation $[Y^c_\xi, Y^s_\xi]$ satisfy*

$$Y^c_\xi = diag[a_l(\xi)], \ \ Y^s_\xi = diag[b_l(\xi)], \quad , \forall \xi \in \mathcal{D},$$

where $max_{l\in\mathcal{L}}\{a_l(\xi); b_l(\xi)\} \leq \kappa$.

then there exists an equilibrium with implicit debts constraint in the economy $\mathcal{E}(\mathcal{D}, \mathcal{C}, \mathcal{H}, \mathcal{J})$.

Sketch of the Proof: First note that Assumptions B, C, and D guarantee that the economy's feasible allocations (that is, those allocations that satisfy the market clearing conditions in the definition of equilibrium) are uniformly bounded at each node of the tree.

Thus, by truncating the original economy into a finite number of periods, and assuming that the collateral coefficients and the margin requirements are strictly positive, and independent of the price level, existence of equilibrium may be shown in a generalized game such that: agents are utility maximizers; there are, at each node, auctioneers whose goal is to maximize the market's excess demand; there are fictitious agents who adjust the anonymous renegotiation rules, taking as given the endogenous renegotiation rules chosen by the agents; finally, there are other fictitious agents who adjust the collateral requirements, taking as given the price levels.

Every equilibrium of the generalized game is an equilibrium of the original economy when it is truncated into a finite number of periods.

Considering an increasing sequence of truncated economies, one shows that the sequence of these finite-dimensional problems' Lagrange multipliers is uniformly bounded at each node. In order to do so one must use the boundedness of the assets' returns along the tree (Assumptions A,B, C, and E).

Thus, since the allocations and the equilibrium prices are also uniformly bounded at each node, it follows from the countability of the set of states of nature that there is a common convergent subsequence, which goes to an allocation in the budget set determined by the limit prices. Such allocations satisfy the feasibility conditions at each node.

Furthermore, since the admissible portfolios must satisfy the implicit debts constraint, it follows that such an allocation is optimal for each agent in the economy. One thus guarantees the existence of equilibrium with implicit debts constraint.

The only goal of the implicit debts constraint is therefore to guarantee that the agents have no incentive to become borrowers "at infinity". □

Now, the existence of equilibrium with implicit debts constraints guarantees that it is always possible to find exogenous limits to the debt levels of agents, $\left(M^h\right)_{h\in\mathcal{H}}$, such that there exists an equilibrium with explicit debts constraints that never bind:

Corollary 2.1: *Under the conditions of the Theorem above, there always exists an equilibrium with explicit debts constraints that never bind.*

Proof: Given an equilibrium with implicit debt constraint $\left[(\overline{p},\overline{q},\overline{\beta});(\overline{x}^h,\overline{y}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\alpha}^h)_{h\in\mathcal{H}}\right]$ consider, for each agent $h\in\mathcal{H}$, a number M^h such that

$$\sup_{\xi\in\mathcal{D}^h-\delta\mathcal{D}^h}\left|\sum_{a\in\mathcal{J}_S(\xi)}\mathrm{PV}_{\xi,a}(\overline{\theta}^h,\overline{\varphi}^h)\right| < M^h. \tag{2.47}$$

Then, given the prices $(\overline{p},\overline{q},\overline{\beta})$, the set of allocations $(\overline{x}^h,\overline{y}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\alpha}^h)$ constitutes an equilibrium with explicit debt constraints $\left(M^h\right)_{h\in\mathcal{H}}$ that never binds. □

Clearly, if there is only a finite number of agents in the economy then the exogenous bound on the debt level may be assumed to be the same for all traders.

As was argued by Magill and Quinzii (1994), the existence of an equilibrium with explicit debt constraint that never binds, guarantees that the introduction of the exogenous debt restriction provides the market with no new imperfections other than those associated to the financial incompleteness. But it is true that equilibrium might not exist without the ad-hoc imposition of this condition, since the agents could then choose portfolios which generate unbounded and indefinitely increasing debts.

It would therefore be interesting to obtain equilibrium results which do not rely on a priori debt constraint on the agents' choices. In this direction, if we assume that the only enforcement in case of default is seizure of the collateral by the lenders, then all of the agents who sell assets in $\mathcal{J}^d$ will deliver, as returns, the minimum between the value of the real promises and the depreciated value of the collateral constituted by them.

Hence a short position in such assets delivers non-negative returns in the following period (since the agents will have the depreciated value of the collateral and have to deliver the minimum between that amount and the value of the real promises).

Thus, if there are no assets in the economy other than those which are either subject to default or constitute collateral tranches, every borrower acts as a lender of assets which only they can buy; they therefore have no interest in becoming borrowers "at infinity". This implies that the implicit debt restrictions are redundant. We have obtained the following result:

Corollary 2.2: *Under the hypotheses of Theorem 1, there is always an equilibrium (with no need of debt constraints on the agents' budgets sets) if any of the following conditions is met:*

a. *There are no infinite-lived agents trading in the economy, or*

b. *The only assets traded in the economy are those belonging to $\mathcal{J}^d \cup \mathcal{J}^t$, and there are no utility penalties for agents who default.*

Proof: The proof of Theorem 2.1 can be adapted to show the existence of equilibrium in either of the two cases above. In fact, the only goal of the implicit debt constraint is to guarantee that the agents have no incentive to become borrowers at infinity.

If there are no infinite-lived agents, then of course no agent may indefinitely accumulate debts by using new capital to pay for past debts, since at the terminal nodes the agents must make good on all their debts (even if this means defaulting).

