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Dicritical holomorphic flows on Stein manifolds

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Abstract. We study holomorphic flows on Stein manifolds. We prove that a holomorphic flow with isolated singularities and a dicritical singularity of the form $\sum_{j=1}^{n} \lambda_j z_j \dfrac{\partial}{\partial z_j} + \ldots,\ \lambda_j \in \mathbb{Q}_+, \forall\, j \in \{1, \ldots, n\}$ on a Stein manifold M^n, $n \geq 2$ with $\overset{\vee}{H}{}^2(M^n, \mathbb{Z}) = 0$, is globally analytically linearizable; in particular M is biholomorphic to $\mathbb{C}^n$. A complete stability result for periodic orbits is also obtained.

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1. Introduction. The study of holomorphic flows on Stein surfaces using techniques of Holomorphic Foliations with singularities was introduced by M. Suzuki [16]. Using techniques also from Potential Theory and Complex Analytic Spaces, Suzuki proves that for a non-trivial complete holomorphic vector field X on a Stein manifold M of dimension $n \geq 2$ there is an invariant subset $\sigma \subset M$ of zero logarithmic capacity such that on $M \backslash \sigma$ all orbits of X are isomorphic to exactly one of the following Riemann surfaces: $\mathbb{C}^* = \mathbb{C} - \{0\}$ or $\mathbb{C}$. We shall refer to this saying that *the generic orbit of X is* (isomorphic to) $\mathbb{C}^*$ or $\mathbb{C}$. Also according to Suzuki we have: (S.i) if the generic orbit is $\mathbb{C}^*$ then there exists a meromorphic function, defined on the full space, which gives the periods of the $\mathbb{C}^*$-orbits; and (S.ii) orbits isomorphic to $\mathbb{C}^*$ are closed on $M \backslash \operatorname{sing}(\mathcal{F})$ where $\operatorname{sing}(\mathcal{F}) \subset M$ denotes the codimension ≥ 2 analytic subset of M which is the set of singularities of the one-dimensional holomorphic foliation $\mathcal{F}$ induced by X on M. Therefore, as a consequence of the classical Remmert-Stein theorem [6], (S.iii) the closure of an orbit isomorphic to $\mathbb{C}^*$ is an analytic curve in M; and finally (S.iv) if M has dimension two then it is proven by Suzuki that a holomorphic complete vector

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field X on M with generic orbit $\mathbb{C}^*$ always admits a (non-constant) meromorphic first integral on M. Regarding the study of singularities of complete holomorphic vector field we have the land-mark work of J. Rebelo et al (see [11], [12] and [4]).

The general motivation for this paper is to study the classification of the pairs (M, X) where M is a Stein surface and X is a complete holomorphic vector field on M having a suitable isolated singularity. Given a vector field X with isolated singularities on a manifold M we denote by $\mathcal{F}(X)$ the one-dimensional holomorphic foliation on M whose leaves are the nonsingular orbits of X and with singular set $\operatorname{sing}(\mathcal{F}(X)) = \operatorname{sing}(X)$. A singularity $P \in \operatorname{sing}(X)$ is *dicritical* if for some neighborhood $P \in V$ there are infinitely many orbits of $X\big|_V$ accumulating only at P. The closure of such a local leaf is an invariant analytic curve called a *separatrix* of X through P. We prove the following global linearization theorem:

Theorem 1.1. *Let X be a complete holomorphic vector field with isolated singularities on a Stein manifold M of dimension $n \geq 3$. Assume that X has isolated singularities and some dicritical singularity with first jet of the form $X_{(\lambda_1,\dots,\lambda_n)} = \sum_{j=1}^n \lambda_j z_j \dfrac{\partial}{\partial z_j}$, where $\lambda_j \in \mathbb{Q}_+$, $\forall\, j$. If $\operatorname{sing}(X)$ is finite and $\overset{\vee}{H}{}^2(M, \mathbb{Z}) = 0$ then X is holomorphically conjugate to $X_{(\lambda_1,\dots,\lambda_n)}$. In particular, M is biholomorphic to $\mathbb{C}^n$.*

Motivated by the proof of Theorem 1.1 we obtain the following complete stability lemma for periodic orbits of holomorphic flows:

Theorem 1.2. *Let X be a complete holomorphic vector field with isolated singularities on the affine space $\mathbb{C}^n$, $n \geq 2$. If for some $p \in \mathbb{C}^n$ the corresponding orbit is periodic (isomorphic to $\mathbb{C}^*$) and has finite holonomy group then the generic orbit of X is isomorphic to $\mathbb{C}^*$.*

In the situation of Theorem 1.2, as it follows from (S.i), (S.ii) and (S.iii) above, the generic orbit of X is contained in an analytic curve and X admits a meromorphic first integral $F\colon \mathbb{C}^n \dashrightarrow \overline{\mathbb{C}}$.

