Non-Periodic Bifurcations of One-Dimensional Maps

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Abstract

We prove that a "positive probability" subset of the boundary of "{uniformly expanding circle transformations}" consists of Kupka-Smale maps. More precisely, we construct an open class of 2-parameter families of circle maps $(f_{a,\theta})_{a,\theta}$ such that for a positive Lebesgue measure subset of values of a, the family $(f_{a,\theta})_\theta$ leaves the uniformly expanding domain at a map for which all periodic points are hyperbolic (expanding) and no critical point is pre-periodic. Furthermore, these maps admit an absolutely continuous invariant measure. We also provide information about the geometry of the boundary of the set of hyperbolic maps.

1 Introduction

We begin by formulating the fundamental problem motivating this work and stating our main results.

1.1 Non-periodic bifurcation

A great deal of understanding of those dynamical systems lying beyond the hyperbolic (Axiom A [24]) domain has been achieved through the following strategy: One considers parameterized families $(\Phi_\theta)_{\theta\in[-1,1]}$ of systems (diffeomorphisms or flows) such that Φ_θ is hyperbolic for initial $\theta = -1$, but crosses the boundary of hyperbolic systems at some parameter, say $\theta = 0$. Then one tries to understand the typical behavior of Φ_θ, for small $\theta > 0$.

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This problem is probably not realistic without some further assumptions about the dynamics at $\theta = 0$. In a series of papers that appeared in the seventies and the early eighties, Newhouse, Palis and Takens were able to develop a detailed theory of the case when the limit set (forward and backward accumulation points of all orbits) of Φ_0 is finite. See [14, 15, 16, 17]. In this case, for a generic (Baire second category) set of arcs of diffeomorphisms, the first bifurcation corresponds to one of the following two situations:

(NH) *there exists a unique periodic orbit that is non-hyperbolic,* and it is either a saddle-node (one eigenvalue equal to 1), a period-doubling (one eigenvalue equal to -1), or a Hopf orbit (two complex conjugate eigenvalues with norm 1);

(NT) all the periodic orbits are hyperbolic, but *there exists a unique non-transverse intersection between some stable and some unstable manifold of periodic orbits*; this intersection is quasi-transverse (codimension 1).

These two bifurcation mechanisms (NH) and (NT) may be responsible for drastic changes of the global dynamics, which have been extensively studied by many authors, both for finite and infinite limit set. See [5, 6, 12, 18, 19, 20, 21, 22, 25], among many other works. Moreover, Newhouse, Palis [15] conjectured that the fist bifurcation of generic families starting from a Morse-Smale system (i.e., with no assumption on the limit set at the bifurcation parameter) is either (NH) or (NT). However, to the best of our knowledge there has been no progress in the direction of this conjecture.

In fact, our results in this work provide a *negative* answer to a natural probabilistic reformulation of the conjecture for 1-dimensional maps. That is, we prove that *there exist large* (in a measure theoretical sense) *subsets of the boundary of the hyperbolic domain constituted by maps whose periodic orbits are all hyperbolic and whose critical points are not pre-periodic.* Let us point out that pre-periodic critical points play the role of homoclinic tangencies in the context of 1-dimensional maps. The reason hyperbolicity breaks down for our maps is the presence of some non-periodic cubic critical point.

More precisely, our main result is

Theorem A. *There exists an open set $\mathcal{U}$ of 2-parameters families $(f_{a,\theta})_{a,\theta}$ of maps of the circle such that for a positive Lebesgue measure set $\mathcal{A}$ of parameters a there exists $\theta_* = \theta_*(a)$ for which*

1. *$f_{a,\theta}$ is uniformly expanding for all $\theta < \theta_*$;*

2. *f_{a,θ_*} has a (cubic) critical point (and so can not be uniformly expanding); and*

3. *all periodic points of f_{a,θ_*} are hyperbolic (expanding), and no critical point is pre-periodic.*

In other words, a "positive probability" subset of the boundary of hyperbolic systems consists of Kupka-Smale maps.

For dimension 2, Sabini, in [23], exhibits dissipative diffeomorphisms with non-periodic bifurcation: all periodic orbits are hyperbolic, their stable and unstable manifolds intersect transversally, but there is a large (in a measure theoretical sense) set of diffeomorphisms with cubic tangencies between stable and unstable manifolds, one of them associated to a non-periodic orbit.

1.2 Absolutely continuous invariant measures

Having proved abundance of non-periodic bifurcations, in Theorem A, one would like to say more about these systems, namely, one would like to describe their dynamics at (and, at a second stage, beyond) the bifurcation.

Our next result is a contribution in this direction: we prove that the maps f_{a,θ_*} have well-defined asymptotics at the statistical level. More precisely,

Theorem B. *For every family $(f_{a,\theta})_{a,\theta} \in \mathcal{U}$ with $a \in \mathcal{A}$ there exists a unique absolutely continuous invariant measure for f_{a,θ_*}. This measure μ is supported in all $\mathbb{S}^1$ and is, in fact, a physical (SRB) measure: the time average $n^{-1}\sum_{j=0}^{n-1}\varphi(f_{a,\theta}^j(x))$ converges to the space average $\int \varphi\, d\mu$ for Lebesgue almost every $x \in \mathbb{S}^1$ and every continuous function φ.*

This statement should be related to the remarkable work of Jakobson [8], where he showed that the quadratic family has an absolutely continuous invariant measure for a positive measure set of parameters, see also [2, 4]. More recently, Lyubich [10] and Avila, Moreira [1] have shown that for generic families of unimodal maps, for almost every parameter either there exists an absolutely continuous invariant measure or the map is hyperbolic (the critical point is in the basin of a periodic attractor). Moreover, the second alternative corresponds to an open and dense set of parameters, by Lyubich [9] and Graczyk-Swiatek [7]. Motived by these results, we propose

Problem 1. Does (NH) ∪ (NT) contain an open and dense subset of the boundary of expanding maps of the circle ?

Problem 2. Has (NH) $\cup$ (NT) $\cup$ {maps with a unique critical (cubic) point} full probability in the boundary ?

1.3 On the boundary of hyperbolic maps

Our construction gives one parameter families $(f_{a,\theta})_\theta$ crossing the boundary of hyperbolic maps at $\theta = \theta_*$ as its first bifurcation, that is, all members of this families with $\theta < \theta_*$ are hyperbolic maps.

A natural question is: *Are there* 1*-parameter families* $(f_{a,\theta})_\theta$ *as above crossing the boundary of hyperbolic maps before* $\theta = \theta_*$ *?* The next theorem gives a positive answer for this question.

Theorem C. *For every family* $(f_{a,\theta})_{a,\theta} \in \mathfrak{U}$ *there exists a sequence* $\{a_k\}$ *of* a*-parameters accumulating in some* $\overline{a}$ *such that* $(f_{a_k,\theta})_\theta$ *crosses the boundary of hyperbolic maps for some* $\theta = \theta(a_k)$ *strictly less than* θ_**.*

In fact, we prove that each $f_{a_k,\theta}$ has a periodic hyperbolic attracting point for some $\tilde{\theta}(a_k) < \theta_*$. Since these points are robust we have an open set of parameters presenting this phenomena inside our family. The fact that $f_{a_k,-1}$ is uniformly expanding permits concluding that $f_{a_k,\theta}$ cross the boundary of the set of expanding maps for some $\theta(a_k) < \tilde{\theta}(a_k) < \theta_*$.

Our results suggest a mental picture of the shape of a piece of the border of hyperbolic maps which is reinforced by computational simulations as in Figure 1. It concerns to Arnold-like family of circle maps

$$f_{a,\theta}(x) = kx + a + \theta \sin 2\pi x \mod 1,$$

with $k = 3$, $-2.5 \leq a \leq 2.5$ and $-0.3 \leq \theta \leq 1.6$. The parameters a and θ are in vertical and horizontal directions, respectively. The dark region corresponds to positive Lyapunov exponent and the light region to negative Lyapunov exponent.

1.4 Cubic critical bifurcation

Let us describe an example of an open set where our theorems hold. We say a C^r-map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ has a *cubic critical point at* $x = a$ if $\varphi'(a) = 0$, $\varphi''(a) = 0$ and $\varphi'''(a) \neq 0$. An 1-parameter C^r-family $\varphi_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ starting inside the set of expanding maps presents a *generic cubic criticality at* $(\theta, x) = (\theta_*, a)$ if

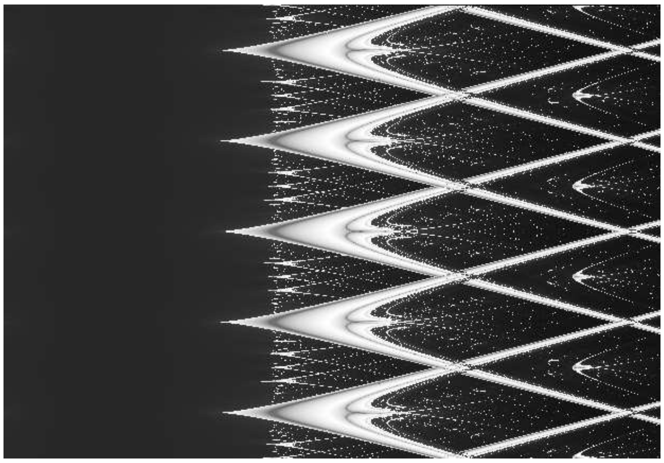

Figure 1: Boundary of hyperbolic maps

- $\varphi'_\theta(x) > 0$ for every $\theta < \theta_*$.
- φ_{θ_*} has a cubic critical point at $x = a$ which is its unique critical point.
- $\partial_\theta \varphi_\theta(a) < 0$.

The parameter θ can be thought as the angle between the graph of φ_θ and the horizontal direction.

We construct 2-parameters families with one parameter having the same hole of θ and another one pointing out for the location of critical points. Let us give the precise definition of the families $(f_{a,\theta})_{a,\theta}$.

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a preserving orientation C^r-expanding map, $r \geq 3$ with a fixed point p with a distinct pre-image $\overline{a}$. We consider a 2-parameter C^r-family $\tilde{f}_{a,\theta} : \mathbb{S}^1 \to \mathbb{S}^1$, $\theta \in [\theta_0, \theta_1]$ and a in some small open interval containing $\overline{a}$ as follows.

(H_1) For every a, $\tilde{f}_{a,\theta_0}$ is uniformly expanding and there is an open interval $I_0 \subset \mathbb{S}^1$ containing $\overline{a}$ such that $\tilde{f}_{a,\theta}$ and f are C^3-close outside I_0.

(H_2) We deform $\tilde{f}_{a,\theta_0}$ inside an open interval $I_a \subset I_0$ containing a in such a way that each family $(\tilde{f}_{a,\theta})_\theta$ presents a generic cubic criticality at $(\theta, x) = (\theta_*, a)$.

(H_3) For every θ, $\tilde{f}_{\overline{a},\theta}(\overline{a}) = p$ and $|\partial_a \tilde{f}_{a,\theta}(x)| > 0$ for every $x \in \mathbb{S}^1$. Moreover, $\tilde{f}'''_{a,0}(\overline{a}) \gg 0$ and $|\tilde{f}'_{a,\theta}(x)| > 1$ for every x outside I_a.

See examples in the next section for this construction. Note that the hypothesis (H_1)–(H_2) are satisfied for an open set of families. Our results apply to every family $(f_{a,\theta})$ in a small C^3-neighborhood $\mathcal{U}$ of $(\tilde{f}_{a,\theta})$.

Up to reparametrization we can take $[\theta_0, \theta_1] = [-1, 1]$ and $\theta_*(a) = 0$ for every a. Thus, a cubic tangency between the graph of $f_{a,\theta}$ and the horizontal direction is created on $(a, f_{a,\theta}(a))$ at $\theta = 0$.

1.5 Examples

The examples introduced in this section can be though as the family satisfying (H_1)–(H_3).

Example 1. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by

$$f(x) = 2x \mod 1.$$

Let $\bar{a} = 1/2$ and $p = 0$. Then $\bar{a}$ is a pre-image of the fixed point p. Let $I_0 = (x_-^0, x_+^0)$ be a small interval centered on $1/2$. Given a close to $1/2$, let $I_a = (x_-^a, x_+^a)$ be a proper subinterval of I_0 containing a.

