LOWER BOUNDS FOR THE HAUSDORFF DIMENSION OF THE GEOMETRIC LORENZ ATTRACTOR: THE HOMOCLINIC CASE

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ABSTRACT. We provide lower bounds for the Geometric Lorenz attractor in terms of the eigenvalues of the singularity and the symbolic dynamics associated to the geometrical distribution of the attractor.

1. INTRODUCTION

In 1963, E. Lorenz [L] introduced the following three-dimensional system of differential equations:

$$(LS) \qquad \begin{aligned} x' &= \sigma(y-x) \\ y' &= rx - y - x\,z \\ z' &= x\,y - b\,z \end{aligned}$$

where σ, r, and b are positive valued parameters, as a simplified model in order to explain the weather behavior. He found that, for the following values of the above parameters: $\sigma = 10$, $b = 8/3$ and $r = 28$, the trajectory of any point tends to the same complicated set, the so-called *Lorenz attractor*.

Since the divergence of the above systems is negative, its follows that the Lebesgue measure of the above set is zero. So the next step is to ask for its Hausdorff dimension. Numerical experiments give that this value is around 2.05.

In order to understand the dynamics of *(LS)* and in particular the geometry of the Lorenz attractor, in late seventies it was introduced a geometric model by Guckenheimer, Williams [GW] and Afraimovich, Bykov and Shilnikov [ABS2]. That is, a three-dimensional flow $\mathcal{L}$ (see next section for a definition), whose dynamics is the same as that of *(LS)*(see [T]).

A first approximation to answer the above question is to ask for the Hausdorff dimension of the attractor in the geometric model. In [AP]and [S], this dimension is characterized in terms of the pressure of the system and in terms of the Lyapunov exponents and the entropy respect to a *good* invariant measure associated to the system.

In this paper, we address the problem to found lower bounds for the attractor associated to $\mathcal{L}$. The first lower bound that can be obtained for the Hausdorff dimension is that it is greater or equal than 2, since there exists a periodic orbit in the attractor with a unstable manifold of dimension two. The questions now is if the dimension can be strictly greater than two and whether it can be obtained

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lower bounds in terms of the characteristic data of the system, in particular, the relationship with the eigenvalues of the unique singularity in the system. Here, we provide answer to these questions for the geometric model in the homoclinic case. The way to do that is reducing the problem for the phase flow $\mathcal{L}$ to a discrete time problem, i.e., a Poincaré return map induced by $\mathcal{L}$ for some cross-section surface. Let us denote that return map by F and let Λ_F be the hyperbolic attractor of F. Then we have the following theorem

Theorem 1. *If $\mathcal{L}$ is homoclinic, then there exists $0 < \gamma < 1$ such that*

$$dim_H(\Lambda_F) \geq 1 + \ln \rho(A) / \ln \left(\frac{1}{\gamma^a}\right) > 1,$$

where $\rho(A)$ is the spectral radius of the matrix A, a $(0,1)$-matrix which describes the geometric distribution of Λ_F and $a = \sum_i u_i v_i$ where u and v are respectively the right and left Perron-Frobenius eigenvectors of A.

Remark 1. It follows from the proof of Theorem (1) that if n is the number of turns of the homoclinic orbit around the singularity O of the flow $\mathcal{L}$, then $\gamma \geq K_1 \min_i\{|m_i|^\alpha\}$, where m_i is the least period orbit ($\geq n$) which *shadows* the intersection of the homoclinic orbit with the cross-section surface used to build the Poincaré return map. In the *monotone case* (see section 5 for a definition) the estimative can be improved using the only two-periodic point of the map F instead of the periodic orbit mentioned previously.

Remark 2. The relationship between α and the eigenvalues of the singularity is given by $\alpha = -\dfrac{\lambda_3}{\lambda_1}$ where λ_1 and λ_3 are respectively, the expanding and the weak contractive eigenvalues of the singularity.

Corollary 2. *The Geometric Lorenz attractor of the flow $\mathcal{L}$ has*

$$dim_H(\Lambda) \geq 2 + \ln \rho(A) / \ln \left(\frac{1}{\gamma^a}\right) > 2.$$

The authors would like to thanks Professor C. Morales for useful discussions about the Geometric Lorenz attractor.

2. The Geometric Lorenz Attractor

In this section we are going to describe a vector field $\mathcal{L}$ in $\mathbb{R}^3$ with a dynamics corresponding to the Lorenz systems for certain parameters.

So let $\mathcal{L}$ be a smooth vector field such that it has a singular attractor Λ, with a singularity at $O = (0,0,0)$. At this singularity, $D\mathcal{L}$ has eigenvalues λ_1, λ_2 and λ_3 satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1$. Also it is assumed that $\lambda_3 + \lambda_1 > 0$.