In the second case, as has already been argued, every borrower will act as a lender, since the assets they sell will always deliver non-negative returns. The transversality condition on the debt is thus trivially satisfied. □

Corollary 2.2 thereby extends the first chapter's results to the case of financial collateral requirements, incomplete participation of the agents, and especially, of financial markets with multiperiod assets.

It should be noted that the transversality conditions and the debt constraints guarantee the *generic* existence of equilibrium in default-free economies whose assets may deliver dividends along several periods, as in Magill and Quinzii (1996) or Levine and Zame (1996). Therefore, the imposition of collateral requirements in an economy with default is not only less ad-hoc than the imposition of debt restrictions, but also guarantees that there are prices which centralize the agents' decisions.

2.3 Asset Pricing and Speculative Bubbles

2.3.1 Admissible Prices

In this section, we will characterize the *admissible prices*, that is, the prices which give a finite optimum to the agents' problems. We suppose that Assumptions A to E are satisfied and that the hypothesis of Theorem 2.1 holds.

It is important to note that the characterization of non-arbitrage prices does not make sense in the context of exogenous collateral models, because the feasibility allocations are bounded, node by node, and this prevents the occurrence of any arbitrage opportunities through allocations that satisfy the feasibility conditions.

Notice that, in equilibrium, the prices $(\overline{p}, \overline{q})$ provide finite solutions for the agents' problems: this follows from both the uniform bound along the event-tree in the consumption allocations, and from the finiteness of the utilities in consumption processes that are bounded from above along the event-tree.

Therefore, given an agent h in $\mathcal{H}$, we are interested in the prices $(p, q) \in \mathcal{P}$ for which there exist anonymous renegotiation rules β that provide problem (2.41) with a finite solution. A necessary condition for this is that, at each node ξ in the event-tree, there do not exist allocations that give with certainty either real positive returns or an improvement of the utility, without any cost.

In other words, one necessary condition for the problem of consumer $h \in \mathcal{H}$ to have a finite solution at prices (p, q, β) is that no allocation $(x, y, \theta, \varphi, \alpha)$ satisfy the following conditions at node $\xi \in \mathcal{D}^h - \delta\mathcal{D}^h$:

$$p_\xi\big[x(\xi) + y(\xi)\big] + \sum_{a \in \mathcal{J}_S(\xi)} \mathrm{PV}_{\xi,a}(\theta, \varphi) \leq 0; \tag{2.48}$$

$$\big[c_\xi^h(x, y, \theta, \varphi)\big]_l \geq 0 \quad ,\forall l \in \mathcal{L}; \tag{2.49}$$

$$W(\mu; x, y, \theta, \varphi, \alpha) \equiv p_\mu\left[Y_\mu^c x(\xi) + Y_\mu^w y(\xi)\right] + \sum_{a \in \mathcal{J}_S(\xi)} \mathrm{R}_{\mu,a}(\theta, \varphi) \geq 0 \quad ,\forall \mu \in \xi^+; \tag{2.50}$$

$$\max_{\{x : p_\mu x \leq W(\mu; x, y, \theta, \varphi, \alpha)\}} \left[u^h(\mu, x) - P^h_{(\mu,\varphi,\alpha)}\right] \geq 0 \quad ,\forall \mu \in \xi^+; \tag{2.51}$$

where at least one of the inequalities is strict.

In fact, if an allocation satisfies the equations above, with at least one strict inequality, then either the associated consumption bundle is non-trivial (equation (2.49)), or there is a node μ, which is an immediate successor of node ξ, at which the allocation delivers positive returns (equation (2.51)), without entailing costs at node ξ (equation (2.48)).

2.3.2 Asset Pricing in Economies without Utility Loss for Default

For the sake of simplicity we will analyze admissible prices (p, q) in the case where the agents' objective functions *include no penalties for default.*

In this particular case, the existence of an allocation $(x, y, \theta, \varphi, \alpha)$ that satisfies the above equations with at least one strict inequality, at the prices (p, q, β), is equivalent to the existence of an allocation (x, y, θ, φ) that satisfies, with at least one strict inequality, the following conditions:

$$p_\xi\big[x(\xi) + y(\xi)\big] + \sum_{a \in \mathcal{J}_S(\xi)} \mathrm{PV}_{\xi,a}(\theta, \varphi) \leq 0; \tag{2.52}$$

$$\big[c_\xi^h(x, y, \theta, \varphi)\big]_l \geq 0 \quad ,\forall l \in \mathcal{L}; \tag{2.53}$$

$$W(\mu; x, y, \theta, \varphi, \beta) \geq 0 \quad ,\forall \mu \in \xi^+; \tag{2.54}$$

In fact, if (x, y, θ, φ) satisfies the conditions (2.52)-(2.54) then $(x, y, \theta, \varphi, \beta)$ satisfies equations (2.48)-(2.51) (because we are assuming that there are no utility penalties for agents who default).

Conversely, if $(x, y, \theta, \varphi, \alpha)$ satisfies equations (2.48)-(2.51) then, since there are no utility losses for agent $h \in \mathcal{H}$ in case of default, we can assume without loss of generality that $\beta_{\mu,j} = 1$ if the value of the promises associated with asset j, at node μ, is *strictly less* than the depreciated value of the collateral constituted at ξ. If the depreciated value of the collateral is *strictly greater* than the value of the promises at μ then we can assume that $\beta_{\mu,j}$ equals zero, and if the promises and the depreciated collateral are worth the same then the value of $\beta_{\mu,j}$ may take any value in the interval $[0, 1]$. Hence allocation $(x, y, \theta, \varphi, \beta)$ also satisfies equations (2.48)-(2.51) and therefore equations (2.52)-(2.54).

