UNUSUAL RIEMANN SOLUTION STRUCTURES FOR THERMAL TWO-PHASE FLOW IN POROUS MEDIA

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Abstract. We consider a nonlinear system of conservation laws arising in petroleum engineering. We are modeling the injection of a mixture of gas and oil, in any proportion, into a porous medium filled with a similar mixture. The two mixtures may have different temperatures. We will focus on a particularly unusual feature found in this model: for open set of Riemann data the solution is given by a single wave group, *i.e.*, there is no constant intermediate state. The key aspect supporting this feature is the existence of structurally stable doubly sonic shock waves, which robustly connect slow rarefaction waves to fast rarefaction waves. The solutions are constructed around a curve of coinciding characteristic speeds, intrinsically associated to most bifurcations in the Riemann solutions for this class of models.

1. Introduction

We consider systems of conservation laws in the form:

$$\varphi \partial_t G(\mathbf{w}) + \partial_x u F(\mathbf{w}) = 0, \tag{1}$$

which models compositional thermal two-phase flow in a porous medium. This class of models is useful in applications such as advanced recovery of gas or oil, see [2], and clean-up of polluted sites, see [8]. Here, the reduced state space consists of oil saturation (s) and temperature (T) above a reference temperature (T_{ref}), which we write as:

$$\mathbf{w} \in \mathbf{\Omega} = \{ \ (s,T) \mid 0 \le s \le 1, \ T > T_{ref}\}.$$

The variable u denotes the Darcy speed and φ is the constant porosity of the medium. To fully describe a state in system (1) one needs to specify $(\mathbf{w}, u) \in \mathbf{\Omega} \times \mathbb{R}^+$. However, the variable u can be recovered from $\mathbf{w}$, as done in [6]; for that reason we study the wave structure in $\mathbf{\Omega}$, which we will call state space. The vector valued accumulation function $G = (G_1, G_2, G_3)^T$ and flux function $F = (F_1, F_2, F_3)^T$ are smooth on the entire state space.

A remarkable property satisfied by models in this class is that they can be written in the following form (after rescaling time and dropping φ):

$$\partial_t \Big(\alpha(T)s + \beta(T)\big(1-s\big) + \gamma(T)\Big) + \partial_x u \Big(\alpha(T) f(s,T) + \beta(T)\big(1 - f(s,T)\big)\Big) = 0.$$

Here, $f = s^2/(s^2 + \nu(T)(1-s)^2)$ is the fractional flow function (ν is the positive viscosity ratio between the two phases) and the vector valued temperature-dependent quantities α, β and γ are obtained from thermodynamical considerations, see [12].

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2. Basic facts

Smooth self-similar solutions of (1), or rarefaction waves, are related to the generalized eigenvalue problem:

$$J(\mathbf{w},u;\lambda_i)\mathbf{r}_i(\mathbf{w},u)=0, \quad \text{with} \quad J(\mathbf{w},u;\lambda)=\big(u\mathrm{D}F(\mathbf{w})-\lambda\mathrm{D}G(\mathbf{w}),\ F(\mathbf{w})\big), \tag{2}$$

where D denotes differentiation relative to $\mathbf{w}$; see more details in [7]. Let $(\mathbf{w}(\xi),u(\xi))$ be such a smooth self-similar solution conveniently parametrized by $\xi=x/t$ and $\mathbf{r}_i$ a right eigenvector defined by (2). Together, they must satisfy the ODE in ξ:

$$\big(\dot{\mathbf{w}}(\xi),\dot{u}(\xi)\big)=\mathbf{r}_i(\mathbf{w}(\xi),u(\xi)), \tag{3}$$

for a suitable initial datum. The family i is one of the two possible choices:

Proposition 1. *There are two eigenpairs associated to the generalized eigenvalue problem (2), namely:*

- *(s)* $\lambda_s(\mathbf{w},u)=u\partial_s f(s,T)$ *and* $\mathbf{r}_s=(1,0,0)^T$;
- *(e)* $\lambda_e(\mathbf{w},u)=u\dfrac{a(T)f(s,T)+b_1(T)}{a(T)s+b_2(T)}$ *and* $\mathbf{r}_e=\big(\widehat{\mathbf{r}}_e(s,T)^T,u\mathcal{X}_3(s,T)\big)^T$,

where $\mathbf{w}=(s,T)$ *and* $\widehat{\mathbf{r}}_e(\mathbf{w})=\big(\mathcal{X}_1(\mathbf{w}),\mathcal{X}_2(\mathbf{w})\big)^T$ *is the restriction of* $\mathbf{r}_e$ *to* $\mathbf{\Omega}$.

