Upper sign-continuity for equilibrium problems

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January, 2013

Abstract

We present the new concept of upper sign-continuity for bifunctions and establish a new existence result for equilibrium problem.

1 Introduction

The equilibrium problem, EP in short, is defined as follows. Given a real Banach space X, a nonempty closed and convex subset K of X and a bifunction $f : X \times X \to \mathbb{R}$, EP consists of:

$$\text{finding } x \in K \text{ such that } f(x, y) \geq 0 \quad \forall y \in K.$$

EP has been extensively studied in recent year (e.g. [11, 12, 17, 15, 16]). A recurrent theme in the analysis of the conditions for the existence of solutions of EP is the connection between them and the solutions of the following feasibility problem (to be denoted CFP), which turns out to be convex under suitable conditions on f, which will be presented later on:

$$\text{find } x \in K \ \text{ such that } \ f(y, x) \leq 0 \quad \forall y \in K.$$

It was proved in [17] that if f is lower semicontinuous in the first argument, convex in the second one and it vanishes on the diagonal of $K \times K$, then every solution of CFP is a solution of EP, and then both solution sets trivially coincide under pseudomonotonicity of f. Bianchi *et al.* in [5] extended this inclusion under a weak continuity property of the bifunction, and they obtained an existence result for EP, adapting the existence result for variational inequality proposed by Aussel *et al.* in [2].

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The classical example of equilibrium problem is the variational inequality problem, which we define next. A Stampacchia variational inequality problem (VIP) is formulated as:

$$\text{find } x \in K \text{ such that there exists } x^* \in T(x) \text{ with } \langle x^*, y - x\rangle \geq 0 \;\forall\; y \in K,$$

where $T : X \to 2^{X^*}$ is a set valued operator, X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. So, if T has compact values, and we define

$$f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle,$$

it follows that every solution of the Equilibrium Problem associated to f_T and K is a solution of the Variational Inequality Problem associated to T and K, and conversely. Now, the CFP associated to f_T is equivalent to

$$\text{find } x \in K \text{ such that } \langle y^*, y - x\rangle \geq 0 \;\forall\; y \in K,\; y^* \in T(y)$$

which is known as Minty variational inequality problem (or dual variational inequality problem).

Recently, it was showed in [18, 14, 7, 8] that any equilibrium problem for which f is lower semicontinuous in the first argument, convex in the second one, monotone, and vanishes on the diagonal of $K \times K$, can be reformulated as a variational inequality. Castellani *et al.* in [6] extended these results to the pseudomonotone case. In this work, we extend them to the quasimonotone case.

2 Preliminaries

First, recall that a bifunction $f : X \times X \to \mathbb{R}$ is said to be

- quasimonotone on a subset K if , for all $x, y \in K$,

$$f(x, y) > 0 \Rightarrow f(y, x) \leq 0,$$

- properly quasimonotone on a subset K if , for all $x_1, x_2, \cdots, x_n \in K$, and all $x \in co(\{x_1, x_2, \cdots, x_n\})$, there exists $i \in \{1, 2, \cdots, n\}$ such that

$$f(x_i, x) \leq 0,$$

- pseudomonotone on a subset K if, for all $x, y \in K$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

Clearly, pseudomonotonicity of f implies properly quasimonotonicity of f, and the latter implies quasimonotonicity of f.

Let us recall that a function $h : X \to \mathbb{R}$ is said to be:

- *quasiconvex* on a subset K if, for all $x, y \in K$ and all $z \in [x, y]$,

$$h(z) \leq \max\{h(x), h(y)\}$$

- *semistrictly quasiconvex* on a subset K if h is quasiconvex on K and

$$h(x) < h(y) \Rightarrow h(z) < h(y), \ \forall z \in [x, y[$$

  for all $x, y \in K$.

The equilibrium problem, EP in short, is defined as follows. Given a nonempty closed and convex subset K of X and a bifunction $f : X \times X \to \mathbb{R}$, EP consists of

$$\text{finding } x \in K \text{ such that } \ f(x, y) \geq 0 \ \ \forall y \in K.$$

The convex feasibility problem (to be denoted CFP), consists of

$$\text{finding } x \in K \text{ such that } \ f(y, x) \leq 0 \ \ \forall y \in K.$$

We denote by $EP(f, K)$ and $CFP(f, K)$ the sets of solutions of the equilibrium and the convex feasibility problem, respectively.

