Entropy-expansiveness and domination

M. J. Pacifico, J. L. Vieitez

May, 5, 2006

Abstract

Let $f : M \to M$ be a C^r-diffeomorphism, $r \geq 1$, defined in a compact boundary-less surface M. We prove that if K is a compact f-invariant subset of M with a dominated splitting then f/K is h-expansive. Reciprocally, if there exists a C^r neighborhood of f, $\mathcal{U}$, such that for $g \in \mathcal{U}$ there exists K_g compact invariant such that g/K_g is h-expansive then there is a dominated splitting for K_g.

2000 Mathematics Subject Classification: 37D30

1 Introduction

To obtain results about the complexity of the dynamics of a discrete or continuous time dynamical system as recurrence, existence of periodic orbits, SRB measures, etc., one usually try to express dynamic properties at the infinitesimal level, i.e.: precise definitions are given prescribing the behavior of the tangent map $Df : TM \to TM$ of a diffeomorphism $f : M \to M$. Examples of that are the concepts of hyperbolicity, partial hyperbolicity and the existence of a dominated splitting. On the other hand a robust dynamic property (i.e. a property that holds for a system and all nearby ones) should leave its *impromptus* in the behavior of the tangent map of those differentiable systems sharing that property. In [PPV], [SV] and [PPSV] it has been studied the influence of expansiveness when it holds in a homoclinic class H associated to a hyperbolic periodic point p such that H and the corresponding homoclinic classes H_g , for all diffeomorphism g nearby f, are expansive. It is proved there that in that case Df/H has a dominated splitting and moreover f/H is hyperbolic in the codimension one case ([PPV], [PPSV]). In the general codimension case we also obtain hyperbolicity adding an extra hypothesis called germ-expansiveness (see [SV]).
In this paper we relax expansiveness asking what should be the properties of the tangent map Df of a diffeomorphism f defined on a surface such that robustly

exhibits h-*expansiveness* (entropy-expansiveness, see definitions below). We obtain that for such maps it exists a dominated splitting. On the other hand we prove that if K admits a dominated splitting then it is h-expansive. Thus robust h-expansiveness is equivalent to the existence of a dominated spitting.

Moreover, we give here an example of a C^∞ diffeomorphism that is not h-expansive. By a result of Buzzi (see [Bu]) such an example is asymptotically h-expansive (see definition below) since it is C^∞. The first examples of a diffeomorphism that is not h-expansive and even not asymptotically h-expansive was given by Misiurewicz in [Mi] answering a question posed by Bowen. We give our example here because of its good properties from various points of view. First it is clear that it has not a dominated splitting. Second it is defined on S^2, is ergodic and even has Bernoulli property. Third it admits analytic models a stronger property than being C^∞.

Let us now give precise definitions. Let M be a compact connected boundaryless Riemannian d-dimensional manifold and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and $\mathrm{dist} : M \times M \to \mathbb{R}^+$ a distance in M compatible with its Riemannian structure. For $E, F \subset K$, $n \in \mathbb{N}$ and $\delta > 0$ we say that E (n, δ) spans F with respect to f if for each $y \in F$ there is $x \in E$ such that $\mathrm{dist}(f^j(x), f^j(y)) \leq \delta$ for all $j = 0, \ldots, n-1$. Let $r_n(\delta, F)$ denote the minimum cardinality of a set that (n, δ) spans F. Since K is compact $r_n(\delta, F) < \infty$. We define

$$h(f, F, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log(r_n(\delta, F))$$

and

$$h(f, F) = \lim_{\delta\to 0} h(f, F, \delta)\,.$$

The last limit exists since $h(f, F, \delta)$ increases as δ decreases to zero.

For $x \in K$ let us define

$$\Gamma_\epsilon(x, f) = \Gamma_\epsilon(x) = \{y \in M \,/\, d(f^n(x), f^n(y)) \leq \epsilon,\ n \in \mathbb{Z}\}\,.$$

Following Bowen (see [Bo]) we say that f/K is **entropy-expansive** or ***h*-expansive** if and only if there exists $\epsilon > 0$ such that

$$h^*_f(\epsilon) = \sup_{x\in K} h(f, \Gamma_\epsilon(x)) \;=\; 0\,.$$

The importance of f being h-expansive is that the topological entropy of f restricted to K, $h(f/K)$, is equal to its estimate using ϵ: $h(f, K) = h(f, K, \epsilon)$. More precisely:

Theorem 1.1. *For all homeomorphism f defined in a compact invariant set K it holds*

$$h(f,K) \leq h(f,K,\epsilon) + h^*_f(\epsilon) \text{ in particular } h(f,K) = h(f,K,\epsilon) \text{ if } h^*_f(\epsilon) = 0\,.$$

Proof. See [Bo], Theorem 2.4. □

A weaker property of that of being h-expansive is that of being **asymptotically h-expansive** ([Mi]). Let K be a compact metric space and $f : K \to K$ an homeomorphism. We say that f is asymptotically h-expansive if and only if

$$\lim_{\epsilon \to 0} h^*_f(\epsilon) = 0\,.$$

Thus we do not require that for a certain $\epsilon > 0$ $h^*_f(\epsilon) = 0$ but that $h^*_f(\epsilon) \to 0$ when $\epsilon \to 0$. It has been proved by Buzzi that any C^∞ diffeomorphism defined on a compact manifold is asymptotically h-expansive. Hence our example although not h-expansive is asymptotically h-expansive.