Now we consider the section $V = \{(x,y,z) : z = 1, |x|, |y| \leq 1\}$, as it is shown in Figure 1, the vector field $\mathcal{L}$ points downward in the interior of this square. So a return map F is well defined except at the intersection points of the stable manifold of the singularity O with the section V. This intersection is given by the segment $D = \{(x,y,z) \in S : x = 0\}$. Using x, y as coordinates on V we request that $F(x,y) = (f(x), g(x,y))$ has the following properties:

(1) $F(-x,-y) = -F(x,y)$;

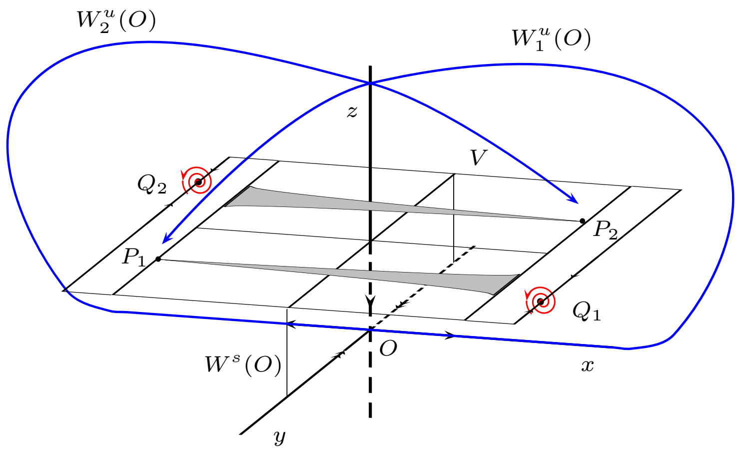

FIGURE 1

(2) There exists two constants $K_1 < K_2 < 1$ such that

$$K_1|x|^\alpha \leq \|\frac{\partial g}{\partial y}(x,y)\| \leq K_2|x|^\alpha;$$

(3) The following limits exist:

$$\lim_{x\to 0^+} F(x,y) = P_1 = (p_{11}, p_{12}) \quad \text{and} \quad \lim_{x\to 0^-} F(x,y) = P_2 = (p_{21}, p_{22});$$

(4) $f'(x) > \sqrt{2}$;

(5) $|\det DF(x,y)| = |f'(x)\dfrac{\partial g}{\partial y}(x,y)| < b < 1$, for $x \neq 0$.

Then image of V under F looks like that in Figure 2, where $V_0 = \{(x,y) \in V : x < 0\}$ and $V_1 = \{(x,y) \in V : x > 0\}$.

Letting $D_n = F^{-n}(D)$ and $\Lambda_F = \overline{\bigcap_0^\infty F^i(V \setminus D_i)}$, then the Geometric Lorenz attractor turns out to be the set

$$\Lambda = \left(\bigcup_{t\in\mathbb{R}} \Lambda_F\right) \cup O.$$

Remark 3. We can think of $F = (f, g)$ as being the following map

$$f(x) = \begin{cases} -1 + Ax^\alpha, & if\ x > 0, \\ \\ 1 - A(-x)^\alpha, & if\ x < 0 \end{cases},$$

where $A \in (1,2)$, $\alpha \in (0,1)$ and $\alpha A > \sqrt{2}$, and

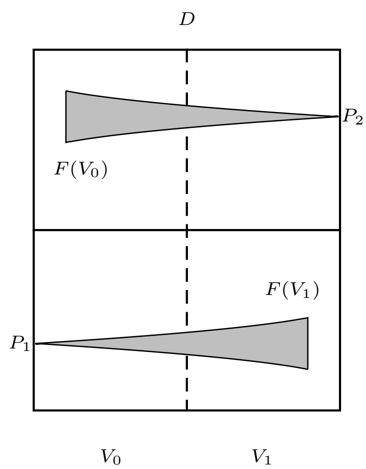

FIGURE 2

$$g(x,y)=\begin{cases} -\dfrac{1}{2}+Bx^{\alpha}+Cy\,x^{\beta}, & if\ x>0 \\ \\ \dfrac{1}{2}-B(-x)^{\alpha}+Cy\,(-x)^{\beta}, & if\ x<0 \end{cases},$$

We have the following relation between α and β above and the eigenvalues of O : $\alpha=-\dfrac{\lambda_3}{\lambda_1}$ and $\beta=-\dfrac{\lambda_2}{\lambda_1}$.

The points P_i are the first intersections of the branches $W_i^u(O)$ of the unstable manifold of O with the transversal section V as can be seen in Figure 1.