Sketch of the proof of Theorem 1.1. The very basic underlying idea is to compare the global dynamics of the vector field X with that of the linear model $X_{(\lambda_1,\dots,\lambda_n)}$ using the very special properties of a dicritical holomorphic flow on a Stein manifold. This idea is already present in [15] though the $n \geq 3$ dimensional case is much more delicate from the technical point of view. Special difficulties arise from the fact that in general it is not possible to extend a holomorphic or meromorphic differential form on an analytic curve to the ambient Stein manifold if the dimension is greater than 2. More precisely, we study the basin of attraction $B(X)$ of the given dicritical singularity and prove this basin is the whole manifold, since the vector fields X and $X_{(\lambda_1,\dots,\lambda_n)}$ are analytically conjugated in the corresponding basins of attraction, this will imply the global linearization of X as well as the analytic equivalence $M \cong \mathbb{C}^n$. The basic argumentation relies on the fact that an orbit of X with non-trivial homology must be contained in an analytic curve in M and this will imply that $\partial B(X)$ is analytic and then empty.

2. Proof of the linearization theorem. In this section we consider the following framework. X is a complete holomorphic vector field with isolated singularities on a Stein manifold M with $\check{H}^2(M^n, \mathbb{Z}) = 0$, $P_0 \in \operatorname{sing}(X)$ is a singularity where the first jet of X is of the form $j_1(X; P_0) = X_{(\lambda_1,\dots,\lambda_n)} = \sum_{j=1}^n \lambda_j z_j \dfrac{\partial}{\partial z_j}$, where $\lambda_j \in \mathbb{Q}_+$, $\forall\, j$. We assume that P_0 is *dicritical*.

The first step toward the proof of Theorem 1.1 is the following lemma:

Lemma 2.1. *The generic orbit of X is holomorphic to $\mathbb{C}^*$.*

Proof. Denote by $\mathcal{C}(\{\lambda_1, \dots, \lambda_n\}) \subset \mathbb{R}^2$ the convex hull of the set $\{\lambda_1, ..., \lambda_n\}$. Then we have $0 \notin \mathcal{C}(\{\lambda_1, \dots, \lambda_n\}) \subset \mathbb{R}^2$ and therefore P_0 is a singularity in the Poincaré domain for X ([1]). Since P_0 is also dicritical, by the Poincaré-Dulac normal form theorem ([1]) X is actually analytically linearizable (analytically conjugate to $X_{(\lambda_1,\dots,\lambda_n)}$) in a neighborhood of P_0 and in particular in this neighborhood its flow is periodic. This implies that:

(i) We have an attraction basin $B_{P_0}(X)$ of the singularity P_0 which is an open neighborhood of P_0 in M^n.

(ii) Every orbit of X in $B_{P_0}(X)$ is diffeomorphic to $\mathbb{C}^*$.

(iii) The flow of X is periodic on $B_{P_0}(X)$.

Since X has an open set of orbits diffeomorphic to $\mathbb{C}^*$, the generic orbit is diffeomorphic to $\mathbb{C}^*$. Indeed, by the Identity Principle we obtain from (iii) above that the flow of X is periodic on M^n and therefore every orbit is diffeomorphic to $\mathbb{C}^*$. Also, flow conjugation gives a biholomorphism taking $X\big|_{B(P_0;X)}$ onto $X_{(\lambda_1,\dots,\lambda_n)}$ on the basin $B(0; X_{(\lambda_1,\dots,\lambda_n)}) \subset \mathbb{C}^n$ which is $\mathbb{C}^n$. □

The proof of Theorem 1.1 is based on the following key proposition:

Proposition 2.2. *Assume that* $\operatorname{sing}(X)$ *is finite. Then the boundary $\partial B_{P_0}(X)$ is a finite union of analytic curves, each curve consists of a non-singular orbit L_0 of X and a single non-dicritical singularity of X at which L_0 accumulates .*

Proposition 2.2 will be proved in several steps. The first is:

Lemma 2.3. *$\partial B_{P_0}(X)$ contains no closed leaf.*

Proof. Suppose by contradiction that $L_0 \subset \partial B_{P_0}(X)$ is a closed leaf. Then $L_0 \subset M^n$ is an analytic curve isomorphic to $\mathbb{C}^*$.

