REGULARITY OF THE FILTRATION FUNCTION INVERSE PROBLEM FOR FLOW IN POROUS MEDIA

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Abstract. In this paper we prove analyticity, stability and monotonicity of the filtration function obtained as the solution of an inverse problem based on experimental measurements of other quantities. The results follow from properties of a functional equation derived from the model equations. Based on these results we propose modifications to a previously presented method for determining the filtration function from effluent concentration history, which make it more robust. Numerical results are presented.

1. Introduction

Models for deep bed filtration during the injection of water with solid inclusions depend on an empirical *filtration function* $\lambda(\sigma)$ that represents the rate of particle retention as a function of deposition σ. This function cannot be measured directly, but must be recovered from the measurements of injected and effluent particle concentrations.

Methods for determining a constant filtration coefficient λ from the effluent particle concentration history at the core outlet were studied in [14, 16, 7, 5]. In [2] a more general method is presented, that determines a variable filtration $\lambda(\sigma)$ based on the effluent and injected particle concentration histories by solving a functional equation; this equation is derived from an invariant along characteristic lines for the particle transport equation.

Some issues, related to the monotonicity and analyticity of the filtration function, were not solved in [2]. The analyticity properties of the recovered solution justified the optimization method utilized in [13], which imposed an analytical expression for the filtration function. Physically, one expects $\lambda(\sigma)$ to be a decreasing function; this was imposed in [2] through assumptions on the data. These properties were also imposed in [13], where the cost function to be optimized was restricted to a space of functions with these nice properties. In this paper, we establish these properties as consequences of hypotheses on the experimental data that can be easily verified.

Another aspect is related to oscillations observed in the numerical solution for $\lambda(\sigma)$ that appear in many situations, even when the outlet effluent concentration is smooth. In [2], an artificial function space was found to allow stable numerical differentiation. Here a stronger stability criterion is established, based on intrinsic properties of the functional equation. These properties are the basis for a more robust method to recover the filtration function.

The numerical method developed in [15] suggests the usage of a system of equations different from the basic equations used in [2]. It has been verified for real data that the suspended particle concentration varies slowly with respect to the deposited particle concentration in deep filtration. Thus it is reasonable to neglect the time derivative of the suspended particle concentration from the mass conservation equation. This modification, called "first approximation" in the fundamental work of Herzig et al. [8], does not introduce significant changes in the solution, while making it numerically easier to calculate. Much of the recovery method developed in [2] is still valid for this case.

This paper is organized as follows. In Section 2, we present the deep bed filtration model as a quasi-linear hyperbolic system of equations. In Section 3, the recovery method for obtaining the filtration function, which utilizes a functional equation, is summarized. In Section 4, we present conditions for the monotonicity of $\lambda(\sigma)$. In Section 5, we prove that $\lambda(\sigma)$ is analytic, assuming that the history of the effluent particle concentration c_e satisfies certain properties, including analyticity. The stability condition of the inverse problem is obtained in Section 6, as a consequence of intrinsic properties of the functional equation. Numerical experiments are presented in section 7. The method used in this case is different from that in [2], where the stability depends on the numerical differentiation method employed.

2. The Direct problem

Our work utilizes the model for deep bed filtration developed in the fundamental work of Herzig et al. [8], which consists of equations expressing the particle mass conservation and the particle retention process [6, 8, 12]. They form a quasi-linear hyperbolic system of equations containing the empirical filtration function $\lambda(\sigma)$, which represents the kinetics of particle retention. For linear flow, this model is given in non-dimensional form by a mass conservation of particles

$$\frac{\partial \sigma}{\partial T} + \frac{\partial c}{\partial X} = 0, \tag{2.1}$$

and the empirical constitutive equation

$$\frac{\partial \sigma}{\partial T} = \lambda(\sigma) c. \tag{2.2}$$

The physical domain is dimensionless position $X \in [0, 1]$ and time $T \geq 0$. The non-dimensional time "unit" is called PVI, from "pore volume injected". The unknowns $c(X, T)$ and $\sigma(X, T)$ are the suspended and deposited particle concentrations, respectively. As boundary data, we assume that the solid particle concentration entering the porous medium is given and constant, i.e.,

$$c(0, T) = c_o > 0, \quad T \geq 0. \tag{2.3}$$

We have taken the inlet concentration $c(0, T)$ as constant just for simplicity. The general case for variable inlet concentration data is studied in [1] and [2]; the results from the current work extend directly to the general case.

As initial data, we assume that the rock contains no deposited particles:

$$\sigma(X, 0) = 0, \quad 0 \leq X \leq 1. \tag{2.4}$$

Remark 2.1. The direct problem of determining $c(X, T)$ and $\sigma(X, T)$ given $\lambda(\sigma)$, i.e., solving the system (2.1)–(2.4), but using the "third approximation" of [8]

$$\frac{\partial}{\partial T}(c + \sigma) + \frac{\partial c}{\partial X} = 0$$

instead of the "first approximation" (2.1), was studied in [2]; see also [3].

The existence and well posedness of the direct problem (2.1)–(2.4) can be established as in [2] under the following:

Assumption 2.2. The filtration $\lambda(\sigma)$ is a positive C^1 function in $0 \leq \sigma \leq 1$.

From (2.2) and (2.3) we obtain the following ordinary differential equation along the line $X = 0$:

$$\frac{d}{dT}\sigma(0,T) = \lambda(\sigma(0,T))c_o, \quad \text{and} \quad \sigma(0,0) = 0. \tag{2.5}$$

Integrating (2.5) provides $\sigma(0,T)$, which under Assumption 2.2 is always positive and increasing. The proof of the following result is similar to the proof of Theorem 2.4 in [2].

Theorem 2.3. *Under Assumption 2.2, there exists a unique, well-posed solution in* $R = \{(X,T) : 0 \le X \le 1; T \ge 0\}$ *for the system* (2.1)–(2.2) *with boundary data* (2.3) *and initial data* (2.4). *This solution is* $C^2(R)$*; it is obtained by solving for each* T *the system of ODE's*

$$\frac{d\sigma}{dX} = -\lambda(\sigma)\sigma \quad \textit{and} \quad \frac{dc}{dX} = -\lambda(\sigma)c, \tag{2.6}$$

with initial conditions for (2.6a) *given by* $\sigma(0,T)$ *calculated in* (2.5) *and for* (2.6b) *given by* $c(0,T)$ *in* (2.3).