Definition 3. *We say that Λ is homoclinic if both $W_i^u(O)$ are included in $W^s(O)$,i.e., $W_i^u(O)$ is an homoclinic orbit.*

3. Shift of finite Type

In this section we recall several notions and results in symbolic dynamics. We will follow closely to [LM]

Let $\mathcal{A}$ be a finite set of symbols which we will call the alphabet. The set $\mathcal{A}^{\mathbb{N}}$ ($\mathbb{N}$ is the set $\{0,1,2,\ldots\}$) is known as the full shift in $\mathcal{A}$, i.e., the set of all sequences of symbols from $\mathcal{A}$. We would write the elements $\mathbf{i}$ of $\mathcal{A}^{\mathbb{N}}$ by

$$\mathbf{i}=i_0i_1i_2...,$$

and denote by $\sigma:\mathcal{A}^{\mathbb{N}}\hookleftarrow$ the shift transformation: $\sigma(\mathbf{i})=\mathbf{j}$ where $j_s=i_{s+1}$. A word (block) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. For $\mathbf{i}$ and $r\leq s$ we let $\mathbf{i}_{[r,s]}$ be the word given by $i_ri_{r+1}...i_s$.

Definition 4. *If $\mathcal{F}$ is a finite collection of words over $\mathcal{A}$, then a subset $X\subset\mathcal{A}^{\mathbb{N}}$ will be called a* shift space of finite type *if for each $\mathbf{i}\in X$ and any pair $r\leq s$ we have that $\mathbf{i}_{[r,s]}\notin\mathcal{F}$.*

Observe that all these subsets of $\mathcal{A}^{\mathbb{N}}$ are compact and invariant under the shift transformation, i.e., $\sigma(X) = X$. One important example for us is the set $X \subset \{0,1\}^{\mathbb{N}}$ given by the restriction $\mathcal{F} = \{\overbrace{000\ldots0}^{n-times}, \overbrace{111\ldots1}^{n-times}\}$. Another class of examples are the subshifts of finite type. Here the collection of restrictions $\mathcal{F}$ is given by a transition matrix A of zeros and ones in the following way: $i_r i_s \in \mathcal{F}$ iff $A_{i_r i_s} = 0$. We observe that not all shifts of finite type are subshifts of finite type: The shift with restrictions $\mathcal{F} = \{00, 11\}$ has the transition matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, but the shift of finite type associated to the restrictions $\mathcal{F} = \{000, 111\}$ cannot be described as a subshift of finite type. On the other hand, it is true that all shifts of finite type are conjugated to a subshift of finite type perhaps with other alphabet, the following proposition establishes this. It is easy to see that for a shift space of finite type all words in $\mathcal{F}$ can be considered of the same length.

Proposition 5. *Let $X \subset A^{\mathbb{N}}$ be a shift of finite type, then there exists $\Phi : X \to Y$ a conjugation ($\Phi \circ \sigma = \sigma \circ \Phi$ and Φ a homeomorphism), where Y is a subshift of finite type over the alphabet built with the words not in $\mathcal{F}$.*

Proof. Let $N+1$ be the length of the words in $\mathcal{F}$ and $\mathcal{B} = \{i_0 i_1 \ldots i_N : i_0 i_1 \ldots i_N \notin \mathcal{F}\}$. Let Y be the subshift of finite type over $\mathcal{B}$ given by the matrix A with $A_{i_0 i_1 \ldots i_N j_0 j_1 \ldots j_N} = 1$ if $i_1 \ldots i_N = j_0 j_1 \ldots j_{N-1}$, otherwise the value of the matrix entry is zero. Now we consider the map $\Phi : X \to \mathcal{B}^{\mathbb{N}}$ defined as

$$\Phi(\mathbf{i}) = \begin{pmatrix} i_N \\ \vdots \\ i_1 \\ i_0 \end{pmatrix} \begin{pmatrix} i_{N+1} \\ \vdots \\ i_2 \\ i_1 \end{pmatrix} \begin{pmatrix} i_{N+2} \\ \vdots \\ i_3 \\ i_2 \end{pmatrix} \cdots,$$

by construction we have that $\phi(X) = Y$. The inverse map of ϕ is given by

$$\begin{pmatrix} i_N \\ \vdots \\ i_1 \\ i_0 \end{pmatrix} \begin{pmatrix} i_{N+1} \\ \vdots \\ i_2 \\ i_1 \end{pmatrix} \begin{pmatrix} i_{N+2} \\ \vdots \\ i_3 \\ i_2 \end{pmatrix} \cdots \to i_0 i_1 i_2 i_3 \ldots$$

∎

Example 1. In order to illustrate, consider the shift space $X \subset \{0,1\}^{\mathbb{N}}$ given by the restriction $\mathcal{F} = \{000, 111\}$, the alphabet associated is given by $\mathcal{B} = \{001, 010, 011, 100, 101, 110\}$ and the transition matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

In fact, the shift space given by the restriction $\mathcal{F} = \{\overbrace{000...0}^{n-times}, \overbrace{111...1}^{n-times}\}$ has a transition matrix of dimension $2^n - 2$ with the same pattern as that of A above.