Remark 1. The quantities $a(T)$, $b_1(T)$ and $b_2(T)$ are positive, as in the examples [6], [13]; see there expressions for $\mathbf{r}_e$ and these quantities in terms of α, β and γ.

The family *(s)* produces pure transport of fluid without changes in temperature and is henceforth called the saturation transport family. Family *(e)* is associated to thermal transport and typically produces waves associated to evaporation/condensation phenomena; it will be called the thermal transport family.

A bounded discontinuous self-similar solution of (1) with a jump connecting the pairs $(\mathbf{w}^-,u^-)$, $(\mathbf{w}^+,u^+)$ must satisfy the Rankine-Hugoniot relation:

$$\mathcal{H}(\mathbf{w}^-,u^-,\mathbf{w}^+,u^+;\sigma)\equiv u^+F(\mathbf{w}^+)-u^-F(\mathbf{w}^-)-\sigma\left(G(\mathbf{w}^+)-G(\mathbf{w}^-)\right)=0, \tag{4}$$

where the wave speed is denoted by σ. For a fixed initial state $(\mathbf{w}^-,u^-)$ a typical issue is the locus of states $(\mathbf{w}^+,u^+)$ that satisfy (4). Here, one first calculates the set of states $\mathbf{w}^+$ in which the determinant of the matrix $\big(F(\mathbf{w}^+),F(\mathbf{w}^-),G(\mathbf{w}^+)-G(\mathbf{w}^-)\big)$ is zero; the resulting set is called the Hugoniot locus of the base state $\mathbf{w}^-$, which will be denoted as $\mathcal{H}(\mathbf{w}^-)\subset\mathbf{\Omega}$. Afterwards, the Darcy speed u^+ can be obtained for any triplet $(\mathbf{w}^-,u^-,\mathbf{w}^+)$, $\mathbf{w}^+\in\mathcal{H}(\mathbf{w}^-)$ by using (4). Here, we call a jump admissible if it satisfies the Liu criterion [10]; admissible jumps are called shock waves. Regarding the shape of the Hugoniot locus, we have [7]:

Proposition 2. *For any pair* $(\mathbf{w}^-,u^-)\in\mathbf{\Omega}\times\mathbb{R}^+$*, the locus* $\mathcal{H}(\mathbf{w}^-)$ *always contains a straight line in state space. On this line, the following equalities hold:*

$$T^+=T^-, \quad u^+=u^- \quad \text{and} \quad \sigma=u^-\frac{f^+-f^-}{s^+-s^-}. \tag{5}$$

Shocks connecting the base state $\mathbf{w}^-$ to points $\mathbf{w}^+$ in this line satisfy the Lax inequalities for the s-family: $\lambda_s(\mathbf{w}^-,u^-)\geq\sigma\geq\lambda_s(\mathbf{w}^+,u^+)$, where $u^+=u^-$, u^- is given and only one inequality is allowed to become an equality. We therefore call this line the saturation branch of the Hugoniot locus, and denote this set by $\mathcal{H}_s(\mathbf{w}^-)$. Regarding the thermal branch of the Hugoniot locus (motivated by Proposition 2) we may ask if it makes sense to take the limit in the expression of the Rankine-Hugoniot relation (4) divided by $T^+ - T^-$, when T^+ tends to T^-, *i.e.*, if the following function is well defined:

$$h_e(\mathbf{w}^+;\mathbf{w}^-) = \begin{cases} \dfrac{\det\big(F(\mathbf{w}^+);\, F(\mathbf{w}^-);\, G(\mathbf{w}^+) - G(\mathbf{w}^-)\big)}{T^+ - T^-}, & T^+ \neq T^-; \\ \lim\limits_{T^+\to T^-} \dfrac{\det\big(F(\mathbf{w}^+);\, F(\mathbf{w}^-);\, G(\mathbf{w}^+) - G(\mathbf{w}^-)\big)}{T^+ - T^-}, & \text{otherwise.} \end{cases} \tag{6}$$

One can prove that the function h_e defined in (6) is smooth. Let $\mathcal{H}_e(\mathbf{w}^-)$ consist of points $\mathbf{w}^+$ in the zero set of $h_e(\,\cdot\,;\mathbf{w}^-)$. We have:

Proposition 3. *For any base state $\mathbf{w}^-$ the Hugoniot locus can be decomposed as:*

$$\mathcal{H}(\mathbf{w}^-) = \mathcal{H}_s(\mathbf{w}^-) \bigcup \mathcal{H}_e(\mathbf{w}^-). \tag{7}$$

Of course, the set $\mathcal{H}_e(\mathbf{w}^-)$ is the thermal branch of the Hugoniot locus; its topology will be discussed in the next section.