In the sequel, the bifunction f of EP is required to satisfy the following assumptions:

P_1^* $f(x, x) = 0$ for all $x \in K$,
P_2^* $f(x, \cdot)$ is semistrictly quasiconvex and lower semicontinuous for all $x \in K$,

3 Upper sign-continuity for bifunctions

Hadjisavvas introduced in [13] the concept of upper sign-continuity for set-valued maps. More precisely, a set-valued map $T : X \to 2^{X^*}$ is said to be upper sign-continuous on a convex subset K if, for any $x, y \in K$, the following implication holds:

$$\left(\forall t \in]0, 1[\ \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0\right) \Rightarrow \ \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0,$$

where $x_t = (1 - t)x + ty$.

All set-valued map T with conic values satifies $\sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0$. Therefore, all set-valued maps with conic values are upper sign-continuous.

Bianchi *et al.* introduced in [5] the concept of upper sign-continuity for bifunctions. More precisely, a bifunction $f : X \times X \to \mathbb{R}$ is said to be BP-upper sign-continuous with respect to the first argument ([5]), on a convex subset K of X, if for all $x, y \in K$ it holds that

$$(f(x_t, y) \geq 0, \ \forall t \in]0,1[) \quad \Rightarrow \ f(x, y) \geq 0,$$

where $x_t = tx + (1-t)y$.

Definition 3.1. *A bifunction $f : X \times X \to \mathbb{R}$ is said to be E-upper sign-continuous on a convex subset K if for any $x, y \in K$ the following implication holds:*

$$(f(x_t, x) \leq 0 \ \forall \ t \in \ ]0,1[\ ) \Rightarrow \ f(x, y) \geq 0,$$

where $x_t = (1-t)x + ty$.

These two notions of upper sign-continuity are related as follows.

Proposition 3.1. *Under P_1^* and P_2^*, if f is BP-upper sign-continuous on K, then f is E-upper sign-continuous on K.*

Proof: Suppose that there exist $x, y \in K$ such that for all $t \in]0,1[$ it holds that $f(x_t, x) \leq 0$ and $f(x, y) < 0$. Since $f(\cdot, y)$ is BP-upper sign-continuous, there exists $t_0 \in]0,1[$ such that $f(x_{t_0}, y) < 0$. Since f satisfies P_1^* ndP_2^*, it follows has that both when $f(x_{t_0}, x) = 0$ or when $f(x_{t_0}, x) < 0$, it holds that $f(x_{t_0}, x_{t_0}) < 0$, which contradicts P_1^*. □

Example 3.1. *Take $K = [0,1]$ and define $f : K \times K \to \mathbb{R}$ as*

$$f(x,y) = \begin{cases} y - 1 & \text{if } \ x = 1 \text{ and } 0 \leq y \leq 1 \\ (x-y)^2 & \text{if } \ 1 \geq y \geq x \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that f satisfies P_1^*, P_2^* and is E-upper sign-continuous. However, f is not BP-upper sign-continuous: consider $x = 1$ and $y = 0$.

The following examples show the both P_1^* and P_2^* are needed for the validity of Proposition 3.1.

Example 3.2. *Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $f(x,y) = -xy$ and take $K = [1,2]$. Clearly f satisfies P_2^* but not P_1^*. Taking $x = 1$ and $y = 2$, one has that $f(x,y) < 0$ and $f(x_t, x) \leq 0$ for all $t \in [0,1]$. Therefore f is not E-upper sign-continuous on K.*

Example 3.3. *Define* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *as* $f(x, y) = \frac{1}{2}x^2 + \frac{3}{2}xy - 2y^2 + y - x$, *and take* $K = [0, 1]$. *Clearly* f *satisfies* P_1^* *but not* P_2^*. *Taking* $x = 0$ *and* $y = 1$, *one has that* $f(0, 1) = -1 < 0$ *and* $f(x_t, 0) = t^2/2 - t \leq 0$ *for all* $t \in [0, 1]$. *Therefore* f *is not E-upper sign-continuous on* K.

The following result relates the concept of E-upper sign-continuity to the inclusion of the solution set of CFP in the solution set of EP.