Definition 1.1. *We say that a compact f-invariant set Λ admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that*

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \;\forall x \in \Lambda,\, n \geq 0. \tag{1}$$

Our main results are the following:

Theorem A. *Let M be a compact boundaryless C^∞ surface and $f : M \to M$ be a C^r diffeomorphism such that $K \subset M$ is a compact f-invariant subset with a dominated splitting $E \oplus F$. Then f/K is h-expansive.*

Since the property of having a dominated splitting is open we may conclude that any g C^1 close to f is such that g/K_g is h-expansive.

In case M is a d-dimensional manifold with $d \geq 3$ the existence of a dominated splitting is not enough to guarantee h-expansiveness as it is shown in the examples presented below.

Observe that the identity map $id : M \to M$ is h-expansive and moreover if the topological entropy of a map $f : M \to M$ vanishes, $h(f) = 0$, then it is h-expansive. Nevertheless, the persistence of h-expansiveness has a dynamical meaning.

Theorem B. *Let M be a compact boundaryless C^∞ surface and $f : M \to M$ be a C^r diffeomorphism. Let $H(p)$ be an f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that there is a continuation $H(p_g)$ of $H(p)$ such that $H(p_g)$ is h-expansive. Then $H(p)$ has a dominated splitting.*

2 Examples

Let us now give an example of an analytic diffeomorphism that is not h-expansive. We consider in $\mathbb{R}^2$ the action given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Since the entries of A are integers and $\det(A) = 1$, the lattice $\mathbb{Z}^2$ is preserved by this action and therefore it passes to the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This gives us a very well known linear Anosov diffeomorphism $a : \mathbb{T}^2 \to \mathbb{T}^2$. Let $[x, y]$ represent the equivalence class of $(x, y) \in \mathbb{R}^2$ in $\mathbb{R}^2/\mathbb{Z}^2$. We define in $\mathbb{R}^2/\mathbb{Z}^2$ the relation $[x, y] \sim [-x, -y] = -[x, y]$. The quotient $\mathbb{T}^2/\sim$ gives the sphere S^2. In order to see this let us take the square in $\mathbb{R}^2$ limited by the straight lines $x = -\frac{1}{2}$, $x = \frac{1}{2}$, $y = -\frac{1}{2}$, $y = \frac{1}{2}$. We obtain a fundamental domain for the torus and we identify it with $\mathbb{T}^2$. In the quotient $\mathbb{T}^2$ the vertices A $(1/2, 1/2)$,B $(-1/2, 1/2)$,C $(-1/2, -1/2)$, D$(1/2, -1/2)$, of the square are all identified. Let us call E to the point $(1/2, 0)$, F to the point $(-1/2, 0)$, G to the point $(0, 1/2)$ and H to the point $(0, -1/2)$. Observe that E is identified with F and G is identified with H in $\mathbb{T}^2$. Now observe that the boundary of the square OEAG is identified with the boundary of the square OEDH (by the relations $(x, y) \sim -(x, y)$ and $(x, y) \sim (x', y')$ if $(x - x', y - y') \in \mathbb{Z}^2$). Hence both squares are two different disks glued in their boundaries by this identification. This gives a sphere. Moreover, the rest of the square ABCD doesn't give more points to the quotient because the squares OEAG and OFCH, and OEDH and OFBG, are identified by the relation $(x, y) \sim -(x, y)$. On the other hand $a([x, y]) \sim -a([x, y]) = a(-[x, y])$ by linearity, and therefore projects to S^2 as a map $g : S^2 \to S^2$, known as a generalized pseudo-Anosov map. If $\Pi : \mathbb{T}^2 \to \S^2$ is the projection defined by the relation $\sim$, we may write $g(x) = \Pi(a(\Pi^{-1}(x)))$. Observe that the projection $\Pi : \mathbb{T}^2 \to S^2$ is a branched covering and that the definition of g doesn't depend on the pre-image of x by Π^{-1}. Therefore periodic points of a projects in periodic points of g and dense orbits of a projects in dense orbits of g. For g there are singular points P where the local ϵ-stable and ϵ-unstable sets are arcs with the point P as an end-point. This local stable (unstable) sets are called 1-prongs (see figure 1 where O is a point with 1-prongs).

Let $O \in S^2$ be the image by Π of $[0, 0]$. Then O is (the unique) fixed point of g. The point O is singular because the unstable manifold of $[0, 0]$ in $\mathbb{T}^2$ projects to S^2 as an arc ending at O (because $[x, y] \sim -[x, y]$). The stable and unstable manifolds of the points in $\mathbb{T}^2$ near $(0, 0)$ projects to points in S^2 near O like in Figure 1. The intersection of the stable and unstable manifolds of the points $(0, x)$ and $(0, -x)$ consists of four points identified by pairs by the relation $[x, y] \sim -[x, y]$. If $[x, y] \in \mathbb{T}^2$ projects to $X \in S^2$, let us call s_X and u_X to the projections of the ϵ-local stable and ϵ-local unstable manifolds respectively of the point $[x, y]$. Hence if a point X is very near to a singular point like O its local stable and unstable sets, s_X and u_x, will intersect twice. Points in s_X are in the ϵ-local stable set of X and points in u_X are in the ϵ-local unstable set of X. Moreover, if $Y \in s_X$ then $\text{dist}(g^n(Y), g^n(X)) \to 0$ when $n \to +\infty$. Similarly for points in u_X replacing $n \to +\infty$ by $n \to -\infty$.