This last remark allows us to study the functional form of the admissible prices at a node $\xi \in \mathcal{D}$, independent of the agents who trade at that node.

Note now that, given the prices of the commodities and of the assets in the economy, then for each node ξ the functions $\mathrm{PV}_{\xi,a}$ and $[\mathrm{R}_{\mu,a}]_{\mu\in\xi^+}$ are linear in (θ,φ) for every asset $a \in \mathcal{J}_S(\xi) \cup \mathcal{J}_B(\xi)$. In fact, for each $a \in \mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi) \cup \mathcal{J}^d(\xi)$ we have

$$\mathrm{PV}_{\xi,a}(\theta,\varphi) = \mathrm{PV}_{\xi,a}(1,0)\,\theta(\xi,a) + \mathrm{PV}_{\xi,a}(0,1)\,\varphi(\xi,a), \tag{2.55}$$

$$\mathrm{R}_{\mu,a}(\theta,\varphi) = \mathrm{R}_{\mu,a}(1,0)\,\theta(\xi,a) + \mathrm{R}_{\mu,a}(0,1)\,\varphi(\xi,a), \quad \forall \mu \in \xi^+. \tag{2.56}$$

And for each tranche $k \in \mathcal{J}^t(\xi)$ we have

$$\mathrm{PV}_{\xi,k}(\theta,\varphi) = \sum_{m=1}^{n_k} q_{\xi,j_k^m}\,\theta(\xi,j_k^m) + \mathrm{PV}_{\xi,a}\,(0;1)\;\varphi(\xi,k), \tag{2.57}$$

$$\mathrm{R}_{\mu,k}(\theta,\varphi) = \sum_{m=1}^{n_k} L_{\mu,k}^{j_k^m}\,\theta(\xi,j_k^m) + \mathrm{R}_{\mu,a}\,(0;1)\;\varphi(\xi,k), \quad \forall \mu \in \xi^+. \tag{2.58}$$

Thus, the conditions given by equations (2.52)-(2.54) may be written in matrix form. That is, for every node ξ there is a matrix $\mathbb{A}_\xi$ such that, if the prices (p,q) are admissible then there is no allocation $(x,y,\theta,\varphi) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}_B(\xi)\times\mathcal{J}_S(\xi)}$ that satisfies[13]

$$\mathbb{A}_\xi \begin{pmatrix} x \\ y \\ \theta \\ \varphi \end{pmatrix} \geq 0. \tag{2.59}$$

It follows from this, by applying Stiemke's lemma (see Hildenbrand (1974)), that if the optimal utility level of the consumers $h \in \mathcal{H}$ is *finite*, at the price level (p,q,β), then there are *strictly positive* state prices $\left[\gamma_\xi, (\tilde{\gamma}_{\xi,l})_{l\in\mathcal{L}}, (\gamma_{\xi,l})_{l\in\mathcal{L}}, (\gamma_{\xi,j}^L)_{j\in\mathcal{J}_B(\xi)-\mathcal{J}^b(\xi)}, (\gamma_{\xi,j}^B)_{j\in\mathcal{J}_S(\xi)-\mathcal{J}^b(\xi)}\right]_{\xi\in\mathcal{D}}$ such that, given a node $\xi \in \mathcal{D}$, we have

- For each commodity $l \in \mathcal{L}$,

$$p_{\xi,l} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+} \gamma_\mu p_\mu (Y_\mu^c)_l + \tilde{\gamma}_{\xi,l}\right), \tag{2.60}$$

- For each asset $j \in \mathcal{J}^b(\xi)$,

$$q_{\xi,j} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+} \gamma_\mu \left(p_\mu A(\mu,j) + q_{\mu,j}\right)\right), \tag{2.61}$$

[13]The matrix $\mathbb{A}_\xi$ depends on the price level (p,q) and on the anonymous renegotiation rules.

- For each asset $j \in \mathcal{J}^m(\xi)$,

$$q_{\xi,j} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu\left(p_\mu A(\mu,j)+q_{\mu,j}\right)+\gamma^L_{\xi,j}\right), \tag{2.62}$$

$$\mathrm{MV}_{\xi,j}-q_{\xi,j} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu\left(\mathrm{DMV}_{\mu,j}-p_\mu A(\mu,j)-q_{\mu,j}\right)+\gamma^B_{\xi,j}\right), \tag{2.63}$$

- For each asset $j \in \mathcal{J}^d(\xi)$,

$$\tilde\beta_{\xi,j}q_{\xi,j} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu\left(L_{\mu,j}+\tilde\beta_{\mu,j}q_{\mu,j}\right)+\sum_{l\in\mathcal{L}}\gamma_{\xi,l}\tilde\beta_{\xi,j}(C^{1,L}_{\xi,j})_l+\gamma^L_{\xi,j}\right), \tag{2.64}$$

$$\begin{aligned}\mathrm{CV}_{\xi,j}-\tilde\beta_{\xi,j}q_{\xi,j} &= \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu\left(\mathrm{DVC}_{\mu,j}-B^h_{\mu,j}-\tilde\beta_{\mu,j}q_{\mu,j}\right)\right)\\ &+\frac{1}{\gamma_\xi}\left(\sum_{l\in\mathcal{L}}\gamma_{\xi,l}\tilde\beta_{\xi,j}\left(C^{1,B}_{\xi,j}+\sum_{j'\in\mathcal{J}^d(\xi)}C^{1,L}_{\xi,j'}\tilde\beta_{\xi,j'}(C^2_{\xi,j})_{j'}\right)_l+\gamma^B_{\xi,j}\right),\end{aligned} \tag{2.65}$$