We can define a 2-parameter family satisfying (H_1)–(H_3) in such a way that

$$\tilde{f}_{a,\theta}(x) = 2a - \theta(x-a) + A(x-a)^3 \mod 1,$$

for every $x \in I_a$.

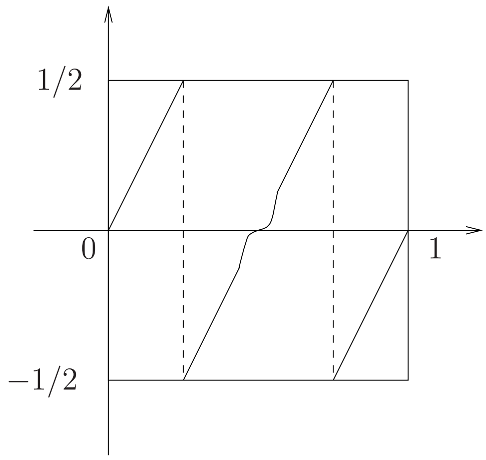

Figure 2: Cubic critical point

Since $\tilde{f}_{a,\theta}$ coincides with f outside I_0, we have $|\tilde{f}'_{a,\theta}(x^0_{\pm})| = 2$. On the other hand, if A is taken large enough,

$$|\tilde{f}'_{a,\theta}(x^a_{\pm})| = -\theta + 3A(x^a_{\pm} - a)^2 > 7/4.$$

Thus, it is possible define $\tilde{f}_{a,\theta}$ in $I_0 \setminus I_a$ such that $|\tilde{f}'_{a,\theta}(x)| \geq \sigma_0 > 1$, for all $x \in I_0 \setminus I_a$.

The next two examples are easier for computational implementation than the previous one and they satisfy conditions (H_1)–(H_3) up to parametrization. In fact, the later example is a C^∞-map defined on the circle, except on a single point.

Example 2 (Arnold-like family). Let $k \geq 2$ be a positive integer. For a close to 0 and θ close to $k/(2\pi)$, we define

$$\tilde{f}_{a,\theta}(x) = kx + a + \theta \sin 2\pi x \mod 1.$$

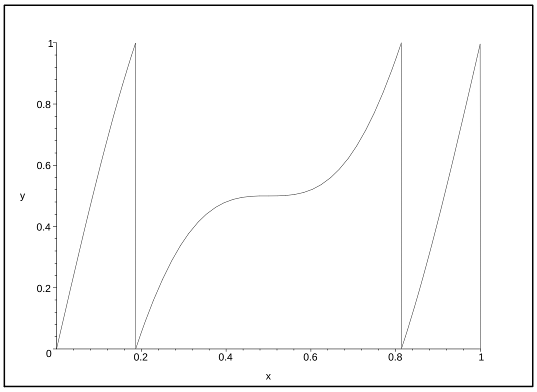

Figure 3: Arnold-like family, $k = 3$, $a = 0$, $\theta = 3/(2\pi)$.

Example 3 (Polynomial family). For a positive integer k we define

$$\tilde{f}_{a,\theta}(x) = 1 + 2k(x^3 + a - \theta x + \theta x^5) \mod 2,$$

where a and θ belong to a neighborhood of 0.

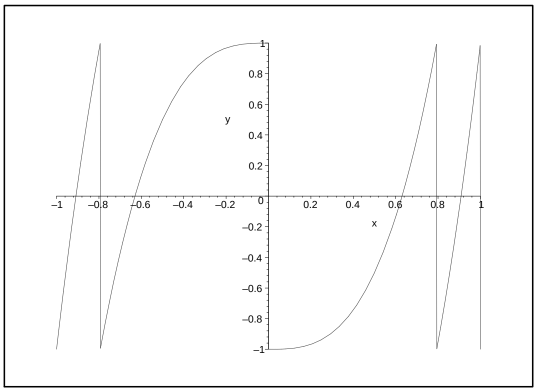

Figure 4: Polynomial family, $k = 1$, $a = 0$, $\theta = 0$.

1.6 Ideas of the proof of Theorem A

We stress that the main difficulty is to prove that our maps in Theorem A are approximated by expanding ones. Next, we give an idea why the proof works.

The proof is by induction based on a Benedicks-Carleson type of argument, see [2, 3, 11]. This construction provides hyperbolicity for elements of the family for certain values of parameters: it can be carried out for a positive Lebesgue measure set of a values. More specifically, uniformly on $\theta < 0$, we get a positive Lebesgue measure set of a values for which $f_{a,\theta}$ is hyperbolic.

Note that, for $\theta < 0$, the map $f_{a,\theta}(x)$ has non-zero derivative for every x in $\mathbb{S}^1$, furthermore the least expansion of the derivative of $f_{a,\theta}$ occurs for $x = a$. This motivates us to call $x = a$ the *critical point* of $f_{a,\theta}$. We want to prove hyperbolicity (i.e., expansion) along the *critical orbit*, that is, the orbit of a. Therefore, an induction process as done in [2] for the critical orbit is what we have in mind.

More specifically, we fix a parameter θ and use a parameter exclusion type of argument to obtain a positive Lebesgue measure set Ω_θ of parameters a such that $f_{a,\theta}$ is a uniformly expanding (hyperbolic) map. The measure of Ω_θ is uniform on θ and so, we get a positive Lebesgue measure subset $\mathcal{A}$ constituted by limit points of the sets Ω_θ as θ goes to zero. Therefore, in this sense, the set of maps $f_{a,\theta}$ such that a belongs to $\mathcal{A}$ is a large subset of the boundary of the expanding maps.

Now we recall basic concepts and issues in [2] we need to deal with. See also [13] for a careful detailed presentation of these ideas.

First of all, let $\xi_n(a) = f^n_{a,\theta}(a)$ stands for the critical orbit. We need to estimate the expansion along this orbit – later we see that expansion for it implies (almost surely) hyperbolicity of $f_{a,\theta}$.

Fixed $\delta > 0$ small, we say that a positive integer n is a *return time* for $f_{a,\theta}$ if $|\xi_n(a) - \xi_0(a)| < \delta$. In this case, as we make precise later, during a certain amount of time called *bound period* the iterates $\xi_{n+1}(a), \xi_{n+2}(a), \ldots$ follow closely the critical orbit (see Section 3). An induction argument can be applied to transport expansion of $f_{a,\theta}$ from $\xi_j(a)$ to $\xi_{n+j}(a)$ in order to recover the possible loss of expansion due to the approximation to the critical point at time n.

After a bound period and before the next return situation there is a period of time called *free period*. The total amount of free periods before n is denoted by $F_n(a)$.

Remark that if $\xi_n(a)$ becomes too much close of $\xi_0(a)$ then a large bound period is required and the induction no longer works. Furthermore, the bound period is intended merely to recover the loss of expansion at time n and so the final expansion is primarily obtained by the total amount $F_n(a)$ of free times. This gives us an idea why the two assumptions below are required for our purpose of induction.

- (BA_n) A *recurrence control:* the returns are not too close to the critical point.

- (FA_n) A *statistical restriction*: a bound of the total amount of time required to recover the loss of derivative at this time and all past times.

At each time n we force (BA_n) and (FA_n) assumptions above to be satisfied by, if necessary, excluding parameters a. The remaining set is denoted by $\Omega_{\theta,n}$ and the intersection for all values of n is Ω_θ.

Exponential growth of derivatives along the critical orbit for those parameters in Ω_θ will be a consequence of the controls inductively imposed.

2 Preliminary results

Here we give some results necessary to prove our theorems.

In what follows we will be working with a fixed $\theta \leq 0$ and so we will write $f_a(x)$ and $\xi_n(a, x)$ instead of $f_{a,\theta}(x)$ and $f^n_{a,\theta}(x)$ whenever possible. Since f_a

is C^1 and from the compactness of $\mathbb{S}^1$ we have $|\partial_a f_a| \leq B$ for some constant $B > 0$.

Lemma 2.1. *Let* $Q_n = \dfrac{\partial_a \xi_n(a,x)}{\partial_x \xi_n(a,x)}$. *Assume that*

i) For each $x \in \mathbb{S}^1$ *and each positive integer* n *there exists a positive number* $M_n(x)$ *such that*

$$|\partial_x \xi_n(a,x)| \geq M_n(x) > 0.$$

ii) For some positive integer N,

$$2B \sum_{n=N}^{\infty} M_n^{-1} \leq |Q_N|.$$

Let $Q = \max\{2|Q_N|^{-1}, 3/2\ |Q_N|\}$. *Then,*

$$Q^{-1} \leq |Q_{n+1}| \leq Q, \quad \textit{for every } n \geq N.$$

Proof. By the chain rule we have

$$\partial_x \xi_n(a,x) = \partial_x[\xi_1(a, \xi_{n-1}(a,x))] = \partial_x \xi_1(a, \xi_{n-1}(a,x)) \partial_x \xi_{n-1}(a,x)$$

and

$$\begin{aligned}\partial_a \xi_n(a,x) &= \partial_a[\xi_1(a, \xi_{n-1}(a,x))] \\ &= \partial_x \xi_1(a, \xi_{n-1}(a,x)) \partial_a \xi_{n-1}(a,x) + \partial_a \xi_1(a, \xi_{n-1}(a,x)).\end{aligned}$$

Thus,

$$Q_n = \frac{\partial_a \xi_n(a,x)}{\partial_x \xi_n(a,x)} = \frac{\partial_a \xi_{n-1}(a,x)}{\partial_x \xi_{n-1}(a,x)} + \frac{\partial_a \xi_1(a, \xi_{n-1}(a,x))}{\partial_x \xi_n(a,x)}.$$

Therefore,

$$Q_n - Q_{n-1} = \frac{\partial_a \xi_1(a, \xi_{n-1}(a,x))}{\partial_x \xi_n(a,x)}.$$

Using (i) and $\xi_1(a,x) = f_a(x)$, we can write

$$|Q_n - Q_{n-1}| \leq \frac{B}{|\partial_x \xi_n(a,x)|} \leq B\, M_n^{-1}(x).$$

By recurrence and assumption (ii), for every $n > N$, we have

$$|Q_n - Q_N| \leq B \sum_{k=N}^{n} M_n^{-1}(x) \leq \frac{|Q_N|}{2}.$$

Therefore,

$$Q^{-1} \leq \frac{1}{2}|Q_N| = |Q_N| - \frac{|Q_N|}{2} \leq |Q_{n+1}| \leq |Q_N| + \frac{|Q_N|}{2} = \frac{3}{2}|Q_N| \leq Q.$$

■

Remark 2.2. If the interval I_a in (H_2) is taken sufficiently small, then for every a close to $\bar{a}$, there exist positive constants $A_1 < A_2 \ll A_3 < A_4$, with $A_2 < 2A_1$ and $A_4 < 2A_3$, such that for every $x \in I_a$ we have

$$-A_1\theta|x-a| + A_3|x-a|^3 \leq |f_{a,\theta}(x) - f_{a,\theta}(a)| \leq -A_2\theta|x-a| + A_4|x-a|^3\,, \quad (1)$$

where the sum is in the circle $\mathbb{S}^1$; and

$$-A_1\theta + 3A_3(x-a)^2 \leq f'_{a,\theta}(x) \leq -A_2\theta + 3A_4(x-a)^2. \quad (2)$$

Moreover, from (H_3), there exists $\sigma_0 > 1$ such that, for every $f_{a,\theta}$ close to $\tilde{f}_{a,\theta}$

$$|f'_{a,\theta}(x)| > \sigma_0\,, \qquad \text{for all} \quad x \in \mathbb{S}^1 \setminus I_a. \quad (3)$$

To the end of this section we consider a fixed parameter $\theta \leq 0$ and so let us write $f_a = f_{a,\theta}$.

Proposition 2.3. *Given $\delta > 0$ sufficiently small there exists $\eta > 0$ and a constant $c > 0$ such that, if*

a) $|f_a^j(x) - a| \geq \delta$ for $0 \leq j < k$; and

b) $|f_a^k(x) - a| < \delta$

then,

$$|(f_a^k)'(x)| \geq e^{ck},$$

whenever $|a - \overline{a}| \leq \eta$ and $-\delta^2 < \theta \leq 0$.

