Existence results for quasi-equilibrium problems

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Abstract

Recently in Castellani-Guili (J. Optim. Th. Appl., 147 (2010), 157-168), it has been showed that the proof of the existence result for quasimonotone Stampacchia variational inequalities developed in Aussel-Hadjisavvas (J. Optim. Th. Appl., 121 (2004), 445-450) can be adapted to the case of equilibrium problem. This proof was based on KKM techniques.

In this paper we define and study the so-called quasi-equilibrium problem, that is an equilibrium problem with a constraint set depending of the current point. Our main contribution consists of an existence result combining fixed point techniques with stability analysis of perturbed equilibrium problems.

1 Introduction

The equilibrium problem, (EP) in short, is defined as follows. Given a real Banach space X, a nonempty subset K of X and a bifunction $f : X \times X \to \mathbb{R}$, (EP) consists of

$$(EP) \qquad \text{find } x \in K \text{ such that } f(x,y) \geq 0 \quad \forall y \in K.$$

Problem (EP) has been extensively studied in recent year (see e.g. [16, 17, 23, 21, 22]). A recurrent theme in the analysis of the conditions for the existence of solutions of (EP) is the connection between them and the solutions of the so-called *Convex Feasibility Problem*, to be denoted (CFP), which turns out to be convex under suitable conditions on f and which corresponds to a sort of dual formulation of (EP),

$$(CFP) \qquad \text{find } x \in K \ \text{ such that } \ f(y,x) \leq 0 \quad \forall y \in K.$$

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It was proved in [23] that if f is lower semicontinuous in the first argument, convex in the second one and it vanishes on the diagonal of $K \times K$, then every solution of (CFP) is a solution of (EP), and moreover both solution sets trivially coincide under pseudomonotonicity of f. Bianchi *et al.* in [9] extended this inclusion under a weak continuity property of the bifunction, and they obtained an existence result for (EP), adapting the existence result for variational inequality proposed by Aussel *et al.* in [6].
The classical example of equilibrium problem is the variational inequality problem, which is defined as follows: a Stampacchia variational inequality problem (VIP) is formulated as

$$(VIP) \qquad \begin{array}{l} \text{find } x \in K \text{ such that there exists } x^* \in T(x) \\ \text{with } \langle x^*, y - x \rangle \geq 0, \ \forall\, y \in K, \end{array}$$

where $T : X \to 2^{X^*}$ is a set-valued operator, X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. So, if T has compact values, and we define the *representative bifunction* f_T of T by

$$f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle,$$

It follows that every solution of the Equilibrium Problem associated to f_T and K is a solution of the Variational Inequality Problem associated to T and K, and conversely. Now, the CFP associated to f_T is equivalent to

$$\text{find } x \in K \text{ such that } \langle y^*, y - x \rangle \geq 0, \ \forall\, y \in K, \ y^* \in T(y)$$

which is known as Minty variational inequality problem (or dual variational inequality problem).

We denote by $S(T, K)$ and $M(T, K)$ the solution sets of the Stampacchia and Minty variational inequality problems respectively.

Recently, it was showed in [24, 20, 11, 12] that any equilibrium problem for which f is lower semicontinuous in the first argument, convex in the second one, monotone, and vanishes on the diagonal of $K \times K$, can be reformulated as a variational inequality. Castellani *et al.* in [10] extended these results to the pseudomonotone and quasimonotone case. In this work we extend them to the quasi-equilibrium problem.

We consider next the problem which is our main object of interest in this paper. The quasi-equilibrium problem, (QEP) in short, is defined as follows. Given a set-valued map $K : X \to 2^X$ and a bifunction $f : X \times X \to \mathbb{R}$, (QEP) consists of

$$(QEP) \qquad \text{find } x \in K(x) \text{ such that } f(x, y) \geq 0, \quad \forall y \in K(x).$$

The associated Minty quasi-equilibrium problem (to be denoted QMEP), consists of

$$(QMEP) \qquad \text{find } x \in K(x) \text{ such that } \ f(y, x) \leq 0, \quad \forall y \in K(x).$$

We denote by QEP(f, K) and QMEP(f, K) the sets of solutions of the quasi-equilibrium and Minty quasi-equilibrium problem, respectively.