Claim 2.4. *The holonomy group of L_0 is a finite (cyclic) group.*

Proof of Claim 2.4. Let $q \in L_0$ and $\Sigma_q \subset M$ be a transverse $(n-1)$-disc to L_0 at $q = \Sigma_q \cap L_0$. Denote by $h_q\colon (\Sigma_q, q) \to (\Sigma_q, q)$ local holonomy diffeomorphism corresponding to the only non-trivial loop in $L_0 \cong \mathbb{C}^*$. The germ h_q corresponds to a germ in the group $\mathrm{Diff}(\mathbb{C}^{n-1}, 0)$ in a natural way. Since $L_0 \subset \partial B_{P_0}(X)$ we have that $q = L_0 \cap \Sigma_q \in \partial B_{P_0}(X) \cap \Sigma_q = \partial(B_{P_0}(X) \cap \Sigma_q)$ so that q is a point of the boundary ∂A of an open subset $\emptyset \neq A \subset \Sigma_q$ such that if $y \in A$ then the orbit $L_y \subset B_{P_0}(X)$.

Since in $B_{P_0}(X)$ the vector field X is conjugate to the linear model $X_{(\lambda_1,\dots,\lambda_n)}$ we conclude that $X\big|_{B_{P_0}(X)}$ admits a primitive meromorphic first integral say $F\colon B_{P_0}(X) \to \overline{\mathbb{C}}^{n-1}$. In particular we have for the holonomy map $F\big|_{\Sigma_q \cap B_{P_0}(X)} \circ h_q = F\big|_{\Sigma_q \cap B_{P_0}(X)}$, i.e., $F \circ h_q = F$ in A. Finally, we recall that A is h_q invariant because $B_{P_0}(X)$ is invariant by X. By classical arguments and the Identity Principle this implies that h_q is a finite order map which can be put in the form $h_q(y_1, \dots, y_{n-1}) = (\xi_1 y_1, \dots, \xi_{n-1} y_{n-1})$ with ξ_j a root of 1, $\forall\, j$, in suitable coordinates $(y_1, \dots, y_{n-1}) \in (\Sigma_q, q)$. This proves the claim. $\square$

Claim 2.5. *There is a holomorphic one-form α in M such that $\oint_{\gamma_0} \alpha\big|_{L_0} = 1$ for a suitable non-trivial cycle $\gamma_0 \in \pi_1(L_0; q)$.*

Proof of Claim 2.5. The flow $\varphi\colon \mathbb{C} \times M \to M$ of X is a periodic say of period $\tau \in \mathbb{C}\backslash\{0\}$. For simplicity we can assume that $\tau = 2\pi\sqrt{-1}$. We introduce therefore an action $\psi\colon \mathbb{C}^* \times M \to M$ by setting $\psi(u, p) = \varphi(\log u, p)$ for any chosen branch of $\log u$ (ψ is well-defined due to the periodicity or φ). Since all orbits of X are isomorphic to $\mathbb{C}^*$, we conclude that for any point $p \in M\backslash \operatorname{sing}(X)$ the map $\psi_p\colon \mathbb{C}^* \to L_p \subset M$ is a biholomorphism. Denote now by Ω the one-form $\Omega(u) = \dfrac{du}{u}$ on $\mathbb{C}^*$ in natural affine coordinates $u \in \mathbb{C}^* \subset \mathbb{C}$. For each $p \in M\backslash \operatorname{sing}(X)$ put $\alpha_p := (\psi_p^{-1})^*(\Omega)$. Then α_p i a (closed) holomorphic one-form in L_p. We claim that if $L_{p_1} = L_{p_2}$ then $\alpha_{p_1} = \alpha_{p_2}$. Indeed, by construction we have $\big(\psi_{p_j}\big)^*\big(\alpha_{p_j}\big) = \Omega$, $j = 1, 2$. Thus, it suffices to show that $\big(\psi_{p_2}^{-1} \circ \psi_{p_1}\big)^*\Omega = \Omega$. Indeed, this is the case because if we write $p_2 = \psi_{p_1}(\lambda) = \psi(\lambda, p_1)$ for some $\lambda \in \mathbb{C}^*$, then $\big(\psi_{p_2}^{-1} \circ \psi_{p_1}\big)(u) = u \cdot \lambda^{-1}$ and therefore

$$\big(\psi_{p_2}^{-1} \circ \psi_{p_1}\big)^*\,\Omega = (u \cdot \lambda^{-1})^*\left(\frac{du}{u}\right) = \frac{du}{u} - \frac{d\lambda}{\lambda} = \frac{du}{u} = \Omega.$$