- For each tranche $k \in \mathcal{J}^t(\xi)$,

$$q_{\xi,j^m_k} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu L^{j^m_k}_k+\gamma^L_{\xi,j^m_k}\right),\quad \forall m\in\{1,\dots,n_k\}, \tag{2.66}$$

$$\mathrm{CV}_{\xi,k}-q_{\xi,k} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu\left(\mathrm{DCV}_{\mu,k}-B_{\mu,k}\right)+\gamma^B_{\xi,k}\right). \tag{2.67}$$

Hence, the admissibility of the prices (p,q) implies that there are state prices which allow us to express a given asset's sale and purchase prices, at a node ξ, as the discounted value of the future dividends generated by the asset.[14] Note that the functional form of the prices of assets in $\mathcal{J}^m\cup\mathcal{J}^d\cup\mathcal{J}^t$ includes non-linearities, which follow from the positiveness conditions imposed on the purchase and sale portfolios.

Due to such non-linearities, which are given by strictly positive deflators, the purchase prices of all assets negotiated at a node ξ will be strictly positive. Furthermore, since in our context

[14]For the sake of simplicity we will in this section treat commodities as assets, because their durability allows us to regard them as Lucas' trees; that is, as rights to benefit from part of a firm's output through the current consumption of certain bundles, as well as through the right to sell the depreciated values of such amounts in future states of nature. The non-linearity of commodity prices follows from the fact that agents cannot buy such "rights".

there are no utility penalties for agents who default on an asset $j \in \mathcal{J}^d$, the sale prices of assets in $\mathcal{J}^d$ will be non-negative, that is

$$\mathrm{CV}_{\xi,j} - \tilde{\beta}_{\xi,j} q_{\xi,j} \geq 0, \quad \forall \xi \in \mathcal{D},\ \forall j \in \mathcal{J}^d(\xi). \tag{2.68}$$

This last inequality is fundamental for obtaining conditions which characterize the existence of speculative bubbles in the prices of assets subject to default.

Speculative Bubbles in Admissible Prices

In the literature on Sequential Equilibrium in Infinite-Horizon Economies, the existence of speculative bubbles in non-arbitrage prices has been studied, among others, by Santos and Woodford (1997) and Magill and Quinzii (1996), in the case where the agents cannot default on their promises and where short sales are bounded by debt restrictions or by exogenous transversality conditions.

In markets where the agents are required to constitute margin requirements or collateral bundles when short selling, whether the aim is to protect buyers in case of future default or to limit the volume of trades through indexation to the total amount of goods in the economy (in models without default), it is important to study speculation in admissible prices, given the possible applications to the macroeconomics literature on financial crises.

It is natural to expect that the admissible price of an asset backed by other assets or goods exhibits bubbles only if the admissible price of one of the corresponding collateral assets or goods differs from its fundamental value. In fact, one might think of an asset that is backed by a (financial or physical) collateral bundle as a derivative, whose underlying assets are those securities or goods that back the short sales. In this sense, one might intuitively think of distortions in a derivative's price relative to its fundamental value as always being caused by bubbles in the prices of one of the underlying assets.

In this section we will discuss such issues and show that the admissible price of an asset protected by collateral, ($j \in \mathcal{J}^d$), exhibits bubbles if and only if the following two conditions hold: the asset is negotiated, at these prices, for an infinite number of periods, and the collateral's depreciated value no tends to zero at infinity (see Lemma 2.4 below).

Using this result and the pyramiding structure given by Assumption D, we show that if an asset $j \in \mathcal{J}^d$ exhibits bubbles, then there is price speculation for at least one of the *default-free assets* in the economy, that is, those assets which belong to $\mathcal{L} \cup \mathcal{J}^b \cup \mathcal{J}^m$. Thus, in order to prevent price distortions relative to the fundamental value of a given asset, it is enough to guarantee that there are no bubbles in the assets that are free of default. Conceivably, one way of doing so could be to adapt the results by Santos and Woodford (1996) and Magill and Quinzii

(1996), since in an infinite-horizon economy without default every $j \in \mathcal{L} \cup \mathcal{J}^b \cup \mathcal{J}^m$ can be regarded as an asset whose short sales are exogenously restricted (see Remark 2.2 below).

Since we are dealing with short-lived (i.e., single-period) tranches, the equations (2.66) - (2.67) guarantee that the sale price of tranche $k \in \mathcal{J}^t(\xi)$ as well as the purchase prices of the assets in family k do not exhibit speculation, because they are equal to the discounted value (using the γ deflators) of the future dividends.

We therefore restrict our attention to durable consumption goods and to assets belonging to $\mathcal{J}^b \cup \mathcal{J}^m \cup \mathcal{J}^d$.

Given admissible prices (p, q) and renegotiation rules β, consider a state prices vector $\Gamma_\xi = \left[\gamma_\mu, (\tilde{\gamma}_{\mu,l})_{l \in \mathcal{L}}, (\gamma_{\mu,l})_{l \in \mathcal{L}}, (\gamma^L_{\mu,j})_{j \in \mathcal{J}_B(\mu) - \mathcal{J}^b(\mu)}, (\gamma^B_{\mu,j})_{j \in \mathcal{J}_S(\mu) - \mathcal{J}^b(\mu)}\right]_{\mu \in \mathcal{D}(\xi)}$ that satisfies equations (2.60)-(2.65) at each node $\mu \in \mathcal{D}(\xi)$.