One of the main reasons for studying quasi-equilibrium problems lies in the relation between them and quasi-variational inequalities (see Section 3.2), which mirrors the well known relation between equilibrium problems and variational inequalities. Quasi-variational inequalities are themselves relevant because they encompass certain problems of interest in various fields of application, which do not fall within the scope of variational inequalities. Perhaps the most important instance of this situation is the generalized Nash equilibrium problem, which models a large number of real life problems in Economics and other areas (see e.g. [14], [27] and references therein). The reduction of generalized Nash equilibrium problems to quasi-variational inequalities has been analyzed, e.g., in [2] and [27]. Existence results for problem (QEP) can be found in [19] and references therein.

The paper is organized as follows. First in Section 2 we describe the main notation and give some preliminary results and comments concerning the mainly used concepts, in particular the upper sign property of a bifunction. Section 3 is devoted to the study of the relationship between (QEP) and (QMEP) on one side and between (QEP) and a variational reformulation on the other hand. Finally in Section 4 we prove an existence result for quasi-equilibrium problem (QEP) using fix point techniques.

2 Preliminaries

All along the paper, X stands for a real Banach space, X^* for its topological dual and $\langle \cdot, \cdot \rangle$ for the associated duality product. The subsets S^* and B^* stands respectively for the unit sphere and unit ball $S^* = \{x^* \in X^*;\ \|x^*\| = 1\}$ and $B^* = \{x^* \in X^*;\ \ \|x^*\| \leq 1\}$ of the dual space X^*.

First, recall that a bifunction $f : X \times X \to \mathbb{R}$ is said to be

- quasimonotone on a subset K if, for all $x, y \in K$,

$$f(x, y) > 0 \Rightarrow f(y, x) \leq 0,$$

- properly quasimonotone on a subset K if, for all $x_1, x_2, \cdots, x_n \in K$, and all $x \in co(\{x_1, x_2, \cdots, x_n\})$, there exists $i \in \{1, 2, \cdots, n\}$ such that

$$f(x_i, x) \leq 0,$$

- pseudomonotone on a subset K if, for all $x, y \in K$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

Clearly, pseudomonotonicity of f implies properly quasimonotonicity of f, and the latter implies quasimonotonicity of f.

Let us recall that a function $h : X \to \mathbb{R}$ is said to be:

- *quasiconvex* on a convex subset K if, for all $x, y \in K$ and all $z \in [x, y]$,
$$h(z) \le \max\{h(x), h(y)\}$$
- *semistrictly quasiconvex* on a convex subset K if h is quasiconvex on K and
$$h(x) < h(y) \Rightarrow h(z) < h(y),\ \forall z \in [x, y[$$
for all $x, y \in K$.

In the sequel, we will sometimes use the following assumptions on the considered bifunctions:

(H_1) $f(x, x) = 0$ for all $x \in K$,

(H_2) $f(x, \cdot)$ is semistrictly quasiconvex for all $x \in K$,

(H_3) $f(x, \cdot)$ is lower semicontinuous for all $x \in K$.

Our forthcoming stability analysis will be based on the following weak continuity property for bifunctions:

Definition 2.1. *A bifunction f is said to have the upper sign property at $x \in K$ if for every $y \in K$ the following implication holds:*
$$\big(f(x_t, x) \le 0,\ \forall\ t \in]0, 1[\ \big) \Rightarrow\ f(x, y) \ge 0, \tag{1}$$
where $x_t = (1 - t)x + ty$.

The above definition is inspired by the analoguous upper sign-continuity for set-valued map, originated in [18], and given as follows

$T : X \rightrightarrows X^*$ is said to be lower sign-continuous at $x \in \operatorname{Dom} T$ if, for any $v \in X$, the following implication holds:
$$\left(\forall t \in]0, 1[,\ \inf_{x_t^* \in T(x_t)} \langle x_t^*, v\rangle \ge 0\right) \Rightarrow \inf_{x^* \in T(x)} \langle x^*, v\rangle \ge 0$$
where $x_t = x + tv$.

A local version of the upper sign property of a bifunction has been considered in [11] by setting that f has the local upper sign property at x if (1) holds for every y in a certain neighbourhood $B(x, \rho)$ of x, with $\rho > 0$. Clearly the upper sign property implies the local upper sign property. Actually, as shown in the forthcoming lemma, the upper sign property and its local counterpart coincide under mild assumptions.