Thus, we can construct a holomorphic one-form α_L on each leaf L of $\mathcal{F}(X)$ on $M\backslash \operatorname{sing}(X)$, by the local trivialization of $\mathcal{F}(X)$ in $M\backslash \operatorname{sing}(X)$. We obtain a well-defined one-form α on $M\backslash \operatorname{sing}(X)$ with the property that $\alpha|_L = \alpha_L$, $\forall\, L \in \mathcal{F}(X)$. A classical result of Hartogs assures that α is holomorphic in $M\backslash \operatorname{sing}(X)$ because it is holomorphic along the leaves $\mathcal{F}(X)$ and also in the transverse directions. Finally, since $\operatorname{cod} \operatorname{sing}(X) \geq 2$, classical Hartogs type extension results ([5]) imply that α admits a unique holomorphic extension to M. This proves the claim. $\square$

Claim 2.6. *The restriction $\alpha|_L$ is exact for every leaf $L \subset B_{P_0}(X)$.*

Proof of Claim 2.6. Indeed, if $L \subset B_{P_0}(X)$ then $\overline{L} = L \cup \{P_0\}$ is an analytic subvariety of M isomorphic to $\mathbb{C}^* \cup \{0\} = \mathbb{C}$ and therefore simply-connected. Moreover, α is defined and holomorphic in M and therefore in a neighborhood of $\overline{L}$. Thus $\alpha\big|_L$ is exact for every leaf $L \subset B_{P_0}(X)$.

In particular we must have $\oint_{\gamma_L} \alpha|_L = 0$ for every closed cycle $\gamma \subset L$ if $L \subset B_{P_0}(X)$. On the other hand for the given cycle $\gamma_0 \in \pi_1(L_0, q)$ the corresponding holonomy map is $h_{\gamma_0} = h_q \colon (\Sigma_q, q) \to (\Sigma_q, q)$ which we have proven to of finite order. This implies that some suitable power γ_0^ℓ of γ_0 has closed lifting $\tilde{\gamma}_y$ to the leaves $L_y \in \mathcal{F}(X)$ with $y \in \Sigma_q \backslash \{q\}$ close enough to q. Thus, for $y \in \Sigma_q$ close enough to q we have $|\int_{\tilde{\gamma}_y} \alpha - \int_{\gamma_0^\ell} \alpha| < \dfrac{1}{2}$ and $\int_{\gamma_0^\ell} \alpha = \ell \int_{\gamma_0} \alpha = \ell \int_{\gamma_0} \alpha|_{L_0} = \ell$. Therefore $\int_{\tilde{\gamma}_y} \alpha \neq 0$, but $\int_{\tilde{\gamma}_y} \alpha = \int_{\tilde{\gamma}_y} \alpha|_{L_y}$ so that $\tilde{\gamma}_y \subset L_y$ is a closed loop with $\int_{\tilde{\gamma}_y} \alpha|_{L_y} \neq 0$. Since we can take $y \in \Sigma_q$ close enough to q and in $A = B_{P_0}(X) \cap \Sigma_q$ we get a contradiction with what we obtained above. This contradiction proves Lemma 2.3. $\square$

Proof of Proposition 2.2. Since the leaves of $\mathcal{F}(X)$ are closed outside $\text{sing}(X)$ we conclude from Lemma 2.3 that each leaf $L \subset \partial B_{P_0}(X)$ must accumulate on some singularity P of X, this singularity is necessarily non-dicritical because L is also accumulated by leaves L' in $B_{P_0}(X)$ and such a leaf cannot accumulate two singularities of X and therefore cannot accumulate on P.

Claim 2.7. *$\partial B_{P_0}(X)$ contains no isolated point.*

Proof of Claim 2.7. Indeed, if $P \in \partial B_{P_0}(X)$ is an isolated point then necessarily $P \in \text{sing}(X)$ and also since $B_{P_0}(X)$ is diffeomorphic to $\mathbb{R}^{2n}$, we have $M \approx \mathbb{R}^{2n} \cup \{\infty\} = S^{2n}$ which is compact, contradiction. $\square$

Summarizing the above discussion we have proved Proposition 2.2. $\square$

The last step before the proof of Theorem 1.1 is:

Lemma 2.8. $M = B_{P_0}(X) \cup \partial B_{P_0}(X)$.