Let us choose the singular point O and given $\epsilon' > 0$ choose $P \neq O$ a periodic point satisfying $\text{dist}(P, 0) < \epsilon'$. Such a point exists since periodic points are dense for the Anosov diffeomorphism a defined on $\mathbb{T}^2$ and projects on S^2 as periodic points for g. Let $\{P, P'\} = s_P \cap u_P$. Then it is not difficult to see that given $\epsilon > 0$ there is $\epsilon' > 0$ small enough such that $P' \in W^u_\epsilon(P) \cap W^s_\epsilon(P)$. Thus we have a homoclinic intersection between ϵ-local stable and ϵ-local unstable arcs of the periodic point P, P' being a homoclinic point such that its orbit is always at a distance less than ϵ from the orbit of P. It follows that for all $\epsilon > 0$ there are points P such that $\Gamma_\epsilon(P)$ contains a small horseshoe. Thus $g : S^2 \to S^2$ is not h-expansive. Moreover, this example is transitive and there are real analytic models for $g : S^2 \to S^2$ (see [Ge], and [LL]).

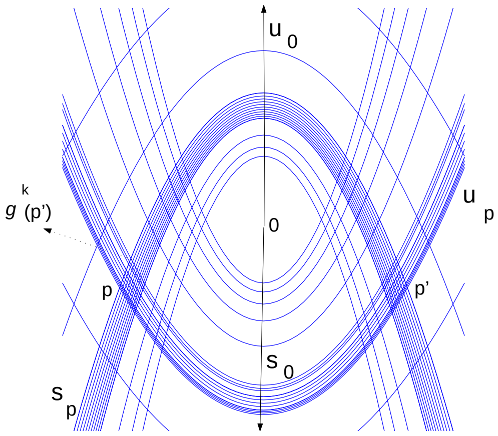

Figure 1: Generalized pseudo-Anosov

Clearly the example is a homoclinic class which has no dominated splitting.

Let us show that property (1) sole does not imply h-expansiveness in dimension 3 or more. Consider the 3-manifold $S^2 \times S^1$ with $g : S^2 \to S^2$ as in the example above, and put in S^1 a diffeomorphism $h : S^1 \to S^1$ with a North-South dynamics, say, $N \in S^1$ is a source and $S \in S^1$ is a sink and the ω-limit of any point in S^1 is S and the α-limit of every point in S^1 is N. We may assume that $|Dh_N| > 2k$ where $k = \sup\{\|Dg(x)\|, x \in S^2\}$. Let us define $f : S^2 \times S^1 \to S^2 \times S^1$ by

$f(x,y) = (g(x), h(y))$. Then if $K = S^2 \times \{N\}$, K is compact invariant and there is a dominated splitting for K, $E \oplus F$, where $E = T_x S^2$, $F = T_N S^1$. By the previous example f is not h-expansive.

This example shows what is the problem; the strongly expanding direction F along S^1 does not interferes on the dynamics of f/S^2. Thus property (1) holds for f defined on $S^2 \times S^1$ albeit does not for $g = f/S^2$.

3 Proof of Theorem A

Here we shall prove

Theorem 3.1. *Let M be a closed smooth surface and $f : M \to M$ be a C^r diffeomorphism such that $K \subset M$ is a compact f-invariant subset with a dominated splitting $E \oplus F$. Then f/K is h-expansive.*

We need the following lemma.

Lemma 3.2 (Pliss). *Let $0 < \lambda_1 < \lambda_2 < 1$ and assume that there exists $n > 0$ arbitrarily large such that*

$$\prod_{j=1}^{n} \|Df/E(f^j(x))\| \leq \lambda_1^n \,.$$

Then there exist a positive integer $N = N(\lambda_1, \lambda_2, f)$, $c = c(\lambda_1, \lambda_2, f) > 0$ such that if $n \geq N$ then there exist numbers

$$0 \leq n_1 \leq n_2 \leq \cdots \leq n_l \leq n$$

such that

$$\prod_{j=n_r}^{h} \|Df/E(f^j(x))\| \leq \lambda_2^{h-n_r} \,,$$

for all $r = 1, 2, \ldots, l$, with $l \geq cn$, and for all h with $n_r \leq h \leq n$.

Proof. The proof of this lemma can be found in [Pl1]. □

Proof of Theorem A. Let M be a surface and $K \subset M$ a compact and f invariant subset such that there is a dominated splitting $E \oplus F$ defined on it. By continuity of f and Df there is $\delta_0 > 0$ such that we may extend the cones defining equation (1) to the closed δ_0 neighborhood of K, $U(K) = \{y \in M \,/\, \mathrm{dist}(y, K) \leq \delta_0\}$. If the orbit of a point y, $orb(y)$, is contained in $U(K)$ then for that point there are defined local center-stable and center-unstable manifolds $W^{cs}_{loc}(y)$ and $W^{cu}_{loc}(y)$

where $loc > 0$ stands for a small real number. Moreover, there is δ_1, $0 < \delta_1 \le \delta_0$ such that if $\mathrm{dist}(f^j(y), f^j(z)) \le \delta_1$ for all $j = 0, \ldots, n$ and $z \in W^{cs}_{loc}(y)$ then $f^j(z) \in W^{cs}_{loc}(f^j(y))$ for all $j = 0, \ldots, n$. Similarly for the local center unstable manifold (see [PS1, Lemma 3.0.4 and Corollary 3.2]).