3. Elementary waves and curves

Since the seminal works [9] and [4], strict hyperbolicity and genuine nonlinearity were mainly recognized as two key assumptions for the well-posedness of the Cauchy problem for systems of conservation laws. In the class of systems we consider both are violated; yet, as far as the Riemann problem is concerned, examples of existence and continuous dependence of the solution with respect to the initial data are known, see [6] and [12]. For our models, strict hyperbolicity is lost when:

Proposition 4. *The two characteristic speeds coincide along a smooth curve, transverse to the constant field $\widehat{\mathbf{r}}_s = (1,0)^T$, which disconnects state space. We denote this coincidence curve by $\mathcal{C}$.*

Away from the coincidence curve we may solve unambiguously the ODE stated in (3), for each characteristic family. However, these orbits can (and typically will) cross the coincidence curve; this is the case we will tackle now. In what follows, we will only focus on the flow induced by the eigenvectors of the thermal family; the other one lies along straight lines. A fundamental observation is that:

Proposition 5. *Along the coincidence curve the kernel of the matrix J, defined in (2b), has dimension 1, generically.*

This and the obvious observation that the dimension of the kernel of J can only be one or two, which can be seen directly from (2b), lead to the following:

Definition 3.1. A point on the coincidence curve is called singular if the dimension of the kernel of J equals 2. We denote the set of singular points by $\mathbb{S}$.

Remark 2. Notice that, by Propositions 4 and 5, the singular points form a discrete subset of the coincidence curve. Researchers interested in models for three-phase flow in porous media should contrast this behavior with those in [5], [1].

Now we are in position to state:

Proposition 6. *The thermal family eigenvector-field can be represented by a smooth, non-vanishing vector field in $\mathbf{\Omega} \setminus \mathbb{S}$.*

Proposition 6 allows us to construct smooth thermal self-similar solutions of (1), provided that genuine nonlinearity holds almost everywhere for this family and the set where it fails is transverse to the thermal vector field. This is the case as will be shown now. We use the notation $\mathcal{I}_e$ for the set of points $(\mathbf{w}, u)$ on which genuine nonlinearity is lost in the thermal family, *i.e.*, the equality $\nabla\lambda_e(\mathbf{w}, u) \cdot \mathbf{r}_e(\mathbf{w}, u) = 0$ is satisfied. The set $\mathcal{I}_e$ is called the inflection locus (of the thermal family); it possesses the following properties:

Proposition 7. *In state space, the thermal inflection locus is a smooth curve such that $\mathcal{I}_e \supset \mathcal{C}$. However, in a neighborhood of a singular point the inflection and the coincidence curves are the same,* i.e., *every singular point possesses a neighborhood V in which $\mathcal{I}_e \cap V = \mathcal{C} \cap V$.*

Together with Propositions 4 and 5, Proposition 7 gives the properties we need for constructing thermal rarefaction waves in a neighborhood of a singular point. Actually, this construction can be extended to the whole state space.

Of course, singular points also have great influence on the thermal branches of the Hugoniot locus. The reader may already expect that:

Proposition 8. *If $\mathbf{w}^*$ is a singular point then* $\partial_{\mathbf{w}^+} h_e(\mathbf{w}^+; \mathbf{w}^*)\Big|_{\mathbf{w}^+=\mathbf{w}^*} = 0.$

The criticality stated in Proposition 8 allows two generic situations: in a neighborhood of a singular point the e-branches of the Hugoniot locus are diffeomorphic either to circles or to hyperbolae. To construct the Riemann solution we propose, the singular points are required to satisfy the following non-degeneracy conditions; both can be verified in physical models such as [6] and [12].