Proof. Put $U = M - \partial B_{P_0}(X)$ and $V = B_{P_0}(X)$. By Proposition 2.2 the boundary $\partial B_{P_0}(X)$ is a thin set, it is a finite union of analytic curves in a complex manifold of dimension $n \geq 3$. Therefore U is connected and we have $V \subset U$ with $\partial V = \partial U = \partial B_{P_0}(X)$. Therefore, $U = V$ and $M = B_{P_0}(x) \cup \partial B_{P_0}(X)$. $\square$

Proof of Theorem 1.1. The case $n = 2$ is proved in [15] (that is the only case where we need $\check{H}^2(M^n, \mathbb{Z}) = 0$). Assume therefore $n \geq 3$. By Lemma 2.8 the flow gives a conjugation $\Psi \colon M \backslash \partial B_{P_0}(X) \to \mathbb{C}^n$ between X and $X_{(\lambda_1,\dots,\lambda_n)}$. The map Ψ writes in coordinate functions as $\Psi = (\Psi_1, \dots, \Psi_n)$ where each $\Psi_j \colon M \backslash \partial B_{P_0}(X) \to \mathbb{C}$ is holomorphic. Since $\partial B_{P_0}(X)$ has codimension $n - 1 \geq 2$, the classical Hartogs Extension Theorem gives a holomorphic extension of each Ψ_j, and therefore of Ψ, to a holomorphic map $\widetilde{\Psi}: M \to \mathbb{C}^n$. Observe that $\widetilde{\Psi}$ is also a local biholomorphism. Indeed, the singular set of $\widetilde{\Psi}$ is given by $\{\text{Det Jac } \widetilde{\Psi} = 0\}$ which is either empty or has codimension one. Finally, since $\Psi_* X = X_{\lambda_j,\dots,\lambda_n}$ on $M \backslash \partial B_{P_0}(X)$ by the Identity Principle we also have $\Psi_* X = X_{(\lambda_1,\dots,\lambda_n)}$ on M. Therefore Ψ takes orbits of X in M onto orbits of $X_{(\lambda_1,\dots,\lambda_n)}$ in $\mathbb{C}^n$. In particular, $\Psi(\partial B_{P_0}(X))$ is a finite union of orbits and singularities of X in $\mathbb{C}^n$. However, every orbit of $X_{(\lambda_1,\dots,\lambda_n)}$ is contained in the attraction basin $B_0(X_{(\lambda_1,\dots,\lambda_n)})$ of the origin and therefore (since $\Psi(B_{P_0}(X)) = B_0(X_{(\lambda_1,\dots,\lambda_n)})$) we conclude that any orbit in $\partial B_{P_0}(X)$ should be contained in $B_{P_0}(X)$, absurd. This implies that $\partial B_{P_0}(X)$ contains only singularities of X and this case we have already excluded above. Hence $\partial B_{P_0}(X) = \emptyset$ and indeed $M = B_{P_0}(X)$ and Ψ defines a conjugation between X on M and $X_{(\lambda_1,\dots,\lambda_n)}$ on $\mathbb{C}^n$. This ends the proof of Theorem 1.1. □

3. Stability for periodic orbits. The proof of Theorem 1.2 requires the following lemma:

Lemma 3.1. *A periodic orbit of a complete holomorphic vector field X on $\mathbb{C}^n$ is contained in an analytic curve which is a (complete) intersection of $(n-1)$ principal analytic subsets of codimension one.*

Proof. Denote by $\varphi: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ the flow of X and write $\varphi = (\varphi^1, \dots, \varphi^n)$ in coordinate functions. Let $p \in \mathbb{C}^n$ be a non-singular point whose orbit L_p is periodic, $L_p \cong \mathbb{C}^*$, say of period $\tau \in \mathbb{C}\backslash\{0\}$. We define the set $\Delta_j := \{z \in \mathbb{C}^n : \varphi^j_\tau(z) = z_j\}$ for $j \in \{1, \dots, n\}$. Then $L_p \subseteq \bigcap\limits_{j=1}^{n} \Delta_j$ and we have:

Claim 3.2. $\bigcap\limits_{j=1}^{n} \Delta_j$ *is an analytic subset of $\mathbb{C}^n$ of dimension ≥ 1 and it is of codimension ≥ 1 except if φ is periodic of period τ.*

Proof of Claim 3.2. We define $\pi_j: \mathbb{C}^n \to \mathbb{C}$ as the projection $\pi_j(z) = z_j$ and $f_j: \mathbb{C}^n \to \mathbb{C}$ by $f_j(z) = \varphi^i_\tau(z) - \pi_j(z)$. Then f_j is an entire function and $\Delta_j = f_j^{-1}(0)$ so that each Δ_j is an analytic subset of dimension $\geq n-1$ of $\mathbb{C}^n$. Moreover clearly φ is periodic of period τ on $\mathbb{C}^n$ if and only if $\Delta_j = \mathbb{C}^n$, $\forall\, j \in \{1, \dots, n\}$. Suppose that φ is not periodic of period τ and let $j_0 \in \{1, \dots, n\}$ be such that $\Delta_{j_0} \subsetneqq \mathbb{C}^n$. Then $\Delta \subset \Delta_{j_0}$ has codimension ≥ 1. □