It is important to note that since markets are incomplete, there might not be a unique state prices vector Γ_ξ at ξ. We therefore define the fundamental value of the various assets negotiated at node ξ as a function of the chosen state prices vector Γ_ξ.

- **Commodities.**

The *fundamental value* at node ξ of a durable good $l \in \mathcal{L}$ is given by

(2.69)
$$\mathrm{F}(\xi, \Gamma_\xi, l) \equiv \frac{1}{\gamma_\xi} \left(\sum_{\mu \geq \xi} \tilde{\gamma}_\mu Y^c_{\xi,\mu} \right)_l ,$$

where $\tilde{\gamma}_\mu = (\tilde{\gamma}_{\mu,l})_{l \in \mathcal{L}}$ and $Y^c_{\xi,\mu}$ is the accumulated depreciation factor of consumption between the nodes ξ and μ (in other words, $Y^c_{\xi,\xi}$ is the identity matrix, and for every $\mu > \xi$, $Y^c_{\xi,\mu}$ is the product of the depreciation matrices Y^c_ν, where $\xi < \nu < \mu$).

We say that the price of good $l \in \mathcal{L}$ is free of bubbles, *in the weak sense*, at node ξ, if there is a deflator vector Γ_ξ such that $p_{\xi,l} = \mathrm{F}(\xi, \Gamma_\xi, l)$.

If for every possible deflator Γ_ξ the good l does not exhibit bubbles, then we say that the price $p_{\xi,l}$ is free of bubbles *in the strong sense*.

Using the deflators Γ_ξ and recursively applying equation (2.60) we have

(2.70)
$$p_{\xi,l} = \frac{1}{\gamma_\xi} \left(\sum_{\mu \in \mathcal{D}(\xi): \tilde{t}(\mu) < T} \tilde{\gamma}_\mu Y^c_{\xi,\mu} \right)_l + \frac{1}{\gamma_\xi} \left(\sum_{\mu \in \mathcal{D}(\xi): \tilde{t}(\mu) = T} \gamma_\mu p_\mu (Y^c_{\xi,\mu}) \right)_l , \quad \forall l \in \mathcal{L},\ \forall T > \tilde{t}(\xi).$$

Denoting by $\mathrm{F}_T(\xi, \Gamma_\xi, l)$ the first term of the right side of equation (2.70), we have that the sequence $(\mathrm{F}_t(\xi, \Gamma_\xi, l))_{t > \tilde{t}(\xi)}$ is increasing and upper-bounded by the commodity price $p_{\xi,l}$ at ξ.

Therefore the sequence converges and its limit is the fundamental value $\mathrm{F}(\xi, \Gamma_\xi, l)$.[15]

Hence the limit of the second term of the right side of (2.70) is well-defined, independently of the choice of deflator. Furthermore we have the following result:

Lemma 2.1: *Given an admissible prices vector (p, q, β), the price of a commodity $l \in \mathcal{L}$ has no bubbles at a node $\xi \in \mathcal{D}$ (in the weak sense) if and only if there is a vector of deflators Γ_ξ, for the successor nodes of ξ, such that*

$$\lim_{T\to\infty} \left(\sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu p_\mu (Y^c_{\xi,\mu}) \right)_l = 0. \tag{2.71}$$

- **Assets in $\mathcal{J}^b$.**

The *fundamental value* at node ξ of an asset $j \in \mathcal{J}^b(\xi)$ is given by the discounted value, using the deflators Γ_ξ, of the dividends delivered at the successors of node ξ,

$$\mathrm{F}(\xi, \Gamma_\xi, j) \equiv \frac{1}{\gamma_\xi} \left[\sum_{\mu\in\mathcal{R}(j):\mu>\xi} \gamma_\mu p_\mu A(\mu, j) \right]. \tag{2.72}$$

We say that the price of an asset $j \in \mathcal{J}^b(\xi)$ is free of bubbles, *in the weak sense*, at node ξ, if there is a deflator vector Γ_ξ such that $q_{\xi,j} = \mathrm{F}(\xi, \Gamma_\xi, j)$. If for every possible deflator Γ_ξ the asset j does not exhibit bubbles, then we say that the price of j is free of bubbles *in the strong sense*.

The next result, which follows directly from the recursive application of equation (2.61) along the sub-tree $\mathcal{D}(\xi)$ and from arguments analogous to those made for commodities $l \in \mathcal{L}$, characterizes the existence of bubbles in the prices of assets in $\mathcal{J}^b$:

Lemma 2.2: *Given an admissible prices vector (p, q, β), the price of an asset $j \in \mathcal{J}^b(\xi)$ has no bubbles at $\xi \in \mathcal{D}$ (in the weak sense) if and only if there is a deflator vector Γ_ξ, for the successor nodes of ξ, such that the following transversality condition holds:*

$$\lim_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu q_{\mu,j} = 0. \tag{2.73}$$

[15] This also proves that, given admissible prices, the fundamental value of a commodity l is always well-defined, independently of the choice of deflator Γ_ξ.

Moreover, if the asset $j \in \mathcal{J}^b$ is finite-lived (that is, there is $T \in \mathbb{N}$ such that $\tilde{t}(\mu) < T$, for every node $\mu \in \mathcal{R}(j)$), then there do not exist bubbles at ξ in the strong sense.