Lemma 3.1. *Let* $f : X \times X \to \mathbb{R}$ *be a bifunction.*

i) Assume that f *is E-upper sign-continuous on* X. *Then* $\mathrm{CEP}(f, K) \subseteq \mathrm{EP}(f, K)$, *for all convex subset* K *of* X.

ii) Assume that f *satisfies* P_1^*, P_2^* *and that* $\mathrm{CEP}(f, [x, y]) \subseteq \mathrm{EP}(f, [x, y])$ *for all* $x, y \in X$. *Then* f *is E-upper sign-continuous.*

Proof: *i*) Take $x \in \mathrm{CEP}(f, K)$. Let $x_t = (1 - t)x + ty$. Observe that $f(x_t, x) \leq 0$ for all $y \in K$ and all $t \in]0, 1[$. Now, since f is E-upper sign-continuous, it holds that $f(x, y) \geq 0$, i.e., $x \in \mathrm{EP}(f, K)$.

ii) Suppose that f is not E-upper sign-continuous. Thus, we can find $x, y \in X$ such that $f(x_t, x) \leq 0$ for all $t \in]0, 1[$, and also $f(x, y) < 0$. Hence, we get from P_1^* and P_2^* that $f(x, x_t) < 0$. Clearly $x \in \mathrm{CEP}(f, [x_t, x])$, so that $x \in \mathrm{EP}(f, [x_t, x])$, implying that $f(x, x_t) \geq 0$, which is a contradiction. □

4 Relations between Variational Inequality and Equilibrium Problems

As mentioned above, the Stampacchia variational inequality problem is formulated as:

$$\text{find } x \in K \text{ such that } \exists\, x^* \in T(x) \text{ with } \langle x^*, y - x \rangle \geq 0 \text{ for all } y \in K,$$

and the Minty variational inequality problem is formulated as:

$$\text{find } x \in K \text{ such that } \langle y^*, y - x \rangle \geq 0 \text{ for all } y \in K \text{ and, } y^* \in T(y).$$

We denote by $S(T, K)$ and $M(T, K)$ the solution sets of the Stampacchia and Minty variational inequality problems, respectively.

Following the classical terminology, a set-valued map $T : X \to 2^{X^*}$ is said to be

- quasimonotone on a subset K if , for all $x, y \in K$,

$$\exists\, x^* \in T(x) \;:\; \langle x^*, y - x\rangle > 0 \Rightarrow \;\forall\, y^* \in T(y) \;:\; \langle y^*, y - x\rangle \geq 0.$$

- properly quasimonotone on a subset K if, for all $x_1, x_2, \cdots, x_n \in K$, and all $x \in co(\{x_1, x_2, \cdots, x_n\})$, there exists $i \in \{1, 2, \cdots, n\}$ such that

$$\forall x_i^* \in T(x_i): \quad \langle x_i^*, x_i - x\rangle \geq 0.$$

- pseudomonotone on a subset K if, for all $x, y \in K$,

$$\exists\, x^* \in T(x) \;:\; \langle x^*, y - x\rangle \geq 0 \Rightarrow \;\forall\, y^* \in T(y) \;:\; \langle y^*, y - x\rangle \geq 0.$$

Clearly, pseudomonotonicity implies proper quasimonotonicity and proper quasimonotonicity implies quasimonotonicity.

Classically, any variational inequality problem with operator T and feasible set K can be reformulated as an equilibrium problem simply by defining the following bifunction $f_T : K \times K \to \mathbb{R}$:

$$f_T(x, y) := \sup_{x^* \in T(x)} \langle x^*, y - x\rangle.$$

Let T be a set-valued map with compact values. Then T is pseudomonotone (properly quasimonotone, quasimonotone) if and only if f_T is pseudomonotone (properly quasimonotone, quasimonotone). We present now an extension of the last remark on upper sign-continuity.