Lemma 2.4. *Consider the solution of* (2.1)–(2.4) *from Theorem 2.3. Then*

$$\frac{\sigma(X,T)}{c(X,T)} = \frac{\sigma(0,T)}{c(0,T)}, \quad \textit{for} \quad T \ge 0. \tag{2.7}$$

Proof: Dividing (2.6a) by (2.6b) for constant T we obtain $d\sigma/dc = \sigma/c$. Integrating this equation, we see that σ/c is invariant along lines $T = const$, hence (2.7) $\Box$.

Remark 2.5. Since the RHS of (2.6b) is negative, the function $c(1,T)$ is C^2 and $c(1,T) < c_o$ in some time interval $[0,A]$, see [1].

As an example, the solution of (2.1)–(2.2) with data (2.3)–(2.4) for constant filtration function $\lambda(\sigma) = \lambda_0$ is:

$$c(X,T) = c_o e^{-\lambda_0 X}, \quad \sigma(X,T) = \lambda_0 c_o T e^{-\lambda_0 X}. \tag{2.8}$$

For sufficiently large times, this solution is almost the same as the solution of the variant of the model described in Remark 2.1, used in [2] and [5]. We expect the two models to give almost identical results for any filtration function $\lambda(\sigma)$ except for short times, which are not relevant in practice, and focus on the system (2.1)–(2.2), because it has numerical advantages (see [15]).

3. The functional equation

Here we summarize a recovery method for the filtration function analogous to that in [1] and [2]. Assume that the effluent concentration $c(1,T)$ is a given C^2 function of T. We introduce the C^3 function in $0 \le T \le A$

$$C(T) \equiv \int_0^T c(1,s)ds, \tag{3.1}$$

where A is the latest dimensionless time for which the data $c_e(T)$ are available.

We now obtain relationships between the deposited and suspended particle concentrations at the inlet and outlet points. From Assumption 2.2, we can define the first integral Ψ of $1/\lambda$ and the quantity m

$$\Psi(\sigma) = \int_0^\sigma \frac{d\eta}{\lambda(\eta)}, \quad m = \int_0^1 \frac{d\eta}{\lambda(\eta)}, \tag{3.2}$$

such that

$$\frac{\partial \Psi(\sigma)}{\partial T} = c, \quad \text{for } \sigma \in [0,1], \tag{3.3}$$

see [2]. Notice that $\Psi(0) = 0$. We integrate equation (3.3) in T: using equation (2.4) we find $\Psi(\sigma(0,T)) = c_o T$, and using (3.1) we find $\Psi(\sigma(1,T)) = C(T)$. From Assumption 2.2 and the definitions in (3.2), we know that $\Psi'(\sigma) = 1/\lambda(\sigma) > 0$, so there exists a function $g : [0,m] \to [0,1]$, with $g(0) = 0$ inverse of the function $\psi = \Psi(\sigma)/c_o$. Setting $X = 1$ in (2.7), we obtain the following functional equation for $\sigma = g(\psi)$:

$$g(C(T)) = \frac{c(1,T)}{c_o} g(c_o T) = \frac{C'(T)}{c_o} g(c_o T) \quad \text{for} \quad T \geq 0.$$

Finally, denoting

$$\tau \equiv c_o T \quad \text{so that} \quad \frac{dT}{d\tau} = \frac{1}{c_o}, \quad B = c_o A \quad \text{and} \quad D(\tau) \equiv C(\tau/c_o), \tag{3.4}$$

equation (3.4) can be rewritten as

$$g(D(\tau)) = D'(\tau) g(\tau) \quad \text{for} \quad \tau \in [0,B], \tag{3.5}$$

which is known as Julia's equation in g for the prescribed D, see [10].

3.1. **Recovery of the filtration function.** In this section we show how to recover the filtration function $\lambda(\sigma)$ using (3.5), based on linear flow experimental data c_o and $c_e(T)$. The effluent concentration c_e can be measured in the laboratory. The inlet concentration c_o is known provided particles are not retained at the injection face, i.e., if there is no cake formation ([1]). Ideally, the recovered $\lambda(\sigma)$ should yield $c(1,T) = c_e(T)$ for the solution of (2.1)–(2.4). In the situation of interest in this work, $c(1,T)$ should approximate well $c_e(T)$. Using (3.1) and (3.4) we redefine

$$D(\tau) = \int_0^{\tau/c_o} c_e(s) ds, \quad D : [0,B] \to \mathbb{R}. \tag{3.6}$$

Motivated by the fact that the filtration function λ should be positive and by Remark 2.5 we make the following:

Assumption 3.1. The function $c_e(\tau)$ is $C^2[0,B]$ and $0 < c_e < c_o$.

The existence and uniqueness of the solution of (3.5) is studied in [2], where the following theorem is proved:

Theorem 3.2. *Consider the Banach space $G_2^0 = \{g \in C^2[0,B], \quad g(0) = 0\}$ with norm $\|g\| = ||g||_\infty + ||g'||_\infty + ||g''||_\infty$. Let $D : [0,B] \to \mathbb{R}$ given by* (3.6) *be a C^2 monotone increasing function, such that $D(0) = 0$, $D(\tau) < \tau$ and $D'(0) < 1$. Then the functional equation* (3.5) *has a solution $g \in G_2^0$, which is uniquely defined by the value of $g'(0)$.*

This unique solution is given by

$$g(\tau) = g'(0) \prod_{n=0}^{\infty} \frac{D(\tau_n)}{D'(\tau_n)\tau_n}, \quad \text{with} \quad \tau_n = D^n(\tau) \equiv D(D^{n-1}(\tau)) \quad \text{and} \quad D^0(\tau) = \tau; \tag{3.7}$$

here, D is given in (3.6) and $\tau \in [0,B]$ is arbitrary. The solution is completed the expressions

$$g'(0) = \lambda(0) = -\log(c_e(0)/c_o). \tag{3.8}$$

and

$$\lambda(\sigma) = \frac{1}{\psi'(\sigma)} = \frac{dg}{d\tau}(\tau) = g'(\tau), \tag{3.9}$$

which can be derived from the definition of g and equation (3.2) (see [2]).

4. Monotonicity

One of the physical premisses of the model is that the retention rate decreases when the deposited particle concentration increases. Thus the filtration function $\lambda(\sigma)$ should be monotone decreasing. In this section we present conditions for the monotonicity of the solution g of (3.5) and the derivative of the effluent concentration history. Differentiating (3.5) and dividing by D' yields

$$g'(D(\tau)) = g'(\tau) + h(\tau) \quad \text{for} \quad \tau \geq 0, \quad \text{with} \quad h(\tau) = \frac{D''}{D'}(\tau)g(\tau). \tag{4.1}$$

If the function h in (4.1) is monotone increasing one obtains sufficient conditions for the filtration function to be monotone decreasing. We will need the following:

Assumption 4.1. We assume that $(\log(c_e(\tau)))'$ is a monotone increasing function in $[0, B]$.