• **Assets with Margin Requirements**

The *fundamental value of the purchase price*, at node ξ, of an asset $j \in \mathcal{J}^m(\xi)$, is given by the discounted value at node ξ of the dividends delivered by the asset to a lender (of one unit of the asset) who maintains his long position in the asset over time. Such dividends include both the value of the promises and the non-linearities that arise from the non-negativity condition on the long position at each successor node of ξ,

$$\mathrm{F}_B(\xi, \Gamma_\xi, j) \equiv \frac{1}{\gamma_\xi}\left[\sum_{\mu \in \mathcal{R}(j):\mu > \xi} \gamma_\mu p_\mu A(\mu, j) + \sum_{\mu \in \mathcal{R}(j):\mu \geq \xi} \gamma^L_{\mu,j}\right], \tag{2.74}$$

Analogously, the *fundamental value of the sell price*, at node ξ, of an asset $j \in \mathcal{J}^m(\xi)$, is given by the discounted value at node ξ of the dividends received by a borrower (of one unit of the asset) who maintains his short position in the asset over time. Such dividends include, at each node μ, the difference between the depreciated value of the margin requirements constituted at μ^- and the new margin requirements that must that must be constituted at μ: $\Delta\mathrm{MV}_{\mu,j} \equiv \mathrm{DMV}_{\mu,j} - \mathrm{MV}_{\mu,j}$. Thus,

$$\mathrm{F}_S(\xi, \Gamma_\xi, j) \equiv \frac{1}{\gamma_\xi}\left[\sum_{\mu \in \mathcal{R}(j):\mu > \xi} \gamma_\mu \left(\Delta\mathrm{MV}_{\mu,j} - p_\mu A(\mu, j)\right) + \sum_{\mu \in \mathcal{R}(j):\mu \geq \xi} \gamma^B_{\xi,j}\right]. \tag{2.75}$$

Analogously to the case of commodities and assets in $\mathcal{J}^b$, we say that there are no bubbles (in the weak sense) in the purchase price (respectively, in the sell price) of asset $j \in \mathcal{J}^m$, at node ξ, if there is a deflator vector Γ_ξ such that $q_{\xi,j} = \mathrm{F}_B(\xi, \Gamma_\xi, j)$ (resp. $\mathrm{MV}_{\xi,j} - q_{\xi,j} = \mathrm{F}_S(\xi, \Gamma_\xi, j)$). The non-existence of bubbles in the strong sense is equivalent to the above equations holding for *every* deflator vector Γ_ξ.

Now, the asset $j \in \mathcal{J}^m$ has no bubbles in the weak sense at the node ξ if and only if there is a vector of deflators such that there is no bubble neither for sales nor for purchases. The asset j has no bubbles in the strong sense if and only if given any deflator vector then there are bubbles neither for sales nor for purchases.

By repeating the arguments used in the case of assets with bounded short-sales, it is easily seen that the (weak) non-existence of bubbles in purchases of an asset $j \in \mathcal{J}^m$ at node ξ is equivalent to the following transversality condition:

$$\lim_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu q_{\mu,j} = 0. \tag{2.76}$$

In the case of an asset's sell price, by repeatedly applying equation (2.63) along the successor nodes to ξ we have that, for all $T > \tilde{t}(\xi)$,

$$\begin{aligned} \mathrm{MV}_{\xi,j} - q_{\xi,j} &= \frac{1}{\gamma_\xi} \sum_{\mu\in\mathcal{R}(j):\mu>\xi,\tilde{t}(\mu)\le T} \gamma_\mu \left(\Delta\mathrm{MV}_{\mu,j} - p_\mu A(\mu,j)\right) \\ &+ \frac{1}{\gamma_\xi} \sum_{\mu\in\mathcal{R}(j):\mu\ge\xi,\tilde{t}(\mu)<T} \gamma^B_{\mu,j} + \frac{1}{\gamma_\xi} \sum_{\mu\in\mathcal{R}(j):\mu\ge\xi,\tilde{t}(\mu)=T} \gamma_\mu \left(\mathrm{MV}_{\mu,j} - q_{\mu,j}\right). \end{aligned} \tag{2.77}$$

So the *fundamental value of the sell price* of the asset j is well-defined, for the vector of deflators Γ_ξ, if and only if the limit, when T goes to infinity, of the third term in the right side of (2.77) exists.

Moreover, there do not exist bubbles in asset j's sell price if and only if the following transversality conditions holds:

$$\lim_{T\to\infty} \sum_{\mu\in\mathcal{R}(j):\mu\ge\xi,\tilde{t}(\mu)=T} \gamma_\mu \left(\mathrm{MV}_{\mu,j} - q_{\mu,j}\right) = 0. \tag{2.78}$$

Note that, in contrast to what happens with the purchase price, the limit given by equation (2.78) might not exist for some deflator vector Γ_ξ. The next result is a direct consequence of the previous one:

Lemma 2.3: *Given admissible prices (p, q, β), a necessary and sufficient condition for the (weak) non-existence of bubbles in the asset $j \in \mathcal{J}^m(\xi)$ is that there exists a vector of deflators Γ_ξ such that the transversality conditions given by equations (2.76) and (2.78) hold.*

Moreover, if the asset j is finite-lived then the fundamental values of the purchase and sell prices are well-defined for every deflator vector Γ_ξ; and there are no bubbles, in the strong sense, in the price of asset j.

• **Asset in** $\mathcal{J}^d$

Analogously to the case of assets which require margin deposits, the assets in $\mathcal{J}^d(\xi)$ might exhibit bubbles both in the sell and in the purchase prices. If there are no bubbles in both the sell and the purchase prices then we say that the asset is free of bubbles at the node ξ (whether in the weak or in the strong sense).