Assumption 3.2. If $\mathbf{w}^* \in \mathbb{S}$ is a singular point then:

A1 $\mathcal{H}_e(\mathbf{w}^*) = \{\mathbf{w}^*\}$. There is a neighborhood V of the singular point such that the curve $\mathcal{H}_e(\mathbf{w}^-)$ is diffeomorphic to a circle for any state $\mathbf{w}^- \in V \setminus \mathbb{S}$. Moreover, for any $\mathbf{w}^- \in V \setminus \mathbb{S}$ the locus $\mathcal{H}_e(\mathbf{w}^-)$ is tangent to the e-family integral curves only at $\mathbf{w}^-$.

A2 The formula for $\widehat{\mathbf{r}}_e$ can be chosen so that $\widehat{\mathbf{r}}_e(\mathbf{w}^*) = 0$, preserving the smoothness of the thermal eigenvector field. The eigenvalues of $\mathrm{D}\widehat{\mathbf{r}}_e(\mathbf{w}^*)$ are a pair of complex conjugate numbers with non-zero real part.

From now on we use bold capital roman letters to denote points of type $\mathbf{P} = (\mathbf{w}^P, u^P) \in \mathbf{\Omega} \times \mathbb{R}^+$. A distinctive property of this class of systems is that, to obtain the solution of the Riemann problem, one just needs to describe the self-similar solutions in $\mathbf{\Omega}$; the variable u can be obtained from its boundary value in the PDE problem and from $\mathbf{w}$, see [6]. Thus we will represent the elementary wave-curves in $\mathbf{\Omega}$; we often use the notation $\mathbf{P}$ instead of $\mathbf{w}^P \in \mathbf{\Omega}$. We illustrate rarefaction curves and shock branches in Figure 1.

A typical Riemann solution can be decomposed into a sequence of *constant states* and *wave-groups*: groups of elementary waves (shocks or rarefactions) that move together as a single entity. We introduce the notation $\mathbf{P_1} \xrightarrow{w} \mathbf{P_2}$ to express the fact that the states $\mathbf{P_1}$, $\mathbf{P_2}$ are connected by an elementary wave w. Here, elementary shocks and rarefactions are saturation or thermal transport waves; they may be slow or fast depending on the ordering between the characteristic speeds, which varies: see Proposition 4. Deciding if a wave is a s-wave or an e-wave for rarefactions is a matter of verifying which is the associated eigenvalue; for shocks one must verify in which branch of the Hugoniot-locus lies the pair of left, right states.

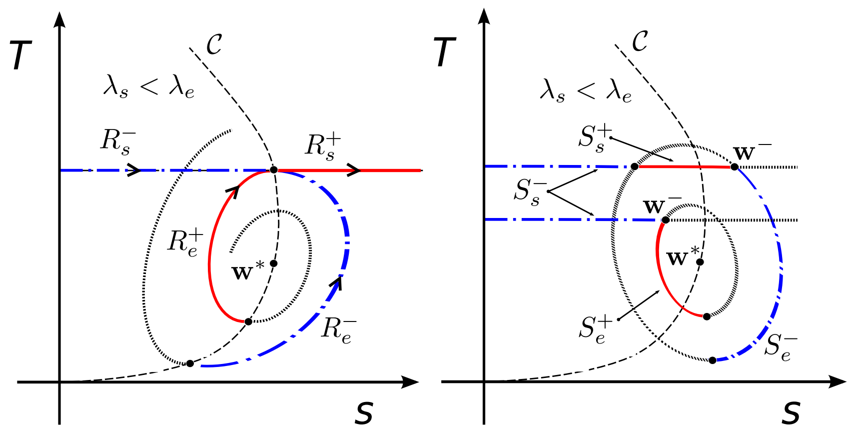

FIGURE 1. Left: rarefaction curves; slow waves are dashed-dotted, fast waves are solid. Right: shocks, the same graphical convention. We use the superscripts $-$, $+$ for slow, fast waves and the subscripts s, e for saturation, thermal waves.