We need the following lemma:

Lemma 3.3. *There is δ_2, $0 < \delta_2 \le \delta_1$ such that if the length of the arc $[y,z]^{cs} \subset W^{cs}_{loc}(y)$ is greater than $\delta > 0$ for $0 < \delta \le \delta_2$, $\ell([y,z]^{cs}) > \delta$, then $\mathrm{dist}(y,z) > \delta/2$. Moreover, there is a constant $L > 0$ such that if $\mathrm{dist}(y,z) \le \delta$ then $\ell([y,z]^{cs}) \le L$. Similarly for an arc $[y,z]^{cu} \subset W^{cu}_{loc}(y)$.*

Proof. Since $E(y)$, $E(z)$ are continuous sub-bundles in $U(K)$ we may find δ_2, $0 < \delta_2 \le \delta_1$ such that given $\eta > 0$ $\angle(E(y), E(w)) < \eta$ for all $w \in B(y, \delta_2) \cap U(K)$ (the number δ_0 can be chosen so small that $B(y, \delta_0)$ is contained in a local chart, so that we may assume locally that we are in $\mathbb{R}^2$). Thus if we parameterize $[y,z]$ by arc-length $\beta : [0,l] \to M$, with $\beta(0) = y$, $\beta(l) = z$, then $\beta'(s) = (\beta_1'(s), \beta_2'(s))$ is parallel to $E(\beta(s))$. Therefore, since $(\beta_1'(s))^2 + (\beta_2'(s))^2 = 1$, we have by the Mean Value Theorem

$$\mathrm{dist}(y,z) = \|\beta(l) - \beta(0)\| =$$

$$= \sqrt{(\beta_1(l) - \beta_1(0))^2 + (\beta_2(l) - \beta_2(0))^2} = \sqrt{((\beta_1'(s_1))^2 + (\beta_2'(s_2))^2} \cdot l =$$

$$= l\left(1 - (\sqrt{((\beta_1'(0))^2 + (\beta_2'(0))^2} - \sqrt{((\beta_1'(s_1))^2 + (\beta_2'(s_2))^2)}\right) =$$

$$= l\left(1 - \frac{(\beta_1'(0))^2 - (\beta_1'(s_1))^2 + (\beta_2'(0))^2 - (\beta_2'(s_2))^2}{1 + \sqrt{((\beta_1'(s_1))^2 + (\beta_2'(s_2))^2)}}\right) \ge$$

$$\ge l\left(1 - |\beta_1'(0) - \beta_1'(s_1)|(\beta_1'(0) + \beta_1'(s_1)) + |\beta_2'(0) - \beta_2'(s_2)|(\beta_2'(0) + \beta_2'(s_2))\right) .$$

But, since $\angle(E(\beta(s)), E(\beta(0))) < \eta$,

$$\|(\beta_1'(s) - \beta_1'(0), \beta_2'(s) - \beta_2'(0))\| \le 2\sin(\eta/2) < \eta, \quad \text{for small } \eta\,.$$

Therefore, taking into account that $\beta_1'(0) + \beta_1'(s_1) \le |\beta_1'(0)| + |\beta_1'(s_1)| \le 2$ and that the same is true with respect to β_2' we have

$$\mathrm{dist}(y,z) \ge l(1 - 4\eta) > l/2$$

if $\eta > 0$ is sufficiently small. The proof that if $\mathrm{dist}(y,z) \le \delta$ then $\ell([y,z]^{cs}) \le L$ is similar. □

Continuing with the proof of Theorem A we observe that taking an iterate f^m of f we may assume that the constant $C > 0$ appearing in the definition of the dominated splitting, equation (1), is one. Since for a compact invariant set X we have that the topological entropy $h(f^m/X) = m \cdot h(f/X)$, if we prove that for some $\epsilon > 0$, $h(f^m/\Gamma_\epsilon(x, f)) = 0$ then the same is true for f. Thus we assume that for f itself $C = 1$.
Let $\lambda_1 = \sqrt[3]{\lambda} < \lambda_2 = \sqrt[4]{\lambda} < \lambda_3 = \sqrt[5]{\lambda} < 1$. If it were necessary we take δ_3, $0 < \delta_3 \leq \delta_2$ such that if $\text{dist}(z, w) \leq \delta_3$ then

$$1 - c < \frac{\|Df/E(z)\|}{\|Df/E(w)\|} < 1 + c \quad \text{and} \quad 1 - c < \frac{\|Df^{-1}/F(z)\|}{\|Df^{-1}/F(w)\|} < 1 + c\,,$$

where $c > 0$ is such that $(1 + c)\lambda_2 \leq \lambda_3$.

We recall that when a dominated splitting $E \oplus F$ is defined in a compact set like $U(K)$ we may find $\gamma > 0$ such that for all $y \in U(K)$ it holds that the angle between $E(y)$ and $F(y)$ is greater than γ, $\angle(E(y), F(y)) > \gamma$. Let us pick a point $x \in U(K)$ and, identifying $I\!R^2$ with a coordinate neighborhood around x, let $l_E(x)$ be the straight line for x with the direction of $E(x)$ and $l_F(x)$ the straight line with the direction of $F(x)$. From a point $y_0 \in l_F(x)$, $y_0 \neq x$, we consider the straight line $y_0 + l_E(x)$ parallel to $E(x)$. Then for any point y in $y_0 + l_E(x)$ we have that the distance between y and x is greater than the distance between y_0 and x multiplied by $\sin\gamma$, $\text{dist}(y, x) \geq \text{dist}(y_0, x) \sin\gamma$, (see figure 2).