□

Lemma 3.3. *Let $p \in \mathbb{C}^n$ be a non-singular point whose orbit L_p is closed and periodic. There is a holomorphic one-form Ω in $\mathbb{C}^n$ such that $\Omega\big|_{L_p} = \alpha$ writes as $\alpha(z) = \dfrac{dz}{z}$ in some suitable coordinate $z \in \mathbb{C}^*$.*

Proof. If $n = 2$ then we refer to the analogous result proven in [15]. Assume therefore that $n \geq 3$. We have $L_p \subset \bigcap_{j=1}^{n} \Delta_j$ as above. In order to fix the ideas we assume that $n = 3$. We have three cases to consider:

1^{st} **case**: $\dim\left(\bigcap_{j=1}^{3} \Delta_j\right) = 1.$

In this case, since each $\Delta_j \in \mathbb{C}^3$ is a principal analytic subset (of codimension ≤ 1) we can suppose that $\Delta_1 \neq \Delta_2$ and $\Delta_1 \subsetneqq \mathbb{C}^3$, $\Delta_2 \subsetneqq \mathbb{C}^3$ but $\Delta_3 \subset \Delta_1 \cap \Delta_2$. So that L_p is an irreducible component of the set $\Delta_1 \cap \Delta_2$.

Since $\Delta_1 \neq \Delta_2$, the restriction $g_2 = f_2\big|_{\Delta_1}$ does not vanish identically and actually L_p is an irreducible component of $\{q \in \Delta_1 : g_2(q) = 0\}$. Now, Δ_1 is a Stein analytic subspace of $\mathbb{C}^3$ and $g_2 \colon \Delta_1 \to \mathbb{C}$ is holomorphic, $g_2 \not\equiv 0$, therefore by classical Cartan's Extension Theorem ([7]) we have a holomorphic extension α_1 to Δ_1 of a one-form α defined on L_P by $\alpha(z) = \dfrac{dz}{z}$ on a suitable coordinate $z \colon L_P \to \mathbb{C}^*$ (notice that, since L_P is closed in $\mathbb{C}^n$, L_P is an analytic curve in Δ_1 and therefore we can suppose that $L_P = g_2^{-1}(0)$). The one-form α_1 on $\Delta_1 = \{q \in \mathbb{C}^3 \colon f_1(q) = 0\}$ also extends to a holomorphic one-form Ω on $\mathbb{C}^3$ by the same theorem of Cartan.

This proves the lemma in this first case.

2^{nd} **case**: $\dim\left(\bigcap_{j=1}^{3} \Delta_j\right) = 3.$

In this case $\Delta_1 = \Delta_2 = \Delta_3 = \mathbb{C}^3$ and the flow φ of X is periodic of period τ on $\mathbb{C}^3$. In this situation we can replace φ by a holomorphic $\mathbb{C}^*$-action $\psi \colon \mathbb{C}^* \times \mathbb{C}^3 \to \mathbb{C}^3$ with the same orbits. Now by arguments of [16] and [13] ($\mathbb{C}^*$ is a reductive Lie group), we know that all the orbits of φ are contained in analytic curves in $\mathbb{C}^3$ and we can define a one-form Ω on $\mathbb{C}^3$ representing the Lie Algebra of $\mathbb{C}^*$.

3^{rd} **case**: $\dim\left(\bigcap_{j=1}^{3} \Delta_j\right) = 2.$

We can assume that $\Delta_3 \subsetneqq \mathbb{C}^3$ but $\Delta_1 = \Delta_2 = \mathbb{C}^3$.