Lemma 2.1. *Let K be a nompty convex subset of X and f be a bifunction satisfying (H_1) and the following property:*

$$\left\{\begin{array}{l} \textit{for every } x, y_1 \textit{ and } y_2 \in X, \textit{ one has} \\ f(x,y_1) \leq 0 \textit{ and } f(x,y_2) < 0 \ \Rightarrow \ f(x,z_t) < 0, \ \forall t \in]0,1[. \end{array}\right. \tag{2}$$

Then f has the local upper sign property on K if and only if f has the upper sign property on K.

Proof. Let us suppose that f has the locally upper sign property on K (with $r > 0$) and let $y \in K$ be such that $f(x_t, x) \leq 0$, for every $x_t = tx + (1-t)y$, $t \in]0,1[$. Then one immediately has that for every $t \in]0,1[$, $f(z_t, x) \leq 0$ where $z_t = tx + (1-t)z$ with $z = 3r/4x + 1/4y \in B(x,r) \cap K$. Now by the local upper sign property $f(x,z) > 0$. This implies that $f(x,y) > 0$ because otherwise, together with (H_1) and property (2) a contradiction occurs. □

Remark 2.1. *As it can be easily shown, any bifunction satisfying assumption (H_2) also verifies property (2). It is in particular the case when $f = f_T$ is a representative bifunction of a set-valued map T.*

Now from the definition of f_T one clearly has the following equivalence for any set-valued map $T : X \to 2^{X^*}$

$$T \text{ is upper sign-continuous at } x \Leftrightarrow \left\{\begin{array}{l} f_T \text{ has the upper} \\ \quad \text{sign property at } x. \end{array}\right. \tag{3}$$

As a direct consequence of the classical Ky Fan intersection theorem [15] one can deduce an existence result for $CFP(f,K)$.

Corollary 2.1. *Let K be weakly compact and f be a properly quasimonotone equilibrium bifunction such that for every $x \in K$ the subset $\{y \in K \ : \ f(x,y) \leq 0\}$ is weakly closed. Then $CFP(f,K)$ is nonempty.*

Proof. It follows the same lines as in [13, Th. 5.1]. Indeed, observe first that $[\cap_{x\in K} F(x)] \subset CFP(f,K)$, where, for every $x \in K$, $F(x)$ stands for the set $F(x) = \{y \in K \ : \ f(x,y) \leq 0\}$. On the other hand, the proper quasimonotonicity of f can be reformulated as

$$\forall\, x_1, \ldots, x_n \in K, \ \forall\, x \in \operatorname{conv}\{x_1, \ldots, x_n\}, \quad x \in \cap_{i=1}^n F(x_i),$$

which is classically expressed as F being a KKM map. The result follows now immediately by applying the Ky Fan intersection theorem. □

Let us end this section with an elementary result on the set $CFP(f,K)$.

Proposition 2.1. *Let K be a convex and closed subset of X and $f : K \times K \to \mathbb{R}$ be a bifunction. If f satisfies (H_2) and (H_3) then $CFP(f,K)$ is convex and closed.*

Proof. Let $x_1,\ x_2 \in CFP(f,K)$, $t \in [0,1]$ and $y \in K$. Since $f(y,x_1) \leq 0$, $f(y,x_2) \leq 0$ and f satisfies (H_2), it holds that $f(y,tx_1+(1-t)x_2) \leq 0$. So $tx_1+(1-t)x_2 \in CFP(f,K)$ for all $t \in [0,1]$, proving the convexity of $CFP(f,K)$.

Now consider a sequence $(x_n)_n \subset \mathrm{CFP}(f,K)$ converging to $\bar{x}$. Since K is closed, it follows that $\bar{x} \in K$. For every $y \in K$, one has $f(y,x_n) \leq 0$ for all $n \in \mathbb{N}$ and thus by (H_3) $f(y,\bar{x}) \leq 0$, thus proving that $\bar{x} \in CFP(f,K)$. □

3 Problem relationships

3.1 Canonical inclusions between (QMEP) and (QEP)

The relations between the solution set $M(T,K)$ and $S(T,K)$ respectively of the Minty and Stampacchia variational inequalities has been extensively studied in the literature (see e.g. [4] for a recent survey), in particular because they play an important role in the proofs of stability and existence results. Our aim in this short subsection is to precise this relations but in the context of quasi-equilibrium problem.