Proof. Since outside I_a the derivatives are uniformly expanding we can assume $x \in I_a$.

For some interval J centered in p, we denote $\sigma_m = \inf_{x \in J} |f_a'(x)|$ and $\sigma_M = \sup_{x \in J} |f_a'(x)|$. One fix J such that $|J| < 1/4$ and

$$l_0 = \frac{\log \sigma_m}{\log \sigma_M} > 2/3.$$

By choosing η small we get $f(I_a) \subset J$ for all a in $(\overline{a} - \eta, \overline{a} + \eta)$.

Since $f_a(x)$ is close to p, during a certain amount of time the orbit of x remains in J. If k_0 is the first iterate of x that escapes from J we must have

$$\sigma_M^{k_0} |f_a(x) - p| \geq |J|. \tag{4}$$

At this moment we have

$$|(f_a^{k_0+1})'(x)| = |(f_a^{k_0})'(f_a(x))||f_a'(x)| \geq \sigma_m^{k_0} |f_a'(x)|. \tag{5}$$

In order to estimate the term $\sigma_m^{k_0}$ we observe from (4) that

$$k_0 \geq \frac{\log(|J|/|f_a(x) - p|)}{\log \sigma_M}$$

and so

$$\log \sigma_m^{k_0} \geq \frac{\log(|J|/|f_a(x) - p|)}{\log \sigma_M} \log \sigma_m = \log(|J|/|f_a(x) - p|)^{l_0}. \tag{6}$$

On the other hand, the derivative in the right side of (5) can be estimated, using (2), as

$$|f_a'(x)| \geq -A_1\theta + 3A_3|x - a|^2 \geq 3A_3|x - a|^2. \tag{7}$$

Therefore, replacing (7) and (6) in (5), we have

$$|(f_a^{k_0+1})'(x)| \geq |J|^{l_0} \frac{3A_3|x - a|^2}{|f_a(x) - p|^{l_0}}. \tag{8}$$

First, we estimate

$$|f_a(x) - p| \leq |f_a(x) - f_a(a)| + |f_a(a) - p|. \tag{9}$$

If f_a is taken close to $\tilde{f}_a$ (see (H3)) and η is sufficiently small, then

$$|f_a(a) - p| \leq |f_a(a) - \tilde{f}_a(a)| + |\tilde{f}_a(a) - \tilde{f}_a(\bar{a})| + |\tilde{f}_a(\bar{a}) - \tilde{f}_{\bar{a}}(\bar{a})| \leq \delta^3. \quad (10)$$

From (1),(9) and (10) and assuming $|x - a| > \delta$ and $\delta^2 > -\theta$, we get

$$\begin{aligned} |f_a(x) - p| &\leq -A_2\theta|x - a| + A_4|x - a|^3 + \delta^3 \\ &\leq (A_2 + A_4 + 1)|x - a|^3 = 3A_3|x - a|^3 . \end{aligned}$$

From this later estimate and (8), we get

$$|(f_a^{k_0+1})'(x)| \geq |J|^{l_0} \frac{3A_3|x - a|^2}{(3A_3|x - a|^3)^{l_0}}.$$

Since $l_0 > 2/3$, by taking A_3 sufficiently large, we can suppose that

$$(3A_3)^{1-l_0}|x - a|^{2-l_0} \geq C|J|^{-l_0},$$

for some constante $C > 1$. Hence, we get

$$|(f_a^{k_0+1})'(x)| \geq C.$$

Furthermore, since $|x - a| > \delta$ we have a uniform bound for k_0 in this last equation which enables us to rewrite it as

$$|(f_a^{k_0+1})'(x)| \geq e^{c(k_0+1)} , \quad (11)$$

for some positive constant c.

Note that we can choose the neighborhood J of p in such a way that $f^{k_0+1}(x) \notin I_a$. To finish the proof we observe that for all $0 \leq j < k$ such that $f_a^j(x)$ belongs to I_a we can use the same arguments before since due to hypothesis (b) we know that $j + k_0 < k$.

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Remark 2.4. Without assuming (b) in the preceding lemma we can easily obtain an estimate for the expanding behavior of f_a like

$$|(f_a^k)'(x)| \geq \delta^2 e^{ck}.$$

Remark 2.5. Note that in the preceding proof we can make k_0 large by shrinking I_a. From this fact we can obtain an estimate for the rate of expansion as, say, $c = 1/10$.

3 The Bound Period

Whenever an iterate of x gets close to a, we have some loss on the expansion of the derivative. On the other hand, by continuity, during some few iterates the orbits of x and a will remain close. Taking into account this basic fact and assuming as an *inductive hypothesis* that the orbit of a is expanding (at least during these iterates) we end up with the trajectory of x recovering its eventual loss on the expansion of its derivative.

In order to present more details about these reasonings, we are going to introduce some terminology.

Fix $\delta = e^{-(\Delta+1)}$, for some $\Delta \in \mathbb{N}$, in such a way that all the results in the previous sections concerning to δ are satisfied.

Given a trajectory $f_a^n(x)$, we can split it using the following approach:

- **Return times.** Those iterates when $f_a^n(x)$ gets close to a. More precisely, we state that n is a return time for the orbit of x if we have $|f_a^n(x) - a| < \delta$.

- **Bound period.** The piece of orbit of x that remains close to the orbit of a. This is accomplished by fixing $\beta > 0$ small and defining

  $$p(x,a) = \max\{k \,:\, |f_a^{n+j}(x) - f_a^j(a)| \le e^{-\beta j}, 0 \le j \le k\}.$$

  The bound period associated to the return n of x is the set of times $n+1, \ldots, n+p$, $p = p(x,a)$.

- **Free period.** Those iterates after a return and its correspondent bound period and before the next return time.

Note that we define these notions for a single orbit. On the other hand, we will need to consider groups of orbits which behave similarly during a certain amount of time. We will use in several places the idea of partitioning the phase space around a critical point in order to achieve this control. We introduce now a model of this type of partitioning which will be slightly reviewed in future sections.

Let a be a parameter under consideration. We can think of a also as a point in the phase space and define a partition of a neighborhood of a by disjoint intervals I_m given by

$$I_m = [a + e^{-(m+1)}, a + e^{-m}), \qquad m \ge \Delta.$$

We extend this definition to negative indexes imposing that I_{-m} and I_m be symmetrical with respect to a. Additionally we write

$$U_m = (a - e^{-m}, a + e^{-m}), \qquad m \geq \Delta.$$

In this partitioning whenever make sense we write

$$U_m^+(a) = I_{-(m-1)} \cup U_m \cup I_{m-1} \quad \text{and} \quad I_m^+(a) = I_{m-1} \cup I_m \cup I_{m+1}\,.$$

As a refinement of this procedure, we divide each interval I_m in m^2 subintervals of equal length which will be denoted by $I_{m,k}, k = 1, \ldots, m^2$.

After all we have,

$$U_\Delta(a) = \bigcup \{I_{m,k} : m \geq \Delta - 1, 1 \geq k \geq m^2\} \cup \{a\}. \tag{12}$$

For special reasons we join to that collection of intervals these two additional ones:

$$I_{\Delta-1,1} = [a + e^{-\Delta}, a + e^{-\Delta} + \tau) \quad \text{and } I_{-(\Delta-1),1} = (a - e^{-\Delta} - \tau, a - e^{-\Delta}],$$

where $\tau = \left|e^{-(\Delta-1)} - e^{-\Delta}\right| / (\Delta - 1)^2$.

Again, taking into account this refinement, we reserve the plus sign $(+)$ to indicate, e.g. by $I_{m,k}^+$, the enlargement of an interval with its two adjacent ones.

At this point let us extend the previous definition of bound period writing

$$p(m, a) = \min_{x \in U_m^+} p(x, a).$$

Note that for some points in U_m^+ we can end up with a shorter bound period than before. Since we need to recover completely the loss of derivatives at return times, we would like to state that this modification does not affect the essential properties of the bound period (see the next proposition). At this point we need to make precise the notion of *recurrence control*: we are assuming

$$|\xi_n(a) - a| \geq e^{-\alpha n}, \tag{BA_n}$$

for some small positive constant α satisfying $\alpha \ll \beta$.

Proposition 3.1. *Let a close to $\overline{a}$, θ negative close to 0, $|m| \geq \Delta$ and $p = p(m, a)$. Then, there is a constant $C_1 = C_1(\beta - \alpha)$ such that:*

a) $\dfrac{1}{C_1} \le \dfrac{|(f_{a,\theta}^j)'(f_{a,\theta}(x))|}{|(f_{a,\theta}^j)'(f_{a,\theta}(a))|} \le C_1\,,$ *for* $j = 1, \ldots, p$ *and every* $x \in U_m^+$*,*

b) $p < 30|m|$*,*

c) $|(f_{a,\theta}^{p+1})'(x)| \ge e^{(1-30\beta)|m|}$ *for every* $x \in I_m^+$*.*

Proof of 3.1(a). Let $x_i = f_{a,\theta}^i(x)$ and $a_i = f_{a,\theta}^i(a)$. Then

$$\frac{|(f_{a,\theta}^j)'(f_{a,\theta}(x))|}{|(f_{a,\theta}^j)'(f_{a,\theta}(a))|} = \prod_{i=1}^{j} \frac{|f'_{a,\theta}(f_{a,\theta}^i(x))|}{|f'_{a,\theta}(f_{a,\theta}^i(a))|} = \prod_{i=1}^{j} \left| 1 + \frac{f'_{a,\theta}(x_i) - f'_{a,\theta}(a_i)}{f'_{a,\theta}(a_i)} \right|. \qquad (13)$$

While the orbit of the critical point avoid the perturbation region I_a, the derivative $f'_{a,\theta}(a_i)$ is uniformly bounded away from zero. The same is true for the derivative of any point between a_i and x_i at least during the bound period (we are assuming $x \in U_m^+$). Thus, if a_i does not belong to the perturbation region, there exists a constant $K > 0$ such that

$$\left| \frac{f'_{a,\theta}(x_i) - f'_{a,\theta}(a_i)}{f'_{a,\theta}(a_i)} \right| \le K e^{-\beta i}.$$

Otherwise, using Taylor's Formula we get, for some positive constant K'

$$\begin{aligned} \left| f'_{a,\theta}(x_i) - f'_{a,\theta}(a_i) \right| &\le \left| f''_{a,\theta}(a)(x_i - a_i) \right| + \left| f'''_{a,\theta}(a)[(x_i - a)^2 - (a_i - a)^2] \right| \\ &\le K' \left| x_i - a_i \right|. \end{aligned}$$

From local form (2), supposing the recurrence control BA_n and the fact that $i \le j \le p$, we get

$$\left| \frac{f'_{a,\theta}(x_i) - f'_{a,\theta}(a_i)}{f'_{a,\theta}(a_i)} \right| \le \frac{K' \left| x_i - a_i \right|}{-A_1\theta + 3A_3(a_i - a)^2} \le \frac{K' e^{-\beta i}}{e^{-2\alpha i}} \le K' e^{-(\beta - 2\alpha) i}.$$

Hence, from (13), we have

$$\begin{aligned} \prod_{i=1}^{j} \left(1 - \left| \frac{f'_{a,\theta}(x_i) - f'_{a,\theta}(a_i)}{f'_{a,\theta}(a_i)} \right| \right) &\le \frac{|(f_{a,\theta}^j)'(f_{a,\theta}(x))|}{|(f_{a,\theta}^j)'(f_{a,\theta}(a))|} \\ &\le \prod_{i=1}^{j} \left(1 + \left| \frac{f'_{a,\theta}(x_i) - f'_{a,\theta}(a_i)}{f'_{a,\theta}(a_i)} \right| \right). \end{aligned}$$

Therefore, as $\beta > 2\alpha$ there is a constant $C_1 > 0$ such that

$$\frac{1}{C_1} \le \frac{|(f^j_{a,\theta})'(f_{a,\theta}(x))|}{|(f^j_{a,\theta})'(f_{a,\theta}(a))|} \le C_1.$$

