Coordinate Percolation on $\mathbb{Z}^3$

Marcelo Hilario
Advisor: Vladas Sidoravicius

February 8, 2012

Resumo

Considere o seguinte processo de percolação dependente na rede Euclidiana tridimensional, $\mathbb{Z}^3$: para cada coluna dessa rede que é paralela a um dos eixos coordenados decidimos removê-la ou não de acordo com um certo parâmetro dependendo apenas da sua direção. As colunas são removidas ou não de maneira independente uma das outras e, após decidir-se o estado de cada uma delas, é obtido um conjunto aleatório de sítios restantes, denominados *sítios abertos*. Esse modelo apresenta dependências de alcance infinito que induzem propriedades interessantes para o conjunto dos sítios abertos. Algumas delas não existem para percolação de Bernoulli ou outros modelos em que as dependências são locais ou mais fracas. É provado que, quando removem-se as colunas com alta probabilidade, não há componentes conexas infinitas, quase certamente. Por outro lado, caso elas sejam removidas com baixa probabilidade então tais componentes passam a exisitir. Isso estabelece uma transição de fase para esse modelo. Também mostra-se que a probabilidade da cauda relativa ao raio da componete conexa contendo a origem decai exponencialmente quando ao menos dois dos parâmetros são fixados grandes. Se, ao contrário, dois parâmetros são tomados relativamente pequenos então a versão truncada dessa cauda tem decaimento, no máximo, polinomial. Também prova-se que o número de componentes conexas na fase supercrítica é, ou um, ou infinito.

Palavras chaves: Percolação dependente, transição de fase, decaimento de conectividade.

Abstract

We consider the following percolation process defined on the $\mathbb{Z}^3$-lattice: For each column that is parallel to one of the coordinate axis we decide whether to remove it or not with a probability (or parameter) depending only on its direction. The columns are removed or not independently, and after establishing the state of each one of them we are left with a random subset of remaining sites called *open sites*. This model contains infinite-range dependencies that induce interesting properties for the set of open sites. Some of them are not present in the Bernoulli percolation or in percolation models having only local or weaker dependencies. It is proven that, if the columns are removed with high probability then there are no infinite components, almost surely. On the other hand, in case they are removed with low probability, then such components indeed exist. This establishes the phase transition for this model. We also show that the tail distribution for the radius of the open cluster containing the origin decays exponentially fast when at least two of the parameters are fixed to be high. However, if two of the parameters are taken relatively small, then the truncated version for this tail decays, at most, polynomially fast. We also prove that the number of infinite connected components in the supercritical phase is either one or infinite, almost surely.

Keywords: Dependent percolation, phase transition, connectivity decay.

Acknowledgements

I would like to thank my advisor Prof. Vladas Sidoravicius for all the support and intense work that resulted in this thesis. Also I would like to thank my friends from IMPA. They were like a family to me and, for sure, they will recognize themselves in this message. I am also grateful to the members of the comittee, Professors Bernado de Lima, Milton Jara, Marcelo Viana, Maria Eulália Vares and Roberto Imbuzeiro for theirs suggestions; and to all professors at IMPA from whom I have learned many things. I am also grateful to all my friends from Belo Horizonte and Rio who were important companies during all those years of work. This thesis is dedicated to my family, Maurício, Beth, Renato e Leandro.

Contents

List of Figures

Chapter 1

Introduction

1.1 Mathematical setting

We consider a site *percolation* process in which we remove the columns of the $\mathbb{Z}^3$ lattice independently with intensity depending on their direction. More precisely, we consider the *coordinate planes*

$$\mathcal{P}_1 = \{(x, y, 0);\ x, y \in \mathbb{Z}\}$$

$$\mathcal{P}_2 = \{(x, 0, z);\ x, z \in \mathbb{Z}\}$$

$$\mathcal{P}_3 = \{(0, y, z);\ y, z \in \mathbb{Z}\}$$

and, for each $i \in \{1, 2, 3\}$ we define $\Omega_i = \{0, 1\}^{\mathcal{P}_i}$ endowed with the σ-field generated by the cylinder sets that we denote by $\mathcal{F}_i$. We write ω_i to indicate the elements of Ω_i. Fix $p_i \in [0, 1]$ and let $\mathbb{P}_{p_i}$ be the probability measure for which $\{\omega_i(v); v \in \mathcal{P}_i\}$ are mutually independent *Bernoulli* random variables with mean p_i. Take now $\Omega_1 \times \Omega_2 \times \Omega_3$ with the product σ-field and the product measure $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$. This corresponds to take independently three two-dimensional *Bernoulli* (or i.i.d.) *site percolation* processes each one defined in the corresponding coordinate plane. A point $v \in \mathcal{P}_i$ is said to be ω_i-*open* if $\omega_i(v) = 1$.

Take $\Omega = \{0, 1\}^{\mathbb{Z}^3}$ with the σ-field generated by the cylinder sets and for a site $(x, y, z) \in \mathbb{Z}^3$ we define

$$\omega(x, y, z) = \omega_1(x, y, 0)\omega_2(x, 0, z)\omega_3(0, y, z). \tag{1.1}$$

Let $\mathbf{p} = (p_1, p_2, p_3)$ and denote by $\mathbb{P}_{\mathbf{p}}$ the distribution of the random element ω under $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$, *i.e.*, for each event $\mathcal{A} \in \Omega$ let

$$\mathbb{P}_{\mathbf{p}}(\mathcal{A}) = \mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}(\{\omega \in \mathcal{A}\}). \tag{1.2}$$

We say that a site $(x, y, z) \in \mathbb{Z}^3$ is ω-*open* or just *open* if $\omega(x, y, z) = 1$. Otherwise (x, y, z) is said to be ω-*closed* or *closed*. The three dimensional percolation process with law $\mathbb{P}_{\mathbf{p}}$ will be called *Coordinate Percolation on* $\mathbb{Z}^3$. This process clearly has infinite-range dependencies since, for instance, knowing that $\omega_1(x, y, 0) = 0$ implies that $\omega(x, y, z) = 0$ for all $z \in \mathbb{Z}$. The numbers $p_i, = 1, 2, 3$ will be called the parameters of the model. Our aim is to study the connectivity properties of the set of open sites in $\mathbb{Z}^3$ as the values of those components are varied.

Remark 1.1.

- We have defined $\omega \in \{0, 1\}^{\mathbb{Z}^3}$ as a random element in $\Omega_1 \times \Omega_2 \times \Omega_3$. However we will also refer to it as a process in Ω sampled by the measure $\mathbb{P}_{\mathbf{p}}$.

- The process ω could be defined without embedding the two-dimensional percolation processes in $\mathbb{Z}^3$. However we prefer to do so in order to make the arguments clearer geometrically.

- Sometimes, in order to simplify the notation, we may not distinguish between $\mathcal{P}_i$ and $\mathbb{Z}^2$. Thus if that is clear that we are referring to an element of $\mathcal{P}_1$ we may write (x, y) instead of $(x, y, 0)$. Similarly we may write (x, z) for $(x, 0, z) \in \mathcal{P}_2$ and (y, z) for $(0, y, z) \in \mathcal{P}_3$.

For understanding the connectivity properties of this model it is useful to have in mind the following picture: For each column that is normal to the plane $\mathcal{P}_i$, toss a coin having probability p_i of landing head up. If it lands tail up then remove all sites lying in that column. After performing all tosses independently and removing the appropriate sites, declare all remaining sites to be open. Then we are left with a random set of open sites that have law $\mathbb{P}_{\mathbf{p}}$.

Since we will need to use results on percolation theory for dimensions 2 and 3, we give some definitions and state some results in general dimension d. We write $\mathbb{P}_p$ to refer to the measure describing *Bernoulli site percolation* with intensity p

in $\mathbb{Z}^d$. Denoting by η an element of $\{0,1\}^{\mathbb{Z}^d}$, then $\mathbb{P}_p$ is the measure for which the random variables $\{\eta(v);\ v \in \mathbb{Z}^d\}$ are mutually independent Bernoulli random variables with mean p.

The origin of $\mathbb{Z}^d$ and of the planes $\mathcal{P}_i$ will be denoted by $\mathbf{0}$. For a nonnegative integer n define $B^d(n;v) = \{w \in \mathbb{Z}^d;\ |w-v|_\infty \leq n\}$ the *box* of radius n around v in $\mathbb{Z}^d$ where $|\cdot|_\infty$ stands for the l_∞-norm in $\mathbb{Z}^d$. We write $B^d(n) = B^d(n;\mathbf{0})$. The boundary of the box $B^d(n;v)$ is $\partial B^d(n;v) = \{w \in \mathbb{Z}^d; |w-v|_\infty = n\}$. A path in $\mathbb{Z}^d$ is a sequence of sites $\Gamma = \{v_0, v_1, \ldots\}$ such that $|v_{i+1} - v_i| = 1$ for all i. We will also consider finite paths $\Gamma = \{v_0, v_1, \ldots v_m\}$ defined in the same way. Given an element $\eta \in \{0,1\}^{\mathbb{Z}^d}$ we say that the path Γ is η-*open* or simply open if $\eta(v) = 1$ for all sites $v \in \Gamma$. For a pair of sites of $\mathbb{Z}^d$, v and w we denote by $\{v \leftrightarrow w\}$ the event that there is an open path starting at v and ending at w. In case this event happens we say that v is connected to w. For a set $A \subset \mathbb{Z}^d$ we define $\{v \leftrightarrow A\} = \cup_{w\in A}\{v \leftrightarrow w\}$. We also define $\{v \leftrightarrow \infty\} = \cap_{n=1}^{\infty}\{v \leftrightarrow \partial B^d(n)\}$.

For a fixed $\eta \in \{0,1\}^{\mathbb{Z}^d}$, the maximal connected components of open sites of $\mathbb{Z}^d$ are called η-*clusters* or simply *clusters*. We denote by $C(v)$ the cluster containing v. Then $\{v \leftrightarrow \infty\} = \{|C(v)| = \infty\}$ corresponds to the event that the site v lies in an infinite cluster (here $|\cdot|$ stands for the cardinality of a set). For the origin we simply write $C = C(\mathbf{0})$. Note that $C(v)$ is empty when $\eta(v) = 0$. Analogously we define $C_i(v)$ the ω_i-cluster in $\mathcal{P}_i$ containing $v \in \mathcal{P}_i$.

For Bernoulli site percolation in $\mathbb{Z}^d$ we define $\theta(p) = \mathbb{P}_p(\{\mathbf{0} \leftrightarrow \infty\})$ and $p_c(\mathbb{Z}^d) = \inf\{p \in [0,1];\ \theta(p) > 0\}$. It is well known that $0 < p_c(\mathbb{Z}^d) < 1$ for all $d \geq 2$ (see for instance, [Gri99, Theorem 1.10, page 14]). We then say that there exists a *phase transition* for this model.

For the percolation model given by (1.1) we define the percolation function by $\theta(\mathbf{p}) = \mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \infty\})$. For a given $\mathbf{p} \in [0,1]^3$ we say that there is percolation if $\theta(\mathbf{p}) > 0$. Otherwise we say that there is no percolation. It is a simple fact that the percolation function is increasing in each parameter, assumes the values zero if some of them is equal to zero, and assume the value one if all of them are equal to one. From now on we will restrict ourselves to the cases in which $p_i > 0$ for all i.

The next theorem establishes the analogous result for our process.

Theorem 1.2. *There exists* $p_c(\mathbb{Z}^2) \leq p^* < 1$ *such that if* $p_i > p^*$ *for all* $i \in \{1,2,3\}$ *then* $\theta(\mathbf{p}) > 0$. *On the other hand, if* $p_i \leq p_c(\mathbb{Z}^2)$ *and* $p_j < 1$ *for two* *indices* $i \neq j \in \{1, 2, 3\}$ *then* $\theta(\mathbf{p}) = 0$.

Suppose that we restrict ourselves to the case $p_1 = p_2 = p_3 = p$ and define as for the Bernoulli percolation,

$$p_c = \inf\{p \in [0, 1];\ \theta((p, p, p)) > 0\}$$

Then, as a consequence of the last theorem we have: $p_c(\mathbb{Z}^2) \leq p_c \leq p^*$.

Theorem 1.2 is proven in Chapter 2. The proof is separated in two steps. In Corollary 2.4 we show that there is no percolation for small values of the parameters, establishing thus the second statement of this theorem. The first statement of this theorem is proven as a consequence of Theorem 2.9. We will also show that $p_c \leq [p_c'(\mathbb{Z}^2)]^{1/3}$ where $p_c'(\mathbb{Z}^2)$ stands for the critical probability for oriented percolation on $\mathbb{Z}^2$. It is an interesting question whether or not $p_c > p_c(\mathbb{Z}^2)$. We believe that this inequality holds, however we do not have a proof.

We say that a parameter p_i is sub-critical (resp. supercritical) if $p_i < p_c(\mathbb{Z}^2)$ (resp. $p_i > p_c(\mathbb{Z}^2)$). The next result establishes bounds for the decay rate of the tail probability for the radius of the cluster containing the origin. Those rates depend on the number of sub-critical and supercritical parameters.

Theorem 1.3. *Consider the coordinate percolation process in* $\mathbb{Z}^3$. *If at least two parameters are sub-critical, then there exists a constant* $\psi(\mathbf{p}) > 0$ *such that*

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \partial B^3(n)\}\right) \leq \exp\left(-\psi(\mathbf{p})n\right). \tag{1.3}$$

On the other hand, if at least two of the parameters are supercritical while a third one is non-zero then there are constants $\alpha(\mathbf{p}) > 0$ *and* $\alpha'(\mathbf{p}) > 0$ *such that*

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \partial B^3(n), |C| < \infty\}\right) \geq \alpha'(\mathbf{p})n^{-\alpha(\mathbf{p})}. \tag{1.4}$$

The proof of equation (1.3) is given in Section 3.1 while the proof of equation (1.4) appears at the end of Section 3.2. For the latter, several block arguments are used in order to construct ω-open paths in $\mathbb{Z}^3$ with high probability. The details are provided throughout Sub-sections 3.2.1 and 3.2.2.

It has been proved by Menshikov [Men86] and by Aizenman and Barsky [AB87] that for Bernoulli percolation in $\mathbb{Z}^d$ there is a constant $\psi(p) > 0$ such that

$$\mathbb{P}_p\left(\{\mathbf{0} \leftrightarrow \partial B^d(n)\}\right) \leq \exp\left(-\psi(p)n\right) \tag{1.5}$$

whenever $p < p_c(\mathbb{Z})$. Equation (1.3) above gives the same behavior for our model when (for instance) all the three parameters of the model are sub-critical. On those cases we say that there is *exponential decay* for the tail distribution for the radius of the cluster of the origin.

On the other hand, if we set two of the parameters to be supercritical (but not equal to one) while the third one is taken sub-critical, then in view of Theorem 1.2 there is no percolation thus equation (1.4) can be rewritten as

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \partial B^3(n)\}\right) \geq \alpha'(\mathbf{p}) n^{-\alpha(\mathbf{p})}. \tag{1.6}$$

This gives a quite different behavior for the sub-critical phase of our model when compared to the Bernoulli percolation processes and we say that we have *sub-exponential decay* for the tail distribution of the radius of the cluster of the origin. In fact the rate of decay for this tail distribution is at most polynomial. In particular, considering the case in which all the three parameters are equal to each other, if $p_c > p_c(\mathbb{Z}^2)$ really holds then equations (1.3) and (1.6) would give us a change in the rate of decay for those tail distributions within the sub-critical phase. This behavior is not present in the Bernoulli case.

Equation (1.4) is also interesting for the supercritical regime of our model. In fact, it follows from the works of Chayes, Chayes and Newman [CCN87] (see also [Gri99, Chapter 8.4]) that for supercritical Bernoulli percolation in $\mathbb{Z}^d$ there are positive constants $A(p,d)$ and $\sigma(p) > 0$ such that

$$\mathbb{P}_p\left(\{\mathbf{0} \leftrightarrow \partial B^d(n), |C| < \infty\}\right) \leq A(p,d) n^d \exp\left(-n\sigma(p)\right).$$

Thus if one sets $p_i > p^*$ for all $i \in \{1,2,3\}$ then by Theorem 1.3 our model exhibits a quite different kind of decay.

Although we expect that $p_c > p_c(\mathbb{Z}^2)$ some techniques used in the proof of Theorems 1.2 and 1.3 can be used to show that if we fix two parameters slightly greater than $p_c(\mathbb{Z}^2)$ and then set the third one to be high enough than there is percolation. More precisely we have:

Theorem 1.4. *Let $p_i, p_j > p_c(\mathbb{Z}^2)$ for two indices $i \neq j \in \{1,2,3\}$. Then there exists a $\epsilon > 0$ (depending on p_i and p_j) such that if the third component of $\mathbf{p}$ is greater than $1 - \epsilon$ then $\theta(\mathbf{p}) > 0$.*

This result add some more information about the phase diagram of the model and is proven in Chapter 4. Note that this is equally good for proving the second statement in Theorem 1.2.

In percolation theory one is usually interested in studding the number of distinct infinite clusters. This is a random variable that we denote by N. We have that

Theorem 1.5. *Consider coordinate percolation on $\mathbb{Z}^3$. Almost surely under $\mathbb{P}_{\mathbf{p}}$, N is a constant random variable that takes values in the set $\{0, 1, \infty\}$.*

The proof for this theorem is given in the Chapter 5. The fact that $N \in \{0, 1, \infty\}$ has been proved before by Newman and Schulman [NS81a, NS81b] for a class of percolation processes that are ergodic under the lattice translations and that satisfies a property known as *finite energy condition*. Intuitively this condition states that it is possible to perform local changes in an event having positive probability obtaining at the end an event that still have positive probability. After that, Aizenman, Kesten and Newman [AKN87] showed that, for a broad class of translation-invariant percolation processes in $\mathbb{Z}^d$, there is at most one infinite cluster. That includes the Bernoulli case and some long range percolation models. Some years later Burton and Keane [BK89] produced a simple proof for the uniqueness of the infinite cluster, when it exists, for all translation-invariant models that have finite energy. This class of models include the Bernoulli percolation process and the Ising model. Due to the fact that the measure $\mathbb{P}_{\mathbf{p}}$ have infinite range dependencies it does not satisfy this condition, so we cannot apply their results directly. However the measure $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$ have finite energy and we use this fact in the proof of the result.

1.2 Related models

In the last section we have introduced the *Coordinate Percolation* and have stated the main results that we will prove about this process. As they were stated we have also compared them to their corresponding results for the *Bernoulli percolation*. Due to the dependencies that the coordinate percolation exhibits, in many situations those results are quite distinct from those available for Bernoulli percolation. Also the proofs of many results such as the existence of a phase transition requires new ideas as we shall see in Chapter 2. That is not peculiar to coordinate percolation. In fact there are many important percolation models in which the state of different sites may not be independent of each other, and usually, even to show that a phase transition occurs or not for those models is a challenging question. In this section we present a quick introduction to some of those models and state some of their properties that have been proved by many different authors.