Remark 4.2. Notice that if $c_e^{''}c_e - (c_e^{'})^2 > 0$, an easy test to perform on the data, then Assumption 4.1 holds. This follows from $(\log(c_e(\tau)))'' = (c_e^{''}c_e - (c_e^{'})^2)/c_e^2$.

The main result of this section is

Theorem 4.3. *Under Assumption 4.1, if the solution g of* (3.5) *is a monotone increasing function then the filtration function is monotone decreasing.*

To prove this theorem, we must first introduce Lemma 4.4, which is a direct application of Theorem 2.3.6, pag. 65 of [10] to our case:

Lemma 4.4. *Let $g \in G_2^0$ in equation* (4.1). *If the function h in* (4.1) *is monotone increasing and $\lim_{\tau\to 0} h(\tau) = 0$, then equation* (4.1) *has a unique one-parameter family of monotone decreasing solutions, which satisfy*

$$g'(\tau) = g'(\xi) - \sum_{n=0}^{\infty} \big(h(\tau_n) - h(\xi_n)\big). \tag{4.2}$$

Here ξ is an arbitrary positive value; τ_n, ξ_n are given in (3.7*b*)*: $\tau_n = D^n(\tau)$, $\xi_n = D^n(\xi)$.*

Remark 4.5. Taking $\xi = 0$ in (4.2) and using that $g(0) = 0$, formula (4.2) can be rewritten as

$$g'(\tau) = g'(0) - \sum_{n=0}^{\infty} h(\tau_n). \tag{4.3}$$

Using (3.5), we define

$$Q_i(\tau) = \sum_{m=0}^{\infty} \frac{D_i^{''}(\tau_{m,i})}{D_i^{'}(\tau_{m,i})} \prod_{j=1}^{m} D_i^{'}(\tau_{m-j,i}) \quad \text{or} \quad Q_i(\tau) = \sum_{m=0}^{\infty} \frac{D_i^{''}(\tau_{m,i})}{D_i^{'}(\tau_{m,i})} \tau_{m,i}^{'}, \ i = 1, 2; \tag{4.4}$$

and (4.3) becomes

$$g'(\tau) = g'(0) - Q(\tau)g(\tau). \tag{4.5}$$

Eq. (4.5) is an explicit formula for $\lambda(\sigma)$ since $\lambda(\sigma) = g'(\tau)$. The quantity $\lambda(0)$ is given in terms of experimental data in (3.8), $Q(\tau)$ in (4.4) and g in (3.7).

Remark 4.6. If the function h in (4.1) is monotone increasing then λ is monotone decreasing. To see this fact, we make $\tau < \xi$ in (4.2) and use $\tau_n < \xi_n$ (see 3.7b), which follows from the fact that D^n is an increasing function. Then $\lambda = g'$ is monotone decreasing.

Now we can prove Theorem 4.3. The solution $g \in G_2^0$ of equation (3.5), given in (3.7), and $D^{''}/D^{'} = (1/c_o)(log(c_e))^{'}$ are both monotone increasing. From (4.1), we have that h is monotone increasing as well. Finally, from Lemma 4.4 (see Remark 4.6), the solution of the functional equation (4.1) is monotone decreasing. Recalling that $\lambda = g^{'}$, we obtain that the filtration function is monotone decreasing. □.

However, if we establish the monotonicity of the solution of the functional equation (3.5), we can derive an alternate form of Theorem 4.3. We use the following:

Assumption 4.7. For $\tau \in [0, B]$ we assume that

$$(D^{'}(\tau))^2\tau - D(\tau)(D^{'}(\tau)\tau)^{'} > 0. \tag{4.6}$$

Lemma 4.8. *Assume that the hypotheses of Theorem* 3.2 *are satisfied and let* D *in* (3.6) *be such that Assumption 4.7 holds. Then the solution* g *of the functional equation* (3.5) *is monotone increasing.*

Proof: We set

$$G(\tau) = \frac{D(\tau)}{D'(\tau)\tau}, \quad \text{so that} \quad G'(\tau) = \frac{(D^{'}(\tau))^2\tau - D(\tau)(D^{'}(\tau)\tau)^{'}}{(D'(\tau)\tau)^2}; \tag{4.7}$$

notice that $G^{'}$ is a fraction with numerator that coincides with the LHS of inequality (4.6) and a positive denominator. Then Assumption 4.7 ensures that $G^{'} > 0$ for all $\tau \in [0, B]$. Let $\tau < \xi$: substituting G from (4.7) in (3.7), we have

$$g(\tau)/g(\xi) = \prod_{j=1}^{\infty}(G(\tau_j)/G(\xi_j)). \tag{4.8}$$

Using the facts that $\tau_j < \xi_j$ and that the function G in (4.7) is increasing, we have $G(\tau_j) < G(\xi_j)$ for $j = 1, 2 \ldots$; and as a consequence of (4.8) we have $g(\tau) < g(\xi)$, so g is increasing. □

This allows us to put Theorem 4.3 in terms of Assumption 4.7:

Theorem 4.9. *Under Assumptions 4.1 and 4.7, the filtration function is monotone decreasing.*

5. Analyticity

We now prove that $\lambda(\sigma)$ is a real analytic function provided the effluent concentration $c_e(\tau)$ is real analytic, i.e.:

Lemma 5.1. *If* $c_e(\tau)$ *is real analytic in* $[0, B]$*, then the solution of* (3.5) *is real analytic.*

Because $c_e(\tau)$ is real analytic, so is its integral D, and $D^{'}$ is also analytic. Since the product and composition of analytic functions are analytic, the reciprocal of an analytic function that is nowhere zero is analytic, then $D(\tau)/D'(\tau)\tau$ is analytic. Using the uniform convergence of the series of the logarithm of (3.7), established in [2] under the hypotheses of Theorem 3.2, we obtain that g is real analytic, and so is its derivative, and from (3.9) we conclude that the filtration function $\lambda(\sigma)$ is real analytic as well.

Remark 5.2. In [2] the filtration function was obtained by solving equation (3.5) first through the iterative procedure (3.7) and then by numerical differentiation. The latter step is intrinsically ill-posed, so that oscillations were obtained in the solution. Formula (4.5) avoids numerical differentiation of g: via Remark 4.5, we only need to replace the data c_e by an analytic approximation, if possible satisfying Assumptions 4.1 and 4.7.