Proposition 3.1. *Let $f : X \times X \to \mathbb{R}$ be any bifunction and consider the following conditions:*

i) *f has the upper sign property on X*

ii) *$QMEP(f,K) \subseteq QEP(f,K)$ for all set-valued map $K : X \to 2^X$ with convex values.*

iii) *$QMEP(f,[x,y]) \subseteq QEP(f,[x,y])$ for all $x,y \in X$.*

Then i) $\Rightarrow$ ii) $\Rightarrow$ iii) and the three conditions are equivalent if f satisfies (H_1) and (H_2).

Proof. *i)* $\Rightarrow$ *ii)* Immediate. Indeed let $K : X \to 2^X$ be a set-valued map with convex values and take $x \in \mathrm{QMEP}(f,K)$. For any $y \in K(x)$ and any $t \in]0,1[$ one clearly has $f(tx+(1-t)y,x) \leq 0$ and the conclusion follows by the upper sign property of f.
Assume now that *iii)* holds and suppose that f doesn't have the upper sign property on X. Therefore, one can find $x,\ y \in X$ such that $f(x,y) < 0$ and $f(tx+(1-t)y,x) \leq 0$, for all $t \in]0,1[$. Hence, we get from (H_1) and (2) that $f(x,x_t) < 0$. Clearly $x \in \mathrm{QMEP}(f,[x_t,x])$, so that $x \in \mathrm{QEP}(f,[x_t,x])$, implying that $f(x,x_t) \geq 0$, which is a contradiction. □

As a direct consequence of Theorem 3.1, we obtain a sufficient condition for the upper-sign continuity of a set-valued map in terms of the inclusions between the solution sets of the Minty and Stampacchia defined by the set-valued map on intervals thus providing a refinement of analoguous relationship given for upper semicontinuity (see [28]).

Corollary 3.1. *Let $T : X \to 2^{X^*}$ be a set-valued map with compact values, and K be a convex subset of X. If $M(T, [x, y]) \subseteq S(T, [x, y])$ for all $x, y \in K$ then T is upper sign-continuous on K.*

Proof. Observing that, for all $x, y \in K$, $S(T, [x, y]) = \mathrm{QEP}(f_T, K)$ and $M(T, [x, y]) = \mathrm{QMEP}(f_T, [x, y])$, one can deduce that $QMEP(f_T, [x, y]) \subseteq QEP(f_T, K)$. Therefore, by Proposition 3.1, f_T has the upper sign property on K and thus, according to (3), T is upper sign-continuous on K. $\square$

3.2 Variational formulation of equilibrium problems

As quoted previously, any Stampacchia variational inequality can be reformulated as an equilibrium problem, thanks to the use of the representative bifunction f_T. This subsection is devoted to the study of the reverse transformation and more precisely, given a bifunction f and a subset K, to give sufficient conditions under which the solution set $EP(f, K)$ of the equilibrium problem coincides with the solution set of a certain Stampacchia variational inequality.

Associated to a convex subset K of X and a bifunction $f : X \times X \to \mathbb{R}$, we define the set-valued map $N : K \to 2^{X^*}$ as

$$N(x) = \{x^* \in X^* : \ \langle x^*, y - x\rangle \le 0, \quad \forall y \in L_f(x)\},$$

where $L_f(x)$ stands for the sublevel set $L_f(x) = \{z \in K : f(x, z) \le 0\}$.

Proposition 3.2. *Let K be a convex subset of X and $f : X \times X \to \mathbb{R}$ be a quasimonotone bifunction. Then the set-valued map N is quasimonotone on K.*

Proof. Let x and y be two elements of K and $x^* \in N(x)$ be such that $\langle x^*, y-x\rangle > 0$. From the definition of N one gets that $f(x, y) > 0$, and hence $f(y, x) \le 0$ by quasimonotonicity of f. Now since x is an element of $L_f(y)$, one has $\langle y^*, x - y\rangle \le 0$ for all $y^* \in N(y)$, proving that N is quasimonotone. $\square$

Following a technique used in [7], let us define now a normalized version of the set-valued map N by $D : K \to 2^{X^*}$:

$$D(x) = \begin{cases} \mathrm{conv}(N(x) \cap S^*) & \text{if } x \notin \arg\min_X f(x, \cdot) \\ B^* & \text{otherwise.} \end{cases}$$

By the same proof as in [5, Lemma 3.1], one can prove that if f satisfies (H_1) and (H_2) then one has $0 \notin D(x)$, for all $x \notin \arg\min_X f(x, \cdot)$.