An example of the physically important dependent percolation is the *Random Cluster model* that couples Bernoulli percolation and the *Potts model*. Dependencies also appears naturally when one wants to study the ocurrence of some events in Bernoulli percolation by using some *block arguments*. For that one starts by dividing the original lattice into some (usually large) blocks in such a way that the probability of observing some set of configuration within those blocks is high. Blocks for which those configurations are observed will be called here *good blocks*. The main idea is to define a block to be good 'carefully' in order to guarantee that if there are sufficiently many good blocks then the original event in the initial scale will happen. We will use some block arguments in Chapter 3.

For the kind of dependent percolation models mentioned in the previou paragraph we have that the dependencies decay with the distance in the sense that the configuration within two set of sites become almost independent as those sets are taken to be far apart from each other. For models containing this kind of dependencies we can use the general methods of Liggett, Schonmann and Stacey [LSS97] and guarantee that some increasing events will occur by dominating the process from below by a Bernoulli percolation process (see Section 2.2.3 for more precise information).

However, as for the coordinate percolation, there are some important models in which the dependencies may not decay with the distance. One notable example is a model introduced by Winkler (see [Win00]) and commonly known as *Winkler* percolation. This model was introduced in terms of colliding random walks as follows: Consider a connected graph G and let $\{X(i)\}_{i\in\mathbb{N}}$ and $\{Y(i)\}_{i\in\mathbb{N}}$ be two independent copies of a single random walk in G. The graph G is said to be *navigable* if there is a positive probability of finding a path γ lying in the positive quadrant of $\mathbb{Z}^2$ such that, for all $(i,j)\in\gamma$ we have that $X(i)\neq Y(j)$. This is equivalent to saying that one can delay or move back one or both random walks in a clever way so that they still move arbitrarily forward on the long rung without colliding to each other.

If we consider $G = k_n$ the complete graph with n vertices, then this problem is equivalent to the following dependent percolation process in $\mathbb{Z}^2$: For each colunm and each row of $\mathbb{Z}^2_+$, select uniformly at random and independently one number in the set $\{1, \ldots, k\}$. Declare a site to be open if the numbers assigned to its column and its row are different. Then look for a path in $\mathbb{Z}^2_+$ composed only of open sites.

It it has been proved independently by Winkler [Win00] and Ballister, Bollobás and Stacey [BBS00] that the graph is navigable when $d \geq 4$. For applications in distributed computation one is mainly interested in restricting the paths γ to be oriented. If there is a postive probability of finding such paths then we say that there is percolation. It has been conjectured by Winkler that there is a positive probability of finding such paths whenever $n \geq 4$ and this conjecture remains open in its full formulation. However, analogously to what is observed for our coordinate percolation process on $\mathbb{Z}^3$ (see equation (1.4)) it has been proved by Gacs [Gác00] that if there is percolation in the oriented case, then the probability for the origin to be connect up to distance k but not to infinity decays at most polynomially in k.

There exists also an example of *bond percolation* model in $\mathbb{Z}^2$ in which the state of each of its edge is determined by random variables indexed by the columns and rows of this lattice: The *Corner Percolation* model. More precisely we consider two independent sequences of $i.i.d.$ Bernoulli random variables with parameter $1/2$: $\{\eta(i)\}_{i\in\mathbb{Z}}$ and $\{\xi_(j)\}_{j\in\mathbb{Z}}$. Let $\{(i, j), (i, j + 1)\}$ be the 'vertical' edge connecting the vertices (i, j) and $(i, j + 1)$. We declare this edge to be open if j is even and $\eta(i) = 1$ or if j is odd and $\eta(i) = 0$. Otherwise we declare this edge to be closed. Similarly we declare each 'horizontal' edge $\{(i, j), (i + 1, j)\}$ to be be open if i is even and $\xi(j) = 1$ or if i is odd and $\xi(j) = 0$. This rules of retention induces a random configuration of open edges such that for each site of $\mathbb{Z}^2$ there are exactely two open edges incident to it. Furthermore those two edges are perpendicular to each other.

Corner percolation was introduced by Balint Tóth as a degenerate case of the *Six Vertex Model* also known as the *Ice-Type Model*, a very popular model in statistical mechanics (see for instance [Bax82]). For that model, it was shown by Pete [Pet08] that there is no percolation. In fact, he shows that all clusters are finite cycles of open edges. He also computes exponents for the rate of decay of the tail probability for the diameter of the cycle containing the origin. In that case the decay is polynomial and the exponent is shown to be equal to $-(5-\sqrt{17})/4$.

Despite from the similarity with the Winkler percolation, the coordinate percolation process that we have introduced in the last section was inspired by a rather different depedent site percolation model called *Random Interlacements*. This model was introduced in [Szn10] by Sznitman and consits of studing the vacant set of sites of a Poisson point process in the space of double-infinite trajectories on $\mathbb{Z}^d$ modulo time shifts. Loosly speaking it is the complement of a *Poisson 'soup'* of doubly-infinte trajectories of a random walk in $\mathbb{Z}^d$ for $d \geq 3$. If the intensity of the Poisson process is high one should expect that the vacant set of the random interlacements contains no infinite connected components whereas for low intensities such a component should appear. This means that the random interlacements should exhibit a phase transition as the intensity of the Possion process is varied. In fact, Sznitman [Szn10] and Sidoravicius and Sznitman [SS09] proves that this phase transition takes place.

This process can be thought as a model for corrosion if one imagine that sites are being removed as some corrosive particle diffuses into a medium performing simple random walks. The vacant set can then be regarded as the set of sites that have not been removed by the corrosive particle. Indeed, in [Win08] it shown that if a random walk runs in a d-dimensional discrete torus up to a time proportinal to its volume then the local picture left by the trace of that random walk converge to the law of a random interlacement in $\mathbb{Z}^d$.

What if instead of removing sites lying in the trace of some random walk, we just remove sites in straight lines? If we remove straight lines that are parallel to the axis independently we will just have the coordinate percolation process discrebed in Section 1.1.

Recently, also inspired by the random interlacements process, Tykeson and Windisch [TW10] have considered the same sort of question in a continuum setting defining a *Poisson cylinder* percolation model. More precisely, they study the percolative properties of the the vacant set of a thickening of a Poisson process defined on the space of lines in $\mathbb{R}^d$. Equivalently, their model consist in studing the set of points of $\mathbb{R}^d$ that has not been covered by the union of all bi-infinite cilynders having diameter one and having their axes given by the lines in a realisation of such a Poisson process. Note that, differently from our model, their model is isotropic (all directions are equivalent) and is defined for all dimension $d \geq 3$. For this model they are able to prove that there is a phase transition for all $d \geq 4$: If the intensity of underlying Poisson process is high then there is no percolation, while there is percolation if the intensity is sufficiently small. Although they are able to prove that there is no percolation for $d = 3$ with high intensity it is still open to prove that there is percolation for small intensities. In fact they also show that for any $d \geq 4$, if the intensities are low enough then there is an infinite connected component already in subspaces of dimension 2. For $d = 3$ this is not the case: They show that the probability of having percolation on any subspace of dimension 2 vanishes.

Chapter 2

Phase transition

In this chapter we prove Theorem 1.2, establishing thus the phase transition for the model described in the previous chapter.

2.1 Absence of percolation for small parameters

Let $\pi_i : \mathbb{Z}^3 \to \mathcal{P}_i$ be the projection from $\mathbb{Z}^3$ into $\mathcal{P}_i$. Then equation (1.1) can be rewritten as:

$$\omega(v) = \prod_{i=1,2,3} \omega_i\left(\pi_i(v)\right). \tag{2.1}$$

Identifying a configuration $\omega \in \mathbb{Z}^3$ with the set of its opens sites $\omega^{-1}(\{1\}) := \{v \in \mathbb{Z}^3;\ \omega(v) = 1\}$ and similarly for the ω_i and $\omega_i^{-1}(\{1\}) \in \mathcal{P}_i$ this equation shows that $\pi_i(\omega) \subset \omega_i$. Furthermore, since the processes ω_i defined in each plane $\mathcal{P}_i$ are independent Bernoulli percolation processes with parameter $p_i > 0$, if w is a ω_i-open site in $\mathcal{P}_i$, then the column $\pi_i^{-1}(w)$ has infinitely many ω-open sites ($\mathbb{P}_{\mathbf{p}}$-almost surely). Thus $\omega_i = \pi_i(\omega)$ almost surely with respect to $\mathbb{P}_{\mathbf{p}}$. Let $B_i^2(n) = \pi_i\left(B^3(n)\right)$ be the boxes of size n in $\mathcal{P}_i$ and define the events:

$$\mathcal{A}_i(n) = \{\omega \in \Omega;\ \pi_i(C(\mathbf{0})) \text{ contains a path connecting } \mathbf{0} \text{ to } \partial B_i^2(n)\}.$$

Note that, if $\omega \in \mathcal{A}_i(n)$ then $\omega_i \in \{\mathbf{0} \leftrightarrow \partial B_i^2(n)\}$.

Lemma 2.1. *For any $v \in \mathbb{Z}^3$, $\pi_i\left(C(v)\right)$ is a connected subset of $C_i\left(\pi_i(v)\right)$. In particular, if $|C_i(\mathbf{0})| < \infty$ then $\{\omega \in \mathcal{A}_i(n)\}$ does not happen for infinitely many indices n.*

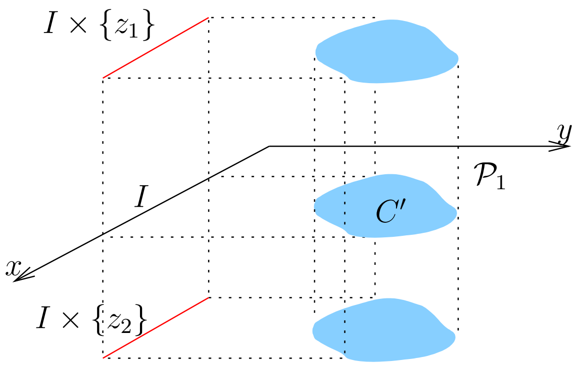

Figure 2.1: The blue set C' in the plane $\mathcal{P}_1$ represents $\pi_1(C(\mathbf{0}))$. In red we have the ω_2-closed segments $I \times \{z_2\}$ and $I \times \{z_3\}$ contained in the plane $\mathcal{P}_2$. The ω-cluster at the origin has to be contained in the *tube* having the other two blue sets as the top and bottom.

Proof. Let $w', w'' \in \pi_i(C(v))$. Pick $v' \in \pi_i^{-1}(\{w'\}) \cap C(v)$ and $v'' \in \pi_i^{-1}(\{w''\}) \cap C(v)$. Since $C(v)$ is connected there exists an open path $\Gamma = \{v_1, \ldots, v_m\}$ with $v_1 = v'$ and $v_2 = v''$. We can consider Γ as being a continuous curve in $\mathbb{R}^3$ by joining each pair of sites (v_{i-1}, v_i) by the line segment that connect them. The projection of this curve into $\mathcal{P}_i$ is a continuous curve connecting w' and w'' and containing only points in $\mathcal{P}_i$ and the segments connecting them so $\pi_i(\Gamma)$ is a path that connects w' to w''. Since, in addition $\pi_i(\omega) \subset \omega_i$ we have that this path is ω_i-open. $\square$

Corollary 2.2. *If* $\omega \in \{\mathbf{0} \leftrightarrow \partial B^3(n)\}$ *then for at least two indices* $i, j \in \{1, 2, 3\}$ $\omega \in \mathcal{A}_i(n) \cap \mathcal{A}_j(n)$. *In particular* $\omega_i \in \{\mathbf{0} \leftrightarrow \partial B_i^2(n)\}$ *and* $\omega_j \in \{\mathbf{0} \leftrightarrow \partial B_j^2(n)\}$.

Proof. Let $\omega \in \{\mathbf{0} \leftrightarrow \partial B^3(n)\}$. Then there is an ω-open path $\Gamma = \{v_1, \ldots, v_m\}$ such that $v_1 = \mathbf{0}$ and $v_m \in \partial B^3(n)$. Let i be so that $\pi_i(v_m) \in \partial B_i^2(n)$. By the previous lemma, $\pi_i(C(\mathbf{0}))$ contains a ω_i-open path connecting $\mathbf{0}$ to $\pi_i(v_m)$ in $\mathcal{P}_i$, so $\omega \in \mathcal{A}_i(n)$. The proof is finished by noting that there are at least two possible choices of indices i such that $\pi_i(v_m) \in \partial B_i^2(n)$. $\square$

We shall use Corollary 2.2 in order to prove that the origin cannot belong to an infinite ω-cluster if some of the ω_i-clusters $C_i(\mathbf{0})$ is finite. The (informal) idea for the proof is the following: Suppose that $C_1(\mathbf{0})$ is finite. By the previous lemma, the projection of $C(\mathbf{0})$ into $\mathcal{P}_1$ is contained in $C_1(\mathbf{0})$, then $C(\mathbf{0})$ must be contained in the *'tube'* $\pi_1^{-1}(C_1(\mathbf{0}))$. However, the projection of this tube into $\mathcal{P}_2$ is a *'strip'* S that have finite width. Then almost surely we will find some ω_2-closed paths traversing S. This means that the tube will be cut above and beneath the plane $\mathcal{P}_1$ preventing $C(\mathbf{0})$ from being infinite (see Figure 2.1). More precisely:

Lemma 2.3. *If either* $p_2 \neq 1$, *or* $p_3 \neq 1$ *then*

$$\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3} \left(\{ \omega \in \{ \mathbf{0} \leftrightarrow \infty \} \} \cap \{ |C_1(\mathbf{0})| < \infty \} \right) = 0.$$

Proof. For simplicity, we fix for this proof $p_2 \neq 1$. We also write $\mathbb{P}$ for the measure $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$. Note that if $C_1(\mathbf{0}) = C'$ for a finite set $C' \subset \mathcal{P}_1$ then, by Lemma 2.1, $\mathcal{A}_1(n)$ cannot happen for n larger than the diameter of C'. If in addition $\{\mathbf{0} \leftrightarrow \partial B^3(n)\}$ happens for n larger than the diameter of C' then, by Corollary 2.2, ω belongs to $\mathcal{A}_2(n) \cap \mathcal{A}_3(n)$. So we have:

$$\begin{aligned} &\mathbb{P} \left(\{ \omega \in \{ \mathbf{0} \leftrightarrow \infty \} \} \cap \{ |C_1(\mathbf{0})| < \infty \} \right) \\ &= \sum_{\substack{C' \subset \mathcal{P}_1 \\ |C'| < \infty}} \mathbb{P} \left(\bigcap_{n=1}^{\infty} \{ \omega \in \{ \mathbf{0} \leftrightarrow \partial B^3(n) \} \} \cap \{ C_1(\mathbf{0}) = C' \} \right) \\ &\leq \sum_{\substack{C' \subset \mathcal{P}_1 \\ |C'| < \infty}} \mathbb{P} \left(\limsup_n \{ \omega \in \mathcal{A}_2(n) \cap \mathcal{A}_3(n) \} \cap \{ C_1(\mathbf{0}) = C' \} \right) \\ &\leq \sum_{\substack{C' \subset \mathcal{P}_1 \\ |C'| < \infty}} \lim_{n \to \infty} \mathbb{P} \left(\{ \omega \in \mathcal{A}_2(n) \} \cap \{ C_1(\mathbf{0}) = C' \} \right). \end{aligned} \tag{2.2}$$

Let us fix $C' \subset \mathcal{P}_1$ a finite connected set. Let $I = \{x \in \mathbb{Z};\ \exists\, y \in \mathbb{Z}$ such that $(x, y) \in C'\}$ and let $S = \{(x, z) \in \mathcal{P}_2; x \in I\}$ (see Figure 2.1). Note that I is a finite set of integers and that, by Lemma 2.1, if $C_1(\mathbf{0}) = C'$ then $\pi_1(C(\mathbf{0})) \subset C'$. Thus we have that

$$\pi_1 \left(\pi_2(C(\mathbf{0})) \right) = \pi_2 \left(\pi_1(C(\mathbf{0})) \right) \subset I \times \{0\} := \{ (x, 0) \in \mathcal{P}_2; x \in I \} \subset \mathcal{P}_2$$

and, in particular, $\pi_2(C(\mathbf{0})) \subset \pi_1^{-1}(I \times \{0\}) = S$. Then it follows that

$$\{ \omega \in \mathcal{A}_2(n) \} \cap \{ C_1(\mathbf{0}) = C' \} \subset \left\{ \mathbf{0} \leftrightarrow \partial B_2^2(n) \text{ in } S \right\},$$

where the event on the right hand side is a cylinder in $\mathcal{P}_2$. Thus

$$\begin{aligned} &\lim_{n \to \infty} \mathbb{P} \left(\{ \omega \in \mathcal{A}_2(n) \} \cap \{ C_1(\mathbf{0}) = C' \} \right) \leq \\ &\lim_{n \to \infty} \mathbb{P}_{p_2} \left(\{ \mathbf{0} \leftrightarrow \partial B_2^2(n) \text{ in } S \} \right) = 0, \end{aligned} \tag{2.3}$$

where the fact that the limit in the right hand side is zero can be justified with the following argument: since $p_2 < 1$, then $\mathbb{P}_{p_2}$-almost surely there are (random) integers $z_1 < 0$ and $z_2 > 0$ such that the *segments* $I \times \{z_1\}$ and $I \times \{z_2\}$ are ω_2-closed so that $\{\mathbf{0} \leftrightarrow \partial B_2^2(n) \text{ in } S\}$ cannot happen if $n > \max\{-z_1, z_2\}$.

The proof is now completed by plugging equation (2.3) into (2.2). □

Of course we can choose any combination of indices other that 1 and 2 for the statement of the last lemma. Recall that if $p_i < p_c(\mathbb{Z}^2)$ then the event $\{|C_i(\mathbf{0})| < \infty\}$ has probability one under $\mathbb{P}_{p_i}$. Then a direct application of the last lemma gives us the proof of the second statement in Theorem 1.2 that we state as a corollary.

Corollary 2.4 (Second statement in Theorem 1.2)**.** *If for* $i \neq j \in \{1, 2, 3\}$ *we have that* $p_i \neq 1$ *and* $p_j < p_c(\mathbb{Z}^2)$ *then* $\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \infty\}) = 0$.

2.2 The existence of the supercritical phase

2.2.1 Directed paths in $\mathcal{P}_i$ and their 'lifts'

An argument similar to that in the proof of Lemma 2.3 can be used in order to show that the probability of percolation within any slab of $\mathbb{Z}^3$ is zero (this is true if, for instance, at least parameters among p_1, p_2 and p_3 are different from one, regardless of how high they are chosen). However, that is not the case for Bernoulli percolation. In fact Grimmett and Marstrand [GM90] have shown that the critical value for the Bernoulli percolation process restricted to a slab of thickness k converges to $p_c(\mathbb{Z}^d)$ as k goes to infinity. The fact that this does not hold for our model restricts the tools that can be used in order to study the properties of the supercritical phase of this model. In particular, in order to prove that this supercritical phase indeed exists it is hopeless to look for infinite ω-components lying in any coordinate plane $\mathcal{P}_i$ or in a slab of $\mathbb{Z}^3$. Instead we construct a sort of oriented subgraph of $\mathbb{Z}^3$ in which we can forget about the strong dependencies of the original process. We begin the construction in this section where we define what is a *directed path* and its *lift*.