Proof of 3.1(b). For simplicity of notation let us suppose $m > 0$ (the other case is analogous). Let $x_m = a + e^{-m}$ be in the boundary of $I_m \subset (U^+_m)$ and let $j = \min\{n, p\} - 1$, where n is a return time for x_m. Then, there exists $y \in (f_{a,\theta}(a), f_{a,\theta}(x_m))$ such that

$$|f_{a,\theta}(x_m) - f_{a,\theta}(a)|\ |(f^j_{a,\theta})'(y)| = |f^{j+1}_{a,\theta}(x_m) - f^{j+1}_{a,\theta}(a)| < e^{-\beta(j+1)}. \tag{14}$$

From the local form (1), we have

$$\begin{aligned} |f_{a,\theta}(x_m) - f_{a,\theta}(a)| &\ge -A_1\theta|x_m - a| + A_3|x_m - a|^3 \\ &\ge A_3|x_m - a|^3 \ge A_3 e^{-3m}. \end{aligned} \tag{15}$$

By Proposition 2.3 and part (a) we have

$$|(f^j_{a,\theta})'(y)| \ge C_1^{-1} e^{cj}.$$

Hence, from inequalities (14) and (15) we have

$$C_1^{-1} e^{cj} A_3 e^{-3m} \le e^{-\beta(j+1)}.$$

Then,

$$\log C_1^{-1} A_3 - 3m + cj \le -\beta(j+1).$$

Thus, as $C_1^{-1} A_3 > 1$ ($A_3 \gg 0$), we have

$$j < \frac{3m - \beta}{c + \beta}. \tag{16}$$

The condition (BA_n) besides the fact that n is a return time for x gives

$$e^{-\alpha n} \le |f^n_{a,\theta}(x_m) - a|. \tag{17}$$

On the other hand, if $f^n_{a,\theta}(x_m)$ belongs to I_m then

$$|f^n_{a,\theta}(x_m) - a| < e^{-m}. \tag{18}$$

Therefore, from (17) and (18), it follows that

$$e^{-\alpha n} < e^{-m} < e^{-\Delta}.$$

Thus,

$$\Delta < m < \alpha n.$$

Hence, from (16), we obtain

$$j < \frac{3\alpha n - \beta}{c + \beta} \ll n.$$

This implies that $j = p - 1$. Thus, from Remark 2.5

$$p \leq \frac{3m - \beta}{c + \beta} + 1 \leq 30m.$$

Proof of 3.1(c). Again, without loss of generality, we suppose $m > 0$. It follows from Mean Value Theorem that there exists $y \in U_m^+$ such that

$$|(f_{a,\theta}^{p+1})(U_m^+)| = |(f_{a,\theta}^{p+1})'(y)|\ |U_m^+| = |(f_{a,\theta}^{p})'(f_{a,\theta}(y))|\ |f'_{a,\theta}(y)|\ |U_m^+|.$$

Let x be an arbitrary point of I_m^+. Then, by part (a),

$$|(f_{a,\theta}^{p+1})'(x)| = |(f_{a,\theta}^{p})'(f_{a,\theta}(x))|\ |f'_{a,\theta}(x)| \geq \frac{1}{C_1}|(f_{a,\theta}^{p})'(f_{a,\theta}(y))|\ |f'_{a,\theta}(x)|.$$

The fact that $|f_{a,\theta}^{p+1}(U_m^+)| \geq e^{-\beta(p+1)}$ and $|U_m^+| < 2e^{-m+1}$ implies that

$$|(f_{a,\theta}^{p+1})'(x)| \geq \frac{1}{C_1}\frac{|f_{a,\theta}^{p+1}(U_m^+)|}{|U_m^+|}\frac{|f'_{a,\theta}(x)|}{|f'_{a,\theta}(y)|} \geq \frac{1}{2C_1}\frac{e^{-\beta(p+1)}}{e^{-m+1}}\frac{|f'_{a,\theta}(x)|}{|f'_{a,\theta}(y)|}. \qquad (19)$$

From the local form (2), we have

$$\frac{f'_{a,\theta}(x)}{f'_{a,\theta}(y)} \geq \frac{-A_1\theta + 3A_3(x-a)^2}{-A_2\theta + 3A_4(y-a)^2} \geq \frac{-A_1\theta + 3A_3e^{-2m}}{-A_2\theta + 3A_4e^{-2m+2}}.$$

For $\theta = 0$, we have

$$\frac{f'_{a,0}(x)}{f'_{a,0}(y)} \geq \frac{A_3}{A_4}e^{-2}.$$

By continuity of $f'_{a,\theta}$ and by reducing θ_0 if necessary, we have, for every negative θ close to 0,

$$\frac{f'_{a,\theta}(x)}{f'_{a,\theta}(y)} \geq \frac{A_3}{A_4}e^{-3}. \tag{20}$$

Finally, it follows from (19), (20) and part(b),

$$|(f_{a,\theta}^{p+1})'(x)| \geq \frac{A_3 e^{-3}}{2A_4 C_1}\frac{e^{-\beta(p+1)}}{e^{-m+1}} \geq e^{(1-30\beta)m}.$$

■

4 The induction process

Since we follow classical technics we just outline the proof when it is analogous to the quadratic case. Emphasis is put on those steps where we deal with the specific cubic setting.

In all what follows, consider $\theta \leq 0$ fixed and so let us write $f_a = f_{a,\theta}$.

Recall we have an initial interval of parameters $\Omega_0 = [\bar{a}-\eta, \bar{a}+\eta]$ (where η satisfies Proposition 2.3 and Proposition 3.1) from which we want to extract a positive Lebesgue measure subset of parameters satisfying the conclusions of Theorem A. In order to obtain such a subset we make some inductive parameter exclusion. For these reason we have to consider the dynamics of interval of parameters instead of a single parameter. Thus, we have to get similar results of previous sections for whole intervals of parameters.

Let us adapt some definitions to get the necessary results. First of all, we put $\mathcal{P}_0 = \{\Omega_0\}$. Now suppose that $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{n-1}$ are defined and we will describe how to refine $\mathcal{P}_{n-1}$ in order to construct $\mathcal{P}_n$. For future references we write

$$\Omega_k = \bigcup_{\omega \in \mathcal{P}_k} \omega.$$

Given $\omega = [\omega_-, \omega_+] \in \mathcal{P}_{n-1}$, we define a partition of the neighborhood of radius δ of ω exactly in the same way as we did before for a single parameter a (see Section 3, specially (12)). For example, we write,

$$U_m(\omega) = (\omega_- - e^{-m}, \omega_+ + e^{-m}) \quad \text{and} \quad I_m(\omega) = [\omega_+ + e^{-(m+1)}, \omega_+ + e^{-m}).$$

Following the referred section the meanings of $I_{m,k}(\omega)$, $I^+_{m,k}(\omega)$ and so on must be clear.

From now on, we omit ω from notation whenever the context permits.

In order to describe the construction of $\mathcal{P}_n$ we have to discuss the fate of such an interval ω in $\mathcal{P}_{n-1}$ when we pass from time $n-1$ to time n. The dynamics of ω will be given by

$$\xi(\omega) = \bigcup_{a\in\omega} \{f_a(a)\}$$

and, more generally,

$$\xi_j(\omega) = \bigcup_{a\in\omega} \{f_a^j(a)\}.$$

At time n, if $\xi_n(\omega)$ does not intersect U_Δ we put $\omega \in \mathcal{P}_n$, otherwise we distinguish two cases. In the first one, if $\xi_n(\omega)$ does not cover completely any $I_{m,k}$ we say that n is an *inessential return time* for ω with *host interval* $I_{\bar{m},\bar{k}}$ where $\xi_n(\omega) \subset I^+_{\bar{m},\bar{k}}$. In the remaining case, we say that n is an *essencial return time* for ω and we have a natural way of inducing a partition of it by considering those sub-intervals of ω whose iterates are intervals $I_{m,k}$ defined early. We write

$$\omega = \bigcup \omega'_{m,k}\,, \tag{21}$$

where each $\omega'_{m,k}$ is an interval mapped by ξ_n into $I_{m,k}$. In the case where the extremal intervals of ω do not cover completely an interval $I_{m,k}$ we join them to its adjacent ones. After all, each $\omega'_{m,k}$ is put in $\mathcal{P}_n$ and assigned to the host interval $I_{m,k}$.

Finally, if n is a return time then each subinterval ω' of ω in the case of an essential return and ω itself in the case of an inessential one will be included in $\mathcal{P}_n$ if, and only if, it satisfies a *recurrence control* and a *statistical restriction* to be explained in a little while.

Hence, when time goes on, successive exclusions are performed in the initial interval Ω_0. Carefully estimating the measure of the excluded set at each stage n we are able to show that these measures decrease exponentially fast and so we obtain a positive measure set of parameters with the desired properties.

We want to describe this process in a more precise way. One already have defined the concept of return times for intervals ω. The notion of *bound period* for ω with a host interval $I_{m,k}$ is defined by

$$p(m,\omega) = \min_{a\in\omega} p(m,a). \tag{22}$$

The *free periods* are the complement of return times and bound periods.

Let us make precise the concepts of recurrence control (BA_n) and statistical restriction (FA_n) for the interval of parameters ω:

- (BA_n): $\xi_n(\omega) \cap U_{[\alpha n]}(\omega) = \emptyset$.
- (FA_n): $F_a(n) \geq (1-\alpha)n$, for all $a \in \omega$,

where $F_a(n)$ stands for the amount of free iterates up to time n. More precisely, each $\omega_n \in \mathcal{P}_n$ has a sequence of ancestors,

$$\omega_1 \supset \omega_2 \supset \ldots \supset \omega_n ,$$

with $\omega_i \in \mathcal{P}_i$ and the concept of free iterates refers to the dynamics of these intervals.

Additionally, let us write

- (EG_n): $|(f_a^j)'(a)| \geq e^{cj}$, for $1 \leq j \leq n$ and for all $a \in \omega$.

Once we have established that (22) is in fact an appropriated definition of bound period for all parameter a in $\omega \in \mathcal{P}_{n-1}$ (that is to say, the content of Proposition 3.1 remains valid when we think of ω instead of a single parameter a) then we can use methods of Benedicks-Carleson in [2, 3, 11]: we suppose at time $n-1$ that all ω in $\mathcal{P}_{n-1}$ satisfy (BA_{n-1}), (FA_{n-1}) and (EG_{n-1}) then one choose for $\mathcal{P}_n$ only the subintervals ω' of ω as in (21) satisfying (BA_n) and (FA_n). From that argument, we have

$$(BA_n) + (FA_n) \implies (EG_n).$$

Remark 4.1. It is important to stress now that the classical arguments guarantee that the exponential growth of the derivative until the time $n-1$ supposed by inductive hypothesis can be extend to the time n imposing the recurrence control and the statistical restriction. Hence, after pass n of our induction, whenever we refer to $\omega \in \mathcal{P}_k$, with $k \leq n$, the reader must be aware that this implies $|(f_a^j)'(a)| \geq e^{cj}$ for all j up to time k and for all $a \in \omega$.

This strategy yields a parameter set

$$\Omega = \Omega_\theta = \bigcap_{n \in \mathbb{N}} \Omega_n ,$$

whose associated maps are all expanding along its critical orbit. Moreover, after we have estimated the total measure of parameter intervals excluded we will be able to conclude that $\mathrm{Leb}(\Omega_\theta) > 0$.

Now we have to prove an analogous of Proposition 3.1 for $p(\omega, m)$ as given in (22). A simple analysis show us that items (a) and (b) remains valid but (c) must be reestablished. A trouble raising in this context refers to steps of the proof that deals with estimates for $|U_m^+|$ (see (19)). Note that, in this new setting

$$|U_m^+| = 2e^{-m} + |\omega|. \tag{23}$$

Thus, it is necessary that $|\omega|$ be very small with respect to e^{-m} in order to we can use the same arguments as before. That will be a consequence of the inductive construction (which enforces (BA_n)) and the next lemma.

Lemma 4.2. *If $\omega \in \mathcal{P}_n$ then $|\omega| \le Qe^{-cn}$.*

Proof. We have, for some $a \in \omega$

$$\deg(f) \ge |\xi_n(\omega)| = |\xi_n'(a)|\,|\omega|.$$

Since we are taking a in $\omega \in \mathcal{P}_n$, we know, from our inductive construction, that $|(f_a^j)'(a)| \ge e^{cj}$, for $1 \le j \le n$. This enables us to use Lemma 2.1 and so

$$|\xi_n'(a)| \ge Q^{-1}|(f_a^n)'(a)|.$$

Therefore,

$$|\omega| \le \deg(f)Qe^{-cn}.$$

Just in order to simplify notations, we take $c > 0$ slightly small or modify the constant Q to have the statement.