In regard to the dynamics of $\sigma|X$ we have that this dynamical system is one-side transitive iff the following property holds: if for every ordered pair of *allowed* blocks u, v in X there exists another one w such that uwv is an allowed one also. This last property is known as irreducibility of $\sigma|X$. When X is a subshift of finite type, this is equivalent to the irreducibility of the matrix A.

Example 2. The shift X given by the restrictions $\mathcal{F} = \{\overbrace{000...0}^{n-times}, \overbrace{111...1}^{n-times}\}$ is irreducible. Since, when

- u ends with a 0 and v begins with a 0, we take $w = 101$.
- u ends with a 0 and v begins with a 1, we take $w = 1010$.
- u ends with a 1 and v begins with a 0, we take $w = 0101$.
- u ends with a 1 and v begins with a 1, we take $w = 010$.

4. Cantor-like Sets

In this section we study the Hausdorff dimension of Cantor-like sets in $\mathbb{R}$ as defined in [P]. So let Λ_0 and Λ_1 two closed intervals and $Q \subset \{0,1\}^{\mathbb{N}}$ a compact set with $\sigma(Q) = Q$, where σ is the shift transformation. The Cantor sets that we are going to consider are given by

$$C = \bigcap_{n=0}^{\infty} \bigcup_{(i_0 i_1 \ldots i_n)} \Lambda_{i_0 i_1 \ldots i_n},$$

where the sets $\Lambda_{i_0 i_1 \ldots i_n}$ are closed intervals and satisfy the following conditions:

(1) The n-tuple $(i_0 i_1 \ldots i_n)$ is Q-admissible: there exists an element $\mathbf{j} \in Q$ such that $j_0 j_1 \ldots j_n = i_0 i_1 \ldots i_n$;
(2) $\Lambda_{i_0 i_1 \ldots i_n j} \subset \Lambda_{i_0 i_1 \ldots i_n}$ for $j = 0, 1$;
(3) $l(\Lambda_{i_0 i_1 \ldots i_n}) \to 0$ as $n \to \infty$, where $l(I)$ denotes the length of the interval I;
(4) $\Lambda_{i_0 i_1 \ldots i_n} \bigcap \Lambda_{j_0 j_1 \ldots j_n} = \emptyset$ for any $(j_0 j_1 \ldots j_n) \neq (i_0 i_1 \ldots i_n)$.

The set C turns out to be perfect, nowhere dense and totally disconnected. We shall say that Q is the symbolic dynamics associated to C. We want to estimate the Hausdorff dimension of C, more concretely to get lower bounds for it. It is a remarkable fact that the Hausdorff dimension of C is determined by the dynamics of Q when the convergence in item (3) above is of the following type: $l(\Lambda_{i_0 i_1 \ldots i_n}) \geq K \lambda_{i_0} \ldots \lambda_{i_n}$ where $0 < \lambda_j < 1$ and $K > 0$. In [P, Theorems 14.1 and 14.3, pag. 135], it is proved the following estimate for the Hausdorff dimension of C.

Proposition 6. *Let C be a Cantor set as above and assume that there exists $0 < \gamma < 1$ such that*

$$\liminf_{n \to \infty} \min\{\frac{1}{n} \log \lambda_{i_0} \ldots \lambda_{i_n}\} \geq \ln \gamma,$$

where the minimum is taken over all admissible n-tuples $i_0 \ldots i_n$. Then for $\epsilon > 0$, we have that $\dim_H F \geq s_{\gamma_\epsilon}$, where s_{γ_ϵ} is the unique zero of the equation

$$P_Q(s\phi) = 0,$$

whit $\phi : Q \to \mathbb{R}$ given by $\phi(\boldsymbol{i}) = \log(\gamma^{1+\epsilon})$ and P_Q is the topological pressure associated to Q.

Now assume that the Cantor set C has an associated symbolic dynamic Q which is a shift of finite type. Then we have

Proposition 7. *s_{γ_ϵ} is the unique zero of the equation*

$$\rho(AM_t) = 1,$$

where A is the matrix associated to the subshift of finite type $\Phi(Q)$, M_t is the matrix $diag(\gamma^{1+\epsilon}, \dots, \gamma^{1+\epsilon})$ and $\rho(B)$ denotes the spectral radius of the matrix B.