We say that a path $\Gamma = \{v_0, v_1, v_2, \dots\} \subset \mathbb{Z}^2$ is a *directed path* if $v_{n+1} \in \{v_n + (0, 1), v_n + (1, 0)\}$ for all $n \geq 0$. Similarly we define the directed paths in each $\mathcal{P}_i$ using the identification with $\mathbb{Z}^2$ as in Remark 1.1.

Let $\Gamma = \{v_0, v_1, v_2, \ldots\} \subset \mathcal{P}_i$ be a infinite directed path having $v_0 = 0$. Fix $n_{-1} = -1$, $n_0 = 0$ and for each integer $j \geq 1$ set

$$n_j = \inf \{n > n_{j-1} \,;\, \langle v_n - v_{n-1}, v_{n+1} - v_n \rangle = 0\}$$

where $\langle \cdot\,, \cdot \rangle$ stands for the restriction of the inner product of $\mathbb{R}^3$ to $\mathcal{P}_i$. Informally, those are the indices for which one have to turn by an angle of 90 degrees when going along the path Γ. Defining $\Gamma_j = \bigcup_{n_{j-1}<n\leq n_j} \{v_n\}$ then $\Gamma = \bigcup_{j=0}^{\infty} \Gamma_j$. Since the Γ_j are disjoint this induces a partition of Γ into *'straight segments'*.

For a finite integer M we say that Γ is M*-directed* if

$$\sup\{n_j - n_{j-1};\ j \in \mathbb{Z}\} \leq M.$$

In this case we have that each 'segment' Γ_j contains at most M sites. If Γ is a infinite M-directed path, then for each integer $t \geq 0$ there is a unique $\Gamma(t) \in \mathbb{Z}$ such that $(t, \Gamma(t)) \in \Gamma$. We say that Γ' is *lower* than Γ if for each t we have $\Gamma'(t) \leq \Gamma(t)$. For any collection of infinite M-directed paths there exists the *lowest path* in this family defined as the path $L = \{(t, L(t)); t \geq 0\}$ where $L(t)$ is the minimum of $\Gamma(t)$ as Γ runs over all paths in the collection.

For each directed path $\Gamma \subset \mathcal{P}_i$ let us define its *lift* $\tilde{\Gamma}$ as being the subgraph of $\mathbb{Z}^3$ given by:

$$\tilde{\Gamma} = \pi_i^{-1}(\Gamma)\,.$$

It can be visualized as a copy of the $\mathbb{Z}^2$-lattice that has been alternately folded left and right at the columns $\pi_i^{-1}(v_{n_j})$ by an angle of 90 degrees and then embedded in $\mathbb{Z}^3$ in such a way that its projection into $\mathcal{P}_i$ is exactly Γ (see Figure 2.2). Defining, for each j, $\tilde{\Gamma}_j = \pi_i^{-1}\Gamma_j$ then $\tilde{\Gamma} = \bigcup_{j=1}^{\infty} \tilde{\Gamma}_j$ is a partition into the disjoint *strips* $\tilde{\Gamma}_j$.

We conclude this section proving that if we choose p_i to be high enough then, under $\mathbb{P}_{p_i}$, the probability that there exists an infinite M-directed open path starting from the origin is positive. For that we use some simple *oriented percolation* arguments in a rescaled lattice. In our context oriented percolation on $\mathbb{Z}^2$ (or in any lattice isomorphic to it) will consist in looking for the existence of infinite directed paths in configurations sampled from $\mathbb{P}_p$ (or the in the corresponding measure for other lattices) and in studying the properties of those paths and of the connected components emerging from the model. As for ordinary Bernoulli percolation, it

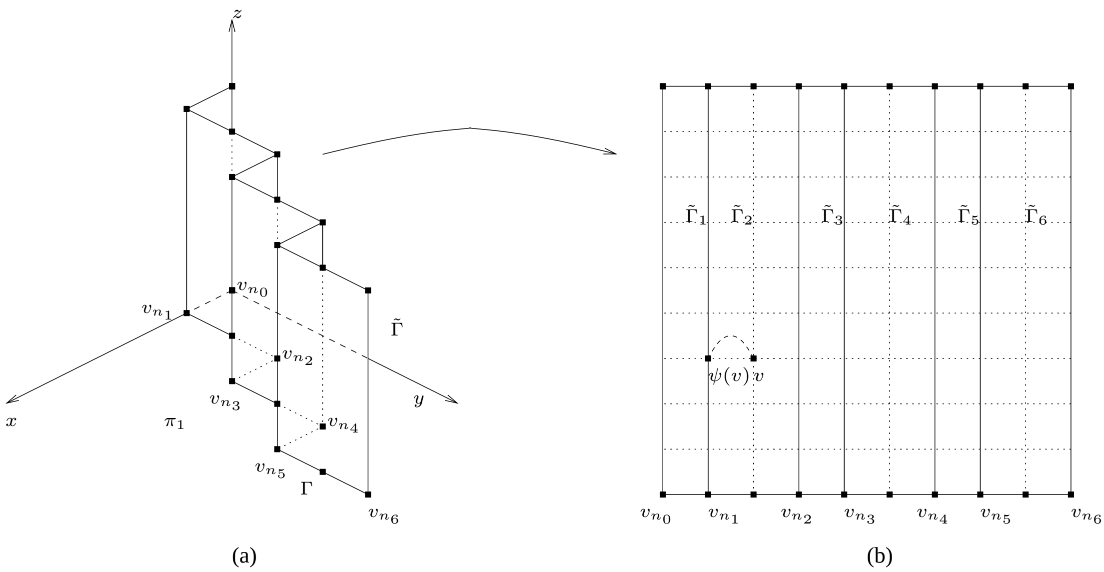

Figure 2.2: (a) The directed path Γ and the graph $\tilde{\Gamma}$; (b) the graph $\tilde{\Gamma}$ is isomorphic to $\mathbb{Z}^2$.

is well known that this model also exhibits a phase transition: there exists a critical value $0 < p'_c(\mathbb{Z}^2) < 1$ such that the probability of finding such paths is zero for each $p < p'_c(\mathbb{Z}^2)$ while it is strictly positive for any $p > p'_c(\mathbb{Z}^2)$. For more information on oriented percolation see [Dur84].

When considering oriented percolation we restrict the kind of allowable paths it is harder for the origin to percolate, so $p'_c(\mathbb{Z}^2) \geq p_c(\mathbb{Z}^2)$. If we increase further the restriction on the paths and look for the existence of infinite M-directed paths then it is intuitively clear that, as we increase the value of the parameter p, there might exist some threshold above which the probability of finding such a path starting from the origin is positive. Next we outline an argument for proving this fact for $M = 2$ that we state in Proposition 2.5. The reader who accepts this fact can skip the discussion below going to Section 2.2.2.

Let $\mathbb{Z}^2_* = \{(x + 1/2, y + 1/2);\ x, y \in \mathbb{Z}\}$ be the *dual lattice* of $\mathbb{Z}^2$. We identify each site of $\mathbb{Z}^2$ with the unique *face* of $\mathbb{Z}^2$ having this site lying in its center. Consider the following subset

$$\mathbb{L} = \left\{(x + 1/2, y + 1/2 \in \mathbb{Z}^2_*;\ x + y = 3k,\ k = 0, 1, 2, \ldots \text{ and } y/2 \leq x \leq 2y\right\}$$

and add directed edges connecting each face $(x + 1/2, y + 1/2) \in \mathbb{L}$ to the faces

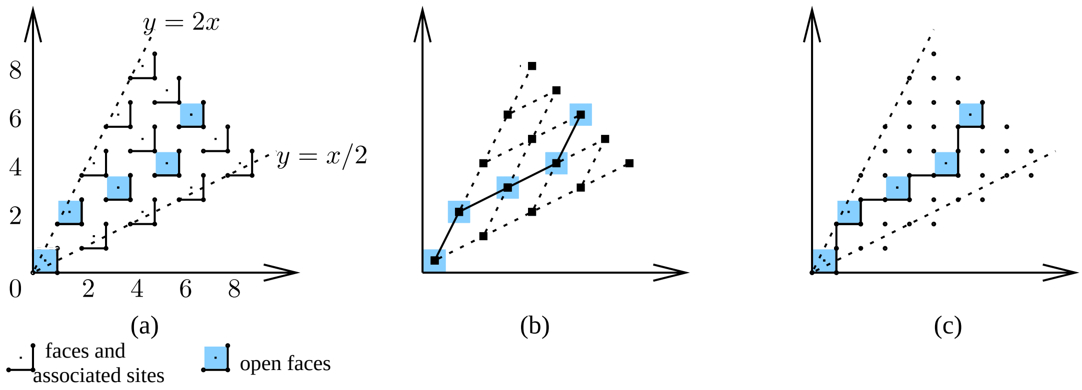

Figure 2.3: (a) The region $\mathbb{L}$ with an oriented path of open faces; (b) The corresponding oriented path on $\mathbb{Z}^2_*$ (c) The corresponding oriented path of associated sites of $\mathbb{Z}^2$.

$((x+2)+1/2,(y+1)+1/2)$ and $((x+1)+1/2,(y+2)+1/2)$. Note that this lattice is isomorphic to the positive quadrant of the $\mathbb{Z}^2$-lattice endowed with oriented arcs connecting its sites (see Figure 2.3,(a) and (b)).

We say that a face $(x+1/2, y+1/2) \in \mathbb{Z}^2_*$ is *open* if the sites (x,y), $(x+1,y)$ and $(x+1,y+1)$ are all open. Those sites will be called the *sites associated* to the given face. Thus we have that $\mathbb{P}_p\left(\{(x+1/2,y+1/2) \text{ is open }\}\right) = p^3$. It follows that if we set $p > [p_c'(\mathbb{Z}^2)]^{1/3}$ then the $\mathbb{P}_p$-probability of finding an infinite directed path of open faces in $\mathbb{L}$ is positive. Now if we fix such an infinite path and look to the set of sites associate to each face in it, then we find a infinite 2-directed open path in $\mathbb{Z}^2$ (see Figure 2.3 (c)). From the discussion in the previous we have proved the following proposition:

Proposition 2.5. *Let* $p > [p_c'(\mathbb{Z}^2)]^{1/3}$. *Then*

$$\mathbb{P}_p\left(\{\exists \text{ an infinite } 2\text{-directed open path starting at } \mathbf{0}\}\right) > 0. \tag{2.4}$$

2.2.2 Restriction to the lift of directed paths

A random element X defined in $\{0,1\}^{\mathbb{Z}^d}$ is called a K-dependent percolation if whenever $A, B \subset \mathbb{Z}^d$ are two sets lying at l_1-distance greater then K apart from each other then the families $\{X(v)\}_{v\in A}$ and $\{X(v)\}_{v\in B}$ are independent. The notion of a K-dependent percolation process is extended to other graphs naturally by considering the graph distance on them.

We will consider the process ω restricted to $\tilde{\Gamma}$. The point in doing so is that if we write $\omega(w) = \omega_1(\pi_1(w))\eta(w)$ where $\eta(w) = \omega_2(\pi_2(w))\omega_3(\pi_3(w))$ then η is a M-dependent percolation on $\tilde{\Gamma}$.

For a matter of concreteness, from now on we fix $\Gamma = \{v_0, v_1, v_2, \ldots\} \subset \mathcal{P}_i$ a M-directed path satisfying

$$v_0 = (0,0,0) \text{ and } v_1 = (1,0,0) \tag{2.5}$$

For each site $w \in \tilde{\Gamma}$ let $j(w)$ be the unique index such that $w \in \tilde{\Gamma}_j$ and $k(w)$ be the unique index such that

$$\pi_1(w) = v_{k(w)} \tag{2.6}$$

A site $w \in \tilde{\Gamma}$ will be uniquely represented as

$$w = (k(w), h(w)) \tag{2.7}$$

where $h(w)$ is the value of the z-coordinate of w, which we call the *height* of w (see Figure 2.2).

Set, on the one hand, $l'(w) = 2$ if $j(w)$ is odd and $l'(w) = 3$ if $j(w)$ is even and, on the other hand, $l''(w) = 2$ if $j(w)$ is even and $l''(w) = 3$ if $j(w)$ is odd. Then define the following random fields in $\tilde{\Gamma}$:

$$\omega'(w) = \omega_{l'(w)}\left(\pi_{l'(w)}(w)\right) \tag{2.8}$$

$$\omega''(w) = \omega_{l''(w)}\left(\pi_{l''(w)}(w)\right) \tag{2.9}$$

Using the definitions of l', l'' and equations (2.6), and (2.8) we can rewrite equation (2.1) as

$$\omega(w) = \omega_1(v_{k(w)})\omega'(w)\omega''(w) = \omega_1(v_{k(w)})\eta(w) \tag{2.10}$$

where η is the percolation process in $\tilde{\Gamma}$ given by

$$\eta(w) = \omega_2(\pi_2(w))\omega_3(\pi_3(w)) = \omega'(w)\omega''(w). \tag{2.11}$$

Next we show that the processes ω' is a Bernoulli percolation process in Γ.

Proposition 2.6. *Let $\Gamma \subset \mathcal{P}_1$ be a M-directed path satisfying the condition (2.5) and set $p_2 = p_3 = p$. Then under $\mathbb{P}_{\mathbf{p}}$, the processes ω' defined on $\tilde{\Gamma}$ by (2.8) is a Bernoulli site percolation with parameter p.*

Proof. Since the mapping $w \mapsto \pi_{l(w)}(w)$ from $\tilde{\Gamma}$ onto $\mathcal{P}_2 \cup \mathcal{P}_3$ is injective, we have that $\{\omega'(w)\}_{w\in\tilde{\Gamma}}$ is a random field defined in terms of a collection of $i.i.d.$ Bernoulli random variables having mean p (under $\mathbb{P}_{\mathbf{p}}$). Thus it is a Bernoulli site percolation with parameter p. □

Using the last result we can characterize η as a M-dependent percolation on $\tilde{\Gamma}$.

Proposition 2.7. *Let $\Gamma \subset \mathcal{P}_1$ be a M-directed path satisfying the condition (2.5), and set $p_2 = p_3 = p$. Then under $\mathbb{P}_{\mathbf{p}}$, the process η defined on $\tilde{\Gamma}$ by (2.11) is a M-dependent percolation process with $\mathbb{P}_{\mathbf{p}}\left(\{\eta(w) = 1\}\right) = p^2$.*

Proof. Recall the representation for the sites in $\tilde{\Gamma}$ given by equation 2.7. Let ψ be the mapping from $\tilde{\Gamma}$ onto itself defined by $\psi(w) = \big(n_{j(w)-1}, h(w)\big)$. In words, if w belongs to the strip $\tilde{\Gamma}_j$ then $\psi(w)$ is the site lying in the *corner* of the strip $\tilde{\Gamma}_{j-1}$ that shares the same height with w. Note that, since Γ is M-directed then the (graph) distance between w and $\psi(w)$ in $\tilde{\Gamma}$ is at most M. The projection of $\psi(w)$ into $\mathcal{P}_{l'(\psi(w))}$ is the same as the projection of w into $\mathcal{P}_{l''(w)}$, thus

$$\omega''(w) = \omega_{l''(w)}\left(\pi_{l''(w)}\left(w\right)\right) = \omega_{l'(\psi(w))}\left(\pi_{l'(\psi(w))}\left(\psi(w)\right)\right) = \omega'(\psi(w)).$$

Plugging the last equation into (2.11) we have that

$$\eta(w) = \omega'(w)\omega'(\psi(w)). \tag{2.12}$$

By equation (2.12) the $\eta(w)$ is determined by the value of ω' in w itself and in $\psi(w)$. Since ω' has been shown to be a Bernoulli percolation process in $\tilde{\Gamma}$ we get that η is a M-dependent percolation process in Γ. Furthermore, we have that

$$\begin{aligned}
\mathbb{P}_{\mathbf{p}}\left(\{\eta(w) = 1\}\right) &= \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}\left(\{\omega'(w)\omega''(w) = 1\}\right) \\
&= \mathbb{P}_{p_{l'(w)}} \times \mathbb{P}_{p_{l''(w)}}\left(\left\{\omega_{l'(w)}\left(\pi_{l'(w)}(w)\right)\omega_{l''(w)}\left(\pi_{l''(w)}(w)\right) = 1\right\}\right) \\
&= \mathbb{P}_{p_{l'(w)}}\left(\left\{\omega_{l'(w)}\left(\pi_{l'(w)}(w)\right) = 1\right\}\right) \times \\
&\quad \mathbb{P}_{p_{l''(w)}}\left(\left\{\omega_{l''(w)}\left(\pi_{l''(w)}(w)\right) = 1\right\}\right) \\
&= p_{l'(w)}p_{l''(w)} = p_2p_3 = p^2.
\end{aligned}$$

□

Remark 2.8.

- If we do not require p_2 to be equal to p_3 then the marginals of ω' are still mutually independent Bernoulli random variables, however they can have mean equal to p_2 or to p_3 depending on which strip $\tilde{\Gamma}_j$ they lie in. Furthermore η is still a M-dependent percolation process with $\mathbb{P}_{\mathbf{p}}(\{\eta(w) = 1\}) = p_2 p_3$.

- The fundamental assumption required in order for the proof of Proposition 2.7 to work is that Γ is M-directed. The condition (2.5) not essential at all and was only introduced in order to enable us to introduce the processes in (2.8). In case Γ does not satisfy this constraint then we would need to interchange the roles of l' and l'' in the definition of these processes and Proposition 2.7 would still hold.

2.2.3 Proof of Theorem 1.2

We are now in the position to prove the existence of percolation if the parameters are taken to be high enough. This will complete the proof of Theorem 1.2.

We can assign a partial order to the set $\{0,1\}^{\mathbb{Z}^d}$ by defining $\eta \leq \eta'$ if $\eta(v) \leq \eta'(v)$ for all $v \in \mathbb{Z}^d$. A random variable X defined on $\{0,1\}^{\mathbb{Z}}$ is said to be increasing if $X(\eta) \leq X(\eta')$ whenever $\eta \leq \eta'$. An event $\mathcal{A} \subset \{0,1\}^{\mathbb{Z}^d}$ is said to be increasing if $\mathbf{1}_{\mathcal{A}}$ is increasing. Instances of increasing events are $\{\mathbf{0} \leftrightarrow \partial B^d(n)\}$ and $\{\mathbf{0} \leftrightarrow \infty\}$. If μ and μ' are two Borel measures on $\{0,1\}^{\mathbb{Z}^d}$ we say that μ' dominates stochastically μ if $\mu'(\mathcal{A}) \geq \mu(\mathcal{A})$ for all increasing event $\mathcal{A}$.

We will use the following result on dependent percolation processes on $\mathbb{Z}^d$ due to Liggett, Schonmann and Stacey: If the process satisfy that, conditioned on what happens outside a given neighborhood of each site, the probability of having that site open is large enough then this process dominates stochastically a Bernoulli percolation process with positive density. Furthermore the density of the dominated Bernoulli percolation can be made arbitrarily close to one provided that the conditional probability referred above is made sufficiently close to one (see [LSS97, Theorem 0.0] for the precise statement). Note that this result apply in particular to the class of M-dependent percolation processes.