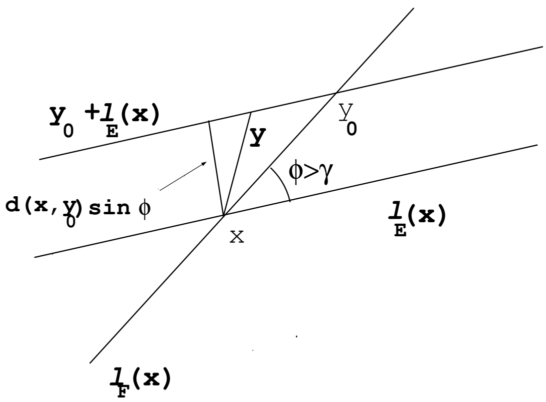

Figure 2: Bounds for the distance between x and $y \in y_0 + l_E(x)$

Since the local center unstable manifold is tangent to F and the local center stable manifold is tangent to E we may assume that δ_3 is so small that

$$\mathrm{dist}(y,x) \geq \mathrm{dist}(y_0,x)(\frac{\sin\gamma}{2+\sin\gamma}) \tag{2}$$

for $y_0 \in W^{cu}_{loc}(x) \cap B(x,\delta_3)$, $y \in W^{cs}_{loc}(y_0) \cap B(x,\delta_3)$.

Let now $\epsilon > 0$ be such that

$$\epsilon < \frac{\delta_3}{(1+2\sin\gamma)}\,. \tag{3}$$

We will prove that for all $x \in K$, $h(f/\Gamma_\epsilon(x)) = 0$. This will prove that f/K is entropy-expansive.

Let us assume first that $y \in W^{cu}_{loc}(x) \cap \Gamma_\epsilon(x)$, $y \neq x$. Then $orb(y) \subset U(K)$ and therefore for all $j \in \mathbb{Z}$ it holds that

$$\|Df/E(f^{j-1}(y))\|\,\|Df^{-1}/F(f^j(y))\| < \lambda$$

and so

$$\prod_{j=1}^{n} \|Df/E(f^{j-1}(y))\|\,\|Df^{-1}/F(f^j(y))\| < \lambda^n,\ \forall\, n \geq 1\,.$$

If it were the case that

$$\prod_{j=1}^{n} \|Df^{-1}/F(f^j(y))\| \leq \lambda_1^n$$

for arbitrarily large $n > 0$ then by Lemma 3.2 there are $N = N(\lambda_1,\lambda_2) \in \mathbb{N}$ and $c = c(\lambda_1,\lambda_2) > 0$ such that if $n \geq N$ there exists $1 \leq n_k < n_{k-1} < \ldots < n_1 \leq n$ with $k > c\cdot n$ and

$$\prod_{j=h}^{n_i} \|Df^{-1}/F(f^j(y))\| \leq \lambda_2^{n_i-h}\,,$$

for $n_i \geq h \geq 1$; $i = 1,\ldots,k$. Observe in particular that $n_1 > c\cdot n$ otherwise we cannot have $k > c\cdot n$. By our choice of δ_3 we then have that

$$\prod_{j=h}^{n_1} \|Df^{-1}/F(f^j(z))\| \leq \lambda_3^{n_1-h}\,,$$

for all $h : n_1 \geq h \geq 1$ if $\mathrm{dist}(f^j(z), f^j(y)) \leq \delta_3$ for all $j : h \leq j \leq n_1$.

If now we have z in the local center unstable arc $[x,y]^{cu}$ joining x and y and $\rho = \mathrm{dist}(x,y) > 0$, we have, taking $h = 1$, that

$$\ell([x,y]^{cu}) \leq \ell([f^{n_1}(x), f^{n_1}(y)]^{cu})\lambda_3^{n_1-1}\,.$$

Since $[f^h(x), f^h(y)]^{cu}$ is tangent to F and $\mathrm{dist}(f^h(x), f^h(y)) \leq \epsilon$, by Lemma 3.3 there is a constant $L > 0$ such that $\ell([f^h(x), f^h(y)]^{cu}) \leq L$. Thus we obtain that

$$\ell([x, y]^{cu}) \leq L \cdot \lambda_3^{n_1 - 1}$$

and since $0 < \lambda_3 < 1$ and $n_1 > c \cdot n \to \infty$ when $n \to \infty$ we conclude that $\rho = 0$ and $x = y$ contradicting our hypothesis.

Hence we have that it is not true that for arbitrarily large $n > 0$

$$\prod_{j=1}^{n} \|Df^{-1}/F(f^j(y))\| \leq \lambda_1^n ,$$

and since

$$\prod_{j=1}^{n} \|Df/E(f^{j-1}(y))\| \, \|Df^{-1}/F(f^j(y))\| < \lambda^n ,$$

we may conclude that

$$\prod_{j=1}^{n} \|Df/E(f^{j-1}(y))\| \leq \lambda_1^n ,$$

for all n large. Thus, in the notation of [PS1], $I = [x, y]^{cu}$ is a ϵ-E-interval. There are two cases: either $\ell(f^n(I)) \to 0$ when $n \to \infty$ or $\ell(f^n(I)) \not\to 0$. In any case we may assume that for all point $z \in I$ we have that $W^{cs}_{loc}(z)$ is a stable manifold. Thus $W^{cs}_{loc}(I)$ attracts a neighborhood in M.
Let us assume first that $\ell(f^n(I)) \to 0$ when $n \to \infty$. Choose $\zeta > 0$ and let us find bounds for $r_n(\zeta, W^{cs}_{loc}(I))$. Since $\ell(f^n(I)) \to 0$ there is $n_0 > 0$ such that $\mathrm{diam}(f^n(W^{cs}_{loc}(I))) \leq \zeta$ for all $n \geq n_0$. Then we may find a finite subset E such that (ζ, n_0)-spans $W^{cs}_{loc}(I)$ and this set also (ζ, n)-spans $W^{cs}_{loc}(I)$ for all $n \geq 0$. It follows readily that

$$h(f, W^{cs}_{loc}(I), \zeta) = \limsup_{n \to \infty} \frac{1}{n} \log(r_n(\zeta, W^{cs}_{loc}(I)) = 0$$

and therefore $h(f, W^{cs}_{loc}(I)) = 0$.