Claim 3.4. *Δ_3 is invariant by X.*

Proof of Claim 3.4. Let $q \in \Delta_3$ and take $\tilde{q} = \varphi_s(q)$ for some $s \in \mathbb{C}$. We have to show that $\tilde{q} \in \Delta_3$, that is, $f_3(\tilde{q}) = 0$, which is equivalent to $\varphi_\tau^3(\tilde{q}) = \tilde{q}_3$ if we write $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) \in \mathbb{C}^3$. Now, $\varphi_\tau^3(\tilde{q}) = \varphi_\tau^3(\varphi_s(q)) = \varphi_s^3(\varphi_\tau(q))$ by the flow condition. Since $q \in \Delta_3$ we have $\varphi_\tau(q) = q$ so that $\varphi_\tau^3(\tilde{q}) = \varphi_s^3(q)$. By choice $\tilde{q} = \varphi_s(q)$ so that $\tilde{q}_3 = \varphi_s^3(q)$ and therefore $\varphi_\tau^3(\tilde{q}) = \tilde{q}_3$ □

Since Δ_3 is invariant by X we have an induced holomorphic action $\tilde{\varphi} = \varphi\big|_{\Delta_3 \times \mathbb{C}} \colon \Delta_3 \times \mathbb{C} \to \Delta_3$ and L_p is an orbit of $\tilde{\varphi}$ of period τ and closed in Δ_3.

Also we can, repeating for $\tilde{\varphi}$ arguments already used for φ, consider only two cases:

(3.i): $\tilde{\varphi}$ is periodic of period τ on Δ_3.

In this case $\tilde{\varphi}$ is associated to a $\mathbb{C}^*$-action $\tilde{\psi}$ on Δ_3 and we can construct a holomorphic one-form $\widetilde{\Omega}$ on Δ_3 corresponding to the Lie Algebra of $\mathbb{C}^*$. The one-form $\widetilde{\Omega}$ extends to a holomorphic one-form Ω on $\mathbb{C}^3$.

(3.ii): $\tilde{\varphi}$ is not periodic on Δ_3 and L_P is given by an analytic equation $\{\tilde{f}_3 = 0\}$ for some holomorphic function $f_3 \colon \Delta_3 \to \mathbb{C}$, $\tilde{f}_3 \not\equiv 0$. Again, by Cartan's Theorem, the one-form $\widetilde{\Omega}$ is obtained on Δ_3 by extension of $\alpha(z) = \dfrac{dz}{z}$ on $L_P \cong \mathbb{C}^*$, and by its turn $\widetilde{\Omega}$ extends to $\mathbb{C}^3$. $\square$

Proof of Theorem 1.2. If $n = 2$ the result is proven in [15]. Assume therefore $n \geq 3$. By Lemma 3.3 there is a holomorphic one-form Ω on $\mathbb{C}^n$ such that $\alpha = \Omega\big|_{L_P}$ writes as $\alpha(z) = dz$ in a coordinate $z \colon L_P \to \mathbb{C}^*$. Now there is a loop $\gamma \colon S^1 \to L_P$ such that

$$\int_\gamma \Omega = \int_\gamma \Omega\big|_{L_P} = \oint_{L_p} \alpha = \oint_{S^1} dz/z = 2\pi\sqrt{-1}, \text{i.e.}, \frac{1}{2\pi\sqrt{-1}} \int_\gamma \Omega = 1.$$

Since by hypothesis the holonomy of the leaf L_P is finite, say of order k, there is a transverse $(n-1)$-disc $\Sigma \subset \mathbb{C}^n$, $\Sigma \cong \mathbb{D}^{n-1} = \{z \in \mathbb{C}^{n-1} \colon |z| < 1\}$, such that $\Sigma \cap L_P = \{P\}$ (notice that L_P is closed so that and the holonomy group we can assume $\#(\Sigma \cap L_P) = 1$).

$\mathrm{Hol}(\mathcal{F}, L_P, \Sigma)$ of the leaf L_P of the foliation $\mathcal{F}$ induced by φ is (conjugate to) a finite subgroup of $\mathrm{Diff}(\mathbb{C}^{n-1}, 0)$ of order k. In particular the k-th power $[\gamma^k] \in \pi_1(L_P, P)$ has a closed holonomy lifting $[\tilde{\gamma}_y^t] \in \pi_1(L_y; y)$ to each leaf L_y of $\mathcal{F}$ through $y \in \Sigma$.

We have $\displaystyle\int_{\tilde{\gamma}_y^k} \Omega = \int_{\tilde{\gamma}_y^k} \Omega\big|_{L_y} = \oint_{\tilde{\gamma}_y^k} \Omega\big|_{L_y}$.