Theorem 2.9. *Suppose that* $p_1 > [p_c'(\mathbb{Z}^2)]^{1/3}$. *Then exists some* $p_0 \in (0,1)$ *such that if* $p_j > p_0$ *for* $j = 2, 3$ *then* $\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \infty\}) > 0$.

Proof. We denote $\mathbb{P} = \mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$ and $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Let $\mathcal{A}_1$ be the event that there is an infinite 2-directed path starting from the origin in $\mathcal{P}_1$ and satisfying condition (2.5). By Proposition 2.5 we have that $\mathbb{P}(\mathcal{A}_1) = \mathbb{P}_{p_1}(\mathcal{A}_1) > 0$. On the event that such paths exist let Γ be the lowest path among them and let $\tilde{\Gamma}$ be its lift. Then

$$\begin{aligned}\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \infty\}\right) &= \mathbb{P}\left(\omega \in \{\mathbf{0} \leftrightarrow \infty\}\right) \\ &= \mathbb{E}\left(\mathbb{P}\left(\omega \in \{\mathbf{0} \leftrightarrow \infty\} \mid \mathcal{F}_1\right)\right) \\ &\geq \mathbb{E}\left(\mathbb{P}\left(\omega \in \left\{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}\right\}, \mathcal{A}_1 \middle| \mathcal{F}_1\right)\right) \\ &= \mathbb{E}\left(\mathbf{1}_{\mathcal{A}_1}\mathbb{P}\left(\omega \in \left\{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}\right\} \middle| \mathcal{F}_1\right)\right)\end{aligned} \tag{2.13}$$

However, for almost all $\omega_1 \in \mathcal{A}_1$,

$$\mathbb{P}\left(\omega \in \left\{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}\right\} \middle| \mathcal{F}_1\right)(\omega_1) = \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}\left(\eta \in \{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}(\omega_1)\}\right), \tag{2.14}$$

where, for each ω_1 the random field $\eta = \eta(\omega_1)$ is defined in $\tilde{\Gamma} = \tilde{\Gamma}(\omega_1)$ by (2.11).

Since, for each $\omega_1 \in \mathcal{A}_1$, $\Gamma(\omega_1)$ is a 2-directed path, by Proposition 2.7, we have that the process η is a 2-dependent percolation process in $\tilde{\Gamma}$ with

$$\mathbb{P}_{p_2} \times \mathbb{P}_{p_3}\left(\{\eta(w) = 1\}\right) = p_2 p_3 \to 1$$

as p_2 and p_3 approach 1 simultaneously. Let $p > p_c(\mathbb{Z}^2)$ and denote by μ the measure corresponding to Bernoulli percolation on $\tilde{\Gamma}(\omega_1)$ with density p. Recall that by the representation $w = (k(w), h(w))$ for $w \in \tilde{\Gamma}$ (see equation 2.7) for each $\omega_1 \in \mathcal{A}_1$ the graph $\tilde{\Gamma}(\omega_1)$ is isomorphic to $\mathbb{Z}^2_+ = \{(x, y) \in \mathbb{Z}^2;\ x \geq 0\}$. In addition, by the fact that $p_c(\mathbb{Z}^2) = p_c(\mathbb{Z}^2_+$ (see [Har60] and [Fis61]) we have that μ describes a supercritical site percolation process. By Theorem 0.0 in [LSS97] one can find a p_0 large enough such that if p_2 and p_3 are both greater than p_0 then the distribution of the process η dominates μ. It follows that

$$\mathbb{P}_{p_2} \times \mathbb{P}_{p_3}\left(\eta \in \{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}(\omega_1)\}\right) \geq \mu\left(\{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}(\omega_1)\}\right) > 0.$$

Plugging this inequality into equation (2.14) and then substituting in equation (2.13) yields that $\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \infty\}\right) > 0$. □

The last result gives:

Proof of Theorem 1.2. The second statement in Theorem 1.2 is given by the Corollary 2.4. Setting $p^* = \max\{[p'_c(\mathbb{Z}^2)]^{1/3}, p_0\}$ and using Theorem 2.9 we see that if all components p_i are greater than p^* then $\mathbb{P}_{\mathbf{p}}(\omega \in \{\mathbf{0} \leftrightarrow \infty\}) > 0$. This proves the first statement in Theorem 1.2. □

2.2.4 Upper bound for p_c

Using the formula for η it is possible to show that the parameter p_0 in the statement Theorem 2.9 can be chosen to be equal to $[p_c(\mathbb{Z}^2)]^{1/3}$. This gives an upper bound for p_c. We outline the argument that leads to this conclusion using the same notation as in the previous proof.

We fix $p_1 > [p'_c(\mathbb{Z}^2)]^{1/3}$ and $p_2 = p_3 = p$ where p will be chosen afterwards. Fix $\omega_1 \in \mathcal{A}_1$ and let $\tilde{\Gamma} = \tilde{\Gamma}(\omega_1)$ be the lift of the lowest infinite 2-directed open path starting at the origin of $\mathcal{P}_1$ and use the representation $w = (k(w), h(w))$ for $w \in \tilde{\Gamma}$. We construct a partition of $\tilde{\Gamma}$ into blocks of three sites $\tilde{\Gamma}^{(3)} = \{R(j,h);\ j \geq 0,\ h \in \mathbb{Z}\}$, where $R(j,h) = \{w \in \tilde{\Gamma};\ h(w) = h,\ 3j \leq k(w) < 3(j+1)\}$. In order to view $\tilde{\Gamma}^{(3)}$ as a subgraph we will add an edge between two blocks $R(j,h)$ and $R(j',h')$ if $|j'-j| + |h'-h| = 1$. For a fixed $\omega \in \Omega$ a block R is said to be good if $\omega'(w) = 1$ for all $w \in R$. Recall the definition of ω' in (2.8). From Proposition 2.6, if $p_2 = p_3 = p$ then this process is a Bernoulli percolation process in $\tilde{\Gamma}$. Define the process $X = \{X(R(j,h));\ j \geq 0,\ h \in \mathbb{Z}\}$ where $X(R(j,h)) = \mathbf{1}_{\{R(j,h) \text{ is good}\}}$. Then X represents the process of good blocks in $\tilde{\Gamma}^{(3)}$ and is described by a Bernoulli percolation measure with density p^3. Thus, setting $p > [p_c(\mathbb{Z}^2)]^{1/3}$, the probability of finding an infinite path of good blocks in $\Gamma^{(3)}$ starting from the block $R(0,0)$ is strictly positive. Recall that the process η, given by equation (2.11) is obtained by multiplying $\omega'(\psi(w))$ and $\omega'(w)$. Since the graph distance between $\psi(w)$ and w in $\tilde{\Gamma}$ is at most 2, and each good block $R(j,h)$ is composed of three neighboring ω'-open sites we have that: to each path of good blocks starting at $R(0,0)$ that corresponds at least one path of η-open sites starting at the site $w_0 = (k(w_0), h(w_0)) := (2, 0)$. Then,

$$\mathbb{P}_{p_2} \times \mathbb{P}_{p_3} \left(\eta \in \left\{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}(\omega_1)\right\}\right) \geq \tag{2.15}$$

$$\left[\mathbb{P}_{p_2} \times \mathbb{P}_{p_3} \left(\eta \in \left\{\mathbf{0} \leftrightarrow w_0 \text{ in } \tilde{\Gamma}(\omega_1)\right\}\right)\right] \times \tag{2.16}$$

$$\left[\mathbb{P}_{p_2} \times \mathbb{P}_{p_3} \left(\eta \in \left\{w_0 \leftrightarrow \infty \text{ in } \tilde{\Gamma}(\omega_1)\right\}\right)\right] > 0 \tag{2.17}$$

where the first inequality comes from the Harris-FKG inequality and the second one comes from the fact that both events in its left hand site have positive probability. Note that this is the same conclusion obtained in equation (2.14).

By the results of Harris [Har60] an Fisher [Fis61], if $p \geq p_c(\mathbb{Z}^2)$ then the configurations of Bernoulli site percolation in $\mathbb{Z}^d$ have, almost surely, the following property: There exists a nested sequence of open semi-circuits in the semi-space $\{(x, y);\ x \geq 0,\ y \in \mathbb{Z}\}$ connecting $\{(y, 0); y > 0\}$ to $\{(y, 0);\ y < 0\}$.

Translating this result to the process $X(R(j, h))$ we see that there exists (almost surely) a sequence of nested open semi-circuits linking the set $\{R(0, y); y > 0\}$ to $\{R(0, y); y < 0\}$ in $\tilde{\Gamma}^{(3)}$. To each such sequence there corresponds a nested sequence of semi-circuits of η-open sites in $\tilde{\Gamma}$ connecting $\{w \in \tilde{\Gamma};\ k(w) = 2,\ h(w) > 0\}$ to $\{w \in \tilde{\Gamma};\ k(w) = 2,\ h(w) < 0\}$. Since any infinite cluster in $\tilde{\Gamma}$ would have to intersect an infinite number of η-open circuits, the existence of such a sequence of circuits assures the uniqueness of the η-cluster in $\tilde{\Gamma}$. In addition, the fact that all sites in $\pi_i(\tilde{\Gamma})$ are ω_1-open, yields that the infinite η-cluster is contained in an infinite ω-cluster. In particular, the infinite open cluster in the restriction of the process ω to $\tilde{\Gamma}$ is also unique.

Remark 2.10. The discussion in the last paragraph does not imply the uniqueness for the infinite ω-cluster when it exists.

Proposition 2.11. *If* $\omega_1 \in \mathcal{A}_1$ *and* $p_i > (p_c(\mathbb{Z}^2))^{1/3}$ *for* $i = 1, 2, 3$ *then*

$$\mathbb{P}_{p_2} \times \mathbb{P}_{p_3} \left(\eta \in \left\{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}(\omega_1)\right\}\right) > 0.$$

Moreover, in the event $\left\{\eta \in \{\mathbf{0} \leftrightarrow \infty \text{ in } \tilde{\Gamma}(\omega_1)\}\right\}$ *the infinite open cluster of* ω *restricted to* $\tilde{\Gamma}(\omega_1)$ *is unique. In particular, if* $p_i > [p'_c(\mathbb{Z}^2)]^{1/3}$ *for all* $i = 1, 2, 3$*, then*

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \infty\}\right) > 0.$$

As a simple consequence of the last proposition and from Proposition 2.3 we highlight the next simple bounds for the critical point p_c.

Corollary 2.12. *For the coordinate percolation process given by (1.1) with* $p_1 = p_2 = p_3$ *the critical parameter* p_c *satisfy:*

$$p_c(\mathbb{Z}^2) \leq p_c \leq \left[p'_c(\mathbb{Z}^2)\right]^{1/3}. \tag{2.18}$$

Chapter 3

The radius of the open cluster at the origin

In this chapter we prove equation (1.3) establishing thus the exponential decay for the tail distribution of the cluster containing the origin for the coordinate percolation process in $\mathbb{Z}^3$ when at least two parameters of the model are sub-critical. We also prove equation (1.4) establishing that the rate of decay is at most polynomial when at least two of the components of $\mathbf{p}$ are super-critical. Those two results give Theorem 1.3.

3.1 Exponential decay

The proof of equation (1.3) is a consequence of Corollary 2.2 and from Menshikov's [Men86] and Aizenman and Barsky [AB87] results on the exponential decay for Bernoulli percolation (see equation (1.5)). The idea is the following: Suppose that the the event $\{\omega \in \{\mathbf{0} \leftrightarrow \partial B^3(n)\}\}$ happens. Then, for at least one index i such that ω_i is a sub-critical Bernoulli percolation in $\mathcal{P}_i$, the event $\{\mathbf{0} \leftrightarrow \partial B_i^2(n)\}$ will also happen. Applying equation (1.5) yields the result. We state that result as the following proposition:

Proposition 3.1 (First statement in Theorem 1.3)**.** *Consider the coordinate percolation model in $\mathbb{Z}^3$. If at least two parameters are sub-critical then there is a constant $\psi(\mathbf{p})$ such that*

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \partial B^3(n)\}\right) \le \exp\left(-\psi(\mathbf{p})n\right). \tag{3.1}$$

Proof. By Corollary 2.2, if $\{\mathbf{0} \leftrightarrow \partial B^3(n)\}$ occurs, then $\omega_j \in \left\{\mathbf{0} \leftrightarrow \partial B_j^2(n)\right\}$ for at least two different indices $j \in \{1, 2, 3\}$. Then, writing $\mathbb{P}$ for $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$ we have:

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \partial B^3(n)\}\right) \leq \sum_{\substack{A \subset \{1,2,3\} \\ |A|=2}} \mathbb{P}\left(\omega_j \in \left\{\mathbf{0} \leftrightarrow \partial B_j^2(n) \text{ for all } j \in A\right\}\right) \quad (3.2)$$

Any fixed subset $A \subset \{1, 2, 3\}$ with $|A| = 2$ must contain an index j such that ω_j is a sub-critical Bernoulli site percolation process in $\mathcal{P}_i$. Fixe, for convenience, p_2 and p_3 the sub-critical parameters of $\mathbf{p}$ and define $\alpha(\mathbf{p}) = \min\{\psi(p_2), \psi(p_3)\}$ where $\psi(p_2)$ and $\psi(p_3)$ are given by equation (1.5) applied to the percolation process in $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively. By equation (1.5) we have that:

$$\mathbb{P}\left(\omega_j \in \left\{\mathbf{0} \leftrightarrow \partial B_j^2(n) \text{ for all } j \in A\right\}\right) \leq \exp\left(-\alpha(\mathbf{p})n\right).$$

Plugging that into equation (3.2) yields:

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0} \leftrightarrow \partial B^3(n)\}\right) \leq 3\exp\left(-\alpha(\mathbf{p})n\right).$$

We can then choose suitably an $0 < \psi(\mathbf{p}) < \alpha(\mathbf{p})$ for which equation (3.1) holds for all n.

□

3.2 Polynomial decay

3.2.1 Crossing events in a block lattice

In this section we derive some results about crossing events in some rescaled lattices isomorphic to $\mathbb{Z}^2$. Those rescaled lattices will be composed of blocks of sites from $\mathbb{Z}^2$ or $\mathbb{Z}^3$. If the configuration within those blocks satisfies some convenient properties (to be defined latter) we will say that the block is *good*. The existence of crossings of good blocks in the rescaled lattices will be important in the next sections where we will use them in order to assure the existence of some long open paths in the original $\mathbb{Z}^3$-lattice.

Let $R(n, m) = \{(x, y) \in \mathbb{Z}^2;\ 0 \leq x \leq n - 1,\ 0 \leq y \leq m - 1\}$ be the rectangle having the origin of $\mathbb{Z}^2$ as its bottom-left corner and horizontal and vertical sides containing n and m vertices respectively. A path of nearest-neighbor sites traversing this rectangle from its left side to its right side will be called a *left-to-right crossing* or a *crossing from left to right* in $R(n,m)$. Similarly we define a *bottom-to-top crossing* or a *crossing from bottom to top* in $R(n,m)$. For a given $\omega \in \Omega$ we say that a crossing is open if all of its sites are ω-open. Define the following crossing events

$$\begin{aligned}\mathcal{A}(n,m) &= \{\text{there is an open left-to-right crossing in } R(n,m)\}\\ \mathcal{B}(n,m) &= \{\text{there is an open bottom-to-top crossing in } R(n,m)\}.\end{aligned} \tag{3.3}$$

For $l,\ k \in \mathbb{Z}$ we define $R(n,m;k,l) = R(n,m) + \{(kn,lm)\}$ and denote by $\mathcal{A}(n,m;k,l)$ and $\mathcal{B}(n,m;k,l)$ the crossing events in $R(n,m;k,l)$ that are analoguous to the ones in (3.3). Moreover, in order to refer to the analogue of the rectangles $R(n,m;k,l)$ that lie on $\mathcal{P}_i$ and the analogue of the crossing events in $\{0,1\}^{\mathcal{P}_i}$ we write $R_i(n,m;k,l)$, $\mathcal{A}_i(n,m;k,l)$ and $\mathcal{B}_i(n,m;k,l)$ respectively. When $n = m$ we may drop the index m and write for instance $R(n;k,l)$ for referring to $R(n,n;k,l)$.

Let a $*$-path in $\mathbb{Z}^2$ be a sequence $\{v_0,v_1,\ldots,v_r\}$ of sites such that $|v_j - v_{j-1}|_\infty = 1$ for all $j = 1,\ldots,r$ (where $|\cdot|_\infty$ stands for the l_∞-distance in $\mathbb{Z}^2$). Denote by $\mathcal{A}^*(n,m;k,l)$ the event that there exists a $*$-path crossing $R(n,m;k,l)$ from left-to-right having all its sites closed. Similarly define the analogous events $\mathcal{B}^*(n,m;k,l)$ for bottom-to-top crossings. It is well known in percolation theory that $\mathcal{B}(n,m;k,l)$ happens if and only if $\mathcal{A}^*(n,m;k,l)$ does not happen.

If now $\mathbb{Z}^2_*$ stands for the graph with vertex set $\mathbb{Z}^2$ and with an edge between each pair of vertices lying at l_∞-distance one from each other and $p_c(\mathbb{Z}^2_*)$ the critical density for Bernoulli percolation on this lattice, then we have that $p_c(\mathbb{Z}^2) + p_c(\mathbb{Z}^2_*) = 1$ (see [Rus81] for a proof). Thus $p > p_c(\mathbb{Z}^2)$ implies that $1-p < p_c(\mathbb{Z}^2_*)$ so by Menshikov's and Aizenman and Barsky Theorem there is a constant $\psi = \psi(p) > 0$ such that for all n,

$$\mathbb{P}_p\left(\{\text{there is a } *\text{-path of closed sites from } \mathbf{0} \text{ to } \partial B(n)\}\right) \le e^{-\psi(p)n}. \tag{3.4}$$

For a real number a let us define $\lceil a \rceil = \min\{n \in \mathbb{Z};\ n \geq a\}$ the least integer greater than a and $\lfloor a \rfloor = max\{n \in \mathbb{Z};\ n \leq a\}$ the greatest integer smaller than a. Fix a constant $c > 0$. Note that the probability that there is a closed $*$-path connecting some fixed site in the left side of $R(\lceil c\log n \rceil, n)$ to some other site lying in the right side of this rectangle is at most equal to the the probability that the origin is connected to the boundary of a box of radius $\lceil c \log n \rceil$ by a closed $*$-path. Then using equation (3.4) we have that:

$$\begin{aligned} \mathbb{P}_p \left(\mathcal{B} \left(\lceil c \log n \rceil, n \right) \right) &= 1 - \mathbb{P}_p \left(\mathcal{A}^* \left(\lceil c \log n \rceil, n \right) \right) \\ &\geq 1 - n e^{-\psi(p) c \log n} = 1 - n^{1 - c\psi(p)} \end{aligned} \tag{3.5}$$

where the factor n appears since we have n choices for the starting points of a crossing in $R \left(\lceil c \log n \rceil, n \right)$. Now if we fix $c > \psi(p)^{-1}$ equation (3.5) yields,

$$\lim_{n \to \infty} \mathbb{P}_p \left(\mathcal{B} \left(\lceil c \log n \rceil, n \right) \right) = 1. \tag{3.6}$$

In particular, for any integer $k > 0$,

$$\lim_{n \to \infty} \mathbb{P}_p \left(\mathcal{A} \left(kn, n \right) \right) = \lim_{n \to \infty} \mathbb{P}_p \left(\mathcal{B} \left(n, kn \right) \right) = 1. \tag{3.7}$$

Let $\tilde{\Gamma}_n(j,l,h) = \{(x,y,z) \in \mathbb{Z}^3;\ jn \leq x \leq (j+1)n - 1,\ ln \leq y \leq (l+1)n - 1,\ hn \leq z \leq (h+1)n - 1\}$. Note that $\tilde{\Gamma}_n(j,l,h) \subset \mathbb{Z}^3$ are *blocks* with side n and satisfy $\pi_1 \left(\tilde{\Gamma}_n(j,l,h) \right) = R_1(n;j,l)$, $\pi_2 \left(\tilde{\Gamma}_n(j,l,h) \right) = R_2(n;j,h)$ and $\pi_3 \left(\tilde{\Gamma}_n(j,l,h) \right) = R_3(n;l,h)$. We define $\Lambda_n = \{\tilde{\Gamma}_n(j,l,h);\ j,l,h \in \mathbb{Z}\}$ and introduce a graph structure to Λ_n by inserting an edge between two blocks $\tilde{\Gamma}_n(j,l,h)$ and $\tilde{\Gamma}_n(j',l',h')$ whenever $|j - j'| + |l - l'| + |h - h'| = 1$. Note that when seen as a graph, Λ_n is isomorphic to the $\mathbb{Z}^3$-lattice.