On the other hand, if $\ell(f^n(I)) \not\to 0$ then by [PS1, Proposition 3.1] we have that for all $z \in I$, the omega -limit set of z, $\omega(z)$, is a periodic orbit or lies in a periodic circle. In the proof of that proposition Pujals and Sambarino use that f is of class C^2. But this is used in the case when $\ell(f^n(I)) \to 0$ when $n \to \infty$ in order to argue as in Schwartz's proof of the Denjoy property ([Sc]). If we already know that $\ell(f^n(I)) \not\to 0$ then it is enough to assume f of class C^1 to ensure that the ω-limit of I is contained in a periodic arc or circle and this is implicit in the proof of [PS1, Proposition 3.1].

In case of $\omega(x)$ being included in a periodic circle $\mathcal{C}$ this circle is normally hyperbolic attracting a neighborhood V of $\mathcal{C}$ and points in V converge exponentially fast to $\mathcal{C}$. If f is C^2 then as in [PS1] we conclude that the dynamics by f^τ (τ being the period of $\mathcal{C}$) in $\mathcal{C}$ is conjugate to an irrational rotation while if f is just C^1 we only have semi-conjugacy (we may have a Cantor set in $\mathcal{C}$ and wandering intervals). In any case (conjugacy or semi-conjugacy with an irrational rotation R_α) we profit from the fact that $h(R_\alpha) = 0$. This implies that if $f^\tau/\mathcal{C}$ is conjugate or semi-conjugate to R_α then $h(f^\tau/\mathcal{C}) = 0$.
On the other hand if $\omega(x)$ is a periodic orbit, say of a point q, since $\ell(f^n(I)) < \delta$ for all $n \geq 0$ we have that there is a periodic point q' in $W^{cu}_{loc}(q)$ such that attracts points in $f^n(I \backslash \{x\})$ (for instance the other end-point of $f^n(I)$ different from $f^n(x)$), see [PS1, Lemma 3.3.1]. Note than since $W^{cu}_{loc}(q)$ is an arc, the period of q' is the same of that of q, or the double of it. Let P be the set of periodic points of f in $W^{cu}_{loc}(q)\backslash\{q\}$. Then all of them have the same period, say τ. The set P divides $W^{cu}_{loc}(q)$ in arcs on which the dynamics by f^τ is monotone. It follows that the topological entropy of $f^\tau/W^{cu}_{loc}(q)$ is zero.
So in both cases, periodic orbit or periodic circle, $f^{\tau n}(W^{cs}_{loc}(I))$ approaches an f^τ invariant one-dimensional manifold $\mathcal{L}$ such that the topological entropy $h(f^\tau, \mathcal{L}) = 0$. Let $\zeta > 0$ and $m \in I\!N$ large be given an find $S' \subset \mathcal{L}$, (m, ζ) spanning $\mathcal{L}$. We may find n_0 and a subset S of $f^n(I)$ for $n \geq n_0$, such that (m, ζ) spans $f^n(I)$ with respect to f^τ. Projecting along the fibers of the local center-stable manifolds which, by equation (1), are dynamically defined ($W^{cs}_{loc}(z)$ is strong stable for all $z \in \mathcal{L}$) we know that there is $n_1 > 0$ such that for any point $z \in I$, $\ell(f^n(W^{cs}_{loc}(z))) < \zeta$. We add points to S in order to ensure that we do have a (m, ζ) spanning set for $f^m(W^{cs}_{loc}(I))$ for $m = 0, 1, \ldots, n_1 - 1$. We conclude that $h(f, W^{cs}_{loc}(I), \zeta) = 0$. Since $\zeta > 0$ is arbitrary we obtain that $h(f, W^{cs}_{loc}(I)) = 0$. By [Bo, Corollary 2.3] we have that if there is a ϵ-E-interval I such that $\Gamma_\epsilon(x) \subset W^{cs}_{loc}(I)$ then $h(\Gamma_\epsilon(x), f) = 0$.

Similarly if $y \in W^{cs}_{loc}(x)$ then $J = [x, y]^{cs}$ is an ϵ-F-interval and reasoning with the α-limit of J we obtain that $h(f, W^{cu}_{loc}(J)) = 0$.

Assume now that $y \notin W^{cs}_{loc}(x)$, $y \notin W^{cu}_{loc}(x)$. By domination

$$\|Df/E(z)\| \, \|Df^{-1}/F(f(z)\| < \lambda, \quad \forall\, z \in K$$

and this still holds for points such that their orbits are in the δ_0-neighborhood of K as is the case of y. Therefore there are defined $W^{cs}_{loc}(y)$ and $W^{cu}_{loc}(y)$ which are embedded arcs. Since the angle between E and F is bounded by $\gamma > 0$ from below, reducing ϵ if it were necessary, we may assume that $W^{cs}_{loc}(y)$ cuts $W^{cu}_{loc}(x)$ and $W^{cs}_{loc}(x)$ cuts $W^{cu}_{loc}(y)$ in points y_F and y_E respectively. By our assumption $y_E \neq x$ and $y_F \neq x$.