6. Stability

Continuous dependence of the functional equation solution g on the given coefficient function D was established in [9]. However, actual bounds were not established. Here we give such bounds that imply stability. Conditions for numerical stability of an implementation of a recovery method for λ given c_o and c_e were presented in [2]; we now establish stability conditions based solely on intrinsic properties of the functional equation (3.5) and on bounds on the data. To do so, in addition to the hypotheses for Theorem (3.2), we impose the following:

Assumption 6.1. We assume that c_o, $c_e(\tau)$ are such that $D(\tau)$ defined in (3.6) is a non-negative C^3 function for $0 \leq \tau \leq B$ with B defined in (3.4) satisfying

$$0 \leq D(\tau) < \tau,\ 0 < D'(\tau) < d \ \text{ for } \ 0 \leq \tau \leq B;\ D(0) = 0 \ \text{ and } \ D''(0) \neq 0, \tag{6.1}$$

where $d < 1$ is a constant. We assume that there exists a constant p such that

$$(D''(\tau) - 2D'(\tau))/D'(0) < p, \ \text{ for all } \ \tau \in [0, B]. \tag{6.2}$$

Assumption 6.2. We assume that the effluent particle concentration $c_e(\tau)$ is restricted to

$$\mathcal{M} = \{c_e \in C^2[0, B] : 0 < r_1 < c_e(\tau) \leq r_2 < c_o,\ 0 \leq r_3 \leq c_e'(\tau) \leq r_4,\ r_5 \leq c_e''(\tau) \leq r_6\}, \tag{6.3}$$

for certain constants $r_1, .., r_6$: notice the strict inequalities $0 < r_1 < c_e$ and $r_2 < c_o$. Analogously, the inlet particle concentration is restricted as follows:

$$0 < r_7 < c_o < r_8, \tag{6.4}$$

for certain constants r_7, r_8.

Remark 6.3. The interval for c_o in (6.4) allows for experimental errors in the c_o measurements to be taken into account, and for the definition of a single stability criterion for data obtained from experiments for which c_o differs. The stability criterion derived here can also be applied to the case where $c(0, T)$ is not constant, as is assumed in this work, but instead varies along time in an interval $[r_7, r_8]$.

Remark 6.4. In the case of constant filtration function, obviously the solution (2.8) of the system satisfies Assumptions 6.1, 6.2 for $X \in [0, 1]$.

Remark 6.5. Recall the expression for $g'(0)$ in (3.8). Assumption 6.2 implies that $g'(0) \neq 0$, and $g'(0) < f$ for a certain constant f. These are consequences of the strict inequalities imposed on r_1 and r_2 respectively. We consider here the class of solutions of the functional equation (3.5) that satisfies the above conditions on $g'(0)$.

The main result of this section is the following stability result.

Theorem 6.6. *Let us consider c_{o1}, c_{o2}, c_{e1}, c_{e2} and define*

$$D_1(\tau) \equiv \int_0^{\tau/c_{o1}} c_{e1}(s)ds, \quad D_2(\tau) \equiv \int_0^{\tau/c_{o2}} c_{e2}(s)ds, \tag{6.5}$$

such that Assumptions 6.1 and 6.2 are satisfied. Then there exist data-independent constants m_1, m_2 such that

$$||\lambda_1 - \lambda_2||_\infty \leq m_1|c_{o1} - c_{o2}| + m_2\Big(||c_{e1} - c_{e2}||_\infty + ||c_{e1}^{'} - c_{e2}^{'}||_\infty\Big). \tag{6.6}$$

In (6.6), the sup-norm is taken on $[0, 1]$ for the filtration function and on $[0, B]$ for the effluent concentration function and its derivative. The proof of Theorem 6.6, found in Appendix A. is built from a number of lemmas. Accurate estimates for the constants appearing in equation (6.6) can be easily collected from the lemmas. They are useful to evaluate to sensitivity of the filtration function to the concentration data.

Remark 6.7. Since $\lambda(\sigma)$ and $c_e(\tau)$ are analytic, inequalities similar to (6.6) hold for derivatives of all orders.

7. Numerical experiments

In [2] examples were shown illustrating that the filtration function $\lambda(\sigma)$ was non–decreasing, contrary to desired physical behaviour, even when the solution g of the functional equation (3.5) was non–decreasing, as expected (recall that $\lambda = g'$). To obtain monotonicity a careful but heuristic pre– and post–processing of data was done to obtain physically nice results. Even so, this data processing allowed us to handle the data given in [11] only up to 70 PVI, and required considerable human intervention.

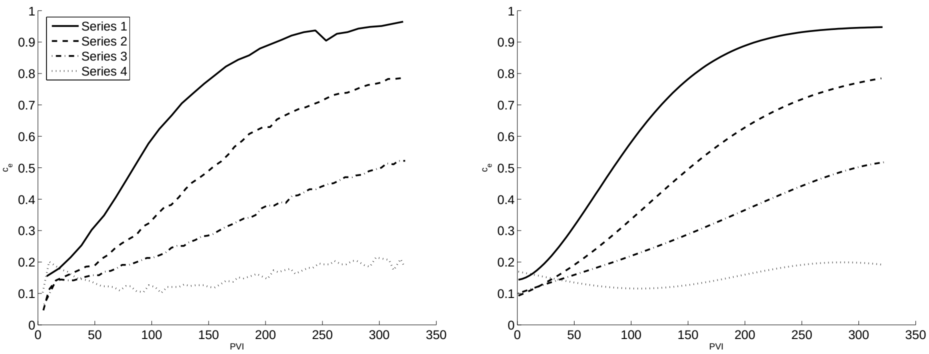
Theorem 4.3 proves that a monotone decreasing filtration function is obtained when input data and the solution of the functional equation (3.5) satisfy certain properties. To illustrate this fact numerically, we take experimental data in a much longer time range than that used in [2] and approximate the data by the real analytic functions given in Table 7.1. The regularized approximations are very close to the original data, as can be seen in Figure 7.1. Results with these approximations are better. We obtain filtration functions for which the solution of the direct problem accurately matches the input data series up to 350 PVI rather than only 70 PVI, i.e, these results are physically plausible for the whole data series. The

Series	Expression	Coefficients
1	$a - b\exp(-cx^d)$	$a = 0.95$, $b = 0.81$, $c = 2.9 \times 10^{-4}$, $d = 1.72$
2	$a(1 + exp(b - cx)^{-1/d}$	$a = 0.83$, $b = 1.22$, $c = 0.01$, $d = 0.67$
3	$a + bx + cx^2 + dx^3 + ex^4$	$a = 0.10$, $b = 1.3 \times 10^{-3}$, $c = -2.8 \times 10^{-6}$ $d = 2.6 \times 10^{-8}$, $e = -5.3 \times 10^{-11}$
4	$a + b\cos(cx + d)$	$a = 0.16$, $b = 0.04$, $c = 0.02$, $d = 1.26$

Table 7.1. Analytic expressions used to approximate the four data series.