To construct the solution of the Riemann problem, we will use a generalization of Liu's wave-curve method; see [10], [3], [11] and [12]. We define $\mathcal{W}^-(\mathbf{L})$, the slow wave curve *emanating* from state $\mathbf{L}$, as the set in state space that can be reached through a speed-increasing succession of admissible elementary slow waves beginning at $\mathbf{L}$. We define $\mathcal{W}^+(\mathbf{R})$, the fast wave curve *reaching* state $\mathbf{R}$, as the set in state space from which one can reach the state $\mathbf{R}$ through a speed-increasing succession of admissible elementary fast waves. Notice the asymmetry in these definitions. In the classical theory for 2×2 Liu systems, the Riemann solution between $\mathbf{L}$ and $\mathbf{R}$ (if it exists) would be obtained by finding the intersection of $\mathcal{W}^-(\mathbf{L})$ with $\mathcal{W}^+(\mathbf{R})$: the solution consists of elementary waves from the slow wave-curve, succeeded by the constant state $\mathbf{M} \in \mathcal{W}^-(\mathbf{L}) \bigcap \mathcal{W}^+(\mathbf{R})$ and, finally, elementary waves from the fast wave-curve. In the specific case we present here there would be no such intersection: a doubly characteristic shock (to be defined in the next paragraph) must be used as a bridge between the two wave-curves.

The construction of the Riemann solution relies on the fact that for any state $\mathbf{w}^-$ in a neighbourhood of the singular point and any $u^- > 0$ there is a companion $\mathbf{w}^+ \in \mathcal{H}_s(\mathbf{w}^-) \bigcap \mathcal{H}_e(\mathbf{w}^-)$ such that the shock between $\mathbf{w}^-$ and $\mathbf{w}^+$ is doubly characteristic with respect to the thermal speed: $\lambda_e(\mathbf{w}^-, u^-) = \sigma = \lambda_e(\mathbf{w}^+, u^+)$, with $u^+ = u^-$ by Proposition 2 and σ is the shock speed in (4). Thus we introduce the following notation: for a set γ (typically, a curve) in state space, the mixed extension set is

$$\mathcal{E}(\gamma) = \left\{ \mathbf{w}^+ \in \mathbf{\Omega} \mid \exists \mathbf{w}^- \in \gamma;\ \mathbf{w}^+ \in \mathcal{H}_s(\mathbf{w}^-) \text{ and } \sigma = \lambda_e(\mathbf{w}^-, u^- = 1) \right\}. \tag{8}$$

One can see that the extension set is independent of the choice $u^- = 1$. Of course, $\mathbf{w}^-$ and $\mathbf{w}^+$ have the same temperature in (8) . We will also need the following:

Proposition 9. *Let $\mathbf{P}^- \equiv (\mathbf{w}^-, u^-) \neq \mathbf{P}^+ \equiv (\mathbf{w}^+, u^+)$ be at the same temperature; $\mathbf{P}^-$, $\mathbf{P}^+$ and σ satisfying (4). Then $\sigma = \lambda_e(\mathbf{P}^+)$ if and only if $\sigma = \lambda_e(\mathbf{P}^-)$.*

4. THE RIEMANN SOLUTION

In this section we provide a sketch for the proof of:

Theorem 4.1. *In a neighborhood W of a singular point $\mathbf{w}^*$ satisfying Assumption 3.2, there are open sets $U, V \subset W$ such that the Riemann solution for any pair $\boldsymbol{L} = (\mathbf{w}^L, u^L)$, $\boldsymbol{R} = (\mathbf{w}^R, -)$ with $\left(\mathbf{w}^L, \mathbf{w}^R\right) \in U \times V$ possesses no intermediate constant state.*

A few remarks are in order: first, we cannot assign both u^L and u^R as Riemann data for system (1) in general, see [7]; this is not a property particular to the Riemann solutions we are showing. Second, this is a part of the full Riemann solution in the neighborhood of a singular point, see [12]: the complete solution satisfies Liu's criterion [10], is L^1_{loc} continuous with respect to the Riemann data and is structurally stable in the sense of [11]. Albeit superficially similar to the doubly sonic (characteristic) transitional shock waves predicted in [11], in our case, the doubly characteristic shock waves vary when we allow the Riemann data to change.

Proof. By Proposition 7 we can restrict ourselves to a neighborhood of the singular point $\mathbf{w}^*$, such that $\mathcal{I}_e = \mathcal{C}$. One can further verify that this neighborhood can be chosen so that $s \mapsto \lambda_s(s, T)$ is monotone (increasing, for definiteness), *i.e.*, the s-family is genuinely nonlinear. Moreover, on isotherms, the saturation s increases (decreases) along s-rarefactions (s-shocks), see Propositions 1 and 2. Both eigenvalues are positive, recall Remark 1, and on the left (right) side of $\mathcal{C}$ we have $\lambda_s < \lambda_e$ ($\lambda_s > \lambda_e$). We choose a $\mathbf{R}$ state on the right side of the coincidence curve $\mathcal{C}$, above the singular point; the set of $\mathbf{L}$ states will be specified soon.