Proof. By Proposition 5, Q is conjugated by a mapping Φ to a subshift of finite type with a transition matrix A over an alphabet $\mathcal{B}$. The topological pressures P_Q and $P_{\Phi(Q)}$ are related as follows $P_Q(\psi) = P_{\Phi(Q)}(\psi \circ \Phi)$. For $\phi(\mathbf{i}) = \log(\gamma^{1+\epsilon})$ it corresponds the function $\widetilde{\phi} : \mathcal{B}^{\mathbb{N}} \to \mathbb{R}$

$$\widetilde{\phi}\left(\begin{pmatrix} i_N \\ \vdots \\ i_1 \\ i_0 \end{pmatrix} \begin{pmatrix} i_{N+1} \\ \vdots \\ i_2 \\ i_1 \end{pmatrix} \begin{pmatrix} i_{N+2} \\ \vdots \\ i_3 \\ i_2 \end{pmatrix} \cdots \right) = \log(\gamma^{1+\epsilon}),$$

that is $\phi \circ \Phi = \widetilde{\phi}$. By Proposition 6, we get that s satisfies

$$0 = P_Q(s\phi) = P_{\Phi(Q)}(s\widetilde{\phi}).$$

By Theorem A2.8 in [P, pag 108] we finally get with M_t that $P_{\Phi(Q)}(\psi \circ \Phi)$ equals $\log(\rho(AM_t))$ as we wanted to show. ∎

In order to get a lower bound for s, we need lower bounds for the spectral radius of a matrix. Consider A a nonnegative matrix as above of order r and let u and v be the positive Perron-Frobenius eigenvectors of A and A^t respectively, associated to the eigenvalue $\rho(A)$ (Perron-Frobenius Theorem). The following lower bound is proved in [FK].

Proposition 8. *If D is a diagonal matrix and A is an irreducible matrix, then*

$$\rho(AD) \geq (\Pi_1^r d_i^{u_i v_i})\, \rho(A).$$

From this Proposition we get then

Theorem 9. *For a Cantor-like set as above we have*

$$\dim_H C \geq \ln \rho(A) / \ln \left(\frac{1}{\gamma^a} \right),$$

where $a = \sum_i u_i\, v_i$ with u and v the Perron-Frobenius eigenvectors associated to the matrix A.

Proof. This follows immediately from Proposition 7 and 8 since for every $\epsilon > 0$ we get

$$\dim_H C \geq \ln \rho(A) / \ln \left(\frac{1}{\gamma^{(1+\epsilon)a}} \right).$$

The conclusion is obtained taking the limit when $\epsilon \to 0$.

∎

Remark 4. Similar lower bounds can be obtained for Cantor-like sets formed beginning with p disjoint sets $\{\Lambda_i\}_0^{p-1}$ instead of the two sets Λ_0 and Λ_1.

Remark 5. This lower bounds is non-trivial only if $\rho(A) > 1$. This fact can be assured in the following context. Let $s = \min_j\{\sum_j a_{i\,j}\}$ and $S = \max_j\{\sum_j a_{i\,j}\}$, if the matrix A is a $(0,1)$-matrix with at least one 1 in each row, is an irreducible one and $s \neq S$ then $\rho(A) > s \geq 1$.

Example 3. Consider a Cantor C with Q the Golden Mean shift, i.e., a subshift of finite type with transition matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $AM_t = \begin{pmatrix} 0 & \lambda_1^t \\ \lambda_0^t & \lambda_1^t \end{pmatrix}$. It is easy to see that the equation $\rho(AM_t) = 1$ is equivalent to the equation

$$\lambda_1^t + (\lambda_0\,\lambda_1)^t = 1.$$

Also we have

$$\begin{aligned} \rho(A) &= \frac{1}{2}(1+\sqrt{5}), \\ u = v &= (\frac{1}{2}(-1+\sqrt{5}), 1), \\ \alpha &= \frac{1}{4}(-1+\sqrt{5})^2 + 1 = \frac{5-\sqrt{5}}{2} \approx 1.38197. \end{aligned}$$

So letting $\lambda = \min\{\lambda_0, \lambda_1\}$, we get that

$$\dim_H C \geq \log(\frac{1}{2}(1+\sqrt{5}))/\log\left(\frac{1}{\lambda^{1.38197}}\right)$$

5. Transversal Cantor like-sets

Let $F : V \setminus D \to V$ the first return map of the Geometric Lorenz flow. The attractor Λ_F is locally a product of a Cantor set by an interval at points different from points in the unstable manifold of the singularity. We are going to describe this Cantor set as a Cantor-like set in the homoclinic case, i.e., for some $n \geq 1$ we have that $F^n(P_i) \in D$.

We are going to consider two cases: the *monotone* and *non-monotone* cases.