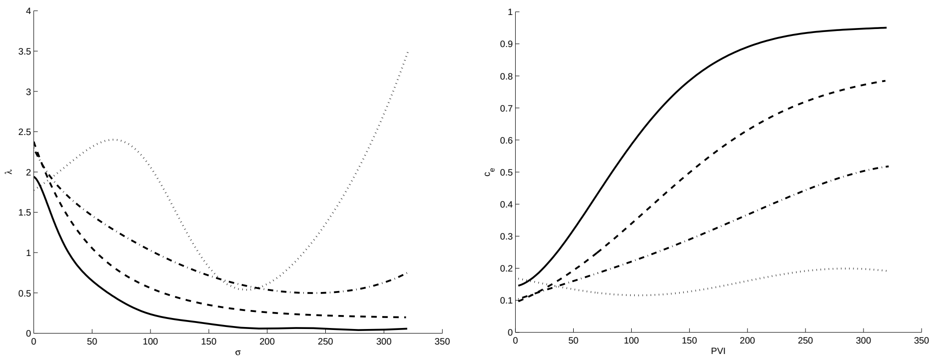
first three experimental series were approximated by functions with positive derivatives (see Figure 7.1b), leading to monotone increasing g and monotone decreasing λ. On the other hand, in the case of the fourth series, we see that when c_e is non-monotone we obtain the strange non-monotone profile for λ shown in Figure 7.2a as the top dotted curve.

For the sake of verification, we solve the direct problem using the recovered filtration function and we calculate the corresponding effluent concentration, shown in Figure 7.2b. These values match accurately the regularized data shown in Figure 7.1b.



a) Raw data series. b) Real analytic fits.

FIGURE 7.1. The left picture shows the data from [11]. The right picture shows the approximations given by the expressions in Table 7.1. Each of the curves here and in Figure 7.2 corresponds to one experiment.



a) Filtration functions. b) Recovered profiles.

FIGURE 7.2. The filtration functions shown on the left figure were obtained by solving the inverse problem with the smooth data presented in Figure 7.1b; the direct problem was then solved using these filtration functions to produce the figure on the right, which shows profiles visually indistinguishable from the input data shown in Figure 7.1b.

These numerical experiments show that the extension defined here to the method introduced in [2] for the calculation of the filtration function, using real analytic approximations for the effluent concentration history, provides a simple, robust and fast algorithm applicable to real data [4].

8. Conclusions

In a previous work [2], we assumed on physical grounds that the recovered filtration function had a monotonicity property. In this work, we establish mathematical conditions that ensure this monotonicity in terms of properties of the experimental data, and provide stability conditions based on intrinsic properties of the functional equation, such as analyticity. The analyticity results presented here justify the use of analytic approximations for the filtration function already used for the recovery method in [13], as well as for the effluent concentration data, which are shown here to generate good results.

Appendix A. Proof of the stability theorem

Theorem 6.6 provides bounds that express the continuous dependency of the solution $g(\tau)$ of (3.5) on the particle concentrations c_o, $c_e(\tau)$, under Assumptions (6.1), (6.2). The proof is built on a number of lemmas. The idea is to first estimate how changes in the concentration c_o and c_e modify D and its derivatives. This step is accomplished with Lemmas A.3 and A.4. Next, we estimate how changes in D affect $g(\tau)$ and $Q(\tau)$ in (4.5). These estimates are obtained in Lemmas A.5 through A.12.

We stress that all following derivations are under Assumptions 6.1 and 6.2. Denoting $s = D(\tau)$, Equation (3.5) can be rewritten as

$$g(s) = D'(D^{-1}(s))g(D^{-1}(s)). \tag{A.1}$$

We denote the inverse of D_i as $\Upsilon_i = D_i^{-1}(s)$ for $i = 1, 2$. All derivatives (e.g., D') are with respect to τ.

Some definitions will be useful as well: constants will be called "data-independent" if they depend on $c_{o1}, c_{o2}, c_{e1}, c_{e2}$ only through the bounds for these quantities expressed in Assumptions (6.1), (6.2), and the bounds for g and g' given in the following theorem, which was proved for d in (6.1b), p in (6.2) and f in Remark 6.5 (see [1, 2]).

Theorem A.1. *The solutions g of the functional equation* (3.5) *are uniformly bounded by $f \exp(1/(1-d))$, and g' are uniformly bounded by $f \exp(p/2(1-d))$.*

Remark A.2. The bound for the derivative g' given in Theorem A.1 is correct. The different bound given in the previous article [2] is a typographical error.

Lemma A.3. *Under Assumptions* (6.1), (6.2)*, there exist constants $N_1, \ldots, N_6$ such that*
(i) $||D_1^{''} - D_2^{''}||_\infty \leq N_1|c_{o1} - c_{o2}| + N_2||c_{e1}^{'} - c_{e2}^{'}||_\infty$,
(ii) $||D_1^{'} - D_2^{'}||_\infty \leq N_3|c_{o1} - c_{o2}| + N_4||c_{e1} - c_{e2}||_\infty$,
(iii) $||D_1 - D_2||_\infty \leq N_5|c_{o1} - c_{o2}| + N_6||c_{e1} - c_{e2}||_\infty$.