The fast wave curve reaching $\mathbf{R}$ from below, $\mathcal{W}^+(\mathbf{R})$, contains a fast thermal rarefaction curve, R_e^+, on the left side of the coincidence curve $\mathcal{C}$ followed by a fast saturation rarefaction curve, R_s^+, on the right side of $\mathcal{C}$ (see Figure 2a). These rarefaction curves are tangent at their intersection $\mathbf{O}$, which lies on $\mathcal{C}$, see Proposition 5; thus these rarefactions form a wave-group.

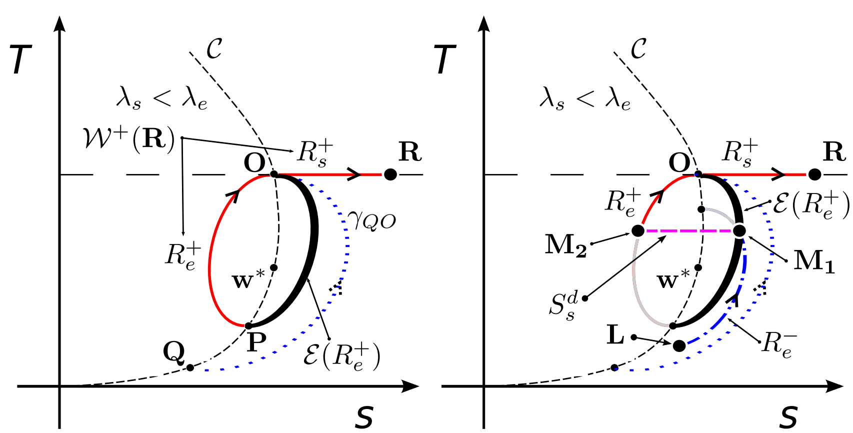

FIGURE 2. Left: $\mathcal{W}^+(\mathbf{R})$, the fast wave curve reaching $\mathbf{R}$ and $\mathcal{E}(R_e^+)$, the (doubly) characteristic extension of R_e^+. Right: R_e^-, slow rarefaction curve emanating from $\mathbf{L}$ and S_s^d, saturation doubly characteristic shock between the slow and fast wave-curves.

Now we focus on the boundaries of the set from which we will select $\mathbf{L}$ states. First: the integral curve containing R_e^+ also contains a slow thermal rarefaction curve beginning at a point $\mathbf{Q} \in \mathcal{C}$ and reaching $\mathbf{O}$, which we will denote as γ_{QO} and, of course, lies entirely on the right side of $\mathcal{C}$, see the dashed curve on Figure 2a. Second: from (8), $\mathcal{E}(R_e^+)$ is the set from which emanate characteristic s-shocks reaching R_e^+. We claim that $\mathcal{E}(R_e^+)$ is a smooth curve (see next section), which is drawn as the bold curve on Figure 2a. Moreover, there is an open set $Z \in \mathbf{\Omega}$ such that $Z \cap \mathcal{E}(R_e^+)$ is transverse to the e-family vector field. By construction, the set Z must be on the right side of $\mathcal{E}(R_e^+)$; one may see that it is on the left side of γ_{QO}.

Finally, we choose a $\mathbf{L} \in Z$ such that there is a slow e-rarefaction R_e^- emanating from it and reaching $Z \cap \mathcal{E}(R_e^+)$ transversally at a point $\mathbf{M_1}$, see Figure 2b. From (8), there is a state $\mathbf{M_2}$ in R_e^+ such that the shock speed σ between $\mathbf{M_1}$ and $\mathbf{M_2}$ satisfies $\sigma = \lambda_e(\mathbf{M_1})$; by Proposition 9, it also satisfies:

$$\lambda_e(\mathbf{M_1}) = \sigma = \lambda_e(\mathbf{M_2}).$$

Under these conditions there is a Riemann solution for the data $\mathbf{L}$, $\mathbf{R}$, built so that the speed admissibility condition is satisfied. The construction is illustrated in Figure 2b, the profiles and the x/t diagram are depicted in Figure 3; the Riemann solution is:

$$\mathbf{L} \xrightarrow{R_e^-} \mathbf{M_1} \xrightarrow{S_s^d} \mathbf{M_2} \xrightarrow{R_e^+} \mathbf{O} \xrightarrow{R_s^+} \mathbf{R}.$$