Monotone Case: Assume that $f^r(\pm 1) = 0$ for $r \geq 2$ and the finite sequence $\{f^i(\pm 1)\}_0^{r-1}$ is monotone. Let $\{Q, F(Q)\}$ be the unique two- periodic orbit of F with $Q \in V_1$ and consider the segments $L^{+(-)}$ given by $\{(x, y) : x = q^{+(-)},\, y > 0 (< 0)\}$ where $Q = (q^+, r)$ and $F(Q) = (q^-, -r)$. We remark that $q^{\pm} \notin \bigcup_{n\geq 0} f^{-n}(0)$.

Proposition 10. *For $L = L^+ \cup L^-$, we have that $\Lambda_F \bigcap L$ is a Cantor-like set with a symbolic dynamics given by the shift of finite type $Q \subset \{0,1\}^{\mathbb{N}}$ with restriction* $\mathcal{F} = \{\overbrace{00\ldots0}^{r+1}, \overbrace{11\ldots1}^{r+1}\}$.

Proof. In order to describe $\Lambda_F \bigcap L$ we need to describe the successive intersections $E_i = F(V \setminus D_1) \cap \cdots \cap F^i(V \setminus D_i)$. In order to do that, we are going to show how the connected components of E_i are codified. We denote the set $F(V_0)$ by $\mathcal{I}_0$ and $F(V_1)$ by $\mathcal{I}_1$ as it is shown in the left picture of Figure 3 below, so $E_1 = \mathcal{I}_0 \cup \mathcal{I}_1$. For E_2, we proceed as follows. We denote $\mathcal{I}_{ij} = F(\mathcal{I}_j \cap V_i)$, then it is easy to check that $E_2 = \cup_{i,j=0}^{1} \mathcal{I}_{ij}$. The right picture of Figure 3 illustrate these sets.

Now let $i_0 i_1 \ldots i_s$ be a finite piece of a sequence of zeros and ones. We let $\mathcal{I}_{i_0 i_1 \ldots i_s} = F(\mathcal{I}_{i_1 \ldots i_s} \cap V_{i_0})$, when $\mathcal{I}_{i_1 \ldots i_s} \cap V_{i_0} \neq \emptyset$.

Now we will show that the admissible sequence $i_0 i_1 i_2 \ldots$ are those which no admit blocks $i_l \ldots i_{l+r+1}$ of the form either $\overbrace{00\ldots0}^{r+1}$ or $\overbrace{11\ldots1}^{r+1}$. In order to do that, we observe that for $i \leq r$, E_i is formed by $2^i - 2$ lobes, those included in $\{y > 0\}$ with the right end at P_2 and the left end at $\{F(P_1), \ldots, F^{i-1}(P_1)\}$ and those included in $\{y < 0\}$ with the left end at P_1 and the right end at $\{F(P_2), \ldots, F^{i-1}(P_2)\}$. Also

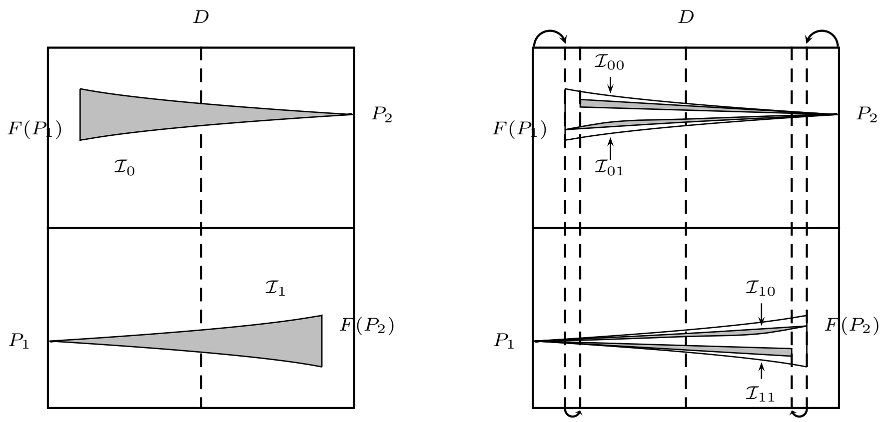

FIGURE 3

we have two wedge with the right end at P_2 and the left end at P_1 respectively, with the side on the segment given by $x = f^i(p_{11})$ and $x = f^i(p_{21})$. This situation is shown in the Figure 4 below. At each step in the construction, each lobe is substituted by two lobes with the same ends and each wedge is substituted by a new wedge and one lobe with right end at P_i and the left end of the lobe at the side of the previous wedge. The left side of the new wedge is contained in the image of the side of the previous wedge.