Proof: The derivatives of D_1, D_2 in (6.5) are $D_1^{'} = c_{e1}/c_{o1}$, $D_1^{''} = c_{e1}^{'}/(c_{o1})^2$, etc.. We have

$$|c_{e1}^{'}/(c_{o1})^2 - c_{e2}^{'}/(c_{o2})^2| \leq (c_{o1}c_{o2})^{-2}((c_{o2})^2|c_{e1}^{'} - c_{e2}^{'}| + |c_{e2}^{'}||(c_{o1})^2 - (c_{o2})^2|). \tag{A.2}$$

Using $N_1 = 2r_2^2/r_7^2$ and $N_2 = r_8^2/r_7^2$ in (A.2) we obtain

$$||D_1^{''} - D_2^{''}||_\infty \leq N_1|c_{o1} - c_{o2}| + N_2||c_{e1}^{'} - c_{e2}^{'}||_\infty.$$

We obtain analogously the inequalities (ii) and (iii).□

Lemma A.4. *Assuming, without loss of generality, that $D_1(B) < D_2(B)$, there exists a data-independent constant M such that the following inequalities hold:*
(i) $||D_1^{-1} - D_2^{-1}||_\infty \leq M||D_1 - D_2||_\infty$,
(ii) $|D_1^{'}(D_1^{-1}(s)) - D_2^{'}(D_2^{-1}(s))| \leq ||D_1^{'} - D_2^{'}||_\infty + M||D_2^{'}||_\infty||D_1 - D_2||_\infty$, $s \in [0, D_1(B)]$.

Proof of (i): We assume that $\tau, \eta \in [0, D_1(B)] \subset [0, D_2(B)]$. Fixed τ, let $\eta = D_1(D_2^{-1}(\tau))$; it follows that $D_1^{-1}(\eta) = D_2^{-1}(\tau)$, and therefore

$$|D_1^{-1}(\tau) - D_2^{-1}(\tau)| = |D_1^{-1}(\tau) - D_1^{-1}(\eta)|$$

We have $D_1' = c_{e1}/c_{o1}$, so $(D_1^{-1}(\tau))' = 1/D_1'(\tau) < (c_{o1}/r_1)$, from Equation (6.3); using this and (6.4) in the expression

$$D_1^{-1}(\tau) - D_1^{-1}(\eta) = \int_0^1 \frac{\partial}{\partial\alpha}(D_1^{-1}(\alpha\tau + (1-\alpha)\eta))d\alpha,$$

we obtain

$$\begin{aligned} |D_1^{-1}(\tau) - D_1^{-1}(\eta)| &\leq (c_{o1}/r_1)|\tau - \eta| \\ &\leq (r_8/r_1)|D_2(D_2^{-1}(\tau)) - D_1(D_2^{-1}(\tau))|. \square \end{aligned} \tag{A.3}$$

Proof of (ii): We apply the mean value theorem to the inequality

$$|D_1'(\Upsilon_1) - D_2'(\Upsilon_2)| \leq |D_1'(\Upsilon_1) - D_2'(\Upsilon_1)| + |D_2'(\Upsilon_2) - D_2'(\Upsilon_1)|$$

to obtain

$$|D_1'(D_1^{-1}(s)) - D_2'(D_2^{-1}(s))| \leq ||D_1' - D_2'||_\infty + ||D_2''||_\infty |D_1^{-1}(s) - D_2^{-1}(s)|; \tag{A.4}$$

using (i) in (A.4) we obtain (ii). $\square$

Lemma A.5. *Let us denote by g_1, g_2 two solutions of Eq.* (3.5) *in $\mathcal{G}$ satisfying* (3.8) *with corresponding D_1, D_2. Then there exist data-independent constants v_1, v_2, such that*

$$||g_1 - g_2||_\infty \leq v_1|c_{o1} - c_{o2}| + v_2||c_{e1} - c_{e2}||_\infty. \tag{A.5}$$

Proof: From (A.1), we write

$$\begin{aligned} |g_1(s) - g_2(s)| &= |D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_2)| \\ &\leq |D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_1)| + \\ &\quad |D_2'(\Upsilon_2)g_2(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_2)|; \end{aligned} \tag{A.6}$$

applying the mean value theorem to the last two lines in the previous expression yields, for the first one,

$$\begin{aligned} &|D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_1)| \\ &\qquad \leq |D_1'(\Upsilon_1)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_1(\Upsilon_1)| + |D_2'(\Upsilon_2)g_1(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_1)| \\ &\qquad = |g_1(\Upsilon_1)||D_1'(\Upsilon_1) - D_2'(\Upsilon_2)| + |D_2'(\Upsilon_2)||g_2(\Upsilon_1) - g_1(\Upsilon_1)| \\ &\qquad \leq ||g_1||_\infty |D_1'(\Upsilon_1) - D_2'(\Upsilon_2)| + ||D_2'||_\infty |(g_2 - g_1)(\Upsilon_1)|, \end{aligned}$$

and for the second,

$$|D_2'(\Upsilon_2)g_2(\Upsilon_1) - D_2'(\Upsilon_2)g_2(\Upsilon_2)| \leq ||D_2'||_\infty ||g_2'||_\infty ||D_2^{-1} - D_1^{-1}||_\infty.$$

Since $||D_2'||_\infty < d < 1$ (Assumption 6.1) we have from the previous inequalities that

$$(1-d)||g_1 - g_2||_\infty \leq ||D_2'||_\infty ||g_2'||_\infty |||D_2^{-1} - D_1^{-1}||_\infty + ||g_1||_\infty |D_1'(\Upsilon_1) - D_2'(\Upsilon_2)|.$$

Finally, (A.5) is obtained applying Theorem A.1 and Lemmas A.3 and A.4 to the expression above. $\square$

Lemma A.6. *Let* $\tau_{m,i} = D_i^m(\tau)$, *(see Equation* (3.7*b*)*) with* $i = 1, 2$ *and* $m = 0, 1, \ldots$ *The following inequalities hold:*

$$|\tau_{m,2} - \tau_{m,1}| \leq ||D_1 - D_2||_\infty \sum_{k=0}^{m} d^k. \tag{A.7}$$

Proof: We prove this Lemma by induction. It is easy to see that the inequality for $m = 0$ is valid, using $\tau_{0,1} = D_1(\tau)$, $\tau_{0,2} = D_2(\tau)$; as induction hypothesis, assume that it is valid for m. From the definitions in (3.7) and (6.1), and the mean value theorem, we write

$$\begin{aligned}
|\tau_{m+1,2} - \tau_{m+1,1}| &\leq |D_2(D_2^m(\tau)) - D_2(D_1^m(\tau))| + |D_2(D_1^m(\tau)) - D_1(D_1^m(\tau))| \\
&\leq ||D_2'||_\infty |\tau_{m,2} - \tau_{m,1}| + ||D_1 - D_2||_\infty \\
&\leq d|\tau_{m,2} - \tau_{m,1}| + ||D_1 - D_2||_\infty \\
&\leq d||D_1 - D_2||_\infty \sum_{k=0}^{m} d^k + ||D_1 - D_2||_\infty \\
&\leq ||D_1 - D_2||_\infty \left(1 + d\sum_{k=0}^{m} d^k\right) = ||D_1 - D_2||_\infty \left(\sum_{k=0}^{m+1} d^k\right).\square
\end{aligned}$$