A block $\tilde{\Gamma}_n(j,l,h)$ is said to be *good* if the following event happens:

$$\left[\mathcal{A}_2(2n, n; j, h) \cap \mathcal{B}_2(n, 2n; j, h) \right] \cap \left[\mathcal{A}_3(2n, n; l, h) \cap \mathcal{B}_3(n, 2n; l, h) \right],$$

By the Harris-FKG inequality and by equation (3.7) we have that

$$\begin{aligned} &\mathbb{P}_{p_2} \times \mathbb{P}_{p_3} \left(\tilde{\Gamma}_n(j,l,h) \text{ is good} \right) \geq \\ &\left[\mathbb{P}_{p_2} \left(\mathcal{A}_2(2n, n; j, h) \right) \right]^2 \times \left[\mathbb{P}_{p_3} \left(\mathcal{A}_3(2n, n; l, h) \right) \right]^2 \longrightarrow 1 \end{aligned}$$

as n gets large. So, choosing n large enough, we can assume that the probability of a block to be good is high. More specifically we have:

Lemma 3.2. *Let $u \in [0,1)$. Suppose that $p_2,\ p_3 > p_c(\mathbb{Z}^2)$. Then there is an integer $n = n(p_2, p_3, u)$ such that*

$$\mathbb{P}_{p_2} \times \mathbb{P}_{p_3} \left(\tilde{\Gamma}_n(j,l,h) \textit{ is good} \right) \geq u. \tag{3.8}$$

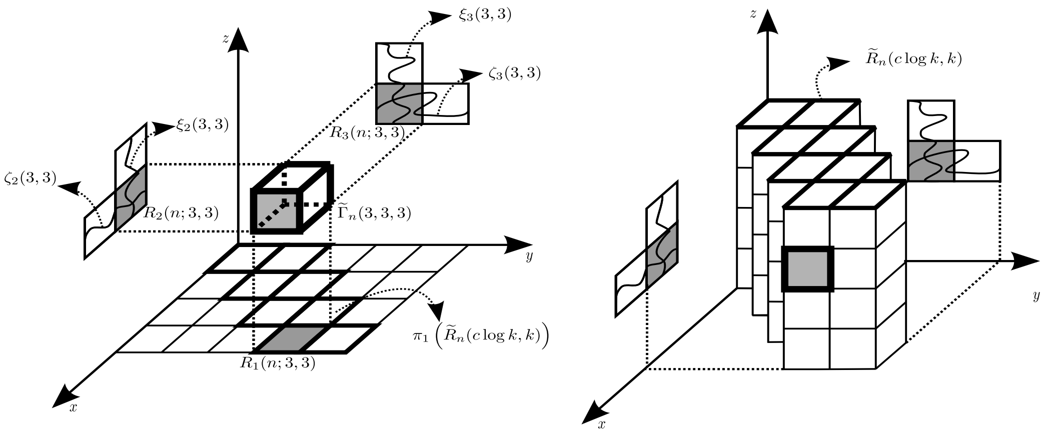

Figure 3.1: The block $\tilde{\Gamma}_n(3,3,3)$ is a good block. In the left picture we see the definitions of the paths $\xi_i(3,3)$ and $\zeta_i(3,3)$ for $i=2,3$. In the right we represent the set $\tilde{R}_n(c\log k,k)$.

For $c>0$ and $k\in\mathbb{Z}$ let

$$\tilde{R}_n(c\log k,k)=\left\{\tilde{\Gamma}_n\left(\lfloor j/2\rfloor,\ \lceil j/2\rceil,\ h\right)\in\Lambda_n;0\le j\le\lceil c\log k\rceil,\ 0\le h\le k\right\}$$
$$=\left\{\tilde{\Gamma}_n(j,l,h)\in\Lambda_n;\ 0\le j\le\lceil c\log k\rceil,\ 0\le h\le k,\ l=j\text{ or }l=j+1\right\}.$$

If we regard the subset Λ_n, $\tilde{R}_n(c\log k,k)$ as a sub-graph of Λ_n, then it is isomorphic to the rectangle $R(2\lceil c\log k\rceil,k)\subset\mathbb{Z}^2$ that have lenght $2\lceil c\log k\rceil$ and height k (see Figure 3.1). Define further,

$$\tilde{\mathcal{B}}_n(c\log k,k)=\left\{\exists\text{ a top-to-bottom crossing of good boxes in }\tilde{R}_n(c\log k,k)\right\}$$

Lemma 3.3. *Let* $p_2,\ p_3>p_c(\mathbb{Z}^2)$ *and* $p_1\neq 0$. *Then there are* $n=n(p_2,p_3)\in\mathbb{Z}_+$, $c=c(p_2,p_3)>0$ *and* $\delta=\delta(p_2,p_3)>0$ *such that for all* $k\in\mathbb{Z}$,

$$\mathbb{P}_{p_2}\times\mathbb{P}_{p_3}\left(\tilde{\mathcal{B}}_n(c\log k,k)\right)\ge\delta. \tag{3.9}$$

Proof. Take p such that $p_i>p>p_c(\mathbb{Z}^2)$ for $i=2,3$. As an application of equation (3.6) (with k playing the role of n) we can choose constants $c=c(p)>0$ and $\delta=\delta(p)>0$ such that

$$\mathbb{P}_p\left(\mathcal{B}_n\left(2\lceil c\log k\rceil,k\right)\right)\ge\delta$$

for all integer $k\ge 0$.

Let $X = \{X(j,h)\}_{(j,h)\in\mathbb{Z}^2}$ be the process on $\{0,1\}^{\mathbb{Z}^2}$ given by

$$X(j,h) = \mathbf{1}_{\{\tilde{\Gamma}_n\left(\lfloor\frac{j}{2}\rfloor,\lceil\frac{j}{2}\rceil,h\right) \text{ is good}\}}$$

and denote by μ its law. The definition of a block to be good depends only on the restriction of the ω_2 and ω_3 processes to the projections of this block and of its neighboring blocks into the planes $\mathcal{P}_2$ and $\mathcal{P}_3$. Thus the we have that the process X is a two-dependent percolation process and that

$$\mathbb{P}_{p_2} \times \mathbb{P}_{p_3}\left(\tilde{\mathcal{B}}_n(c\log k, k)\right) = \mu\left(\mathcal{B}_n\left(2\lceil c\log k, \rceil k\right)\right). \tag{3.10}$$

Then, by [LSS97, Theorem 0.0] we have that there exists $u \in (0,1)$ such that, if

$$\mu\left(\{X(j,h) = 1\}\right) > u, \tag{3.11}$$

then X dominates stochastically a Bernoulli site percolation with parameter p. By Lemma 3.2 we can choose n large enough so that equation (3.11) holds. Then using the fact that $\mathcal{B}_n\left(2\lceil c\log k\rceil, k\right)$ is an increasing event and the stochastic domination, we have that:

$$\mu\left(\mathcal{B}_n\left(2\lceil c\log k\rceil, k\right)\right) \geq \mathbb{P}_p\left(\mathcal{B}_n\left(2\lceil c\log k\rceil, k\right)\right) \geq \delta.$$

Plugging this last inequality into equation (3.10) finishes the proof. □

3.2.2 Constructing paths from projections

For this section we will fix $p_2,\ p_3 > p_c(\mathbb{Z}^2)$, $c = c(p_2,p_3)$, and $n = n(p_2,p_3)$ as given by Lemma 3.3.We will also drop the subscript n that refer to the size of the renormalized blocks in Γ_n, $\tilde{\Gamma}_n$, $\mathcal{B}_n$, $\tilde{\mathcal{B}}_n$, $\tilde{R}_n$, Λ_n and others. Then, the previous lemma assures that the probability of existence of paths of good blocks in $\tilde{R}(c\log k, k)$ is bounded by below uniformly as k increases. In order easy the notation, let us not distinguish between $\tilde{R}(c\log k, k) \subset \Lambda$ and $\bigcup_{j,l,h}\tilde{\Gamma}(j,l,h) \subset \mathbb{Z}^3$ where the union is taken over

$$\{(j,l,k);\ 0 \leq j \leq \lceil c\log k\rceil,\ 0 \leq h \leq k,\ l = j \text{ or } l = j+1\}.$$

The point in considering crossings of good blocks in $\tilde{R}(c\log k, k)$ is that to each such a crossing there corresponds a path $\gamma = \{v_0, v_1, \ldots, v_r\}$ of sites of $\mathbb{Z}^3$ having the following properties

- γ is a path contained in $\tilde{R}(c \log k, k)$
- v_0 belongs to $\pi_1(\tilde{R}(c \log k, k))$ and $v_r \in \{(x, y, z) \in \tilde{R}(c \log k, k); z = (k-1)n\}$.
- The projections of γ into $\mathcal{P}_2$ and $\mathcal{P}_3$ are respectively ω_2 and ω_3-open paths.

In order to prove the existence of such a path γ we present, as a lemma, a procedure that enables us to create paths in $\mathbb{Z}^3$ having their projections lying in crossings of rectangles of $\mathcal{P}_2$ and $\mathcal{P}_3$ previously defined. Before that we introduce some more notation.

For a site $v = (x, y, z) \in \mathbb{Z}^3$ we define $h(z) = z$ the value of its third coordinate that we call the *height* of v. For two sites $v = (x, 0, z) \in \mathcal{P}_2$ and $w = (0, y, z) \in \mathcal{P}_3$ with $h(v) = h(w) = z$ we define

$$v \times w = (x, 0, z) \times (0, y, z) = (x, y, z) \tag{3.12}$$

In words, $v \times w$ is the unique site having v and w as its projections onto $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively. Let

$$\gamma = \{v_0, v_1, \dots, v_m\} \subset \mathbb{Z}^3 \tag{3.13}$$

be a path. We define its *variation* as being

$$h(\gamma) = h(v_m) - h(v_0). \tag{3.14}$$

For any $0 \le k \le m$ we denote

$$\gamma^{(k)} = \{v_0, \dots, v_k\} \tag{3.15}$$

the path γ stopped at its k-th step. Denoting for any $k \in \mathbb{Z}$

$$\tau_k = \tau_k(\gamma) = \inf\{j \ge 0;\ h(v_j) = k\} \tag{3.16}$$

we can define the path stopped at the first time it hits height k by:

$$\gamma \wedge k = \gamma^{(\tau_k)}. \tag{3.17}$$

If the infimum in equation (3.16) is taken over an empty set we simply take $\gamma \wedge k = \gamma$. For a path γ as in (3.13) we define its *reversal* as being the path

$$\overline{\gamma} = \{v_m, v_{m-1}, \dots, v_0\}. \tag{3.18}$$

If we now take γ as in (3.13) and another path

$$\gamma' = \{w_0, \dots, w_{m'}\} \tag{3.19}$$

having $v_m = w_0$ we define their *concatenation* by:

$$\gamma * \gamma' = \{v_0, \dots, v_m, w_1, \dots, w_{m'}\}. \tag{3.20}$$

Now, if γ and γ' given by (3.13) and (3.19) intersect theirselves however not necessarily in their endpoints we define their *juxtaposition* $\gamma \circ \gamma'$ as being the unique path described as follows: It starts at v_0, goes along γ until it first hits γ_2 and then, from that point on, it goes along γ_2 until its final point $w_{m'}$.

We say that two paths γ and γ' given as in (3.13) and (3.19) given are *compatible* if

$$\gamma \subset \mathcal{P}_2 \text{ and } \gamma' \subset \mathcal{P}_3 \tag{3.21}$$

$$h(v_0) = h(w_0) \text{ and } h(v_m) = h(w_{m'}) \tag{3.22}$$

$$\begin{gathered} h(\gamma^{(k)})h(\gamma) \geq 0 \text{ for all } 0 \leq k \leq m \text{ and} \\ h(\gamma'^{(l)})h(\gamma') \geq 0 \text{ for all } 0 \leq l \leq m' \end{gathered} \tag{3.23}$$

$$\tau_h(\gamma) = m \text{ and } \tau_h(\gamma') = m', \text{ where } h = h(\gamma) = h(\gamma'). \tag{3.24}$$

Condition (3.22) states that both γ and γ' start and finish at the same height, and it implies that $h(\gamma) = h(\gamma')$. Condition (3.23) requires that as one goes forward along the paths, the variation does not change signs, meaning that the paths will always lie above or beneath their initial height. Finally, condition(3.24) guarantees that γ and γ' only hit the final height at their respective end points v_m and $w_{m'}$.

The point in defining the notion of compatible paths is that whenever γ and γ' are compatible it is possible to find a path in $\mathbb{Z}^3$ having γ and γ' as its projections into $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively. This is the content of the following lemma:

Lemma 3.4. *Let γ and γ' given as in* (3.13) *and* (3.19) *be two compatible paths. There is a path $\gamma \times \gamma' \subset \mathbb{Z}^3$ starting at $v_0 \times w_0$ and ending at $v_m \times w_{m'}$ satisfying:*

$$\pi_2(\gamma' \times \gamma) = \gamma \textit{ and } \pi_3(\gamma \times \gamma') = \gamma'. \tag{3.25}$$

Remark 3.5. Usually there can be more than one path connecting $v_0 \times w_0$ and $v_m \times v_m$ and satisfying (3.25). So whenever we write $\gamma \times \gamma'$ we are referring to one of those paths arbitrarily selected.

Proof of Lemma 3.4: Let $h = |h(\gamma)| = |h(\gamma')|$. We will use induction in h. We also restrict ourselves to the case $h(v_0) = h(w_0) = 0$ and $h(v_m) = h(w_{m'}) = h > 0$. The proof of any other case is similar.

We first consider $h = 1$. Let $\gamma_H = \{v_0, \dots, v_{m-1}\}$ and $\gamma'_H = \{w_0, \dots, w_{m'-1}\}$ the horizontal parts of the paths γ and γ' (note that they can be a single point if $n = 1$ or $m' = 1$). We can thus define:

$$\gamma_H \times w_0 = \{v_0 \times w_0, v_1 \times w_0, \dots, v_{m-1} \times w_0\} \text{ and} \tag{3.26}$$

$$v_{m-1} \times \gamma'_H = \{v_{m-1} \times w_0, v_{m-1} \times w_1, \dots, v_{m-1} \times w_{m'-1}\}. \tag{3.27}$$

The paths above are well defined, since $h(v_j) = h(w_i) = 0$ for any v_j and w_j appearing at the right hand side of those equations. Since the ending point of $\gamma_H \times w_0$ is equal to the starting point of $v_{m-1} \times \gamma'_H$ we can define

$$\gamma_H \times \gamma'_H = (\gamma_h \times w_0) * (v_{m-1} \times \gamma'_H). \tag{3.28}$$

It is then straightforward to check that $\pi_2(\gamma_H \times \gamma'_H) = \gamma_H$ and that $\pi_3(\gamma_H \times \gamma'_H) = \gamma'_H$ and that $\gamma_H \times \gamma'_H$ starts at $v_0 \times w_0$ and ends at $v_{m-1} \times w_{m'-1}$. If we now let $\gamma_V = \{v_{m-1}, v_m\}$ and $\gamma'_V = \{w_{m'-1}, w_{m'}\}$ be the vertical parts of γ and γ' respectively, and define

$$\gamma_V \times \gamma'_V = \{v_{m-1} \times w_{m'-1}, v_m \times w_{m'}\} \tag{3.29}$$

then $\pi_2(\gamma_V \times \gamma'_V) = \gamma_V$ and $\pi_3(\gamma_V \times \gamma'_V) = \gamma'_V$. Finally let us set

$$\gamma \times \gamma' = (\gamma_H \times \gamma'_H) * (\gamma_V \times \gamma'_V) \tag{3.30}$$

which is a path starting at $v_0 \times w_0$, ending at $v_m \times w_{m'}$ and satisfying (3.25). This finishes the proof for $h = 1$.

Now let us consider the case $h = h_0 + 1$ where $h_0 \geq 1$ is fixed. Assuming that the lemma holds for any pair of compatible paths having height no greater than $n_0 \geq 1$ we are going to show that the lemma holds for γ and γ' finishing thus the proof.

We begin by splitting the paths γ and γ' into several up and down-excursions having variation n_0. For that let $t_0 = t'_0 = 0$ and define inductively for all $n \geq 1$:

$$\begin{aligned} t_{2n-1} &= \inf\{j > t_{2n-2}; h(v_j) = h_0\} \\ t_{2n} &= \inf\{j > t_{2n-1}; h(v_j) = 0\}, \end{aligned} \tag{3.31}$$

with the convention that $\inf \emptyset = \infty$. Let f be defined so that $2f-1$ is the number of finite elements in the sequence $t_0, t_1, t_2, \ldots$. Then f represents the number of excursions from height zero to height h_0. Similarly we define t'_{2n-1}, t'_{2n} and f', the analogous indices for the path γ'. We will assume that $f > 1$ and $f' > 1$. The other cases are simpler to deal with.

Then we have the following sequences:

$$t_0 < t_1 < \cdots < t_{2f-1} \text{ and } t'_0 < t'_1 < \cdots < t'_{2f'-1} \tag{3.32}$$

and the paths:

$$\begin{aligned} \gamma_j &= \{v_{t_j}, \ldots, v_{t_{j+1}}\} \text{ for } j = 0, \ldots, 2f-2 \\ \gamma'_j &= \{w_{t'_j}, \ldots, w_{t'_{j+1}}\} \text{ for } j = 0, \ldots, 2f'-2. \end{aligned} \tag{3.33}$$

Note that, if j is even, then γ_j and γ'_j are paths with variation equal to h_0 with the starting point having height equal to zero and the ending point having height equal to h_0.