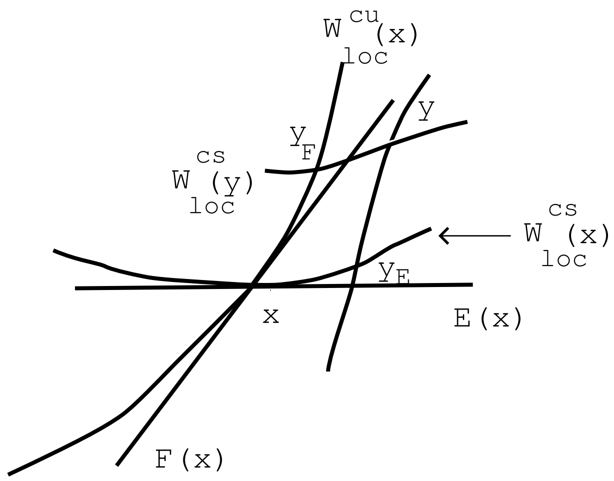

Figure 3: Case when $y \notin W^{cs}_{loc}(x)$, $y \notin W^{cu}_{loc}(x)$.

Suppose that there are $n > 0$ arbitrarily large such that for λ_1 it holds that

$$\prod_{j=1}^{n} \|Df/E(f^{-j}(y_E))\| \leq \lambda_1^n \, .$$

Then, choosing λ_2 and λ_3 as we did above, by Pliss ' Lemma there is $N = N(\lambda_1, \lambda_2) \in I\!N$ and $c = c(\lambda_1, \lambda_2) > 0$ such that if $n > N$ there is $n_1 > c \cdot n$ such that

$$\prod_{j=1}^{h} \|Df/E(f^{-j}(y_E))\| \leq \lambda_2^h \quad \forall \, 1 \leq h \leq n_1 \, ,$$

and changing λ_2 by λ_3 the same holds for points z in $[x, y_E]^{cs}$. It follows that $\mathrm{dist}(x, y_E) \leq \mathrm{dist}(f^{-n_1}(x), f^{-n_1}(y_E))\lambda_3^{n_1-1}$. Therefore

$$\mathrm{dist}(f^{-n_1}(x), f^{-n_1}(y_E)) \geq \frac{\mathrm{dist}(x, y_E)}{\lambda_3^{n_1}} \, .$$

Since by (2)

$$\mathrm{dist}(f^{-n_1}(x), f^{-n_1}(y)) \geq \mathrm{dist}(f^{-n_1}(x), f^{-n_1}(y_E)) \frac{\sin\gamma}{2 + \sin\gamma}$$

we conclude, taking into account that $0 < \lambda_3 < 1$, that

$$\mathrm{dist}(f^{-n_1}(x), f^{-n_1}(y)) \geq \frac{\mathrm{dist}(x, y_E)}{\lambda_3^{n_1}} \cdot \frac{\sin\gamma}{2 + \sin\gamma} > \epsilon$$

if n_1 is large enough contradicting the fact that $y \in \Gamma_\epsilon(x)$. We conclude in this case that y_E must coincide with x contradicting our hypothesis.

So, we cannot have arbitrarily large contraction from time $-n$ to 0 and as a consequence we have that $[x, y_E]^{cs}$ is a δ-F-interval for some $0 < \delta < \delta_0$. So the arguments employed above in the case when $y \in W^{cu}_{loc}(x)$ apply.

In any case we have proved that

$$\Gamma_\epsilon(x) \subset W^{cs}_{loc}(J) \cup W^{cu}_{loc}(I)$$

for a δ-E-interval I and a δ-F-interval J and that

$$h(f, W^{cs}_{loc}(J)) = h(f, W^{cu}_{loc}(I)) = 0$$

so that $h(f, \Gamma_\epsilon(x)) = 0$. □

4 Proof of Theorem B

In this section we prove the following

Theorem 4.1. *Let M be a compact boundaryless C^∞ surface and $f : M \to M$ be a C^r diffeomorphism. Let $H(p)$ be an f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that there is a continuation $H(p_g)$ of $H(p)$ such that $H(p_g)$ is h-expansive. Then $H(p)$ has a dominated splitting.*

In order to prove this theorem we will use results of Downarowicz and Newhouse (see [DN] and [Nh2]). Recall that a subshift (g, Y) is the restriction of the full shift in a finite alphabet to a closed invariant subsystem.

Definition 4.1. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. A symbolic extension of the pair (f, X) is a pair (g, Y), where (g, Y) is a subshift with a continuous surjection $\pi : Y \to X$ such that $f\pi = \pi g$. A symbolic extension is principal if the topological entropy of the extension coincides with that of the original system, that is, $h(g, Y) = h(f, X)$.*

In [DN] the following theorems are proved.

Theorem 4.2. *Fix $2 \leq r < \infty$. There is a residual subset $\mathcal{R}$ of the space Diff$^r(M)$ of C^r-diffeomorphisms of a closed surface M such that if $f \in \mathcal{R}$ and f has a homoclinic tangency, then f has no principal symbolic extension.*

Proof. See [DN, Theorem 1.4]. □

Moreover, if f has no principal symbolic extension then f cannot be asymptotically h-expansive as has been proved by M. Boyle, D. Fiebig and U. Fiebig (see [BFF]).

Proof of Theorem B. Let M and $f : M \to M$ be as in Theorem A and $H(p)$ an f-homoclinic class associated to the f-hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that there is a continuation $H(p_g)$ of $H(p)$ such that $H(p_g)$ is h-expansive. Let $x \in W^s(p) \cap W^u(p)$ be a transverse homoclinic point associated to the periodic point p. We define $E(x) = T_xW^s(p)$ and $F(x) = T_xW^u(p)$. Since p is hyperbolic we have that $E(x) \oplus F(x) = T_xM$. Moreover, $E(x)$ and $F(x)$ are Df-invariant, i.e.: $Df(E(x)) = E(f(x))$ and $Df(F(x)) = F(f(x))$.