∎

From the previous lemma and (23) we can state that

$$|U_m^+| \le 4e^{-m}$$

and so the reasonings in the proof of Proposition 3.1 which rely on the size of U_m^+ can be reproduced in this new context.

In order to state that item (c) holds in this new general setting we begin by noting that $p(\omega, m) = p(a, m)$ for some $a \in \omega$. Therefore, writing $p = p(\omega, m)$, we have

$$|(f_a^{p+1})'(x)| \ge e^{(1-30\beta)|m|}.$$

We have to prove that a similar estimate still holds with another arbitrary parameter b in place of a. To accomplish this, we need to prove that derivatives during the bound period are comparable in space parameter.

Let $b \in \omega$ be arbitrary. Then

$$\frac{(f_a^{p+1})'(x)}{(f_b^{p+1})'(y)} = \prod_{j=1}^{p+1} \frac{f_a'(f_a^j(x))}{f_b'(f_b^j(y))} = \prod_{j=1}^{p+1} \left(1 + \frac{f_a'(f_a^j(x)) - f_b'(f_b^j(y))}{f_b'(f_b^j(y))} \right).$$

We write $x_{a,j} = f_a^j(a)$ and $y_{b,j} = f_b^j(b)$. And taking into account that the maps f_a and f_b are C^2 we can choose z_j between $x_{a,j}$ and $y_{b,j}$, $t \in [a,b] \subset \omega$ and a positive constant K such that

$$\begin{aligned}
\frac{|f_a'(x_{a,j}) - f_b'(y_{b,j})|}{|f_b'(y_{b,j})|} &= \frac{|f_a'(x_{a,j}) - f_a'(y_{b,j}) + f_a'(y_{b,j}) - f_b'(y_{b,j})|}{|f_b'(y_{b,j})|} \\
&\leq \frac{|f_a''(z_j)|\ |x_{a,j} - y_{b,j}| + |\partial_a f_t'(y_{b,j})|\ |a-b|}{|f_b'(y_{b,j})|} \\
&\leq \frac{K(|x_{a,j} - y_{b,j}| + |a-b|)}{|f_b'(y_{b,j})|}.
\end{aligned}$$

We have $|f_b'(y_{b,j})| \geq C|y_{b,j} - b|^2$ and

$$\begin{aligned}
|y_{b,j} - b| &\geq |f_b^j(b) - b + y_{b,j} - f_b^j(b)| \\
&\geq e^{-\alpha j} - e^{-\beta j} \geq (1 - e^{-(\beta-\alpha)j})e^{-\alpha j}.
\end{aligned}$$

Moreover, from our definition of bound period, and since $x, y \in I_m^+$ and $a, b \in \omega$, we have

$$|x_{a,j} - y_{b,j}| + |a-b| \leq \tilde{C}(e^{-\beta j} + e^{-cn}) < 2\tilde{C}e^{-\beta j} \qquad j = 1, \ldots, p,$$

for some $\tilde{C} > 0$.

From these last two estimates we have

$$\frac{|f_a'(x_{a,j}) - f_b'(y_{b,j})|}{|f_b'(y_{b,j})|} \leq 4KC^{-1}\tilde{C}e^{-(\beta-2\alpha)j}$$

and so

$$\sum_{j=1}^{\infty} \frac{|f_a'(x_{a,j}) - f_b'(y_{b,j})|}{|f_b'(y_{b,j})|} < \infty,$$

what it is suffices to get bounded distortion as announced.

For future references let us state what we have just proved.

Proposition 4.3. *Let $\omega \subset \mathcal{P}_n$, $|m| \geq \Delta$ and $p = p(m, \omega)$. Then, there are constants $C_1 = C_1(\beta - \alpha)$ and $C_3 > 0$ such that for all $a \in \omega$ we have*

a) $\dfrac{1}{C_1} \leq \dfrac{|(f^j_{a,\theta})'(f_{a,\theta}(x))|}{|(f^j_{a,\theta})'(f_{a,\theta}(a))|} \leq C_1,$ *for $j = 1, \ldots, p$ and every $x \in U^+_m(\omega)$,*

b) $p < 30|m|$,

c) $|(f^{p+1}_{a,\theta})'(x)| \geq e^{(1-30\beta)|m|}$ *for every $x \in I^+_m(\omega)$.*

Remark 4.4. Note that item (b) and (c) above imply that there exists a constant $\sigma_1 > 1$ such that

$$|(f^{p+1}_{a,\theta})'(x)| \geq \sigma_1^{p+1}, \qquad \text{for all } x \in I^+_m(\omega).$$

Another useful information about the dynamics of parameter intervals is a comparison between the sizes of its iterates in different moments. This is the content of the next lemma.

Lemma 4.5. *Let $\omega \in \mathcal{P}_n$ and $i < j \leq n$ where $|\xi_n(\omega)| < 1$. Then, for some $t \in \xi_i(\omega)$, we have*

$$\frac{1}{Q^2}|(f^{j-i}_t)'(t)| \leq \frac{|\xi_j(\omega)|}{|\xi_i(\omega)|} \leq Q^2|(f^{j-i}_t)'(t)|.$$

Proof. We choose arbitrarily $a, b \in \omega$ and write $a_j = f^j_a(a)$ and $b_j = f^j_b(b)$. Since by induction hypothesis we have expansion up to n and $\xi_n(\omega)$ does not cover $\mathbb{S}^1$ then $\xi_i : \omega \to \xi_i(\omega)$ and $\xi_j : \omega \to \xi_j(\omega)$ are diffeomorphisms. Thus,

$$|a_j - b_j| = |\xi_j \circ \xi_i^{-1}(a_i) - \xi_j \circ \xi_i^{-1}(b_i)|.$$

By the Mean Value Theorem, there exists t between a and b such that

$$|a_j - b_j| = |(\xi_j \circ \xi_i^{-1})'(t)|\ |a_i - b_i| = \frac{|\xi_j'(t)|}{|\xi_i'(t)|}\ |a_i - b_i|.$$

Since $a, b \in \omega$ are taken arbitrarily the claim follows from Lemma 2.1.

■

5 Parameter exclusions

In order to estimate the measure of the subset of excluded parameters we have to deal with the size of the iterates of the intervals we are following as well as the frequency for which these returns are *well* sized.

This is accomplished by considering the components of intervals ω that are mapped to extremes of the critical region. More precisely, consider $\omega \in \mathcal{P}_{n-1}$ and its associated partition $\{I_{m,k}\}_{m,k}$. Suppose that n is a return time for ω. If $\omega' \subset \omega$ has a host interval satisfying $m = \Delta - 1$ then we call n an *escape time* for all $a \in \omega'$ and the interval ω' itself will be referred to as an *escape component*.

The next lemma state that an escape component will return large. This implies that with high probability new escape components will be created.

Lemma 5.1. *Let $\omega \in \mathcal{P}_n$ an escape component and μ its next return time. Then, for some $\rho > 0$*

$$|\xi_\mu(\omega)| \geq e^{-\rho\Delta}.$$

Proof. We can suppose $|\xi_n(\omega)| < 1$ since otherwise the claim is obvious.

Recall that $\Delta + 1 = -\log\delta$ and note $\tilde{\delta} = \omega_+ + \delta$ or $\tilde{\delta} = \omega_- - \delta$ belongs to $\xi_n(\omega)$. In order to simplify our proof let us consider the first case since the other is similar.

Let p_a be the fixed point of f_a which is the analytical continuation of p. Let $a \in \mathcal{P}_n$ be a parameter for which $\xi_n(a) = \tilde{\delta}$. Then, using the local form (1), we have

$$|f_a(\tilde{\delta}) - f_a(a)| \leq -A_2\theta(\delta + |\omega|) + A_4(\delta + |\omega|)^3\,.$$

From (10) and from the fact that $p_a \approx p$ we have

$$\begin{aligned} |f_a(\tilde{\delta}) - p_a| &\leq |f_a(\tilde{\delta}) - f_a(a)| + |f_a(a) - p| + |p - p_a| \\ &\leq -2A_2\theta\delta + 2A_4\delta^3 + 2\delta^3. \end{aligned}$$

Since $-\theta \leq \delta^2$,

$$|f_a(\tilde{\delta}) - p_a| \leq C_0\delta^3,$$

for some constant $C_0 > 0$.

Hence, if $M = \sup\{f_a'(x)\}$, then

$$|f_a^k(\tilde{\delta}) - f_a^k(p_a)| \leq M^{k-1}|f_a(\tilde{\delta}) - p_a| \leq C_0M^{k-1}\delta^3.$$

Therefore, taking $C = C_0 M^{-1}$,

$$|f_a^{\mu-n}(\tilde{\delta}) - p_a| \leq CM^{\mu-n}\delta^3.$$

If we suppose that the next return μ occurs while

$$CM^{\mu-n}\delta^3 < \frac{1}{4},$$

then $\xi_\mu(\omega)$ has an extremal point, $f_a^{\mu-n}(\tilde{\delta})$, far from the return region, and contains some point inside the return region. Otherwise,

$$CM^{\mu-n}\delta^3 \geq \frac{1}{4} \tag{24}$$

implies

$$\mu - n \geq \frac{-\log(4C\delta^3)}{\log M}.$$

Moreover, by Proposition 4.3 and Lemma 4.5, we have

$$\frac{|\xi_\mu(\omega)|}{|\xi_n(\omega)|} \geq Q^{-2}|(f_a^{\mu-n})'(\xi_n(a))| \geq Q^{-2}C_1|(f_a^{\mu-n})'(\xi_n(\tilde{\delta}))|.$$

Since n is an escape time for ω, we have $\xi_n(\omega)$ outside $(a-\delta, a+\delta)$ and so, by Proposition 2.3, all orbits starting at $\xi_n(\omega)$ expand uniformly from a return situation to the next one. Using Lemma 2.1 the same holds on parameter space. Thus,

$$\frac{|\xi_\mu(\omega)|}{|\xi_n(\omega)|} \geq C_1 Q^{-2} e^{c(\mu-n)}. \tag{25}$$

By hypothesis, n is an escape time for ω. Then

$$|\xi_n(\omega)| \geq \frac{e^{-\Delta}}{\Delta^2}.$$

Therefore from (24) and (25), there exists a constant $\rho > 0$ such that (recall $e^{-(\Delta+1)} = \delta$)

$$\begin{aligned}|\xi_\mu(\omega)| &\geq C_1 Q^{-2} e^{-c\frac{\log(4C\delta^3)}{\log M}}\,|\xi_n(\omega)| \geq C_1 Q^{-2} e^{-c\frac{\log(4C\delta^3)}{\log M}}\frac{e^{-\Delta}}{\Delta^2} \\ &\geq \tilde{C} e^{\frac{3c(\Delta+1)}{\log M}}\frac{e^{-\Delta}}{\Delta^2} \geq \tilde{C}\frac{e^{\frac{3c-1}{\log M}\Delta}}{\Delta^2} \geq e^{-\rho\Delta},\end{aligned}$$

where $\tilde{C}$ is a positive constant. ■

The next lemma points toward an exponentially fast growth of the intervals of the partition $\mathcal{P}_n$. In fact, it provides the essential estimates that enable us conclude that.