Proposition 4.1. *Let T be a set-valued map and f_T define as above. Then T is upper sign-continuous if and only if f_T is E-upper sign-continuous.*

Proof: *i*) Suppose that T is upper sign-continuous, and take $x, y \in K$ such that $f_T(x_t, x) \leq 0$ for all $t \in]0, 1[$. Now, since $f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle$, it follows that $\sup_{x_t^* \in T(x_t)} \langle x_t^*, x - x_t\rangle \leq 0$ for all $t \in]0, 1[$, which is equivalent to $\inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0$ for all $t \in]0, 1[$ and we obtain, from the upper sign-continuity of T, $\sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0$, so that

$$f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0.$$

Conversely, assume that f_T is E-upper sign-continuous and take $x, y \in K$ such that $\inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0$ for all $t \in]0, 1[$. Since $t(y - x) = x_t - x$, it follows that $\inf_{x_t^* \in T(x_t)} \langle x_t^*, x_t - x\rangle \geq 0$ for all $t \in]0, 1[$. Consequently,

$$f_T(x_t, x) = \sup_{x_t^* \in T(x_t)} \langle x_t^*, x - x_t\rangle = -\inf_{x_t^* \in T(x_t)} \langle x_t^*, x_t - x\rangle \leq 0$$

for all $t \in]0,1[$, and hence, we obtain, from the upper sign-continuity* of f_T,

$$\sup_{x^* \in T(x)} \langle x^*, y - x \rangle = f_T(x,y) \geq 0,$$

which completes the proof. □

Next, we exhibit a set-valued map T which is upper sign-continuous, but such that f_T is not BP-upper sign-continuous.

Example 4.1. *Define $T : [1,2] \to 2^{\mathbb{R}}$ as*

$$T(x) = \begin{cases} \{-1\} & \text{if } x = 1 \\ \{-1, 1\} & \text{if } 1 < x \leq 2. \end{cases}$$

Clearly, T is upper sign-continuous, but the bifunction $f_T(x,y)$ is defined as

$$f_T(x,y) = \begin{cases} x - y & \text{if } 1 \leq y < x \\ 0 & \text{if } y = x \\ y - x & \text{if } x < y \leq 2, \end{cases}$$

so that it is not BP-upper sign-continuous.

Along the line of J. Reinhard's results in [20], we present the equivalence between upper sign-continuity and the inclusion of the solution set of Minty's variational inequality problem in the solution set of the Stampacchia's one.

Corollary 4.1.1. *Let $T : X \to 2^{X^*}$ be a set-valued map with compact values, and K a convex subset of X. If $M(T,[x,y]) \subseteq S(T,[x,y])$ for all $x, y \in K$ then T is upper sign-continuous on K.*

Proof: Observe that, for all $x, y \in K$, $S(T,[x,y]) = \mathrm{EP}(f_T, K)$ and $M(T,[x,y]) = \mathrm{CFP}(f_T,[x,y])$. So, one gets $\mathrm{CFP}(f_T,[x,y]) \subseteq EP(f_T, K)$ and thus f_T is E-upper sign-continuous on K (Lemma 3.1 ii). Therefore, T is upper sign-continuous on K. by Proposition 4.1. □

It follows from Theorem 2 in [20] and Corollary 4.1.1 that T is pseudomonotone and upper sign-continuous on K if and only if $M(T,K) = S(T,K)$.

Associated to a convex subset K of X and a bifunction $f : X \times X \to \mathbb{R}$, we define the set-valued map $N : K \to 2^{X^*}$ as:

$$N(x) = \{x^* \in X^* : \ \langle x^*, y - x \rangle \leq 0, \ \ \forall y \in L_{f(x,\cdot)}(x)\},$$

where $L_{f(x,\cdot)}(x) = \{z \in K : f(x,z) \leq 0\}$.

Proposition 4.2. *Under P_1^*, P_2^* and quasimonotonicity of f on K, it holds that N is quasimonotone on K.*

Proof: Take $x, y \in K$ and $x^* \in N(x)$ such that $\langle x^*, y - x\rangle > 0$. Thus, $f(x, y) > 0$, and hence $f(y, x) \leq 0$ by quasimonotonicity of f. So, we get from P_1^* that $x \in L_{f(y,\cdot)}(y)$, and as consequence of P_2^*, $\langle y^*, x - y\rangle \leq 0$ for all $y^* \in N(y)$, which is equivalent to $\langle y^*, y - x\rangle \geq 0$, i.e., N is quasimonotone. □

Let $S = \{x^* \in X^*;\ \|x^*\| = 1\}$ and $B = \{x^* \in X^*;\ \ \|x^*\| \leq 1\}$. Define $D : K \to 2^{X^*}$ as

$$D(x) = \begin{cases} \operatorname{conv}(N(x) \cap S) & \text{, if } x \notin \arg\min_X f(x, \cdot) \\ B & \text{otherwise.} \end{cases}$$

We remark that $0 \notin D(x)$ for all $x \notin \arg\min_X f(x, \cdot)$, because f satisfies P_1^*, P_2^*.