By definition $H(p) = \text{clos}(\text{hom}(p))$ where $\text{hom}(p)$ is the set of transverse homoclinic points associated to p so if we prove that there is a dominated splitting for $\text{hom}(p)$ we are done since then we can extend by continuity the splitting to the closure $H(p)$.

Let us prove that there is a dominated splitting for $\text{hom}(p)$. To do so it is enough to prove that there exists $m > 0$ such that for some $k : 0 \leq k \leq m$ it holds for all $x \in \text{hom}(p)$ that

$$\|Df^k/E(x)\| \, \|Df^{-k}/F(f^k(x))\| \leq \frac{1}{2}\,.$$

Hence arguing by contradiction let us assume that for all $m > 0$ there is $x_m \in \text{hom}(p)$ such that for all $k : 0 \leq k \leq m$ we have

$$\|Df^k/E(x_m)\| \, \|Df^{-k}/F(f^k(x_m))\| > \frac{1}{2}\,.$$

Using the arguments developed by Mañé for periodic points in [Ma1] modified as in [SV] for homoclinic points, for any $\gamma > 0$ and $\epsilon > 0$ we may find $m > 0$, depending on ϵ and γ, such that with an ϵ-C^1-perturbation g' of f we obtain a homoclinic point $x_{g'}$ associated to $p_{g'}$ such that the angle at $x_{g'}$ between $W^s_{loc}(x_{g'}, g')$ and $W^u_{loc}(x_{g'}, g')$ is less than γ. Since C^2-diffeomorphisms are dense in C^1-topology we may assume that g' is C^2. Since γ is arbitrarily small we may C^1-perturb g' obtaining g of class C^2 with a tangency at x_g between $W^s_{loc}(x_g)$ and $W^u_{loc}(x_g)$. Moreover this perturbation can be assumed to give us a C^2-robust tangency of Hènon-like type (see [Nh1]). By the results of [DN] and [Nh2] we conclude that there is no symbolic extension for $g/H(p_g)$. Therefore, by [BFF], $g/H(p_g)$ is not asymptotic h-expansive and *a fortiori* it is not h-expansive contradicting our hypotheses. This finishes the proof of Theorem B. $\square$

References

[BFF] Boyle, M., Fiebig, D., Fiebig, U., *Residual entropy, conditional entropy, and subshift covers,* Forum Math., **Vol. 14** (2002), p. 713-757.

[Bo] R. Bowen, *Entropy-expansive maps,* Transactions of the American Mathematical Society, **vol 164** (February 1972), p. 323-331.

[Bu] J. Buzzi, *Intrinsic ergodicity for smooth interval map,* Israel J. Math., **100** (1997), p. 125-161.

[DN] T.A. Downarowicz, S.A. Newhouse, *Symbolic extensions and smooth dynamical systems,* Inventiones Mathematicae, **160, No.3,** (2005), p. 453-499.

[Ge] Marlies Gerber, *Conditional stability and real analytic pseudo-Anosov maps,* Mem. Amer. Math. Soc., **54** 1985.321

[LL] J. Lewowicz, E. Lima de Sá, *Analytic models of pseudo-Anosov maps,* Erg. Th. Dynam. Sys, **6** (1986), p. 385-392.

[Ma1] R. Mañé, *An ergodic closing lemma,* Annals of Mathematics, **116** (1982), p. 503-540.

[Mi] M. Misuriewicz, *Diffeomorphisms without any measure with maximal entropy,* Bull. Acad. Polon. Sci., **21** (1973), p. 903-910.

[Nh1] S. Newhouse, *The abundance of wild hypebolic sets and non-smooth stable sets for diffeomorphisms,* Inst. Haute Études Sci. Publ. Math., **50** (1979), p. 101-151.

[Nh2] S. Newhouse, *New phenomena associated with homoclinic tangencies,* Ergod. Th. & Dynam. Sys., **24** (2004), p. 1725-1738.

[Pl1] V. A. Pliss, *Analysis of the necessity of the conditions of Smale and Robbin for structural stability of periodic systems of differential equations,* Diff. Uravnenija, **8** (1972), p. 972-983.

[PS1] E. Pujals, M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms,* Annals of Mathematics, **151** (2000), p. 961-1023.

[PS2] E. Pujals, M. Sambarino, *On the Dynamics of Dominated Splitting,* Preprint, (2002), p. -.

[PPV] M. J. Pacifico, E. R. Pujals, J. L. Vieitez, *Robust expansive homoclinic classes,* Ergodic Theory and Dynamical Systems, **25** (2005), p. 271-300.

[PPSV] M. J. Pacifico, E. Pujals, M. Sambarino, J. Vieitez, *Robustly expansive codimension-one homoclinic classes are hyperbolic,* submitted for publication, (preprint IMPA, série D), p. -.

[Sc] Schwartz, A. J., *A generalization of a Poincaré-Bendixson theorem to closed two-dimensional manifolds,* Amer. J. Math., **85** (1963), p. 453-458.; Errata, *ibid* **85**, (1963),p. 753.

[SV] M. Sambarino, J. Vieitez, *On C^1-persistently Expansive Homoclinic Classes,* Discrete and Continuous Dynamical Systems, **14, No.3** (2006), p. 465-481.

M. J. Pacifico,
Instituto de Matematica,
Universidade Federal do Rio de Janeiro,
C. P. 68.530, CEP 21.945-970,
Rio de Janeiro, R. J. , Brazil.

pacifico@im.ufrj.br

J. L. Vieitez,
Instituto de Matematica,
Facultad de Ingenieria,
Universidad de la Republica,
CC30, CP 11300,
Montevideo, Uruguay
jvieitez@fing.edu.uy