Since $\displaystyle\oint_{\gamma^k} \Omega = k \oint_\gamma \Omega = 2\pi k\sqrt{-1}$ if $q \in \Sigma$ is close enough to P then also $\displaystyle\int_{\tilde{\gamma}_y^k} \Omega \neq 0$ and therefore $\Omega\big|_{L_y}$ is a (closed) holomorphic one-form on the Riemann surface L_y whose line integral along the closed path $\tilde{\gamma}_y^k \in \pi_1(L_y, y)$ is not zero. This implies that L_y is not simply-connected and therefore necessarily L_y is isomorphic to $\mathbb{C}^*$. We also claim that L_y is closed analytic in $\mathbb{C}^n$. Indeed, otherwise since a periodic orbit is closed in $\mathbb{C}^n \backslash (\mathcal{F})$, L_y accumulates a single singularity say Q of $\mathcal{F}$. In particular $\overline{L}_y = L_y \cup \{Q\}$ is a subvariety of $\mathbb{C}^n$ isomorphic to $\mathbb{C}$ and is simply-connected. Since Ω is defined in the space $\mathbb{C}^n$ it also admits a restriction to $\overline{L}_P$ and therefore necessarily $\displaystyle\int_{\tilde{\gamma}_y^k} \Omega = 0$ yielding a contradiction in the above notation.

This shows that there is an invariant open subset W containing L_P and no singularity of $\mathcal{F}$ such that every orbit (leaf) in W is isomorphic to $\mathbb{C}^*$ and is a closed analytic subset of $\mathbb{C}^n$. This implies that the generic orbit of φ is isomorphic to $\mathbb{C}^*$. □

As a straightforward corollary of the proof of Theorem 1.2 we obtain:

Corollary 3.5. *Let X be a complete holomorphic vector field with isolated singularities on $\mathbb{C}^n$, $n \geq 2$. Given a closed orbit $L_P \subset \mathbb{C}^n$ (for some $P \in \mathbb{C}^n$) isomorphic to $\mathbb{C}^*$ and of finite holonomy there is an invariant open neighborhood W of L_P in $\mathbb{C}^n$ such that W contains no singularity of X and each orbit in W is closed isomorphic to $\mathbb{C}^*$ and of finite holonomy group. In other words the set*

$$\mathcal{H} = \{P \in \mathbb{C}^n \colon L_P \text{ is closed isomorphic to } \mathbb{C}^* \text{ and has finite holonomy group}\}$$

is an invariant open subset of $\mathbb{C}^n \setminus \operatorname{sing}(X)$.

4. Some questions and conjectures. We give a general construction:

Example 4.1. Let $D \subset \mathbb{C}^n$ be a bounded domain and, $(n \geq 1)$, and denote by $\operatorname{Aut}(D)$ the (topological) group of automorphisms of D endowed with the compact open topology. A classical result of H. Cartan ([8]) states that $\operatorname{Aut}(D)$ is a real Lie group whose Lie algebra $\operatorname{Aut}(D)$ consists of all complete holomorphic vector fields $X\colon D \to \mathbb{C}^n$. In particular we can have $D \subset \mathbb{C}^n$ bounded and Stein. This shows that there is a large collection of Stein manifolds equipped with holomorphic flows and our main result shows that the existence of a non-degenerate dicritical singularity is actually a strong restriction on both, the complete vector field and the ambient Stein manifold.

4.1. Conjectures. We think that (1) and (2) below are true:

(1) *On a Stein surface M^2 with $\overset{\vee}{H}{}^2(M,\mathbb{Z}) = 0$ a complete holomorphic vector field without singularities and generic orbit $\mathbb{C}^*$ has all orbits isomorphic to $\mathbb{C}^*$.*

As a consequence of the work of Nishino ([9], [10]), Saito ([14]) and Suzuki ([17]) the above conjecture is true for complete holomorphic vector fields on $\mathbb{C}^2$.

(2) *A complete holomorphic vector field X on $\mathbb{C}^2$ with isolated singularities and having a singularity at the origin of the form $X(x,y) = nx\,\dfrac{\partial}{\partial x} - my\,\dfrac{\partial}{\partial y}$, $n, m \in \mathbb{N}$; is analytically linearizable on $\mathbb{C}^2$. The same holds if we replace $\mathbb{C}^2$ by a Stein surface M^2 with $\overset{\vee}{H}{}^2(M,\mathbb{Z}) = 0$.*

This is proved in [3] for complete *polynomial* vector fields.

4.2. Questions. Our study motivates the following questions.

(1) Given a holomorphic flow φ on $\mathbb{C}^2$, is it possible to have an exceptional minimal subset? Can an orbit diffeomorphic to $\mathbb{C}$ accumulate on a non-closed orbit also diffeomorphic to $\mathbb{C}$?

(2) Is is true that a periodic flow on a Stein manifold M^n, $n \geq 2$, always admits a meromorphic first integral $M \dashrightarrow \overline{\mathbb{C}}^{n-1}$?

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