Lemma A.7. *Let* $\tau_{m,i}$ *be as in Lemma (A.6). The following inequalities hold:*

(i) $|D_1''(\tau_{m,1}) - D_2''(\tau_{m,2})| \leq ||D_1'' - D_2''||_\infty + ||D_2'''||_\infty ||D_1 - D_2||_\infty \sum_{k=0}^{m} d^k.$

(ii) $|D_1'(\tau_{m,1}) - D_2'(\tau_{m,2})| \leq ||D_1' - D_2'||_\infty + ||D_2''||_\infty ||D_1 - D_2||_\infty \sum_{k=0}^{m} d^k.$

Proof: We prove only (i), since the proof of (ii) is analogous. Using the mean value theorem, we derive

$$\begin{aligned}
|D_1''(\tau_{m,1}) - D_2''(\tau_{m,2})| &\leq |D_1''(\tau_{m,1}) - D_2''(\tau_{m,1})| + |D_2''(\tau_{m,1}) - D_2''(\tau_{m,2})| \\
&\leq ||D_1'' - D_2''||_\infty + ||D_2'''||_\infty |\tau_{m,1} - \tau_{m,2}|.
\end{aligned}$$

Applying Lemma A.6 to the expression above yields (i). □

Lemma A.8. *Let* $\tau_{m,i}$ *be as in Lemma (A.6), but* $m = 1, 2, \ldots$ *The following inequalities hold:*

$$\sum_{j=1}^{m} |D_1'(\tau_{m-j,1}) - D_2'(\tau_{m-j,2})| \leq ||D_2''||_\infty \left(\sum_{j=1}^{m} \sum_{k=0}^{m-j} d^k\right) ||D_1 - D_2||_\infty + m||D_1' - D_2'||_\infty. \tag{A.8}$$

Proof: Using the mean value theorem, we write

$$\begin{aligned}
|D_1'(\tau_{m-j,1}) - D_2'(\tau_{m-j,2})| &\leq |D_1'(\tau_{m-j,1}) - D_2'(\tau_{m-j,1})| + |D_2'(\tau_{m-j,2}) - D_2'(\tau_{m-j,1})| \\
&\leq ||D_1' - D_2'||_\infty + ||D_2''||_\infty |\tau_{m-j,2} - \tau_{m-j,1}|.
\end{aligned}$$

Applying Lemma A.6 to the summation of these inequalities from $j = 1$ to m yields (A.8).□

Lemma A.9. *Let* $\tau_{m,i}$ *be as in Lemma (A.8). The following inequalities hold:*

$$0 < \tau_{m,i}' = \prod_{j=1}^{m} D_i'(\tau_{m-j,i}) \leq d^m, \quad i = 1, 2. \tag{A.9}$$

Proof: It is an immediate application of the definition of d given in Equation (6.1), i.e. $0 < D_i' < d$.$\square$

Lemma A.10. *Recall the definition of Q_i in Remark 4.5: the following inequalities hold*

$$0 < Q_i \le K, \quad i = 1, 2, \tag{A.10}$$

for some constant K.

Proof: Using Lemma (A.9) in (4.4) we obtain

$$|Q_i(\tau)| \le \left\| \frac{D_i''}{D_i'} \right\|_\infty \sum_{m=0}^{\infty} d^m = \left\| \frac{D_i''}{D_i'} \right\|_\infty \frac{1}{1-d} \le K, \quad i = 1, 2.$$

(Recall that $D_i'' = c_{e,i}'/(c_{o,i})^2$ and $D_i' = c_{e,i}/c_{o,i}$. Since $c_{e,i}$ in $\mathcal{M}$ and $c_{o,i}$ satisfy (6.4) then D_i''/D_i' with $i = 1, 2$ are uniformly bounded).$\square$

Lemma A.11. *For $m = 1, 2, \dots$ the following inequalities hold:*

$$(\tau_{m,1} - \tau_{m,2})' \le d^{m-1} \sum_{j=1}^{m} |D_1'(\tau_{j,1}) - D_2'(\tau_{j,2})|. \tag{A.11}$$

Proof: Notice that $(\tau_{m,1} - \tau_{m,2})' = \prod_{l=1}^m D_1'(\tau_{l,1}) - \prod_{l=1}^m D_2'(\tau_{l,2})$, we prove (A.11) by induction. It is obvious for $m = 1$; assuming it is valid for $m - 1$, we write

$$\begin{aligned} &\left| \prod_{l=1}^{m} D_1'(\tau_{l,1}) - \prod_{l=1}^{m} D_2'(\tau_{l,2}) \right| \\ &\quad \le |D_1'(\tau_{m,1})| \left| \prod_{l=1}^{m-1} D_1'(\tau_{l,1}) - \prod_{l=1}^{m-1} D_2'(\tau_{l,2}) \right| + \left(\prod_{l=1}^{m-1} |D_2'(\tau_l^2)| \right) |D_1'(\tau_{m,1}) - D_2'(\tau_{m,2})| \\ &\quad \le D_1'(\tau_{m,1}) d^{m-2} \sum_{j=1}^{m-1} |D_1'(\tau_{j,1}) - D_2'(\tau_{j,2})| + \left(\prod_{l=1}^{m-1} |D_2'(\tau_{l,2})| \right) |D_1'(\tau_{m,1}) - D_2'(\tau_{m,2})|. \end{aligned} \tag{A.12}$$

Using (6.1b) and (A.9) in (A.12) we obtain (A.11). $\square$

Lemma A.12. *Let us denote by g_1, g_2 two solutions of Eq.* (3.5) *in $\mathcal{G}$ satisfying* (3.8) *and corresponding to D_1, D_2. Then there exist data-independent constants v_1, v_2, v_3 such that*

$$|Q_1(\tau) g_1(\tau) - Q_2(\tau) g_2(\tau)| \le v_1 |c_{o1} - c_{o2}| + v_2 ||c_{e1} - c_{e2}||_\infty + v_3 ||c_{e1}' - c_{e2}'||_\infty. \tag{A.13}$$

Proof: Let us define for $0 \le \alpha \le 1$:

$$G_n(\tau, \alpha) = \alpha D_1''(\tau_{n,1}) + (1 - \alpha) D_2''(\tau_{n,2}), \;\; F_n(\tau, \alpha) = \alpha D_1'(\tau_{n,1}) + (1 - \alpha) D_2'(\tau_{n,2}).$$