Let us also define the following paths:

$$\begin{aligned} \eta &= \overline{(\overline{\gamma_0} \wedge 0)} \text{ and} \\ \zeta &= \overline{(\overline{\gamma_{2f-2}} \wedge 0)}. \end{aligned} \tag{3.34}$$

We also define the paths η' and ζ' as the analogues of η and ζ fo the path γ'. In words, η can be described as the set of sites that would be travessed when one travels along γ_0 after visiting height zero for the last time. Note that η connects a site lying at height zero to a site lying at height h_0 without ever touching these two heights in between. The paths ζ, η' and ζ' can be described in a similar fashion.

Having already defined the paths $\gamma_0, \ldots, \gamma_{2f-2}$ and $\gamma'_0, \ldots, \gamma'_{2f'-2}$ let us now define:

$$\begin{aligned} \gamma_{2f-1} &= \overline{\zeta} \wedge 1 \quad \text{and} \quad \gamma_{2f} = \gamma \setminus (\gamma_0 * \cdots * \gamma_{2f-1}) \\ \gamma'_{2f'-1} &= \overline{\zeta'} \wedge 1 \quad \text{and} \quad \gamma_{2f'} = \gamma \setminus (\gamma'_0 * \cdots * \gamma_{2f'-1}). \end{aligned} \tag{3.35}$$

Thus we can write

$$\begin{aligned} \gamma &= \gamma_0 * \gamma_1 * \cdots * \gamma_{2f-2} * \gamma_{2f-1} * \gamma_{2f} \quad \text{and} \\ \gamma' &= \gamma'_0 * \gamma'_1 * \cdots * \gamma_{2f'-2} * \gamma_{2f'-1} * \gamma_{2f'}. \end{aligned} \tag{3.36}$$

Roughly speaking, this equation express the decomposition of γ (and similarly for γ') as follows: Go up along γ until hitting height h_0. Then go down back to height zero. Repeat it for $f-1$ times and then go up again until hitting height h_0. At that point the path $\overline{\gamma}_{2f-2}$ has just been traversed from bottom to top. Now go along its reversal $\overline{\gamma}_{2f-2}$ stopping at the step just after reaching height zero. This corresponds to γ_{2f-1}. Then follow γ from this point on until hitting its last site v_m.

Note that γ_0 and γ_0' are two compatible paths of variatin h_0 so, by the induction hypothesis there is a path

$$\gamma_0 \times \gamma_0' \tag{3.37}$$

starting at $v_0 \times w_0$ and ending at $v_{t_1} \times w_{t_1'}$ and such that $\pi_2(\gamma_0 \times \gamma_0') = \gamma_0$ and $\pi_3(\gamma_0 \times \gamma_0') = \gamma_0'$. Also, for each $0 < j < 2f-2$ odd, γ_j and $\overline{\eta'}$ are compatible paths of height h_0. Similarly for each $0 < j \leq 2f-2$ even, γ_j and η' also constitute a pair of compatible paths. So, we can pick the paths $\gamma_j \times \eta'$ for j even and $\gamma_j \times \overline{\eta'}$ for j odd. All those paths have their projections into $\mathcal{P}_2$ equal to γ_j and their projections into $\mathcal{P}_3$ equal to η' or $\overline{\eta'}$. Also the ending point of each one of them is the starting point of the following one. So we can define:

$$\gamma \times \eta' = (\gamma_1 \times \overline{\eta'}) * (\gamma_2 \times \eta') * \ldots * (\gamma_{2f-2} \times \eta') \tag{3.38}$$

and it follows that $\pi_2(\gamma \times \eta') = \gamma$ and $\pi_3(\gamma \times \eta') = \eta'$.

Following an analogous procedure we can pick the paths $\zeta \times \gamma_j'$ for j even and $\overline{\zeta} \times \gamma_j'$ for j odd ($1 \leq j \leq 2f'-2$) and then define:

$$\zeta \times \gamma' = (\overline{\zeta} \times \gamma_1') * (\zeta \times \gamma_2') * \cdots * (\zeta \times \gamma_{2f'-2}'). \tag{3.39}$$

Note that this path starts at $v_{t_{2f-2}} \times w_{t_1'}$ and ends at $v_{t_{2f-2}} \times w_{t_{2f'-2}'}$. Also they satisfy that $\pi_2(\zeta \times \gamma') = \zeta$ or $\overline{\zeta}$ and $\pi_3(\zeta \times \gamma') = \gamma'$.

Noting also that γ_{2f-1} and $\gamma_{2f'-1}'$ are compatible paths with variation equal to $h_0 - 1$ starting at $v_{t_{2f-2}}$ and $w_{t_{2f'-2}}'$ respectively and that γ_{2f} and $\gamma_{2f'}'$ are compatible paths of variation h_0, we can then pick

$$\gamma_{2f-1} \times \gamma_{2f'-1} \text{ and } \gamma_{2f} \times \gamma_{2f'}' \tag{3.40}$$

and concatenate then in order to have a path starting at $v_{t_{2f-2}} \times w_{t_{2f'-2}'}$ and finishing at $v_m \times w_{m'}$ and having:

$$\begin{aligned} &\pi_2\left((\gamma_{2f-1} \times \gamma_{2f'-1}') * (\gamma_{2f} \times \gamma_{2f'})\right) \subset \gamma_{2f-1} \cup \gamma_{2f} \text{ and} \\ &\pi_3\left((\gamma_{2f-1} \times \gamma_{2f'-1}') * (\gamma_{2f} * \gamma_{2f'}')\right) \subset \gamma_{2f'-1}' \cup \gamma_{2f'}'. \end{aligned} \tag{3.41}$$

Finally let us define:

$$\gamma \times \gamma' = (\gamma_0 \times \gamma_0') * (\gamma \times \eta') * (\zeta \times \gamma') * (\gamma_{2f-1} \times \gamma'_{2f'-1}) * (\gamma_{2f} \times \gamma_{2f'}). \quad (3.42)$$

which is a path in $\mathbb{Z}^3$ starting at $v_0 \times w_0$, ending at $v_m \times w_{m'}$ and satisfying (3.25). ☐

When $\mathcal{A}_i(2n, n; j, h) \cap \mathcal{B}_i(n, 2n; j, h)$ happens we denote by $\xi_i(j, h)$ a bottom-to-top ω_i-open crossing in $R_i(n, 2n; j, h)$ and by $\zeta_i(j, h)$ a left-to-right ω_i-open crossing in $R_i(2n, n; j, h)$ both being arbitrarily selected among the existing possible crossings (for instance ξ_i could be taken to be the left-most crossing and ζ_i the lowest one). We write $(\xi_i(j, h))_0$ and $(\zeta_i(j, h))_0$ in order to refer to the starting points of those paths. Now suppose that $\tilde{\Gamma}(j, l, h)$ and $\tilde{\Gamma}(j', l', h')$ are good blocks that are neighbors in the graph Λ. We will use Lemma 3.4 in order to construct paths traversing the union of those blocks while connecting $(\xi_2(j, h))_0 \times (\xi_3(l, h))_0$ to $(\xi_2(j', h'))_0 \times (\xi_3(l', h'))_0$ and, in addition, having its projections into $\mathcal{P}_i$ (for $i = 2, 3$) always contained in the union of the paths ξ_i, ζ_i corresponding to the events of those blocks to be good.

Lemma 3.6. *Suppose that $\tilde{\Gamma}(j, l, h)$ and $\tilde{\Gamma}(j', l', h')$ are good neighboring blocks in the graph Λ. Then there is a path $\gamma = \{v_0, v_1, \ldots, v_m\} \subset \mathbb{Z}^3$ satisfying:*

1. $\gamma \subset \tilde{\Gamma}(j, l, h) \cup \tilde{\Gamma}(j', l', h')$;
2. $v_0 = (\xi_2(j, h))_0 \times (\xi_3(l, h))_0$ *and* $v_m = (\xi_2(j', h'))_0 \times (\xi_3(l', h'))_0$;
3. $\pi_2(\gamma) \subset \zeta_2(j, h) \cup \xi_2(j, h) \cup \zeta_2(j', h') \cup \xi_2(j', h')$;
4. $\pi_3(\gamma) \subset \zeta_3(l, h) \cup \xi_3(l, h) \cup \zeta_3(l', h') \cup \xi_3(l', h')$.

In particular by the items 3 and 4 we have that every site in γ is simultaneously ω_2 and ω_3-open.

Proof. Since $\tilde{\Gamma}(j, l, h)$ and $\tilde{\Gamma}(j', l', h')$ are neighboring boxes we have that $|j' - j| + |l - l'| + |h - h'| = 1$. Then split the proof into six cases (each one corresponding one of the indices changing ± 1 units) and use Lemma 3.4 in each of those cases. We only prove the cases $h' - h = \pm 1$ and $j' - j = \pm 1$, the remaining cases $l' - l = \pm 1$ are analogous.

1. $h' - h$ = 1 (the basic strategy is depicted in Figure 3.2)

   For convenience let us fix $h = 0$, $j = j' = 0$ and $l = l' = 0$. Since $\xi_2(0,0)$ and $\xi_3(0,0)$ are respectively bottom-to-top crossings of $R_2(n,2n)$ and $R_3(n,2n)$ they are compatible. By Lemma 3.4 we can pick a path $\xi = \xi_2(0,0) \times \xi_3(0,0) \subset \mathbb{Z}^3$ starting at $(\xi_2(0,0))_0 \times (\xi_3(0,0))_0$ and having $\pi_i(\xi) = \xi_i(0,0)$ for $i = 2,3$. In particular ξ is contained in $\tilde{\Gamma}(0,0,0) \cup \tilde{\Gamma}(0,0,1)$.

   Now, for $i = 2,3$ let $\beta_i(0,1) = \overline{\xi_i(0,0)} \circ \overleftrightarrow{\zeta_i(0,1)} \circ \overline{\xi_i(0,1)}$ be the path in $\mathcal{P}_i$ defined the following way: First start at the final point of $\xi_i(0,0)$, then go down along its reversal $\overline{\xi_i(0,0)}$ until hitting $\zeta_i(0,1)$. After hitting $\zeta_i(0,1)$ go along this path in the appropriate sense in order to hit the path $\xi_i(0,1)$. Note that either $\zeta_i(0,1)$ or its reversal should be taken in order to hit $\xi_i(0,1)$. Finally, after hitting $\xi_i(0,1)$ take its reversal $\xi_i(0,1)$ until getting to its starting point $(\xi_i(0,1))_0$.

   Remark 3.7. The arrow is placed on the top of $\zeta_i(0,1)$ in order to indicate that one should goes along either $\zeta_i(0,1)$ or $\overline{\zeta_i(0,1)}$ depending on which one of these paths will lead to $\xi_i(0,1)$. We prefer not to give a formal definition and trust that this description is enough for making the construction clear.

   Note that $\beta_2(0,1)$ and $\beta_3(0,1)$ are top-to-bottom crossings of $R_2(n;0,1)$ and $R_3(n;0,1)$ then they are compatible paths and by Lemma 3.4 we can pick a path $\beta = \beta_2(0,1) \times \beta_3(0,1) \subset \tilde{\Gamma}(0,1)$ connecting the ending point of ξ to the site $(\xi_2(0,1))_0 \times (\xi_3(0,1))_0$ and having $\pi_i(\beta) \subset \xi_i(0,0) \cup \zeta_i(0,1) \cup \xi_i(0,1)$ for $i = 2,3$.

   Let us define $\gamma = \xi * \beta$. Then this path starts at $(\xi_2(0,0))_0 \times (\xi_3(0,0))_0$ finishes at $(\xi_2(0,1))_0 \times (\xi_3(0,1))_0$. The properties 1, 3 and 4 in the statement are satisfied since they hold for both ξ and β.

2. $h' - h = -1$. By the previous case if we interchange the roles of h and h' we can pick a path satisfying the properties 1, 3 and 4, however starting at $(\xi_2(j',h'))_0 \times (\xi_3(l',h'))_0$ and finishing at $(\xi_2(j,h))_0 \times (\xi_3(l,h))_0$. The reversal of this path satisfy all the required properties.

3. $j' - j = 1$ (the basic strategy is depicted in Figure 3.3)

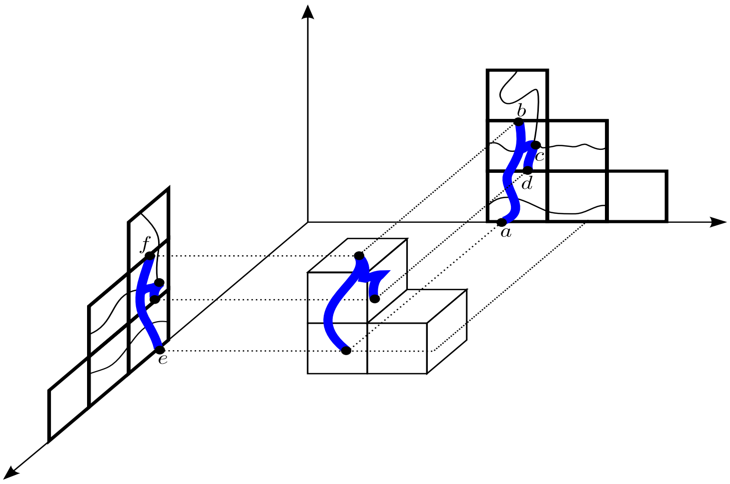

Figure 3.2: The paths $\xi_2(0,0)$ and $\xi_3(0,0)$ are the blue ones connecting a to b and e to f respectively. The path $\beta_2(1,0)$ is the blue one starting at b, passing through c and ending at d, and $\beta_3(1,0)$ is defined similarly. The path $\gamma = (\xi_2 \times \xi_3) * (\beta_2 \times \beta_3)$ is contained in the union of $\tilde{\Gamma}(j,l,0) \cup \tilde{\Gamma}(j,l,1)$.

In order to simplify the notation we fix $j = 0$. Let $\alpha_2 = \xi_2(0,0) \circ \zeta_2(0,0) \circ (\xi_2(1,0) \wedge n) \subset \mathcal{P}_2$ be the following path: Start at $(\xi_2(0,0))_0$ and go along $\xi_2(0,0)$ until it hits $\zeta_2(0,0)$. After hitting $\zeta_2(0,0)$ go along this path until hitting $\xi_2(1,0)$. Finally go along $\xi_2(1,0)$ up to height n. Note that α_2 is bottom-to-top crossing of the rectangle $R_2(2n,n;0,0)$.

Define now $\alpha_3 = \xi_3(0,0) \wedge n$. Then α_3 is the bottom-to-top crossing of the rectangle $R_3(n;0,0)$ that starts at $(\xi_3(0,0))_0$, goes along $\xi_3(0,0)$ up to the time it first hits height n.

Since α_2 and α_3 are crossings of blocks with same hight they are compatible. We can apply Lemma 3.4 in order to pick a path $\alpha = \alpha_2 \times \alpha_3$ that starts at $(\xi_2(0,0))_0 \times (\xi_3(0,0))_0$, goes up to height n and that has projections $\pi_2(\alpha) = \alpha_2 \subset \xi_2(0,0) \cup \zeta_2(0,0) \cup \xi_2(1,0)$ and $\pi_3(\alpha) = \alpha_3 \subset \xi_3(0,0)$. In particular α is contained in $\tilde{\Gamma}(0,0,0) \cup \tilde{\Gamma}(1,0,0)$.

Let us now define $\beta_3 = \overline{\xi_3(0,0) \wedge n}$ which is the path starting at the ending point of $\xi_3(0,0) \wedge n$ and going along its reversal $\overline{\xi_3(0,0)}$ until it hits $(\xi_3(0,0))_0$. We also define $\beta_2 = \overline{\xi_2(1,0) \wedge n}$ to be the top-to-bottom crossing of $R_2(n,n;1,0)$ that starts at the ending point of $\xi_2(0,1) \wedge n$ and goes along $\xi_2(0,1)$ all the way down to the site $(\xi_2(0,1))_0$.

Note that β_2 and β_3 are compatible and so, once more, Lemma 3.4 enables us to select a path $\beta = \beta_2 \times \beta_3$ connecting the ending point of α to the site $(\xi_2(1,0))_0 \times (\xi_3(1,0))_0$ and having $\pi_2(\beta) = \beta_2 \subset \xi_2(1,0)$ and $\pi_3(\beta) = \beta_3 \subset \xi_3(0,0) \cup \zeta_3(0,0) \cup \xi_3(1,0)$. In particular $\beta \subset \tilde{\Gamma}(0,0,0) \cup \tilde{\Gamma}(1,0,0)$.

Then the concatenation $\gamma = \alpha * \beta$ is a path satisfying the properties 1 to 4 above.

4. $j' - j = -1$

   Interchange the roles of j and j', use the previous case and then reverse the obtained path.

□

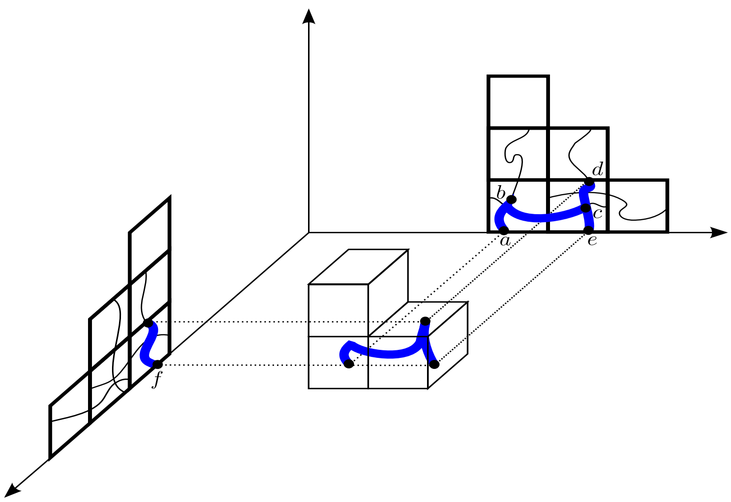

Figure 3.3: The path α_2 is the blue one starting at a, passing through b and c, and finishing at d. The path α_3 starts at f and goes up until hitting height n, β_3 is simply its reversal. β_2 is the path connecting d to e. The path $\gamma = (\alpha_2 \times \alpha_3) * (\beta_2 \times \beta_3)$ is contained in $\tilde{\Gamma}(0,l,h) \cup \tilde{\Gamma}(1,l,h)$.