Proposition 3.3. *Let K be a convex subset of X and $f : X \times X \to \mathbb{R}$ be a quasimonotone bifunction satisfying (H_1), $(H_2)$and (H_3). If* $\operatorname{int}(L_f(x)) \neq \emptyset$ *for any $x \notin \arg\min_X f(x, \cdot)$, then $EP(f, K) = S(D, K)$.*

Proof. If $x \in EP(f, K)$ then x is actually a minimizer of $f(x, \cdot)$ on K. Now, if $x \in \arg\min_X f(x, \cdot)$ then $0 \in D(x)$, i.e., $x \in S(D, K)$. On the other hand, if $x \notin \arg\min_X f(x, \cdot)$ then, in view of the first order necessary optimality condition in quasiconvex programming (see Theorem 4.1 in [8]), there exists $x^* \in D(x)$ such that x is a solution of $S(D, K)$.
Conversely, assume that $x \in S(D, K)$. If $0 \in D(x)$ then x belongs to $\arg\min_X f(x, \cdot)$, so that $f(x, y) \geq 0$ for all $y \in K$, i.e., $x \in EP(f, K)$. Now if $0 \notin D(x)$ then there exists a non zero element x^* of $N(x)$ such that $\langle x^*, y - x\rangle \geq 0$, for every $y \in K$ and thus, according to the sufficient optimality condition given in [7, Corollary 4.5], one gets that x is a minimizer of $f(x, \cdot)$ and thus $x \in EP(f, K)$. □

4 Existence of quasi-equibriums

Contrary to the proofs used for quasimonotone variational inequalities in [6], for quasimonotone equilibrium problems in [11] and for quasi-equilibrium problems in [19] which are based on KKM techniques, the existence results proposed in the forthcoming Theorem 4.1 is proved by mixing a Kakutani fixed point theorem with an adapted study of the stability (regularity) of the solution set (map) of a perturbed Convex Feasible Problem.

So let us first consider the following perturbation of the Convex Feasibility Problem: let $K : \mathbb{R}^m \to 2^{\mathbb{R}^n}$ be a set-valued map and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a bifunction. Then for any $\mu \in \mathbb{R}^m$, we define

$$(CFP_\mu) \qquad \text{find } x \in K(\mu) \text{ such that } f(y, x) \leq 0 \quad \forall y \in K(\mu).$$

Let us denote by $CFP(f, K(\cdot))$ the solution map which associates to any μ the solution set $CFP(f, K(\mu))$ of problem (CPP_μ). The following proposition, which is an adaptation of Proposition 4.4 in [4], gives sufficient conditions for the set-valued map $CFP(f, K(\cdot))$ to have a closed graph.

Proposition 4.1. *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction and $K : \mathbb{R}^m \to 2^{\mathbb{R}^n}$ be a set-valued map which satisfy the following properties*

i) *f verify (H_3);*

ii) *for all sequence $\{y_n\} \subset \mathbb{R}^n$ converging to $y \in \mathbb{R}^n$ and such that $\liminf_{n\to+\infty} f(y_n, x) \leq 0$, one has $f(y, x) \leq 0$;*

iii) *K is closed, lower semicontinuous and convex valued, with* $\operatorname{int}(K(\mu)) \neq \emptyset$*, for all $\mu \in \operatorname{dom}K$.*

Then the solution map $CFP(f, K(\cdot))$ is closed.

In order to prove the above proposition let us recall from [1] and [4] the following interesting properties.