Lemma 5.2. *Let μ be a return for $\omega \in \mathcal{P}_n$ with host interval $I_{m,k}$ and p its binding period. Then, if $\mu' \leq n$ is the next return after μ and $q = \mu' - \mu - p - 1$, we have*

(a) $|\xi_{\mu'}(\omega)| \geq e^{cq} e^{(1-50\beta)|m|)} |\xi_\mu(\omega)| \geq 2|\xi_\mu(\omega)|$;

(b) $|\xi_{\mu'}(\omega)| \geq e^{cq} e^{-50\beta|m|}$, *if μ is an essential return;*

Proof. Write
$$\frac{|\xi_{\mu'}(\omega)|}{|\xi_\mu(\omega)|} = \frac{|\xi_{\mu'}(\omega)|}{|\xi_{\mu+p+1}(\omega)|} \frac{|\xi_{\mu+p+1}(\omega)|}{|\xi_\mu(\omega)|}.$$
Then, from Lemma 4.5 and Proposition 4.3, we get
$$\begin{aligned}\frac{|\xi_{\mu'}(\omega)|}{|\xi_\mu(\omega)|} &\geq Q^{-4} |(f_t^q)'(\xi_{\mu+p+1}(t))| \; |(f_{t'}^{p+1})'(\xi_\mu(t'))| \\ &\geq Q^{-4} |(f_t^q)'(\xi_{\mu+p+1}(t))| e^{(1-30\beta)|m|}.\end{aligned}$$
Part (a) follows from
$$|(f_t^q)'(\xi_{\mu+p+1}(t))| \geq e^{cq}.$$
Furthermore, if μ is an essential return then part (b) is a consequence of
$$|\xi_\mu(\omega)| \geq \frac{e^{-|m|}}{m^2}.$$
■

Remark 5.3. It follows from the preceding lemma and the induction hypothesis (FA_n) that if μ is a return time for $\omega \in \mathcal{P}_n$ then
$$|\xi_\mu(\omega)| \geq e^{c(1-\alpha)\mu} \, |\omega| \, .$$

On returns we prove $|\xi_\mu(\omega)|$ is very large when compared with $|U_{[\alpha n]}|$ which is an important result to estimate exclusions due to (BA_n) condition.

Lemma 5.4. *Suppose n is a return time for $\omega \in \mathcal{P}_{n-1}$. Then*

$$|\xi_n(\omega)| \geq e^{-\frac{1}{2}\alpha n}.$$

Proof. Take the first $\mu \leq n-1$ satisfying $\omega_\mu = \omega$. Then $\xi_\mu(\omega)$ contains some host interval $I_{m,k}$ by construction. If we suppose that μ is an escape time, that is, $|m| = \Delta - 1$ then, from Lemma 5.1, $|\xi_n(\omega)| \geq e^{-\rho\Delta} \geq e^{-\frac{1}{2}\alpha n}$ if n is large enough.

Otherwise, due to the (BA_n) condition, $|\alpha n - 1| \geq |m| \geq \Delta$. Let $\mu_0 = \mu$, $\mu_{s+1} = n$ and μ_i the returns of ω after μ then, from Lemma 5.2, we have

$$|\xi_n(\omega)| \geq |\xi_{\mu_1}(\omega)| \prod_{1}^{s-1} \frac{|\xi_{\mu_{i+1}}(\omega)|}{|\xi_{\mu_i}(\omega)|} \frac{|\xi_{\mu_{s+1}}(\omega)|}{|\xi_{\mu_s}(\omega)|} \geq e^{-50\beta|m|} 2^{s-1}.$$

Therefore,

$$|\xi_n(\omega)| \geq e^{-50\beta(\alpha n)} \geq e^{-\frac{1}{2}\alpha n}.$$

∎

Proposition 5.5. *There exists a constant C depending only on α and β such that if $\omega \in \mathcal{P}_{n-1}$ and n is a return time for ω then for all $a, b \in \omega$ we have*

$$\frac{|(f_a^n)'(a)|}{|(f_b^n)'(b)|} \leq C \qquad \text{and} \qquad \frac{|\xi_n'(a)|}{|\xi_n'(b)|} \leq C.$$

Proof. By Lemma 2.1 it is suffices to prove the left inequality.

Denote by μ_i, $i = 1, \ldots, s$, the returns of ω up to time n with corresponding bound periods of length p_i, host intervals $I_{m_1,k_1}, \ldots, I_{m_s,k_s}$ and $\sigma_i = \xi_{\mu_i}(\omega)$. Let $\mu_0 = 1$, $p_0 = -1$.

We have

$$\frac{(f_a^n)'(a)}{(f_b^n)'(b)} = \prod_{j=0}^{n-1} \frac{f_a'(f_a^j(a))}{f_b'(f_b^j(b))} = \prod_{j=0}^{n-1} \left(1 + \frac{f_a'(f_a^j(a)) - f_b'(f_b^j(b))}{f_b'(f_b^j(b))}\right).$$

Hence, it is sufficient to show that

$$\sum_{j=0}^{\infty} \frac{f_a'(f_a^j(a)) - f_b'(f_b^j(b))}{f_b'(f_b^j(b))} < \infty. \tag{26}$$

Let us write $a_j = f_a^j(a)$ and $b_j = f_b^j(b)$. The fact that $f_a(x)$ is a C^2-map implies that there exist $t \in [a,b] \subset \omega$ and z_j between a_j and b_j such that

$$
\begin{aligned}
S_j = \frac{|f_a'(a_j) - f_b'(b_j)|}{|f_b'(b_j)|} &= \frac{|f_a'(a_j) - f_a'(b_j) + f_a'(b_j) - f_b'(b_j)|}{|f_b'(b_j)|} \\
&\leq \frac{|f_a''(z_j)|\ |a_j - b_j| + |\partial_a f_t'(b_j)|\ |a-b|}{|f_b'(b_j)|}.
\end{aligned}
$$

For estimating $\sum S_j$ on return times, assuming that $j = \mu_i$ for some $i = 1, \ldots, s$ then $b_j \in I_{m_i,k_i}$ and so the derivative $|f_b'(b_j)| \geq (e^{-|m_i|})^2 \geq |I_{m_i}|^2$. On the other hand, the factors $|f_a''(z_j)|$ and $|\partial_a f_t'(b_j)|$ are bounded simultaneously by $C|I_{m_i}|$ for some constant C. Furthermore by Lemma 5.2 the size of $\xi_j(\omega)$ gets expanded between returns and so $|a-b| < |a_j - b_j|$ for all j. Hence we can write

$$
S_{\mu_i} \leq \frac{C|I_{m_i}||a_j - b_j|}{|I_{m_i}|^2} \leq \frac{C\sigma_i}{|I_{m_i}|}.
$$

We group the remaining terms of $\sum S_j$ taking into account the bound periods and free times. One consider the partial sums for $i = 0, \ldots, s-1$

$$
S_i^b = \sum_{j=\mu_i+1}^{\mu_i+p_i} S_j \qquad \text{and} \qquad S_i^f = \sum_{j=\mu_i+p_i+1}^{\mu_{i+1}-1} S_j.
$$

Let us estimate S_i^f for $i = 0, \ldots s-1$ and $\mu_i + p_i + 1 \leq j \leq \mu_{i+1}$. Again by Lemma 5.2 we have $|a-b| < |a_j - b_j|$ for all such index j. Additionally we have a uniform bound K for $|f_a''(z_j)| + |\partial_a f_t'(b_j)|$ and so

$$
S_j \leq K \frac{|a_j - b_j|}{|f_b'(b_j)|}. \tag{27}
$$

From Lemma 4.5 there exists $t_j \in \omega$ such that

$$
\frac{|a_{\mu_i+1} - b_{\mu_i+1}|}{|a_j - b_j|} \geq \frac{1}{Q^2} |(f_{t_j}^{\mu_{i+1}-j})'(\xi_j(t_j))| \geq Q^{-2} e^{c(\mu_{i+1}-j)},
$$

where the last inequality is due to the fact that j is a free time. Therefore, since j is a free time (and then $|f_b'(b_j)| \geq M\delta^2$), we have

$$
\frac{|a_j - b_j|}{|f_b'(b_j)|} \leq \frac{Q^2}{M\delta^2} e^{-c(\mu_{i+1}-j)}\ |\sigma_{i+1}|.
$$

Hence, there exists a positive constant $\tilde{C}_0$ such that

$$S_i^f \leq K\frac{Q^2}{\delta^2}|\sigma_{i+1}|\sum_{j=\mu_i+p_i+1}^{\mu_{i+1}-1} e^{c(j-\mu_{i+1})} \leq \tilde{C}_0|I_{m_{i+1}}|\frac{|\sigma_{i+1}|}{|I_{m_{i+1}}|} \leq c_f\frac{|\sigma_{i+1}|}{|I_{m_{i+1}}|},$$

for some positive constant c_f.

In order to estimate S_i^b we first observe that if $\mu_i + 1 \leq j \leq \mu_i + p_i$ then we can estimate using Remarks 2.4 and 5.3 we get

$$|a_j - b_j| \geq \delta^2 e^{-2(\mu_i+1)}\,|a_{\mu_i} - b_{\mu_i}| \geq \delta^2 e^{-2(\mu_i+1)} e^{c(1-\alpha)\mu_i}\,|a-b|\,.$$

Hence $|a_j - b_j| \geq |a-b|$ and we have the same bound for S_j as given in (27).

We write

$$\frac{|a_j - b_j|}{|f_b'(b_j)|} = \frac{|a_j - b_j|}{|b_j - b_{j-\mu_i}|}\frac{|b_j - b_{j-\mu_i}|}{|f_b'(b_j)|}. \tag{28}$$

From the definition of bound period we have

$$|b_j - b_{j-\mu_i}| \leq e^{-\beta(j-\mu_i)}.$$

Moreover, as $|f_b'(b_j)| \geq M|b_j - b|^2$ and

$$\begin{aligned}|b_j - b| &\geq |b_{j-\mu_i} - b + b_j - b_{j-\mu_i}| \geq |b_{j-\mu_i} - b| - e^{-\beta(j-\mu_i)}\\ &\geq e^{-\alpha(j-\mu_i)} - e^{-\beta(j-\mu_i)} \geq (1 - e^{-(\beta-\alpha)})e^{-\alpha(j-\mu_i)}.\end{aligned}$$

Hence, if $M_1 = (M(1 - e^{-(\beta-\alpha)})^2)^{-1}$, then

$$\frac{|b_j - b_{j-\mu_i}|}{|f_b'(b_j)|} \leq M_1\ e^{-(\beta-2\alpha)(j-\mu_i)}.$$

In order to bound the other factor in (28), we use Lemma 4.5 to get, for some $t \in \omega$,

$$\frac{|a_j - b_j|}{|a_{\mu_i} - b_{\mu_i}|} \leq Q^2|(f_t^{j-\mu_i-1})'(f_t(t_{\mu_i}))|\ |f_t'(t_{\mu_i})|. \tag{29}$$

There exists $x \in U_m^+$ such that

$$|b_j - b_{j-\mu_i}| = |(f_b^{j-\mu_i-1})'(f_b(x))|\ |f_b(b_{\mu_i}) - f_b(b)|. \tag{30}$$

From the local form (1), we have

$$|f_t'(t_{\mu_i})| \leq -\theta A_2 + 3A_4(t_{\mu_i} - t)^2$$

and

$$\begin{aligned}|f_b(b_{\mu_i}) - f_b(b)| &\geq -\theta A_1 \, |b_{\mu_i} - b| + A_3 \, |b_{\mu_i} - b|^3 \\ &= |b_{\mu_i} - b| \; (-\theta A_1 + A_3(b_{\mu_i} - b)^2).\end{aligned}$$

Thus, for $\theta \leq 0$, we have

$$\begin{aligned}\frac{|f_t'(t_{\mu_i})|}{|f_b(b_{\mu_i}) - f_b(b)|} &\leq \frac{1}{|b_{\mu_i} - b|} \frac{-\theta A_1 + 3A_3(t_{\mu_i} - t)^2}{-\theta A_2 + A_4(b_{\mu_i} - b)^2} \\ &= \frac{1}{|b_{\mu_i} - b|} \left(\frac{-\theta A_1}{-\theta A_2 + A_4(b_{\mu_i} - b)^2} + \frac{3A_3(t_{\mu_i} - t)^2}{-\theta A_2 + A_4(b_{\mu_i} - b)^2} \right) \\ &\leq \frac{1}{|b_{\mu_i} - b|} \left(\frac{A_1}{A_2} + \frac{4A_3|t_{\mu_i} - t|^2}{A_4|b_{\mu_i} - b|^2} \right),\end{aligned}$$

whenever θ is sufficiently small. Since μ_i is a return time for ω, b_{μ_i} and t_{μ_i} belongs to the same host interval $I_{m_i,k}$. The distance from $I_{m_i,k}$ to ω is greater than e^{-m+1} and less than $e^{-m} + |\omega|$. Hence, there is a positive constant A such that

$$\frac{|f_t'(t_{\mu_i})|}{|f_b(b_{\mu_i}) - f_b(b)|} \leq \frac{A}{|b_{\mu_i} - b|}. \tag{31}$$

Thus, from (29) and (31) we have

$$\frac{|a_j - b_j|}{|f_b(b_{\mu_i}) - f_b(b)|} \leq AQ^2 |(f_t^{j-\mu_i-1})'(f_t(t_{\mu_i}))| \, \frac{|a_{\mu_i} - b_{\mu_i}|}{|b_{\mu_i} - b|}. \tag{32}$$

Therefore, from using (30) and (32) we obtain

$$\frac{|a_j - b_j|}{|b_j - b_{j-\mu_i}|} \leq AQ^2 \frac{|(f_t^{j-\mu_i-1})'(f_t(t_{\mu_i}))|}{|(f^{j-\mu_i-1})'(f_b(x))|} \frac{|a_{\mu_i} - b_{\mu_i}|}{|b_{\mu_i} - b|}.$$

Since $f_t(t_{\mu_i})$ and $f_b(x)$ are far from the perturbation region, we conclude that there is a positive constant A_0 such that

$$\frac{|a_j - b_j|}{|b_j - b_{j-\mu_i}|} \leq A_0 \frac{|a_{\mu_i} - b_{\mu_i}|}{|b_{\mu_i} - b|}.$$

Hence,

$$S_i^b \leq c_b \frac{|\sigma_i|}{|I_{m_i}|},$$

for some positive constant c_b.