Corollary 3.8. *For each path of good blocks*

$$\tilde{\gamma} = \left\{ \tilde{\Gamma}(j_0, l_0, h_0), \ldots, \tilde{\Gamma}(j_m, l_m, h_m) \right\} \subset \Lambda$$

there exists a path of sites $\gamma = \{v_0, \ldots, v_r\} \subset \mathbb{Z}^3$ *satisfying:*

1. *γ is contained in $\bigcup_{i=0}^{m} \tilde{\Gamma}(j_i, l_i, h_i)$;*

2. *$h(v_0) = nh_0$ and $h(v_r) = nh_m$;*

3. *all sites in $\pi_2(\gamma)$ and $\pi_3(\gamma)$ are ω_2 and ω_3-open respectively.*

Proof. We apply Lemma 3.6 to each pair $\tilde{\Gamma}(j_i, l_i, h_i)$ and $\tilde{\Gamma}(j_{i+1}, l_{i+1}, h_{i+1})$ that are neighboring sites in $\tilde{\gamma}$ in order to find a sequence of paths γ_i contained in $\gamma_i \subset \tilde{\Gamma}(j_i, l_i, h_i) \cup \tilde{\Gamma}(j'_{i+1}, l'_{i+1}, h'_{i+1})$; starting at $(\xi_2(j_i, h_i))_0 \times (\xi_3(l_i, h_i))_0$ and finishing at $(\xi_2(j_{i+1}, h_{i+1}))_0 \times (\xi_3(l_{i+1}, h_{i+1}))_0$ that have the projections into $\mathcal{P}_2$ and $\mathcal{P}_3$ being ω_2 and ω_3-open respectively. Since the ending point of each of the γ_i is the starting point of each of γ_i we can concatenate them all obtaining a path $\gamma = \gamma_0 * \ldots * \gamma_m$ having the desired properties. □

Corollary 3.9. *If $\tilde{\mathcal{B}}(c \log k, k)$ occurs and all sites in $\pi_1\left(\tilde{R}(c \log k, k)\right)$ are ω_1-open then there is a ω-open path $\gamma = \{v_0, v_i, \ldots, v_r\} \subset \tilde{R}(c \log k, k)$ such that $h(v_0) = 0$ and $h(v_m) = kn$.*

Proof. Recall that $\tilde{\mathcal{B}}(c \log k, k)$ is the event that there is a path

$$\tilde{\gamma} = \left\{\tilde{\Gamma}(j_0, l_0, h_0), \ldots, \tilde{\Gamma}(j_m, l_m, h_m)\right\} \subset \Lambda$$

crossing $R(c \log k, k)$ from bottom-to-top. In particular, $h_0 = 0$ and $h_m = kn$ By the previous corollary, there is a path $\gamma = \{v_0, \ldots, v_r\}$ with $h(v_0) = 0$ and $h(v_r) = kn$ having both its projections into $\mathcal{P}_2$ and $\mathcal{P}_3$ being ω_2 and ω_3-open respectively. In addition all sites in γ are contained in $\tilde{R}(c \log k, k)$, and since all sites in $\pi_1\left(\tilde{R}(c \log k, k)\right)$ are ω_1 open it guarantees that the projection of γ into $\mathcal{P}_1$ is also composed of ω_1-open sites. It follows that γ is ω-open finishing the proof. □

3.2.3 Proof of the polynomial decay rate

We are now in the position to proof that the tail probability for the radius of the open cluster at the origin decays at most in a polynomial rate if at least two of the components of the vector $\mathbf{p}$ are chosen to be high. Recall that the constants c, n and δ are held fixed as in Lemma 3.3. We remark that the term kn appearing in inequality (3.43) is playing the role of n in inequality (1.4). This change is due to the fact that n has been previously fixed for denoting the side lenghs of the boxes in the renormalized block lattice.

Theorem 3.10. *Suppose that $p_2, p_3 > p_c(\mathbb{Z}^2)$ and that $p_1 \neq 1$. Then there exists constants $\alpha = \alpha(\mathbf{p}) > 0$ and $\alpha' = \alpha'(\mathbf{p}) > 0$ such that for all integer $k \geq 1$,*

$$\mathbb{P}_{\mathbf{p}} \left(\left\{ \mathbf{0} \leftrightarrow \partial B^3 \left(kn \right) \right\}, |C| < \infty \right) \geq \alpha'(\mathbf{p}) k^{-\alpha(\mathbf{p})}. \tag{3.43}$$

Proof. By the choices of n, c and δ and by Lemma 3.3 we have that

$$\mathbb{P}_{\mathbf{p}} \left(\tilde{\mathcal{B}}(c \log k, k) \right) \geq \delta$$

for all positive integers $k \geq 1$. We define the following events:

$$\mathcal{C}_2 = \{\text{all sites } (x,0,0) \in \mathcal{P}_2 \text{ such that } 0 \leq y \leq n \lceil c \log k \rceil \text{ are } \omega_2\text{-open}\}$$

$$\mathcal{C}_3 = \{\text{all sites } (0,y,0) \in \mathcal{P}_3 \text{ such that } 0 \leq x \leq n \lceil c \log k \rceil \text{ are } \omega_3\text{-open}\}$$

$$\mathcal{D}_1 = \left\{ \text{all sites in } \pi_1 \left(\tilde{R}(c \log k, k) \right) \text{ are } \omega_1\text{-open} \right\}$$

$$\mathcal{E}_1 = \{\text{there is a } *\text{-circuit of } \omega_1\text{-closed sites surrounding the origin in } \mathcal{P}_1\}$$

where, a $*$-circuit is defined to be a $*$-path that starts and finishes at the same site without intersecting itself before it finishes.

The event that all sites of $\mathcal{P}_1$ lying at l_∞-distance 1 from $\pi_1 \left(\mathcal{R}(c \log k, k) \right)$ are ω_1-closed is contained in $\mathcal{E}_1$. Furthermore it is independent of $\mathcal{D}_1$. Thus we have that

$$\mathbb{P}_{p_1} \left(\mathcal{D}_1 \cap \mathcal{E}_1 \right) \geq {p_1}^{2n^2(c \log k + 1)} (1 - p_1)^{4n(c \log k + 1)} \tag{3.44}$$

Now, if $\mathcal{D}_1 \times \tilde{\mathcal{B}}(c \log k, k)$ happens then by Corollary 3.9 there is a ω-open path starting at a (random) site v_0 in $\mathcal{P}_1$ and finishing at a site v_r in $\partial B^3 \left(kn \right)$. Then, in order to have the origin connected to $\partial B^3(kn)$ it is enough to guarantee that it is connected to v_0. This can be accomplished by simply requiring further that $\mathcal{C}_2 \times \mathcal{C}_3$ occurs. In fact, if $\mathcal{D}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ happens then all sites in $\pi_1 \left(\tilde{R}(c \log k, k) \right)$ are ω-open. Thus we have that:

$$\mathcal{D}_1 \times \left(\tilde{\mathcal{B}}(c \log k, k) \cap (\mathcal{C}_2 \times \mathcal{C}_3) \right) \subset \left\{ \omega \in \left\{ \mathbf{0} \leftrightarrow \partial B^3 \left(kn \right) \right\} \right\}$$

It is well known that for a site percolation process in the square lattice (and hence on $\mathcal{P}_1$) the open cluster at the origin is finite if and only if there exists a closed $*$-circuit surrounding the origin. So, by Lemma 2.3 we have that, on the event $\mathcal{E}_1$ for $\mathbb{P}_{\mathbf{p}}$ almost all configurations there is no percolation. Then:

$$\begin{aligned}&\mathbb{P}_{\mathbf{p}}\left(\left\{\mathbf{0}\leftrightarrow\partial B^3\,(kn),|C|<\infty\right\}\right)\geq\\&\mathbb{P}_{p_1}\times\mathbb{P}_{p_2}\times\mathbb{P}_{p_3}\left((\mathcal{D}_1\cap\mathcal{E}_1)\times\left(\tilde{\mathcal{B}}(c\log k,k)\cap(\mathcal{C}_2\times\mathcal{C}_3)\right)\right)=\\&\mathbb{P}_{p_1}\left(\mathcal{D}_1\cap\mathcal{E}_1\right)\mathbb{P}_{p_2}\times\mathbb{P}_{p_3}\left(\left(\tilde{\mathcal{B}}(c\log k,k)\cap(\mathcal{C}_2\times\mathcal{C}_3)\right)\right).\end{aligned}\tag{3.45}$$

By the Harris-FKG inequality and by equation (3.43) we have that

$$\begin{aligned}&\mathbb{P}_{p_2}\times\mathbb{P}_{p_3}\left(\tilde{\mathcal{B}}(c\log k,k)\cap(\mathcal{C}_2\times\mathcal{C}_3)\right)\geq\\&\mathbb{P}_{p_2}\times\mathbb{P}_{p_3}\left(\tilde{\mathcal{B}}(c\log k,k)\right)\mathbb{P}p_2\left(\mathcal{C}_2\right)\mathbb{P}_{p_3}\left(\mathcal{C}_3\right)\geq\\&\delta\;p_2^{n(c\log k+1)}\;p_3^{n(c\log k+2)}.\end{aligned}\tag{3.46}$$

Now if we plug equations (3.46) and (3.44) into equation (3.45) we get:

$$\begin{aligned}&\mathbb{P}_{\mathbf{p}}\left(\left\{\mathbf{0}\leftrightarrow\partial B^3\,(kn),|C|<\infty\right\}\right)\geq\\&\delta\;{p_1}^{2n^2(c\log k+1)}(1-p_1)^{4n(c\log k+1)}\;p_2^{n(c\log k+1)}\;p_3^{n(c\log k+2)}=\\&\alpha'(\mathbf{p})k^{-\alpha(\mathbf{p})},\end{aligned}$$

where the constants α' and α depend on $\mathbf{p}$. □

We conclude this section providing a quick remark about the:

Proof of the second statement in Theorem 1.3. All the work has been done. In fact, from equation (3.43), we have that for all $k\geq 2$ that

$$\begin{aligned}&\mathbb{P}_{\mathbf{p}}\left(\left\{\mathbf{0}\leftrightarrow\partial B^3\,(k),|C|<\infty\right\}\right)\geq\\&\mathbb{P}_{\mathbf{p}}\left(\left\{\mathbf{0}\leftrightarrow\partial B^3\,(kn),|C|<\infty\right\}\right)\geq\\&\alpha'(\mathbf{p})k^{-\alpha(\mathbf{p})}\end{aligned}$$

which is equation (1.4) with k playing the role of n. □

Chapter 4

More about the supercritical phase

The existence of a supercritical phase was settled in Chapter 2. As a by-product, in Corollary 2.12, we have shown that if all the components of $\mathbf{p}$ are larger than $[p'_c(\mathbb{Z}^2)]^{1/3}$ then there is percolation. Assume that we now decrease one of the components of $\mathbf{p}$ while we increase another one, say p_2 and p_1 respectively. Intuitively, as we decrease p_2 we create new closed columns thus removing all the sites on those columns that were open before. On the other hand, increasing p_1 we will sprinkle more open sites all over $\mathbb{Z}^3$. It turns out that, as long as p_2 stays larger than $p_c(\mathbb{Z}^2)$, can be increased p_1 quickly enough in order to still guarantee the existence of percolation. In fact, Theorem 1.4 yields more than that: As long as p_2 and p_3 remain supercritical, it will always be possible to fix p_1 large enough so that there will still be percolation.

The aim of this section is to present the proof of Theorem 1.4. We use two lemmas that are based on Lemma 3.4 and ideas similar to those of the proof of Theorem 2.9.

Recall that $R_1(n,n;j,l)$ are squares of side n contained in $\mathcal{P}_1$ and consider the set $\mathcal{P}_1^{(n)} = \{R_1(n,n;j,l);\ j \in \mathbb{Z}, l \in \mathbb{Z}\}$ that can be seen as a graph by adding an edge between $R_1(n,n;j,l)$ and $R_1(n,n;j',l')$ if, and only if, $|j'-j|+|l'-l| = 1$. Let for the moment $\gamma_n \subset \mathcal{P}_1^{(n)}$ be a fixed 2-directed path of rectangles and consider $\tilde{\gamma}_n = \left\{\tilde{\Gamma}_n(j,l,h) \in \Lambda_n; \pi_1(\tilde{\Gamma}_n(j,l,h)) \in \gamma_n\right\}$.

The following lemma shows that we can choose n large enough so that the probability of finding infinite paths of good boxes in $\tilde{\gamma}_n$ is positive. We use the fact that if n is large, then a block in $\tilde{\gamma}_n$ is good with high probability and that the event that a block in $\tilde{\gamma}_n$ is good does only depend on the state of blocks within a fixed (graph)-distance of this block in $\tilde{\gamma}_n$. Those ideas have been used before in the proof of Theorem 2.9, so the reader can skip the proof if he accepts the statement as being true.

Lemma 4.1. *Let $p_2, p_3 > p_c(\mathbb{Z}^2)$. Then there is $n = n(p_2, p_3)$ for which:*

$$\mathbb{P}_{2,3}\left(\left\{\exists \text{ an infinite path of good blocks starting from } \tilde{\Gamma}_n(0,0,0) \text{ in } \tilde{\gamma}_n\right\}\right) > 0, \tag{4.1}$$

where $\mathbb{P}_{2,3} = \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$.

Proof. Consider the set of indices $I = \{(j,l) \in \mathbb{Z}^2;\ R_1(n,n;j,l) \in \gamma_n\}$ and $\tilde{I} = \{(j,l,h) \in \mathbb{Z}^3;\ \tilde{\gamma}_n(j,l,h) \in \Gamma_n\}$. Then I can be regarded as a 2-directed path in $\mathbb{Z}^2$ and $\tilde{I}$ can regarded as its lift in $\mathbb{Z}^3$. Consider the process $\{X(j,l,k)\}_{(j,l,k)\in I}$ where $X(j,l,K) = \mathbf{1}_{\{\tilde{\Gamma}_n(j,l,h) \text{ is good}\}}$ and let μ denote its law on $\{0,1\}^I$. The event $\left\{\tilde{\Gamma}_n(j,l,h) \text{ is good}\right\}$ only depends on the ω_2 and ω_3 processes restricted to the projections $\pi_2\left(\tilde{\Gamma}_n(j',l',h')\right)$ and $\pi_3\left(\tilde{\Gamma}_n(j',l',h')\right)$ of rectangles $\tilde{\Gamma}_n(j',l',h')$ satisfying $|j'-j|+|l'-l|+|h'-h| \leq 1$. Moreover, since the path I is 2-directed, it follows that there is a positive integer M large enough such that the projections of each rectangle $\tilde{\Gamma}_n(j,l,h)$ into $\mathcal{P}_2$ and $\mathcal{P}_3$ overlap with at most the projection of M other rectangles in Λ_n. Then it follows that the process X is a M-dependent percolation process with

$$\mu\left(\{\, X(j,l,h) = 1 \,\}\right) = \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}\left(\left\{\tilde{\Gamma}_n(j,l,h) \text{ is good}\right\}\right).$$

Applying once more [LSS97, Theorem 0.0] there is a $u \in (0,1)$ such that if $\mu\left(\{X(j,l,h) = 1\}\right) > u$ then $\mu\left(\{\mathbf{0} \leftrightarrow \infty\}\right) > 0$. By Lemma 3.2 we can chose a positive integer n large enough (and depending on p_2 and p_3) so that $\mathbb{P}_{p_2} \times \mathbb{P}_{p_3}\left(\left\{\tilde{\Gamma}_n(j,l,h) \text{ is good}\right\}\right) > u$. So for that choice of n it follows that

$$\mathbb{P}_{2,3}\left(\left\{\exists \text{ an infinite path of good blocks starting from } \tilde{\Gamma}_n(0,0,0) \text{ in } \tilde{\gamma}_n\right\}\right) = \mu\left(\{\mathbf{0} \leftrightarrow \infty\}\right) > 0.$$

□

From now on let us fix n as in the statement of the previous lemma. The next lemma proves that the probability of finding an infinite ω-open path starting somewhere in the block $\tilde{\Gamma}_n(0,0,0)$ is positive if we keep p_2 and p_3 fixed and set p_1 to be large enough. Before stating this result and proving it we give an idea of the proof. First we choose p_1 high enough in order to guarantee that the event that are infinite 2-directed paths of ω_1-open blocks in $\mathcal{P}_1^n$ have positive probability under $\mathbb{P}_{p_1}$. Then conditioning on the realization of this path, we consider the percolation processes of good blocks on the lift of the resulting paths in Λ_n. This process will be supercritical so the (conditional) probability of finding an infinite path of good blocks in Λ_n is positive (almost surely). Finally we use Corollary 3.8 in order to relate such a path of good blocks to an infinite path of ω-open sites.

Lemma 4.2. *Suppose that $p_2, p_3 > p_c(\mathbb{Z}^2)$. Then there is $\epsilon = \epsilon(p_2, p_3) > 0$ such that, if $p_1 > 1 - \epsilon$, than:*

$$\mathbb{P}_{\mathbf{p}}\left(\left\{\exists \text{ a infinite } \omega\text{-open path starting at } \tilde{\Gamma}_n(0,0,0)\right\}\right) > 0. \qquad (4.2)$$

Proof. Define the process $X_1 = \{X_1(j,l)\}_{(j,l)\in\mathbb{Z}^2}$ where

$$X_1(j,h) = \mathbf{1}_{\{\text{all sites in } R_1(n,n;j,l) \text{ are } \omega_1\text{-open}\}}$$

Due to the independence of the process ω_1 in each rectangle we have that the law of X_1 as a process is the same as the Bernoulli percolation $\mathbb{P}_{p_1^{n^2}}$. Since n has been fixed before, we can choose an $\epsilon > 0$ so that whenever $p_1 > 1 - \epsilon$ then $p_1^{n^2} > [p_c'(\mathbb{Z}^2)]^{1/3}$. Fix such a $\epsilon > 0$ and $p_1 > 1 - \epsilon$.

Let $\mathcal{A}_1 = \{\exists$ an infinite 2-directed path of open rectangles in $\mathcal{P}_1^n\}$. If follows from Proposition 2.5 that

$$\mathbb{P}_{p_1}(\{\mathcal{A}_1\}) = \mathbb{P}_{p_1^{n^2}}(\{\exists \text{ an infinite 2-directed path starting at } \mathbf{0}\}) > 0. \qquad (4.3)$$

For each $\omega_1 \in \mathcal{A}_1$ let $\gamma_n = \gamma_n(\omega_1)$ be the lowest 2-directed path of open rectangles in $\mathcal{P}_1^n$ and let $\tilde{\gamma}_n = \tilde{\gamma}_n(\omega_1) \subset \Lambda_n$ be its lift.