For $i \geq r+1$ the pattern varies since we loose the images of the wedges, since they are totally contained in V_0 or V_1. Until here the finite sequence $i_0 i_1 \ldots i_s$ with $s \leq r$ have no restriction. At this point the sequences $\overbrace{00\ldots0}^{r+1}$ and $\overbrace{11\ldots1}^{r+1}$ are missing since to have $\mathcal{I}_{\underbrace{00\ldots0}_{r+1}}$ we would have the wedge $\mathcal{I}_{\underbrace{00\ldots0}_{r}}$ meeting V_0 but this is no the case. The same happens with the other wedge corresponding to $\mathcal{I}_{\underbrace{11\ldots1}_{r}}$. Now the process goes on, the wedges are substituted by one lobes (no more wedges from now on) and the new sets $\mathcal{I}_{i_0 i_1 \ldots i_s}$ avoid sequence having the blocks $\{\overbrace{00\ldots0}^{r+1}, \overbrace{11\ldots1}^{r+1}\}$. The reason to avoid these blocks is that

$$
\begin{aligned}
\mathcal{I}_{\underbrace{00\ldots0}_{r+1} i_{r+2}\ldots i_s} &= F(\mathcal{I}_{\underbrace{00\ldots0}_{r} i_{r+2}\ldots i_s} \cap V_0) \\
&= F(\mathcal{I}_{\underbrace{00\ldots0}_{r-1} i_{r+2}\ldots i_s} \cap V_0) \\
&\;\vdots \\
&= F(\mathcal{I}_{i_{r+2}\ldots i_s} \cap V_0)
\end{aligned}
$$

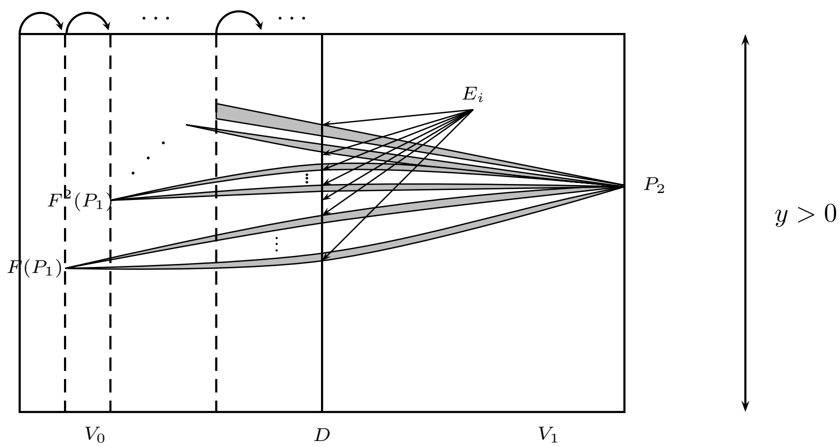

FIGURE 4

and this would imply that after r iterated the image of $\mathcal{I}_{i_{r+2}\dots i_s}$ still meets V_0 which is an absurd. With a similar argument the case corresponding to the block $\overbrace{11\dots1}^{r+1}$ is proved.

Now let $I_{i_0 i_1 i_2 \dots i_s} = \mathcal{I}_{i_0 i_1 i_2 \dots i_s} \bigcap L$, then

$$\Lambda_F \cap L = \bigcap_0^\infty \bigcup_{(i_0 i_1 \dots i_n)} \Lambda_{i_0 i_1 \dots i_n}$$

with $i_0 i_1 \dots i_n$ admissible, i.e, no admitting the blocks $\{\overbrace{00\dots0}^{r+1}, \overbrace{11\dots1}^{r+1}\}$.

Now we are going to estimate the length of the intervals $I_{i_0 i_1 \dots i_s}$. Let $\underline{\lambda} = K_1|q^+|^\alpha$ and $\overline{\lambda} = K_2|q^+|^\alpha$. Since $\mathcal{I}_{i_0 i_1 \dots i_s} = F(\mathcal{I}_{i_1 \dots i_s} \cap V_{i_0})$ we obtain that

$$2\underline{\lambda}^{s+1} \leq l(\mathcal{I}_{i_0 i_1 \dots i_s}) \leq 2\overline{\lambda}^{s+1}, \tag{1}$$

so $\lim_{s\to\infty} l(\mathcal{I}_{i_0 i_1 \dots i_s}) = 0$.