□

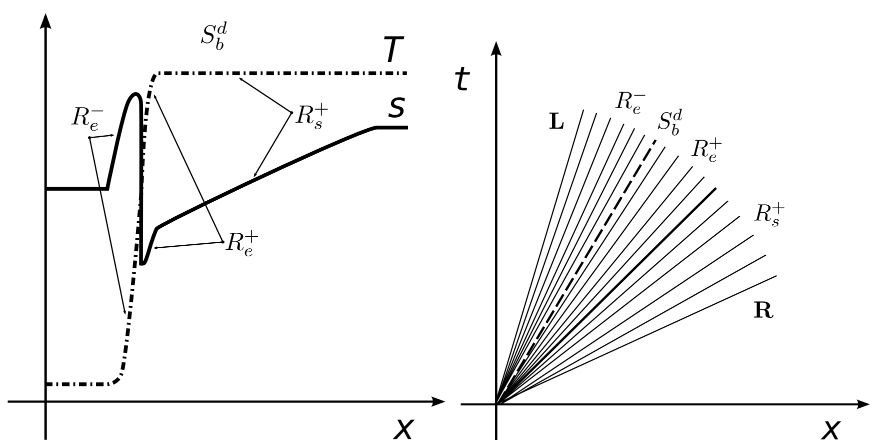

FIGURE 3. Riemann solution. Left: s and T profiles (out of scale) for a fixed time. Right: diagram of characteristics in x/t plane; R_e^+ and R_s^+ join at the bold line.

The construction outlined above is fairly generic; in the next section we will address why the extension (8) maps integral curves on the left side of $\mathcal{C}$ to smooth curves on the right side of $\mathcal{C}$.

5. FINAL FACTS

For condition (4) to hold in an open set of state space it is necessary that the secondary bifurcation of the Hugoniot locus does exist generically, [3]. We reserve the capital letter W for a suitably chosen neighborhood of the singular point in $\mathbf{\Omega}$ and define:

$$\mathcal{D} = \Big\{W \times W \times \mathbb{R}\Big\} \setminus \Delta, \quad \Delta = \{(\mathbf{w}^-, \mathbf{w}^+, u^+) \in \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbb{R}^+ \mid \mathbf{w}^- = \mathbf{w}^+\}. \tag{9}$$

Using Proposition 9, one can see that the set of pairs of states between which doubly characteristic shocks can be constructed is exactly the zero level of the smooth map:

$$\mathcal{F} : \mathcal{D} \to \mathbb{R}^4, \quad (\mathbf{w}^-, \mathbf{w}^+, u^+) \mapsto \Big(\mathcal{H}(\mathbf{w}^-, u^- = 1, \mathbf{w}^+, u^+; \lambda_e^+),\ \mathbf{l}_e(\mathbf{w}^+)[G]\Big), \tag{10}$$

where $\lambda_e^+ = \lambda_e(\mathbf{w}^+, u^+)$, $[G] = G(\mathbf{w}^+) - G(\mathbf{w}^-)$, $\mathbf{l}_e$ is the left eigenvector of the thermal family and the function $\mathcal{H}$ is given in (4). Such zero level is the secondary bifurcation locus $\mathcal{B} = \mathcal{F}^{-1}\{0\}$. From $\mathcal{B}$ we can define the projection maps $\pi_+, \pi_- : \mathcal{B} \to W$, $\pi_+(\text{resp. } \pi_-) : (\mathbf{w}^-, \mathbf{w}^+, u^+) \mapsto \mathbf{w}^+(\text{resp. } \mathbf{w}^-)$. We are ready to state:

Theorem 5.1. *The secondary bifurcation locus $\mathcal{B}$ is a smooth two dimensional manifold in $\mathcal{D}$. The projections π_+, π_- map $\mathcal{B}$ diffeomorphically onto $W \setminus \mathcal{C}$.*

Since one can use the map $\pi_- \circ \pi_+^{-1}$ in $W \setminus \mathcal{C}$ as the extension map of the fast e-rarefactions, Theorem 5.1 gives immediately the properties we sought.

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