Then writing $\mathbb{P}$ for $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$, $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$, $\mathbb{P}_{2,3}$ for $\mathbb{P}_{p_2} \times \mathbb{P}_{p_3}$ and $\left\{\tilde{\Gamma}_n(0,0,0) \leftrightarrow \infty\right\}$ for the event

$$\left\{\exists \text{ a infinite } \omega\text{-open path starting at } \tilde{\Gamma}_n(0,0,0)\right\},$$

we have that:

$$\mathbb{P}_{\mathbf{p}}\left(\left\{\tilde{\Gamma}_n(0,0,0)\leftrightarrow\infty\right\}\right)=$$
$$\mathbb{E}\left[\mathbb{P}\left(\left\{\omega\in\left\{\tilde{\Gamma}_n(0,0,0)\leftrightarrow\infty\right\}\right\}\Big|\mathcal{F}_1\right)\right]\geq$$
$$\mathbb{E}\left[\mathbf{1}_{\mathcal{A}_1}\mathbb{P}\left(\left\{\omega\in\left\{\tilde{\Gamma}_n(0,0,0)\leftrightarrow\infty\text{ in }\tilde{\gamma}_n\right\}\right\}\Big|\mathcal{F}_1\right)(\omega_1)\right]\geq$$
$$\mathbb{E}\left[\mathbf{1}_{\mathcal{A}_1}\mathbb{P}_{2,3}\left(\left\{\exists\text{ an infinite path of good blocks starting from }\tilde{\Gamma}_n(0,0,0)\text{ in }\tilde{\gamma}_n\right\}\right)\right],$$

where the last inequality follows from Corollary 3.8. By Lemma 4.1 the term into the brackets in the right hand side is positive for $\omega_1\in\mathcal{A}_1$. This finishes the proof. □

Using the two previous lemmas we are now in the position to prove theorem 1.4:

Proof of Theorem 1.4. We can without any loss of generality fix $p_2,p_3>p_c(\mathbb{Z}^2)$ and take ϵ as given by the last lemma. Consider for each $i\in\{1,2,3\}$, the increasing event $\mathcal{B}_i=\left\{\pi_i\left(\tilde{\Gamma}_n(0,0,0)\right)\text{ is }\omega_i\text{-open}\right\}$. If $\mathcal{B}_1\times\mathcal{B}_2\times\mathcal{B}_3$ happens then all sites in $\tilde{\Gamma}_n(0,0,0)$ are ω-open. Then $\{\mathbf{0}\leftrightarrow\infty\}$ is contained in $\left\{\omega\in\{\tilde{\Gamma}_n(0,0,0)\leftrightarrow\infty\}\right\}\cap\mathcal{B}_1\times\mathcal{B}_2\times\mathcal{B}_3$. Using the Harris-FKG inequality and the last lemma we have that:

$$\mathbb{P}_{\mathbf{p}}\left(\{\mathbf{0}\leftrightarrow\infty\}\right)\geq$$
$$\mathbb{P}_{\mathbf{p}}\left(\left\{\tilde{\Gamma}_n(0,0,0)\leftrightarrow\infty\right\}\right)\prod_{i=1,2,3}\mathbb{P}_{p_i}\{\mathcal{B}_i\}>0.$$

This finishes the proof. □

Chapter 5

The number of infinite clusters

Define the random variable N:= the number of infinite clusters. By standard ergodicity arguments N is constant almost surely. More than that we show that this constant can only assume the values 0, 1 or ∞ almost surely. For that we use a procedure similar to that of Newman and Schulman in [NS81a] and [NS81b]. However their methods do not apply directly to the measure $\mathbb{P}_{\mathbf{p}}$ due to fact that this measure fails to satisfy the so-called *finite energy condition*, introduced in those papers. Thus a non-trivial extension is needed. We use the fact that for a translation-invariant measure on $\{0,1\}^{\mathbb{Z}^d}$ all infinite clusters have a well defined density (see [BK89]) and that the percolation processes ω_i satisfy the finite energy condition. The question whether $N \in \{0,1\}$ is still open. We conjecture that it should be the case at least when the components of $\mathbf{p}$ are high and conclude by giving an informal argument of why this should be so.

5.1 Translation-invariance, ergodicity and density

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$, and $\mathcal{F}$ be the sigma-field generated by the *cylinder subsets* of Ω. For each $v \in \mathbb{Z}^d$, let $T_v : \mathbb{Z}^d \to \mathbb{Z}^d$ be the translation by v, i.e., $T_v(w) = w + v$ for each $w \in \mathbb{Z}^d$. We also let T_v act in the *configuration* space ω by defining for each $\omega \in \Omega$, $T_v\omega$ the configuration given by $(T_v\omega)_x = \omega_{x-v}$. For a random variable X defined on Ω we define (T_vX) as the random variable satisfying $T_vX(\omega) = X(T_{-v}\omega)$. The random variable X is said to be invariant under T_v if $T_vX = X$. An event $\mathcal{A} \in \mathcal{F}$ is called *invariant* under T_v if $\mathbf{1}_{\mathcal{A}}$ is invariant under this transformation, or equivalently if $T_v^{-1}\mathcal{A} = \mathcal{A}$.

A measure μ on $\mathcal{F}$ is invariant under T_v if $\mu(T_v^{-1}\mathcal{A}) = \mu(\mathcal{A})$ for each $\mathcal{A} \in \mathcal{F}$. An invariant measure μ defined on the Borel sigma-field of $\{0,1\}^{\mathbb{Z}^d}$ is said to be *ergodic* with respect to T_v if all invariant events $\mathcal{A}$ have measure 0 or 1. This is equivalent to say that all invariant random variables in Ω are μ-almost surely constant.

An invariant measure μ as above is said to be *mixing* for the transformation T_v if for all pair of events $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{F}$,

$$\lim_{n\to\infty} \mu(\mathcal{A} \cap T_v^{-n}\mathcal{B}) = \mu(\mathcal{A})\mu(\mathcal{B}). \tag{5.1}$$

Intuitively the condition of being mixing for the transformation T_v means that, as we iterate this transformation, the initial conditions get more and more irrelevant. In a probabilistic point of view this means that the events $\{T_v^n(\omega) \in \mathcal{B}\}$ and $\{\omega \in \mathcal{A}\}$ are asymptotically independent. It is an standard fact in ergodic theory that every mixing transformation is ergodic. Moreover one usually checks that an invariant measure is ergodic by verifying the mixing condition (5.1).

We say that a subset $A \subset \mathbb{Z}^d$ have density ρ if for any sequence of rectangles $R_1 \subset R_2 \subset \ldots$ with $\cup_{i\geq 1} R_i = \mathbb{Z}^d$ the limit

$$\lim_{i\to\infty} \frac{|A \cap R_i|}{|R_i|}$$

exists and is equal to ρ.

If we fix a configuration $\omega \in \Omega$ then it is clear that each one of its finite components have zero density. It has been proved by Burton and Keane [BK89, Theorem 1] that if μ is a translation-invariant probability on $\mathcal{F}$ then all the open clusters have a well defined density.

Let us fix now μ a translation invariant measure that is ergodic with respect to T_v. Suppose that $N \geq 1$, μ-almost surely. Then we can create a ranked density vector, by inserting at each coordinate the value of the density of one of the infinite clusters in a non-increasing way. More specifically define:

$$\rho = \begin{cases} (\rho_1, \ldots, \rho_N), & \text{if } N < \infty \\ (\rho_1, \rho_2, \ldots), & \text{if } N = \infty, \end{cases} \tag{5.2}$$

where $\rho_i \geq \rho_{i+1}$ are the densities of the infinite clusters. Since ρ is invariant under T_v we have that ρ is almost surely constant.

Remark 5.1.

- If there is more than one cluster with the same density then ρ will not depend on the order their density will appear.

Of course there can be infinite clusters having zero density. The next proposition says that those clusters cannot exist if $N < \infty$.

Proposition 5.2. *Suppose that μ is a translation-invariant ergodic probability measure on Ω for which $0 < N < \infty$ almost surely and let ρ be the ranked density vector given by (5.2). Then all entries of ρ are strictly positive constants, μ-almost surely.*

Proof. As mentioned before, by ergodicity, it follows that each entry of ρ is constant. Suppose, in order to find a contradiction, that there is an index k for which $\rho_k = 0$. By the definition of ρ it follows that $\rho_j = 0$ for all $k \le j \le N$.

Let, for each $j \in \{1, \ldots, N\}$, C_j stand for the cluster corresponding to the j-th entry of ρ. Define $C' = \cup_{j=k}^{N} C_j$. Then C' is a non-empty random infinite subset of $\mathbb{Z}^d$ having distribution invariant under lattice translations. In particular, $\mu(\{\mathbf{0} \in C'\}) = \mu(\{v \in C'\})$ for all $v \in \mathbb{Z}^d$.

Let $R_1 \subset R_2 \subset \cdots$ be any increasing sequence of rectangles such that $\cup_{j=1}^{\infty} R_j = \mathbb{Z}^d$. Then, since the densities indeed exist, we have that:

$$\lim_{n\to\infty} \frac{1}{|R_n|} \sum_{v\in R_n} \mathbf{1}_{\{v\in C'\}} = (N-k+1)\rho_k = 0.$$

Integrating the left-hand side with respect to μ, using the Bounded Convergence Theorem and translation-invariance we have that $\mu(\{\mathbf{0} \in C'\}) = 0$, so that, $\mu(\{C' = \emptyset\}) = 1$, a contradiction. □

Corollary 5.3. *Under the hypothesis of the last proposition, if the probability of finding an infinite cluster of density zero is positive, then there exists infinitely many of them almost surely.*

5.2 The number of clusters is either 0, 1, or ∞

In this section we prove Theorem 1.5. We start defining the notion of finite energy. For a site $v \in \mathbb{Z}^d$, let $\Omega^{(v)} = \{0,1\}^{\mathbb{Z}^d \setminus \{v\}}$. Denote by $\omega^{(v)}$ a configuration in $\Omega^{(v)}$.

We say that a probability measure $\mathbb{P}$ have *finite energy* if for a site $v \in \mathbb{Z}^d$:

$$0 < \mathbb{P}\left(\{\omega(v) = 1\}\Big|\omega|_{\mathbb{Z}^d\setminus\{v\}} = \omega^{(v)}\right) < 1, \text{ with probability one.} \tag{5.3}$$

In words, this condition means that, the conditional expectation of the random variable $\omega(v)$ given the σ-field generated by the configurations outside the site v is bounded away from 0 and 1. Note that the finite energy condition holds trivially for the Bernoulli percolation processes (if the density p is not equal to zero or one). For the Ising model, verifying this condition is related to verifying that the energy shift due to a single spin flip is finite. Intuitively, it means that if an event have positive probability, then modifying the state of the site v will not change the probability of this event to zero. Thus after performing local modifications in the configurations of an event of positive probability we still obtain an event of positive probability as we explain below.

A measurable transformation $\phi : \Omega \to \Omega$ is said to be *local* if there is a finite set $V \subset \mathbb{Z}^d$ such that $\phi(\omega)(v) = \omega(v)$ for all $v \in \mathbb{Z}^d\setminus\{V\}$. If $\mathbb{P}$ has finite energy, then for any event $\mathcal{A}$ having $\mathbb{P}(\mathcal{A}) > 0$ and any local transformation ϕ, $\mathbb{P}(\phi(\mathcal{A})) > 0$ (see [NS81a, Proposition 9]) for a proof). This fact is easy to verify in the Bernoulli case.

For ergodic probability measures having finite energy the proof that $N \in \{0, 1, \infty\}$ is based on this fact. The idea is the following: assume that N is a finite number strictly greater then one. Take a box large enough so that the probability of intersecting all the infinite clusters is positive. Then perform the local modification that consists in opening all sites in that box. This transformation connects all the clusters in a unique one. This implies that the probability of having a unique cluster is also positive. However this is in contradiction to the fact that N is constant almost surely.

As stated above, more than that is known to hold for probability measures having finite energy: it has been shown in [BK89] that there can be at most one infinite component.

From now on we fix $d = 3$ and the measure $\mathbb{P}_{\mathbf{p}}$ with $p_i \neq 0, 1$ for all i. For this measure, it is not the case that any T_v is ergodic regardless of the chosen vector v. In fact if we take $v = (1, 0, 0)$ then we have that the event $\{\omega(kv) = 0 \text{ for all } k \in \mathbb{Z}\}$ is invariant under T_v, however it has probability equal to $1 - p_1 \notin \{0, 1\}$.

On the other hand, when $v = (x, y, z)$ have at least two nonzero coordinates then it is the case that T_v is mixing. In order to see that, let $\mathcal{A}$ and $\mathcal{B}$ be two cylinders in $\mathcal{F}$ and suppose that $A \subset \mathbb{Z}^d$ and $B \subset \mathbb{Z}^d$ are finite sets of sites such that the state of the sites in A and B determines the occurrence of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then there exists a n_0 (depending on A and B) such that $\pi_i(T_v^n B) \cap \pi_i(A) = \emptyset$ for all $i = 1, 2, 3$ and for all $n > n_0$. So, for all such indices $n > n_0$, we have that $\{\omega(w);\ w \in \mathcal{A}\}$ and $\{\omega(w);\ w \in \mathcal{B}\}$ are independent sets of random variables, which implies the mixing condition. It is a standard fact that in order to verify that a system (μ, T_z) satisfy the mixing condition it is enough to check equation (5.1) for each pair of cylinder sets. This follows from the fact that any set in $\mathcal{F}$ can be *approximated* by a cylinder in the sense that the measure of the symmetric difference between them can be taken to be arbitrarily small (see, for instance [Bil78] for a proof).

For the vector $e = (1, 1, 1)$ we denote simply $T = T_e$. Since $e = (1, 1, 1)$ has all its three components different from zero, we have that $\mathbb{P}_{\mathbf{p}}$ is ergodic with respect to T. In particular, it follows that $TN = N$ for all $\omega \in \Omega$, so that N is a random variable that is invariant with respect to T and then N is constant almost surely.

However for this measure it is not true that we can perform local modifications. To see that, notice that on the event that all neighbors of the origin are open the origin itself is open with probability one. In particular flipping the state of the origin to zero would yield an event of probability zero. This implies that the finite energy condition does not hold. Also the proof of Newman and Schulman sketched above does not hold: If we want to have all the sites of a given box to be open then we need to modify the state of vertices in all the columns intersecting this box. We could do that in order to glue components together, however it could be that case that other infinite components would appear elsewhere. This shows that their proof do not apply directly. However instead of trying to preserve the number of cluster we could try to modify the configurations preserving the density of the clusters. This proof uses the same ideas as those in a work in preparation by Hilário and Teixeira [HT].

Proof of Theorem 1.5: Suppose that $1 < N < \infty$ and let $\rho = (\rho_1, \ldots, \rho_N)$ be the ranked density vector defined in (5.2) and $C(1), \ldots C(N)$ be the infinite clusters corresponding to each of the entries of ρ.

From proposition 5.2 we have that all the entries of ρ are strictly positive. Thus we can take a positive n_0 such that for all $n > n_0$ the probability that $B^3(n)$ intersects all the clusters $C(1), \ldots, C(N)$ is positive. So, fixing $n > n_0$ and denoting by $\mathcal{A}$ the event in $\Omega_1 \times \Omega_2 \times \Omega_3$ given by

$$\mathcal{A} = \big\{\omega \in \{B^3(n) \text{ intersects all the clusters } C(1), \ldots, C(N)\}\big\}$$

we have that $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}(\mathcal{A}) > 0$.

Define the mapping $\phi : \Omega_1 \times \Omega_2 \times \Omega_3 \to \Omega_1 \times \Omega_2 \times \Omega_3$ by setting

$$\phi(\omega_1, \omega_2, \omega_3) = (\phi_1(\omega_1), \phi_2(\omega_2), \phi_3(\omega_3)),$$

where,

$$\phi_i(\omega_i)(v) = \begin{cases} 1, & \text{if } v \in B_i^2(n) \\ \omega_i(v), & \text{if } v \notin B_i^2(n). \end{cases}$$

On the event $\{(\omega_1, \omega_2, \omega_3) \in \phi(\mathcal{A})\}$ the vector ρ have an entry with value at least $\rho_1 + \ldots + \rho_N > \rho_1$. This follows from the fact that the event $\mathcal{A}$ is increasing and the also from the fact that for any configuration in this set, if declaring all sites in $B_i^2(n) = \pi_i\left(B_i^3(n)\right)$ to be ω_i-open we will get a configuration in for which all the box B_i^3 will be ω-open. Then we will have all the clusters $C(1), \ldots C(n)$ merged in a single ω-cluster. Since $\mathbb{P}_{p_1} \times \mathbb{P}_{p_2} \times \mathbb{P}_{p_3}(\mathcal{A}) > 0$ and this measure satisfy the finite energy condition, $\mathbb{P}_{p_1} \times \mathbb{P}_{p_1} \times \mathbb{P}_{p_3}(\phi(\mathcal{A})) > 0$. This implies that the vector ρ have a component equal to $\rho_1 + \ldots + \rho_N$ with positive probability. A contradiction with the fact that its first component should be constant equal to ρ_1 almost surely.

$\square$

Bibliography

[AB87] Michael Aizenman and David J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987.

[AKN87] M. Aizenman, H. Kesten, and C. M. Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.*, 111(4):505–531, 1987.

[Bax82] Rodney J. Baxter. *Exactly solved models in statistical mechanics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982.

[BBS00] P. N. Balister, B. Bollobás, and A. M. Stacey. Dependent percolation in two dimensions. *Probab. Theory Related Fields*, 117(4):495–513, 2000.

[Bil78] Patrick Billingsley. *Ergodic theory and information*. Robert E. Krieger Publishing Co., Huntington, N.Y., 1978. Reprint of the 1965 original.

[BK89] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.

[CCN87] J. T. Chayes, L. Chayes, and C. M. Newman. Bernoulli percolation above threshold: an invasion percolation analysis. *Ann. Probab.*, 15(4):1272–1287, 1987.

[Dur84] Richard Durrett. Oriented percolation in two dimensions. *Ann. Probab.*, 12(4):999–1040, 1984.

[Fis61] Michael E. Fisher. Critical probabilities for cluster size and percolation problems. *J. Mathematical Phys.*, 2:620–627, 1961.

[Gác00] Peter Gács. The clairvoyant demon has a hard task. *Combin. Probab. Comput.*, 9(5):421–424, 2000.

[GM90] G. R. Grimmett and J. M. Marstrand. The supercritical phase of percolation is well behaved. *Proc. Roy. Soc. London Ser. A*, 430(1879):439–457, 1990.

[Gri99] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 1999.

[Har60] T. E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.*, 56:13–20, 1960.

[HT] M. R. Hilário and A. Teixeira. *in preparation*.

[LSS97] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25(1):71–95, 1997.

[Men86] M. V. Menshikov. Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR*, 288(6):1308–1311, 1986.

[NS81a] C. M. Newman and L. S. Schulman. Infinite clusters in percolation models. *J. Statist. Phys.*, 26(3):613–628, 1981.

[NS81b] C. M. Newman and L. S. Schulman. Number and density of percolating clusters. *J. Phys. A*, 14(7):1735–1743, 1981.

[Pet08] Gábor Pete. Corner percolation on $\mathbb{Z}^2$ and the square root of 17. *Ann. Probab.*, 36(5):1711–1747, 2008.

[Rus81] Lucio Russo. On the critical percolation probabilities. *Z. Wahrsch. Verw. Gebiete*, 56(2):229–237, 1981.

[SS09] V. Sidoravicius and A-S. Sznitman. Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math*, 62(6):831–858, 2009.

[Szn10] Alain-Sol Sznitman. Vacant set of random interlacements and percolation. *Ann. of Math. (2)*, 171(3):2039–2087, 2010.

[TW10] J. Tykesson and D. Windisch. Percolation in the vacant set of poisson cylinders. *ArXiv Mathematics e-prints*, 2010.

[Win00] Peter Winkler. Dependent percolation and colliding random walks. *Random Structures Algorithms*, 16(1):58–84, 2000.

[Win08] David Windisch. Random walk on a discrete torus and random interlacements. *Electron. Commun. Probab.*, 13:140–150, 2008.