Using the convention that $(\alpha \tau_{0,1} + (1 - \alpha) \tau_{0,2})' = 1$ we define as in (4.4)

$$Q(\tau, \alpha) = \sum_{n=0}^{\infty} \frac{G_n(\tau, \alpha)}{F_n(\tau, \alpha)} (\alpha \tau_{n,1} + (1 - \alpha) \tau_{n,2})', \tag{A.14}$$

so that $Q(\tau, 1) = Q_1(\tau)$ and $Q(\tau, 0) = Q_2(\tau)$. Now for each τ

$$|Q_1 g_1 - Q_2 g_2| \le |Q_1 g_1 - Q_1 g_2| + |Q_2 g_2 - Q_1 g_2| \le |Q_1| |g_1 - g_2| + |g_2| |Q_2 - Q_1|.$$

Thus it is necessary to bound $|Q_2(\tau) - Q_1(\tau)|$. From the mean value theorem there exists $\alpha \in (0, 1)$ such that

$$Q_2(\tau) - Q_1(\tau) = \frac{\partial Q(\tau, \alpha)}{\partial \alpha}. \tag{A.15}$$

We have from (A.14) and (A.15) that for $Q(\tau, \alpha)$ and $F_n = F_n(\tau, \alpha)$ and $G_n = G_n(\tau, \alpha)$:

$$\begin{aligned}\frac{\partial Q}{\partial \alpha} &= \sum_{n=0}^{\infty}\left(\frac{(D_1^{''}(\tau_{n,1}) - D_2^{''}(\tau_{n,2}))F_n - (D_1^{'}(\tau_{n,1}) - D_2^{'}(\tau_{n,2}))G_n}{(F_n)^2}\right)(\alpha\tau_{n,1} + (1-\alpha)\tau_{n,2})^{'} + \\ &\quad \sum_{n=0}^{\infty}(G_n/F_n)(\tau_{n,1} - \tau_{n,2})^{'}, \end{aligned} \tag{A.16}$$

where $(\tau_{0,1} - \tau_{0,2})^{'} = 0$. Using Lemma (A.11) the following inequality holds:

$$|(\tau_{n,1} - \tau_{n,2})^{'}| \leq d^{n-1}\sum_{j=1}^{n}|D_1^{'}(\tau_{n-j,1}) - D_2^{'}(\tau_{n-j,2})|.$$

Finally, using Lemma A.8 we obtain the following inequality:

$$\begin{aligned}|(\tau_{n,1} - \tau_{n,2})^{'}| &\leq d^{n-1}\left(||D_2^{''}||_\infty\left(\sum_{j=1}^{n}\sum_{k=0}^{n-j} d^k\right)||D_1 - D_2||_\infty + n||D_1^{'} - D_2^{'}||_\infty\right) \\ &\leq d^{n-1}(n/(1-d))||D_2^{''}||_\infty||D_1 - D_2||_\infty + nd^{n-1}||D_1^{'} - D_2^{'}||_\infty. \end{aligned}\tag{A.17}$$

Notice that Lemma A.9 immediately gives for $0 \leq \alpha \leq 1$:

$$|(\alpha\tau_{n,1} + (1-\alpha)\tau_{n,2})^{'}| \leq d^n \quad \text{for } n = 1, \ldots. \tag{A.18}$$

From Lemma A.7, Equations (A.15), (A.16), (A.17), (A.18), and the convergence of the series $\sum_{n=1}^{\infty} d^{n-1}$ and $\sum_{n=0}^{\infty} nd^n$, the functions $D_i^{'}$, $D_i^{''}$, $D_i^{'''}$ with $i = 1, 2$ are uniformly bounded and the functions $|G_n|, |F_n|, n = 0, 1, \ldots$ are uniformly bounded below and above by positive constants. Therefore, there exist data-independent constants t_1, t_2, t_3 and s_1, s_2 such that

$$\begin{aligned}|Q_2(\tau) - Q_1(\tau)| &\leq t_1||D_1^{''} - D_2^{''}||_\infty + t_2||D_1^{'} - D_2^{'}||_\infty + t_3||D_1 - D_2||_\infty + \\ &\quad \max_{\tau,\alpha,n}\{G_n/F_n\}(s_1||D_1^{'} - D_2^{'}||_\infty + s_2||D_1 - D_2||_\infty).\end{aligned}$$

Thus there exist data-independent constant p_1, p_2, p_3 such that

$$|Q_2(\tau) - Q_1(\tau)| \leq p_1||D_1^{''} - D_2^{''}||_\infty + p_2||D_1^{'} - D_2^{'}||_\infty + p_3||D_1 - D_2||_\infty.$$

From Lemmas (A.3) and (A.10), we have that (A.13) holds.□

Using previous lemmas we can prove Theorem (6.6). Using (4.5) we obtain

$$|g_1'(\tau) - g_2'(\tau)| \leq |g_1'(0) - g_2'(0)| + |Q_1(\tau)g_1(\tau) - Q_2(\tau)g_2(\tau)|.$$

Now, using (3.8) and the mean value theorem we obtain

$$|g_1'(0) - g_2'(0)| = |\lambda_1(0) - \lambda_2(0)| \leq (1/\xi_1)||c_{e1} - c_{e2}||_\infty + (1/\xi_2)|c_{o1} - c_{o2}|$$

where $\xi_1 \in (c_{e1}, c_{e2})$ and $\xi_2 \in (c_{o1}, c_{o2})$. Thus, since $c_e \in \mathcal{M}$ and we take c_o satisfying (6.4) there exist data-independent constants n_1, n_2 such that

$$|g_1'(0) - g_2'(0)| = |\lambda_1(0) - \lambda_2(0)| \leq n_1|c_{o1} - c_{o2}| + n_2||c_{e1} - c_{e2}||_\infty.$$

Thus

$$|g_1'(\tau) - g_2'(\tau)| \le n_1|c_{o1} - c_{o2}| + n_2||c_{e1} - c_{e2}||_\infty + |Q_1(\tau)g_1(\tau) - Q_2(\tau)g_2(\tau)|. \qquad \text{(A.19)}$$

Finally using (3.9) and (A.13) in (A.19), we obtain inequality (6.6). □

Acknowledgments We are grateful to Prof. Pavel Bedrikovetsky for having introduced us to the area of deep bed filtration. Moreover, we are grateful to Eng. A. G. de Siqueira, Dr. Eng. F. Shecaira and Dr. Eng. A. L. Serra for encouragement, support and many discussions.

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