Hence, the fundamental value of the purchase price is given by

$$\mathrm{F}_B(\xi,\Gamma_\xi,j)=\frac{1}{\gamma_\xi}\left[\sum_{\mu\in\mathcal{R}(j):\mu>\xi}\gamma_\mu L_{\mu,j}+\sum_{\mu\in\mathcal{R}(j):\mu\geq\xi}\left(\sum_{l\in\mathcal{L}}\gamma_{\mu,l}\tilde{\beta}_{\mu,j}(C^{1,L}_{\mu,j})_l+\gamma^L_{\mu,j}\right)\right], \tag{2.79}$$

The fundamental value of the sell price is

$$\begin{aligned}\mathrm{F}_S(\xi,\Gamma_\xi,j) &= \frac{1}{\gamma_\xi}\left[\sum_{\mu\in\mathcal{R}(j):\mu>\xi}\gamma_\mu\left(\Delta\mathrm{CV}_{\mu,j}-B^h_{\mu,j}\right)\right]+\frac{1}{\gamma_\xi}\left[\sum_{\mu\in\mathcal{R}(j):\mu\geq\xi}\gamma^B_{\mu,j}\right]\\ &+\frac{1}{\gamma_\xi}\left[\sum_{\mu\in\mathcal{R}(j):\mu\geq\xi}\sum_{l\in\mathcal{L}}\gamma_{\mu,l}\tilde{\beta}_{\mu,j}\left(C^{1,B}_{\mu,j}+\sum_{j'\in\mathcal{J}^d(\mu)}C^{1,L}_{\mu,j'}\tilde{\beta}_{\mu,j'}(C^2_{\mu,j})_{j'}\right)_l\right],\end{aligned} \tag{2.80}$$

where $\Delta\mathrm{CV}_{\mu,j}=\mathrm{DVC}_{\mu,j}-\mathrm{CV}_{\mu,j}$.

As in the case of other asset types, the existence of bubbles in an asset $j\in\mathcal{J}^d(\xi)$ depends on whether a certain transversality condition holds.

The next result follows from equation (2.65) (via its repeated application along the successors to node ξ) and from the fact that, at each node $\mu\in\mathcal{R}(j)\cap\mathcal{D}(\xi)$, the sell price of asset j: $\mathrm{CV}_{\mu,j}-\tilde{\beta}_{\mu,j}q_{\mu,j}$ is positive, due to the absence of utility penalties for agents who default.

Lemma 2.4 *An asset $j\in\mathcal{J}^d(\xi)$ does not have bubbles (in the weak sense), at node ξ, if and only if there exists a vector of deflators Γ_ξ such that the discounted value, at ξ, of asset j's collateral requirements at period T goes to zero, as T goes to infinity. That is,*

$$\lim_{T\to\infty}\sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T}\gamma_\mu\mathrm{CV}_{\mu,j}=0. \tag{2.81}$$

The assets in $\mathcal{J}^d(\xi)$ which are finite-lived (or which go to default in finite time) do not have bubbles, in the strong sense, at the node ξ.[16]

We have thus characterized necessary and sufficient conditions for the existence of bubbles in assets negotiated at a node $\xi\in\mathcal{D}$.

[16]We say that an infinite-lived asset in $\mathcal{J}^d$ *goes to default in finite time* if there is an integer T such that $\beta_{\mu,j}=0$ for every $\mu\in\mathcal{D}_t,\ t>T$.

The main result of this section is the following Theorem, which sums up the lemmas above:

Theorem 2.2: *Given admissible prices (p, q, β), all finite-lived assets (and all assets which go to default in finite time) are free of bubbles, in the strong sense, at the nodes where they are traded.*

Moreover, a necessary and sufficient condition for the non-existence of bubbles, in the weak sense, for those assets that are traded at node ξ, is that the following two transversality conditions hold for a vector of deflators Γ_ξ,

$$\lim_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu p_\mu = 0, \tag{2.82}$$

$$\lim_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu q_{\mu,j} = 0, \quad \forall j \in \mathcal{J}^b(\xi) \cup \mathcal{J}^m(\xi). \tag{2.83}$$

The proof follows from the lemmas above and from Assumptions B and C; see the Appendix.

Remark 2.1: It follows from the Theorem above that, in order to prevent the existence of bubbles in the assets which are negotiated at a node ξ, it is sufficient to guarantee the absence of speculation in the prices of those durable goods and assets which are not subject to default.

Within the context of the literature on infinite-horizon economies without default, Santos and Woodford (1997) and Magill and Quinzii (1996) have characterized the conditions that guarantee the non-existence of bubbles in asset prices. They show that if aggregated endowment of the economy is bounded by a portfolio trading plan and furthermore the asset has positive net supply, then the equilibrium prices exhibit no bubbles. If the markets are incomplete, then the occurrence of bubbles in equilibrium prices of assets with zero net supply might have real effects on the economy; that is, there might not be other bubble-free equilibrium prices which support the same allocations.

In our case, since there are assets which are subject to default, the markets might become more incomplete as time goes by (due to the closure of markets in assets which default), and therefore it might become more difficult to guarantee the existence of a finite-cost portfolio that bounds the aggregated endowment.

Hence one might expect that in our model the sufficiency conditions for the validity of equations (2.82) and (2.83) turn out to be more restrictive than those imposed by Santos and Woodford (1997) or Magill and Quinzii (1996), and therefore that there might be in a sense more speculation in equilibrium.

2.4 Concluding Remarks

In this chapter we study an infinite-horizon sequential economy whose assets are either subject to default and protected by collateral or not subject to default but with restricted short sales. Such restrictions consist either of exogenous bounds on the amount that may be sold at each node, or of margin requirements which index the amount of short sales to the supply of goods and assets with bounded short-sales.