Thus, there exists a positive constant C, such that

$$S^f_{i-1} + S_{\mu_i} + S^b_i \leq C \frac{|\sigma_i|}{|I_{m_i}|}.$$

The fact that $|\sigma_{j+1}| \geq 2|\sigma_j|$ implies that, for each fixed m the sequence of all $|\sigma_{m_i}|$, $\sigma_{m_i} \subset I_m$, is geometrically increasing. Hence, the sum of $|\sigma_{m_i}|/|I_m|$ over the corresponding values of i is bounded by a multiple of the last term:

$$\sum_{i=1}^{s} \frac{|\sigma_i|}{|I_{m_i}|} = \sum_{m>0} \sum_{i:m_i=m} \frac{|\sigma_i|}{|I_m|} \leq \sum_{m>0} \frac{|\sigma_{i(m)}|}{|I_m|},$$

where, by definition, $i(m)$ is the largest value of i for which $m_i = m$. By construction, σ_i belongs to some $I^+_{m,k}$. Then, there is a positive constant C_0 such that

$$\frac{|\sigma_{i(m)}|}{|I_m|} \leq \frac{|I^+_{m,k}|}{|I_m|} \leq \frac{C_0}{m^2}.$$

Therefore,

$$S = \sum_{j=1}^{s} S^f_{i-1} + S_{\mu_i} + S^b_i \leq C \sum_{i=1}^{s} \frac{|\sigma_i|}{|I_{m_i}|} \leq C_0 \sum_{m>0} \frac{1}{m^2} < \infty.$$

and the proof is complete. ■

6 Estimating parameter exclusions

Recall that Ω_n is the set of the remaining parameters at stage n. Let us write Ω'_{n+1} for the set of parameters just obtained by imposing the (BA_n) condition to Ω_n as described before.

Lemma 6.1. *For n sufficiently large it holds*

$$\mathrm{Leb}(\Omega_n \setminus \Omega'_{n+1}) \leq e^{-\epsilon n}\, \mathrm{Leb}(\Omega_n).$$

for some fixed $\epsilon > 0$.

Proof. Suppose n is a return time for ω where we exclude an interval $\bar{\omega}$ imposing the (BA_n) restriction. Then $|\xi_n(\bar{\omega})| \leq 2e^{-\alpha n+1}$.

Therefore

$$\frac{|\bar{\omega}|}{|\omega|} \leq \frac{|\xi_n'(t)|\ |\xi_n(\bar{\omega})|}{|\xi_n'(\bar{t})|\ |\xi_n(\omega)|},$$

for some $t \in \omega$ and $\bar{t} \in \bar{\omega}$.

Using the bounded distortion given by Proposition 5.5 and Lemma 5.4, we have

$$\frac{|\bar{\omega}|}{|\omega|} \leq \frac{2Ce^{1-\alpha n}}{e^{-\frac{1}{2}\alpha n}} = 2Ce^{1-\frac{1}{2}\alpha n} \leq e^{-\frac{1}{3}\alpha n},$$

whenever n is sufficiently large. This implies the statement. ∎

Since we construct Ω_{n+1} from Ω'_{n+1} by imposing the (FA_n) condition, all that remains to be proved is that $\mathrm{Leb}(\Omega'_{n+1} \setminus \Omega_{n+1})$ decreases exponentially fast with n. We use a large deviation argument that can be carried out exactly in the same way as in [3, 11] as soon as we have some basic results. Among these facts, we already have established estimates for the growth of $|\xi_n(\omega)|$ between returns (see Lemma 5.2). Now we turn our attention to the frequency of returns. This is motivated by the following considerations: the assumption (FA_n) is related to the amount of time the orbits remain inside free periods. Good trajectories are those that spend a small proportion of time recovering losses of derivative. Among the iterates which contribute with real expansion, we have those pieces of orbits which are escape periods, that is to say, those iterates between successive escape times. In order to estimate the total amount of time spent in escape periods, we need to estimate the expected time between escape situations and this can be carried out if we can relate the length of orbit between successive returns to the distance of a return to the critical point. This is the content of the next lemma.

Lemma 6.2. *Suppose n is an essential return for the interval $\omega \in \mathcal{P}_n$ with $I_{m,k} \subset |\xi_n(\omega)|$. Then the next return situation ν satisfies*

$$\nu < n + 50|m|.$$

Proof. In order to simplify notations we suppose $m > 0$. First of all, let $n = \mu_0 < \mu_1 < \ldots < \mu_s < \mu_{s+1} = \nu$ be the intermediate inessential returns with host intervals I_{m_i,k_i}. Moreover, for all $i = 0, \ldots, s$, we write p_i for the bound period and $q_i = \mu_{i+1} - (\mu_i + p_i + 1)$ for the free period. Again, we write simply m_i instead of $|m_i|$.

By Lemma 5.2, we have

$$|\xi_{\mu_1}(\omega)| \geq e^{-50\beta m} e^{cq_0}.$$

and

$$\frac{|\xi_{i+1}(\omega)|}{|\xi_i(\omega)|} \geq e^{cq_i} e^{(1-50\beta)m_i}.$$

Therefore,

$$|\xi_{\mu_{s+1}}(\omega)| = |\xi_{\mu_1}(\omega)| \prod_{i=1}^{s} \frac{|\xi_{\mu_{i+1}}(\omega)|}{|\xi_{\mu_i}(\omega)|} \geq e^{-50\beta m} e^{cq_0} \prod_{i=1}^{s} e^{cq_i} e^{(1-50\beta)m_i}.$$

Now, observe we have a natural bound

$$|\xi_{\mu_{s+1}}(\omega)| \leq \deg(f).$$

By taking Δ sufficiently large we can suppose $m \gg \deg(f)$ $(m \geq \Delta)$. Taking logarithms above, we get

$$\sum_{i=0}^{s} cq_i + \sum_{i=1}^{s} (1-50\beta)m_i \leq \log(\deg(f)) + 50\beta m. \tag{33}$$

Furthermore, from Remark 2.5 we have $c > 1/25$. Then,

$$\nu - n = \sum_{i=0}^{s} (p_i + q_i + 1) \leq \sum_{i=0}^{s} (p_i + 50cq_i).$$

But, from Proposition 4.3, we have

$$p_i \leq 30m_i \leq 50(1-50\beta)m_i.$$

Therefore, from (33),

$$\begin{aligned} \nu - n &\leq 40m + 50\left(\sum_{i=0}^{s} cq_i + \sum_{i=1}^{s}(1-5\beta)m_i\right) \\ &\leq 40m + 50(\log(\deg(f)) + 50\beta m) \leq 50m. \end{aligned}$$

■

After this result we can follow the same standard large deviation arguments present in [3] and finally get:

$$\text{Leb}(\{a \in \Omega'_{n+1} : F_n(a) \leq (1-\alpha)n\}) \leq e^{-\epsilon n}\,\text{Leb}(\Omega_0)$$

for some fixed $\epsilon > 0$.

7 Growth of derivatives

Let $\theta \leq 0$ and $a \in \Omega_\theta$ be fixed and write $f = f_{a,\theta}$. Given $x \in \mathbb{S}^1$ we put $x_0 = x$ and $x_j = f(x_{j-1})$, for all $j \geq 1$. Let us introduce the following notation

$$D_{k,n}(x) = \prod_{j=k}^{n} |f'(x_j)|\,, \qquad \text{and} \qquad D_n(x) = \prod_{j=0}^{n} |f'(x_j)|.$$

We can extend, in a natural way, to the orbit of x the notions of return time and bound period of the critical orbit.

For every $x \in \mathbb{S}^1$ such that $f'(x_n) \neq 0$ for all $n \geq 0$, let $(n_k)_k$ be the sequence of return times of x by the map f with associated bound periods $(p_k)_k$. Since during the bound period the orbit of x inherits the same expanding behavior of the critical orbit (Proposition 4.3, part (a)), we have

$$D_{n_k,n_k+j}(x) \geq |f'(x_{n_k})|\, C_1 \sigma_1^j\,, \qquad 0 \leq j < p_k\,,$$

for some constante $\sigma_1 > 1$. At the end of the bound period, by adapting $\sigma_1 > 1$, if necessary, Remark 4.4 gives

$$D_{n_k,n_k+p_k}(x) \geq \sigma_1^{p_k+1}. \tag{34}$$

Using Proposition 2.3 and Remark 2.4, we get

$$D_{n_k,n_k+p_k+j}(x) \geq \sigma_1^{p_k+1}\delta^2\sigma_1^j\,, \qquad \text{for all } 1 \leq j < n_{k+1} - p_k - n_k, \tag{35}$$

and also

$$D_{n_k,n_{k+1}-1}(x) \geq \sigma_1^{n_{k+1}-n_k}.$$

Given $n \geq 0$, let $n_k \leq n$ be the last return time of x. Then

$$D_n(x) = D(x)\left(\prod_{j=1}^{k-1} D_{n_j,n_{j+1}-1}(x)\right) D_{n_k,n}(x), \tag{36}$$

where if $n_1 = 0$ then $D(x) = 1$ otherwise $D(x) = D_{n_1-1}(x)$. In the later case, from (34) and (35), we have

$$D_j(x) \geq \delta^2\sigma_1^j\,, \qquad \text{for all } 0 \leq j < n_1,$$

and also

$$D_{n_1}(x) \geq \sigma_1^{n_1}.$$

When $\theta < 0$, we have $|f'(x)| > C\theta$, for all $x \in \mathbb{S}^1$ and so, in view of all the previous estimates, we can state that there exists a constant $c = c(\theta) > 0$ such that

$$D_j(x) \geq c\sigma_1^j, \quad \text{for all } x \in \mathbb{S}^1.$$

Therefore, f is a uniformly expanding map. We have proved the content of the next proposition.

Proposition 7.1. *For each fixed negative θ close to 0 there exists $c = c(\theta) > 0$ and $\sigma_1 > 1$ such that for all $a \in \Omega_\theta$ and $n \in \mathbb{N}$, we have for all $x \in \mathbb{S}^1$*

$$|(f_a^k)'(x)| \geq c\,\sigma_1^k\,, \quad \textit{for all } k \geq 1.$$

When $\theta = 0$ we have $D_n(a) = 0$ for all $n \geq 0$. In this case, we turn our attention to Lyapunov exponents of orbits by f_a. That is, we are interested in estimating

$$\lambda_{\inf}(x) = \liminf_{n\to\infty} \frac{1}{n+1} \log D_n(x) \tag{37}$$

From (36) and estimates we carried out before, we only have to worry about the terms $D_{n_k}(x)$.