Proposition 4.3. *Under P_1^*, P_2^* and quasimonotonicity of f on K, it holds that $EP(f, K) = S(D, K)$.*

Proof: Take $x \in EP(f, K)$. Thus $f(x, y) \geq 0$ for all $y \in K$, i.e., x is a minimizer of $f(x, \cdot)$ on K. Now, if $x \in \arg\min_X f(x, \cdot)$ then $0 \in D(x)$, i.e., $x \in S(D, K)$. On the other hand, if $x \notin \arg\min_X f(x, \cdot)$ then, in view of the first order optimality condition in quasiconvex programming (see Theorem 4.1 in [4]), there exist $x^* \in D(x)$ such that x is a solution of $S(D, K)$.

Conversely, assume that $x \in S(D, K)$. If $0 \in D(x)$ then x belongs to $\arg\min_X f(x, \cdot)$, so that $f(x, y) \geq 0$ for all $y \in K$, i.e., $x \in EP(f, K)$. On the other hand, if $0 \notin D(x)$, then from $N \setminus \{0\}-$pseudoconvexity of $f(x, \cdot)$, one gets that x is a minimizer of $f(x, \cdot)$, i.e., $x \in EP(f, K)$. □

5 Existence result

The following example shows that we cannot apply either Theorem 3.2 of [5] or Theorem 2.1 of [2] in their variational formulation (in the sense [14, 8, 6]), for ensuring the existence of solutions of EP in our current context.

Example 5.1. *Let f and K be as in Example 3.1. Then $EP(f, K) = [0, 1[$, $f(x, \cdot)$ is convex on K for all $x \in K$. Indeed,*

$$\begin{aligned} f(0, y) &= y^2, \\ f(x, y) &= \begin{cases} 0 & \text{if } 0 \leq y < x \\ (x - y)^2 & \text{if } x \leq y \leq 1, \end{cases} \qquad \text{for all } x \in]0, 1[\\ f(1, y) &= y - 1. \end{aligned}$$

It is clear that $f(x,\cdot)$ is differentiable on K. Now, if we use the reformulation in [14, 6, 18], then the diagonal operator T_f defined in [14, 18, 6] is given by

$$T_f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

However, T_f is not upper sign-continuous on K.

Proposition 5.1. *Let X be a Banach space, K a compact convex subset of X and $f : K \times K \to \mathbb{R}$ bifunction on K. If f is quasimonotone and E-upper sign-continuous on K, and it satisfies P_1^*, P_2^*, then $EP(f,K)$ admits at least one solution.*

Proof: If f is properly quasimonotone then by Ky Fan's Lemma (see [10]), there exists $x \in \mathrm{CFP}(f,K)$. So, from them E-upper sign continuity of f and Lemma 3.1, one gets that $x \in \mathrm{EP}(f,K)$. Now, suppose that f is not properly quasimonotone. Then, there exist $x_1, \cdots, x_n \in K$ and $x \in co(\{x_1, x_2, \cdots, x_n\})$ such that $f(x_i, x) > 0$ for all $i \in \{1, 2, \cdots, n\}$. Now, by P_2^* there exists a convex neighborhood V of x such that $f(x_i, z) > 0$ for all $z \in V \cap K$ and all i. By quasimonotonicity of f, $f(z, x_i) \leq 0$ for all i. It follows from P_2^* that $f(z,x) \leq 0$ for all $z \in V \cap K$, and therefore $x \in CFP(f, V \cap K)$. From E-upper sign-continuity of f, $x \in EP(f, V \cap K)$, so that $f(x,z) \geq 0$ for all $z \in V \cap K$. Take $y \in K$. Then, there exists $z \in]x,y[\cap V \cap K$. By quasiconvexity of $f(x,\cdot)$, $0 \leq f(x,z) \leq \max\{f(x,x), f(x,y)\}$. Since $f(x,\cdot)$ satisfies P_2^*, it holds that $f(x,y) \geq 0$. Thus, $x \in EP(f,K)$. □

Since BP-upper sign-continuity implies E-upper sign continuity, Proposition 5.1 provides a stronger result than those in Theorem 3.2 and Theorem 4.1 in [5].

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