Proposition 4.2. *[1, Prop. 3.2] Let S be a subset of X and $(S_n)_n$ be a sequence of convex subsets of X. Assume that $\mathrm{int}(S_n) \neq \emptyset$ for all n and that $\mathrm{int}(S) \neq \emptyset$. Then $i) \Rightarrow ii)$ where*

- *$i)$* $\mathrm{w-}\limsup_n S_n \subset S \subset \liminf_n S_n$
- *$ii)$* $\mathrm{w-}\limsup_n S_n \subset S,\ \mathrm{int}(S) \subset \liminf_n \mathrm{int}(S_n)$ *and S is convex.*

Proposition 4.3. *[4, Lemma 4.1] Let C and $(C_n)_n$ be a convex set and a sequence of convex subsets of $\mathbb{R}^n$ such that $C \subset \liminf_n \mathrm{int}(C_n)$. If $\mathrm{int}(C) \neq \emptyset$ and $\mathrm{int}(C_n) \neq \emptyset$, for all $n \in \mathbb{N}$, then for each $y \in \mathrm{int}(C)$ there exists $n_0 \in \mathbb{N}$ such that $y \in \mathrm{int}(C_n)$, for all $n \geq n_0$.*

Proof of Proposition 4.1. Let $((\mu_n, x_n))_n \subset \mathrm{Gr}\, CFP(f, K(\cdot))$ be a sequence converging to (μ, x). Since K is closed, one has that $x \in K(\mu)$.

We claim that any $y \in \mathrm{int}(K(\mu))$ belongs to $\mathrm{int}(K(\mu_n))$ for n large enough. Observe that, according to the lower semicontinuity of K, $K(\mu) \subset \liminf_{n\to+\infty} K(\mu_n)$. Since the sets $K(\mu)$ and $K(\mu_n)$ are convex with nonempty interior, it follows from Proposition 4.2 that

$$y \in \mathrm{int}(K(\mu)) \subset K(\mu) \subset \liminf_{n\to+\infty} \mathrm{int}(K(\mu_n)),$$

and thus the claim is now a consequence of Proposition 4.3.

Since, for all n we have $x_n \in CFP(f, K(\mu_n))$, it follows that

$$f(y, x_n) \leq 0,$$

and therefore $f(y, x) \leq 0$, by property (H_3). Now, for any $y \in K(\mu)$ there exists a sequence $(y_n)_n \subset \mathrm{int}(K(\mu))$ converging to y. So

$$f(y_n, x) \leq 0, \quad \text{for any } n \text{ large enough.}$$

By assumption $ii)$ on the bifunction f, one has that $f(y, x) \leq 0$ and therefore $x \in CFP(f, K(\mu))$. □

Remark 4.1. *An analoguous to property $i)$ of the bifunction f in Proposition 4.1 was introduced in [1, 4] for set-valued mappings, and it was called* dually lower semicontinuity *in [4].*

We are now in a position to establish our existence result for quasimonotone quasi-equilibrium problems.

Theorem 4.1. *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction, C a nonempty convex compact subset of $\mathbb{R}^n$ and $K : C \to 2^C$ be a set-valued map, and suppose that the following properties hold:*

i) *the map K is closed and lower semicontinuous, with convex values, and $\operatorname{int}(K(x)) \neq \emptyset$ for all $x \in C$,*

ii) *f has the upper sign property on C,*

iii) *$\operatorname{int}(K(x)) \neq \emptyset$ for all x, and for all $x, y \in X$ and all sequence $(y_n)_n \subset X$ converging to y it holds that*

$$\liminf_{n\to+\infty} f(y_n, x) \le 0 \;\Rightarrow\; f(y, x) \le 0,$$

or, for all $x, y \in X$ and all sequences $(x_n)_n$, $(y_n)_n \subset X$ converging to x, y respectively, it holds that

$$\liminf_{n\to+\infty} f(y_n, x_n) \le 0 \;\Rightarrow\; f(y, x) \le 0,$$

iv) *f is properly quasimonotone and satisfies (H_2) and (H_3).*

Then the quasi-equilibrium problem $QEP(f, K)$ admits at least a solution.

Proof. Since f is properly quasimonotone and $K(x)$ is nonempty, convex and compact, it follows from Corollary 2.1 that $CFP(f, K(x)) \neq \emptyset$ for all $x \in C$. Now the map $CFP(f, K(\cdot)) : C \to 2^C$ is closed and has convex and closed values by Propositions 4.1 and 2.1. Hence, due to the compactness of C, this solution map is upper semicontinuous. Now according to the Kakutani Fixed Point Theorem (see e.g. [25]), we conclude that the map $CFP(f, K(\cdot))$ admits at least a fixed point x, or in other words $x \in CFP(f, K(x))$. By Proposition 3.1(i) we get that $x \in EP(f, K(x))$, and therefore x is a solution of $QEP(f, K)$. □

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