Since

$$\frac{1}{n_k+1} \log D_{n_k}(x) \geq \frac{n_k}{n_k+1}\left(\log\sigma_1 + \frac{1}{n_k}\log|f'(x_{n_k})|\right),$$

we realize that if x is not a pre-image of the critical point and

$$|f'(x_{n_k})| > e^{-C\alpha n_k}, \quad \text{for some constant } C > 0, \tag{38}$$

for all returns n_k occurring after a iterate $N = N(x)$, then we will have

$$\lambda_{\inf}(x) \geq \log\sigma_1 - 2\alpha > 0.$$

In fact, we can prove there exists a full measure subset X of $\mathbb{S}^1$ such that, for each point $x \in X$, we have $N(x)$ well defined, that is to say, such points have orbits that fail to satisfy (38) only a finite number of times. Clearly X contains all periodic points of f.

We start by defining, for $k \geq 1$,

$$X_k = \{x \in \mathbb{S}^1; \left|f^j(x)\right| \geq e^{-C\beta j},\ \text{for all } j \geq k\}.$$

We also define

$$X_{k,n} = \{x \in \mathbb{S}^1; \left|f^j(x)\right| \geq e^{-C\beta j},\, k \leq j \leq n\} \quad \text{and} \quad X_{k,\infty} = \bigcup_{n=k}^{\infty} X_{k,n}\,.$$

Clearly,

$$X = \bigcup_{k=0}^{\infty} X_k \quad \text{and} \quad X_{k,\infty} \subset X_k,\ \text{for all } k \geq 0.$$

Let us sketch the steps to be followed now. From the results obtained in the previous sections and using well known lines of reasoning we proceed by defining partitions in the phase space characterized by the sharing of indistinguishable dynamical evolution in finite pieces of orbits. We end up with global control of distortion of small intervals of $\mathbb{S}^1$. Besides that, the defining condition of X_k is related to avoiding an interval whose length decreases exponentially fast. Altogether, it is not difficult to get, for some $\varepsilon > 0$ (at least for k sufficiently large),

$$\operatorname{Leb}(X_{k,n} \setminus X_{k,n+1}) \leq e^{-\varepsilon(n+1)} \operatorname{Leb}(X_{k,n})$$

and so

$$\operatorname{Leb}(X_{k,\infty}) = \left(\prod_{n\geq k}(1 - e^{-\varepsilon(n+1)})\right) \operatorname{Leb}(X_{k,k})$$

Since $\operatorname{Leb}(\mathbb{S} \setminus X_{k,k})$ tends to zero when k increases, we have

$$\lim_{k\to\infty} \operatorname{Leb}(X_{k,\infty}) = 1.$$

This yields the desired result.

8 Proof of Theorem A

We proved in Section 6 that for every $\theta < 0$ sufficiently small there exists a positive Lebesgue measure set Ω_θ such that each $f_{a,\theta}$, $a \in \Omega_\theta$, presents expansion along its critical orbit. Furthermore, Proposition 7.1 proves the statement of part (1) of Theorem A. Part (2) is a direct consequence of definition of our families.

Since the measure of Ω_θ has a uniform lower bound (for negative θ close to 0) and the methods applied to get these sets imply that them vary continuously with θ we can define $\mathcal{A}'$ as the limit of Ω_θ as θ goes to 0. Naturally

$\mathcal{A}'$ has positive Lebesgue measure and $f_{a,0}$ has a cubic critical point, by construction, for all $a \in \mathcal{A}'$.

The bifurcations constructed above are not of the (NH) type due to the recurrence control imposed while constructing Ω_θ: if the orbit of a critical point a returns too close to itself this parameter is excluded. Therefore, the critical point is non-periodic. Futhermore, from Section 7 we know that periodic points of $f_{a,0}$, $a \in \mathcal{A}'$, are hyperbolic since their Lyaponov exponents are positive.

Moreover, if we exclude a set of zero measure of $\mathcal{A}'$, the bifurcations are not of the (NT) type: fixed an interval I_n centered in the critical point of radius $1/n$, for n large enough, the set of parameters $\mathcal{B}_n$ such that the critical orbit never goes into I_n has zero measure.

Therefore, the set of parameters $\mathcal{A}$ such that the critical orbit is pre-periodic (NT) is contained in the countable union of the sets $\mathcal{B}_n$ has zero measure (recall if a orbit is non-wandering then it can not be pre-periodic). This proves part (3) and conclude the proof of the theorem.

9 Proof of Theorem B (Invariant measures)

From now on we fix $\theta \leq 0$ and a parameter $a \in \Omega_\theta$ and we are going to show how to construct an invariant and absolutely continuous measure for such a map. For simplicity we consider $f = f_{a,\theta}$.

We follows the strategy of inducing Lebesgue measures in small pieces of our phase space.

Taking into account how much of this subject is classical, we restrict ourselves in stating the realm of conditions which yields a point where getting such a measure is a matter of applying very known arguments.

First of all, let us reintroduce some terminology and notation in a slightly different way. We begin with a sequence of partitions in the phase space which will be denoted by $(\mathcal{Q}_n)_n$ and which resembles very closely the partition $(\mathcal{P}_n)_n$ in the parameter space introduced in Section 3. We define

$$\mathcal{Q}_0 = \{I_{m,k}; m \geq \Delta; 1 \leq k \leq m^2\}.$$

where $I_{m,k}$ is defined in exactly the same way as in section 3.

Successive refinements on $\mathcal{Q}_0$ will be made by taking into account the dynamic of f.

Let λ denote the Lebesgue measure normalized in U_Δ and consider the pull-back of λ given by

$$f_*^j(\lambda)(A) = \lambda(f^{-j}(A)) = \mathrm{Leb}(f^{-j}(A) \cap U_\Delta)/\,\mathrm{Leb}(U_\Delta).$$

We define

$$\mu_n = \frac{1}{n}\sum_0^{n-1} f_*^j(\lambda). \tag{39}$$

For each n we have that μ_n is a finite measure, absolutely continuous with respect to Lebesgue. Putting μ as any limit point of this sequence in the weak*-topology, it is easy to see that such a measure is invariant under f.

In order to establish that μ is absolutely continuous with respect to Lebesgue, we follows the strategy in [4]: analyzing (39) we divide the dynamics of each ω taking into account the concepts of returns, bound periods and free iterates. We consider four sequences of measures $(\mathcal{X}_n)$, $(\mathcal{Y}_n)$, $(\mathcal{Z}_n)$ and $(\mathcal{W}_n)$ such that

$$\mu_n \leq \frac{1}{n}\sum_{j=0}^{n-1}(\mathcal{X}_n + \mathcal{Y}_n + \mathcal{Z}_n + \mathcal{W}_n).$$

The proof then relies on the statement that these four measures are uniformly bounded in L^1-norm. To accomplish this we need two main results. The first one is a complete control of the distortion while iterating the intervals ω (Lemma 1.5 in [4]). Its contents is present in the next lemma.

Lemma 9.1. *Let ω be an interval on $\mathbb{S}^1$ whose points have the same dynamical evolution up to time n: the same returns, host intervals and bound and free periods. Then we have:*

$$\left|\frac{(f^n)'(x)}{(f^n)'(y)}\right| \leq C \qquad \textit{for some constant } C \textit{ independent of } n \textit{ and } \omega.$$

The proof of this lemma relies on the same estimates we have already established in the preceding sections an so we will omit it. (See lemma 5.1 and [4]).

The second auxiliary result we need is also classical and can be stated as follows: the probability of an orbit to avoid a fixed region around the criticality decreases exponentially fast. More precisely:

Lemma 9.2. *Let $\epsilon > 0$ be sufficiently small. Then there is a constant $C > 0$ and $\eta_1 > 0$ such that*

$$\mathrm{Leb}\{x : f^j(x) \notin (a-\epsilon, a+\epsilon), j = 1, \ldots, k\} \leq C\eta_1^k.$$

For the proof, see [4, Sublemma 3.6].

10 Proof of Theorem C

For each pair (a, θ) let us denote by $p_{a,\theta}$ the analytical continuation of the fixed point p of f. The next lemma asserts that it is possible to extract from each 2-parameter family $(f_{a,\theta})$ an 1-parameter family whose members have their critical point mapped onto its fixed point $p_{a,\theta}$.

Lemma 10.1. *There exists an 1-parameter family $f_{\bar{a}(\theta),\theta}(x)$ such that $\bar{a}(\cdot)$ is a C^2-map defined for $\theta \leq 0$ and such that $f_{\bar{a}(\theta),\theta}(\bar{a}(\theta)) = p_{\bar{a}(\theta),\theta}$. Moreover, there exists $\epsilon > 0$ such that $f_{a,\theta}(a) \neq p_{a,\theta}$ for every $0 < |a - \bar{a}(\theta)| < \epsilon$.*

Proof. Let us define

$$H(a, \theta) = f_{a,\theta}(f_{a,\theta}(a)) - f_{a,\theta}(a).$$

For $a = \bar{a}$ and $\theta = -1$ we have $H(a, \theta) = 0$. Moreover,

$$\begin{aligned}
\partial_a H(a, \theta) &= \partial_a\big(f_{a,\theta}(f(a, \theta, a)) - f_{a,\theta}(a)\big) \\
&= \partial_a f_{a,\theta}(f_{a,\theta}(a)) + \partial_x f_{a,\theta}(f_{a,\theta}(a))[\partial_a f_{a,\theta}(a) + \partial_x f_{a,\theta}(a)] - \\
&\quad - \partial_a f_{a,\theta}(a) - \partial_x f_{a,\theta}(a) \\
&= \partial_a f_{a,\theta}(f_{a,\theta}(a)) + [\partial_x f_{a,\theta}(f_{a,\theta}(a)) - 1][\partial_x f_{a,\theta}(a) + \partial_a f_{a,\theta}(a)].
\end{aligned}$$

By assumption (H_3), for every family $(f_{a,\theta})$ sufficiently close to $(\tilde{f}_{a,\theta})$, we have $\partial_a f_{a,\theta}(f_{a,\theta}(a)) > 0$. From the local form (2), for all $\theta \leq 0$, we have $\partial_x f_{a,\theta}(a) + \partial_a f_{a,\theta}(a) > 0$. Assumption (H_1) implies $\partial_x f_{a,\theta}(f_{a,\theta}(a)) - 1 > 0$. Therefore,

$$\partial_a H(a, \theta) > 0, \text{ for every } (a, \theta).$$

By Implicit Function Theorem there exists a function $\bar{a}(\theta)$ such that for all θ close to -1 we have $H(\bar{a}(\theta), \theta) = 0$. It is not difficult to see that we can extend $\bar{a}(\theta)$ for every $\theta \leq 0$. Moreover, the elements of this family are unique with this property in a neighborhood of $\bar{a}(\theta)$. This proves the lemma. ∎

Proof of Theorem C. By a reparametrization, we can suppose that our central family satisfies

$$f_{\bar{a},\theta}(\bar{a}) = p_{\bar{a},\theta},$$

for every $\theta \leq 0$.

The points a and $p_{a,\theta}$ determine a split of $\mathbb{S}^1$ in two open arcs I_+ and I_-. By construction the critical point a is mapped close to $p_{a,\theta}$. From Lemma 10.1, $f_{a,\theta}(a) \neq p_{a,\theta}$, for all $\theta \leq 0$ and a close to $\bar{a}$. Let us suppose that $f_{a,\theta}(a)$ belongs to I_+ (otherwise the proof is similar).

Fixed $\theta \leq 0$, for $N = N(a,\theta)$ sufficiently large and a close to $\bar{a}$, $a \neq \bar{a}$, we define

$$I_\theta(N) = \{a \ : \ \xi_j(a) \in I_+, j = 1, \dots N-1 \text{ and } f^N_{a,\theta}(a) \in I_-\}.$$

These intervals are well defined and their extremal points are not mapped onto $p_{\bar{a}}$ because $\bar{a}$ is the unique point in a neighborhood of itself going onto a fixed point of $(f_{a,\theta})$. Moreover, $\xi_N(I_\theta(N)) = I_-$. In particular $\xi_N(I_\theta(N))$ covers $I_\theta(N)$.

By continuity, ξ has a periodic point a_N of period N, that is, $f^N_{a,\theta}(a) = a$. By construction, the orbit of a_N has a unique iterate inside the perturbation region I_0. Since the orbit of a_N is not modified by $f_{a_N,\theta}$, if θ is sufficiently small, a_N is a periodic sink. This proves Theorem C. ∎

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