∎

Non-monotone Case: Firstly, as in the monotone case we show how to codify the connected components of E_i. Until $f^i(-1) \leq 0$ the codification is the same as in the monotone case. After the first time where $f^{j_0}(-1) > 0$ we loose the blocks $\overbrace{00\dots0}^{j_0+1}$ and $\overbrace{11\dots1}^{j_0+1}$. Now, from here until $j = r+1$, we loose those blocks $j_0 j_1 \dots j_s$ with $s \leq r+1$ for which $\mathcal{I}_{j_1 \dots j_s}$ is a subset of $V \setminus V_{j_0}$. Finally, when we arrive at $r+1$ we loose a finite set of blocks of length $s \leq r+1$. So, as in the monotone case, we have a symbolic dynamics Q which is a subshift of finite type where the set of forbidden blocks $\mathcal{F}$ include the blocks $\{\overbrace{00\dots0}^{j_0+1}, \overbrace{11\dots1}^{j_0+1}\}$ where j_0 is the first time when $f^j(-1) > 0$.

Now, we introduce several vertical segments in order to capture the transversal behavior of Λ_F. We proceed in the following way. Since the periodic points are dense , we choose one of them of period s and denote it by M. This point has to be near enough to P_1, and such that its orbit shadows that of P_1, during its existence. Let $m_i = \pi(F^i(M))$ where $\pi(M)$ is the first coordinate of M.For $i \in [0, r]$ we choose vertical segments L_{m_i} such that the size of these segments is enough to catch those components of E_j that begin at $F^i(P_1)$. More concretely, $L_{m_i} = F^i(m_{s-1} \times [-1, 1])$.

Proposition 11. *We have that $\Lambda_F \bigcap L_{m_1} \bigcap \cdots \bigcap L_{m_r}$ is a Cantor-like set with a symbolic dynamics given by the shift of finite type $Q \subset \{0,1\}^{\mathbb{N}}$ with restriction $\mathcal{F}$ which include the blocks $\{\overbrace{00\dots0}^{j_0+1}, \overbrace{11\dots1}^{j_0+1}\}$, where j_0 is the first time when $f^{j_0}(-1) >$ 0.*

Proof. For the estimative of the length of the intervals $I_{i_0 i_1 \dots i_s}$, let $\underline{\lambda} = K_1 \min_i\{|m_i|^\alpha\}$ and $\overline{\lambda} = K_2 \max_i\{|m_i|^\alpha\}$. Since $\mathcal{I}_{i_0 i_1 \dots i_s} = F(\mathcal{I}_{i_1 \dots i_s} \cap V_{i_0})$ we obtain, as in the monotone case, that

$$2\underline{\lambda}^{s+1} \leq l(\mathcal{I}_{i_0 i_1 \dots i_s}) \leq 2\overline{\lambda}^{s+1}, \tag{2}$$

so $\lim_{s\to\infty} l(\mathcal{I}_{i_0 i_1 \dots i_s}) = 0$. ∎

Remark 6. We mention, since we need to know it in the next section, that the symbolic set Q associated to the transversal Cantor sets above is one-side transitive. That is so, since the whole dynamics in the geometric Lorenz attractors is itself transitive.

One could ask if the monotone case is the only possible, the following example shows that this is not so.

Example 4. Consider the following configuration for the orbit of ± 1:

$f^3(-1)$

-1 $f(-1)$ d_- $f^2(-1)$ 0 $f^2(1)$ d_+ $f(1)$ 1

Here $f(d_\pm) = 0$. In this configuration we have that the following blocks are missing: $\{0000, 1111, 01000, 10111\}$.

6. Proof of Theorem 1

Proof. It is well known that at points different from P_i and its finite orbit $O(P_i)$, the set Λ_F is locally the product of an interval by a Cantor set. More concretely it is proved in [ABS1] that for any point in $\Lambda_F \setminus (O(P_1) \cup O(P_2))$ its unstable manifold goes from one point of $O(P_1)$ to other point of $O(P_2)$. Since these sets are finite the size of the unstable manifolds above are bounded below away from zero. Also the tangent space of these manifold varies smoothly in $\bigcup_i (\Lambda_F \bigcap L_{m_i})$, since the points there are far away from D. So we can build a Lipchitz homeomorphism between a neighborhood $\mathcal{U}_i \subset \Lambda_F$ of $\Lambda_F \bigcap L_{m_i}$ and $(\Lambda_F \bigcap L_{m_i}) \times (m_i - \epsilon, m_i + \epsilon)$, so

$$dim_H(\mathcal{U}_i) \geq dim_H((\Lambda_F \cap L_{m_i}) \times (m_i - \epsilon, m_i + \epsilon)) \geq 1 + dim_H(\Lambda_F \cap L_{m_i}).$$

Applying now Theorem (9) to the sets $\Lambda_F \cap L_{m_i}$ we get

$$dim_H(\Lambda_F) \geq \max\{dim_H(\mathcal{U}_i)\} \geq 1 + \log \rho(A)/\log\left(\frac{1}{\gamma^a}\right)$$

∎

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