We study the existence of equilibrium and generalize the result of Araujo, Páscoa and Torres-Martíınez (2000) to the case of multiperiod assets, incomplete participation of the agents, and financial collateral requirements. That is, if we assume that the only enforcement in case of default is the seizure of the collateral requirements by the lenders, we show that the collateral requirements (which might depend on prices but are taken as given by the agents) guarantee the existence of equilibrium without the need of a priori ad-hoc conditions that bound indefinite debt accumulation by the traders.

We also study speculation in the economy's admissible prices. We show that the existence of bubbles is caused by speculation either with durable goods or with default-proof assets. We therefore guarantee that the non-existence of bubbles at a given node is assured by the validity of transversality conditions on the prices of durable goods and of assets free of default.

It would be interesting to study restrictions which prevent bubbles in the default-proof assets or in other words, restrictions that guarantee that the aforementioned transversality conditions hold. Such restrictions (which we believe might be analogous to those imposed by Magill and Quinzii (1996) and Santos and Woodford (1997)) might have to guarantee that the economy's aggregate endowment is replicated with finite cost at each node of the tree.

On the other hand the fact that we allow the collateral requirements to depend on the commodity- and asset-prices, is in a certain sense, an endogenization of such restrictions. It would nonetheless be interesting to allow the markets to have personalized collateral requirements which depend on the default level of each agent. We could in this way exclude the utility penalties, which are less natural than credit restrictions that are proportional to default levels.

In order to allow such personalized collateral requirements it might be necessary to allow a continuum of agents, which would avoid the non-convexities caused by the dependence of future collaterals on current renegotiation rules. All of these questions will be the subject of future research.

2.5 Appendix

Proof of Theorem 2.2 Clearly the validity of condition (2.83) guarantees the non-existence of bubbles in both the price of assets in $\mathcal{J}^b(\xi)$ and the purchase prices of assets in $\mathcal{J}^m(\xi)$.

Now, it follows from condition (d.) of Theorem 2.1 that

$$0 < \lim_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu p_\mu Y^c_{\xi,\mu} \leq \lim_{T\to\infty} \kappa^{T-\tilde{t}(\xi)} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu \|p_\mu\|_\Sigma (1,1,\dots,1). \tag{2.84}$$

It therefore follows from Lemma 2.1 that the validity of the transversality condition (2.82) guarantees the non-existence of bubbles in durable good prices.

Thus it is sufficient to prove that if an asset in $\mathcal{J}^d(\xi)$ exhibits bubbles or if there are bubbles in the sell price (but not in the purchase price) of some asset in $\mathcal{J}^m(\xi)$, then at least one of the conditions (2.82)-(2.83) does not hold.

Assume that there is an asset $j \in \mathcal{J}^m(\xi)$ with bubbles in the sell price *but not in the purchase price*. Then it follows from Lemma 3 and from the positiveness of the margin requirements that

$$\limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu \mathrm{MV}_{\mu,j} > 0. \tag{2.85}$$

Now, Assumption B implies that there is an upper bound $\overline{M}_j$ on the margin requirements of the asset j that depends neither on the price nor on the states of nature. Hence, since the commodities and the securities that are used as margin requirements for the asset j are the same along the sub-tree $\mathcal{N}(j)$ (Assumption B), we have

$$\begin{aligned} 0 &< \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu \mathrm{MV}_{\mu,j} \\ &\leq \left(\sum_{l\in\mathcal{L}} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{M}_{j,l}\gamma_\mu p_{\mu,l} + \sum_{j'\in\mathcal{J}^b(\xi)} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{M}_{j,j'}\gamma_\mu q_{\mu,j'} \right). \end{aligned}$$

Since every term on the right side of this last inequality is non-negative, there is a consumption good $l \in \mathcal{L}$ or an asset $j \in \mathcal{J}^b(\xi)$ for which one of the conditions (2.82)-(2.83) does not hold.

If there is some asset $j \in \mathcal{J}^d(\xi)$ at node ξ with bubbles, then by Lemma 2.4 we have

$$\limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu \mathrm{CV}_{\mu,j} > 0. \tag{2.86}$$

Therefore Assumptions C and D imply that if $j \in \mathcal{A}_k$ then

$$\begin{aligned} 0 &< \sum_{l\in\mathcal{L}} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{C}_{j,l}\gamma_\mu p_{\mu,l} + \sum_{j'\in\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{C}_{j,j'}\gamma_\mu q_{\mu,j'} \\ &\quad + \sum_{j'\in\mathcal{A}_r\cup\mathcal{J}^d(\xi),\ r<k} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{C}_{j,j'}\gamma_\mu q_{\mu,j'}. \end{aligned}$$

Hence, either one of the transversality conditions given by equations (2.82)-(2.83) does not hold, or there is an asset $j' \in \mathcal{A}_r$, where $r < k$, which exhibits bubbles at ξ.

Repeating the argument above for the assets in $\mathcal{A}_r \cap \mathcal{J}^d(\xi)$ it follows that the existence of bubbles in an asset $j \in \mathcal{J}^d(\xi)$ implies either that one of the transversality conditions given by the Theorem 2.2 does not hold, or that there is an asset $j' \in \mathcal{A}_0 \cap \mathcal{J}^d(\xi)$ which exhibits bubbles.

Since the collateral of the assets j' in $\mathcal{A}_0$ consists of consumption goods or of assets in $\mathcal{J}^b \cup \mathcal{J}^m$, the existence of bubbles in j' implies that one of the transversality conditions does not hold. This ends the proof. □

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